Surreal Numbers Presentation Outline

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Abstract

Surreal numbers are a field that contain both the reals as well as infinitely large and infinitely small numbers. Similar to Dedekind cuts, surreals are cosntructed using two sets of previously created numbers, and the subsequent properties that emerge create a rich set of properties. While a still largely unexplored area, this paper discusses some elementary properties of surreals, construction, comparisons, and operations. In addition, interesting computations involving various forms of infinity will be described, as well as applications in the field of games and combinatorial game theory.

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1 Introduction to Surreal Numbers

All of us are familiar with the very basic number systems; these would include the integers, rationals, reals, complex, etc. However, there is another set of numbers introduced recently by John Horton Conway via Donald Knuth's 1974 Mathematical novelette, "Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness."

More formally, these numbers form a field that contain reals as well as infinite and infinitesimal numbers. They have interesting properties as well as useful applications that we will talk about today.

1.1 Construction

Every number in the surreals corresponds to two sets of previously created numbers with one restriction: no member of the left set is greater than or equal to any members of the right set:

$$x = (X_L, X_R)$$

with the restriction that

$$X_L \ngeq X_R$$

This simply means that given an surreal x_L in X_L, and for any number x_R in X_R, we must have $x_L \ngeq x_R$ for this to be a valid surreal. Note: for the purposes of notation, I will choose to use lowercase letters for numbers, while uppercase letters will usually mean sets of numbers.

The obvious question is what numbers exist in the first place to get these set constructed numbers started. Conway realized that the natural start would be when both X_L and X_R are taken to be the empty set $\emptyset$. Therefore we can see that zero can be defined as follows:

$$0 = (\emptyset, \emptyset)$$

We must be careful that this is indeed a valid surreal number. Based on the definition, we know it must satisfy $X_L \ngeq X_R$. Since both sets are empty, we see that this is immediately true, as no element of the left set is greater than an element of the right set as there are no elements at all.

From here, it is not hard to build more and more surreals. We now have 0 available to us to use as part of the left or right set. For example, we can define -1 as follows:

$$-1 = (\emptyset, \{0\})$$

and similarly we can define 1 as follows:

$$1 = (\{0\}, \emptyset)$$

We do a quick check with each of these numbers to confirm they are well formed. Since both of them have an empty set, we again see that the condition is satisfied, as there are no elements in either the left or right set for comparison.

One interesting note is that if either X_L or X_R is the empty set, the condition $X_L \ngeq X_R$ is always true, and therefore we can see immediately that infinitely many numbers can be created in this fashion.

1.2 Comparison Definition

We now define what it means for surreal numbers to be less than or equal. The usual notation for something like this is $x \leq y$. For this to be true in the surreal numbers, we must have:

$$X_L \ngeq y$$

and

$$x \ngeq Y_R$$

Specifically, every element of X_L satisfies the condition $x_L \ngeq y$, and that x is not greater than or equal to any element of Y_R.

With this definition, we can show some elementary properties of the numbers we have shown to exist so far. For example, using our definitions we can show that $0 \leq 0$. Since $0 = (\emptyset, \emptyset)$, we see that indeed no number of the left set of 0 is greater than or equal to 0, and no number of the right set of 0 is less than or equal to 0. Therefore we have shown that $0 \leq 0$.

In addition, we should be able to show other properties of these numbers that make intuitive sense, such as -1 not being greater than or equal to 0. That is, we aim to show $0 \nleq -1$ where $-1 = (\emptyset, \{0\})$.

For $0 \leq -1$ to be true, we require that $x \ngeq Y_R$, where Y_R in this case is $\{0\}$. But we have shown earlier that indeed, $0 \geq 0$, so therefore this is not true and we have that -1 is not greater than or equal to 0.

One final example to convince that these rules produce an intuitive ordering for the surreals, we will show that -1 is less than 1. By the inequality rule, we see that $-1 \leq 1$. We want to also show that $1 \nleq -1$. $1 \leq -1$ implies that $0 \ngeq -1$, but we have shown that this is false. Therefore it must be that $-1 < 1$.

1.3 Transitive Law

In comparing numbers so far, we have taken the general approach of making sure that, for example, every element of X_L is not greater than or equal to y.

One observation we can make is that it seems logical that when comparing numbers, it is unnecessary to go through all of them—what seems to matter is the largest element of X_L. That is, if any particular element in X_L is $\geq y$, then the largest one should be as well. This applies as well in the case where we are comparing x with Y_R; we should only have to examine the smallest element of Y_R.

If this result holds, then it means that any surreal (X_L, X_R) will act exactly the same in the comparison sense if the left set is replaced by the largest element and the right set replaced by the smallest.

In essence, what we need to prove is transitivity. That is, for surreal numbers x, y, and z,

$$\text{if } x \leq y \text{ and } y \leq z, \text{ then } x \leq z.$$

For the purposes of this presentation, I will show this informally and sketch a proof by contradiction. Let us assume that there exist three numbers such that:

$$x \leq y, \text{ and } y \leq z, \text{ and } x \nleq z.$$

From our inequality rules, we can see that if these numbers exist, then the following is also true:

$$X_L \ngeq y$$

$$x \ngeq Y_R$$

$$Y_L \ngeq z$$

$$y \ngeq Z_R$$

We also know that $y \leq z$, $z \leq x_L$ and $y \not\leq x_L$, so this is yet another set of numbers that do not work with transitivity. More specifically, it can be shown that there are only two possible cases for every "bad" x, y, z triple:

Case 1: There exists some $x_L \geq z$, and therefore (y, z, x_L) are another triplet of numbers that do not satisfy transitivity.

Case 2: There exists some $z_R \leq x$, and therefore (z_R, x, y) are another bad triplet.

Where does this get us? Well, in either case, the newly constructed triplet is a simpler version of the original (x, y, z). In other words, using induction we can show that if a set of numbers like this does exist, then it is impossible to keep finding simpler and simpler sets of these. Since we cannot keep doing this indefinitely, there cannot exist any numbers that do not follow transitivity at all.

1.4 Reasoning Directly/Indirectly

To further our understanding of surreal numbers, we will now show a few interesting properties. First we ask the question, is it true that two numbers are always related to each other? In other words, could it ever be the case that

$$x \not\leq y \text{ and } y \not\leq x$$

To show that this cannot happen, I first present the following theorem without proof:

Theorem A: *Let x be a surreal number. Then $x \leq x$*

Now assume x and y are numbers that are not related to each other in the sense we have defined above. Then for some x_L or y_R, one of the following must be true:

$$x_L \geq y \text{ or } x \geq y_R$$

We examine this case by case. If $y \leq x_L$ and if $x_L \leq x$, we then would have $y \leq x$ by transitivity, and by assumption $y \not\leq x$. Therefore $x_L \not\leq x$.

For the other case where $y_R \leq x$, the same logic would show that this implies $y \not\leq y_R$. Now all that is left to show is an important concept within surreal numbers, that of every number lying in between all the elements of its left and right sets. In other words:

Theorem B: $X_L \leq x$ *and* $x \leq X_R$

We prove one half of this—the other half follows by an analogous argument. For the sake of contradiction, assume that $x_L \nleq x$. Then there is either a number x_{LL} in X_{LL} such that $x_{LL} \geq x$, or there exists a number x_R in X_R such that $x_L \geq x_R$. We know this latter case is impossible from the construction of surreal numbers.

As for the first case, we notice that since x_{LL} is an element of the left set of x_L, we can see that $x_{LL} \leq x_L$ by induction. We then have:

$$x \leq x_{LL} \text{ and } x_{LL} \leq x_L$$

which by transitivity implies that $x \leq x_L$. But that is impossible, because it would imply that $X_L \ngeq x_L$, which would mean that x_L is contained in X_L. Along with Theorem A, this reasoning implies that$X_L \leq x$. The other part that we need to show, $x \leq X_R$ can also be proved by a similar argument.

With these results, we no longer have to write and reason indirectly in comparisons. In other words, $x \ngeq y$ is identical to $x < y$.

2 Equivalence and Arithmetic

2.1 Equivalence

As mentioned before, an interesting insight is that when comparing numbers, it is unnecessary to go through every one in a set. In particular, all that seems to matter is the largest element of X_L, and the smallest element of Y_R. In this part of the presentation, we discuss this issue in more depth.

Take a given number $x = (X_L, X_R)$. Also given are arbitrary sets of numbers Y_L and Y_R, with the constraint that:

$$Y_L < x < Y_R$$

Now define a new number z as follows:

$$z = (Y_L \cup X_L, X_R \cup Y_R)$$

The claim is that x "is like" z. This appeals to our theory that creating larger sets X_L and X_R by adding numbers that do not affect the max/min of the appropriate sets does not significantly change x in the comparison sense.

Clearly z is a well formed number. We now want to show two things:

$$z \leq x \text{ and } x \leq z$$

For the first part, we need to demonstrate that:

$$Y_L \cup X_L < x \text{ and } z < X_R$$

However, it is easy to see why this is true. By assumption, $Y_L < x$ and we know $X_L < x$ and $z < X_R \cup Y_R$ from the previous section. The same argument can be used to prove that $x \leq z$. Therefore we have shown the following:

Theorem C: *if* $Y_L < x < Y_R$, *then* $x \equiv (Y_L \cup X_L, X_R \cup Y_R)$

Where we use the notation $x \equiv z$ to denote "x is like z".

2.2 Addition and Subtraction with Surreals

The concept of addition with surreal numbers is a very natural one. Informally, the left set of the sum of two numbers is the sum of all left parts of each number, and the right set of the sum is the sum of all right parts of each number.

More formally, we can write the sum of two surreals as:

$$x + y = ((X_L + y) \cup (Y_L + x), (Y_R + x) \cup (X_R + y))$$

Conway also defined the negative of a number very precisely as having as its sets the negatives of the number's opposite sets. Formally:

$$-x = (-X_R, -X_L)$$

Subtraction was naturally defined to be the addition of the negative.

Now we can do an example to show how addition works with surreals:

Example 1: Say we want to calculate $1+1$. We recall that $1 = (\{0\}, \emptyset)$. Using Conway's rule, we get:

$$1 + 1 = (\{0+1, 0+1\}, \emptyset)$$

To further simplify this, we also calculate what $0+1$ is. Recalling that $0 = (\emptyset, \emptyset)$, we get that:

$$0 + 1 = (\{0 + 0\}, \emptyset)$$

One last calculation to do, and that is to find out what $0 + 0$ is. Once again, using the addition rule we find that:

$$0 + 0 = (\emptyset, \emptyset) = 0$$

Substituting back, we see that $0 + 1 = (\{0\}, \emptyset) = 1$. From this, we finally see that:

$$1 + 1 = (\{1\}, \emptyset)$$

We can call this number $(\{1\}, \emptyset) = 2$. We've just proved that $1 + 1 = 2$!

Example 2: What is $1 + \frac{1}{2}$?

Given that $1 = \{0, \emptyset\}$ and $\frac{1}{2} = \{0, 1\}$ we get:

$$1 + \frac{1}{2} = \{(0 + \frac{1}{2}) \cup (1 + 0), (\emptyset + \frac{1}{2}) \cup (1 + 1)\}$$

$=\{(0 + \frac{1}{2}) \cup (1 + 0), 1 + 1\}$

Simplifying the sums, we see that:

$=\{(\frac{1}{2}) \cup (1), 2\}$
$=\{1, 2\}$
$=1\frac{1}{2}$

2.3 Multiplication

Although operations with multiplication are somewhat complex, we will go over the formula for the sake of completeness. If you have to surreals x and y, then their product is the following:

$$xy = \{(X_L y + x Y_L - X_L Y_L) \cup (X_R y + x Y_R - X_R Y_R), (X_L y + x Y_R - X_L Y_R) \cup (X_R y + x Y_L - X_R Y_L)\}$$

where

$$XY = \{xy | x \text{ element in } X \text{ and } y \text{ element in } Y\}$$

$$Xy = X\{y\}$$

$$xY = \{x\}Y$$

Example 1: What is $0 \times x$?

For this example of Surreal multiplication, we proceed given the formula above. We see that this becomes:

$0x = \{(\emptyset x + 0X_L - \emptyset X_L) \cup (\emptyset x + 0X_R - \emptyset X_R), (\emptyset x + 0X_R - \emptyset X_R) \cup (\emptyset x + 0X_L - \emptyset X_L)\}$
$= \{\emptyset, \emptyset\}$
$= 0$

Example 2: What is $1 \times x$?

Again, using the definition we get:

$1x = \{(0x + 1X_L - 0X_L) \cup (\emptyset x + 1X_R - \emptyset X_R), (0b + 1X_R - 0X_R) \cup (\emptyset x + 1X_L - \emptyset X_L)\}$
$= \{1X_L, 1X_R\}$
$= x$

3 Examples and Applications

3.1 Numerical Examples

As mentioned earlier in the presentation, the surreals contain the real numbers. This, however, is not yet clear from the discussion so far. To see how the surreals can represent every real number, we recall that every real number is just an infinite decimal or binary expansion. I will illustrate this idea with a few examples:

Example 1: How do we represent the real number $\frac{1}{3}$?

We know we can represent this as the following binary expansion:

$$\frac{1}{3} = .0101010101...$$

Which is of course non terminating. So far we have only given examples of surreal numbers with finite left and right sets. To represent $\frac{1}{3}$, we introduce the idea of choosing either X_L or X_R as infinite sets.

For example, we can choose the left set X_L to approach the binary expansion from below, as such:

$$X_L = \{.01, .0101, .010101, .01010101, ...\}$$

Similarly, we also need an infinite set of numbers that approach the binary expansion from above for the right set:

$$X_R = \{.1, .011, .01011, .0101011, .010101011, ...\}$$

Since we proved that the number lies between the left and right sets, we see that $x = (X_L, X_R)$ will represent the number $\frac{1}{3}$.

Example 2: How would we represent π?

Similar to the previous example, we first represent π as a binary expansion:

$$\pi = 11.00100100001111...$$

We can now represent the surreal number with the following left and right sets:

$$\Pi_L = \{11.001, 11.001001, 11.00100100001, ...\}$$

which continues indefinitely by stopping at every 1.

$$\Pi_R = \{11.1, 11.01, 11.0011, 11.00101, ...\}$$

which comes about by stopping at every 0 and then adding a 1 to the sequence. These sequences converge to π from above and below.

An important observation is that there are many ways to choose these sets to represent the same real number. In fact, there are infinitely many ways to choose X_L and X_R such that the surreal number is equivalent.

3.2 Surreals and Infinity

One of the most interesting properties of surreal numbers is the way the concept of infinity is defined and manipulated. The definition is in fact very natural. We let:

$$X_L = \{1, 2, 3, 4, 5, 6, ...\}$$

and for the right set, we define:

$$X_R = \emptyset$$

Clearly, this number, denoted as $\omega = (X_L, X_R)$ is larger than all other numbers and is the infinity of surreal numbers.

Let us explore a related concept in a different sense. Consider the number $\epsilon = (X_L, X_R)$ where:

$$X_L = \{0\}$$
$$X_R = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$$

We can see that based on the rules for surreal numbers, ϵ is a number greater than zero and less than all positive real numbers. This is an extremely important idea, and shows how surreal numbers can fill in the "gaps" between the real numbers.

Even more interesting is how our rules allows us to manipulate numbers such as ω and ϵ in ways that are not intuitive to us. We can even see things such as which infinity is larger, and how to add and subtract from them.

Example 1: What is $\omega + 1$?

Our rules show us that $\omega + 1 = (\{\omega, 2, 3, 4, 5, ...\}, \emptyset)$. This can simplify to:

$$\omega + 1 \equiv (\{\omega\}, \emptyset)$$

Example 2: What is $\omega + 2$?

It is easy to show that this is equivalent to:

$$\omega + 2 \equiv (\{\omega + 1\}, \emptyset)$$

Example 3: What is $\omega - 1$?

Applying Conway's subtraction rule, we get:

$$\omega - 1 \equiv (\{1, 2, 3, 4, ...\}, \{\omega\})$$

What is the intuition for a number such as $\omega - 1$? It is basically the first surreal number created that is larger than every integer, but still less than ω. In other words, this is an infinite number less than infinity.

3.3 Surreal Numbers and Game Theory

Surreal numbers have their origins within the theory of games. It has been said that Conway originally came up with the idea for surreal number construction after intensely studying "Go."

In particular, the restriction that every element of the left set must be less than every element of the right set is dropped, and then this more general class of elements is known as games.

The construction rule for a game is as follows:

> **Construction:** If L and R are two sets of games then $\{L \mid R\}$ is a game.

All the familiar rules for addition, subtraction, multiplication, etc. that we have defined earlier for surreal numbers is valid in the context of games. It is important to note that since games are less restricted than surreals, every surreal is a game but not every game is a surreal number. While we showed that surreals have a total order, games only have a partial order.

Technically, a "move" in a game is when the player whose turn it is chooses a game from the left set (if he is the left player) or the right set (if he is the right player), and then passing the game to the opponent. The loss condition comes when the player whose turn it is cannot move because his choice is from an empty set.

References

[1] D.E. Knuth. *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness: a Mathematical Novelette.* Addison-Wesley, 1974.

[2] J.H. Conway. *On Numbers and Games.* A.K. Peters, 2001.

[3] Wikipedia Foundation, *Surreal Numbers*, viewed 9 April 2006, <http://en.wikipedia.org/wiki/Surreal_numbers>.