12.009 Problem Set 2

Due Thursday, 17 February 2011

1. *Thermal diffusion in the seafloor* As discussed in class, a section of seafloor has a thermal structure which is determined by the diffusion of heat from the hot mantle, through the seafloor, to the cold ocean. We model this as a slab which has two temperature boundary conditions at the top and the bottom.

   (a) We start with the example of the infinite half plane discussed in class which is equivalent to making the seafloor slab of infinite thickness. Using Matlab, plot the temperature profile $T(z,t)$ in the slab as a function of z , for a few specific choices of time t and a thermal diffusivity of unity. This plot shows how heat diffuses into a uniform, cold slab. Put these profiles together on the same graph.

   (b) Define a characteristic penetration depth $d(t)$ for the diffusion length and plot it on the graph from part 1a. How does $d(t)$ scale with time?

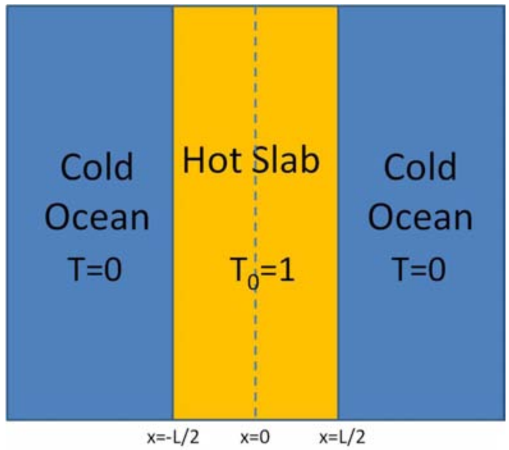
   (c) Let us return to the case of a seafloor slab of finite thickness. Suppose that the slab has thickness L and with boundary conditions T_1 and T_2 for the bottom and top of the slab, respectively:

   What is the temperature profile $T(z)$ after steady state is reached?

   (d) Alright, so this is the steady state solution, but how exactly does the system approach steady state? If we imagine that the slab itself starts at T_2, then heat would diffuse upwards from the hot mantle into the cold slab. Estimate how long it will take for the system to reach steady state.

2. *Separation of variables.* Now for the long and involved part. In class we solved the diffusion equation in the infinite half-plane case by seeking a similarity solution which turned the partial differential equation into an ordinary differential equation. This worked because there was no length scale in the problem. Here we address a problem in which a clear length scale exists (as in, e.g., the case of a slab of rock with a finite thickness). To do this we will introduce a classic method called *separation of variables.* We would like to answer the question, "How does the seafloor reach thermal equilibrium between the cold seawater and the hot mantle?" However, the hot-cold asymmetry adds an additional twist, which to be frank, makes it quite hard. Instead of the more realistic configuration expressed in 1c above, we will instead solve a toy problem expressed by this diagram:



As you will see, we shall not only introduce separation of variables, but also provide this course's first taste of Fourier Series...

(a) State the diffusion equation in one dimension.

(b) What makes the diffusion equation harder to solve than a normal ordinary differential equation? Explain why the similarity solution discussed in class fails here.

(c) Suppose that we can write temperature in the following form:

$$T = X(t)Y(x), \tag{1}$$

where X depends only on t and Y only on x. Show that by assuming this form for T we can express the diffusion equation as

$$\frac{1}{X}\frac{\partial X}{\partial t} = D\frac{1}{Y}\frac{\partial^2 Y}{\partial x^2}. \tag{2}$$

(d) We can now demonstrated something very cool. Equation (2) allows us to write

$$\frac{1}{X}\frac{\partial X}{\partial t} = \pm\lambda^2 \tag{3}$$

$$D\frac{1}{Y}\frac{\partial^2 Y}{\partial x^2} = \pm\lambda^2 \tag{4}$$

where $\pm\lambda^2$ is either a positive or negative *separation constant*. Why does equation (2) force equations (3) and (4) to hold? Give this some thought. It is the conceptual crux of the entire method.

(e) At this point we have two choices. Either the separation constant is positive or negative giving us two pairs of equations. (You can ignore the possibility that $\lambda = 0$) Write the general solution to each of the four differential equations. (For the second order ODE, where the solution is a sum of sines and cosines, ignore the sine solutions.)

(f) We seek the temperature profile $T(z, t)$ inside the slab, assuming the two cold oceans are maintained at constant temperture with an initial condition $T_0 = 1$ inside the slab. State the boundary conditions formally.

(g) Show that the boundary conditions require that $\pm\lambda^2 < 0$ (i.e., $+\lambda^2$ is disallowed). Do not worry for the moment about satisfying the initial temperature condition inside the slab.

(h) Write down the solutions to the two differential equations (one for t and one for x) and show that the boundary conditions force

$$\lambda = \frac{n\pi\sqrt{D}}{L}, \qquad (n = 1, 2, 3 \ldots). \tag{5}$$

(i) You have now solved for all of the $X(t)$ and $Y(x)$ pairs which satisfy the boundary conditions. This gives you an infinite set of temperature profiles T which each satisfy the boundary conditions. Show that the sum

$$T(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(\frac{-(n\pi)^2 Dt}{L^2}\right) \cos\left(\frac{n\pi x}{L}\right), \tag{6}$$

where the C_n are constants, is both a solution to the diffusion equation and satisfies the boundary conditions on T.

(j) Now we are ready to look at the initial condition for the temperature profile. At $t = 0$, $T = T_0$ in the slab, our solution must match the initial temperature profile. The coefficients C_n must therefore satisfy

$$T_0 = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right). \tag{7}$$

To solve for C_n, first multiply both sides of equation (7) by

$$\cos\left(\frac{n'\pi x}{L}\right), \tag{8}$$

where n' is a new index, and integrate from $-L/2$ to $L/2$:

$$\int_{-L/2}^{L/2} T_o \cos\left(\frac{n'\pi x}{L}\right) dx = \sum_{n=1}^{\infty} C_n \int_{-L/2}^{L/2} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n'\pi x}{L}\right) dx. \tag{9}$$

Solve this equation for C_n and use this to write the solution for $T(x,t)$ in terms of a summation. (Hint: start by integrating

$$\int_{-L/2}^{L/2} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n'\pi x}{L}\right) dx, \qquad n = 1, 2, 3 \ldots, \qquad n' = 1, 2, 3 \ldots. \tag{10}$$

If this is causing you grief try plotting

$$\cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n'\pi x}{L}\right) \tag{11}$$

for $n = 1$, $n' = 1$ and $n = 1$, $n' = 2$.)

(k) Use Matlab to plot the temperature profile, $T(x, t)$, as a function of x for a few different times t between 0 and 1 to show how heat diffuses out of the slab. To keep things clear, let $D = 1$ and $L = 2$. How does the number of terms you use from the series qualitatively influence what you observe? Why do we not require any of the sine solutions?

(l) Return for a moment to problem 1c. What will happen if we try to apply this method in order determine how the system approaches steady state?

12.009 Theoretical Environmental Analysis
Spring 2011