12.009 Problem Set 5

Due Tuesday April 5

1. *Perturbations in Eccentricity*: Earth orbits around the sun in an elliptical orbit with the Sun as one of the foci, (to a very very good approximation). The orbital eccentricity ϵ varies with a period of $\sim 100,000$ years.

   (a) What are the equations for the semi-major and semi-minor axis as a function of the orbital energy, E, the angular momentum l, and orbital constants?

   (b) The fluctuations in eccentricity are mostly attributed to interactions with other planets.

      i. If these interactions were to provide a single impulse to the Earth, under what conditions would the energy of the orbit stay the same?

      ii. Assuming that the size of an impulse is small, where on the orbit and in what direction must the impulse be applied to maintain the orientation of the major axis? There is more than one answer. Find the simplest solution. Draw a diagram and explain your answer.

2. *Rotation of the Major Axis*: Without any perturbations from the outside, a planet orbiting in an elliptical orbit will remain in the same orbit forever. It turns out, however, that the stability of these fixed ellipses is tied very closely to the $1/r$ potential in which the planets sit. In this problem, we will explore what happens if we add a small perturbing force,

$$f \propto \frac{1}{r^4}, \tag{1}$$

   to our system. This higher order term comes about both from interactions with other planets and through the results of General Relativity. To make our lives easier we will first convert our orbital equation from one in terms of the potential, to one centered on the force.

   (a) Starting with the equation for the orbital energy derived in class,

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r), \tag{2}$$

   show that an equivalent expression can be written in terms of the force field, $F(r)$,

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r). \tag{3}$$

   Remember that

$$F(r) = -\frac{\partial U}{\partial r}. \tag{4}$$

   (b) By enforcing the gravitational potential and including the following definitions:

$$\frac{1}{\alpha} \equiv \frac{Gm_1^2 m_2}{l^2}, \qquad u = \frac{1}{r} \tag{5}$$

show that equation 3 reduces to

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha}. \tag{6}$$

(c) Show that the conic section solution for the gravitational potential,

$$u = \frac{1}{\alpha}\left(1 + \epsilon \cos(\theta)\right), \tag{7}$$

is a solution to equation 6.

(d) At this point, we are ready to explore the consequences of including a small term which depends on r^{-4} to $F(r)$. This allows us to alter equation 6 to get

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \tag{8}$$

where δ is a very small constant. Because the δu^2 term is small it can be solved through a series of successive guesses where we start by guessing the solution of equation 6 and plug it into equation 8. Successive guesses are changed to get us closer to the real solution. For this problem, we find that

$$u \cong \frac{1}{\alpha}\left(1 + \epsilon \cos(\theta)\right) + \frac{\delta\epsilon}{\alpha^2}\theta \sin(\theta). \tag{9}$$

i. How is this solution consistent with the successive guessing approach?

ii. Show, remembering the small angle approximation, that our solution can be simplified to

$$u \cong \frac{1}{\alpha}\left[1 + \epsilon \cos\left(\theta - \frac{\delta}{\alpha}\theta\right)\right]. \tag{10}$$

iii. Make a few plots, using MATLAB, of $r(\theta)$ for varying values of δ, α and ϵ. To start off with, try $\delta = \pi/40$, $\alpha = 1$ and $\epsilon = .5$. You might find `polar` a useful MATLAB function.

iv. Using your plots and the information contained in equation 10 explain how changes in δ effect both the quantitative and qualitative behavior of eliptic orbits. Provide physical reasoning.

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