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Restriction to hypersurfaces of non-isotropic Sobolev spaces

by

MOHAMED MEKIAS

M.S., Massachusetts Institute of Technology (1987)

D.E.S., Universite de Setif, Algeria (1984)

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Signature redacted

Signature of author ______________________

Department of Mathematics
March 11, 1993

Signature redacted

Certified by ______________________

David S. Jerison
Professor of Mathematics
Thesis Supervisor

Signature redacted

Accepted by ______________________

Sigurdur Helgason, Chairman
Departmental
Graduate Committee

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Abstract

Let $X_1, ..., X_{2n}$ be smooth linearly independent vector fields in an open set $\Omega \subset R^{2n+1}$. We form the nonisotropic Sobolev spaces S^2_α for $\alpha > 0$, by measuring smoothness in terms of the X_j's. These function spaces are the natural ones to consider when dealing with operators of the form $X_1^2 + \cdots + X_{2n}^2$. In particular, in the Dirichlet problem associated with these operators the problem of restriction to the boundary comes up naturally. Let M be a smooth hypersurface of Ω. In this thesis the restriction problem is investigated. It is shown that many results that hold, concerning the restriction problem, for the isotropic Sobolev space L^2_α have analogues in the nonisotropic setting, in particular the result $L^2_\alpha|_M = L^2_{\alpha-\frac{1}{2}}$. When the index α is small $\frac{1}{2} < \alpha < \frac{3}{2}$, we have complete characterization of the space of restrictions; $S^2_\alpha|_M = F^2_{\alpha-\frac{1}{2}}(M)$, this latter space is described by similar smoothness conditions to the classical Sobolev spaces using the first diffrences. We merely replace the distance function by the nonisotropic one and the surface measure by a weighted measure, this weight is precisely the angle made by the tangent space and the span of the vector fields X_j's. If $d\sigma$ denotes the surface measure on M, we show also that the space S^2_1 admits restrictions to M that are members of $L^2(d\sigma)$, this result is sharp; in the case of surfaces where the span of the X_j's is nowhere tangent the condition is $\alpha > \frac{1}{2}$. For higher indices α, we use a method due to Jonsson and Wallin to describe the restriction spaces.

Thesis Supervisor : David Jerison
Title : Professor of Mathematics

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بِسمِ اللهِ الرَّحمٰنِ الرَّحِيمِ وَ صَلَّى اللهُ علَى سَيِّدِنَا مُحَمَّد

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وَ الحَمدُ لِلهِ رَبِّ العَالَمِين .

Contents

Chapter 1

Introduction

Let $X_1, \cdots, X_m$ be smooth vector fields defined on some manifold Ω. Assume further that this system of vector fields satisfies a finite step Hörmanders' condition, that is $X_1; \cdots; X_m$ and commutators of finite length span the tangent space at each point . Identifying the X_j's with the directional derivatives, we will measure smoothness of functions on Ω in terms of these vector fields. We form the Sobolev type spaces denoted by S_k^p whose members f are characterized by the requirement that $f \in S_{k-1}^p$ and $X_j f \in S_{k-1}^p$, locally, for $1 \leq j \leq m$ and $k = 1; 2; \cdots$. We, of course, adopt the convention that $S_0^p = L^p$ and the derivatives are in the sense of distributions. These spaces are the natural ones to consider when dealing with differential operators of the form $X_1^2 + \cdots + X_m^2$ and its variants, see for example [DE]. Now in the same way as the classical Sobolev spaces L_k^p, the question of restriction to the boundary of the members of S_k^p is a natural one and it is of importance, in particular, to the Dirichlet problem and other boundary value problems, in [JE2] the Dirichlet problem for the Kohn laplacian was studied from a nonisotropic point view using the nonisotropic Lipshitz spaces. More precisely, one may formulate the restriction problem in the following way: Let M be a lower dimensional submanifold of Ω, which we assume to be of codimension one and we ask

Question 1. What is the space $S_k^p|_M$?

An answer to this question consists of characterizing smoothness conditions for functions defined on M, so that members of S_k^p satisfy them on M and conversely, given a function g defined on M satisfying these conditions we should be able to extend it so as to lie in S_k^p. To try to answer this question in its full generalities as stated above would be an ambitious task and even a foolish one.

In this thesis we examine the particular case of what is called a contact manifold (or sometimes refered to as CR manifold) and the situation is as follows:

1. We assume that there are $m = 2n$ linearly independent vector fields $X_1, \cdots, X_{2n}$ and that Hörmander condition is a step 2 one.

2. The dimension of Ω is equal $2n + 1$

The system of vector fields $X_1, \cdots, X_{2n}$ span a hyperplane of the tangent space of Ω, and thus, the data given by 1. and 2. are equivalent to a given nondegenerate field of hyperplanes i.e.; a contact structure on Ω. Locally, this situation is modelled by the Heisenberg group and the function spaces S_k^p as well as S_α^p for α not integer are now realized as potential spaces, that is, there is a function J_α such that $S_\alpha^p = L^p * J_\alpha$, convolution is group convolution. Once we realize these spaces in this manner one may approach the restriction problem by trying to follow the same lines as the classical restriction problem see [ST2] or [JW] and the litterature listed there. Let us limit our discusion to the case $p = 2$, all that follow have analogue to $p \neq 2$. Recall that we have (see [ST2])

$$L_\alpha^2(R^n)|_{R^{n-1}} = \Lambda_{\alpha-\frac{1}{2}}^2(R^{n-1}) \quad for \quad \alpha > \frac{1}{2} \tag{1.1}$$

the spaces on the right hand side of (1.1) are the classical Besov spaces (which in this case ($p = 2$) coincide with the Sobolev spaces themselves) see [ST] for more on these spaces, the result holds for general smooth hypersurface M. For $0 < \alpha < 1$, a norm on the space $\Lambda_\alpha^2(R^n)$ is

$$||f||_{L^2} + \left(\int\int_{|x-y|\leq 1} \frac{|f(x) - f(y)|^2}{|x-y|^{2\alpha+n}} dxdy\right)^{1/2} \tag{1.2}$$

A similar version of (1.2) involving derivatives is used to define $\Lambda_\alpha^2(R^n)$ for higher α's. We will seek similar norms as (1.2) to charactrize the space of restrictions. In the noncharacteristic case (see definition next chapter) similar expression can be used to identify the space of restrictions of S_α^2 when α is small i.e.; $0 < \alpha < 1$, see [ME]. We merely replace the distance function in the denominator of the integrand of (1.2) by the appropriate one, namely the nonisotropic one (see next chapter), in particular the restriction of S_α^2 to noncharacteristic hypersurfaces is contained in $L^2(d\sigma)$ when $\alpha > \frac{1}{2}$. This leads us to another question, namely Question 2 : If $d\sigma$ is the surface measure, when do we have the embedding $S_\alpha^2|_M \subset L^2(d\sigma)$? or what is the relationship (if any) between the space of restrictions $S_\alpha^2|_M$ and the classical Sobolev spaces on M?

For general M, it turns out that the answer to the question 2 is when $\alpha \geq 1$, this is theorem 5 chapter 3, the reason for this is precisely the possibilty of tangency of the surface to the field of hyperplanes at some points. The result for $\alpha > 1$ can be gotten cheaply by the following method, embed S_α^2 in $L_{\alpha/2}^2$ according to proposition 5 of chapter 2, and then restrict using (1.1), the condition $\alpha > 1$ is necessary by this method. This makes the case $\alpha = 1$ interesting.

Now we go back to Question 1. Because it is important how M sits inside Ω, by seeking norms similar to (1.2) to characterize the space of restrictions we have to have a norm that incorporates this information. It turns out that there is a natural object to consider namely, the angle made by M (i.e.; its tangent space) and the field of hyperplanes giving us the contact structure. This function, which we denote by w, has the right weighting as well as scaling properties see (next chapter). Now set $d\mu = wd\sigma$. This new measure $d\mu$ is the right measure to combine with the system of nonisotropic balls in the same way the surface measure $d\sigma$ is the right one to consider with the system of Euclidian balls, it gives a certain homogeneous dimension to the surface. Resuts of this thesis are best expressed in terms of some function spaces denoted by F_α^2 for small values of α and B_α^2 for large values of α. For $0 < \alpha < 1$ we take expression (1.2) above and replace the denominator of the integrand by

the nonisotropic distance and the surface measure by the measure $d\mu$ we get the norm for the space F^2_α. Now we are able to prove the analogue to (1.1) above

$$S^2_\alpha|_M = F^2_{\alpha-\frac{1}{2}} \quad for \quad \frac{1}{2} < \alpha < \frac{3}{2} \tag{1.3}$$

The defintion of F^2_α for $\alpha \geq 1$ is available to us by the use of local polynomial approximation (see definition next chapter). These polynomials are taken in some coordinate system in which M is just the hyperplane. We should mention that we do have $S^2_\alpha|_M \subset F^2_{\alpha-\frac{1}{2}}$, the trouble lies in the extension part i.e.; the reverse of this inclusion. Let us indicate how a Whitney extension theorem is proved in the Euclidian case. First take the so called Whitney decomposition (see chapter 4 or [ST1]) of the complement of M in the ambient space. On each Whitney ball B define the extension of a function f defined on M to be equal the polynomial, in local coordinates, that best approximates f on an appropriate subset of M(say for example $B^* \cap M$, where B^* denotes a ball with same center as B but radius 10 times the radius of B) in the $L^2(d\sigma)$-sense, we should mention that the polynomials are extended in a natural way to the ambient space to be constant along the vertical direction. We can make this more precise by using a partition of unity to get an extension $\mathbf{E}(f)$ of f to the ambient space. In proving $||\mathbf{E}(f)||_{L^2_{\alpha+\frac{1}{2}}} \leq C||f||_{\Lambda^2_\alpha}$ the following trivial fact is used

$$\int_B |p|^2 dvol \leq Cr \int_{B\cap M} |p|^2 d\sigma \tag{1.4}$$

To be able to carry on in the nonisotropic case we need an inequality of the type (1.4) with B in (1.4) replaced by nonisotropic ball and $d\sigma$ by $d\mu$. Unfortunetly, in this setting (1.4) is false, except in two cases 1. when M is noncharacteristic and 2. when p is a constant. Heuristically, the reason for this is the fact that our balls are tilted and as we approach the characteristic set they become flat on M and thus if we try to extend polynomials to be constant along the vertical direction they exit too quickly from the balls before they get known inside the solid ball.

In the case of small α's we do have extension this way, and that's why we were able to prove (1.3). To remedy this crisis we have to pay a price by leaving the spaces F^2_α and replacing them by somewhat smaller ones , namely the spaces B^2_α. These spaces consists of system of functions (see the definition at the end of chapter 2) and when considered on the ambient space each system in B^2_α consists of a function and all its derivatives up to a certain order, and this means that the single function determines uniquely the whole system. In the Euclidian setting the spaces F^2_α and B^2_α are equivalent since we can prove restriction and extension theorems to both spaces. The question as to whether these spaces on M coincide in the nonisotropic setting is not trivial. Combining theorems 6 and 8 we state the result in the following form

$$S^2_\alpha|_M = B^2_{\alpha-\frac{1}{2}} \tag{1.5}$$

An extension for the spaces F^2_α, for $\alpha \geq 1$, to the ambient space would be possible if we could somehow extend polynomials on M (in some local coordinate system) to the ambient space by allowing them to lose their character of being polynomials, but we should be able to control them, for example in (1.4) above if the p on the left denotes the extension of p on the right we may still have that inequality. In the case of the characteristic hyperplane of the Heisenberg group, this idea is currently under investigation and progress seems promising at least to extend functions from F^2_α to $S^2_{\alpha+\frac{1}{2}}$ in the case where $0 < \alpha < 2$. This method consists of extending the coordinate functions to be constant along a certain field direction, and hence giving an extension to polynomials forcing them to stay longer in balls. Generalization to arbitrary M may be possible if it has isolated characteristic points.

Let us sketch the plan of this thesis. In chapter 2 we present the preliminary background, section 2.1 is meant to be expository and gives various definitions and notions that are equivalent in different setting and it essentially justifies the transfer of the original question to the setting of the Heisenberg group. An important object of this work is the weight function w. In section 2.2 we prove interesting (elementary) facts about this function. Section 2.3 represents known definitions and facts about the nonisotropic Sobolev and Besov spaces, the main references for this are [FO],[FS], [NS] and [SA], we also introduce the spaces on the boundary F^p_α and we follow the construction of [JW] to define the spaces B^p_α. Chapter 3 is the restriction theorem, the main theorems are theorems 5 and 6 and chapter 4 is the converse to the restriction theorem. Finally we close by mentioning how possible extensions of our results are possible to the case fo $p \neq 2$, it contains essentially the L^p analogue to theorem 5.

Chapter 2

Preliminary Background

2.1 The nonisotropic distance

We collect the background necessary to desribe the nonisotropic geometry of the problem; we also present the notation used in this thesis. This section is meant to be only expository and therefore no proofs will be given, we refer to [NA],[NS] and [NSW]. We assume that $\Omega \subset R^{2n+1}$ is some smooth open set, since the results are local. Let $\{X_1, \cdots, X_{2n}\}$ be a set of linearly independent smooth vector fields defined on a neighborhood of $\bar{\Omega}$, and satisfying the step-two Hörmanders' condition i.e.; the system of vectors $\{X_1|_p, \cdots, X_{2n}|_p, [X_j, X_k]|_p, 1 \leq j, k \leq 2n\}$ spans $R^{2n+1} (\cong T_p\Omega)$ for all $p \in \Omega$. The objective of this section is to define the nonisotropic distance reflecting basically the noncommutativity of the vector fields, as well as Hörmanders' condition. There are two main approaches to defining suitable distances associated with a set of vector fields. The first one is global and it is the well known control distance d. The second definition is local and based on the exponential map, and thus depends on the base point. Let us discuss briefly these two constructions. We denote by X_{2n+1} the missing direction i.e.; we choose from the commutators $\{[X_j, X_k]\}$ one that completes the system $\{X_1, \cdots, X_{2n}\}$.

2.1.1 Global definitions

Definition 1 *Let $r > 0$; x_0 and x_1 be two points of Ω. We say that $d_1(x_0, x_1) \leq r$ if there is a (piecewise smooth) map $\phi : [0,1] \to \Omega$ satisfying:*
(i) $\phi(0) = x_0$ *and* $\phi(1) = x_1$ *;*
(ii) $\frac{d\phi}{dt} = \sum_{j=1}^{2n+1} a_j(t) X_j(\phi(t))$ *for almost every* $t \in [0,1]$
with $|a_j(t)| < r$ *for* $1 \leq j \leq 2n$ *and* $|a_{2n+1}(t)| < r^2$.

The following proposition gives the first comparison of the distance d and the Euclidian distance which we denote by $||.||$, for a proof of this see [NS].

Proposition 1 *d is a metric on Ω, and for every compact set $\omega \subset\subset \Omega$, there are constants C_1 and C_2 such that*

$$C_1||x_0 - x_1|| \leq d(x_0, x_1) \leq C_2||x_0 - x_1||^{1/2} \tag{2.1}$$

There are several variants of the control distance. One of them for example we get by restricting the right hand side of (ii) in definition 1 above to be a constant linear combinations. Another construction is by requiring the vector $\frac{d\phi}{dt}$ be only in the field of hyperplanes which is not all of the space. The fact that d is distance is easy to verify but what is the volume of a corresponding ball of radius r? In general it is a hard question to tell just from the definition of d what shape these balls take, and what their volume is. In order to solve this problem we define a local metric based on canonical coordinates (i.e.; the exponential map), that would make the shape as well as the volume of these balls transparent.

2.1.2 Local definitions

Let $p \in \Omega$ be fixed, and let $\vec{v} \in T_p\Omega \cong R^{2n+1}$ be a tangent vector, since $\{X_1|_p, \cdots, X_{2n+1}|_p\}$ spans, $\vec{v}$ can be written as $\sum_{j=1}^{2n+1} a_j X_j|_p$. Now the vector field $\sum_{j=1}^{2n+1} a_j X_j$ is smooth near p and coinciding with $\vec{v}$ at p. So, we may flow along the integral curves of this vector field for unit time if the coefficients a_j are small enough, we get this way the exponential map based at p,

$$Exp_p : U_0 \to V_p \qquad (a_1, \cdots, a_{2n+1}) \to Exp_p(\sum_{j=1}^{2n+1} a_j X_j) \tag{2.2}$$

U_0 and V_p are respectively, a'small neighborhood of the origin in R^{2n+1} and of p in Ω, we choose them small enough so that the map (2.2) above is a diffeomorphism. The Jacobian of the transformation is the determinant of $(X_1, \cdots, X_{2n+1})$, i.e.; the volume of the parallelopiped spanned by the vector fields. We give weight 1 to the vector fields $X_1, \cdots, X_{2n}$ and weight 2 for X_{2n+1}. Set

$$Box(r) =: \{\vec{v} = \sum_{j=1}^{2n+1} \lambda_j X_j : ((\sum_{j=1}^{2n} \lambda_j^2)^2 + \lambda_{2n+1}^2)^{\frac{1}{4}} \leq r\} \tag{2.3}$$

and let, for r small enough, $B_p(r)$ be defined by

$$B_p(r) =: Exp_p(Box(r)) \tag{2.4}$$

Let us denote by $x(p)$ the coordinate representation of the point p, then it is worth noticing that the system of balls defined by equation (2.4) is equivalent to the following system of balls

$$\tilde{B}_p(r) := \{x(p) + s_1 X_1|_p + \cdots + s_{2n+1} X_{2n+1}|_p : |s_1|, \cdots, |s_{2n}| \leq r \ \ and \ |s_{2n+1}| \leq r^2\} \tag{2.5}$$

From (2.4) and (2.5) above we easily see that the balls are sets that looks like tilted ellipsoids that sits on the span of the X_j's of size r^{2n} and of thickness r^2. And hence the volume of a ball of radius r is equal to Cr^{2n+2}. The number Q= 2n+2 is termed the homogeneous dimension of Ω. As we have said in the introduction everything we said so far can be rephrased in the language of contact structure and the Heisenberg group, we include a brief discussion about these.

2.1.3 The contact structure and the Heisenberg group

Let Ω be any smooth manifold of dimension N.

Definition 2 *A contact structure on Ω is a given smooth nonintegrable field of hyperplanes of the tangent space of Ω, satisfying a nondegeneracy condition described below.*

The nonintegrability condition means that there is no integral hypersurface to the field. An example of a field of hyperplanes may be given by the zero set of a 1-form, and conversely, every smooth field of hyperplanes is locally given by the zero set of a 1-form. This form is unique up to a nonvanishing smooth factor. If we impose a normalizing condition on the form, it becomes uniquely determined by the field of hyperplanes. Let us denote by θ this form. Now we state the nondegeneracy condition by saying that the bilinear form $d\theta$ restricted to the field of hyperplanes is nondegenerate i.e.;

$$rank(d\theta|_{\theta=0}) = N - 1$$

This condition forces the manifold Ω to be odd dimensional, this is because the bilinear form $d\theta$ is skew symmetric and if N is even then $N-1$ is odd, and as is well known from linear algebra, there are no nondegenerate skew symmetric forms on an odd dimensional space. Set $N = 2m+1$, the nondegeneracy condition may also be stated in a fancy way by the condition

$$\theta \wedge (d\theta)^m \neq 0$$

Sometimes Ω with a contact structure is called a CR manifold. The standard and important example of a contact structure is the Euclidian space R^{2m+1} with coordinates $(x_1, \cdots, x_m, y_1, \cdots, y_m, t) = (x, y, t)$ and the field of hyperplanes given by the one form

$$\theta = dt + 2(xdy - ydx) := dt + 2\sum_{j=1}^{m}(x_j dy_j - y_j dx_j) \tag{2.6}$$

(In [AR] it is given by $dt + xdy$, but it is clear that they are equivalent.) The following is the analogue to the well known Darboux's theorem in symplectic geometry. It states that every point of Ω has a neighborhood and a coordinate system where the contact form takes the form (2.6) identically, unlike [FS2] where they proved that the form coincides with one of the form (2.6) only at that point.

Theorem 1 *Every differential 1-form defining a nondegenerate field of hyperplanes (i.e.; a contact structure) on some odd dimensional manifold, can be written in a local coordinate system (x, y, t) in its canonical form (2.6) above.*

This theorem is in appendix 4 page 362 of [AR]. It justifies the fact that it suffices to transfer our problem to the Heisenberg group whose definition is the following.

Definition 3 *The Heisenberg group H_n is the Lie group whose manifold realization is the Euclidian space R^{2n+1} with standard coordinate functions $(x_1, \cdots, x_n, y_1, \cdots, y_n, t) = (x, y, t)$ and whose group law is given by*

$$(x,y,t)(x',y',t) = (x+x', y+y', t+t'+2(yx'-xy'))$$

The identity element of H_n is $(0, \cdots, 0)$, and the inverse to a general element $u = (x, y, t)$ is $u^{-1} = (-x, -y, -t)$.
The vector fields,

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t} \qquad X_{j+n} = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t} \quad 1 \le j \le n \tag{2.7}$$

form a basis for the left invariant vector fields (i.e.; the Lie algebra) The commutation relations are $[X_{j+n}, X_j] = 4T$ and all others are zero. Thus the system of vector fields $X_1, \cdots, X_{2n}$ satisfies the step-two condition. The corresponding 1-form is equal to θ in (2.6) and satisfies : $\theta(X_j) = 0$ for $j = 1, \cdots, 2n$ and $\theta(T) = 1$
Next we define the nonisotropic distance to the origin $|.|$. It is equivalent to the one given as a control distance with respect to the vector fields X_j given by (2.7). For $u = (x, y, t)$ we set

$$|u| := ((\sum_{j=1}^{n}(x_j^2 + y_j^2))^2 + t^2)^{1/4}$$

and

$$d(u,v) = |v^{-1}u| = |u^{-1}v|$$

We have the following triangle inequality

Lemma 1 *There is a constant $C > 1$ such that:*
1. $|uv| \le C(|u| + |v|)$
2. $|u+v| \le C(|u| + |v|)$
3. $||u|| \le |u| \le ||u||^{1/2}$ *for* $|u| \le 1$

Next we define dilations by which scaling is performed. For $r > 0$ and $u \in H_n$ we denote by ru, dilation by r,

$$ru = r(x,y,t) = (rx, ry, r^2t) \tag{2.8}$$

this change of variables gives the change in the volume by

$$dV(ru) = r^Q dV(u) \tag{2.9}$$

the number Q is the homogeneous dimension. Another important formula we'll be needing is the following

$$\int_{a\le|u|\le b} |u|^{\alpha-Q}dV(u) = \begin{cases} C(b^\alpha - a^\alpha) & \text{if } \alpha \ne 0 \\ C\log(b/a) & \text{if } \alpha = 0 \end{cases} \tag{2.10}$$

from which it follows that $|u|^{\alpha-Q}$ is locally integrable if $\alpha > 0$ and integrable at infinity if $\alpha < 0$. The number $-Q$ is the critical power index.
We need the following notations, J will always stand for a multiindex $J = (j_1, \cdots, j_{2n+1})$, let us also denote by X_{2n+1} the vector field T, for $u = (x_1, \cdots, x_{2n+1}) \in H_n$ set

$$u^J = x_1^{j1} \cdots x_{2n+1}^{j_{2n+1}} \quad X_J = X_1^{j_1} \cdots X_{2n+1}^{j_{2n+1}}, and \quad |J| = j_1 + j_2 + \cdots + j_{2n} + 2j_{2n+1}$$

The left invariant differential operators are linear combinations of the X_J's. Another important notion is that of convolution, when we say convolution of two functions f and g it is always meant convolution in the Heisenberg group, and this is defined as follow

$$f * g(u) := \int_{H_n} f(v)g(v^{-1}u)dV(v)$$

it is important to mention that, unlike the ordinary convolution, it is not a commutative operation. The interaction of the left invariant differential operators with the convolution we just defined is the following

$$X_J(f * g) = f * (X_J g)$$

2.2 The weight function w

Let now M be a smooth hypersurface of H_n given locally as the zero set of a smooth function ρ, that is $M = \{u \in H_n : \rho(u) = 0\}$ and $\nabla\rho \neq 0$ on M. The tangent space TM of M sits in a natural way in the tangent space of H_n as a smooth integrable field of hyperplanes and thus makes an angle $w(x)$ with the one defining the contact structure. This angle is thought of as a function on M, and the purpose of this section is to measure it. Let us make the following important definition

Definition 4 *A point x of the hypersurface M is said to be characteristic if the tangent space of M, T_xM, coincides with the given field of hyperplanes defining the contact structure i.e.; characteristic points are those where w vanishes.*

A surface M may or may not have characteristic sets. This set can be a point a curve or in general a lower dimensional set of M, and hence if there are any, characteristic points form a set of surface measure zero. The geometric structure of the characteristic set can take a complicated form, however lemma 5 below or rather its proof suggests that the Hausdorff dimension is at most equal to $2n - 1 = dimension - 2$.
An example of characteristic M is what we call in this thesis the characteristic hyperplane, in the canonical coordinates (x, y, t) it is the hyperplane $\{t = 0\}$, it is easy to see that the vector fields $X_j, \cdots, X_{2n}$ given by (2.7) are tangent at 0. This is an example of an isolated characteristic set. The family of surfaces $\{t = c(||x||^2 + ||y||^2)\}$ provides with examples of surfaces having isolated characteristic point, namely the origin. This family plays the role of cones in the nonisotropic sense. Another example is given by, in the Heisenberg group,

the surface $\{t = x_j^2 - y_j^2\}$, the characteristic set here is the manifold $\{x_j = y_j\}$. Next we compute this angle to be roughly equal to

$$w(x) = (\sum_{j=1}^{2n}(X_j\rho)^2(x)))^{1/2} \tag{2.11}$$

it important to note that this function is not smooth.
The following series of lemmas give the size of the function w. Lemma 2 gives the pointwise size of the function w, lemma 3 gives the ordinary surface measure of balls intersected with M. Lemma 4 is one of the most important result it says that the weighted measure $wd\sigma$ respects the system of nonisotropic balls in the same way the surface measure respects the ordinary Euclidian balls, that is the measure of any nonisotropic ball of radius r is a constant power of the radius throughout the hypersurface M, and thus giving the hypersurface M *a homogenous dimension* equal to one less of that of the ambient space . All of these lemmas are proved using Taylor expansion arguments. We work in a general coordinate system $(x_1, \cdots, x_{2n+1})$ where now M is the hyperplane $\{x_{2n+1} = 0\}$, we write the expression of the vector fields in these coordinate system

$$X_j = \sum_{i=1}^{2n} a_{j,i}\frac{\partial}{\partial x_i} + a_j\frac{\partial}{\partial x_{2n+1}} \tag{2.12}$$

We could have used the canonical coordinates (i.e.; the vector fields given by (2.7)).

Lemma 2 *Let* $x_0 \in M$ *and set* $\eta = w(x_0) \geq 0$, *then*
(a) $w(x) \leq C(r + \eta)$ *for all* $x \in B_{x_0}(r) \cap M$
(b) *If* $\eta > 0$, *then* $w(x) > \eta/2$ *for all* $x \in B_{x_0}(c\eta)$
(c) $\max_{x \in B(r)\cap M} w(x) \geq Cr$

Proof:

The function $w(x)^2$ is smooth and equal

$$w(x)^2 = \sum_{j=1}^{2n} a_j(x_1, \cdots, x_{2n}, 0)^2$$

each a_j is the $\frac{\partial}{\partial x_{2n+1}}$-component of the vector field X_j. Let $x^0 \in M$ which we assume to be the origin. By Taylor formula applied to the function $w(x)^2$ we have

$$w(x)^2 = \sum_{j=1}^{2n}(a_j(0) + \varepsilon_j(x))^2$$

with $\varepsilon_j(x) \leq C|x|$, and thus

$$w(x)^2 \leq \sum_{j=1}^{2n} a_j^2(0) + C|x|\sum_{j=1}^{2n}|a_j(0)| + C|x|^2$$

from which it follows that $w(x)^2 \le C(\eta + r)^2$, proving (a). As for (b), we have that

$$w(x)^2 \ge \sum_{j=1}^{2n} a_j(0)^2 + 2\sum_{j=1}^{2n} a_j(0)\epsilon_j(x) \ge \eta^2 - C|x|\eta \ge \eta^2/2$$

as soon as $|x| \le 3\eta/4C$.
To prove (c) let us make the following important remark.

Remark 1 *Suppose that $0 \in M$ is close to the characteristic set, so that $w(0)$ is small, then there is at least one $j \in \{1, \cdots, 2n\}$ such that $||\nabla_{R^{2n}} a_j(0)|| \ge C > 0$ (where $\nabla_{R^{2n}}$ is the 2n-dimensional gradient), otherwise Hörmander condition would be violated. To see this, since the X_j's and the commutators span there is a uniform constant C such that:*

$$0 < C \le \sum_{j=1}^{2n}(X_j(x_{2n+1}))^2 + \sum_{j,k}([X_j, X_k](x_{2n+1}))^2 = w^2(x) + \sum_{j,k}(X_j a_k - X_k a_j)^2$$

$$= w^2(x) + \sum_{j,k}(\sum_{l=1}^{2n}(a_{j,l}\frac{\partial a_k}{\partial x_l} - a_{k,l}\frac{\partial a_j}{\partial x_l}) + (a_j\frac{\partial a_k}{\partial x_{2n+1}} - a_k\frac{\partial a_j}{\partial x_{2n+1}}))^2$$

$$\le Cw^2(x) + C\sum_{l=1}^{2n}||\nabla_{R^{2n}} a_l||^2$$

from which the remark follows.

It is obvious, in proving (c) it suffices to assume that $w(x) \le Cr$, for all points in $B(r) \cap M$, and this means that the ball B is near the characteristic set. Pick any point which again we assume to be the origin such that $a_j(0)$ is small, by Taylor expansion up to order 1 now we have

$$w(x)^2 = \sum_{j=1}^{2n}(a_j(0) + \nabla a_j(0).(x) + \varepsilon_j(x))^2$$

Because the point 0 is near being characteristic, by the previous remark there is at least one $j \in \{1, \cdots, 2n\}$ such that $\max_{|x| \le \epsilon} |\nabla a_j(x)| > 0$, and thus we have

$$w(x)^2 \ge \sum_{j=1}^{2n} a_j^2(0) + \sum_{j=1}^{2n}(\nabla a_j(0).x)^2 - C|x|^2 \sum_{j=1}^{2n}|\nabla a_j(0).x|$$

passing to the maximum we get

$$\max_{x \in B(r)} w(x)^2 \ge Cr^2 - C'r^3 \ge Cr^2$$

and this proves (c) and hence the lemma. □
The next lemma is an estimate on the ordinary surface measure of nonisotropic balls.

Lemma 3 *(a)* *There are constants C_1 and C_2 such that:* $C_1 r^{2n+1} \leq \sigma(B \cap M) \leq C_2 r^{2n}$
(b) *If we set* $\max_{x \in B(r) \cap M} w(x) = \eta$ *then we have* $\sigma(B(r) \cap M) \cong r^{2n+1}/\eta$
in particular if M has no characteristic points we have $\sigma(B(r) \cap M) \cong r^{2n+1}$

Proof: To prove this lemma we fix the point x^0 and we look at the ball in terms of normal coordinates. Since the map (2.2) above is a diffeomorphism the surface measure $\sigma(B(r) \cap M)$ is essentially the $2n$-dimensional Lebesgue measure of the intersection of the hyperplane $T_{x^0}M$ and $Box(r)$. Now estimates in (a) become obvious, the extreme cases are when the tangent space $T_{x^0}M$ coincides with the hyperplane $\{\lambda_{2n+1} = 0\}$, in which case the surface is Cr^{2n}, and the other case is when $T_{x^0}M$ coincides with one of the hyperplanes $\{\lambda_j = 0\}$, $j \leq 2n$, in which case the surface is Cr^{2n+1}, and (a) is proved. To prove (b) we write down the equation of the hyperplane $T_{x^0}M$ in the coordinates $\lambda = (\lambda_1, \cdots, \lambda_{2n+1})$, which is given by

$$(\nabla \rho|_{\lambda=0}).\lambda = 0$$

where ρ is a defining function of M in the coordinates λ i.e.; $M = \{\lambda : \rho(\lambda) = 0\}$. We need to notice that

$$w(x^0)^2 = \sum_{j=1}^{2n} (\frac{\partial \rho}{\partial \lambda_j}(0))^2 \tag{2.13}$$

Now if H denotes any hyperplane

$$\lambda_{2n+1} = a_1\lambda_1 + \cdots + a_{2n}\lambda_{2n}$$

by a simple rotation argument it easy to see that

$$L^{2n}(Box(r) \cap H) \cong \frac{r^{2n+1}}{(a_1^2 + \cdots + a_{2n}^2)^{\frac{1}{2}}}$$

as long as $(a_1^2 + \cdots + a_{2n}^2)^{1/2} \geq Cr$. L^{2n} is the $2n$-dimensional Lebesgue measure. Now taking into account (2.11) above (b) follows.
□

The upper bound in the estimate given to us by the following lemma is used in the proof of the restriction theorem and the lower bound is used in the extension theorem.

Lemma 4

$$\int_{B(r) \cap M} w(x) d\sigma(x) \cong r^{2n+1}$$

Proof: According to lemma 2(a) and lemma 3(b) we have that

$$\int_{B(r) \cap M} w(x) d\sigma(x) \leq C(r + \eta)\sigma(B(r) \cap M) \leq Cr^{2n+1} + C(\frac{r}{\eta})r^{2n+1}$$

now since η can be at least Cr the ($\leq$) part follows. For the reverse inequality, let $x^0 \in B(r) \cap M$ be such that $\eta = w(x^0) = \mathbf{max}_{x \in B(r/2) \cap M} w(x)$, then by lemma 2(b) $\eta \geq Cr$ and

by lemma 1 (b) $w(x) \geq C\eta$ for all points x in $B_{x^0}(\eta/10) \cap M$. We have two cases either
1. $B_{x^0}(\eta/100) \subset B(r)$ in which case they are actually equivalent and we have

$$\int_{B(r)\cap M} w(x)d\sigma(x) \geq C\int_{B_{x^0}(r/10)\cap M} w(x)d\sigma(x) \geq Cr\sigma(B_{x^0}(r/10) \cap M) \cong Cr^{2n+1}$$

or
2. $B(r) \subset B_{x^0}(C\eta)$ in which case $w(x) \geq C\eta$ throughout the ball $B(r) \cap M$, and in this case too we have

$$\int_{B(r)\cap M} w(x)d\sigma(x) \cong C\eta\sigma(B(r) \cap M)$$

using lemma 2 (b) we get the desired result. □ The following lemma is key in proving theorem 5.

Lemma 5 *For all* $0 < \varepsilon < 1$, *the function* $w(x)^{-\varepsilon}$ *is locally integrable, and have the following estimate*

$$\int_{B\cap M} w^{-\varepsilon}(x)d\sigma(x) \leq C(\max_{x\in B\cap M} w(x))^{-\varepsilon}\sigma(B \cap M)$$

Proof:
The fact that if w vanishes at some point it can only do so to first order by the preceding remark implies that $w^{-\varepsilon}$ is locally integrable, also the ($\geq$) part is obvious. Let $x^0 \in B(r) \cap M$ be a point, which we can assume to be the origin, such that the maximum is reached there i.e.; $w(0) = \eta = \max_{x\in B(r)\cap M} w(x)$. We have to consider two cases like before:
case 1. $\eta > 10r$ in which case the function w continues to be larger than r, in fact it is almost constant ($\cong \eta$) on the ball $B(r)$, and therefore we get in this case

$$\int_{B(r)\cap M} w^{-\varepsilon}d\sigma \cong \eta^{-\varepsilon}\sigma(B(r) \cap M)$$

case 2. $\eta \cong r$, in which case the ball $B(r) \cap M$ is a Euclidian ball. Write $w(x)^2$ in its Taylor formula:

$$w(x)^2 = \sum_{j=1}^{2n}(a_j(0) + \nabla a_j(0) \cdot x + \varepsilon_j)^2$$

Choose a j such that $|\nabla a_j(0)| \geq C > 0$ and let H be the hyperplane whose equation:

$$a_j + \nabla a_j(0) \cdot x = 0$$

We have

$$w(x)^2 \geq ||a_j(0) + \nabla a_j(0) \cdot x|^2 - 2|\varepsilon_j||a_j(0) + \nabla a_j(0)||$$

from which it follows that

$$w(x) \geq \frac{1}{2}|a_j(0) + \nabla a_j(0) \cdot x| = \frac{1}{2}|\nabla a_j(0)|dist(x, H) \tag{2.14}$$

so long as $\frac{1}{2}|\nabla a_j(0)|dist(x,H) > 2C|x|^2$, where the constant C is the minimum of all constants such that $\varepsilon_j(x) \leq C|x|^2$, which is the case if $|\nabla a_j(0)|dist(x,H) > 2Cr^2$. Let $C' = 2C/|\nabla a_j(0)|$ and define the set $E(B(r))$ to be

$$E(B(r)) := \{x \in B(r) \cap M : dist(x,H) \leq C'diam(B(r))^2\} \tag{2.15}$$

And thus, we have

$$\int_{B(r)\cap M} w^{-\varepsilon} d\sigma \leq C \int_{B(r)\cap M} dist(x,H)^{-\varepsilon} d\sigma + \int_{E(B(r))} w^{-\varepsilon} d\sigma$$

by a linear change of variable making the hyperplane H horizontal we see that the first integral is equal to $Cr^{-\varepsilon+2n} \cong C\eta^{-\varepsilon}\sigma(B(r) \cap M)$. As for the other term we either have $\sigma(E(B(r))) = 0$ in which case that's the end of it or else it can be at most a small corridor of size $C\underbrace{r \times r \times \cdots \times r}_{2n-1} \times r^2$. Cover it with balls of radius r^2, there can be at most Cr^{-2n+1} of them $B_i(r^2)$. And so we have,

$$\int_{E(B(r))} w^{-\varepsilon} d\sigma \leq C \sum_i \int_{B_i(r^2)} w^{-\varepsilon} d\sigma$$

do the same procedure as above to each $B_i(r^2)$ and we arrive at

$$\int_{B_i(r^2)} w^{-\varepsilon} d\sigma \leq Cr^{-2\varepsilon+4n} + \int_{E(B_i(r^2))} w^{-\varepsilon} d\sigma$$

summing over i we get

$$\int_{E(B(r))} w^{-\varepsilon} d\sigma \leq Cr^{-2\varepsilon+2n+1} + \sum_i \int_{E(B_i(r^2))} w^{-\varepsilon} d\sigma$$

if we continue this procedure again to each term in the sum we arrive finally at

$$\int_{B(r)\cap M} w^{-\varepsilon} d\sigma \leq Cr^{-\varepsilon+2n}(1 + r^{-\varepsilon+1} + r^{3(-\varepsilon+1)} + \cdots) \leq Cr^{-\varepsilon+2n}$$

since $\varepsilon < 1$. And this finishes the proof of lemma 4 as well as this is the end of this section.

2.3 The spaces S^p_α, Γ^p_α, F^p_α and B^p_α

Next we discuss the function spaces of interest to us i.e.; the nonisotropic Sobolev spaces denoted here by S^p_α defined on the ambient space, the nonisotropic Besov spaces Γ^p_α and the expected spaces of restrictions to M $F^p_\beta(M)$ and $B^p_\beta(M)$.

2.3.1 The Spaces S^p_α

These spaces were studied exclusively by few authors, [FG]; [FS]; [NS] and [SA]. Recall that in the Euclidian space R^n with standard coordinate functions $(x_1, \cdots, x_n)$ the basic differentiation $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ are used to define the classical Sobolev spaces, we denote them here by L^p_k. Analogously, if we replace the $\frac{\partial}{\partial x_j}$ by the directional derivatives X_j for $j = 1; \cdots; 2n$ we obtain the spaces S^p_k namely we have the following:

Definition 5 *We say that the function (or to be more precise the Lebesgue class) f belongs to the nonisotropic Sobolev space S^p_k , if it and its distributional derivatives $X_J f$, with $|J| \leq k$ all belong to L^p. The norm of f in S^p_k is given by*

$$||f||_{S^p_k} := \sum_{|J| \leq k} ||X_J f||_{L^p} \tag{2.16}$$

One may follow various classical constuctions to define the spaces S^p_α for α noninteger. The approach we follow here is by realizing these spaces as potentials. We let $\mathrm{L} = -\sum_{j=1}^{2n} X_j^2$ be the sublaplacian we denote by L_p its extension to the space L^p and we define the spaces S^p_α as the range of the operator $(I + \mathrm{L}_p)^{-\alpha/2}$. The operator $(I + \mathrm{L}_p)^{-\alpha/2}$ is realized as a convolution operator in the Heisenberg group acting on L^p. What this means for us is that there is a function J_α, the kernel of $(I + L_p)^{-\alpha/2}$, such that

$$S^p_\alpha = \{f * J_\alpha, f \in L^p\} \tag{2.17}$$

The function J_α has the following properties

Proposition 2 *The function J_α is smooth away from the origin and*
1. As $\quad |x| \to 0$

$$|X_J J_\alpha(x)| \leq C|x|^{\alpha - |J| - Q} \quad if \quad \alpha \neq |J| + Q$$

2. $X_J J_\alpha$ is continuous if $\alpha > |J| + Q$
3. If $\alpha = |J| + Q$ we have instead

$$|X_J J_\alpha(x)| \leq C \log(\frac{1}{|x|})$$

4. Away from the origin the function J_α is rapidly decreasing, that is as $|x| \to \infty$, for any N and J there is a constant $C = C_{N,J}$ such that

$$|X_J J_\alpha(x)| \leq C|x|^{-N}$$

In particular $X_J J_\alpha$ is integrable for all J such that $\alpha - |J| > 0$. The norm we take for a function $f = g * J_\alpha \in S^p_\alpha$ is

$$||f||_{S^p_\alpha} := ||g||_{L^p} \tag{2.18}$$

The following proposition says that in the case where α is integer the second definition (i.e.; the potential definition) is equivalent to the one above. This is tedious task already in the classical case see [ST1] for this the generalization is due to Folland, its second part also states that it is enough to define the fractional spaces only for small α .

Proposition 3 *When α is an integer, definition (2) above and (11) are equivalent in the sense that the norms (10) and (12) are equivalent. Moreover if $\alpha \geq 1$, a function f belongs to S^p_α if and only if f and $X_j f$ for $j = 1, \cdots, 2n$ all belong to $S^p_{\alpha-1}$. An equivalent norm is*

$$||f||_{S^p_{\alpha-1}} + \sum_{j=1}^{2n} ||X_j f||_{S^p_{\alpha-1}}$$

Another result that we use later is the following

Proposition 4 *The operator defined by convolution with J_β is an isomorphism between the spaces S^p_α and S^p_γ, where $\gamma = \alpha + \beta$.*

Next we compare the spaces S^p_α and the classical L^p_β. It is clear that if $\alpha = 2m$ is an even integer then we have locally

$$S^p_\alpha \subset L^p_{\alpha/2}$$

and by complex interpolation this is true for all $\alpha \geq 2$, the next proposition tells us that this embedding is true for all $\alpha \geq 0$.

Proposition 5 *For all $\alpha \geq 0$, we have*

$$S^p_\alpha \subset L^p_{\alpha/2} \qquad (locally)$$

2.3.2 The spaces $\Gamma^{p,q}_\alpha$

In this section we also include a short discussion on the nonisotropic Besov spaces and we indicate various connections to the spaces S^p_α. These spaces are the analogues to the classical Besov spaces $\Lambda^{p,q}_\alpha$, see [ST1]. The reference for this is Saka's paper [SA]. This paper is essentially the nonisotropic analog to Taibleson's classical paper [TA]. The theory of nonisotropic Besov spaces and Sobolev spaces is almost entirely parallel to the isotropic ones, in the sense that all the results that hold for latter spaces hold in an appropriate sense for the former ones. Also we show below that these spaces can be characterized by local polynomial approximations.

Definition 6 *Let $\alpha > 0$ and $1 < p, q < \infty$. We say that f belongs to the space $\Gamma^{p,q}_\alpha$ if its norm defined below is finite 1. If $0 < \alpha < 1$*

$$||f||_{\Gamma^p_\alpha} := ||f||_{L^p} + (\int_{|y|\leq 1} \frac{||f(\cdot y) - f(\cdot)||^q_{L^p}}{|y|^{Q+\alpha q}} dy)^{1/q}$$

2. If $\alpha = 1$, the second difference is used instead,

$$||f||_{\Gamma^{p,q}_1} := ||f||_{L^p} + (\int_{|y|\leq 1} \frac{||f(\cdot y) + f(\cdot y^{-1}) - 2f(\cdot)||^q_{L^p}}{|y|^{Q+q}} dy)^{1/q}$$

3. More generally we let k be the integer such that $0 \leq k < \alpha \leq k+1$ and we set

$$||f||_{\Gamma^{p,q}_\alpha} := \sum_{|J|\leq k} ||X_J f||_{\Gamma^{p,q}_{\alpha-k}}$$

In the same way the classical Besov spaces can be characterized by means of higher differences, the nonisotropic ones also enjoy similar characterization. For $y \in H_n$ we let

$$\Delta_h f(x) =: f(xh) - f(x)$$

and set

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) \tag{2.19}$$

Let $k > \alpha$, then an equivalent definition of the norm above is

$$||f||_{\Gamma_\alpha^{p,q}} = ||f||_{L^p} + (\int_{|y|\leq 1} \frac{||\Delta_y^k f||_{L^p}^q}{|y|^{Q+\alpha q}} dy)^{1/q} \tag{2.20}$$

The next results are Theorem 20 and 23 of [SA]. The first, giving various imbeddings with the spaces S_α^p, is of interest to us because it says in particular that in the case $p = 2$ the spaces $\Gamma_\alpha^{2,2}$ and S_α^2 are equivalent. The second realizes the spaces $\Gamma_\alpha^{p,q}$ as real interpolation spaces of S_α^p .

Theorem 2 *Let $\alpha > 0$, then*

$$\begin{aligned} &(1) \quad \Gamma_\alpha^{p,p} \subset S_\alpha^p \subset \Gamma_\alpha^{p,2} \quad for \quad 1 < p \leq 2 \\ &(2) \quad \Gamma_\alpha^{p,2} \subset S_\alpha^p \subset \Gamma_\alpha^{p,p} \quad for \quad 2 \leq p < \infty \end{aligned}$$

in particular we have $\Gamma_\alpha^{2,2} = S_\alpha^2$.

Theorem 3 *The spaces S_α^p form a scale of real interpolation and we have for $0 < \theta < 1$, $1 < p < \infty$ $1 \leq q \leq \infty$ and $\gamma = (1-\theta)\alpha + \theta\beta$,*

$$(S_\alpha^p, S_\beta^p)_{\theta,q} = \Gamma_\gamma^{p,q}$$

The next characterization of the Besov spaces $\Gamma_\alpha^{p,q}$ is going to be in terms of local polynomial approximation. For simplicity we discuss only the case $p = q$ and denote the resulting spaces by Γ_α^p. It is well known in the classical setting that in many instances differentiabilty properties of functions can be desribed by how much we can approximate the function by polynomials see e.g. the appendix of [ST] where the existence of derivatives in the L^p is discussed. In [NI], [BR1] and [BR2] and many other papers, similar ideas were exploited. It turns out that membership to the Besov spaces (and their generalizations the Lizorkin-Triebel spaces) , can be expressed in terms of the following quantities:
for any function f and a ball B and N an integer we denote by

$$\omega_N(f,B)^p := \inf_{d^o P \leq N} \int_B |f - P|^p \tag{2.21}$$

Following [JW], by a net $\mathcal{F}_k$ of mesh 2^{-k} we mean a collection of balls B of radius 2^{-k} and covering the ambient space satisfying the requirement that if we shrink them enough we get a disjoint family of balls. For each intege k, let $\mathcal{F}_k$ be a net of mesh 2^{-k}, then we have the following result

Proposition 6 *There is an integer $N > \alpha$ such that the following quantity*

$$||f||_{L^p} + (\sum_{k=0}^{\infty} 2^{kp\alpha} \sum_{B \in \mathcal{F}_k} \omega_N(f,B)^p)^{1/p} \tag{2.22}$$

is an equivalent norm for the space Γ^p_α.

The following important result is well known, it reflects the interpolation property theorem 2 above and can be obtained from general interpolation (see [BL] lemma 3.2.3). This result is used quite often in this thesis.

Proposition 7 *Let $f \in \Gamma^p_\alpha$, and N be an integer larger than α.Then there are smooth functions f_j, $j = 0, 1 \cdots$ such that the following hold:*

$$f = \sum_{j=0}^{\infty} f_j \tag{2.23}$$

convergence is in the sense of $\mathcal{S}'$

$$||X_J f_j||_{L^p} \leq 2^{-j(\alpha - |J|)} a_j \qquad \text{for} \quad \text{all} \qquad |J| \leq N \tag{2.24}$$

for some sequence of positive real numbers a_j satisfying $\sum_{j=0}^{\infty} a_j^p \leq C < \infty$.

Conversely,given a family of functions satisfying (2) *for some a_j's then the formal infinite sum $\sum_{j=0}^{\infty} f_j$ defines a distribution that is an element of Γ^p_α.*

We give a proof below that propositions 6 and 7 are equivalent. We basically follow the proof of [NS1] given for this same proposition but for nonisotropic Lipschitz spaces (Γ_α). For simplicity we considere only the case $p = 2$ since that is the case we are going to need in the sequel anyway.

Proof:

Let φ be defined by:

$$\varphi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 2 \end{cases}$$

and $0 \leq \varphi(t) \leq 1$ for $\xi \in R^{2n+1}$ set

$$\Phi_0(\xi) = \varphi(|\xi|)$$

and for $j \neq 0$

$$\Phi_j(\xi) = \varphi(2^{-j}|\xi|) - \varphi(2^{-j+1}|\xi|)$$

Denote by K_j the inverse Fourier transform of Φ_j we easily verify that

$$\sum_{j=0}^{\infty} K_j = \delta \tag{2.25}$$

and we obtain decomposition of functions f into a sum of smooth functions $f_j = f * K_j$ where convolution here again is the group convolution. The following estimates on the K_j's, which express the size of K_j and its rapid decrease, are needed.

Lemma 6 *(a)* $|K_j(x)| \leq C2^{jQ}$
(b) For each integer N there is a constant C_N such that

$$|K_j(x)| \leq C_N 2^{jQ-mN} \quad for \quad 2^{-j+m} \leq |x| \leq 2^{-j+m+1}$$

A property, that is easy to verify (because the Fourier transform is supported away from the origin), that the K_j's have for $j \neq 0$ is they orthogonal to polynomials. We begin by showing that the f_j defined satisfy the conclusions of the proposition. We will prove the estimates only for the L^2-norm of f_j, because the estimates for the derivatives follow exactly the same way. It is obvious that

$$||f_0||_{L^2} \leq C||f||_{L^2}$$

For $j \neq 0$, using the orthogonality of polynomials to K_j, the L^2 norm of f_j is equal to

$$||f_j||^2 = \sum_{B\in\mathcal{F}_j} \int_B |f_j(x)|^2 dx = \sum_{B\in\mathcal{F}_j} \int_B |\int_{R^{2n+1}} (f(y) - P_B(y))K_j(y^{-1}x)dy|^2 dx \tag{2.26}$$

the right hand side of (2.26) is less than or equal to $I_1 + I_2$, where

$$I_1 = \sum_{B\in\mathcal{F}_j} \int_B (\int_B |f(y) - P_B(y)||K_j(y^{-1}x)|dy)^2 dx \tag{2.27}$$

and

$$I_2 = \sum_{B\in\mathcal{F}_j} \int_B (\int_{R^{2n+1}\setminus B} |f(y) - P_B(y)||K_j(y^{-1}x)|dy)^2 dx \tag{2.28}$$

and we show that each I_i is less than or equal to $2^{-2j\alpha}b_j^2$ for some numbers such that $\sum_{j=0}^{\infty} b_j^2 < \infty$. I_1 well controlled and it is less than

$$C \sum_{B\in\mathcal{F}_j} \int_B |f(y) - P_B(y)|^2 dy \leq C2^{-2j\alpha}a_2$$

The trouble is to handle the other part I_2. For each B in $\mathcal{F}_j$ we denote by $2B$ the ball in $\mathcal{F}_{j-1}$ containing it, or to be precise one of the balls in $\mathcal{F}_{j-1}$ containing it. And inductively we denote by 2^kB a ball in $\mathcal{F}_{j-k}$ containing $2^{k-1}B$, with this we can write the inner integral in I_2 as

$$\int_{R^{2n+1}\setminus B} |f(y) - P_B(y)||K_j(y^{-1}x)|dy$$

$$\leq \int_{2B\setminus B} |P_{2B}(y) - P_B(y)||K_j(y^{-1}x)|dy + \int_{R^{2n+1}\setminus 2B} |f(y) - P_B(y)||K_j(y^{-1}x)|dy$$

$$\leq$$

$$\sum_{k=0}^{j} \int_{2^{k+1}B\setminus 2^kB} |P_{2^{k+1}B}(y) - P_{2^kB}(y)||K_j(y^{-1}x)|dy + \int_{R^{2n+1}\setminus 2^jB} |f(y) - P_{2^jB}(y)||K_j(y^{-1}x)|dy = A_1 + A_2$$

By lemma 6 we have,

$$|K_j(y^{-1}x)| \le C_N 2^{jQ} 2^{-jN}$$

and thus

$$A_1 \le C_N 2^{jQ} \sum_{k=0}^{j} 2^{-kN} \int_{2^{k+1}B \backslash 2^k B} |P_{2^{k+1}B}(y) - P_{2^k B}(y)| dy$$

$$\le C_N 2^{jQ/2} \sum_{k=0}^{j} 2^{-k(N-Q/2)} (\int_{2^{k+1}B \backslash 2^k B} |P_{2^{k+1}B}(y) - P_{2^k B}(y)|^2)^{1/2} dy$$

$$\le$$

$$C_N 2^{jQ/2} \sum_{k=0}^{j} 2^{-k(N-Q/2)} (\int_{2^{k+1}B} |P_{2^{k+1}B}(y) - f(y)|^2)^{1/2} dy$$

$$+ C_N 2^{jQ/2} \sum_{k=0}^{j} 2^{-k(N-Q/2)} (\int_{2^k B} |f(y) - P_{2^k B}(y)|^2)^{1/2} dy$$

integrating this string of inequalities over B and then summing over B in $\mathcal{F}_j$ using Minkowski's inequality, and also keeping in mind that when we sum over $B \in \mathcal{F}_j$ quantities such as $\int_{2^k B} |P_{2^k B}(y) - f(y)|^2 dy$ get repeated 2^{kQ} times, we get

$$\sum_{B \in \mathcal{F}_j} \int_B A_1^2 dx \le \sum_{B \in \mathcal{F}_j} (\sum_{k=0}^{j} 2^{-k(N-Q/2)} (\int_{2^k B} |P_{2^k B}(y) - f(y)|^2 dy)^{1/2})^2 \le C 2^{-2j\alpha} b_j^2$$

where

$$b_j^2 = 2^{-2j(N-Q/2-\alpha)} (\sum_{k=0}^{j} 2^{k(N-Q/2-\alpha)} a_k)^2$$

choose $N > \alpha + Q/2$ and use Hardy's inequality, lemma 8 below, to get

$$\sum_{j=0}^{\infty} b_j^2 < \infty$$

Now we turn to A_2.

$$A_2 = \int_{R^{2n+1} \backslash 2^j B} |f(y) - P_{2^j B}(y)| |K_j(y^{-1}x)| dy$$

$$\le \int_{R^{2n+1} \backslash 2^j B} |f(y)| |K_j(y^{-1}x)| dy + \int_{R^{2n+1} \backslash 2^j B} |P_{2^j B}(y)| |K_j(y^{-1}x)| dy$$

and so we have

$$\sum_{B \in \mathcal{F}_j} \int_B (\int_{R^{2n+1} \backslash 2^j B} |f(y)| |K_j(y^{-1}x)| dy)^2 dx \le C \int_{R^{2n+1}} (\int_{|y^{-1}x| \ge 1} |f(y)| |K_j(y^{-1}x)| dy)^2 dx \le$$

$$C_N 2^{-2j(N-Q)} ||f||_{L^2}^2 \le 2^{-2j\alpha} (2^{-2j(N-Q-\alpha)} ||f||_{L^2}^2)$$

The other term is estimated in the same way

$$\sum_{B\in\mathcal{F}_j}\int_B(\int_{R^{2n+1}\setminus 2^jB}|P_{2^jB}(y)||K_j(y^{-1}x)|dy)^2dx = \sum_{B\in\mathcal{F}_j}\int_B(\sum_{k=j}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}|P_{2^jB}(y)||K_j(y^{-1}x)|dy)^2dx$$

$$\leq C_N\sum_{B\in\mathcal{F}_j}\int_B(\sum_{k=j}^{\infty}2^{-kN}2^{jQ}\int_{2^{k+1}B\setminus 2^kB}|P_{2^jB}(y)dy)^2dx = C_N\sum_{B\in\mathcal{F}_j}2^{-jQ}(\sum_{k=j}^{\infty}2^{-kN}2^{jQ}\int_{2^kB}|P_{2^jB}(y)|dy)^2$$

$$\leq C_N2^{-2j(N-Q)}\sum_{B\in\mathcal{F}_j}\int_{2^jB}|P_{2^jB}(y)|^2dy = C_N2^{-2j(N-Q)}\sum_{B\in\mathcal{F}_0}\int_B|P_B(y)|^2dy$$

$$\leq$$

$$2^{-2j\alpha}(C_N2^{-2j(N-Q-\alpha)}(||f||_{L^2}^2+\sum_{B\in\mathcal{F}_0}\int_B|f(y)-P_B(y)|^2dy))$$

Choose $N>\alpha+Q$, and conclude by Hardy's inequality.
The converse is easier. But before we prove it we need Taylor's formula in the setting of the Heisenberg group. Because we need it later in the restriction theorem let us show how to derive it. Let f be a smooth function. For simplicity we indicate the calculations in the three dimensional space. We want to expand the function f arround a point $y=(y_1,y_2,y_3)$. Set

$$F(s):=f(y(s(y^{-1}x)))$$

we have $F(0)=f(y)$ and $F(1)=f(x)$. Taking ordinary Taylor expansion of F arround 0 we get

$$F(1)=\sum_{j=0}^{N}\frac{F^{(j)}(0)}{j!}+R_N$$

$$R_N=\frac{1}{N!}\int_0^1(1-s)^NF^{(N+1)}(s)ds$$

we calculate $F'(s)$

$$F'(s)=\frac{d}{ds}\{f(y_1+s(x_1-y_1),y_2+s(x_2-y_2),y_3+s^2(x_3-y_3)+2s(s-1)(y_1x_2-y_2x_1))\}$$

$$=$$

$$(x_1-y_1)\frac{\partial f}{\partial x_1}(y(s(y^{-1}x)))+(x_2-y_2)\frac{\partial f}{\partial x_2}(y(s(y^{-1}x)))+(2s(x_3-y_3)+(4s-2)(y_1x_2-y_2x_1))\frac{\partial f}{\partial x_3}(y(s(y^{-1}x))$$

evaluating F' at 0 we get

$$F'(0)=(x_1-y_1)\frac{\partial f}{\partial x_1}(y)+(x_2-y_2)\frac{\partial f}{\partial x_2}(y)-2(y_1x_2-y_2x_1)\frac{\partial f}{\partial x_3}(y)$$

$$=(y^{-1}x)^{(1,0,0)}(X_1f)(y)+(y^{-1}x)^{(0,1,0)}(X_2f)(y)$$

More generally we have

$$F^{(m)}(s) = \sum_{m\leq|J|\leq 2m} c_{J,m} s^{|J|-m}(y^{-1}x)^J (X_J f)(y(s(y^{-1}x)))$$

and evaluating at $s = 0$ we get the left Taylor polynomial

$$F^{(m)}(0) = \sum_{|J|=m} c_{J,m}(y^{-1}x)^J (X_J f)(y) \tag{2.29}$$

The remainder is written in its integral form as

$$R_N = \frac{1}{N!}\sum_{N\leq|J|\leq 2N} c_{J,N}(y^{-1}x)^J \int_0^1 s^{|J|-N}(1-s)^N (X_J f)(y(s(y^{-1}x)))ds \tag{2.30}$$

Let us go back now to the proof of the converse, let f_j be given to satisfy (2.24) of proposition 7, from which it follows at once that

$$||\sum_{k=0}^{\infty} f_k||_{L^2} \leq \sum_{k=0}^{\infty} ||f_k||_{L^2} \leq \sum_{k=0}^{\infty} 2^{-k\alpha} a_k \leq C(\sum_{k=0}^{\infty} a_k^2)^{1/2}$$

next we prove

$$\sum_{k=0}^{\infty} 2^{2k\alpha} \sum_{B\in\mathcal{F}_k} \omega(\sum_{j=0}^{\infty} f_j, B)^2 \leq C\sum_{j=0}^{\infty} a_j^2 \tag{2.31}$$

the left hand side of (2.31) is less than

$$\sum_{k=0}^{\infty} 2^{2k\alpha} \sum_{B\in\mathcal{F}_k} \omega(\sum_{j=0}^{k} f_j, B)^2 + C\sum_{k=0}^{\infty} 2^{2k\alpha}(\sum_{j=k+1}^{\infty} ||f_j||_{L^2})^2 \tag{2.32}$$

replacing $||f_j||_{L^2}$ with its estimates and applying Hardy's inequality we get that the second term of (2.32) is less than $\sum_{j=0}^{\infty} a_j^2$. As for the first term we have

$$\sum_{B\in\mathcal{F}_j} \omega(\sum_{j=0}^{k} f_j, B)^2 \leq (\sum_{j=0}^{k}(\sum_{B\in\mathcal{F}_k} \int_B |R_j(x, x_B)|^2 dx)^{1/2})^2 \tag{2.33}$$

where $R_j(x, x_B)$ is the remainder of the left Taylor expansion of f_j arround any point x_B of B. Now by the estimates satisfied by the remainder we see that the right hand side of (2.33) is less than or equal to

$$C\sum_J 2^{-2k|J|}(\sum_{j=0}^{k} ||X_J f_j||_{L^2})^2 \leq C\sum_J 2^{-2k|J|}(\sum_{j=0}^{k} 2^{-j(\alpha-|J|)} a_j)^2$$

Now conclude the proof by Hardy.

2.3.3 The Spaces F^p_α and B^p_α

Now we come to define what we expect to be the boundary values of the spaces S^p_α. The definition we are going to use is the local polynomial approximation on M. Let $\mathcal{F}_k$ be a net of mesh 2^{-k} covering the surface M (not the ambient space). Let us also fix a coordinate system $(x_1, \cdots, x_{2n+1})$ in which M is described by $M = \{(x_1, \cdots, x_{2n+1}) : x_{2n+1} = 0\}$, with this we can speak of polynomials in M. Now the definition of F^p_α is as follow: we let as in the case of the ambient space

$$\omega_N(f, B)^2 := \inf_{d^\circ P \leq N} \int_B |f - P|^p d\mu \tag{2.34}$$

Definition 7 *Let $N > \alpha$, a function f defined on M is said to belong to the space $F^p_{\alpha,N}$ if*

$$||f||_{F^p_{\alpha,N}} := ||f||_{L^p(d\mu)} + (\sum_{k=0}^{\infty} 2^{kp\alpha} \sum_{B \in \mathcal{F}_k} \omega_N(f, B)^p)^{1/p} \tag{2.35}$$

is finite for some $N > \alpha$.

The expressions in the right hand side of (2.34) and hence (2.35) depend on the choice of local coordinates but membership to the space $F^p_{\alpha,N}$ is independent of the coordinate system chosen. The case $0 < \alpha < 1$ is of special interest because if we use approximation by constants instead of polynomials in (2.34) i.e.; $\omega_0(f, B)^2$ and form the corresponding quantities (2.35) we get that this space, which we denote from now on by F^2_α may be characterized by the first difference, that is if $0 < \alpha < 1$ an equivalent expression to the one in (2.35) is given by

$$||f||_{F^p_\alpha} := ||f||_{L^p(d\mu)} + (\int \int_{d(x,y) \leq 1} \frac{|f(x) - f(y)|^p}{d(x, y)^{Q-1+\alpha p}} d\mu(x) d\mu(y))^{1/p} \tag{2.36}$$

Since we will be only interested in this space, because this is so far the space that we proved restriction and extension theorems, for higher α we unfortunetly have to replace them with the spaces B^p_α whose definition is next. We would like however before we move to next definition relate these spaces to known spaces such as $L^2(d\sigma)$ for example, and thus we ask the question for which values of α ($0 < \alpha < 1$) do we have the following imbedding?

$$F^2_\alpha \subset L^2(d\sigma)$$

For this we have the following

Proposition 8 *For all α such that $\frac{1}{2} < \alpha < 1$ we have the following*

$$F^2_\alpha \subset L^2(d\sigma)$$

Proof: Suppose that $f \in F^2_\alpha$ which means $f \in L^2(d\mu)$ and

$$\sum_{k \geq 0} 2^{2k\alpha} \sum_{Q \in \mathcal{F}_k} \omega_0(Q, f)^2 < \infty$$

For $Q \in \mathcal{F}_k$ we denote by $\mathcal{F}_l(Q)$ balls in $\mathcal{F}_l$ touching the ball Q ie;

$$\mathcal{F}_l(Q) = \{Q' \in \mathcal{F}_l : Q'^* \cap Q \neq \emptyset\}$$

where Q'^* stands for the ball concentric with Q' and with radius 10 times the radius of Q'. We note that $\#\mathcal{F}_l(Q) \leq C$ this constant is independent of Q; k and l, it depends of course on surface M under consideration and other absolute constants. Let c_Q be the constant that best approximates f in Q inthe $L^2(d\mu)$ sense i.e.; $c_Q = \frac{1}{\mu(Q)} \int_Q f d\mu$. Let φ_Q^k be a partition of unity subbordinate to the cover $\mathcal{F}_k$, and define the function f_k by:

$$f_k := \sum_{Q \in \mathcal{F}_k} \varphi_Q c_Q$$

because of the local finiteness of the partition of unity the function f_k is well defined even smooth and it easy to see that the sequence of functions f_k approaches f in $L^2(d\mu)$. For ,

$$||f - f_k||_{L^2(d\mu)} \approx (\sum_{Q \in \mathcal{F}_k} || \sum_{Q \in \mathcal{F}_k} \varphi_Q (f - c_Q)||^2_{L^2(d\mu, Q)})^{\frac{1}{2}} \leq$$

$$(\sum_{Q \in \mathcal{F}_k} (\sum_{Q' \in \mathcal{F}_k(Q)} ||f - c_{Q'}||^2_{L^2(d\mu, Q')})^{\frac{1}{2}} \leq$$

$$C(\sum_{Q \in \mathcal{F}_k} ||f - c_Q||^2_{L^2(d\mu, Q)})^{\frac{1}{2}} \leq C 2^{-k\alpha} a_k \to 0$$

Next, we estimate $||f_k||_{L^2(d\sigma)}$. For $k = 0$ we have,

$$||f_0||^2_{L^2(d\sigma)} \leq C \sum_{Q_0 \in \mathcal{F}_0} ||f_0||^2_{L^2(Q, d\sigma)} = \sum_{Q_0 \in \mathcal{F}_0} || \sum_{Q' \in \mathcal{F}_0} \varphi_{Q'} c_{Q'} ||^2_{L^2(Q, d\sigma)}$$

$$\leq C \sum_{Q \in \mathcal{F}_0} |c_Q|^2 \sigma(Q) \leq C \sum_{Q \in \mathcal{F}_0} \frac{\sigma(Q)}{\mu(Q)} \int_Q |c_Q|^2 d\mu \leq ||f||^2_{L^2(d\mu)} + \sum_{Q \in \mathcal{F}_0} \omega(Q, f)^2$$

we have used the fact that for $Q \in \mathcal{F}_0$, then $\frac{\sigma(Q)}{\mu(Q)} \approx 1$, and more generally by lemma 3(a) and lemma 4 we have,

$$C \leq \frac{\sigma(Q)}{\mu(Q)} \leq C' Diam(Q)^{-1}$$

Now for $k > 0$ we write

$$f_k = (f_k - f_{k-1}) + \cdots + (f_1 - f_0) + f_0$$

and

$$f_j - f_{j-1} = \sum_{Q \in \mathcal{F}_j} \sum_{Q' \in \mathcal{F}_{j-1}} \varphi_Q \varphi_{Q'} (c_Q - c_{Q'})$$

and so

$$||f_k||_{L^2(d\sigma)} \leq ||f_0||_{L^2(d\sigma)} + \sum_{j=1}^{k} ||f_j - f_{j-1}||_{L^2(d\sigma)}$$

$$||f_j - f_{j-1}||^2_{L^2(d\sigma)} \leq C \sum_{\tilde{Q}\in\mathcal{F}_j} \int_{\tilde{Q}} \sum_{Q\in\mathcal{F}_j(\tilde{Q})} \sum_{Q'\in\mathcal{F}_{j-1}(\tilde{Q})} |\varphi_Q\varphi_{Q'}|^2 |c_Q - c_{Q'}|^2 d\sigma \leq$$

$$C 2^j \sum_{\tilde{Q}\in\mathcal{F}_j} \sum_{Q\in\mathcal{F}_j(\tilde{Q})} \sum_{Q'\in\mathcal{F}_{j-1}(\tilde{Q})} \int_{\tilde{Q}} |c_Q - c_{Q'}|^2 d\mu$$

because also of the finiteness of the sums over $\mathcal{F}_j(\tilde{Q})$ and $\mathcal{F}_{j-1}(\tilde{Q})$ we get

$$||f_j - f_{j-1}||^2_{L^2(d\sigma)} \leq C 2^j (\sum_{Q\in\mathcal{F}_j} \omega(Q,f)^2 + \sum_{Q'\in\mathcal{F}_{j-1}} \omega(Q',f)^2) \leq$$

$$2^j (2^{-2j\alpha} a_j^2 + 2^{-2(j-1)\alpha} a_{j-1}^2)$$

for some numbers a_j such that $\sum_{j=0}^{\infty} a_j^2 < \infty$. Finally,

$$||f_k||_{L^2(d\sigma)} \leq C(||f||_{L^2(d\mu)} + \sum_{j=0}^{k} 2^{-j(\alpha-\frac{1}{2})} a_j) \leq C(||f||_{L^2(d\mu)} + (\sum_{j=0}^{k} 2^{-2j(\alpha-\frac{1}{2})})^{\frac{1}{2}} (\sum_{j=0}^{k} a_j^2)^{\frac{1}{2}}$$

from which it follows, if $\alpha > \frac{1}{2}$, that $||f||_{L^2(d\sigma)} \leq C||f||_{F^2_\alpha}$.

Remark 2 *We could have gotten this result cheaply by combining both the classical restriction theorem and the extension theorem (theorem 8 in chapter 4), in the following way extend the function $f \in F^2_\alpha$ to a function $\mathbf{E}(f) \in S^2_{\alpha+\frac{1}{2}}$ this function belongs in particular to $L^2_{\frac{\alpha}{2}+\frac{1}{4}}$ by proposition 5 above and then restrict to get that the restriction is possible if $\alpha > 1/2$. Also the proposition doesn't say anything about the range $0 < \alpha \leq \frac{1}{2}$. The case $\alpha = 1/2$ is actually theorem 5, a result that at present seems to be unprovable by direct methods as the case of $\alpha > \frac{1}{2}$. As for the range $0 < \alpha < \frac{1}{2}$ the proposition is false.*

Finally we define the spaces B^p_α, that would replace the spaces F^2_α for large α. We follow the definition of Jonsson and Wallin.

Definition 8 *Let k, α, N be such that $0 \leq k < \alpha \leq k+1$ and $N > \alpha$. A system of functions $f := \{f_J : |J| \leq k\}$ is said to belong to the space B^p_α, if for every net $\mathcal{F}_m$ of mesh 2^{-m}*

and every $Q \in \mathcal{F}_m$ there is a polynomial P_Q in the ambient space such that the following hold :

1. For all multiindices J with $|J| \leq k$, we have

$$\sum_{Q\in\mathcal{F}_m} \int_Q |f_J - X_J P_Q|^p d\mu \leq (c_m 2^{-m(\alpha-|J|)})^p$$

2. For $k+1 \le |J| \le N$, we have

$$\sum_{Q \in \mathcal{F}_m} \sum_{Q' \in \mathcal{F}_m(Q)} \int_Q |X_J(P_Q - P_{Q'})|^p d\mu \le (c_m 2^{-m(\alpha-|J|)})^p$$

3. For unit size Q's we have

$$\sum_{Q \in \mathcal{F}_m} \int_Q |X_J P_Q|^p d\mu \le C$$

It is clear that the requirements of definition 8 are very strong and difficult to fulfill. Intuitively, the system of functions $\{f_J\}$ represents the whole history of f, i.e.; all the derivatives, in fact when considered in the ambient space the system . This definition was used in the restriction to very general sets (the so called d-sets). In this case it is clear also why one wants to consider systems of functions instead of single one.

2.4 Two classical inequalities

Before we close this chapter we record two main inequalities used extensively in this thesis and they are Young's inequality and Hardy's inequality. The latter inequality had a great impact on the development of the theory of function spaces, in particular the Besov and Sobolev spaces. We also use the standard ones such as Holder's and Minkowski's inequalities.

Lemma 7 *Let (X,μ) and (Y,ν) be two measure spaces, and let $k(x,y)$ be a $\mu \times \nu$-measurable function on $X \times Y$ such that*

$$(i) \ \sup_{x \in X} \int_Y |k(x,y)| d\nu(y) \le C_1$$

and

$$(ii) \ \sup_{y \in Y} \int_X |k(x,y)| d\mu(x) \le C_2$$

then, for all $f \in L^2(Y,\nu)$ we have

$$\left(\int_X \left| \int_Y k(x,y) f(y) d\nu(y) \right|^2 d\mu(x)\right)^{1/2} \le C_1^{1/2} C_2^{1/2} \left(\int_Y |f(y)|^2 d\nu(y)\right)^{1/2}$$

Lemma 8 *Let $a_k \ge 0$. Then, for $0 < p < \infty$*

$$(i) \ \sum_{k=0}^{\infty} 2^{sk} \left(\sum_{i=0}^{k} a_i\right)^p \le C \sum_{k=0}^{\infty} 2^{sk} a_k^p \qquad if \qquad s < 0$$

$$(ii) \ \sum_{k=0}^{\infty} 2^{sk} \left(\sum_{i=k}^{\infty} a_i\right)^p \le C \sum_{k=0}^{\infty} 2^{sk} a_k^p \qquad if \qquad s > 0$$

Chapter 3

The Restriction Theorem

3.1 The L^2 estimates

At this preliminary stage of the restriction theorem we resolve first question 2 raised in the introduction i.e.; when is the restriction operator bounded from S^2_α to L^2? First of all, there are two L^2-spaces that we are concerned with. The first is the ordinary $L^2(d\sigma)$, $d\sigma$ is the ordinary surface measure, and the second is $L^2(d\mu)$, where we have set $d\mu = wd\sigma$. The following two theorems are the main results of this section.

Theorem 4 *The restriction operator is bounded from S^2_α to $L^2(d\mu)$ if $\alpha > \frac{1}{2}$.*

The next theorem tells us when is the restriction operator bounded from S^2_α to $L^2(d\sigma)$. Recall that, by proposition (5) chapter 2, we have

$$S^2_\alpha \subset L^2_{\frac{\alpha}{2}} \tag{3.1}$$

where the spaces on the right hand side of (3.1) are the standard Sobolev-potential spaces, furthermore one cannot do better than this. On the one hand, by the classical restriction theorem we have that $L^2_{\frac{\alpha}{2}}$ and therefore S^2_α restrict to $L^2_{\frac{\alpha-1}{2}}(d\sigma)$ if $\alpha > 1$, on the other hand theorem 4 tells us that restriction cannot possibly be bounded from S^2_α to $L^2(d\sigma)$ if $\alpha \leq 1/2$. So the problem is really when $\frac{1}{2} < \alpha \leq 1$. Theorem 5 below that says that there is boundedness for $\alpha = 1$. We will give at the end of this section an example of a function belonging to $S^2_{1-\varepsilon}$ for all $\varepsilon > 0$ but fails to have restriction to the characteristic hyperplane. So theorem 5 is a borderline result, making it a sharp and interesting result.

Theorem 5 *The restriction operator is a bounded linear operator from S^2_1 to $L^2(d\sigma)$.*

3.1.1 Proof of theorem 4 :

We note first that it is part of the theorem that the restriction is well defined almost everywhere. Since both theorems above are local, let us fix once and for all a bounded open set

U in the ambient space and let us denote by M the intersection of some smooth hypersurface with the open set U, we will ignore the edges by assuming that all our functions have compact support inside U. We may, also without loss, assume that M is given by the graph of some smooth function, say to fix the notation

$$M = \{(t, z) \in R^{2n+1} = R \times R^{2n} : t = \psi(z)\} \cap U \tag{3.2}$$

In order to prove the theorem we have to prove the inequality

$$||f|_M||_{L^2(d\mu)} \leq C||g||_{L^2} \tag{3.3}$$

where $f = g * J_\alpha$, and the constant C may depend on the support . Theorem 4 says not only the restriction of f to M is in $L^2(d\mu)$ but also the restriction of all derivatives up to order k, where $k < \alpha \leq k+1$, are in L^2. Now let $f \in S^2_\alpha$ and let J be a multiindex. Write $X_J f$ as

$$X_J f(x) = f_1(x) + f_2(x)$$

where

$$f_1(x) = \int_{|x^{-1}y|\geq 1} (X_J J_\alpha)(y^{-1}x) g(y) dy$$

and

$$f_2(x) = \int_{|x^{-1}y|<1} (X_J J_\alpha)(y^{-1}x) g(y) dy$$

Let us assume first that $\alpha \neq Q + |J|$. By Schwarz inequality

$$|f_1(x)| \leq (\int_{|y^{-1}x|\geq 1} |g(y)|^2 dy)^{1/2} (\int_{|y^{-1}x|\geq 1} (X_J J_\alpha)(y^{-1}x)^2 dy)^{1/2}$$

which is less than or equal to $C||g||_{L^2}$ because of the rapid decrease of J_α away from the origin. And thus we have

$$||f_1||_{L^2(d\mu)} \leq C||g||_{L^2} = C||f||_{S^2_\alpha} \tag{3.4}$$

We turn to f_2 and write it

$$f_2(x) = \sum_{m=0}^{\infty} \int_{|y^{-1}x|\sim 2^{-m}} (X_J J_\alpha)(y^{-1}x) g(y) dy$$

by Minkowski's inequality we get

$$||f_2||_{L^2(wd\sigma)} \leq \sum_{m=0}^{\infty} (\int_M (\int_{|y^{-1}x|\sim 2^{-m}} (X_J J_\alpha)(y^{-1}x) g(y) dy)^2 w(x) d\sigma(x))^{1/2} \tag{3.5}$$

Let $\chi_m(x, y)$ be the characteristic function of the set

$$A_m(x, y) = \{(x, y) \in M \times R^{2n+1} / 2^{-m-1} \leq |y^{-1}x| < 2^{-m}\}$$

and set $\tilde{J}_\alpha(x,y) = \chi_m(x,y)(X_J J_\alpha)(y^{-1}x)$ in order to apply Young's inequality to each term in the sum of (3.5) we need to check that

$$\sup_{y\in M}\int_U |\tilde{J}_\alpha(x,y)|dx \le C_m \tag{3.6}$$

and

$$\sup_{x\in U}\int_M |\tilde{J}_\alpha(x,y)d\mu(y) \le C'_m \tag{3.7}$$

For any fixed $x \in M$ we have

$$\int_U |\tilde{J}_\alpha(y^{-1}x)|dy \le C\int_{|y^{-1}x|\sim 2^{-m}} |y^{-1}x|^{\alpha-|J|-Q}dy \le C2^{-m(\alpha-|J|)} \tag{3.8}$$

similarly , we have for a fixed $y \in U$,

$$\int_{|y^{-1}x|\sim 2^{-m}} |y^{-1}x|^{\alpha-|J|-Q}w(x)d\sigma(x) \le C2^{-m(\alpha-|J|-1)} \tag{3.9}$$

from which it follows that

$$(\int_M(\int_{|y^{-1}x|\sim 2^{-m}} X_J J_\alpha(y^{-1}x)g(y)dy)^2 w(x)d\sigma(x))^{1/2} \le C2^{-m(\alpha-|J|-\frac{1}{2})}||g||_{L^2}$$

finally we obtain

$$||f_2||_{L^2(d\mu)} \le C||g||_{L^2}\sum_{m=0}^{\infty} 2^{-m(\alpha-|J|-\frac{1}{2})} \le C||g||_{L^2}$$

as long as $\alpha - |J| > 1/2$

Suppose now that $\alpha = |J| + Q$, then the only difference from the previous case is now the estimate we use for $X_J J_\alpha$ is

$$|X_J J_\alpha(y)| \le C\log(\frac{1}{|y|}) \quad as \quad |y| \to 0$$

and the constants in (3.8) and (3.9) are respectively $Cm2^{-mQ}$ and $Cm2^{-m(Q-1)}$.
□

3.1.2 Proof of theorem 5 :

Now we turn to prove theorem 5. We will prove the theorem first in the special case of the characteristic hyperplane $\{(t,z) : t = 0\}$. The following lemma, from the general theory of Hilbert spaces, is key in proving the estimates needed .

Lemma 9 *Let H_1 and H_2 be two Hilbert spaces ; T be a linear operator from H_1 to H_2, and define the adjoint operator T^* from H_2 to H_1 by*

$$(T^*u, w)_{H_1} := (u, Tw)_{H_2}$$

If T is bounded then TT^* (resp. T^*T) is a bounded operator from H_2 to H_2 (resp. from H_1 to H_1) . And conversely, if either TT^* or T^*T is bounded operator, then T is also bounded.

Proof:

It is clear that T is bounded if and only if T^* and the boundedness of TT^* and T^*T follows. Conversely, if TT^* is bounded then by Cauchy Schwarz

$$||T^*u||_{H_1}^2 = (T^*u, T^*u)_{H_1} = (u, TT^*u) \leq ||u||_{H_2}||TT^*u||_{H_1} \leq C||u||_{H_1}^2$$

□

1. The case of the characteristic hyperplane

If $f \in S_1^2$

$$f(t,z) = \int_{H_n} I((s,u)^{-1}(t,z))g(s,u)dsdu \tag{3.10}$$

where I is the Riesz kernel i.e.; the kernel function of the operator $(-\mathrm{L})^{-\frac{1}{2}} := (\sum_{j=1}^{2n} X_j^2)^{-\frac{1}{2}}$, and $g \in L^2$. Recall that we have

$$|I(t;z)| \leq C|(t;z)|^{-Q+1} \tag{3.11}$$

So proving that $f(0,z) \in L^2(dz)$ ammounts to proving that the operator $\mathbf{R}(-\mathrm{L})^{-\frac{1}{2}}$ ($\mathbf{R}$ is the restriction operator to M i.e.; $\mathbf{R}f := f|_M$) is bounded from $L^2(dtdz)$ to $L^2(dz)$, which according to lemma 9 above is equivalent to showing that the operator

$$T = (\mathbf{R}(-\mathrm{L})^{-\frac{1}{2}})(\mathbf{R}(-\mathrm{L})^{-\frac{1}{2}})^* \tag{3.12}$$

is bounded from $L^2(dz)$ to $L^2(dz)$. The kernel function of T is given by

$$K(z,z') = \int_{H_n} I((s,u)^{-1}(0,z))I((s,u)^{-1}(0,z'))dsdu \tag{3.13}$$

it is easy to check that the kernel function $K(z,z')$ is symmetric and homogeneous of degree -2 that is

$$K(rz, rz') = r^{-2}K(z,z')) \tag{3.14}$$

Let us assume for simplicity that $n = 1$ and change variables to polar coordinates.

$$z = re^{i\theta}, z' = r'e^{i\theta'} \tag{3.15}$$

the action of T is seen as a group convolution on $]0,\infty[\times S^1$ with its obvious group structure

$$Tf(re^{i\theta}) = \int_0^{2\pi}\int_0^{\infty} F(\frac{r}{r'}, \theta - \theta')f(r'e^{i\theta'})\frac{dr'}{r'}d\theta'$$

where

$$F(r,\theta) = K(1, re^{i\theta})$$

the properties of $F(r,\theta)$ can be read off directly from the estimates on the kernel K which is lemma 11 below.

Lemma 10

$$(a)\quad F(r,\theta)=O((|r-1|+\theta^{\frac{1}{2}})^{-2})\quad as\quad (r,\theta)\to(1,0)$$

$$(b)\quad F(r,\theta)=O(1)\quad as\quad r\to 0$$

$$(c)\quad F(r,\theta)=O(r^{-2})\quad as\quad r\to\infty$$

Now in order to prove that T maps $L^2(rdrd\theta,(0,\infty)\times S^1)$ to itself we need to show:

$$\int_0^{2\pi}\int_0^{\infty}(\int_0^{2\pi}\int_0^{\infty}F(\frac{r}{r'},\theta-\theta')f(r'e^{i\theta'})\frac{dr'}{r'}d\theta')^2 rdrd\theta$$

$$\leq$$

$$C\int_0^{2\pi}\int_0^{\infty}|f(re^{i\theta})|^2 rdrd\theta$$

and this is the same as

$$\int_0^{2\pi}\int_0^{\infty}(\int_0^{2\pi}\int_0^{\infty}r^{\frac{1}{2}}r'^{\frac{-3}{2}}F(\frac{r}{r'},\theta-\theta')(r'^{\frac{1}{2}}f(r'e^{i\theta'}))dr'd\theta')^2 drd\theta$$

$$\leq$$

$$C\int_0^{2\pi}\int_0^{\infty}|r^{\frac{1}{2}}f(re^{i\theta})|^2 drd\theta$$

For Young's inequality to apply , we need to show two things

$$\sup_{r,\theta}\int_0^{2\pi}\int_0^{\infty}r^{\frac{1}{2}}r'^{\frac{-3}{2}}F(\frac{r}{r'},\theta-\theta')dr'd\theta'\leq C \tag{3.16}$$

$$\sup_{r',\theta'}\int_0^{2\pi}\int_0^{\infty}r^{\frac{1}{2}}r'^{\frac{-3}{2}}F(\frac{r}{r'},\theta-\theta')drd\theta\leq C \tag{3.17}$$

but via a change of variables the two integrals are amenable to each other, and this is because the kernel K is symmetric, so we need only consider one of them, say (3.16). Fix r and θ and break the domain of integration in () into three parts

$$\int_0^{2\pi}\int_0^{r/2}+\int_0^{2\pi}\int_{r/2}^{2r}+\int_0^{2\pi}\int_{2r}^{\infty}=A_1+A_2+A_3$$

using lemma 10 we have

$$A_1\leq C\int_0^{r/2}r^{\frac{1}{2}}r'^{\frac{-3}{2}}(\frac{r}{r'})^{-2}dr'\leq C$$

while

$$A_3\leq C\int_{2r}^{\infty}r^{\frac{1}{2}}r'^{\frac{-3}{2}}dr'\leq C$$

$$A_2\leq C\int_0^{2\pi}\int_{r/2}^{2r}r^{\frac{1}{2}}r'^{\frac{-3}{2}}(|\frac{r}{r'}-1|+|\theta-\theta'|^{\frac{1}{2}})^{-2}dr'd\theta'=$$

$$C\int_0^{2\pi}\int_{1/2}^{2} s^{\frac{1}{2}}(|s-1|+|\theta-\theta'|^{\frac{1}{2}})^{-2}\frac{ds}{s}d\theta' \leq$$
$$C\int_0^{2\pi}\int_{1/2}^{2}(|s-1|+\theta^{\frac{1}{2}})^{-2}dsd\theta \leq C$$

2. The general case.

Let us settle some notation to avoid cumbersome expressions. The notation $z \in M$ has the obvious interpretation to mean $(\psi(z), z) \in M$, also the notation $d(z, z')$ should be taken to mean the nonisotropic distance between the points $(\psi(z), z)$ and $(\psi(z'), z')$, similarly if $x \in U$ and $z \in M$ the notation $d(x, z)$ should be clear. Finally, denote by $d(z)$ the distance to the origin.

We describe now the general case. Let $K(z, z')$ be the kernel function of T given by (3.12), it is a symmetric kernel and defined in the same way as in (3.13) by

$$K(z,z') = \int_U I((s,u)^{-1}(\psi(z),z))I((s,u)^{-1}(\psi(z'),z'))dsdu \tag{3.18}$$

Let $q(z) \geq 0$ be a nonnegative function defined on M and define the measure $d\nu$ by the equation

$$q(z)d\nu(z) = d\sigma(z)$$

so that

$$Tf(z) \in L^2(d\sigma) \;\; if\ and\ only\ if \;\; q(z)^{1/2}Tf(z) \in L^2(d\nu(z))$$

and so the inequality to prove

$$\int_M |\int_M K(z,z')f(z')d\sigma(z')|^2 d\sigma(z) \leq C\int_M |f(z)|^2 d\sigma(z)$$

is equivalent to

$$\int_M |\int_M q(z')^{1/2}K(z,z')q(z)^{1/2}(q(z')^{1/2}f(z'))d\nu(z')|^2 d\nu(z) \leq C\int_M |q(z)^{1/2}f(z)|^2 d\nu(z)$$

Set

$$\tilde{K}(z,z') = q(z)^{1/2}K(z,z')q(z')^{1/2} \tag{3.19}$$

By Young's inequality we need to show

$$\sup_{z\in M}\int_M \tilde{K}(z,z')d\nu(z') \leq C \tag{3.20}$$

This is exactly what we did in the special case of the hyperplane, there the function $q(z)$ is equal to $r^{1/2}$. The function r in the previous case is equal to the weight function w. Here too we choose the function q to be equal some power of the weigth function w, $q(z) := w(z)^{\varepsilon}$, for any $0 < \varepsilon < 1$.

The following lemma gives the estimates for K, which is lemma (2) in the case of the hyperplane.

Lemma 11 *The kernel $K(z;z')$ satisfies*

$$K(z;z') \leq Cd(z,z')^{-Q+2}$$

as $d(z;z')$ tends to zero

Proof: $K(z,z')$ is essentially

$$K(z;z') \approx \int_U d(u;z)^{-Q+1} d(u;z')^{-Q+1} du$$

By the triangle inequality, it is easily seen that there is a constant $C > 1$ such that

$$d(u;z') \leq \frac{d(z;z')}{C}$$

implies

$$d(u;z) \geq \frac{d(z;z')}{C}$$

so we can break the integral defining K into two parts, one is

$$\int_{d(u,z') \leq \frac{d((z;z')}{C}} d(u;z)^{-Q+1} d(u;z')^{-Q+1} du \leq Cd(z;z')^{-Q+2}$$

the other part is

$$\sum_{k=0}^{\infty} \int_{d(u;z') \sim 2^k (\frac{d(z;z')}{C})} d(u;z)^{-Q+1} d(u;z')^{-Q+1} du$$

but if $d(u;z') \sim 2^k d(z;z')$, then also we have $d(u;z) \sim 2^k d(z;z')$, so the sum is less than or equal to $d(z;z')^{-Q+2}$.

□

(3.20) follows if we prove

$$w(z')^{\varepsilon} \int_M d(z,z')^{-Q+2} w(z)^{-\varepsilon} d\sigma(z) \leq C < \infty \tag{3.21}$$

with constant independent of z'. Fix $z' \in M$ which we can assume to be the origin 0 and assume that $w(0) = \eta > 0$. Split the the integration in (3.21) into two parts, and we prove

$$I_1 = \eta^{\varepsilon} \int_{d(z) < \eta/10} d(z)^{-Q+2} w(z)^{-\varepsilon} d\sigma \leq C < \infty \tag{3.22}$$

and

$$I_2 = \eta^{\varepsilon} \int_{d(z) \geq \eta/10} d(z)^{-Q+2} w(z)^{-\varepsilon} d\sigma \leq C < \infty \tag{3.23}$$

$$I_2 = \eta^{\varepsilon} \sum_{k=1}^{\infty} \int_{d(z) \sim 2^k (\eta/10)} d(z)^{-Q+2} w(z)^{-\varepsilon} d\sigma(z)$$

$$\leq C\eta^{\varepsilon}\sum_{k=1}^{\infty}(2^k\eta/10)^{-Q+2}\int_{d(z)\leq 2^k\eta/10} w(z)^{-\varepsilon}d\sigma(z)$$

by lemma 5 and 3(b) of chapter 2.2 we have

$$\int_{d(z)\leq 2^k\eta} w(z)^{-\varepsilon} \leq (2^k\eta)^{Q-1}(\max_{z\in B(2^k\eta)} w(z))^{-1-\varepsilon} \leq (2^k\eta)^{Q-2-\varepsilon}$$

and thus,

$$I_2 \leq C\eta^{\varepsilon}\sum_{k=1}^{\infty}(2^k\eta/10)^{-Q+2}(\max_{d(z)\leq 2^k\eta/10} w(z))^{-\varepsilon}\sigma(B(0,2^k\eta/10)\cap M)$$

$$\leq$$

$$C\eta^{\varepsilon}\sum_{k=1}^{\infty}(2^k\eta)^{-Q+2}(2^k\eta)^{Q-1}(2^k\eta)^{-1-\varepsilon} = C\sum_{k=1}^{\infty}2^{-k\varepsilon} < \infty$$

and (3.23) follows.

Now we turn to I_1,by lemma 2(c) chapter 2, since $\eta > 0$, $w(z) \geq \eta/10$ for all z such that $d(z) \leq \eta/10$ and so

$$I_1 \leq C\int_{d(z)<\eta/10} d(z)^{-Q+2}d\sigma(z) \tag{3.24}$$

Change variables to

$$z' = z/\eta \qquad s = t/\eta^2$$

In this coordinate system the surface M transforms to

$$M' = \{(s,z') : s = \psi'(z') = \frac{1}{\eta^2}\psi(\eta z')\}$$

Now the bounds on $\psi'(z')$ are : if we take the Taylor expansion of ψ we get that

$$\psi(z) = \nabla\psi(0)\cdot z + q(z)$$

$$\eta = |\nabla\psi(0)|$$

with this we have

$$\psi'(z') = \frac{1}{\eta^2}(\nabla\psi(0)\cdot(\eta z')) + \frac{1}{\eta^2}q(\eta z')$$

the function $q'(z') := \frac{1}{\eta^2}q(\eta z')$ is bounded with all its derivatives, for since $q(z) = O(|z|^2)$ it follows that $q'(z')$ is bounded and if take derivatives things get better, for example for the first derivatives we have

$$\nabla q'(z') = \frac{1}{\eta^2}\times\eta^2(\nabla q)(\eta z') = \nabla q(z) = O(|z|)$$

and so on. The new corresponding weight function w' is comparable to 1 because $w'(0) = \frac{1}{\eta}|\nabla\psi(0)| = 1$. That is, the new surface is noncharacteristic which implies in particular that $\sigma(B(0,r)\cap M') \cong Cr^{Q-1}$. The distance $d(z)$ transforms to

$$d(z) = \eta d(z')$$

the surface measure expressed in the new coordinates is just

$$d\sigma_{M'}(z') \cong \eta^{-Q+2}dz'$$

and therefore

$$I_1 \leq C\int_{d(z')<1/10} \eta^{-Q+2}d(z')^{-Q+2}\eta^{Q-2}d\sigma(z') \leq C\int_{d(z')<1/10} d(z')^{-Q+2}d\sigma_{M'}$$

$$\leq C\sum_{k=0}^{\infty}\int_{d(z')\sim 2^{-k}} d(z')^{-Q+2}dz'$$

$$\leq$$

$$C\sum_{k=0}^{\infty}2^{(Q-2)k}\sigma(B(0,2^{-k})\cap M') \leq C\sum_{k=0}^{\infty}2^{-k} < \infty$$

□

3.1.3 S^2_α doesn't restrict to $L^2(d\sigma)$ for $\alpha < 1$

In this section we provide with an example of a function belonging to $S^2_{1-\varepsilon}$ for all $\varepsilon > 0$ but fails to have restriction in $L^2(d\sigma)$. For simplicity we place ourselves in the three dimensional space R^3. Set $\rho = (t^2 + (x^2+y^2))^{1/4}$ and let

$$f(t,x,y) = f(\rho) = \rho^{-1}\phi(\rho)$$

where ϕ is a smooth function identically equal to 1 for $\rho \leq 1$ and vanishes for $\rho \geq 2$. Clearly, $f|_{t=0} = (x^2+y^2)^{-1/2}$ near 0 fails to belong to $L^2(dxdy)$. That f belongs to S^2_α follows if we show that

$$f * I_{1-\alpha} \in S^2_1$$

for all $\alpha < 1$. Because $I_{1-\alpha}$ is a homogeneous function of degree $1-\alpha-Q = -3-\alpha$, and f is homogeneous of degree -1 near the origin , $f * I_{1-\alpha}$ is homogeneous of degree $-\alpha$ (this follows in gneral from the fact that if two functions are homogeneous of degree s and l their convolution, if it makes sense, is homogeneous of degree $s+l+Q$.) Now $X(f * I_{1-\alpha})$ is homogeneous of degree $-1-\alpha$, which is in $L^2(d\sigma)$ if and only if $\alpha < 1$.

3.1.4 Some consequences of theorem 4& 5

Before we move on to the main estimates for the restriction we discuss some consequences of Theorem 4& 5. If B is any unit size ball, then according to Theorem 5 we have the inequality

$$\int_{B\cap M} |f(t,z)|^2 d\sigma(t,z) \leq C \int_{B^*} (|f(t,z)|^2 + |\tilde{\nabla} f(t,z)|^2) dtdz \tag{3.25}$$

we would like to have this inequality at all scales. Let $r > 0$, and $B(r)$ a ball centered at the point $(\psi(z_0), z_0) \in M$. By left translation we can assume that $z_0 = 0$ and $\psi(z_0) = 0$ i.e.; $0 \in M$. Next we assume that the weight function w assumes its maximum on the ball B at the point $(0,0)$, for otherwise we look for one point where the maximum is achieved and take a ball centered there of a larger radius so as to contain the original ball B, and then left translate to the origin. Now the situation we are in is a ball B of radius $r > 0$, centered at the origin and we would like to rescale (3.25) above. Change variables to

$$t' = t/r^2 \qquad\qquad z' = z/r$$

the ball B gets transformed into the unit ball B' and the surface into

$$M' = \{(t', z') : t' = \psi'(z') := r^{-2}\psi(rz')\}$$

if we denote by $d\sigma_M(t,z)$ and $d\sigma_{M'}(t',z')$ the surface measures on M and M' respectively, and if we set $f'(t',z') := f(r^2t', rz') = f(t,z)$, we get upon applying (3.25) above

$$\int_{M'\cap B'} |f'(t',z')|^2 d\sigma_{M'}(t',z') \leq C \int_{B'^*} (|f(t',z')|^2 + |\tilde{\nabla} f'(t',z')|^2) dt'dz' \tag{3.26}$$

The right hand side of (3.26) is equal to

$$Cr^{-Q} \int_{B^*} |f(t,z)|^2 dtdz + Cr^{-Q+2} \int_{B^*} |\tilde{\nabla} f(t,z)|^2 dtdz \tag{3.27}$$

the surface measure $d\sigma_{M'}(t',z')$ on M' is, in terms of the graph coordinates, equal to

$$d\sigma_{M'}(t',z') = (1 + |\nabla\psi'(z')|^2)^{1/2} dz' = (1 + r^{-2}|\nabla\psi(rz')|^2)^{1/2} dz' =$$

$$r^{-Q+1} \left(\frac{r^2 + |\nabla\psi(rz')|^2}{1 + |\nabla\psi(rz')|^2}\right)^{1/2} d\sigma_M(r^2t', rz')$$

By Taylor expansion

$$\psi(z) = \nabla\psi(0) \cdot z + O(|z|^2)$$

and so

$$\nabla\psi(z) = \nabla\psi(0) + O(|z|)$$

and thus,

$$|\nabla\psi(z)| \geq C|w_B - Cr|$$

now because w_B is at least Cr according to lemma 2(c), we get

$$\left(\frac{r^2+|\nabla\psi(rz')|^2}{1+|\nabla\psi(rz')|^2}\right)^{1/2} \geq Cw_B$$

and hence,

$$d\sigma_{M'}(t',z') \geq Cw_B r^{-Q+1} d\sigma_M(r^2t',rz') \tag{3.28}$$

Now using this we have

$$\int_{B\cap M} |f(t,z)|^2 d\sigma_M(t,z) = \int_{B'\cap M'} |f'(t',z')|^2 d\sigma_M(r^2t',rz') \leq \frac{r^{Q-1}}{w_B}\int_{B'\cap M'} |f'(t',z')|^2 d\sigma_{M'}(t',z') \leq$$

$$C\frac{r^{Q-1}}{w_B}\int_{B'^*} |f'(t',z')|^2 dt'dz' + C\frac{r^{Q+1}}{w_B}\int_{B'^*} |\tilde{\nabla} f'(t',z')|^2 dt'dz' \leq$$

$$C(r^{-1}w_B^{-1}\int_{B^*} |f(t,z)|^2 dtdz + rw_B^{-1}\int_{B^*} |\tilde{\nabla} f(t,z)|^2 dtdz)$$

Therefore,

$$\int_{B\cap M} |f(t,z)|^2 d\mu \leq C(r^{-1}\int_{B^*} |f(t,z)|^2 dtdz + r\int_{B^*} |\tilde{\nabla} f(t,z)|^2 dtdz) \tag{3.29}$$

Next we recall Poincare inequality due to Jerison [JE1] in this setting. Let $B(r)$ be a ball of radius $r > 0$, then we have

$$\inf_c \int_B |f-c|^2 dtdz \leq Cr^2 \int_{B^*} |\tilde{\nabla} f|^2 dtdz \tag{3.30}$$

We can obtain a similar inequality on the surface as follows. Apply inequality (3.29) to the function $(f-c)$ and then inequality (3.30) to have at once

$$\inf_c \int_{B\cap M} |f-c|^2 d\mu \leq Cr\int_{B^*} |\tilde{\nabla} f|^2 dtdz$$

We summarize all of the above in the following lemma.

Lemma 12 *Let B be any ball of radius $r > 0$, and M a smooth hypersurface. Then,*

$$(a) \quad \int_{B\cap M} |f(t,z)|^2 d\sigma(t,z) \leq C(r^{-1}w_B^{-1}\int_{B^*} |f(t,z)|^2 dtdz + rw_B^{-1}\int_{B^*} |\tilde{\nabla} f(t,z)|^2 dtdz)$$

$$(b) \quad \int_{B\cap M} |f(t,z)|^2 d\mu \leq C(r^{-1}\int_{B^*} |f(t,z)|^2 dtdz + r\int_{B^*} |\tilde{\nabla} f(t,z)|^2 dtdz)$$

$$(c) \quad \inf_c \int_{B\cap M} |f-c|^2 d\mu \leq Cr\int_{B^*} |\tilde{\nabla} f|^2 dtdz$$

3.2 The main estimates

The following theorem is the main result of this section.

Theorem 6 *Let $f \in S^2_\alpha$ and let $k \geq 0$ be the integer such that $k < \alpha - \frac{1}{2} \leq k+1$. Let M be a smooth hypersurface, then the system of functions $\{f_J := X_J f|_M, |J| \leq k\}$ is an element of the space $B^2_{\alpha-\frac{1}{2}}(M)$. In particular if $k = 0$ i.e.; when $\frac{1}{2} < \alpha < \frac{3}{2}$, then the restriction operator is bounded from S^2_α to $F^2_{\alpha-\frac{1}{2}}$.*

3.2.1 The case $0 < \beta = \alpha - \frac{1}{2} < 1$

According to the definitions, the spaces $B^2_\beta(M)$ and $F^2_\beta(M)$ coincide when $0 < \beta < 1$. And so we can use the first difference characterization (2.36) to prove the theorem in this case. The following inequality is what we have to prove:

$$A_\beta(g)^2 := \int\int_{|y^{-1}x|\leq 1} \frac{|f(x)-f(y)|^2}{|y^{-1}x|^{2\beta+Q-1}} d\mu(x)d\mu(y) \leq C||g||^2$$

for $f = g * J_\alpha$. We that we note that if $\beta < 1/2$, we can give a straightforward proof. Decompose the function f according to Proposition 7 like;

$$f = \sum_{j=0}^{\infty} f_j \tag{3.31}$$

with the f_j's satisfying the required estimates. We prove the inequality,

$$\sum_{m=0}^{\infty} 2^{2m\beta} \sum_{B\in\mathcal{F}_m} \inf_c \int_{B\cap M} |f-c|^2 d\mu \leq C||f||^2_{S^2_\alpha} \tag{3.32}$$

By lemma 12(c) we have,

$$\sum_{m=0}^{\infty} 2^{2m\beta} \sum_{B\in\mathcal{F}_m} \inf_c \int_{B\cap M} |f-c|^2 d\mu \leq$$

$$C\sum_{m=0}^{\infty} 2^{2m\beta} \sum_{B\in\mathcal{F}_m} \inf_c \int_{B\cap M} |\sum_{j=0}^{m} f_j - c|^2 d\mu + \sum_{m=0}^{\infty} 2^{2m\beta} \sum_{B\in\mathcal{F}_m} \int_{B\cap M} |\sum_{j=m+1}^{\infty} f_j|^2 d\mu \leq I + II$$

$$I = \sum_{m=0}^{\infty} 2^{2m(\beta-\frac{1}{2})} \sum_{B\in\mathcal{F}_m} \int_{B^*} |\sum_{j=0}^{m} \tilde{\nabla} f_j|^2 dvol$$

$$\leq \sum_{m=0}^{\infty} 2^{2m(\alpha-1)} (\sum_{j=0}^{m} (\sum_{B\in\mathcal{F}_m} \int_{B^*} |\tilde{\nabla} f_j|^2 dvol)^{1/2})^2 \leq$$

$$C\sum_{m=0}^{\infty} 2^{2m(\alpha-1)} (\sum_{j=0}^{m} 2^{-j(\alpha-1)} a_j)^2 \leq C\sum_{m=0}^{\infty} a_m^2$$

since $\alpha < 1$. The estimate for II is also similar, we use lemma 12(b) instead applied to each individual f_j we get,

$$II \leq C \sum_{m=0}^{\infty} 2^{2m\beta} (\sum_{j=m+1}^{\infty} 2^{-j\beta} a_j)^2 \leq C \sum_{m=0}^{\infty} a_m^2$$

The following proof is due to Jonsson and Wallin [JW], it is based on real interpolation (vector valued version), and so we state the theorem and refer to [BL]; Theorem 5.6.1; page 122 for a proof and more on interpolation. If X is a banach space we denote by $l_p^s(X)$ the space of sequences $(a_j)_{j=0}^{\infty} \subset X$ such that:

$$||(a_j)||_{l_p^s(X)} := (\sum_{j=0}^{\infty} 2^{jps} ||a_j||_X^p)^{1/p} < \infty \tag{3.33}$$

Theorem 7 *Let $0 < q_0, q_1 \leq \infty$ and $s \neq s_0$. Then for all q we have*

$$(l_{q_0}^{s_0}(X), l_{q_1}^{s_1}(X))_{\theta,q} = l_q^s(X)$$

Let $f = g * J_\alpha \in S_\alpha^2$, we would like to show the inequality :

$$A_\alpha(f)^2 \leq C||g||_{L^2}^2 \tag{3.34}$$

the left hand side of (3.34) is clearly equivalent to the expression

$$\sum_{m=0}^{\infty} 2^{2m\beta} \int\int_{2^{-m-1} \leq |y^{-1}x| < 2^{-m}} |f(x) - f(y)|^2 \frac{d\mu(x) d\mu(y)}{|y^{-1}x|^{Q-1}} \tag{3.35}$$

Set $d\mu'(x,y) := \frac{d\mu(x)d\mu(y)}{|y^{-1}x|^{Q-1}}$ and let $X = L^2(M \times M, d\mu'(x,y)$, and define the operator $T := (T_j)_{j=0}^{\infty}$ by

$$T_j f(x,y) = \chi_j(x,y)(f(x) - f(y))$$

where $\chi_j(x,y)$ denotes the characteristic function of the set

$$\{(x,y) \in M \times M : 2^{-j-1} \leq |y^{-1}x| < 2^{-j}\}$$

Then another way of restating what we want to prove is that the operator T is bounded from S_α^2 to $l_2^\beta(X)$. For this to be true it suffices, according to the interpolation theorem above, that the operator T be bounded from S_α^2 to $l_\infty^\beta(X)$ i.e.;

$$2^{2m\beta} \int\int_{2^{-m-1} \leq |y^{-1}x| < 2^{-m}} |f(x) - f(y)|^2 d\mu'(x,y) \leq C \tag{3.36}$$

the constant C is independent of m. To show (3.36) let $\varepsilon > 0$ and set

$$|J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)| = |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^\varepsilon |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{1-\varepsilon}$$

by Schwarz inequality

$$\int\int_{|y^{-1}x|\cong 2^{-m}} |f(x)-f(y)|^2 d\mu'(x,y) \leq$$

$$\int\int_{|y^{-1}x|\cong 2^{-m}} \left(\int |J_\alpha(z^{-1}x)-J_\alpha(z^{-1}y)|^{2\varepsilon}|g(z)|^2 dz\right)\left(\int |J_\alpha(z^{-1}x)-J_\alpha(z^{-1}y)|^{2(1-\varepsilon)}dz\right)d\mu'(x,y) \leq$$

$$C_1C_2||g||^2_{L^2}$$

where

$$C_1 = \sup_{|y^{-1}x|\cong 2^{-m}} \int |J_\alpha(z^{-1}x)-J_\alpha(z^{-1}y)|^{2(1-\varepsilon)}dz$$

and

$$C_2 = \sup_z \int\int_{|y^{-1}x|\cong 2^{-m}} |J_\alpha(z^{-1}x)-J_\alpha(z^{-1}y)|^{2\varepsilon} d\mu'(x,y)$$

$$C_1 = \sup_{|y^{-1}x|\cong 2^{-m}} \int |J_\alpha(z^{-1}x)-J_\alpha(z^{-1}y)|^{2(1-\varepsilon)}dz = I + II$$

$$I \leq C\int_{|z^{-1}x|\leq 2^{-m}} (|z^{-1}x|^{\alpha-Q}+|z^{-1}y|^{\alpha-Q})^{2(1-\varepsilon)}dz$$

but we note that if $|z^{-1}x| \leq 2^{-m}$ and $|y^{-1}x| \cong 2^{-m}$, then also $|z^{-1}y| \leq 2^{-m+10}$, and thus

$$I \leq C\int_{|z^{-1}x|\leq 2^{-m}} |z^{-1}x|^{2(1-\varepsilon)(\alpha-Q)} \leq C2^{-2m(1-\varepsilon)(\alpha-Q)}2^{-mQ}$$

as long as

$$2(1-\varepsilon)(\alpha-Q)+Q > 0 \tag{3.37}$$

by the mean value theorem we have

$$II \leq C2^{-2m(1-\varepsilon)}\int_{|z^{-1}x|\geq 2^{-m}} |z^{-1}x|^{2(\alpha-Q-1)(1-\varepsilon)}dz \leq C2^{-2m(1-\varepsilon)(\alpha-Q)}2^{-mQ}$$

provided that

$$2(\alpha-Q-1)(1-\varepsilon)+Q < 0 \tag{3.38}$$

combining (3.37) and (3.38) we get that ε has to be chosen such that

$$1-\frac{Q}{2(Q+1-\alpha)} < \varepsilon < 1-\frac{Q}{2(Q-\alpha)} \tag{3.39}$$

The estimates for C_2 are similar to C_1. Let $E_1 = \{(x,y) : |y^{-1}x| \cong 2^{-m}, |z^{-1}x| \leq 2^{-m}\}$ and E_2 the complement of E_1 in $|y^{-1}x| \cong 2^{-m}$, then we have

$$C_2 \leq I' + II'$$

$$I' = \int\int_{E_1} |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{2\varepsilon} d\mu'(x,y) \leq$$

$$C2^{m(Q-1)} \int_{|z^{-1}x|\leq 2^{-m}} |z^{-1}x|^{2(\alpha-Q)\varepsilon} \mu(B(x,2^{-m}) \cap M) d\mu(x) \leq$$

$$C\int_0^{2^{-m}} r^{2(\alpha-Q)\varepsilon} d(\mu(B(z,r)\cap M)) \leq C2^{-2m(\alpha-Q)\varepsilon}2^{-m(Q-1)}$$

as long as $2(\alpha - Q)\varepsilon + Q - 1 > 0$ i.e.;

$$\varepsilon < \frac{Q-1}{2(Q-\alpha)} \tag{3.40}$$

The same estimate is satisfied by II except the condition on ε is now

$$\varepsilon > \frac{Q-1}{2(Q+1-\alpha)} \tag{3.41}$$

combining (3.40) and (3.41) we get that ε must be such that

$$\frac{Q-1}{2(Q+1-\alpha)} < \varepsilon < \frac{Q-1}{2(Q-\alpha)} \tag{3.42}$$

Now it suffices to note that

$$C_1C_2 = C2^{-2m\beta}$$

which is what we have set to prove provided that ε can be chosen so as to satisfy both requirements (3.39) and (3.42) . Solving this system of inequalities we see that it is the same as $\frac{1}{2} < \alpha < \frac{3}{2}$.
□

3.2.2 The case $\beta \geq 1$

Let k be the integer such that $0 \leq k < \beta \leq k+1$, then we want to prove that the system of functions $\{f_J = X_J f|_M, |J| \leq k\}$ belongs to B^2_β. Let us note the functions f_J are well defined, this is because $X_J f \in S^2_{\alpha-|J|}$ and $\alpha - |J| > \frac{1}{2}$. Recall that this means that for each net $\mathcal{F}_m$ of size 2^{-m} , and each set $Q \in \mathcal{F}_m$ there is a polynomial $P_Q(t,z)$ in the ambiant space of degree $N \geq k+1$ such that

1. For all $|J| \leq k$ we have

$$\sum_{Q\in\mathcal{F}_m} \int_Q |f_J - X_J P_Q|^2 d\mu \leq (c_m 2^{-m(\beta-|J|)})^2$$

2. For $k+1 \leq |J| \leq N$

$$\sum_{Q\in\mathcal{F}_m} \sum_{Q'\in\mathcal{F}_{m+1}(Q)} \int_Q |X_J(P_Q - P_{Q'})|^2 d\mu \leq (c_m 2^{-m(\beta-|J|)})^2$$

and
3. For $|J| \leq N$ we have

$$\sum_{Q\in\mathcal{F}_0} \int_Q |X_J P_Q|^2 d\mu \leq c_0$$

the constants c_m satisfy $\sum_{m=0}^{\infty} c_m^2 < \infty$. We start by the proof of 1. For $Q \in \mathcal{F}_m$, we let P_Q be the left Taylor polynomial, of degree N, given by the formula (2.29) and (2.30) of chapter 2, of the finite sum

$$F_m = \sum_{j=0}^{m} f_j$$

taken arround any fixed point of Q, say its center x_Q. Let us write those formulas again in this context

$$P_Q(x) = \sum_{|J|\leq N} C_J(x_Q^{-1}x)^J (X_J F_m)(x_Q)$$

denote by $R_Q(x)$ the remainder $F_m - P_Q$, then

$$R_Q(x) = \frac{1}{N!} \sum_{N+1\leq|J|\leq 2(N+1)} C_J(x_Q^{-1}x)^J \int_0^1 s^{|J|-N-1}(1-s)^N (X_J F_m)(x_Q(s(x_Q^{-1}x)))ds$$

Estimates in 1. follows if we prove
(1').

$$\sum_{Q\in\mathcal{F}_m} \int_Q |X_J R_Q(x)|^2 d\mu(x) \leq (c_m 2^{-m(\beta-|J|)})^2$$

and
(1").

$$\sum_{Q\in\mathcal{F}_m} \int_Q |X_J(\sum_{j=m+1}^{\infty} f_j(x))|^2 d\mu(x) \leq (c_m 2^{-m(\beta-|J|)})^2$$

Using lemma 12(b) and Minkowski;s inequality we have

$$\sum_{Q\in\mathcal{F}_m} \int_Q |X_J(\sum_{j=m+1}^{\infty} f_j)|^2 d\mu(x) \leq C2^m(\sum_{j=m+1}^{\infty} ||X_J f_j||)^2 + C2^{-m}(\sum_{j=m+1}^{\infty} ||\tilde{\nabla}(X_J f_j)||)^2 \leq$$

$$C2^m(\sum_{j=m+1}^{\infty} 2^{-j(\alpha-|J|)}a_j)^2 + C2^{-m}(\sum_{j=m+1}^{\infty} 2^{-j(\alpha-|J|-1)}a_j)^2 \leq$$

$$(c_m 2^{-m(\beta-|J|)})^2$$

by Hardy's inequality we have

$$\sum_{m=0}^{\infty} c_m^2 = C\sum_{m=0}^{\infty} 2^{2m(\beta-|J|+\frac{1}{2})}(\sum_{j=m+1}^{\infty} 2^{-j(\alpha-|J|)}a_j)^2 \leq$$

$$C\sum_{m=0}^{\infty} a_m^2 = C||f||_{S_\alpha^2}^2$$

and this takes care of (1").
Next, $X_J R_Q(x)$ is a sum of terms of the type

$$A(x) = C\sum_{J'} C_{J'} X_{J_1}(x_Q^{-1}x)^{J'} \int_0^1 s^{(|J'|-N-1)}(1-s)^{(N+|J_2|)}(X_{J_2}X_{J'}F_m)(x_Q(s(x_Q x)))ds$$

with $J = J_1 + J_2$.

$$|A(x)| \le C\sum_{J'} 2^{-m(|J'|-|J_1|)}(\int_0^1 |X_{J_2}X_{J'}F_m(x_Q(s(x_Q^{-1}x)))|^2 ds)^{1/2}$$

invoking lemma 12(b), it follows that

$$\sum_{Q\in\mathcal{F}_m}\int_Q |A(x)|^2 d\mu(x) \le$$

$$C\sum_{J'} 2^{-2m(|J'|-|J_1|-\frac{1}{2})}\sum_{Q\in\mathcal{F}_m}\int_{Q^*}|X_{J_2}X_{J'}F_m|^2 dvol + \sum_{J'} 2^{-2m(|J'|-|J_1|+\frac{1}{2})}\sum_{Q\in\mathcal{F}_m}\int_{Q^*}|\tilde{\nabla}X_{J_2}X_{J'}F_m|^2 dvol$$

$$\le C\sum_{J'} 2^{-2m(|J'|-|J_1|-\frac{1}{2})}(\sum_{j=0}^{m} 2^{-j(\alpha-|J'|-|J_2|)}a_j)^2 + C\sum_{J'} 2^{-2m(|J'|-|J_1|+\frac{1}{2})}(\sum_{j=0}^{m} 2^{-j(\alpha-|J'|-|J_2|-1)}a_j)^2 \le$$

$$(c_m 2^{-m(\beta-|J|)})^2$$

by keeping in mind that $|J'| \ge N+1$ we can apply Hardy's inequality to conclude, and this takes care of (1').
3. is straightforward we have,

$$\sum_{Q\in\mathcal{F}_0}\int_Q |X_J P_Q|^2 d\mu \le$$

$$C\sum_{Q\in\mathcal{F}_0}\int_Q |X_J(f_0 - P_Q)|^2 d\mu + C\sum_{Q\in\mathcal{F}_0}\int_Q |X_J f_0|^2 d\mu \le$$

$$C\sum_{Q\in\mathcal{F}_0}\int_{Q^*}|X_J(f_0 - P_Q)|^2 dvol + C\sum_{Q\in\mathcal{F}_0}\int_{Q^*}|\tilde{\nabla}X_J(f_0 - P_Q)|^2 dvol +$$

$$C\sum_{Q\in\mathcal{F}_0}\int_{Q^*}|X_J f_0|^2 dvol \le C\sum_{Q\in\mathcal{F}_0}\int_{Q^*}|\tilde{\nabla}X_J f_0|^2 dvol \le c_0$$

Finally we prove 2. For $Q \in \mathcal{F}_m$ and $Q' \in \mathcal{F}_{m+1}$, we write the difference $P_Q - P_{Q'}$ like,

$$P_Q(x) - P_{Q'}(x) = \sum_{j=0}^{m}(r_j(x_Q, x) - r_j(x_{Q'}, x)) + r_{m+1}(x_{Q'}, x) - f_{m+1}(x) \tag{3.43}$$

where $r_j(z,x) := f_j(x) - p_j(z,x)$ is the Taylor remainder of the function f_j taken arround the point z. And so,

$$\sum_{Q\in\mathcal{F}_m} \int_Q |X_{J'}(P_Q - P_{Q'})|^2 d\mu$$

$$\leq \sum_{Q\in\mathcal{F}_m} \int_Q (\sum_{j=0}^{m} X_{J'}(r_j(x_Q,x) - r_j(x_{Q'},x)))^2 d\mu + \sum_{Q\in\mathcal{F}_m} \int_Q |X_{J'} r_{m+1}(x_{Q'},x)|^2 d\mu + \sum_{Q\in\mathcal{F}_m} \int_Q |X_{J'} f_{m+1}|^2 d\mu \tag{3.44}$$

The last two terms in the right hand side of (3.44) are dealt with in the same way as before by invoking lemma 12(b), and they are both less than or equal to

$$(c_{m+1} 2^{-(m+1)(\beta - |J'|)})^2$$

By Taylor's formula we can write the difference $r_j(z,x) - r_j(y,x)$ as ;

$$r_j(z,x) - r_j(y,x) = \sum_{1\leq|J_1|\leq 2} C_{J_1}(z^{-1}y)^{J_1} \int_0^1 s^{|J_1|-1}(X^w_{J_j}(r_j(w,x)))|_{w=(z(s(z^{-1}y)))} ds \tag{3.45}$$

the notation X^w means that the action is on the w variable. Writing out the integral representation of $r_j(w,x)$ and taking the derivatives $X^w r_j(w,x)$ we get that $r_j(z,x) - r_j(y,x)$ is a sum of terms of the type

$$C(z^{-1}y)^{J_1} \int_0^1 \int_0^1 s^{|J_1|-1} t^{|J|-N-1}(1-t)^{N+|J_2|}(X_{J_3}((w^{-1}x)^J))(X_{J_2} X_J f_j(w(t(w^{-1}x)))|_{w=(z(s(z^{-1}x)))} ds dt \tag{3.46}$$

with $J_1 = J_3 + J_2$. Applying $X_{J'}$ to this typical term we get that $X_{J'}(r_j(z,x) - r_j(y,x))$ is a sum of terms like;

$$C(z^{-1}y)^{J_1} \int_0^1 \int_0^1 s^{|J_1|-1} t^{|J|-N-1}(1-t)^{N+|J_2|}(X_{J_1'} X_{J_3}((w^{-1}x)^J))(X_{J_2'} X_{J_2} X_J f_j(w(t(w^{-1}x)))) ds dt \tag{3.47}$$

with $J' = J_1' + J_2'$ and the expression in the integrand of (3.47) is supposed to be evaluated at $w = (z(s(z^{-1}x)))$. Taking into account

$$|z^{-1}y| \leq 2^{-m+2}$$

and that

$$|w^{-1}x| = |(s(z^{-1}y))y^{-1}x| \leq C(s|z^{-1}y| + |y^{-1}x|) \leq 2^{-m+10}$$

also

$$|X_{J_1'} X_{J_3}((w^{-1}x)^J)| \leq C 2^{-m(|J|-|J_1'|-|J_3|)}$$

if $|J| - |J_1' - |J_3| \geq 0$ which is the case. The terms of the type (3.47) are in absolute value less than or equal to

$$C 2^{-m(|J_1|+|J|-|J_1'|-|J_3|)} \left(\int_0^1 \int_0^1 |X_{J_2'} X_{J_2} X_J f_j(w(t(w^{-1}x))))|^2|_{w=(z(s(z^{-1}x)))} ds dt\right)^{1/2} \tag{3.48}$$

summing over j and square it integrating it over Q, using lemma 12(b) and the estimates on the f_j's over Q, and then summing over $Q \in \mathcal{F}_m$ using Minkowski's inequality we get finally

$$\sum_{Q\in\mathcal{F}_m}\int_Q\Big(\sum_{j=0}^{m} X_{J'}(r_j(x_Q,x)-r_j(x_{Q'},x))\Big)^2 d\mu \le$$

$$C2^{-2m(|J|+|J_1|-|J_1'|-|J_3|+\frac{1}{2})}\Big(\sum_{j=0}^{m} 2^{-j(\alpha-|J|-|J_2'|-|J_2|)}a_j\Big)^2+$$

$$C2^{-2m(|J|+|J_1|-|J_1'|-|J_3|-\frac{1}{2})}\Big(\sum_{j=0}^{m} 2^{-j(\alpha-|J|-|J_2'|-|J_2|-1)}a_j\Big)^2 \le$$

$$(c_m 2^{-m(\beta-|J'|)})^2$$

and the proof of the restriction theorem is now complete.
□

Chapter 4

The Extension Theorem

This chapter is concerned with the extension of functions, or rather systems of functions, defined on a surface M, to the ambient space. It is the converse to the restriction theorem. It was our goal to prove an extension theorem for the spaces F^2_β instead of B^2_β, of course that would have been a stronger result. We still do not know wether it is possible or not. Another simple way to ask the question is: are the spaces B^2_β and F^2_β equivalent? The answer is yes for $0 < \beta < 1$ and would be yes if we can prove an extension to the ambient space for the space F^2_β, since the restriction is bounded from the space $S^2_{\beta+\frac{1}{2}}$ to F^2_β. The extension described below is based on two main ingredients, the first objects are polynomials and the second is the Whitney decomposition of open sets in the nonisotropic metric. This technique of extending functions is classical and goes back to Whitney (1930's) who used it to extend Lipschitz functions defined on an arbitray closed set of R^n see [WH] and [ST1] for more on this. Recall that in the classical case (isotropic) the extension using the Whitney decomposition is as follows. First flatten the hypersurface M by a local coordinate system $(x_1, \cdots, x_n)$ such that $M = \{x_n = 0\}$. Now we can talk about polynomials in the coordinates $(x_1, \cdots, x_{n-1})$ and use them to define the spaces F^2_β. Polynomials p in R^{n-1} have obvious extension to R^n, just extend them to be constant along the vertical direction (the x_n-axis). These extended polynomials satisfy the following trivial but important property: let B be any Euclidian ball of radius $r > 0$, and centered on M then,

$$\int_B |p(x_1, \cdots, x_{n-1})|^2 dvol \leq Cr \int_{B\cap M} |p(x_1, \cdots, x_{n-1})|^2 d\sigma \tag{4.1}$$

Once we have this, given $f \in F^2_\beta$ we define its extension on a Whitney cube to be the best polynomial approximation (in the $L^2(d\sigma)$-sense) of$f$ on a corresponding ball on M. With (4.1) and the so called Markov's inequality (see [JW]) we can prove that the extension is bounded etc$\cdots$

Of course in the Euclidian case there are other possible ways of extensions for example the reflection method, see [AD] and [ST1] [ST2] for example, is applicable for smooth hypersurfaces, there is also the method of Calderon that one can use when we have a surface that is smooth. The extension of Whitney type is a powerful tool when we don't have smoothness of the surface. In [JO], one of the most recent (1984) papers dealing with extendibility of

classical Sobolev spaces defined on "almost arbitrary open sets", the so called (ε, δ)-domains, this method is used successfully.
Let us return to the nonisotropic setting. The first thing to say is that (1) is false if we replace the Euclidian ball B by a nonisotropic one and the surface measure by the the measure $d\mu$, except if the surface is noncharacteristic or the polynomial is actually a constant, in the former case it is true because one can choose a coordinate system $(x_1, \cdots, x_{2n+1})$ such that $M = \{x_{2n+1} = 0\}$ and for which a transverse vector field say X_{2n} is given by $\frac{\partial}{\partial x_{2n+1}}$, and this of course makes (4.1) valid . We don't have a canonical way of extending polynomials to the ambient space and still have (4.1) valid in this coordinate system, the reason being the fact that when we extend by constant along the vertical direction we might exit the balls too quickly especially when the balls are centered at characteristic points. We can use normal coordinates to extend polynomials, but this requires us to fix a base point on M and this means that the same polynomial has different extensions a property that we certainly do not want. One may seek a universal vector field along which we can perform constant extension. This idea is in fact under investigation and it seems to work at least for $0 < \alpha < 2$, for the characteristic hyperplane and hence would work for hypersurfaces with isolated characteristic points. Which is an improvement, unfortunetly the details of this technique have not been worked out and therefore cannot be included in here, they will appear elsewhere in the future. This technical difficulty led us to seek other methods to deal with the higher derivatives case. The method of Jonsson and Wallin is applicable in our context. This method requires the knowledge of all derivatives even transverse ones to be able to construct extensions, this of course in their setting was necessary, because they deal with complicated d-sets. Let us state the main theorem.

Theorem 8 *Let $0 \leq k < \beta \leq k+1$. There is an extension operator* $\mathbf{E}$ *taking elements $f := \{f_J : |J| \leq k\}$ of $B^2_\beta(M)$ to functions defined on the ambient space such that:*

$$\mathbf{E} : B^2_\beta(M) \longrightarrow S^2_\alpha$$

is bounded and such that $X_J \mathbf{E}(f)|_M = f_J$ almost everywhere, where $\alpha = \beta + \frac{1}{2}$.

After constructing the extension operator $\mathbf{E}$, we plan the proof of the theorem as follows. First, because the difficulty is only technical, we treat the case $0 < \beta < 1$, in this case constants are enough to use. This case is important because the spaces F^2_β and B^2_β are the same. We start with $\beta = \frac{1}{2}$, and show that the first derivatives are in L^2 and then use other possible characterizations of the spaces S^2_α.

4.1 The Whitney decomposition

The construction of the extension operator is achieved using the Whitney decomposition and a partition of unity. Recall that this cover is obtained by covering Ω by balls of radius 2^{-k} and then selecting from this balls that are at distance 10×2^{-k} away from M, for $k = 0, 1, \cdots$, integer. This cover is required to meet the condition that the balls do not get pilled up too much, this means that if B and B' are two distinct balls with nonempty intersection then if we shrink them enough they end up disjoint. We summarize all what we need to know about the Whitney decomposition in the following theorem. We refer to [ST1] for the proofs.

Theorem 9 *Let M be a closed subset of Ω, then there are nonisotropic balls W covering $\Omega \setminus M$ with the following properties:*
(a)
$\Omega \setminus M \subset \cup_{B\in W} B$
(b)
$\frac{1}{10} r(B) \leq dist(B, M) \leq 10 r(B)$
(c)
$B_* \cap B'_* = \emptyset$*, where the notation B_* means the ball with the same center as B but with radius $\frac{1}{10}$th the radius of B.*
(d)If B and B' are two balls such that $B^* \cap B'^* \neq \emptyset$, then

$$\frac{1}{10} \leq \frac{r(B)}{r(B')} \leq 10$$

(e)Finally No point in $\Omega \setminus M$ belongs to more than 100 balls of W

In particular by property (d) we can cover the surface M by sets of the form $B^{**} \cap M$. Recall also that membership to the spaces F^2_α or B^2_α is independent of the covering $\mathcal{F}_k$, and hence we can use W_k to define a grid on M, $\mathcal{F}_k := \{B^{**} \cap M, B \in W_k\}$. In particular for small β a possible norm for the spaces B^2_β and F^2_β is given by:

$$||f||^2_{F^2_\beta} = ||f||^2_{L^2(d\mu)} + \sum_{k=0}^{\infty} 2^{2k\beta} \sum_{B\in W_k} \inf_c \int_{B^{**}\cap M} |f-c|^2 d\mu \tag{4.2}$$

Next we let φ_B be a partition of unity subbordinate to the cover W. The support of φ_B is contained in B^*, it is identically equal to 1 in B_*. The two properties essential in what follows about the φ_B's are:
For all $x \in \Omega \setminus M$ we have

$$\sum_{B\in W} \varphi_B(x) = 1 \tag{4.3}$$

and

$$|X_J \varphi_B(x)| \leq C r(B)^{-|J|} \tag{4.4}$$

4.2 Proof of theorem 8

4.2.1 The case $0 < \beta < 1$

Let $f \in F^2_\beta$, for each ball $B \in W$ we denote by c_B the constants in the definition of F^2_β, such that

$$\sum_{B\in W_k} \int_{B^{**}\cap M} |f - c_B|^2 d\mu \leq (c_k 2^{-k\beta})^2 \tag{4.5}$$

we can be more precise with the constant c_B to take it the best constant approximating f in the $L^2(d\mu)$-norm i.e.; its average over the set $B^{**} \cap M$, but that is not necessary. The extension $\mathbf{E}$ is given by,

$$\mathbf{E}(f)(x) = \sum_{B\in W} \varphi_B(x) c_B \quad for \quad x \in \Omega \setminus M \tag{4.6}$$

The sum defining the extension is locally finite and hence defines a smooth function away from M. Let us check that the resulting function is an element of $L^2(\Omega)$. Indeed, we have

$$||\mathbf{E}f||^2 \leq \sum_{B\in W} \sum_{B'\in W(B)} \int_B |c_{B'}|^2 = C \sum_{B\in W} |B| \sum_{B'\in W(B)} |c_{B'}|^2$$

where $|B|$ denotes the volume of B. But by theorem (2)-e the number of balls in the inner sum is bounded by 100, hence

$$||\mathbf{E}f||^2 \leq C \sum_{B\in W} |B||c_B|^2 = \sum_{B\in W} \frac{|B|}{\mu(B^{**}\cap M)} \int_{B^{**}\cap M} c_B^2 d\mu$$

$$\leq$$

$$C \sum_{k=0}^{\infty} 2^{-k} \sum_{B\in W_k} \int_{B^{**}\cap M} |f - c_B|^2 d\mu + C||f||^2_{L^2(d\mu)} \leq C||f||^2_{F^2_\beta} \quad for \ any \quad 0 < \beta < 1$$

Suppose now that $\beta = \frac{1}{2}$, then we need to check that the first order derivatives $X(\mathbf{E}f)$ are in L^2. By (3) we have

$$\sum_{B\in W} X\varphi_B(x) \equiv 0 \tag{4.7}$$

and so

$$||X(\mathbf{E}f)||^2 \leq$$

$$\sum_{B\in W} r(B)^{-2}|B| \sum_{B'\in W(B)} |c_B - c_{B'}|^2 \leq C \sum_{B\in W} r(B)^{-2} \frac{|B|}{\mu(B^{**}\cap M)} \sum_{B'\in W(B)} \int_{B^{**}\cap M} |c_B - c_{B'}|^2 d\mu$$

$$\leq$$

$$C \sum_{k=0}^{\infty} 2^{k} \sum_{B\in W_k} \int_{B^{**}\cap M} |f - c_B|^2 d\mu \leq C||f||^2_{F^2_{\frac{1}{2}}}$$

To prove the estimates needed for the full range $0 < \beta < 1$ we distinguish between two cases. The first case is when $0 < \beta < \frac{1}{2}$ and the second when $\frac{1}{2} < \beta < 1$. Instead of proving directly that the extended function is in the Sobolev space S^2_α, we may, in the first case, prove that given any tiling $\mathcal{F}_k$ of the ambient space (not to be confused with the tiling of the surface M which in the present situation is given to us by the Whitney decomposition), we have

$$\sum_{B\in\mathcal{F}_k} \inf_c \int_B |F(x) - c|^2 dx \leq 2^{-2k(\beta+\frac{1}{2})} c_k^2$$

for some c_k satisfying

$$\sum_{k=0}^{\infty} c_k^2 \leq C||f||^2_{F^2_\beta}$$

in the second case we have to check that the same holds for the derivatives of first order.

Now let $\mathcal{F}_k$ be a net of mesh of 2^{-k} for the ambient space, and let $B \in \mathcal{F}_k$, suppose also that it is contained in a Whitney ball Q_0, then by Poincare inequality, we have

$$\inf_c \int_B |F(x) - c|^2 dx \leq Cr(B)^2 \int_B |\tilde{\nabla} F(x)|^2 dx$$

but by (4.7)

$$\tilde{\nabla} F(x) = \sum_{Q \in W} c_Q \tilde{\nabla} \varphi_Q(x) = \sum_{Q \in W} (c_{Q_0} - c_Q) \tilde{\nabla} \varphi_Q(x)$$

and thus we have,

$$\inf_c \int_B |F(x) - c|^2 dx \leq C(\frac{r(B)}{r(Q_0)})^2 |B| \sum_{Q \in W(Q_0)} |c_Q - c_{Q_0}|^2$$

and since for $Q \in W(Q_0)$ we have $r(Q) \sim r(Q_0)$, we get that

$$\inf_c \int_B |F(x) - c|^2 dx \leq C(\frac{r(B)}{r(Q)})^2 \frac{|B|}{\mu(Q_0 \cap M)} \sum_{Q \in W(Q_0)} \int_{Q^* \cap M} |f(x) - c_Q|^2 d\mu(x) \tag{4.8}$$

now the balls in $\mathcal{F}_k$ that are contained in Whitney balls are those that are far away from M, denote those balls by $\mathcal{F}_k(W)$ i.e.,

$$\mathcal{F}_k(W) = \{B \in \mathcal{F}_k : B \subset Q, for\ some\ \ Q \in W\}$$

with this we have

$$\sum_{B \in \mathcal{F}_k} \inf_c \int_B |F(x) - c|^2 dx = \sum_{B \in \mathcal{F}_k(W)} + \sum_{B \notin \mathcal{F}_k} = S_{k,1} + S_{k,2}$$

Let the letter s (instead of Q) denote temporarily the homogeneous dimension (s=2n+2), using (4.8)

$$S_{k,1} \leq C \sum_{Q \in W} \sum_{B \in \mathcal{F}_k(Q)} \int_B (\frac{r(B)}{r(Q)})^2 \frac{|B|}{\mu(Q^* \cap M)} \sum_{Q' \in W(Q)} \int_{Q'^* \cap M} |f(x) - c_{Q'}|^2 d\mu(x)$$

$$\leq$$

$$C \sum_{m=0}^{k} 2^{-2(k-m)} 2^{-ks} 2^{m(s-1)} \sum_{Q \in W_m} \sum_{B \in \mathcal{F}_k(Q)} \sum_{Q' \in W(Q)} \int_{Q'^* \cap M} |f(x) - c_{Q'}|^2 d\mu(x)$$

note also that

$$\sum_{B \in \mathcal{F}_k(Q)} 1 \leq C 2^{(k-m)s}$$

using this we finally get

$$S_{k,1} \leq C2^{-2k} \sum_{m=0}^{k} 2^m \sum_{Q \in W_m} \int_{Q^* \cap M} |f(x) - c_Q|^2 d\mu(x) \leq C2^{-2k(\beta+\frac{1}{2})} c_k^2$$

where

$$c_k^2 = 2^{-2k(\frac{1}{2}-\alpha)} \sum_{m=0}^{k} 2^{2m(\frac{1}{2}-\alpha)} a_m^2$$

and as usual by Hardy's inequality we conclude that

$$\sum_{k=0}^{\infty} c_k^2 \leq C \sum_{k=0}^{\infty} a_k^2$$

Now we turn to balls $B \notin \mathcal{F}_k(W)$, pick any cube $Q \in W_{k+1}$ that is contained in B and set

$$c_B := c_Q$$

$$\int_B |F(x) - c_B|^2 dx = \sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} \int_Q |F(x) - c_B|^2 dx$$

$$\int_Q |F(x) - c_B|^2 dx \leq C|Q| \sum_{Q' \in W(Q)} |c_{Q'} - c_B|^2 \leq$$

$$C2^{-m} \int_{Q^* \cap M} |f(x) - c_B|^2 d\mu(x) + \sum_{Q' \in W(Q)} C2^{-m} \int_{Q'^* \cap M} |f(x) - c_{Q'}|^2 d\mu(x)$$

$$=$$

$$I(Q) + II(Q)$$

$$\sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} I(Q) \leq C \sum_{m=k+1}^{\infty} 2^{-m} \int_{B^{**} \cap M} |f(x) - c_B|^2 d\mu(x) \leq C2^{-k-1} \int_{Q^{**} \cap M} |f(x) - c_Q|^2 d\mu(x)$$

and summing over $B \notin \mathcal{F}_k(W)$ we get

$$\sum_{B \notin \mathcal{F}_k(W)} \sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} I(Q) \leq C2^{-k-1} \sum_{Q \in W_{k+1}} \int_{Q^* \cap M} |f(x) - c_Q|^2 d\mu(x) \leq C2^{-2(k+1)(\alpha+\frac{1}{2})} a_{k+1}^2$$

summing $II(Q)$ over $Q \in W_m(B)$ and then over m and then over $B \in \mathcal{F}_k$ we get

$$\sum_{B \in \mathcal{F}_k} \sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} II(Q) \leq C \sum_{m=k+1}^{\infty} 2^{-m} \sum_{Q \in W_m} \int_{Q^* \cap M} |f(x) - c_Q|^2 d\mu(x)$$

$$\leq C \sum_{m=k+1}^{\infty} 2^{-2m(\beta+\frac{1}{2})} a_m^2 \leq 2^{-2k(\alpha+\frac{1}{2})} c_k^2$$

where

$$c_k^2 = C \sum_{2k(\alpha+\frac{1}{2})} \sum_{m=k+1}^{\infty} 2^{-2m(\alpha+\frac{1}{2})} a_m^2$$

and by the second part of Hardy's inequality we get that

$$\sum_{k=0}^{\infty} c_k^2 \leq C \sum_{k=0}^{\infty} a_k^2$$

By the same method we get the estimates needed for the derivatives of first oredr in the case where $\frac{1}{2} < \beta < 1$.
□

4.2.2 The case $\beta \geq 1$

Now we come to the description of the extension operator in the general case. Unfortunetly we have to leave the spaces F_β^2 and deal with B_β^2 whose elements are systems of functions. Here we assume that the integer $N >> \beta$. Let $0 \leq k < \beta \leq k+1$ and $f := \{f_J : |J| \leq k\} \in B_\beta^2$, recall that this means that for each $B \in W$ there is a polynomial P_B of degree $\leq N$ such that:
(a)

$$\sum_{B \in \mathcal{F}_m} \int_{B^{**} \cap M} |f_J - X_J P_B|^2 d\mu \leq (c_m 2^{-m(\beta-|J|)})^2, \quad for\ all \quad |J| \leq k$$

(b)

$$\sum_{B \in \mathcal{F}_m} \sum_{B' \in W(B)} \int_{B^{**} \cap M} |X_J(P_B - P_{B'})|^2 d\mu \leq (c_m 2^{-m(\beta-|J|)})^2, \quad for \quad k+1 \leq |J| \leq N$$

(c)

$$\sum_{B \in \mathcal{F}_0} \int_{B^{**} \cap M} |X_J P_B|^2 d\mu \leq c_0, \quad for \quad all \quad |J| \leq N$$

The extension of $f = \{f_J\}$ is now given by

$$F(x) := \mathbf{E}(f)(x) = \sum_{Q \in W} \varphi_Q(x) P_Q(x), \quad for \quad x \notin \Omega \setminus M \tag{4.10}$$

Let $\alpha = \beta + \frac{1}{2}$, The first thing that needs to be checked is that

$$||X_J F|| \leq C(\sum_{m=0}^{\infty} c_m^2)^{1/2} \qquad for \quad |J| \leq \alpha \tag{4.11}$$

let $x \notin M$, then $X_J F(x)$ is a sum of two types of terms:

$$A_{J',J''}(x) = \sum_{Q \in W} X_{J'} \varphi_Q(x) X_{J''} P_Q(x), \quad J' + J'' = J, |J'| \neq 0$$

and

$$B_J(x) = \sum_{Q\in W} \varphi_Q(x) X_J P_Q(x)$$

We need the following lemma

Lemma 13 *Let $P(x)$ be a polynomial and Q a nonisotropic ball such that, $\mu(Q^{**} \cap M) \cong r(Q)^{s-1}$ then*

$$\int_Q |P(x)|^2 dx \leq C \sum_J r(Q)^{1+2|J|} \int_{Q^{**}\cap M} |X_J P_Q(x)|^2 d\mu \tag{4.12}$$

Proof:
It suffices to note that if Q is unit size then both quantities of (4.12) define norms and they are equivalent. Rescale to get the result.
□
By (4.7) and the lemma

$$||A_{J',J"}||^2 \leq C \sum_{Q\in W} \int_Q |A_{J',J"}(x)|^2 dx \leq C \sum_{Q\in W} r(Q)^{-2|J'|} \sum_{Q'\in W(Q)} \int_Q |X_{J"}(P_Q - P_{Q'})|^2 dx$$

$$\leq$$

$$C \sum_{\tilde{J}} \sum_{Q\in W} r(Q)^{1+2(|\tilde{J}|-|J'|)} \sum_{Q'\in W(Q)} \int_Q |X_{\tilde{J}} X_{J"}(P_Q - P_{Q'})|^2 d\mu$$

it easy to realize that

$$\sum_{\tilde{J}} r(Q)^{1+2(|\tilde{J}|-|J'|)} \int_{Q^*\cap M} |X_{\tilde{J}} X_{J''} P|^2 d\mu \leq C \sum_{|\tilde{J}|\geq|J''|} r(Q)^{1+2(|\tilde{J}|-|J|)} \int_{B^*\cap M} |X_{\tilde{J}} P|^2 d\mu$$

And hence

$$||A_{J',J"}||^2 \leq C \sum_{|J"|\leq|\tilde{J}|\leq k} \sum_{Q\in W} r(Q)^{1+2(|\tilde{J}|-|J|)} \sum_{Q'\in W(Q)} \int_Q |X_{\tilde{J}}(P_Q - P_{Q'})|^2 d\mu$$

$$+$$

$$C \sum_{|\tilde{J}|\geq k+1} \sum_{Q\in W} r(Q)^{1+2(|\tilde{J}|-|J|)} \sum_{Q'\in W(Q)} \int_Q |X_{\tilde{J}}(P_Q - P_{Q'})|^2 d\mu$$

by (a) and (b) we get at once

$$||A_{J',J"}||^2 \leq \sum_{m=0}^{\infty} 2^{-2m(\frac{1}{2}-|J|+\beta)} c_m^2 \leq \sum_{m=0}^{\infty} c_m^2$$

as long as $|J| \leq \beta + \frac{1}{2} = \alpha$.

Next we turn to B_J. This term is estimated in a different way.

$$||B_J||^2 \leq C \sum_{Q\in W} \int_B |B_J|^2 dx \leq C \sum_{Q\in W} \int_B |X_J P_Q|^2 dx = \sum_{m=0}^{\infty} \sum_{Q\in W_m} \int_B |X_J P_Q|^2 dx$$

For each $Q \in W_m$, $W_{m-1}(Q)$ denotes the balls in W_{m-1} touching the ball Q, the immediate ancestors of Q. By theorem 2-d there are at most a finite number of them say less than 100 for each ball Q, let us enumerate them keeping in mind that this enumeration depends on Q.

$$W_{m-1}(Q) = \{Q_1^1, Q_1^2, \cdots, Q_1^{N_1}\}, \quad where \quad N_1 = N_1(Q)$$

Put $Q_0^{N_0} = Q$, inductively we enumerate the set $W_{m-j}(Q_{j-1}^{N_{j-1}})$,

$$W_{m-j}(Q_{j-1}^{N_{j-1}}) = \{Q_j^1, Q_j^2, \cdots, Q_j^{N_j}\} \qquad for \qquad j = 1, 2, \cdots, m$$

with this we can write P_Q as a telescopic sum:

$$P_Q = (P_Q - P_{Q_1^1}) + (P_{Q_1^1} - P_{Q_1^2}) + \cdots + (P_{Q_1^{N_1-1}} - P_{Q_1^{N_1}}) + P_{Q_1^{N_1}}$$

$$=$$

$$(P_Q - P_{Q_1^1}) + (P_{Q_1^1} - P_{Q_1^2}) + \cdots + (P_{Q_1^{N_1-1}} - P_{Q_1^{N_1}}) + (P_{Q_1^{N_1}} - P_{Q_2^1}) + \cdots (P_{Q_2^{N_2-1}} - P_{Q_2^{N_2}}) + P_{Q_2^{N_2}}$$

$$=$$

$$(P_Q - P_{Q_1^1}) + \sum_{l=1}^{N_1-1} (P_{Q_1^l} - P_{Q_1^{l+1}}) + (P_{Q_1^{N_1}} - P_{Q_2^1}) + \sum_{l=1}^{N_2-1} (P_{Q_2^l} - P_{Q_2^{l+1}}) + \cdots + (P_{Q_{m-1}^{N_{m-1}}} - P_{Q_m^1})$$

$$+ \sum_{l=1}^{N_m-1} (P_{Q_m^l} - P_{Q_m^{l+1}}) + P_{Q_m^{N_m}}$$

$$=$$

$$\sum_{j=1}^{m} \sum_{l=1}^{N_j-1} (P_{Q_j^l} - P_{Q_j^{l+1}}) + \sum_{j=0}^{m-1} (P_{Q_j^{N_j}} - P_{Q_{j+1}^1}) + P_{Q_m^{N_m}}$$

Now use Minkowski's inequality to get

$$(\sum_{Q\in W_m} \int_B |X_J P_Q|^2 dx)^{1/2} \leq \sum_{j=1}^{m} (\sum_{Q\in W_m} \sum_{l=1}^{N_j-1} \int_Q |X_J (P_{Q_j^l} - P_{Q_j^{l+1}})|^2 dx)^{1/2}$$

$$+$$

$$\sum_{j=0}^{m-1} (\sum_{Q\in W_m} \int_Q |X_J (P_{Q_j^{N_j}} - P_{Q_{j+1}^1})|^2 dx)^{1/2} + (\sum_{Q\in W_m} \int_Q |X_J P_{Q_m^{N_m}}|^2 dx)^{1/2}$$

By the lemma we get

$$I=\sum_{j=1}^{m}(\sum_{Q\in W_m}\sum_{l=1}^{N_j-1}\int_Q |X_J(P_{Q_j^l}-P_{Q_j^{l+1}})|^2dx)^{1/2}\leq$$

$$C\sum_{\tilde{J}}2^{-m(\frac{1}{2}+|\tilde{J}|)}\sum_{j=1}^{m}(\sum_{Q\in W_m}\sum_{l=1}^{N_j-1}\int_{Q^*\cap M}|X_{\tilde{J}}X_J(P_{Q_j^l}-P_{Q_j^{l+1}})|^2d\mu)^{1/2}\leq$$

$$C\sum_{|\tilde{J}|\geq|J|}2^{-m(\frac{1}{2}+|\tilde{J}|-|J|)}\sum_{j=1}^{m}(\sum_{Q\in W_m}\sum_{l=1}^{N_j-1}\int_{Q^*\cap M}|X_{\tilde{J}}(P_{Q_j^l}-P_{Q_j^{l+1}})|^2d\mu)^{1/2}$$

Let us look at the double sum

$$\sum_{Q\in W_m}\sum_{l=1}^{N_j(Q)-1}\int_{Q^*\cap M}|X_{\tilde{J}}(P_{Q_j^l}-P_{Q_j^l})|^2d\mu$$

Since the balls Q_j^l in the second sum are the jth ancestors of the balls $Q\in W_m$ the first sum can be replaced by a sum over balls in $Q\in W_m$ whose projections on M $Q^*\cap M$ are contained on the projection of balls in W_{m-j} then followed by a sum over W_{m-j}. There might be repetition in this way but as we said they get repeated only a finite number of times and thus we have

$$\sum_{Q_m}\sum_{l=1}^{N_j(Q)-1}\int_{Q^*\cap M}|X_{\tilde{J}}(P_{Q_j^l}-P_{Q_j^l})|^2d\mu\leq C\sum_{Q\in W_{m-j}}\sum_{Q'\in W_{m-j}(Q)}\int_{Q^*\cap M}|X_{\tilde{J}}(P_Q-P_{Q'})|^2d\mu$$

and therefore we get

$$I\leq C\sum_{|\tilde{J}|\geq|J|}2^{-m(\frac{1}{2}+|\tilde{J}|-|J|)}\sum_{j=1}^{m}(\sum_{Q\in W_{m-j}}\sum_{Q'\in W_{m-j}(Q)}\int_{Q^*\cap M}|X_{\tilde{J}}(P_Q-P_{Q'})|^2d\mu)^{1/2}$$

By (a) (if $|\tilde{J}|\leq k$) and (b) (if $|\tilde{J}|\geq k+1$) we get that the last term is less than or equal to

$$C\sum_{|\tilde{J}|\geq|J|}2^{-m(\frac{1}{2}+|\tilde{J}|-|J|)}\sum_{j=1}^{m}2^{-j(\beta-|\tilde{J}|)}c_j$$

which when squared and summed over m, using Hardy's inequality, is less than or equal to

$$C\sum_{m=0}^{\infty}2^{-2m(\alpha-|J|)}c_m^2\leq C\sum_{m=0}^{\infty}c_m^2 \qquad for \qquad all \qquad |J|\leq\alpha$$

the other terms are dealt with similarly we

$$II=\sum_{j=0}^{m-1}(\sum_{Q\in W_m}\int_Q|X_J(P_{Q_j^{N_j}}-P_{Q_{j+1}^1})|^2dx)^{1/2}\leq$$

$$C\sum_{\bar{J}|\geq|J|}2^{-m(\frac{1}{2}+|\bar{J}|-|J|)}\sum_{j=0}^{m-1}(\sum_{Q\in W_m}\int_{Q^*\cap M}|X_{\bar{J}}(P_{Q_j^{N_j}}-P_{Q_{j+1}^1})|^2d\mu)^{1/2}\leq$$

$$C\sum_{|\bar{J}|\geq|J|}2^{-m(\frac{1}{2}+|\bar{J}|-|J|)}\sum_{j=0}^{m-1}(\sum_{Q\in W_{m-j}}\sum_{Q'\in W_{m-j+1}(Q)}\int_{Q^*\cap M}|X_{\bar{J}}(P_Q-P_{Q'})|^2d\mu)^{1/2}$$

Now square it and sum it over m like before to it less than or equal to

$$\sum_{m=0}^{\infty}2^{-2m(\alpha-|J|)}c_m^2$$

the last term is simpler we use estimate (c) to get it less than or equal to c_0. Thus we have proved that for all $|J|\leq\alpha$

$$||X_JF||\leq C(\sum_{m=0}^{\infty}c_m^2)^{1/2}\leq C||\{f_{J'}\}||_{B_\beta^2}$$

and hence if α is integer the extension theorem is proved.

Next we treat the case α noninteger. Let l be the integer such that $\beta-\frac{1}{2}<l<\beta+\frac{1}{2}=\alpha$, and set $\lambda=\alpha-l$, then like the proof in the case of small β we are going to prove that the derivatives X_JF belongs to S_λ^2 by showing that

$$\sum_{B\in\mathcal{F}_m}\inf_c\int_B|X_JF-c|^2dx\leq 2^{-2m\lambda}c_m^2\quad |J|\leq l \tag{4.13}$$

We take as before for each integer m a grid $\mathcal{F}_m$ of the ambient space. We prove (13) proving it for each of the terms $A_{J',J''}$ and B_J. Let B be a ball that is contained in a Whitney ball Q, then by Poincare inequality we have

$$\inf_c\int_B|A_{J',J''}-c|^2dx$$

$$\leq$$

$$Cr(B)^2\int_B|\tilde{\nabla}A_{J',J''}|^2\leq$$

$$Cr(B)^2\int_B|\sum_{Q'\in W(Q)}(\tilde{\nabla}X_{J'}\varphi_{Q'})(X_{J''}P_{Q'})|^2dx+Cr(B)^2\int_B|\sum_{Q'\in W(Q)}(X_{J'}\varphi_{Q'})(\tilde{\nabla}X_{J''}P_{Q'})|^2dx$$

$$=$$

$$I+II$$

both terms I and II satisfy the same estimates so we treat only one of them. Summing over the balls that are contained in Whitney balls we get

$$\sum_{B\in\mathcal{F}_m(W)}\inf_c\int_B|A_{J',J''}-c|^2dx\leq C2^{-2m}\sum_{j=0}^{m}2^{2j(|J'|+1)}\sum_{Q\in W_j}\sum_{B\in\mathcal{F}_m(Q)}\sum_{Q'\in W(Q)}\int_B|X_{J''}(P_Q-P_{Q'})|^2dx$$

$$\leq$$

$$C2^{-2m}\sum_{j=0}^{m}2^{2j(|J'|+1)}\sum_{Q\in W_j}\sum_{Q'\in W(Q)}\int_Q |X_{J''}(P_Q-P_{Q'})|^2dx$$

$$\leq$$

$$C2^{-2m}\sum_{|\bar{J}|\geq|J''|}\sum_{j=0}^{m}2^{2j(-|\bar{J}|+|J|+\frac{1}{2})}\sum_{Q\in W_j}\sum_{Q'\in W(Q)}\int_{Q^*\cap M}|X_{\bar{J}}(P_Q-P_{Q'})|^2d\mu \leq C2^{-2m}\sum_{j=0}^{m}2^{2j(|J|-\beta+\frac{1}{2})}c_j^2$$

$$\leq 2^{-2m\lambda}a_m^2$$

where

$$\sum_{m=0}^{\infty}a_m^2 \leq C\sum_{m=0}^{\infty}2^{-2m(l-|J|)}c_m^2 \leq C\sum_{m=0}^{\infty}c_m^2 \quad for \quad |J|\leq l$$

Next we turn to balls that are near M i.e. $B\notin\mathcal{F}_m(W)$. For these we show that

$$\sum_{B\notin\mathcal{F}_m(W)}\int_B |A_{J',J''}|^2dx \leq 2^{-2m\lambda}c_m^2$$

$$\int_B |A_{J',J''}|^2dx \leq C\sum_{j=m}^{\infty}\sum_{Q\in W_j(B)}\int_Q|\sum_{Q'\in W(Q)}(X_{J'}\varphi_{Q'})(X_{J''}(P_Q-P_{Q'}))|^2dx \leq$$

$$C\sum_{j=m}^{\infty}2^{2j|J'|}\sum_{Q\in W_j(B)}\sum_{Q'\in W(Q)}\int_Q|X_{J''}(P_Q-P_{Q'})|^2dx \leq$$

$$C\sum_{|\bar{J}|\geq|J''|}\sum_{j=m}^{\infty}2^{2j(|J|-|\bar{J}|-\frac{1}{2})}\sum_{Q\in W_j(B)}\sum_{Q'\in W(Q)}\int_{Q^*\cap M}|X_{\bar{J}}(P_Q-P_{Q'})|^2d\mu$$

After we sum over $B\notin\mathcal{F}_m(W)$ and rearranging the terms we finally get

$$\sum_{B\notin\mathcal{F}_m(W)} \leq 2^{-2m\lambda}(C2^{2m\lambda}\sum_{j=m}^{\infty}2^{-2j\lambda}a_j^2) \leq 2^{-2m\lambda}c_m^2$$

The other term B_J is estimated in the same manner except that we would have to use the method by which we proved it to be in L^2 , we do not wish to repeat it here.

Thus the extension theorem is proved granted we prove finally that $(X_JF)|_M = f_J$. Set $G(x) := X_JF(x)$. First we should note that since $|J|\leq k<\beta$ and $F\in S^2_\alpha$ $(\beta+\frac{1}{2}=\alpha)$ then $G\in S^2_\lambda$ for some $\frac{1}{2}<\lambda<1$. Set

$$G(Q) := \frac{1}{|Q|}\int_Q G(x)dx$$

Let W_m^* be balls of W_m expanded enough to cover the surface M, and let ψ_{Q^*} be a partition of unity subbordinate to this cover and finally set

$$G_m(x) := \sum_{Q\in W_m}\psi_{Q^*}G(Q)$$

then this function is smooth and hence its restriction to M is well defined. To prove that the restriction is what we started with it suffices to prove two things:
1.

$$\int_M |f_J - G_m|^2 d\mu \to 0 \qquad m \to \infty$$

and
2.

$$\int_M |G - G_m|^2 d\mu \to 0 \qquad m \to \infty$$

to prove 1. we have

$$\int_M |f_J - G_m|^2 d\mu = \int_M |\sum_{Q\in W_m} \psi_{Q^*}(f_J - G(Q'))|^2 d\mu \le \sum_{Q\in W_m} \int_{Q^*\cap M} |\sum_{Q'\in W_m(Q)} \psi_{Q'^*}(f_J - G(Q'))|^2 d\mu$$

$$\le \sum_{Q\in W_m} \sum_{Q^*\in W_m(Q)} |f_J - G(Q')|^2 d\mu \le C \sum_{Q\in W_m} \int_{Q^*\cap M} |f_J - G(Q)|^2 d\mu$$

$$\le C \sum_{Q\in W_m} \int_{Q^*\cap M} |f_J - X_J P_Q|^2 d\mu + C \sum_{Q\in W_m} \int_{Q^*\cap M} |G(Q) - X_J P_Q|^2 d\mu$$

the first term is by definition less than $2^{-2m(\beta-|J|)} c_m^2 \to 0$. As for the second term it is easily proved that if P is a polynomial and Q any ball then we have

$$\int_{Q^*\cap M} |P|^2 d\mu \le C 2^m \int_Q |P|^2 dvol$$

(if we use a linear change of variables so as the class of polynomials is preserved and the surface becomes horizontal then the assertion is true by finite dimensionality of polynomials, then rescale to get the general version, it is also worht noticing that the reverse inequality is not true.) And thus we have that the second term is less than

$$C 2^m \sum_{Q\in W_m} \int_Q |G(Q) - X_J P_Q|^2 dvol \le C 2^m \sum_{Q\in W_m} \int_Q |G(Q) - G|^2 dvol + C 2^m \sum_{Q\in W_m} \int_Q |G - X_J P_Q|^2 dvol$$

by (1.) the first term is less than $c_m^2 2^{-2m(\lambda-\frac{1}{2})} \to 0 \;\; as \;\; m \to \infty$
write $G - X_J P_Q$ as a sum of terms of the form

$$c_{J',J''} \sum_{Q'\in W} X_{J'} \varphi_{Q'} X_{J''}(P_{Q'} - P_Q),\, J' + J'' = J, |J'| \ne 0$$

plus

$$\sum_{Q'\in W} \varphi_{Q'} X_J (P_{Q'} - P_Q)$$

if we use now lemma (11) we get at once

$$\int_Q |G - X_J P_Q|^2 dvol$$

$$\leq C \sum_{J',J''} \sum_{\tilde{J}} 2^{-2m(|\tilde{J}|-|J'|+\frac{1}{2})} \int_{Q^*\cap M} |X_{\tilde{J}} X_{J''}(P_Q - P_{Q'})|^2 d\mu + \sum_{\tilde{J}} 2^{-2m(|\tilde{J}|+\frac{1}{2})} \int_{Q^*\cap M} |X_{\tilde{J}} X_J(P_Q - P_{Q'})|^2 d\mu$$

$$\leq C \sum_{J',J''} \sum_{|\tilde{J}|\geq |J''|} 2^{-2m(|\tilde{J}|-|J|+\frac{1}{2})} \int_{Q^*\cap M} |X_{\tilde{J}}(P_Q - P_{Q'})|^2 d\mu + \sum_{|\tilde{J}|\geq |J|} 2^{-2m(|\tilde{J}|-|J|+\frac{1}{2})} \int_{Q^*\cap M} |X_{\tilde{J}}(P_Q - P_{Q'})|^2 d\mu$$

summing over $Q \in W_m$ and using the definition of B^2_β we get the desired result. To prove 2. it suffices to note that from the proof of the restriction theorem that we presented for $\frac{1}{2} < \alpha < 1$, we see that if Q is a Whitney ball and if $f \in S^2_\lambda$, and if we set $f(Q) := |frac1|Q| \int_Q f dvol$, then we ahve

$$\sum_{Q\in W_m} \int_{Q^{**}\cap M} |f - f(B)|^2 d\mu \leq C_m^2 2^{-2m(\lambda-\frac{1}{2})}$$

from which 2. follows immediately.

Chapter 5

Some Generalizations to $p \neq 2$

In this note we shall give indications on how to obtain generalizations to $p \neq 2$ whenever possible. Now, the spaces we are concerned with are $S^p_\alpha := L^p * J_\alpha$. Theorem 1 extends to the $p \neq 2$ case trivilally, the limitation there for p and α is that $\alpha > \frac{1}{p}$, and $1 \leq p < \infty$. The same proof can be carried out in this case, the only difference is, whenever we used Schwarz's inequality we use Hölder's. The restriction theorem is also true if we adopt the Jonsson and Wallin proof to our situation. Our proof extends to a restriction theorem for the nonisotropic Besov spaces instead; the reason is because we used decomposition of functions that is valid for Besov spaces and not for Sobolev spaces in the case $p \neq 2$. The extension is also true from B^p_β spaces on the hypersurface to the ambiant space to land in S^p_α, $\alpha = \beta + \frac{1}{p}$, but now p must be $1 < p \leq \infty$. The only difference and "not obvious" extension to $p \neq 2$ is theorem 2. Recall that we have the embedding

$$S^p_\alpha \longrightarrow L^p_{\frac{\alpha}{2}} \qquad (locally) \tag{5.1}$$

where the spaces on the right hand side of (5.1) are the classical potential spaces.

$$\mathbf{R} : S^p_\alpha \longrightarrow \Lambda^{p,p}_{\frac{\alpha}{2}-\frac{1}{p}}(M) \tag{5.2}$$

provided that $\frac{\alpha}{2} - \frac{1}{p} > 0$; i.e.; $\alpha > \frac{2}{p}$. The spaces on the right hand side of (5.2) are the classical Besov spaces see [ST]; chapter V. In particular (5.2) tells us that the restriction spaces of S^p_α to M, is in $L^p(d\sigma)$ if $\alpha > \frac{2}{p}$. Let M $= \{t = 0\}$ be the characteristic hyperplane, and let $\rho = (|z|^4 + t^2)^{\frac{1}{4}}$; then the function $\rho^{-\frac{Q+2}{p}}$, belongs to S^p_α near the origin for all $\alpha < \frac{2}{p}$, but clearly its restriction to $\{t = 0\}$ fails to belong to $L^p(d\sigma)$. Thus, the question remains unsettled only for $\alpha = \frac{2}{p}$ and this ammounts to checking whether or not we have boundedness of the operator

$$\mathbf{R} \circ J_{\frac{2}{p}} : L^p(R^{2n+1}) \longrightarrow L^p(M, d\sigma) \tag{5.3}$$

We will employ the same weighted methods as in the case of $p = 2$, except here we need two weight functions $w_1(x)$ and $w_2(y)$ on M and R^{2n+1} respectively. So, let $w_1(x) \geq 0$ and

$w_2(y) \geq 0$ be two functions not identically zero on M and R^{2n+1} respectively; and denote by T the operator $\mathbf{R} \circ J_{\frac{2}{p}}$. Let $g \in L^p(R^{2n+1})$ having support in, say the unit ball, then for $x \in M$,

$$Tg(x) = \int_{|y|\leq 1} J_{\frac{2}{p}}(y^{-1}x)g(y)dy$$

Set $d\mu(x) = w_1(x)^{-p}d\sigma(x)$ (sorry if it causes confusion this $d\mu$ is not the we've been calling in this thesis) and $d\nu(y) = w_2(y)^{-p}dy$, then it is easily verified that the inequality

$$\| \, Tg \, \|_{L^p(d\sigma)} \leq C \, \| \, g \, \|_{L^p(dy)} \tag{5.4}$$

is equivalent to :

$$\int_M (\int_{|y|\leq 1} K(x,y)f(y)d\nu(y))^2 d\mu(x) \leq C \int_{|y|\leq 1} | \, f(y) \, |^p \, d\nu(y). \tag{5.5}$$

where $K(x,y) = w_2(y)^{p-1}J_{\frac{2}{p}}(y^{-1})w_1(x)$. According to Young's inequality, to have (5) it suffices that we check two estimates:

$$(a) \quad \sup_{x\in M} \int_{|y|\leq 1} |K(x,y)|d\nu(y) \leq C$$

and

$$(b) \quad \sup_{|y|\leq 1} \int_M |K(x,y)|d\mu(x) \leq C.$$

in terms of the Lebesgue measure and ordinary surface measure (taking into account the estimate for $J_{\frac{2}{p}}$) (a) and (b) follow from

$$(a)' \quad \sup_{x\in M} w_1(x) \int_{|y|\leq 1} | \, y^{-1}x \, |^{\frac{2}{p}-Q} \, w_2(y)^{-1}dy \leq C$$

and

$$(b)' \quad \sup_{|y|\leq 1} w_2(y)^{p-1} \int_M | \, y^{-1}x \, |^{\frac{2}{p}-Q} \, w_1(x)^{1-p}d\sigma(x) \leq C.$$

From our experience with the case $p = 2$, we know what $w_1(x)$ ought to be; $w_1(x) = w(x)^\varepsilon$, for some $\varepsilon > 0$ small enough. The problem is to find $w_2(y)$ satisfying (a)' and (b)'. From (b)' we can take $w_2(y)$ to be equal to,

$$(w_2(y))^{-p+1} = \int_M | \, y^{-1}x \, |^{\frac{2}{p}-Q} \, w_1(x)^{1-p}d\sigma(x) \tag{5.6}$$

and we will be left with (a)'. Let us analyze the integral

$$\int_M |y^{-1}x|^{\frac{2}{p}-Q}w(x)^{\varepsilon(1-p)}d\sigma(x). \tag{5.7}$$

The factor $|y^{-1}x|^{\frac{2}{p}-Q}$ is at its worst (blowing up) for those $x \in M$ that are near enough to y; i.e., within a $d(y, M)$ from M. The factor $w(x)^{1-\varepsilon}$ is worst when x is near the characteristic set Γ and the integrand is worst when the singularities are combined, i.e., when x is near y and near the characteristic point that is when $d(y,\Gamma)$ is comprable to $d(y,M)$. The integral in (5.7) is therefore going to be expressed (or estimated) solely in terms of two quantities, $d(y,M)$ the distance of y to M and $d(y,\Gamma)$, the distance of y to Γ. Let us treat the case $M=\{t=0\}$ first. In this case integral in (5.7) is

$$I=\int_M |y^{-1}x|^{\frac{2}{p}-Q}|x|^{\varepsilon(1-p)}d\sigma(x)=I_1+I_2+I3$$

$$I_1=\int_{|x|\le\frac{|y|}{2C}}|x|^{\varepsilon(1-p)}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)\quad,\quad I_2=\int_{\frac{|y|}{2C}\le|x|\le C|y|}|x|^{\varepsilon(1-p)}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)$$

$$I_3=\int_{|x|\le C|y|}|x|^{\varepsilon(1-p)}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)$$

We need the following easily proved lemma whose statements are just variants of the triangle inequality.

Lemma 14 *Let $0\neq x\in M$, $y\in R^{2n+1}$ (a) If $|y|<|x|$ and $|y|\cong d(y,M)$ then $|y^{-1}x|\cong|x|$. If $|y|\gg|x|$ then $|y^{-1}x|\cong|y|$. (b) If $|x|\le\frac{|y|}{2C}$ then $|y^{-1}x|>\frac{|y|}{2C}$. (c) If $d(y,M)\ll|y|$ and $x_0\in M$ is such that $d(y,M)=|x_0^{-1}y|$, then $|x_0|\cong|y|$.*

By lemma (b)

$$I_1\le C|y|^{\frac{2}{p}-Q}\int_{|x|\le\frac{|y|}{2C}}|x|^{\varepsilon(1-p)}d\sigma(x)\le C|y|^{(\frac{2}{p}+\varepsilon)(1-p)}$$

$$I_3=\int_{|x|\le C|y|}|x|^{\varepsilon(1-p)}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)$$

By lemma (a) $|y^{-1}x|\cong|x|$ and thus

$$I_3\le C\int_{|x|\ge C|y|}|x|^{-Q+\frac{2}{p}+\varepsilon(1-p)}d\sigma(x)\le C|y|^{(\frac{2}{p}+\varepsilon)(1-p)}$$

I_2 is more subttle and requires a careful analysis

$$I_2\le C|y|^{\varepsilon(1-p)}\int_{|x|\le C|y|}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)$$

If $|y|\cong d(y,M)$, we have

$$I_2\le C|y|^{(\frac{2}{p}+\varepsilon)(1-p)}$$

If $|y|\gg d(y,M)$, pick a point $x_0\in M$ such that $|x_0^{-1}y|\cong d(y,M)$ and write

$$\int_{|x|\le C|y|}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)=\int_{|x_0^{-1}x|\le d(y,M)}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)+\sum_{k=1}^{\infty}A_k$$

where

$$A_k=\int_{\Delta_k}|y^{-1}x|^{\frac{2}{p}-Q}d\sigma(x)$$

where

$$\Delta_k = \{2^{k-1}d(y,M) < |x_0^{-1}x| < 2^k d(y,M)\}$$

$$\int_{|x_0^{-1}x|\le d(y,M)} |y^{-1}x|^{\frac{2}{p}-Q} d\sigma(x) \le d(y,M)^{\frac{2}{p}-Q}\sigma(\{|x_0^{-1}x| \le d(y,M)\})$$

By lemma 3(b) of chapter 2 we have

$$\sigma(\{|x_0^{-1}x| \le d(y,M)\}) \le C\frac{d(y,M)^{Q-1}}{|x_0|} \quad \text{and} \quad \sigma(\Delta_k) = \frac{(2^k d(y,M))^{Q-1}}{|y|}$$

and by lemma (c)

$$A_k \le (2^k d(y,M))^{\frac{2}{p}-Q}\sigma(\{|x_0^{-1}x| \le 2^k d(y,M)\}) \le C(2^k d(y,M))^{\frac{2}{p}-Q}\frac{(2^k d(y,M))^{Q-1}}{|y|}$$

and hence,

$$\int_{|x|\le C|y|} |y^{-1}x|^{\frac{2}{p}-Q} d\sigma(x) \le C|y|^{-1}d(y,M)^{\frac{2}{p}-1}\sum_{k=0}^{\infty}(2^k)^{\frac{2}{p}-1} \le C|y|^{-1}d(y,M)^{\frac{2}{p}-1}$$

provided that $p > 2$. Therefore

$$I_2 \le C|y|^{\epsilon(1-p)-1}d(y,M)^{\frac{2}{p}-1}$$

which also contains the estimate for I_1 and I_3, so,

$$I \le C|y|^{\epsilon(1-p)-1}d(y,M)^{\frac{2}{p}-1} \qquad \textit{for } p > 2 \tag{5.8}$$

What about $p \le 2$? If $p = 2$ the integral to estimate is

$$\int_M |y^{-1}x|^{-Q+1}|x|^{-\epsilon}d\sigma(x)$$

Case 1: $|y| \cong d(y,M)$, the same estimate as I_3 above yields

$$\int_M |y^{-1}x|^{-Q+1}|x|^{-\epsilon}d\sigma(x) \le C|y|^{-\epsilon-1}$$

Case 2: $|y| \gg d(y,M)$; break the integral into three parts

$$\int_M |y^{-1}x|^{-Q+1}|x|^{-\epsilon}d\sigma(x) = I_1 + I_2 + I_3$$

$$I_1 = \int_{|x_0^{-1}x|<d(y,M)} |y^{-1}x|^{-Q+1}|x|^{-\epsilon}d\sigma(x) \quad , \quad I_2 = \int_{d(y,M)\le|x_0^{-1}x|\le|y|} |y^{-1}x|^{-Q+1}|x|^{-\epsilon}d\sigma(x)$$

$$I_3 = \int_{|x_0^{-1}x|>|y|} |y^{-1}x|^{-Q+1}|x|^{-\epsilon}d\sigma(x)$$

$$I_1 \leq C|y|^{-\varepsilon} d(y,M)^{-Q+1} \sigma(\{|x_0^{-1}x| < d(y,M)\}) \leq C|y|^{-\varepsilon-1}$$

$$I_2 \leq C|y|^{-\varepsilon} \int_{d(y,M) \leq |x_0^{-1}x| \leq |y|} |y^{-1}x|^{-Q+1} d\sigma(x) \leq C|y|^{-\varepsilon} \int_{d(y,M) \leq |x_0^{-1}x| \leq |y|} |x_0^{-1}x|^{-Q+1} d\sigma(x) =$$

$$= C|y|^{-\varepsilon} (\int_{d(y,M)}^{|y|} r^{-Q+1} d\sigma(\{|x_0^{-1}x| < r\})) \leq C|y|^{-\varepsilon}(|y|^{-1} + |y|^{-1} \log(\frac{|y|}{d(y,M)}))$$

$$I_2 \leq C|y|^{-\varepsilon-1}(1 + \log(\frac{|y|}{d(y,M)}))$$

I_3 is easy, we notice that $|y^{-1}x|$ and $|x|$ are exchangeable and compute that $I_3 \leq C|y|^{-\varepsilon-1}$ and thus

$$I \leq C|y|^{-\varepsilon-1}(1 + \log(\frac{|y|}{d(y,M)})) \tag{5.9}$$

$p < 2$: Case 1: $|y| \cong d(y,M)$, then, like before

$$\int_M |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x) = I_1 + I_2$$

$$I_1 = \int_{|x|<|y|} |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x) \quad , \quad I_2 = \int_{|x|>|y|} |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x)$$

$$I_2 = C \int_{|x|>|y|} |x|^{\frac{2}{p}-Q+\varepsilon(1-P)} d\sigma(x) = C|y|^{(\frac{2}{p}+\varepsilon)(1-p)}$$

By lemma (a)

$$I_1 = \int_{|x|<|y|} |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x) \leq C|y|^{\frac{2}{p}-Q} \int_{|x|<|y|} |x|^{\varepsilon(1-p)} d\sigma(x) \leq C|y|^{(\frac{2}{p}+\varepsilon)(1-p)}$$

Case 2: $|y| \gg d(y,M)$, we have

$$\int_M |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x) = I_1 + I_2 + I_3$$

$$I_1 = \int_{|x_0^{-1}x| < \frac{|y|}{C}} |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x) \quad , \quad I_2 = \int_{|x| < \frac{|y|}{C}} |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x)$$

$$I_3 = \int_{|x_0^{-1}x| > \frac{|y|}{C}, |x| > \frac{|y|}{C}} |y^{-1}x|^{\frac{2}{p}-Q} |x|^{\varepsilon(1-p)} d\sigma(x)$$

integration bu parts gives

$$I_1 \leq C|y|^{\varepsilon(1-p)} \int_{|x_0^{-1}x| < \frac{|y|}{C}} |x_0^{-1}x|^{\frac{2}{p}-Q} d\sigma(x) = C|y|^{\varepsilon(1-p)} \int_0^{\frac{|y|}{C}} r^{\frac{2}{p}-Q} d(\sigma(|x_0^{-1}x| < r))$$

$$= C|y|^{\varepsilon(1-p)} r^{\frac{2}{p}-Q} \left. \frac{r^{Q-1}}{|x_0|} \right|_0^{|y|} + C \int_0^{\frac{|y|}{C}} r^{\frac{2}{p}-Q-1} \sigma(|x_0^{-1}x| < r) dr$$

$$\leq C|y|^{\varepsilon(1-p)}\frac{|y|^{\frac{2}{p}-1}}{|x_0|} \qquad \text{since} \quad \frac{2}{p}-1>0$$

$$\leq C|y|^{(\varepsilon+\frac{2}{p})(1-p)} \qquad \text{since} \quad |x_0|\cong|y|$$

I_2 is similar to I_1

$$I_2 \leq C|y|^{\frac{2}{p}-Q}\int_{|x|\leq|y|}|x|^{\varepsilon(1-p)}d\sigma(x) \leq C|y|^{(\varepsilon+\frac{2}{p})(1-p)}$$

The tail I_3 is also easy, in this case the quantities $|x|$ and $|y^{-1}x|$ are exchangeable, and hence

$$I_3 \leq C\int_{|x|\geq|y|}|x|^{\frac{2}{p}-Q+\varepsilon(1-p)}d\sigma(x) = C\int_{|y|}^{\infty}r^{\frac{2}{p}-Q+\varepsilon(1-p)}r^{Q-3}dr$$

$$\leq C|y|^{(\frac{2}{p}+\varepsilon)(1-p)}$$

Hence, if $p<2$ we have a better estimate

$$\int_M |y^{-1}x|^{\frac{2}{p}-Q}|x|^{\varepsilon(1-p)}d\sigma(x) \leq C|y|^{(\frac{2}{p}+\varepsilon)(1-p)} \tag{5.10}$$

In summary, by (5.6),(5.8),(5.9) and (5.10) $w_2(y)$ ought to be

$$w_2(y) = \begin{cases} C|y|^{(-\varepsilon-\frac{1}{p-1})}d(y,M)^{\frac{2-p}{p(p-1)}} & if \quad p>2 \\ C|y|^{-\varepsilon-1}(1+\log(\frac{|y|}{d(y,M)})) & if \quad p=2 \\ C|y|^{-(\varepsilon+\frac{2}{p})} \qquad if \quad p<2 \end{cases}$$

In order to conclude, we need to check estimate (a)'

$$\sup_{x\in M}|x|^{\varepsilon}\int_{|y|\leq 1}|y^{-1}x|^{\frac{2}{p}-Q}w_2(y)^{-1}dy \leq C$$

Assume $|x|\neq 0$ and $p>2$

$$\int_{|y|\leq 1}|y^{-1}x|^{\frac{2}{p}-Q}|y|^{-(\varepsilon+\frac{1}{p-1})}d(y,M)^{\frac{2-p}{p(p-1)}}dy \leq$$

$$\leq \int_{|y|\cong d(y,M)}|y^{-1}x|^{\frac{2}{p}-Q}|y|^{-(\varepsilon+\frac{2}{p})}dy + \int_{|y|\gg d(y,M)}|y^{-1}x|^{\frac{2}{p}-Q}|y|^{-(\varepsilon+\frac{1}{p-1})}d(y,M)^{\frac{2-p}{p(p-1)}}dy = I+II$$

Let $E_1=\{y:|y|\cong d(y,M)\,,\,|y|\leq|x|\}$ and $E_2=\{y:|y|\cong d(y,M)\,,\,|y|>|x|\}$

$$I \leq \int_{E_1}|y^{-1}x|^{\frac{2}{p}-Q}|y|^{-\varepsilon-\frac{2}{p}}dy + \int_{E_2}|y^{-1}x|^{\frac{2}{p}-Q}|y|^{-\varepsilon-\frac{2}{p}}dy$$

By lemma 14(a)

$$I \leq C|x|^{\frac{2}{p}-Q} \int_{|y|\leq|x|} |y|^{-\varepsilon-\frac{2}{p}} dy + \int_{|y|>|x|} |y|^{-\varepsilon-Q} dy \leq C|x|^{-\varepsilon}$$

Let $E_k = \{y : |y| \gg d(y,M)\, and\, d(y,M) \cong 2^k|x|\}$ and $F = \{y : |y| \gg d(y,M)\, and\, d(y,M) \leq |x|\}$

$$II = \sum_{k=0}^{\infty} \int_{E_k} |y^{-1}x|^{\frac{2}{p}-Q} |y|^{-\varepsilon-\frac{1}{p-1}} d(y,M)^{\frac{2-p}{p(p-1)}} dy + \int_F |y^{-1}x|^{\frac{2}{p}-Q} |y|^{-\varepsilon-\frac{1}{p-1}} d(y,M)^{\frac{2-p}{p(p-1)}} dy$$

$$\int_{E_k} |y^{-1}x|^{\frac{2}{p}-Q} |y|^{-\varepsilon-\frac{1}{p-1}} d(y,M)^{\frac{2-p}{p(p-1)}} dy \leq (2^k|x|)^{\frac{2-p}{p(p-1)}} \int_{|y|>2^k|x|} |y^{-1}x|^{\frac{2}{p}-Q} |y|^{-\varepsilon-\frac{1}{p-1}} dy$$

$$\leq (2^k|x|)^{\frac{2-p}{p(p-1)}} \int_{|y|>2^k|x|} |y|^{\frac{2}{p}-Q-\varepsilon-\frac{1}{p-1}} dy = (2^k|x|)^{\frac{2-p}{p(p-1)}} \int_{|y|>2^k|x|} |y|^{-\varepsilon-\frac{p-2}{p(p-1)}} \frac{dy}{|y|^Q} \leq C(2^k|x|)^{-\varepsilon}$$

Summing over k we get

$$\sum_{k=0}^{\infty} \int_{E_k} |y^{-1}x|^{\frac{2}{p}-Q} |y|^{-\varepsilon-\frac{1}{p-1}} d(y,M)^{\frac{2-p}{p(p-1)}} dy \leq C|x|^{-\varepsilon}$$

The other term of II is estimated in a similar manner and the case $p \leq 2$ is easy to check. The estimates for general M follow the same ideas, and because of the torture they give, we do not want them to be here.

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