Restriction to hypersurfaces of non-isotropic Sobolev spaces

by

MOHAMED MEKIAS

M.S., Massachusetts Institute of Technology (1987)
D.E.S., Universite de Setif, Algeria (1984)

Submitted to the
Department of Mathematics
in partial fulfilment of the requirements
for the degree of
Doctor of Philosophy
at the
Massachusetts Institute of Technology
March 11, 1993

© Massachusetts Institute of Technology 1993

Signature redacted

Signature redacted

Certified by

Signature redacted

Accepted by

ARChives
Massachusetts Institute of Technology
JUL 27 1993
Restriction to hypersurfaces of non-isotropic Sobolev spaces
by
Mohamed Mekias
Submitted to the Department of Mathematics
on March 11, 1993 in partial fulfillment of the
requirements for the Degree of Doctor of Philosophy in
Mathematics

Abstract

Let $X_1, \ldots, X_{2n}$ be smooth linearly independent vector fields in an open set $\Omega \subset \mathbb{R}^{2n+1}$. We form the nonisotropic Sobolev spaces $S^2_\alpha$ for $\alpha > 0$, by measuring smoothness in terms of the $X_j$'s. These function spaces are the natural ones to consider when dealing with operators of the form $X_1^2 + \cdots + X_{2n}^2$. In particular, in the Dirichlet problem associated with these operators the problem of restriction to the boundary comes up naturally. Let $M$ be a smooth hypersurface of $\Omega$. In this thesis the restriction problem is investigated. It is shown that many results that hold, concerning the restriction problem, for the isotropic Sobolev space $L^2_\alpha$ have analogues in the nonisotropic setting, in particular the result $L^2_\alpha|_M = L^2_{\alpha-\frac{1}{2}}$. When the index $\alpha$ is small $\frac{1}{2} < \alpha < \frac{3}{4}$, we have complete characterization of the space of restrictions; $S^2_\alpha|_M = F^2_{\alpha-\frac{1}{2}}(M)$, this latter space is described by similar smoothness conditions to the classical Sobolev spaces using the first differences. We merely replace the distance function by the nonisotropic one and the surface measure by a weighted measure, this weight is precisely the angle made by the tangent space and the span of the vector fields $X_j$'s. If $d\sigma$ denotes the surface measure on $M$, we show also that the space $S^2_1$ admits restrictions to $M$ that are members of $L^2(d\sigma)$, this result is sharp; in the case of surfaces where the span of the $X_j$'s is nowhere tangent the condition is $\alpha > \frac{1}{2}$. For higher indices $\alpha$, we use a method due to Jonsson and Wallin to describe the restriction spaces.

Thesis Supervisor: David Jerison
Title: Professor of Mathematics
Acknowledgement

Praise be to Allah the Most Gracious the Most Merciful. I witness that there is no one worthy of worship except Him, and that Muhammad peace be upon him is the true messenger of Allah.

I would like to thank my advisor Professor David Jerison for his help and generosity with his time. Most of all I am grateful to his teaching me many techniques in analysis. I also greatly appreciate his patience with me, he was always ready to discuss things with me even if it meant for the 100th times. I would like to thank Professor Richard Melrose who read this thesis and commented on it. I appreciated his sense of humour. I extend thanks also to Professor Victor Guillemin.

There are numerous people whom I met over all these years and whom I am indebted to, some of whom are still in close contact with me, others I would probably never see again. I would like to thank those who offered me help and encouragement, among them are my brothers Hocine and Abdelkader. I am grateful to my friends Amr Mohamed; Hisham Hassanein; Youcef Shatilla and Assad Mahboub Ali who assisted me financially and morally. I would like to thank all those members of the MITMSA who were concerned about me. I would like to thank some of my acquaintances, among them are my friend Symphore and his wife whose help is acknowledged. To the Austin family in particular Ann, who provided me with a roof over my head and treated me as one of her own member of her family, Bliss and Andrew; Mariam; John and Ann I am grateful. Special warm thanks go to my dear Heather, who provided me with warm confort and who shared with me many things, and her family, Donita, Andrew and Sherill and Mat. Lastly but by no mean least I take this occasion to thank my parents who have been supportive of me at all time and who worked very hard to bring up the children providing them with guidance as well as putting food on the table, yamma wa baba I love very much and I am sorry for taking a long time. I’d like to extend thanks to my brothers and sisters.
Contents

1 Introduction ............................................................................................................. 1

2 Preliminary Background ......................................................................................... 5

  2.1 The nonisotropic distance .................................................................................. 5
    2.1.1 Global definitions ....................................................................................... 5
    2.1.2 Local definitions ......................................................................................... 6
    2.1.3 The contact structure and the Heisenberg group ......................................... 7

  2.2 The weight function $w$ .................................................................................... 9

  2.3 The spaces $S^p$, $\Gamma^p$, $F^p$ and $B^p$. ......................................................... 14
    2.3.1 The spaces $S^p$ ......................................................................................... 14
    2.3.2 The spaces $\Gamma^p$ .................................................................................. 16
    2.3.3 The spaces $F^p$ and $B^p$ ......................................................................... 22

  2.4 Two classical inequalities .................................................................................... 25

3 The Restriction Theorem ......................................................................................... 27

  3.1 The $L^2$ estimates ............................................................................................ 27
    3.1.1 Proof of theorem 4 ..................................................................................... 27
    3.1.2 Proof of theorem 5 ..................................................................................... 29
    3.1.3 $S^2_\alpha$ doesn't restrict to $L^2(d\sigma)$ for $\alpha < 1$ .................................. 35
    3.1.4 Some consequences of theorems 4 & 5 ....................................................... 36
Chapter 1

Introduction

Let $X_1, \ldots, X_m$ be smooth vector fields defined on some manifold $\Omega$. Assume further that this system of vector fields satisfies a finite step Hörmanders' condition, that is $X_1; \ldots; X_m$ and commutators of finite length span the tangent space at each point. Identifying the $X_j$'s with the directional derivatives, we will measure smoothness of functions on $\Omega$ in terms of these vector fields. We form the Sobolev type spaces denoted by $S^p_k$ whose members $f$ are characterized by the requirement that $f \in S^p_{k-1}$ and $X_j f \in S^p_{k-1}$, locally, for $1 \leq j \leq m$ and $k = 1; 2; \ldots$. We, of course, adopt the convention that $S^p_0 = L^p$ and the derivatives are in the sense of distributions. These spaces are the natural ones to consider when dealing with differential operators of the form $X_1^2 + \cdots + X_m^2$ and its variants, see for example [DE].

Now in the same way as the classical Sobolev spaces $L^p_k$, the question of restriction to the boundary of the members of $S^p_k$ is a natural one and it is of importance, in particular, to the Dirichlet problem and other boundary value problems, in [JE2] the Dirichlet problem for the Kohn laplacian was studied from a nonisotropic point view using the nonisotropic Lipshitz spaces. More precisely, one may formulate the restriction problem in the following way: Let $M$ be a lower dimensional submanifold of $\Omega$, which we assume to be of codimension one and we ask

Question 1. What is the space $S^p_k|_M$?

An answer to this question consists of characterizing smoothness conditions for functions defined on $M$, so that members of $S^p_k$ satisfy them on $M$ and conversely, given a function $g$ defined on $M$ satisfying these conditions we should be able to extend it so as to lie in $S^p_k$.

To try to answer this question in its full generalities as stated above would be an ambitious task and even a foolish one.

In this thesis we examine the particular case of what is called a contact manifold (or sometimes referred to as CR manifold) and the situation is as follows:

1. We assume that there are $m = 2n$ linearly independent vector fields $X_1, \ldots, X_{2n}$ and that Hörmander condition is a step 2 one.
2. The dimension of $\Omega$ is equal $2n + 1$

The system of vector fields $X_1, \ldots, X_{2n}$ span a hyperplane of the tangent space of $\Omega$, and thus, the data given by 1. and 2. are equivalent to a given nondegenerate field of hyperplanes.
i.e.: a contact structure on $\Omega$. Locally, this situation is modelled by the Heisenberg group and the function spaces $S^p_\alpha$ as well as $S^p_\alpha$ for $\alpha$ not integer are now realized as potential spaces, that is, there is a function $J_\alpha$ such that $S^p_\alpha = L^p \ast J_\alpha$, convolution is group convolution. Once we realize these spaces in this manner one may approach the restriction problem by trying to follow the same lines as the classical restriction problem see [ST2] or [JW] and the litterature listed there. Let us limit our discussion to the case $p = 2$, all that follow have analogue to $p \neq 2$. Recall that we have (see [ST2])

$$L^2(R^n)|_{R^{n-1}} = \Lambda^2_{\alpha-\frac{1}{2}}(R^{n-1}) \quad \text{for} \quad \alpha > \frac{1}{2}$$

(1.1)

the spaces on the right hand side of (1.1) are the classical Besov spaces (which in this case ($p = 2$) coincide with the Sobolev spaces themselves) see [ST] for more on these spaces, the result holds for general smooth hypersurface $M$. For $0 < \alpha < 1$, a norm on the space $\Lambda^2_\alpha(R^n)$

$$||f||_{L^2} + \left( \int \int_{|x-y| \leq 1} \frac{|f(x) - f(y)|^2}{|x-y|^{2\alpha+n}} \, dx \, dy \right)^{1/2}$$

(1.2)

A similar version of (1.2) involving derivatives is used to define $\Lambda^2_\alpha(R^n)$ for higher $\alpha$'s. We will seek similar norms as (1.2) to characterize the space of restrictions. In the noncharacteristic case (see definition next chapter) similar expression can be used to identify the space of restrictions of $S^2_\alpha$ when $\alpha$ is small i.e.; $0 < \alpha < 1$, see [ME]. We merely replace the distance function in the denominator of the integrand of (1.2) by the appropriate one, namely the nonisotropic one (see next chapter), in particular the restriction of $S^2_\alpha$ to noncharacteristic hypersurfaces is contained in $L^2(\sigma)$ when $\alpha > \frac{1}{2}$. This leads us to another question, namely Question 2: If $\sigma$ is the surface measure, when do we have the embedding $S^2_\alpha \subset L^2(\sigma)$ or what is the relationship (if any) between the space of restrictions $S^2_\alpha|_M$ and the classical Sobolev spaces on $M$?

For general $M$, it turns out that the answer to the question 2 is when $\alpha \geq 1$, this is theorem 5 chapter 3, the reason for this is precisely the possibility of tangency of the surface to the field of hyperplanes at some points. The result for $\alpha > 1$ can be gotten cheaply by the following method, embed $S^2_\alpha$ in $L^2_{\alpha/2}$ according to proposition 5 of chapter 2, and then restrict using (1.1), the condition $\alpha > 1$ is necessary by this method. This makes the case $\alpha = 1$ interesting.

Now we go back to Question 1. Because it is important how $M$ sits inside $\Omega$, by seeking norms similar to (1.2) to characterize the space of restrictions we have to have a norm that incorporates this information. It turns out that there is a natural object to consider namely, the angle made by $M$ (i.e.; its tangent space) and the field of hyperplanes giving us the contact structure. This function, which we denote by $w$, has the right weighting as well as scaling properties see (next chapter). Now set $d\mu = w \, d\sigma$. This new measure $d\mu$ is the right measure to combine with the system of nonisotropic balls in the same way the surface measure $d\sigma$ is the right one to consider with the system of Euclidian balls, it gives a certain homogeneous dimension to the surface. Results of this thesis are best expressed in terms of some function spaces denoted by $F^2_\alpha$ for small values of $\alpha$ and $B^2_\alpha$ for large values of $\alpha$. For $0 < \alpha < 1$ we take expression (1.2) above and replace the denominator of the integrand by
3.2 The main estimate .................................................. 38
  3.2.1 The case $0 < \beta = \alpha - \frac{1}{2} < 1$ ......................... 38
  3.2.2 The case $\beta \geq 1$ ........................................ 41
4 The Extension Theorem ................................................. 46
  4.1 The Whitney decomposition ....................................... 47
  4.2 Proof of theorem 8 ................................................. 48
    4.2.1 The case $0 < \beta < 1$ .................................... 48
    4.2.2 The case $\beta \geq 1$ ....................................... 52
5 Some Generalizations to $p \neq 2$ ................................... 60
Bibliography .............................................................. 67
the nonisotropic distance and the surface measure by the measure $d\mu$ we get the norm for
the space $F^2_\alpha$. Now we are able to prove the analogue to (1.1) above

$$S^2_\alpha|_M = F^2_{\alpha-\frac{1}{2}} \text{ for } \frac{1}{2} < \alpha < \frac{3}{2}$$  \hspace{1cm} (1.3)

The definition of $F^2_\alpha$ for $\alpha \geq 1$ is available to us by the use of local polynomial approximation
(see definition next chapter). These polynomials are taken in some coordinate system in
which $M$ is just the hyperplane. We should mention that we do have $S^2_\alpha|_M \subset F^2_{\alpha-\frac{1}{2}}$, the
trouble lies in the extension part i.e.; the reverse of this inclusion. Let us indicate how a
Whitney extension theorem is proved in the Euclidian case. First take the so called Whitney
decomposition (see chapter 4 or [ST1]) of the complement of $M$ in the ambient space. On
each Whitney ball $B$ define the extension of a function $f$ defined on $M$ to be equal the
polynomial, in local coordinates, that best approximates $f$ on an appropriate subset of
$M$(say for example $B^* \cap M$, where $B^*$ denotes a ball with same center as $B$ but radius 10
times the radius of $B$) in the $L^2(do\cdot)$-sense, we should mention that the polynomials are
extended in a natural way to the ambient space to be constant along the vertical direction.
We can make this more precise
by using a partition of unity to get an extension $E(f)$ of $f$ to the ambient space. In proving $||E(f)||_{L^2_{\alpha+\frac{1}{2}}} \leq C||f||_{A^2_\alpha}$ the following trivial fact is used

$$\int_B |p|^2 d\nuol \leq C \int_{B\cap M} |p|^2 d\sigma$$  \hspace{1cm} (1.4)

To be able to carry on in the nonisotropic case we need an inequality of the type (1.4) with
$B$ in (1.4) replaced by nonisotropic ball and $d\sigma$ by $d\mu$. Unfortunately, in this setting (1.4)
is false, except in two cases 1. when $M$ is noncharacteristic and 2. when $p$ is a constant.
Heuristically, the reason for this is the fact that our balls are tilted and as we approach
the characteristic set they become flat on $M$ and thus if we try to extend polynomials to
be constant along the vertical direction they exit too quickly from the balls before they get
known inside the solid ball.

In the case of small $\alpha$'s we do have extension this way, and that's why we were able to
prove (1.3). To remedy this crisis we have to pay a price by leaving the spaces $F^2_\alpha$ and
replacing them by somewhat smaller ones , namely the spaces $B^2_\alpha$. These spaces consists of
system of functions (see the definition at the end of chapter 2) and when considered on the
ambient space each system in $B^2_\alpha$ consists of a function and all its derivatives up to a certain
order, and this means that the single function determines uniquely the whole system. In the
Euclidian setting the spaces $F^2_\alpha$ and $B^2_\alpha$ are equivalent since we can prove restriction and
extension theorems to both spaces. The question as to whether these spaces on $M$ coincide
in the nonisotropic setting is not trivial. Combining theorems 6 and 8 we state the result in
the following form

$$S^2_\alpha|_M = B^2_{\alpha-\frac{1}{2}}$$  \hspace{1cm} (1.5)

An extension for the spaces $F^2_\alpha$, for $\alpha \geq 1$, to the ambient space would be possible if we
could somehow extend polynomials on $M$ (in some local coordinate system) to the ambient
space by allowing them to lose their character of being polynomials, but we should be able to control them, for example in (1.4) above if the \( p \) on the left denotes the extension of \( p \) on the right we may still have that inequality. In the case of the characteristic hyperplane of the Heisenberg group, this idea is currently under investigation and progress seems promising at least to extend functions from \( F_\alpha^2 \) to \( S_{\alpha+\frac{1}{2}}^2 \) in the case where \( 0 < \alpha < 2 \). This method consists of extending the coordinate functions to be constant along a certain field direction, and hence giving an extension to polynomials forcing them to stay longer in balls. Generalization to arbitrary \( M \) may be possible if it has isolated characteristic points.

Let us sketch the plan of this thesis. In chapter 2 we present the preliminary background, section 2.1 is meant to be expository and gives various definitions and notions that are equivalent in different setting and it essentially justifies the transfer of the original question to the setting of the Heisenberg group. An important object of this work is the weight function \( w \). In section 2.2 we prove interesting (elementary) facts about this function. Section 2.3 represents known definitions and facts about the nonisotropic Sobolev and Besov spaces, the main references for this are [FO],[FS], [NS] and [SA], we also introduce the spaces on the boundary \( F_\alpha^p \) and we follow the construction of [JW] to define the spaces \( B_\alpha^p \). Chapter 3 is the restriction theorem, the main theorems are theorems 5 and 6 and chapter 4 is the converse to the restriction theorem. Finally we close by mentioning how possible extensions of our results are possible to the case for \( p \neq 2 \), it contains essentially the \( L^p \) analogue to theorem 5.
Chapter 2
Preliminary Background

2.1 The nonisotropic distance

We collect the background necessary to describe the nonisotropic geometry of the problem; we also present the notation used in this thesis. This section is meant to be only expository and therefore no proofs will be given, we refer to [NA],[NS] and [NSW]. We assume that $\Omega \subset \mathbb{R}^{2n+1}$ is some smooth open set, since the results are local. Let $\{X_1, \cdots, X_{2n}\}$ be a set of linearly independent smooth vector fields defined on a neighborhood of $\Omega$, and satisfying the step-two Hörmanders' condition i.e.; the system of vectors $\{X_1|_p, \cdots, X_{2n}|_p, [X_j, X_k]|_p, 1 \leq j, k \leq 2n\}$ spans $\mathbb{R}^{2n+1}(\cong T_p\Omega)$ for all $p \in \Omega$. The objective of this section is to define the nonisotropic distance reflecting basically the noncommutativity of the vector fields, as well as Hörmanders' condition. There are two main approaches to defining suitable distances associated with a set of vector fields. The first one is global and it is the well known control distance $d$. The second definition is local and based on the exponential map, and thus depends on the base point. Let us discuss briefly these two constructions. We denote by $X_{2n+1}$ the missing direction i.e.; we choose from the commutators $\{[X_j, X_k]\}$ one that completes the system $\{X_1, \cdots, X_{2n}\}$.

2.1.1 Global definitions

Definition 1 Let $r > 0; x_0$ and $x_1$ be two points of $\Omega$. We say that $d_1(x_0, x_1) \leq r$ if there is a (piecewise smooth) map $\phi: [0, 1] \to \Omega$ satisfying:

(i) $\phi(0) = x_0$ and $\phi(1) = x_1$;

(ii) $\frac{d\phi}{dt} = \sum_{j=1}^{2n+1} a_j(t)X_j(\phi(t))$ for almost every $t \in [0, 1]$ with $|a_j(t)| < r$ for $1 \leq j \leq 2n$ and $|a_{2n+1}(t)| < r^2$.

The following proposition gives the first comparison of the distance $d$ and the Euclidian distance which we denote by $||\cdot||$, for a proof of this see [NS].
Proposition 1 \(d\) is a metric on \(\Omega\), and for every compact set \(\omega \subset \subset \Omega\), there are constants \(C_1\) and \(C_2\) such that

\[
C_1||x_0 - x_1|| \leq d(x_0, x_1) \leq C_2||x_0 - x_1||^{1/2}
\]  

(2.1)

There are several variants of the control distance. One of them for example we get by restricting the right hand side of (ii) in definition 1 above to be a constant linear combinations. Another construction is by requiring the vector \(\frac{de}{dt}\) be only in the field of hyperplanes which is not all of the space. The fact that \(d\) is distance is easy to verify but what is the volume of a corresponding ball of radius \(r\)? In general it is a hard question to tell just from the definition of \(d\) what shape these balls take, and what their volume is. In order to solve this problem we define a local metric based on canonical coordinates (i.e.; the exponential map), that would make the shape as well as the volume of these balls transparent.

2.1.2 Local definitions

Let \(p \in \Omega\) be fixed, and let \(\vec{v} \in T_p \Omega \cong R^{2n+1}\) be a tangent vector, since \(\{X_1|_p, \ldots, X_{2n+1}|_p\}\) spans, \(\vec{v}\) can be written as \(\sum_{j=1}^{2n+1} a_j X_j|_p\). Now the vector field \(\sum_{j=1}^{2n+1} a_j X_j\) is smooth near \(p\) and coinciding with \(\vec{v}\) at \(p\). So, we may flow along the integral curves of this vector field for unit time if the coefficients \(a_j\) are small enough, we get this way the exponential map based at \(p\),

\[
\text{Exp}_p: U_0 \to V_p \quad (a_1, \ldots, a_{2n+1}) \to \text{Exp}_p(\sum_{j=1}^{2n+1} a_j X_j)
\]  

(2.2)

\(U_0\) and \(V_p\) are respectively, a small neighborhood of the origin in \(R^{2n+1}\) and of \(p\) in \(\Omega\), we choose them small enough so that the map (2.2) above is a diffeomorphism. The Jacobian of the transformation is the determinant of \(\{X_1, \ldots, X_{2n+1}\}\), i.e.; the volume of the parallelepiped spanned by the vector fields. We give weight 1 to the vector fields \(X_1, \ldots, X_{2n}\) and weight 2 for \(X_{2n+1}\). Set

\[
\text{Box}(r) =: \left\{ \vec{v} = \sum_{j=1}^{2n+1} \lambda_j X_j : \left(\sum_{j=1}^{2n} \lambda_j^2\right)^{1/2} + \lambda_{2n+1}^2 \leq r \right\}
\]  

(2.3)

and let, for \(r\) small enough, \(B_p(r)\) be defined by

\[
B_p(r) =: \text{Exp}_p(\text{Box}(r))
\]  

(2.4)

Let us denote by \(x(p)\) the coordinate representation of the point \(p\), then it is worth noticing that the system of balls defined by equation (2.4) is equivalent to the following system of balls

\[
B_p(r) := \{ x(p) + s_1 X_1|_p + \cdots + s_{2n+1} X_{2n+1}|_p : |s_1|, \ldots, |s_{2n}|, |s_{2n+1}| \leq r, \text{ and } |s_{2n+1}| \leq r^2 \}
\]  

(2.5)

From (2.4) and (2.5) above we easily see that the balls are sets that looks like tilted ellipsoids that sits on the span of the \(X_j\)'s of size \(r^{2n}\) and of thickness \(r^2\). And hence the volume of a ball
of radius \( r \) is equal to \( C r^{2n+2} \). The number \( Q = 2n+2 \) is termed the homogeneous dimension of \( \Omega \). As we have said in the introduction everything we said so far can be rephrased in the language of contact structure and the Heisenberg group, we include a brief discussion about these.

### 2.1.3 The contact structure and the Heisenberg group

Let \( \Omega \) be any smooth manifold of dimension \( N \).

**Definition 2**  
*A contact structure on \( \Omega \) is a given smooth nonintegrable field of hyperplanes of the tangent space of \( \Omega \), satisfying a nondegeneracy condition described below.*

The nonintegrability condition means that there is no integral hypersurface to the field. An example of a field of hyperplanes may be given by the zero set of a 1-form, and conversely, every smooth field of hyperplanes is locally given by the zero set of a 1-form. This form is unique up to a nonvanishing smooth factor. If we impose a normalizing condition on the form, it becomes uniquely determined by the field of hyperplanes. Let us denote by \( \theta \) this form. Now we state the nondegeneracy condition by saying that the bilinear form \( d\theta \) restricted to the field of hyperplanes is nondegenerate i.e.;

\[
\text{rank}(d\theta|_{\theta=0}) = N - 1
\]

This condition forces the manifold \( \Omega \) to be odd dimensional, this is because the bilinear form \( d\theta \) is skew symmetric and if \( N \) is even then \( N - 1 \) is odd, and as is well known from linear algebra, there are no nondegenerate skew symmetric forms on an odd dimensional space. Set \( N = 2m + 1 \), the nondegeneracy condition may also be stated in a fancy way by the condition

\[
\theta \wedge (d\theta)^m \neq 0
\]

Sometimes \( \Omega \) with a contact structure is called a CR manifold. The standard and important example of a contact structure is the Euclidian space \( \mathbb{R}^{2m+1} \) with coordinates \((x_1, \ldots, x_m, y_1, \ldots, y_m, t) = (x, y, t)\) and the field of hyperplanes given by the one form

\[
\theta = dt + 2(xdy - ydx) := dt + 2 \sum_{j=1}^{m} (x_j dy_j - y_j dx_j) \tag{2.6}
\]

(In [AR] it is given by \( dt + xdy \), but it is clear that they are equivalent.) The following is the analogue to the well known Darboux's theorem in symplectic geometry. It states that every point of \( \Omega \) has a neighborhood and a coordinate system where the contact form takes the form (2.6) identically, unlike [FS2] where they proved that the form coincides with one of the form (2.6) only at that point.

**Theorem 1** Every differential 1-form defining a nondegenerate field of hyperplanes (i.e.; a contact structure) on some odd dimensional manifold, can be written in a local coordinate system \((x, y, t)\) in its canonical form (2.6) above.
This theorem is in appendix 4 page 362 of [AR]. It justifies the fact that it suffices to transfer our problem to the Heisenberg group whose definition is the following.

**Definition 3** The Heisenberg group $H_n$ is the Lie group whose manifold realization is the Euclidian space $R^{2n+1}$ with standard coordinate functions $(x_1, \cdots, x_n, y_1, \cdots, y_n, t) = (x, y, t)$ and whose group law is given by

$$(x, y, t)(x', y', t) = (x + x', y + y', t + t' + 2(yx' - xy'))$$

The identity element of $H_n$ is $(0, \cdots, 0)$, and the inverse to a general element $u = (x, y, t)$ is $u^{-1} = (-x, -y, -t)$. The vector fields,

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad X_{j+n} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad 1 \leq j \leq n$$

(2.7)

form a basis for the left invariant vector fields (i.e.; the Lie algebra). The commutation relations are $[X_{j+n}, X_j] = 4T$ and all others are zero. Thus the system of vector fields $X_1, \cdots, X_{2n}$ satisfies the step-two condition. The corresponding 1-form is equal to $\theta$ in (2.6) and satisfies: $\theta(X_j) = 0$ for $j = 1, \cdots, 2n$ and $\theta(T) = 1$. Next we define the nonisotropic distance to the origin $|.|$. It is equivalent to the one given as a control distance with respect to the vector fields $X_j$ given by (2.7). For $u = (x, y, t)$ we set

$$|u| := ((\sum_{j=1}^n (x_j^2 + y_j^2))^2 + t^2)^{1/4}$$

and

$$d(u, v) = |v^{-1}u| = |u^{-1}v|$$

We have the following triangle inequality

**Lemma 1** There is a constant $C > 1$ such that:

1. $|uv| \leq C(|u| + |v|)$
2. $|u + v| \leq C(|u| + |v|)$
3. $||u|| \leq |u| \leq ||u||^{1/2}$ for $|u| \leq 1$

Next we define dilations by which scaling is performed. For $r > 0$ and $u \in H_n$ we denote by $ru$, dilation by $r$,

$$ru = r(x, y, t) = (rx, ry, r^2t)$$

(2.8)

this change of variables gives the change in the volume by

$$dV(ru) = r^Q dV(u)$$

(2.9)

the number $Q$ is the homogeneous dimension. Another important formula we'll be needing is the following

$$\int_{a \leq |u| \leq b} |u|^{\alpha - Q} dV(u) = \begin{cases} C(b^\alpha - a^\alpha) & \text{if } \alpha \neq 0 \\ C \log(b/a) & \text{if } \alpha = 0 \end{cases}$$

(2.10)
from which it follows that $|u|^{-\alpha}$ is locally integrable if $\alpha > 0$ and integrable at infinity if $\alpha < 0$. The number $-Q$ is the critical power index.

We need the following notations. $J$ will always stand for a multiindex $J = (j_1, \cdots, j_{2n+1})$, let us also denote by $X_{2n+1}$ the vector field $T$, for $u = (x_1, \cdots, x_{2n+1}) \in H_n$ set

$$u^J = x_1^{j_1} \cdots x_{2n+1}^{j_{2n+1}} \quad X_J = X_1^{j_1} \cdots X_{2n+1}^{j_{2n+1}}, \text{ and } |J| = j_1 + j_2 + \cdots + j_{2n} + 2j_{2n+1}$$

The left invariant differential operators are linear combinations of the $X_J$'s. Another important notion is that of convolution, when we say convolution of two functions $f$ and $g$ it is always meant convolution in the Heisenberg group, and this is defined as follow

$$f \ast g(u) := \int_{\mathbb{H}_n} f(v) g(v^{-1}u) dV(v)$$

it is important to mention that, unlike the ordinary convolution, it is not a commutative operation. The interaction of the left invariant differential operators with the convolution we just defined is the following

$$X_J(f \ast g) = f \ast (X_Jg)$$

### 2.2 The weight function $w$

Let now $M$ be a smooth hypersurface of $H_n$ given locally as the zero set of a smooth function $\rho$, that is $M = \{u \in H_n : \rho(u) = 0\}$ and $\nabla \rho \neq 0$ on $M$. The tangent space $TM$ of $M$ sits in a natural way in the tangent space of $H_n$ as a smooth integrable field of hyperplanes and thus makes an angle $w(x)$ with the one defining the contact structure. This angle is thought of as a function on $M$, and the purpose of this section is to measure it. Let us make the following important definition

**Definition 4** A point $x$ of the hypersurface $M$ is said to be characteristic if the tangent space of $M$, $T_xM$, coincides with the given field of hyperplanes defining the contact structure i.e.; characteristic points are those where $w$ vanishes.

A surface $M$ may or may not have characteristic sets. This set can be a point a curve or in general a lower dimensional set of $M$, and hence if there are any, characteristic points form a set of surface measure zero. The geometric structure of the characteristic set can take a complicated form, however lemma 5 below or rather its proof suggests that the Hausdorff dimension is at most equal to $2n - 1 = \text{dimension} - 2$.

An example of characteristic $M$ is what we call in this thesis the characteristic hyperplane, in the canonical coordinates $(x, y, t)$ it is the hyperplane $\{t = 0\}$, it is easy to see that the vector fields $X_1, \cdots, X_{2n}$ given by (2.7) are tangent at 0. This is an example of an isolated characteristic set. The family of surfaces $\{t = c(\|x\|^2 + \|y\|^2)\}$ provides with examples of surfaces having isolated characteristic point, namely the origin. This family plays the role of cones in the nonisotropic sense. Another example is given by, in the Heisenberg group,
the surface \( \{ t = x_1^2 - y_1^2 \} \), the characteristic set here is the manifold \( \{ x_j = y_j \} \). Next we compute this angle to be roughly equal to

\[
w(x) = \left( \sum_{j=1}^{2n} (X_j \rho)^2(x) \right)^{1/2}
\]

it important to note that this function is not smooth.

The following series of lemmas give the size of the function \( w \). Lemma 2 gives the pointwise size of the function \( w \), lemma 3 gives the ordinary surface measure of balls intersected with \( M \). Lemma 4 is one of the most important result it says that the weighted measure \( w d\sigma \) respects the system of nonisotropic balls in the same way the surface measure respects the ordinary Euclidean balls, that is the measure of any nonisotropic ball of radius \( r \) is a constant power of the radius throughout the hypersurface \( M \), and thus giving the hypersurface \( M \) a homogenous dimension equal to one less of that of the ambient space. All of these lemmas are proved using Taylor expansion arguments. We work in a general coordinate system \( (x_1, \ldots, x_{2n+1}) \) where now \( M \) is the hyperplane \( \{ x_{2n+1} = 0 \} \), we write the expression of the vector fields in these coordinate system

\[
X_j = \sum_{i=1}^{2n} a_{ji} \frac{\partial}{\partial x_i} + a_j \frac{\partial}{\partial x_{2n+1}}
\]

We could have used the canonical coordinates (i.e.; the vector fields given by (2.7)).

**Lemma 2** Let \( x_0 \in M \) and set \( \eta = w(x_0) \geq 0 \), then

(a) \( w(x) \leq C(r + \eta) \) for all \( x \in B_{x_0}(r) \cap M \)

(b) \( \text{If } \eta > 0, \text{ then } w(x) > \eta/2 \) for all \( x \in B_{x_0}(c\eta) \)

(c) \( \max_{x \in B(r) \cap M} w(x) \geq Cr \)

**Proof:**

The function \( w(x)^2 \) is smooth and equal

\[
w(x)^2 = \sum_{j=1}^{2n} a_j(x_1, \ldots, x_{2n}, 0)^2
\]

each \( a_j \) is the \( \frac{\partial}{\partial x_{2n+1}} \)-component of the vector field \( X_j \). Let \( x^0 \in M \) which we assume to be the origin. By Taylor formula applied to the function \( w(x)^2 \) we have

\[
w(x)^2 = \sum_{j=1}^{2n} (a_j(0) + \varepsilon_j(x))^2
\]

with \( \varepsilon_j(x) \leq C|x| \), and thus

\[
w(x)^2 \leq \sum_{j=1}^{2n} a_j^2(0) + C|x| \sum_{j=1}^{2n} |a_j(0)| + C|x|^2
\]
from which it follows that \( w(x)^2 \leq C(\eta + r)^2 \), proving (a). As for (b), we have that

\[
w(x)^2 \geq \sum_{j=1}^{2n} a_j(0)^2 + 2 \sum_{j=1}^{2n} a_j(0)\epsilon_j(x) \geq \eta^2 - C|x|\eta \geq \eta^2/2
\]
as soon as \( |x| \leq 3\eta/4C \).

To prove (c) let us make the following important remark.

**Remark 1** Suppose that \( 0 \in M \) is close to the characteristic set, so that \( w(0) \) is small, then there is at least one \( j \in \{1, \cdots, 2n\} \) such that \( \|\nabla_{R^{2n}} a_j(0)\| \geq C > 0 \) (where \( \nabla_{R^{2n}} \) is the \( 2n \)-dimensional gradient), otherwise Hörmander condition would be violated. To see this, since the \( X_j \)'s and the commutators span there is a uniform constant \( C \) such that:

\[
0 < C \leq \sum_{j=1}^{2n} (X_j(x_{2n+1}))^2 + \sum_{j,k} (\{X_j, X_k\}(x_{2n+1}))^2 = w^2(x) + \sum_{j,k} (X_ja_k - X_ka_j)^2
\]

\[
= w^2(x) + \sum_{j,k} \left( \sum_{i=1}^{2n} \left( \frac{\partial a_k}{\partial x_i} - a_k \frac{\partial a_j}{\partial x_i} \right) + \left( \frac{\partial a_k}{\partial x_{2n+1}} - a_k \frac{\partial a_j}{\partial x_{2n+1}} \right) \right)^2
\]

\[
\leq Cw^2(x) + C \sum_{i=1}^{2n} \|\nabla_{R^{2n}} a_i\|^2
\]

from which the remark follows.

It is obvious, in proving (c) it suffices to assume that \( w(x) \leq Cr \), for all points in \( B(r) \cap M \), and this means that the ball \( B \) is near the characteristic set. Pick any point which again we assume to be the origin such that \( a_j(0) \) is small, by Taylor expansion up to order 1 now we have

\[
w(x)^2 = \sum_{j=1}^{2n} \left( a_j(0) + \nabla a_j(0).x + \epsilon_j(x) \right)^2
\]

Because the point 0 is near being characteristic, by the previous remark there is at least one \( j \in \{1, \cdots, 2n\} \) such that \( \max_{|x| \leq r} |\nabla a_j(x)| > 0 \), and thus we have

\[
w(x)^2 \geq \sum_{j=1}^{2n} a_j^2(0) + 2 \sum_{j=1}^{2n} (\nabla a_j(0).x)^2 - C|x|^2 \sum_{j=1}^{2n} |\nabla a_j(0).x|
\]

passing to the maximum we get

\[
\max_{x \in B(r)} w(x)^2 \geq Cr^2 - C'r^3 \geq Cr^2
\]

and this proves (c) and hence the lemma. \( \square \)

The next lemma is an estimate on the ordinary surface measure of nonisotropic balls.
Lemma 3  (a) There are constants $C_1$ and $C_2$ such that: $C_1 r^{2n+1} \leq \sigma(B \cap M) \leq C_2 r^{2n}$

(b) If we set $\max_{x \in B(r) \cap M} w(x) = \eta$ then we have $\sigma(B(r) \cap M) \cong r^{2n+1}/\eta$

in particular if $M$ has no characteristic points we have $\sigma(B(r) \cap M) \cong r^{2n+1}$

**Proof:** To prove this lemma we fix the point $x^0$ and we look at the ball in terms of normal coordinates. Since the map (2.2) above is a diffeomorphism the surface measure $\sigma(B(r) \cap M)$ is essentially the $2n$-dimensional Lebesgue measure of the intersection of the hyperplane $T_{x^0}M$ and $Box(r)$. Now estimates in (a) become obvious, the extreme cases are when the tangent space $T_{x^0}M$ coincides with the hyperplane $\{\lambda_{2n+1} = 0\}$, in which case the surface is $C_r r^{2n}$, and the other case is when $T_{x^0}M$ coincides with one of the hyperplanes $\{\lambda_j = 0\}$, $j \leq 2n$, in which case the surface is $C_r r^{2n+1}$, and (a) is proved. To prove (b) we write down the equation of the hyperplane $T_{x^0}M$ in the coordinates $\lambda = (\lambda_1, \cdots, \lambda_{2n+1})$, which is given by

$$(\nabla \rho|_{\lambda=0}) \lambda = 0$$

where $\rho$ is a defining function of $M$ in the coordinates $\lambda$ i.e.; $M = \{\lambda : \rho(\lambda) = 0\}$. We need to notice that

$$w(x^0)^2 = \sum_{j=1}^{2n} \left( \frac{\partial \rho}{\partial \lambda_j}(0) \right)^2$$

(2.13)

Now if $H$ denotes any hyperplane

$$\lambda_{2n+1} = a_1 \lambda_1 + \cdots + a_{2n} \lambda_{2n}$$

by a simple rotation argument it easy to see that

$$L^{2n}(Box(r) \cap H) \cong r^{2n+1} \frac{r^{2n+1}}{(a_1^2 + \cdots + a_{2n}^2)^{1/2}}$$

as long as $(a_1^2 + \cdots + a_{2n}^2)^{1/2} \geq Cr$. $L^{2n}$ is the $2n$-dimensional Lebesgue measure. Now taking into account (2.11) above (b) follows.

The upper bound in the estimate given to us by the following lemma is used in the proof of the restriction theorem and the lower bound is used in the extension theorem.

**Lemma 4**

$$\int_{B(r) \cap M} w(x) d\sigma(x) \cong r^{2n+1}$$

**Proof:** According to lemma 2(a) and lemma 3(b) we have that

$$\int_{B(r) \cap M} w(x) d\sigma(x) \leq C(r + \eta) \sigma(B(r) \cap M) \leq C r^{2n+1} + C(\frac{r}{\eta}) r^{2n+1}$$

now since $\eta$ can be at least $Cr$ the $(\leq)$ part follows. For the reverse inequality, let $x^0 \in B(r) \cap M$ be such that $\eta = w(x^0) = \max_{x \in B(r/2) \cap M} w(x)$, then by lemma 2(b) $\eta \geq Cr$ and
by lemma 1 (b) \( w(x) \geq C \eta \) for all points \( x \) in \( B_{\varepsilon^0}(\eta/10) \cap M \). We have two cases either

1. \( B_{\varepsilon^0}(\eta/100) \subset B(r) \) in which case they are actually equivalent and we have

\[
\int_{B(r) \cap M} w(x) d\sigma(x) \geq C \int_{B_{\varepsilon^0}(r/10) \cap M} w(x) d\sigma(x) \geq C r \sigma(B_{\varepsilon^0}(r/10) \cap M) \cong C r^{2n+1}
\]

or

2. \( B(r) \subset B_{\varepsilon^0}(C \eta) \) in which case \( w(x) \geq C \eta \) throughout the ball \( B(r) \cap M \), and in this case too we have

\[
\int_{B(r) \cap M} w(x) d\sigma(x) \cong C \eta \sigma(B(r) \cap M)
\]

using lemma 2 (b) we get the desired result. \( \square \) The following lemma is key in proving theorem 5.

**Lemma 5** For all \( 0 < \varepsilon < 1 \), the function \( w(x)^{-\varepsilon} \) is locally integrable, and have the following estimate

\[
\int_{B \cap M} w^{-\varepsilon}(x) d\sigma(x) \leq C (\max_{x \in B \cap M} w(x))^{-\varepsilon} \sigma(B \cap M)
\]

**Proof:**

The fact that if \( w \) vanishes at some point it can only do so to first order by the preceding remark implies that \( w^{-\varepsilon} \) is locally integrable, also the \((\geq)\) part is obvious. Let \( x^0 \in B(r) \cap M \) be a point, which we can assume to be the origin, such that the maximum is reached there i.e.; \( w(0) = \eta = \max_{x \in B(r) \cap M} w(x) \). We have to consider two cases like before:

case 1. \( \eta > 10r \) in which case the function \( w \) continues to be larger than \( r \), in fact it is almost constant \((\cong \eta)\) on the ball \( B(r) \), and therefore we get in this case

\[
\int_{B(r) \cap M} w^{-\varepsilon} d\sigma \cong \eta^{-\varepsilon} \sigma(B(r) \cap M)
\]

case 2. \( \eta \cong r \), in which case the ball \( B(r) \cap M \) is a Euclidian ball. Write \( w(x)^2 \) in its Taylor formula:

\[
w(x)^2 = \sum_{j=1}^{2n} (a_j(0) + \nabla a_j(0) \cdot x + \varepsilon_j)^2
\]

Choose a \( j \) such that \( |\nabla a_j(0)| \geq C > 0 \) and let \( H \) be the hyperplane whose equation:

\[
a_j + \nabla a_j(0) \cdot x = 0
\]

We have

\[
w(x)^2 \geq ||a_j(0) + \nabla a_j(0) \cdot x||^2 - 2 |\varepsilon_j| ||a_j(0) + \nabla a_j(0)||
\]

from which it follows that

\[
w(x) \geq \frac{1}{2} |a_j(0) + \nabla a_j(0) \cdot x| = \frac{1}{2} ||\nabla a_j(0)|| \text{dist}(x, H) \tag{2.14}
\]
so long as $\frac{1}{2}|\nabla a_j(0)| \text{dist}(x, H) > 2C|x|^2$, where the constant $C$ is the minimum of all constants such that $\epsilon_j(x) \leq C|x|^2$, which is the case if $|\nabla a_j(0)| \text{dist}(x, H) > 2Cr^2$. Let $C' = 2C/|\nabla a_j(0)|$ and define the set $E(B(r))$ to be

$$E(B(r)) := \{ x \in B(r) \cap M : \text{dist}(x, H) \leq C'diam(B(r))^2 \}$$

(2.15)

And thus, we have

$$\int_{B(r) \cap M} w^{-\epsilon} d\sigma \leq C \int_{B(r) \cap M} \text{dist}(x, H)^{-\epsilon} d\sigma + \int_{E(B(r))} w^{-\epsilon} d\sigma$$

by a linear change of variable making the hyperplane $H$ horizontal we see that the first integral is equal to $C r^{-\epsilon+2n} \cong C \eta^{-\epsilon} \sigma(B(r) \cap M)$. As for the other term we either have $\sigma(E(B(r))) = 0$ in which case that’s the end of it or else it can be at most a small corridor of size $C r \times r \times \cdots \times r \times r^2$. Cover it with balls of radius $r^2$, there can be at most $C r^{-2n+1}$ of them $B_i(r^2)$. And so we have,

$$\int_{E(B(r))} w^{-\epsilon} d\sigma \leq C \sum_i \int_{B_i(r^2)} w^{-\epsilon} d\sigma$$

do the same procedure as above to each $B_i(r^2)$ and we arrive at

$$\int_{B_i(r^2)} w^{-\epsilon} d\sigma \leq C r^{-2\epsilon+4n} + \int_{E(B_i(r^2))} w^{-\epsilon} d\sigma$$

summing over $i$ we get

$$\int_{E(B(r))} w^{-\epsilon} d\sigma \leq C r^{-2\epsilon+2n+1} + \sum_i \int_{E(B_i(r^2))} w^{-\epsilon} d\sigma$$

if we continue this procedure again to each term in the sum we arrive finally at

$$\int_{B(r) \cap M} w^{-\epsilon} d\sigma \leq C r^{-\epsilon+2n}(1 + r^{-\epsilon+1} + r^{3(-\epsilon+1)} + \cdots) \leq C r^{-\epsilon+2n}$$

since $\epsilon < 1$. And this finishes the proof of lemma 4 as well as this is the end of this section.

2.3 The spaces $S^p_{\alpha}$, $\Gamma^p_{\alpha}$, $F^p_{\alpha}$ and $B^p_{\alpha}$

Next we discuss the function spaces of interest to us i.e.; the nonisotropic Sobolev spaces denoted here by $S^p_{\alpha}$ defined on the ambient space, the nonisotropic Besov spaces $\Gamma^p_{\alpha}$ and the expected spaces of restrictions to $M$ $F^p_{\beta}(M)$ and $B^p_{\beta}(M)$.
2.3.1 The Spaces $S^p_\alpha$

These spaces were studied exclusively by few authors, [FG]; [FS]; [NS] and [SA]. Recall that in the Euclidean space $\mathbb{R}^n$ with standard coordinate functions $(x_1, \ldots, x_n)$ the basic differentiation $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are used to define the classical Sobolev spaces, we denote them here by $L^p_k$. Analogously, if we replace the $\frac{\partial}{\partial x_j}$ by the directional derivatives $X_j$ for $j = 1; \ldots; 2n$ we obtain the spaces $S^p_\alpha$ namely we have the following:

**Definition 5** We say that the function (or to be more precise the Lebesgue class) $f$ belongs to the nonisotropic Sobolev space $S^p_\alpha$, if it and its distributional derivatives $X_J f$, with $|J| \leq k$ all belong to $L^p$. The norm of $f$ in $S^p_\alpha$ is given by

$$||f||_{S^p_\alpha} := \sum_{|J| \leq k} ||X_J f||_{L^p}$$  \hspace{1cm} (2.16)

One may follow various classical constructions to define the spaces $S^p_\alpha$ for $\alpha$ noninteger. The approach we follow here is by realizing these spaces as potentials. We let $L = -\sum_{j=1}^{2n} X_j^2$ be the sublaplacian we denote by $L_p$ its extension to the space $L^p$ and we define the spaces $S^p_\alpha$ as the range of the operator $(I + L_p)^{-\alpha/2}$. The operator $(I + L_p)^{-\alpha/2}$ is realized as a convolution operator in the Heisenberg group acting on $L^p$. What this means for us is that there is a function $J_\alpha$, the kernel of $(I + L_p)^{-\alpha/2}$, such that

$$S^p_\alpha = \{ f * J_\alpha, f \in L^p \}$$  \hspace{1cm} (2.17)

The function $J_\alpha$ has the following properties

**Proposition 2** The function $J_\alpha$ is smooth away from the origin and

1. As $|x| \to 0$

$$|X_J J_\alpha(x)| \leq C|x|^{\alpha - |J| - Q} \quad \text{if} \quad \alpha \neq |J| + Q$$

2. $X_J J_\alpha$ is continuous if $\alpha > |J| + Q$

3. If $\alpha = |J| + Q$ we have instead

$$|X_J J_\alpha(x)| \leq C \log(\frac{1}{|x|})$$

4. Away from the origin the function $J_\alpha$ is rapidly decreasing, that is as $|x| \to \infty$, for any $N$ and $J$ there is a constant $C = C_{N,J}$ such that

$$|X_J J_\alpha(x)| \leq C|x|^{-N}$$

In particular $X_J J_\alpha$ is integrable for all $J$ such that $\alpha - |J| > 0$. The norm we take for a function $f = g * J_\alpha \in S^p_\alpha$ is

$$||f||_{S^p_\alpha} := ||g||_{L^p}$$  \hspace{1cm} (2.18)

The following proposition says that in the case where $\alpha$ is integer the second definition (i.e.; the potential definition) is equivalent to the one above. This is tedious task already in the classical case see [ST1] for this the generalization is due to Folland, its second part also states that it is enough to define the fractional spaces only for small $\alpha$. 

15
Proposition 3 When $\alpha$ is an integer, definition (2) above and (11) are equivalent in the sense that the norms (10) and (12) are equivalent. Moreover if $\alpha \geq 1$, a function $f$ belongs to $S^p_\alpha$ if and only if $f$ and $X_j f$ for $j=1,\ldots,2n$ all belong to $S^p_{\alpha-1}$. An equivalent norm is

$$\|f\|_{S^p_\alpha} + \sum_{j=1}^{2n} \|X_j f\|_{S^p_{\alpha-1}}$$

Another result that we use later is the following

Proposition 4 The operator defined by convolution with $J_\beta$ is an isomorphism between the spaces $S^p_\alpha$ and $S^p_\gamma$, where $\gamma = \alpha + \beta$.

Next we compare the spaces $S^p_\alpha$ and the classical $L^p_\gamma$. It is clear that if $\alpha = 2m$ is an even integer then we have locally

$$S^p_\alpha \subset L^p_{\alpha/2}$$

and by complex interpolation this is true for all $\alpha \geq 2$, the next proposition tells us that this embedding is true for all $\alpha \geq 0$.

Proposition 5 For all $\alpha \geq 0$, we have

$$S^p_\alpha \subset L^p_{\alpha/2} \quad \text{(locally)}$$

2.3.2 The spaces $\Gamma^{p,q}_\alpha$

In this section we also include a short discussion on the nonisotropic Besov spaces and we indicate various connections to the spaces $S^p_\alpha$. These spaces are the analogues to the classical Besov spaces $\Lambda^{p,q}_\alpha$, see [ST1]. The reference for this is Saka’s paper [SA]. This paper is essentially the nonisotropic analog to Taibleson's classical paper [TA]. The theory of nonisotropic Besov spaces and Sobolev spaces is almost entirely parallel to the isotropic ones, in the sense that all the results that hold for latter spaces hold in an appropriate sense for the former ones. Also we show below that these spaces can be characterized by local polynomial approximations.

Definition 6 Let $\alpha > 0$ and $1 < p, q < \infty$. We say that $f$ belongs to the space $\Gamma^{p,q}_\alpha$ if its norm defined below is finite. If $0 < \alpha < 1$

$$||f||_{\Gamma^{p,q}_\alpha} := ||f||_{L^p} + \left( \int_{|y|\leq 1} \frac{||f(y) - f(\cdot)||_{L^p}^q}{|y|^{Q+q}} \ dy \right)^{1/q}$$

2. If $\alpha = 1$, the second difference is used instead,

$$||f||_{\Gamma^{p,q}_1} := ||f||_{L^p} + \left( \int_{|y|\leq 1} \frac{||f(y) + f(y^{-1}) - 2f(\cdot)||_{L^p}^q}{|y|^{Q+q}} \ dy \right)^{1/q}$$

3. More generally we let $k$ be the integer such that $0 \leq k < \alpha \leq k + 1$ and we set

$$||f||_{\Gamma^{p,q}_\alpha} := \sum_{|j| \leq k} \|X_j f\|_{\Gamma^{p,q}_{\alpha-k}}$$
In the same way the classical Besov spaces can be characterized by means of higher differences, the nonisotropic ones also enjoy similar characterization. For \( y \in H_\alpha \) we let

\[
\Delta_h f(x) = f(xh) - f(x)
\]

and set

\[
\Delta_h^k f(x) = \Delta_h (\Delta_h^{k-1} f(x))
\]  

(2.19)

Let \( k > \alpha \), then an equivalent definition of the norm above is

\[
\|f\|_{\Gamma_\alpha^p,q} = \|f\|_{L^p} + \left( \int_{|y| \leq 1} \frac{\|\Delta_h^k f\|_{L^p}^q}{|y|^{q + \alpha q}} \, dy \right)^{1/q}
\]  

(2.20)

The next results are Theorem 20 and 23 of [SA]. The first, giving various embeddings with the spaces \( S_\alpha^2 \), is of interest to us because it says in particular that in the case \( p = 2 \) the spaces \( \Gamma_\alpha^{2,2} \) and \( S_\alpha^2 \) are equivalent. The second realizes the spaces \( \Gamma_\alpha^{p,q} \) as real interpolation spaces of \( S_\alpha^p \).

**Theorem 2** Let \( \alpha > 0 \), then

1. \( \Gamma_\alpha^{p,p} \subset S_\alpha^p \subset \Gamma_\alpha^{p,2} \) for \( 1 < p \leq 2 \)
2. \( \Gamma_\alpha^{p,2} \subset S_\alpha^p \subset \Gamma_\alpha^{p,p} \) for \( 2 < p < \infty \)

in particular we have \( \Gamma_\alpha^{2,2} = S_\alpha^2 \).

**Theorem 3** The spaces \( S_\alpha^p \) form a scale of real interpolation and we have for \( 0 < \theta < 1 \), \( 1 < p < \infty \) \( 1 \leq q \leq \infty \) and \( \gamma = (1 - \theta)\alpha + \theta \beta \),

\[
(S_\alpha^p, S_\alpha^q)_{\theta, \gamma} = \Gamma_\gamma^{p,q}
\]

The next characterization of the Besov spaces \( \Gamma_\alpha^{p,q} \) is going to be in terms of local polynomial approximation. For simplicity we discuss only the case \( p = q \) and denote the resulting spaces by \( \Gamma_\alpha^p \). It is well known in the classical setting that in many instances differentiability properties of functions can be described by how much we can approximate the function by polynomials see e.g. the appendix of [ST] where the existence of derivatives in the \( L^p \) is discussed. In [NI], [BR1] and [BR2] and many other papers, similar ideas were exploited. It turns out that membership to the Besov spaces (and their generalizations the Lizorkin-Triebel spaces), can be expressed in terms of the following quantities:

for any function \( f \) and a ball \( B \) and \( N \) an integer we denote by

\[
\omega_N(f, B)^p := \inf_{d(P, B) \leq N} \int_B |f - P|^p
\]  

(2.21)

Following [JW], by a net \( \mathcal{F}_k \) of mesh \( 2^{-k} \) we mean a collection of balls \( B \) of radius \( 2^{-k} \) and covering the ambient space satisfying the requirement that if we shrink them enough we get a disjoint family of balls. For each integer \( k \), let \( \mathcal{F}_k \) be a net of mesh \( 2^{-k} \), then we have the following result.
Proposition 6 There is an integer \( N > \alpha \) such that the following quantity

\[
||f||_{L^p} + \left( \sum_{k=0}^{\infty} 2^{k \alpha} \sum_{B \in \mathcal{F}_k} \omega_N(f, B)^p \right)^{1/p}
\]  

is an equivalent norm for the space \( \Gamma_\alpha^p \).

The following important result is well known, it reflects the interpolation property theorem 2 above and can be obtained from general interpolation (see [BL] lemma 3.2.3). This result is used quite often in this thesis.

Proposition 7 Let \( f \in \Gamma_\alpha^p \), and \( N \) be an integer larger than \( \alpha \). Then there are smooth functions \( f_j, j = 0, 1 \cdots \) such that the following hold:

\[
f = \sum_{j=0}^{\infty} f_j
\]  

convergence is in the sense of \( S' \)

\[
||X_j f_j||_{L^p} \leq 2^{-j|\alpha - |J||} a_j \quad \text{for all} \quad |J| \leq N
\]  

for some sequence of positive real numbers \( a_j \) satisfying \( \sum_{j=0}^{\infty} a_j^p \leq C < \infty \).

Conversely, given a family of functions satisfying (2) for some \( a_j \)'s then the formal infinite sum \( \sum_{j=0}^{\infty} f_j \) defines a distribution that is an element of \( \Gamma_\alpha^p \).

We give a proof below that propositions 6 and 7 are equivalent. We basically follow the proof of [NS1] given for this same proposition but for nonisotropic Lipschitz spaces \( (\Gamma_\alpha) \). For simplicity we consider only the case \( p = 2 \) since that is the case we are going to need in the sequel anyway.

Proof:

Let \( \varphi \) be defined by:

\[
\varphi(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 1 \\
0 & \text{if } t > 2
\end{cases}
\]

and \( 0 \leq \varphi(t) \leq 1 \) for \( \xi \in R^{2n+1} \) set

\[
\Phi_0(\xi) = \varphi(|\xi|)
\]

and for \( j \neq 0 \)

\[
\Phi_j(\xi) = \varphi(2^{-j}|\xi|) - \varphi(2^{-j+1}|\xi|)
\]

Denote by \( K_j \) the inverse Fourier transform of \( \Phi_j \) we easily verify that

\[
\sum_{j=0}^{\infty} K_j = \delta
\]

and we obtain decomposition of functions \( f \) into a sum of smooth functions \( f_j = f * K_j \) where convolution here again is the group convolution. The following estimates on the \( K_j \)'s, which express the size of \( K_j \) and its rapid decrease, are needed.
Lemma 6  (a) \(|K_j(x)| \leq C2^{jQ}\)
(b) For each integer \(N\) there is a constant \(C_N\) such that
\[|K_j(x)| \leq C_N 2^{jQ-mN} \text{ for } 2^{-j+m} \leq |x| \leq 2^{-j+m+1}\]

A property, that is easy to verify (because the Fourier transform is supported away from the origin), that the \(K_j\)'s have for \(j \neq 0\) is they orthogonal to polynomials. We begin by showing that the \(f_j\) defined satisfy the conclusions of the proposition. We will prove the estimates only for the \(L^2\)-norm of \(f_j\), because the estimates for the derivatives follow exactly the same way. It is obvious that
\[\|f_0\|_{L^2} \leq C\|f\|_{L^2}\]

For \(j \neq 0\), using the orthogonality of polynomials to \(K_j\), the \(L^2\) norm of \(f_j\) is equal to
\[\|f_j\|^2 = \sum_{B \in \mathcal{F}_j} \int_B |f_j(x)|^2 \, dx = \sum_{B \in \mathcal{F}_j} \int_B \int_{R^{n+1}} \left| f(y) - P_B(y) \right| |K_j(y^{-1}x)| \, dy \, dx \tag{2.26}\]

the right hand side of (2.26) is less than or equal to \(I_1 + I_2\), where
\[I_1 = \sum_{B \in \mathcal{F}_j} \int_B \left( \int_B |f(y) - P_B(y)||K_j(y^{-1}x)| \, dy \right)^2 \, dx \tag{2.27}\]

and
\[I_2 = \sum_{B \in \mathcal{F}_j} \int_B \left( \int_{R^{n+1} \setminus B} |f(y) - P_B(y)||K_j(y^{-1}x)| \, dy \right)^2 \, dx \tag{2.28}\]

and we show that each \(I_i\) is less than or equal to \(2^{-2j\alpha} b_j^2\) for some numbers such that \(\sum_{j=0}^{\infty} b_j^2 < \infty\). \(I_1\) well controlled and it is less than
\[C \sum_{B \in \mathcal{F}_j} \int_B |f(y) - P_B(y)|^2 \, dy \leq C 2^{-2j\alpha} a_2\]

The trouble is to handle the other part \(I_2\). For each \(B\) in \(\mathcal{F}_j\) we denote by \(2B\) the ball in \(\mathcal{F}_{j-1}\) containing it, or to be precise one of the balls in \(\mathcal{F}_{j-1}\) containing it. And inductively we denote by \(2^k B\) a ball in \(\mathcal{F}_{j-k}\) containing \(2^{k-1} B\), with this we can write the inner integral in \(I_2\) as
\[\int_{R^{n+1} \setminus B} |f(y) - P_B(y)||K_j(y^{-1}x)| \, dy \leq \int_{2B \setminus B} |P_2B(y) - P_B(y)||K_j(y^{-1}x)| \, dy + \int_{R^{n+1} \setminus 2B} |f(y) - P_B(y)||K_j(y^{-1}x)| \, dy \leq \sum_{k=0}^{j} \int_{2^k B \setminus 2^{k+1} B} |P_{2^k+1}B(y) - P_{2^k}B(y)||K_j(y^{-1}x)| \, dy + \int_{R^{n+1} \setminus 2^j B} |f(y) - P_{2^j}B(y)||K_j(y^{-1}x)| \, dy = A_1 + A_2\]
By lemma 6 we have,
\[ |K_j(y^{-1}x)| \leq C N^{2jQ} 2^{-jN} \]
and thus
\[ A_1 \leq C N^{2jQ} \sum_{k=0}^{j} 2^{-kN} \int_{2k+1B \setminus 2^kB} |P_{2^{k+1}B}(y) - P_{2kB}(y)| \, dy \]
\[ \leq C N^{2Q/2} \sum_{k=0}^{j} 2^{-k(N-Q/2)} \left( \int_{2^{k+1}B \setminus 2^kB} |P_{2^{k+1}B}(y) - P_{2kB}(y)|^2 \right)^{1/2} \, dy \]
\[ \leq C N^{2Q/2} \sum_{k=0}^{j} 2^{-k(N-Q/2)} \left( \int_{2^kB} |P_{2kB}(y) - f(y)|^2 \right)^{1/2} \, dy \]
\[ + C N^{2Q/2} \sum_{k=0}^{j} 2^{-k(N-Q/2)} \left( \int_{2^kB} |f(y) - P_{2kB}(y)|^2 \right)^{1/2} \, dy \]
integrating this string of inequalities over \( B \) and then summing over \( B \) in \( F_j \) using Minkowski's inequality, and also keeping in mind that when we sum over \( B \in F_j \) quantities such as \( \int_{2^kB} |P_{2kB}(y) - f(y)| \, dy \) get repeated \( 2^kQ \) times, we get
\[ \sum_{B \in F_j} \int_B A_1^2 \, dx \leq \sum_{B \in F_j} \left( \sum_{k=0}^{j} 2^{-k(N-Q/2)} \left( \int_{2^kB} |P_{2kB}(y) - f(y)|^2 \right)^{1/2} \right)^2 \leq C 2^{-2j_0 b_j^2} \]
where
\[ b_j^2 = 2^{-2j(N-Q/2-a)} \left( \sum_{k=0}^{j} 2^{k(N-Q/2-a)} a_k \right)^2 \]
choose \( N > \alpha + Q/2 \) and use Hardy's inequality, lemma 8 below, to get
\[ \sum_{j=0}^{\infty} b_j^2 < \infty \]

Now we turn to \( A_2 \).
\[ A_2 = \int_{R^n \setminus 2^kB} |f(y) - P_{2kB}(y)||K_j(y^{-1}x)| \, dy \]
\[ \leq \int_{R^n \setminus 2^kB} |f(y)||K_j(y^{-1}x)| \, dy + \int_{R^n \setminus 2^kB} |P_{2kB}(y)||K_j(y^{-1}x)| \, dy \]
and so we have
\[ \sum_{B \in F_j} \int_B \left( \int_{R^n \setminus 2^kB} |f(y)||K_j(y^{-1}x)| \, dy \right)^2 \, dx \leq C \int_{R^n \setminus \{y^{-1}x \geq 1\}} \left( \int_{|y^{-1}x| \geq 1} |f(y)||K_j(y^{-1}x)| \, dy \right)^2 \, dx \leq C N^{-2Q} \|f\|_{L^2}^2 \leq 2^{-2j_0} \left( 2^{-2j(N-Q/2-a)} \|f\|_{L^2}^2 \right) \]
The other term is estimated in the same way

\[
\sum_{B \in \mathcal{F}_j} \int_B \left( \int_{R^{n+1 \setminus 2B}} |P_{2B}(y)||K_j(y^{-1}x)|dy \right)^2dx = \sum_{B \in \mathcal{F}_j} \int_B \left( \sum_{k=j}^\infty \int_{2^{k+1}B \setminus 2kB} |P_{2B}(y)||K_j(y^{-1}x)|dy \right)^2dx
\]

\[
\leq C_N \sum_{B \in \mathcal{F}_j} \int_B \left( \sum_{k=j}^\infty 2^{-kN2^Q} \int_{2^{k+1}B \setminus 2kB} |P_{2B}(y)|dy \right)^2dx = C_N \sum_{B \in \mathcal{F}_j} 2^{-jQ} \left( \sum_{k=j}^\infty 2^{-kN2^Q} \int_{2kB} |P_{2B}(y)|dy \right)^2dx
\]

\[
\leq C_N 2^{-2j(N-Q)} \sum_{B \in \mathcal{F}_j} \int_{2B} |P_{2B}(y)|^2dy = C_N 2^{-2j(N-Q)} \sum_{B \in \mathcal{F}_0} \int_B |P_B(y)|^2dy
\]

\[
\leq 2^{-2j\alpha} (C_N 2^{-2j(N-Q-\alpha)} (||f||_2^2 + \sum_{B \in \mathcal{F}_0} \int_B |f(y) - P_B(y)|^2dy))
\]

Choose \( N > \alpha + Q \), and conclude by Hardy’s inequality.

The converse is easier. But before we prove it we need Taylor’s formula in the setting of the Heisenberg group. Because we need it later in the restriction theorem let us show how to derive it. Let \( f \) be a smooth function. For simplicity we indicate the calculations in the three dimensional space. We want to expand the function \( f \) around a point \( y = (y_1, y_2, y_3) \).

Set

\[
F(s) := f(y(s(y^{-1}x)))
\]

we have \( F(0) = f(y) \) and \( F(1) = f(x) \). Taking ordinary Taylor expansion of \( F \) around 0 we get

\[
F(1) = \sum_{j=0}^N \frac{F^{(j)}(0)}{j!} + R_N
\]

\[
R_N = \frac{1}{N!} \int_0^1 (1 - s)^N F^{(N+1)}(s)ds
\]

we calculate \( F'(s) \)

\[
F'(s) = \frac{d}{ds} \{ f(y_1 + s(x_1 - y_1), y_2 + s(x_2 - y_2), y_3 + s(x_3 - y_3) + 2s(s-1)(y_1x_2 - y_2x_1)) \}
\]

\[
= \left( 2s(x_3 - y_3) + 4s(2s-2)(y_1x_2 - y_2x_1) \right) \frac{\partial f}{\partial x_3}(y(s(y^{-1}x)))
\]

evaluating \( F' \) at 0 we get

\[
F'(0) = (x_1 - y_1) \frac{\partial f}{\partial x_1}(y) + (x_2 - y_2) \frac{\partial f}{\partial x_2}(y) - 2(y_1x_2 - y_2x_1) \frac{\partial f}{\partial x_3}(y)
\]

\[
= (y^{-1}x)^{(1,0,0)}(X_1f)(y) + (y^{-1}x)^{(0,1,0)}(X_2f)(y)
\]
More generally we have

\[ F^{(m)}(s) = \sum_{m \leq |J| \leq 2m} c_{J,m} s^{|J| - m}(y^{-1}x)^J(X_J f)(y(s(y^{-1}x))) \]

and evaluating at \( s = 0 \) we get the left Taylor polynomial

\[ F^{(m)}(0) = \sum_{|J| = m} c_{J,m}(y^{-1}x)^J(X_J f)(y) \]  \( (2.29) \)

The remainder is written in its integral form as

\[ R_N = \frac{1}{N!} \sum_{N \leq |J| \leq 2N} c_{J,N}(y^{-1}x)^J \int_0^1 s^{|J| - N}(1 - s)^N(X_J f)(y(s(y^{-1}x)))ds \]  \( (2.30) \)

Let us go back now to the proof of the converse, let \( f_j \) be given to satisfy (2.24) of proposition 7, from which it follows at once that

\[ \| \sum_{k=0}^{\infty} f_k \|_{L^2} \leq \sum_{k=0}^{\infty} \| f_k \|_{L^2} \leq \sum_{k=0}^{\infty} 2^{-k\alpha} a_k \leq C(\sum_{k=0}^{\infty} a_k^2)^{1/2} \]

next we prove

\[ \sum_{k=0}^{\infty} 2^{2k\alpha} \sum_{B \in \mathcal{F}_k} \omega(\sum_{j=0}^{k} f_j, B)^2 \leq C \sum_{j=0}^{\infty} a_j^2 \]  \( (2.31) \)

the left hand side of (2.31) is less than

\[ \sum_{k=0}^{\infty} 2^{2k\alpha} \sum_{B \in \mathcal{F}_k} \omega(\sum_{j=0}^{k} f_j, B)^2 + C \sum_{k=0}^{\infty} 2^{2k\alpha} (\sum_{j=k+1}^{\infty} \| f_j \|_{L^2})^2 \]  \( (2.32) \)

replacing \( \| f_j \|_{L^2} \) with its estimates and applying Hardy’s inequality we get that the second term of (2.32) is less than \( \sum_{j=0}^{\infty} a_j^2 \). As for the first term we have

\[ \sum_{B \in \mathcal{F}_j} \omega(\sum_{j=0}^{k} f_j, B)^2 \leq (\sum_{j=0}^{k} \left( \sum_{B \in \mathcal{F}_k} \int_B |R_j(x, x_B)|^2dx \right)^{1/2})^2 \]  \( (2.33) \)

where \( R_j(x, x_B) \) is the remainder of the left Taylor expansion of \( f_j \) around any point \( x_B \) of \( B \). Now by the estimates satisfied by the remainder we see that the right hand side of (2.33) is less than or equal to

\[ C \sum_{j=0}^{k} 2^{-2\alpha|J|}(\sum_{j=0}^{k} \| X_J f_j \|_{L^2})^2 \leq C \sum_{j=0}^{k} 2^{-2\alpha|J|}(\sum_{j=0}^{k} 2^{-j(\alpha-|J|)}a_j)^2 \]

Now conclude the proof by Hardy.
2.3.3 The Spaces $F^p_\alpha$ and $B^p_\alpha$

Now we come to define what we expect to be the boundary values of the spaces $S^p_\alpha$. The definition we are going to use is the local polynomial approximation on $M$. Let $\mathcal{F}_k$ be a net of mesh $2^{-k}$ covering the surface $M$ (not the ambient space). Let us also fix a coordinate system $(x_1, \cdots, x_{2n+1})$ in which $M$ is described by $M = \{(x_1, \cdots, x_{2n+1}) : x_{2n+1} = 0\}$, with this we can speak of polynomials in $M$. Now the definition of $F^p_\alpha$ is as follow: we let as in the case of the ambient space

$$\omega_N(f, B)^2 := \inf_{d\in \mathcal{F}_N} \int_B |f - P|^p d\mu \quad (2.34)$$

**Definition 7** Let $N > \alpha$, a function $f$ defined on $M$ is said to belong to the space $F^p_{\alpha,N}$ if

$$\|f\|_{F^p_{\alpha,N}} := \|f\|_{L^p(d\mu)} + \left( \sum_{k=0}^{\infty} 2^{kp\alpha} \sum_{B \in \mathcal{F}_k} \omega_N(f, B)^p \right)^{1/p} \quad (2.35)$$

is finite for some $N > \alpha$.

The expressions in the right hand side of (2.34) and hence (2.35) depend on the choice of local coordinates but membership to the space $F^p_{\alpha,N}$ is independent of the coordinate system chosen. The case $0 < \alpha < 1$ is of special interest because if we use approximation by constants instead of polynomials in (2.34) i.e.; $\omega_0(f, B)^2$ and form the corresponding quantities (2.35) we get that this space, which we denote from now on by $F^2_\alpha$ may be characterized by the first difference, that is if $0 < \alpha < 1$ an equivalent expression to the one in (2.35) is given by

$$\|f\|_{F^2_\alpha} := \|f\|_{L^2(d\mu)} + \left( \int \int_{d(x,y) \leq 1} \frac{|f(x) - f(y)|^p}{d(x,y)^{2-1+\alpha p}} d\mu(x) d\mu(y) \right)^{1/p} \quad (2.36)$$

Since we will be only interested in this space, because this is so far the space that we proved restriction and extension theorems, for higher $\alpha$ we unfortunately have to replace them with the spaces $B^p_\alpha$ whose definition is next. We would like however before we move to next definition relate these spaces to known spaces such as $L^2(d\sigma)$ for example, and thus we ask the question for which values of $\alpha$ ($0 < \alpha < 1$) do we have the following imbedding?

$$F^2_\alpha \subset L^2(d\sigma)$$

For this we have the following

**Proposition 8** For all $\alpha$ such that $\frac{1}{2} < \alpha < 1$ we have the following

$$F^2_\alpha \subset L^2(d\sigma)$$

**Proof:** Suppose that $f \in F^2_\alpha$ which means $f \in L^2(d\mu)$ and

$$\sum_{k \geq 0} 2^{2\alpha k} \sum_{Q \in \mathcal{F}_k} \omega_0(Q, f)^2 < \infty$$

23
For $Q \in \mathcal{F}_k$ we denote by $\mathcal{F}_i(Q)$ balls in $\mathcal{F}_i$ touching the ball $Q$ ie;

$$\mathcal{F}_i(Q) = \{Q' \in \mathcal{F}_i : Q' \cap Q \neq \emptyset\}$$

where $Q'^*$ stands for the ball concentric with $Q'$ and with radius 10 times the radius of $Q'$.

We note that $\#\mathcal{F}_i(Q) \leq C$ this constant is independent of $Q$, $k$, and $l$, it depends of course on surface $M$ under consideration and other absolute constants. Let $c_Q$ be the constant that best approximates $f$ in $Q$ in the $L^2(d\mu)$ sense i.e.; $c_Q = \frac{1}{\mu(Q)} \int_Q f d\mu$. Let $\varphi^k_Q$ be a partition of unity subordinate to the cover $\mathcal{F}_k$, and define the function $f_k$ by:

$$f_k := \sum_{Q \in \mathcal{F}_k} \varphi_Q c_Q$$

because of the local finiteness of the partition of unity the function $f_k$ is well defined even smooth and it easy to see that the sequence of functions $f_k$ approaches $f$ in $L^2(d\mu)$. For,

$$\|f - f_k\|_{L^2(d\mu)} \approx \left( \sum_{Q \in \mathcal{F}_k} \| \sum_{Q' \in \mathcal{F}_k} \varphi_Q (f - c_Q) \|_{L^2(d\mu,Q')}^2 \right)^{\frac{1}{2}} \leq$$

$$\left( \sum_{Q \in \mathcal{F}_k} \left( \sum_{Q' \in \mathcal{F}_k} \|f - c_Q\|_{L^2(d\mu,Q')}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq$$

$$C \left( \sum_{Q \in \mathcal{F}_k} \|f - c_Q\|_{L^2(d\mu,Q)}^2 \right)^{\frac{1}{2}} \leq C 2^{-ka} a_k \rightarrow 0$$

Next, we estimate $\|f_k\|_{L^2(d\sigma)}$. For $k = 0$ we have,

$$\|f_0\|_{L^2(d\sigma)} \leq C \sum_{Q \in \mathcal{F}_0} \|f_0\|_{L^2(Q,d\sigma)} = \sum_{Q \in \mathcal{F}_0} \| \sum_{Q' \in \mathcal{F}_0} \varphi_Q c_Q \|_{L^2(Q,d\sigma)}$$

$$\leq C \sum_{Q \in \mathcal{F}_0} |c_Q|^2 \sigma(Q) \leq C \sum_{Q \in \mathcal{F}_0} \frac{\sigma(Q)}{\mu(Q)} \int_Q |c_Q|^2 d\mu \leq \|f\|_{L^2(d\mu)}^2 + \sum_{Q \in \mathcal{F}_0} \omega(Q,f)^2$$

we have used the fact that for $Q \in \mathcal{F}_0$, then $\frac{\sigma(Q)}{\mu(Q)} \approx 1$, and more generally by lemma 3(a) and lemma 4 we have,

$$C \leq \frac{\sigma(Q)}{\mu(Q)} \leq C' \text{Diam}(Q)^{-1}$$

Now for $k > 0$ we write

$$f_k = (f_k - f_{k-1}) + \cdots + (f_1 - f_0) + f_0$$

and

$$f_j - f_{j-1} = \sum_{Q \in \mathcal{F}_j} \sum_{Q' \in \mathcal{F}_{j-1}} \varphi_Q \varphi_{Q'} (c_Q - c_{Q'})$$
and so
\[ ||f_k||_{L^2(d\sigma)} \leq ||f_0||_{L^2(d\sigma)} + \sum_{j=1}^{k} ||f_j - f_{j-1}||_{L^2(d\sigma)} \]
\[ ||f_j - f_{j-1}||_{L^2(d\sigma)}^2 \leq C \sum_{Q \in \mathcal{F}_j} \sum_{Q' \in \mathcal{F}_{j-1}(Q)} \sum_{Q'' \in \mathcal{F}_{j-1}(Q')} |c_Q - c_{Q''}|^2 d\sigma \leq \]
\[ C2^j \sum_{Q \in \mathcal{F}_j} \sum_{Q' \in \mathcal{F}_{j-1}(Q)} \int_Q |c_Q - c_{Q'}|^2 d\mu \]
because also of the finiteness of the sums over \( \mathcal{F}_j(Q) \) and \( \mathcal{F}_{j-1}(Q) \) we get
\[ ||f_j - f_{j-1}||_{L^2(d\sigma)}^2 \leq C2^j \left( \sum_{Q \in \mathcal{F}_j} \omega(Q, f)^2 + \sum_{Q' \in \mathcal{F}_{j-1}} \omega(Q', f)^2 \right) \leq \]
\[ 2^j \left( 2^{-2j^2} a_j^2 + 2^{-2(j-1)^2} a_{j-1}^2 \right) \]
for some numbers \( a_j \) such that \( \sum_{j=0}^{\infty} a_j^2 < \infty \). Finally,
\[ ||f_k||_{L^2(d\sigma)} \leq C(||f||_{L^2(d\omega)} + \sum_{j=0}^{k} 2^{-j(\alpha - \frac{1}{2})} a_j) \leq C(||f||_{L^2(d\omega)} + \left( \sum_{j=0}^{k} 2^{-2j(\alpha - \frac{1}{2})} \right) \frac{k}{\frac{1}{2}} (\sum_{j=0}^{k} a_j^2)^{\frac{1}{2}} \]
from which it follows, if \( \alpha > \frac{1}{2} \), that \( ||f||_{L^2(d\sigma)} \leq C||f||_{F^2_\alpha} \).

**Remark 2** We could have gotten this result cheaply by combining both the classical restriction theorem and the extension theorem (theorem 8 in chapter 4), in the following way extend the function \( f \in F^2_\alpha \) to a function \( E(f) \in S^{2^2 + \frac{1}{2}} \), this function belongs in particular to \( L^2(\frac{\alpha}{2} + \frac{1}{2}) \) by proposition 5 above and then restrict to get that the restriction is possible if \( \alpha > 1/2 \). Also the proposition doesn't say anything about the range \( 0 < \alpha \leq \frac{1}{2} \). The case \( \alpha = 1/2 \) is actually theorem 5, a result that at present seems to be unprovable by direct methods as the case of \( \alpha > \frac{1}{2} \). As for the range \( 0 < \alpha < \frac{1}{2} \) the proposition is false.

Finally we define the spaces \( B^g_\alpha \), that would replace the spaces \( F^2_\alpha \) for large \( \alpha \). We follow the definition of Jonsson and Wallin.

**Definition 8** Let \( k, \alpha, N \) be such that \( 0 \leq k < \alpha \leq k + 1 \) and \( N > \alpha \). A system of functions \( f := \{ f_j : |J| \leq k \} \) is said to belong to the space \( B^g_\alpha \), if for every net \( \mathcal{F}_m \) of mesh \( 2^{-m} \) and every \( Q \in \mathcal{F}_m \) there is a polynomial \( P_Q \) in the ambient space such that the following hold:
1. For all multiindices \( J \) with \( |J| \leq k \), we have
\[ \sum_{Q \in \mathcal{F}_m} \int_Q |f_j - X_j P_Q|^p d\mu \leq (c_m 2^{-m(\alpha - |J|)})^p \]
2. For $k + 1 \leq |J| \leq N$, we have

$$
\sum_{Q \in \mathcal{F}_m} \sum_{Q' \in \mathcal{F}_m(Q)} \int_{Q} |X_{J}(P_{Q} - P_{Q'})|^p d\mu \leq \left(c_{m}2^{-m(\alpha - |J|)}\right)^p
$$

3. For unit size $Q$'s we have

$$
\sum_{Q \in \mathcal{F}_m} \int_{Q} |X_{J}PQ|^p d\mu \leq C
$$

It is clear that the requirements of definition 8 are very strong and difficult to fulfill. Intuitively, the system of functions $\{f_{J}\}$ represents the whole history of $f$, i.e.; all the derivatives, in fact when considered in the ambient space the system. This definition was used in the restriction to very general sets (the so called $d$-sets). In this case it is clear also why one wants to consider systems of functions instead of single one.

### 2.4 Two classical inequalities

Before we close this chapter we record two main inequalities used extensively in this thesis and they are Young’s inequality and Hardy’s inequality. The latter inequality had a great impact on the development of the theory of function spaces, in particular the Besov and Sobolev spaces. We also use the standard ones such as Holder’s and Minkowski’s inequalities.

**Lemma 7** Let $(X, \mu)$ and $(Y, \nu)$ be two measure spaces, and let $k(x, y)$ be a $\mu \times \nu$-measurable function on $X \times Y$ such that

(i) \[ \sup_{x \in X} \int_{Y} |k(x, y)| d\nu(y) \leq C_1 \]

and

(ii) \[ \sup_{y \in Y} \int_{X} |k(x, y)| d\mu(x) \leq C_2 \]

then, for all $f \in L^2(Y, \nu)$ we have

$$
(\int_{X} |\int_{Y} k(x, y)f(y) d\nu(y)|^2 d\mu(x))^{1/2} \leq C_1^{1/2}C_2^{1/2}(\int_{Y} |f(y)|^2 d\nu(y))^{1/2}
$$

**Lemma 8** Let $a_k \geq 0$. Then, for $0 < p < \infty$

(i) \[ \sum_{k=0}^{\infty} 2^{sk} \left(\sum_{i=0}^{k} a_i\right)^p \leq C \sum_{k=0}^{\infty} 2^{sk} a_k^p \quad \text{if} \quad s < 0 \]

(ii) \[ \sum_{k=0}^{\infty} 2^{sk} \left(\sum_{i=k}^{\infty} a_i\right)^p \leq C \sum_{k=0}^{\infty} 2^{sk} a_k^p \quad \text{if} \quad s > 0 \]
Chapter 3

The Restriction Theorem

3.1 The $L^2$ estimates

At this preliminary stage of the restriction theorem we resolve first question 2 raised in the introduction i.e.; when is the restriction operator bounded from $S^2_\alpha$ to $L^2$? First of all, there are two $L^2$-spaces that we are concerned with. The first is the ordinary $L^2(d\sigma)$, $d\sigma$ is the ordinary surface measure, and the second is $L^2(d\mu)$, where we have set $d\mu = w d\sigma$. The following two theorems are the main results of this section.

**Theorem 4** The restriction operator is bounded from $S^2_\alpha$ to $L^2(d\mu)$ if $\alpha > 1/2$.

The next theorem tells us when is the restriction operator bounded from $S^2_\alpha$ to $L^2(d\sigma)$.

Recall that, by proposition (5) chapter 2, we have

$$S^2_\alpha \subset L^2_{3/2}$$

where the spaces on the right hand side of (3.1) are the standard Sobolev-potential spaces, furthermore one cannot do better than this. On the one hand, by the classical restriction theorem we have that $L^2_{3/2}$ and therefore $S^2_\alpha$ restrict to $L^2_{(2\alpha-1)/\alpha}$ if $\alpha > 1$, on the other hand theorem 4 tells us that restriction cannot possibly be bounded from $S^2_\alpha$ to $L^2(d\sigma)$ if $\alpha \leq 1/2$. So the problem is really when $\frac{1}{2} < \alpha \leq 1$. Theorem 5 below that says that there is boundedness for $\alpha = 1$. We will give at the end of this section an example of a function belonging to $S^2_{1-\varepsilon}$ for all $\varepsilon > 0$ but fails to have restriction to the characteristic hyperplane. So theorem 5 is a borderline result, making it a sharp and interesting result.

**Theorem 5** The restriction operator is a bounded linear operator from $S^2_1$ to $L^2(d\sigma)$.

3.1.1 Proof of theorem 4:

We note first that it is part of the theorem that the restriction is well defined almost everywhere. Since both theorems above are local, let us fix once and for all a bounded open set
in the ambient space and let us denote by $M$ the intersection of some smooth hypersurface with the open set $U$, we will ignore the edges by assuming that all our functions have compact support inside $U$. We may, also without loss, assume that $M$ is given by the graph of some smooth function, say to fix the notation

$$M = \{(t, z) \in \mathbb{R}^{2n+1} = R \times R^{2n} : t = \psi(z)\} \cap U$$

(3.2)

In order to prove the theorem we have to prove the inequality

$$||f||_{L^2(d\sigma)} \leq C||g||_{L^2}$$

(3.3)

where $f = g * J_\alpha$, and the constant $C$ may depend on the support. Theorem 4 says not only the restriction of $f$ to $M$ is in $L^2(d\mu)$ but also the restriction of all derivatives up to order $k$, where $k < \alpha \leq k + 1$, are in $L^2$. Now let $f \in S^\infty$ and let $J$ be a multiindex. Write $X_J f$ as

$$X_J f(x) = f_1(x) + f_2(x)$$

where

$$f_1(x) = \int_{|x - y| \geq 1} (X_J J_\alpha)(y^{-1}x)g(y)dy$$

and

$$f_2(x) = \int_{|x - y| < 1} (X_J J_\alpha)(y^{-1}x)g(y)dy$$

Let us assume first that $\alpha \neq Q + |J|$. By Schwarz inequality

$$|f_1(x)| \leq \left( \int_{|x - y| \geq 1} |g(y)|^2 dy \right)^{1/2} \left( \int_{|x - y| \geq 1} (X_J J_\alpha)(y^{-1}x)^2 dy \right)^{1/2}$$

which is less than or equal to $C||g||_{L^2}$ because of the rapid decrease of $J_\alpha$ away from the origin. And thus we have

$$||f_1||_{L^2(d\sigma)} \leq C||g||_{L^2} = C||f||_{S^\infty}$$

(3.4)

We turn to $f_2$ and write it

$$f_2(x) = \sum_{m=0}^{\infty} \int_{|y^{-1}x| \sim 2^{-m}} (X_J J_\alpha)(y^{-1}x)g(y)dy$$

by Minkowski’s inequality we get

$$||f_2||_{L^2(wd\sigma)} \leq \sum_{m=0}^{\infty} \left( \int_M \left( \int_{|y^{-1}x| \sim 2^{-m}} (X_J J_\alpha)(y^{-1}x)g(y)dy \right)^2 w(x)d\sigma(x) \right)^{1/2}$$

(3.5)

Let $\chi_m(x, y)$ be the characteristic function of the set

$$A_m(x, y) = \{(x, y) \in M \times R^{2n+1}/2^{-m-1} : |y^{-1}x| < 2^{-m}\}$$
and set \( \tilde{J}_\alpha(x,y) = \chi_m(x,y)(X_JJ_\alpha)(y^{-1}x) \) in order to apply Young's inequality to each term in the sum of (3.5) we need to check that

\[
\sup_{y \in M} \int_U |\tilde{J}_\alpha(x,y)| \, dx \leq C_m
\]  

and

\[
\sup_{x \in U} \int_M |\tilde{J}_\alpha(x,y)| \, d\mu(y) \leq C'_m
\]  

For any fixed \( x \in M \) we have

\[
\int_U |\tilde{J}_\alpha(y^{-1}x)| \, dy \leq C \int_{|y^{-1}x|^{-2-m}} |y^{-1}x|^{\alpha - |J| - Q} \, dy \leq C 2^{-m(\alpha - |J|)}
\]  

(3.8)

similarly, we have for a fixed \( y \in U \),

\[
\int_{|y^{-1}x|^{-2-m}} |y^{-1}x|^{\alpha - |J| - Q} w(x) \, d\sigma(x) \leq C 2^{-m(\alpha - |J| - 1)}
\]  

(3.9)

from which it follows that

\[
(\int_M (X_JJ_\alpha(y^{-1}x)g(y) \, dy)^2 w(x) \, d\sigma(x))^{1/2} \leq C 2^{-m(\alpha - |J| - \frac{1}{2})} ||g||_{L^2}
\]

finally we obtain

\[
||f_2||_{L^2(d\mu)} \leq C ||g||_{L^2} \sum_{m=0}^{\infty} 2^{-m(\alpha - |J| - \frac{1}{2})} \leq C ||g||_{L^2}
\]

as long as \( \alpha - |J| > 1/2 \)

Suppose now that \( \alpha = |J| + Q \), then the only difference from the previous case is now the estimate we use for \( X_JJ_\alpha \) is

\[
|X_JJ_\alpha(y)| \leq C \log\left(\frac{1}{|y|}\right) \text{ as } |y| \to 0
\]

and the constants in (3.8) and (3.9) are respectively \( Cm2^{-mQ} \) and \( Cm2^{-m(Q-1)} \).

\[\square\]

3.1.2 Proof of theorem 5:

Now we turn to prove theorem 5. We will prove the theorem first in the special case of the characteristic hyperplane \( \{(t,z): t = 0\} \). The following lemma, from the general theory of Hilbert spaces, is key in proving the estimates needed.

**Lemma 9** Let \( H_1 \) and \( H_2 \) be two Hilbert spaces; \( T \) be a linear operator from \( H_1 \) to \( H_2 \), and define the adjoint operator \( T^* \) from \( H_2 \) to \( H_1 \) by

\[
(T^*u, w)_{H_1} := (u, Tw)_{H_2}
\]

If \( T \) is bounded then \( TT^* \) (resp. \( T^*T \)) is a bounded operator from \( H_2 \) to \( H_2 \) (resp. from \( H_1 \) to \( H_1 \) ). And conversely, if either \( TT^* \) or \( T^*T \) is bounded operator, then \( T \) is also bounded.
Proof:

It is clear that \( T \) is bounded if and only if \( T^* \) and the boundedness of \( TT^* \) and \( T^*T \) follows. Conversely, if \( TT^* \) is bounded then by Cauchy Schwarz

\[
||T^*u||^2_{H_1} = (T^*u, T^*u)_{H_1} = (u, TT^*u) \leq ||u||_{H_2}||TT^*u||_{H_1} \leq C||u||^2_{H_1}
\]

\( \square \)

1. The case of the characteristic hyperplane

If \( f \in S^1 \)

\[
f(t, z) = \int_{H_n} I((s, u)^{-1}(t, z))g(s, u)dsdu \tag{3.10}
\]

where \( I \) is the Riesz kernel i.e.; the kernel function of the operator \((-L)^{-\frac{1}{2}} := (\sum_{j=1}^{2n} X_j^2)^{-\frac{1}{2}}\), and \( g \in L^2 \). Recall that we have

\[
|I(t; z)| \leq C|(t; z)|^{-Q+1} \tag{3.11}
\]

So proving that \( f(0, z) \in L^2(dz) \) ammounts to proving that the operator \( R(-L)^{-\frac{1}{2}} \) (\( R \) is the restriction operator to \( M \) i.e.; \( Rf := f|_M \) ) is bounded from \( L^2(dt dz) \) to \( L^2(dz) \), which according to lemma 9 above is equivalent to showing that the operator

\[
T = (R(-L)^{-\frac{1}{2}})(R(-L)^{-\frac{1}{2}})^* \tag{3.12}
\]

is bounded from \( L^2(dz) \) to \( L^2(dz) \). The kernel function of \( T \) is given by

\[
K(z, z') = \int_{H_n} I((s, u)^{-1}(0, z))I((s, u)^{-1}(0, z'))dsdu \tag{3.13}
\]

it is easy to check that the kernel function \( K(z, z') \) is symmetric and homogeneous of degree \(-2\) that is

\[
K(rz, rz') = r^{-2}K(z, z') \tag{3.14}
\]

Let us assume for simplicity that \( n = 1 \) and change variables to polar coordinates.

\[
z = re^{i\theta}, z' = r'e^{i\theta'} \tag{3.15}
\]

the action of \( T \) is seen as a group convolution on \([0, \infty[ \times S^1 \) with its obvious group structure

\[
Tf(re^{i\theta}) = \int_0^{2\pi} \int_0^{\infty} F(r', \theta - \theta')f(r'e^{i\theta'})\frac{dr'}{r'}d\theta' \tag{3.16}
\]

where

\[
F(r, \theta) = K(1, re^{i\theta})
\]

the properties of \( F(r, \theta) \) can be read off directly from the estimates on the kernel \( K \) which is lemma 11 below.
Lemma 10

(a) \( F(r, \theta) = O((|r - 1| + \theta^{\frac{1}{2}})^{-2}) \) as \((r, \theta) \to (1, 0)\)

(b) \( F(r, \theta) = O(1) \) as \( r \to 0 \)

(c) \( F(r, \theta) = O(r^{-2}) \) as \( r \to \infty \)

Now in order to prove that \( T \) maps \( L^2(rdrd\theta, (0, \infty) \times S^1) \) to itself we need to show:

\[
\int_0^{2\pi} \int_0^\infty \left( \int_0^{2\pi} \int_0^\infty F\left( \frac{r}{r'}, \theta - \theta' \right) f(r'e^{i\theta'}) \left( \frac{dr'}{r'} \right) d\theta' \right)^2 rdrd\theta 
\]

\[
\leq C \int_0^{2\pi} \int_0^\infty |f(re^{i\theta})|^2 rdrd\theta
\]

and this is the same as

\[
\int_0^{2\pi} \int_0^\infty \left( \int_0^{2\pi} \int_0^\infty r^{\frac{1}{2}}r'^{\frac{3}{2}} F\left( \frac{r}{r'}, \theta - \theta' \right) (r^{\frac{1}{2}} f(r'e^{i\theta'})) dr'd\theta' \right)^2 drd\theta
\]

\[
\leq C \int_0^{2\pi} \int_0^\infty |r^{\frac{1}{2}} f(re^{i\theta})|^2 drd\theta
\]

For Young's inequality to apply, we need to show two things

\[
\sup_{r, \theta} \int_0^{2\pi} \int_0^\infty r^{\frac{1}{2}}r'^{\frac{3}{2}} F\left( \frac{r}{r'}, \theta - \theta' \right) dr'd\theta' \leq C \tag{3.16}
\]

\[
\sup_{r', \theta'} \int_0^{2\pi} \int_0^\infty r^{\frac{1}{2}}r'^{\frac{3}{2}} F\left( \frac{r}{r'}, \theta - \theta' \right) drd\theta \leq C \tag{3.17}
\]

but via a change of variables the two integrals are amenable to each other, and this is because the kernel \( K \) is symmetric, so we need only consider one of them, say (3.16). Fix \( r \) and \( \theta \) and break the domain of integration in ( ) into three parts

\[
\int_0^{2\pi} \int_0^{r/2} + \int_0^{2\pi} \int_{r/2}^{2r} + \int_0^{2\pi} \int_{2r}^\infty = A_1 + A_2 + A_3
\]

using lemma 10 we have

\[
A_1 \leq C \int_0^{r/2} r^{\frac{1}{2}}r'^{\frac{3}{2}} (\frac{r}{r'})^{-2} dr' \leq C
\]

while

\[
A_3 \leq C \int_0^{2\pi} r^{\frac{1}{2}}r'^{\frac{3}{2}} dr' \leq C
\]

\[
A_2 \leq C \int_0^{2\pi} \int_{r/2}^{2r} r^{\frac{1}{2}}r'^{\frac{3}{2}} (|\frac{r}{r'} - 1| + |\theta - \theta'|^{\frac{1}{2}})^{-2} dr'd\theta' =
\]
2. The general case.
Let us settle some notation to avoid cumbersome expressions. The notation \( z \in M \) has the obvious interpretation to mean \((\psi(z), z) \in M\), also the notation \( d(z, z') \) should be taken to mean the nonisotropic distance between the points \((\psi(z), z) \) and \((\psi(z'), z') \) , similarly if \( x \in U \) and \( z \in M \) the notation \( d(x, z) \) should be clear. Finally, denote by \( d(z) \) the distance to the origin.

We describe now the general case. Let \( K(z, z') \) be the kernel function of \( T \) given by (3.12), it is a symmetric kernel and defined in the same way as in (3.13) by

\[
K(z, z') = \int_U I((s, u)^{-1}(\psi(z), z))I((s, u)^{-1}(\psi(z'), z'))dsdu
\]

Let \( q(z) \geq 0 \) be a nonnegative function defined on \( M \) and define the measure \( d\nu \) by the equation

\[
q(z)d\nu(z) = d\sigma(z)
\]

so that

\[
Tf(z) \in L^2(d\sigma) \text{ if and only if } q(z)^{1/2}Tf(z) \in L^2(d\nu(z))
\]

and so the inequality to prove

\[
\int_M |\int_M K(z, z')f(z')d\sigma(z')|^2d\sigma(z) \leq C \int_M |f(z)|^2d\sigma(z)
\]

is equivalent to

\[
\int_M |\int_M q(z')^{1/2}K(z, z')q(z)^{1/2}q(z')^{1/2}f(z')d\nu(z')|^2d\nu(z) \leq C \int_M |q(z')^{1/2}f(z')|^2d\nu(z)
\]

Set

\[
\tilde{K}(z, z') = q(z)^{1/2}K(z, z')q(z')^{1/2}
\]

By Young's inequality we need to show

\[
\sup_{z \in M} \int_M \tilde{K}(z, z')d\nu(z') \leq C
\]

This is exactly what we did in the special case of the hyperplane, there the function \( q(z) \) is equal to \( r^{1/2} \). The function \( r \) in the previous case is equal to the weight function \( w \). Here too we choose the function \( q \) to be equal some power of the weight function \( w \), \( q(z) := w(z) \)^\( \varepsilon \), for any \( 0 < \varepsilon < 1 \).

The following lemma gives the estimates for \( K \), which is lemma (2) in the case of the hyperplane.
Lemma 11  The kernel $K(z; z')$ satisfies

$$K(z; z') \leq C d(z, z')^{-Q+2}$$

as $d(z; z')$ tends to zero.

**Proof:** $K(z, z')$ is essentially

$$K(z; z') \approx \int_U d(u; z)^{-Q+1} d(u; z')^{-Q+1} du$$

By the triangle inequality, it is easily seen that there is a constant $C > 1$ such that

$$d(u; z') \leq \frac{d(z; z')}{C}$$

implies

$$d(u; z) \geq \frac{d(z; z')}{C}$$

so we can break the integral defining $K$ into two parts, one is

$$\int_{d(u, z') \leq \frac{d(z; z')}{C}} d(u; z)^{-Q+1} d(u; z')^{-Q+1} du \leq C d(z; z')^{-Q+2}$$

the other part is

$$\sum_{k=0}^{\infty} \int_{d(u, z') \sim 2^k \left( \frac{d(z; z')}{C} \right)} d(u; z)^{-Q+1} d(u; z')^{-Q+1} du$$

but if $d(u; z') \sim 2^k d(z; z')$, then also we have $d(u; z) \sim 2^k d(z; z')$, so the sum is less than or equal to $d(z; z')^{-Q+2}$.

(3.20) follows if we prove

$$w(z') \eta \int_M d(z, z')^{-Q+2} w(z)^{-\eta} d\sigma(z) \leq C < \infty$$

(3.21)

with constant independent of $z'$. Fix $z' \in M$ which we can assume to be the origin 0 and assume that $w(0) = \eta > 0$. Split the the integration in (3.21) into two parts, and we prove

$$I_1 = \eta \int_{d(z) < \eta/10} d(z)^{-Q+2} w(z)^{-\eta} d\sigma(z) \leq C < \infty$$

(3.22)

and

$$I_2 = \eta \int_{d(z) \geq \eta/10} d(z)^{-Q+2} w(z)^{-\eta} d\sigma(z) \leq C < \infty$$

(3.23)

$$I_2 = \eta \sum_{k=1}^{\infty} \int_{d(z) \sim 2^k \left( \eta/10 \right)} d(z)^{-Q+2} w(z)^{-\eta} d\sigma(z)$$
\[
\leq C\eta^\varepsilon \sum_{k=1}^{\infty} (2^k \eta/10)^{-Q+2} \int_{d(z) \leq 2^k \eta/10} w(z)^{-\varepsilon} d\sigma(z)
\]
by lemma 5 and 3(b) of chapter 2.2 we have
\[
\int_{d(z) \leq 2^k \eta} w(z)^{-\varepsilon} \leq (2^k \eta)^{Q-1} (\max_{z \in B(2^k \eta)} w(z))^{-1-\varepsilon} \leq (2^k \eta)^{Q-2-\varepsilon}
\]
and thus,
\[
I_2 \leq C\eta^\varepsilon \sum_{k=1}^{\infty} (2^k \eta/10)^{-Q+2} \left( \max_{d(z) \leq 2^k \eta/10} w(z)^{-\varepsilon} \sigma(B(0, 2^k \eta/10) \cap M) \right)
\]
\[
\leq C\eta^\varepsilon \sum_{k=1}^{\infty} (2^k \eta)^{-Q+2} (2^k \eta)^{Q-1} (2^k \eta)^{-1-\varepsilon} = C \sum_{k=1}^{\infty} 2^{-k\varepsilon} < \infty
\]
and (3.23) follows.

Now we turn to \(I_1\), by lemma 2(c) chapter 2, since \(\eta > 0\), \(w(z) \geq \eta/10\) for all \(z\) such that \(d(z) \leq \eta/10\) and so
\[
I_1 \leq C \int_{d(z) < \eta/10} d(z)^{-Q+2} d\sigma(z) \tag{3.24}
\]
Change variables to
\[
z' = z/\eta \quad s = t/\eta^2
\]
In this coordinate system the surface \(M\) transforms to
\[
M' = \{(s, z') : s = \psi'(z') = \frac{1}{\eta^2} \psi(\eta z')\}
\]
Now the bounds on \(\psi'(z')\) are: if we take the Taylor expansion of \(\psi\) we get that
\[
\psi(z) = \nabla \psi(0) \cdot z + q(z)
\]
\[
\eta = |\nabla \psi(0)|
\]
with this we have
\[
\psi'(z') = \frac{1}{\eta^2} (\nabla \psi(0) \cdot (\eta z')) + \frac{1}{\eta^2} q(\eta z')
\]
the function \(q'(z') := \frac{1}{\eta^2} q(\eta z')\) is bounded with all its derivatives, for since \(q(z) = O(|z|^2)\) it follows that \(q'(z')\) is bounded and if take derivatives things get better, for example for the first derivatives we have
\[
\nabla q'(z') = \frac{1}{\eta^2} \times \eta^2 (\nabla q)(\eta z') = \nabla q(z) = O(|z|)
\]
34
and so on. The new corresponding weight function \( w' \) is comparable to 1 because \( w'(0) = \frac{1}{r} |\nabla \psi(0)| = 1 \). That is, the new surface is noncharacteristic which implies in particular that \( \sigma(B(0, r) \cap M') \approx C r^{Q-1} \). The distance \( d(z) \) transforms to

\[
d(z) = \eta d'(z')
\]

the surface measure expressed in the new coordinates is just

\[
d\sigma_{M'}(z') \approx \eta^{-Q+2} dz'
\]

and therefore

\[
I_1 \leq C \int_{d(z')<1/10} \eta^{-Q+2} d(z')^{-Q+2} \eta^{Q-2} d\sigma(z') \leq C \int_{d(z')<1/10} d(z')^{-Q+2} d\sigma_{M'}
\]

\[
\leq C \sum_{k=0}^{\infty} \int_{d(z')<2^{-k}} d(z')^{-Q+2} dz'
\]

\[
\leq C \sum_{k=0}^{\infty} 2^{(Q-2)k} \sigma(B(0, 2^{-k}) \cap M') \leq C \sum_{k=0}^{\infty} 2^{-k} < \infty
\]

\( \square \)

### 3.1.3 \( S^2_\alpha \) doesn’t restrict to \( L^2(d\sigma) \) for \( \alpha < 1 \)

In this section we provide with an example of a function belonging to \( S^2_{1-\varepsilon} \) for all \( \varepsilon > 0 \) but fails to have restriction in \( L^2(d\sigma) \). For simplicity we place ourselves in the three dimensional space \( \mathbb{R}^3 \). Set \( \rho = (t^2 + (x^2 + y^2))^{1/4} \) and let

\[
f(t, x, y) = f(\rho) = \rho^{-1} \phi(\rho)
\]

where \( \phi \) is a smooth function identically equal to 1 for \( \rho \leq 1 \) and vanishes for \( \rho \geq 2 \). Clearly, \( f|_{t=0} = (x^2 + y^2)^{-1/2} \) near 0 fails to belong to \( L^2(dx dy) \). That \( f \) belongs to \( S^2_{\alpha} \) follows if we show that

\[
f \ast I_{1-\alpha} \in S^2_1
\]

for all \( \alpha < 1 \). Because \( I_{1-\alpha} \) is a homogeneous function of degree \( 1 - \alpha - Q = -3 - \alpha \), and \( f \) is homogeneous of degree \(-1\) near the origin, \( f \ast I_{1-\alpha} \) is homogeneous of degree \(-\alpha\) (this follows in general from the fact that if two functions are homogeneous of degree \( s \) and \( l \) their convolution, if it makes sense, is homogeneous of degree \( s + l + Q \).) Now \( X(f \ast I_{1-\alpha}) \) is homogeneous of degree \(-1 - \alpha\), which is in \( L^2(d\sigma) \) if and only if \( \alpha < 1 \).
3.1.4 Some consequences of theorem 4& 5

Before we move on to the main estimates for the restriction we discuss some consequences of Theorem 4& 5. If \( B \) is any unit size ball, then according to Theorem 5 we have the inequality

\[
\int_{B \cap M} |f(t, z)|^2 d\sigma(t, z) \leq C \int_B (|f(t, z)|^2 + |\nabla f(t, z)|^2) dt dz
\]

(3.25)

we would like to have this inequality at all scales. Let \( r > 0 \), and \( B(r) \) a ball centered at the point \((\psi(z_0), z_0) \in M\). By left translation we can assume that \( z_0 = 0 \) and \( \psi(z_0) = 0 \) i.e.; \( 0 \in M \). Next we assume that the weight function \( w \) assumes its maximum on the ball \( B \) at the point \((0, 0)\), for otherwise we look for one point where the maximum is achieved and take a ball centered there of a larger radius so as to contain the original ball \( B \), and then left translate to the origin. Now the situation we are in is a ball \( B \) of radius \( r > 0 \), centered at the origin and we would like to rescale (3.25) above. Change variables to

\[
t' = t/r^2 \\
z' = z/r
\]

the ball \( B \) gets transformed into the unit ball \( B' \) and the surface into

\[M' = \{(t', z') : t' = \psi'(z') := r^{-2}\phi(rz')\} \]

if we denote by \( d\sigma_M(t, z) \) and \( d\sigma_{M'}(t', z') \) the surface measures on \( M \) and \( M' \) respectively, and if we set \( f'(t', z') := f(r^2 t', rz') = f(t, z) \), we get upon applying (3.25) above

\[
\int_{M' \cap B'} |f'(t', z')|^2 d\sigma_{M'}(t', z') \leq C \int_{B'} (|f(t', z')|^2 + |\nabla f'(t', z')|^2) dt' dz' \]

(3.26)

The right hand side of (3.26) is equal to

\[
Cr^{-Q} \int_{B'} |f(t, z)|^2 dt dz + Cr^{-Q+2} \int_{B'} |\nabla f(t, z)|^2 dt dz
\]

(3.27)

the surface measure \( d\sigma_{M'}(t', z') \) on \( M' \) is, in terms of the graph coordinates, equal to

\[
d\sigma_{M'}(t', z') = (1 + |\nabla \psi'(z')|^2)^{1/2} dz' = (1 + r^{-2}|\nabla \psi(rz')|^2)^{1/2} dz' =
\]

\[
r^{-Q+1}(r^2 + |\nabla \psi(rz')|^2)^{1/2} d\sigma_M(r^2 t', rz')
\]

By Taylor expansion

\[\psi(z) = \nabla \psi(0) \cdot z + O(|z|^2)\]

and so

\[\nabla \psi(z) = \nabla \psi(0) + O(|z|)\]

and thus,

\[|\nabla \psi(z)| \geq C|w_B - Cr|\]
now because $w_B$ is at least $Cr$ according to lemma 2(c), we get

$$\left(\frac{r^2 + |\nabla \psi(rz')|^2}{1 + |\nabla \psi(rz')|^2}\right)^{1/2} \geq C w_B$$

and hence,

$$d\sigma_M(t', z') \geq C w_B r^{-Q+1} d\sigma_M(r^2 t', r z') \quad (3.28)$$

Now using this we have

$$\int_{B \cap M} |f(t, z)|^2 d\sigma_M(t, z) = \int_{B \cap M'} |f(t', z')|^2 d\sigma_M(r^2 t', r z') \leq \frac{r^{Q-1}}{w_B} \int_{B \cap M'} |f(t', z')|^2 d\sigma_M(t', z') \leq$$

$$C \frac{r^{Q-1}}{w_B} \int_{B^*} |f(t', z')|^2 dt' dz' + C \frac{r^{Q+1}}{w_B} \int_{B^*} |\nabla f(t', z')|^2 dt' dz' \leq$$

$$C (r^{-1} w_B^{-1}) \int_{B^*} |f(t, z)|^2 dt dz + rw_B^{-1} \int_{B^*} |\nabla f(t, z)|^2 dt dz$$

Therefore,

$$\int_{B \cap M} |f(t, z)|^2 d\mu \leq C (r^{-1} \int_{B^*} |f(t, z)|^2 dt dz + r \int_{B^*} |\nabla f(t, z)|^2 dt dz) \quad (3.29)$$

Next we recall Poincare inequality due to Jerison [JE1] in this setting. Let $B(r)$ be a ball of radius $r > 0$, then we have

$$\inf_c \int_B |f - c|^2 d\mu \leq Cr^2 \int_{B^*} |\nabla f|^2 dt dz$$

We can obtain a similar inequality on the surface as follows. Apply inequality (3.29) to the function $(f - c)$ and then inequality (3.30) to have at once

$$\inf_c \int_{B \cap M} |f - c|^2 d\mu \leq Cr \int_{B^*} |\nabla f|^2 dt dz$$

We summarize all of the above in the following lemma.

**Lemma 12** Let $B$ be any ball of radius $r > 0$, and $M$ a smooth hypersurface. Then,

(a) $\int_{B \cap M} |f(t, z)|^2 d\sigma(t, z) \leq C (r^{-1} w_B^{-1}) \int_{B^*} |f(t, z)|^2 dt dz + rw_B^{-1} \int_{B^*} |\nabla f(t, z)|^2 dt dz$

(b) $\int_{B \cap M} |f(t, z)|^2 d\mu \leq C (r^{-1} \int_{B^*} |f(t, z)|^2 dt dz + r \int_{B^*} |\nabla f(t, z)|^2 dt dz)$

(c) $\inf_c \int_{B \cap M} |f - c|^2 d\mu \leq Cr \int_{B^*} |\nabla f|^2 dt dz$
3.2 The main estimates

The following theorem is the main result of this section.

**Theorem 6** Let \( f \in S^2_0 \) and let \( k \geq 0 \) be the integer such that \( k < \alpha - \frac{1}{2} \leq k + 1 \). Let \( M \) be a smooth hypersurface, then the system of functions \( \{ f_j := X_j f|_M, |J| \leq k \} \) is an element of the space \( B^2_{\alpha - \frac{1}{2}}(M) \). In particular if \( k = 0 \) i.e.; when \( \frac{1}{2} < \alpha < \frac{3}{2} \), then the restriction operator is bounded from \( S^2_0 \) to \( F^2_{\alpha - \frac{1}{2}} \).

3.2.1 The case \( 0 < \beta = \alpha - \frac{1}{2} < 1 \)

According to the definitions, the spaces \( B^2_{\beta}(M) \) and \( F^2_{\beta}(M) \) coincide when \( 0 < \beta < 1 \). And so we can use the first difference characterization (2.36) to prove the theorem in this case. The following inequality is what we have to prove:

\[
A_\beta(g)^2 := \int \int_{|y^{-1}x| \leq 1} \frac{|f(x) - f(y)|^2}{|y^{-1}x|^{2\beta + Q - 1}} d\mu(x) d\mu(y) \leq C||g||^2
\]

for \( f = g \ast J_\alpha \). We that we note that if \( \beta < 1/2 \), we can give a straightforward proof. Decompose the function \( f \) according to Proposition 7 like;

\[
f = \sum_{j=0}^{\infty} f_j
\]

with the \( f_j \)'s satisfying the required estimates. We prove the inequality,

\[
\sum_{m=0}^{\infty} 2^{2m\beta} \sum_{B \in \mathcal{F}_m} \inf \int_{B \cap M} |f - c|^2 d\mu \leq C||f||^2_{B^2_{\beta}}
\]

By lemma 12(c) we have,

\[
C \sum_{m=0}^{\infty} 2^{2m\beta} \sum_{B \in \mathcal{F}_m} \inf \int_{B \cap M} |f - c|^2 d\mu \leq
\]

\[
I = \sum_{m=0}^{\infty} 2^{2m(\beta - \frac{1}{2})} \sum_{B \in \mathcal{F}_m} \left( \sum_{j=0}^{m} |\hat{\nabla} f_j|^2 d\text{vol} \right) \]

\[
\leq \sum_{m=0}^{\infty} 2^{2m(\alpha - 1)} \left( \sum_{j=0}^{m} \left( \int_{B^*} |\hat{\nabla} f_j|^2 d\text{vol} \right)^{1/2} \right)^2 \leq
\]

\[
C \sum_{m=0}^{\infty} 2^{2m(\alpha - 1)} \left( \sum_{j=0}^{m} 2^{-j(\alpha - 1)} a_j \right)^2 \leq C \sum_{m=0}^{\infty} a_m^2
\]
since $\alpha < 1$. The estimate for $II$ is also similar, we use lemma 12(b) instead applied to each individual $f_j$ we get,

$$II \leq C \sum_{m=0}^{\infty} 2^{2m\beta} \left( \sum_{j=m+1}^{\infty} 2^{-j\beta} a_j \right)^2 \leq C \sum_{m=0}^{\infty} a_m^2$$

The following proof is due to Jonsson and Wallin [JW], it is based on real interpolation (vector valued version), and so we state the theorem and refer to [BL]; Theorem 5.6.1; page 122 for a proof and more on interpolation. If $X$ is a banach space we denote by $l^p_\beta(X)$ the space of sequences $(a_j)_{j=0}^{\infty} \subset X$ such that:

$$\|\sum_{j=0}^{\infty} 2^{jp\beta} \|a_j\|_X^{p} \| < \infty$$

**Theorem 7** Let $0 < q_0, q_1 \leq \infty$ and $s \neq s_0$. Then for all $q$ we have

$$(l^{q_0}_0(X), l^{q_1}_1(X))_{\theta,q} = l^\theta_q(X)$$

Let $f = g \ast J_\alpha \in S^2_\alpha$, we would like to show the inequality:

$$A_\alpha(f)^2 \leq C\|f\|_{L^2}^2$$

the left hand side of (3.34) is clearly equivalent to the expression

$$\sum_{m=0}^{\infty} 2^{2m\beta} \int \int_{2^{-m-1} \leq |y^{-1}x| < 2^{-m}} |f(x) - f(y)|^2 \frac{d\mu(x)d\mu(y)}{|y^{-1}x|^{Q-1}}$$

Set $d\mu'(x,y) := \frac{d\mu(x)d\mu(y)}{|y^{-1}x|^{Q-1}}$ and let $X = L^2(M \times M, d\mu'(x,y)$, and define the operator $T := (T_j)_{j=0}^{\infty}$ by

$$T_jf(x, y) = \chi_j(x, y)(f(x) - f(y))$$

where $\chi_j(x, y)$ denotes the characteristic function of the set

$$\{(x, y) \in M \times M : 2^{-j-1} \leq |y^{-1}x| < 2^{-j}\}$$

Then another way of restating what we want to prove is that the operator $T$ is bounded from $S^2_\alpha$ to $l^\theta_q(X)$. For this to be true it suffices, according to the interpolation theorem above, that the operator $T$ be bounded from $S^2_\alpha$ to $l^\theta_{\infty}(X)$ i.e.;

$$2^{2m\beta} \int \int_{2^{-m-1} \leq |y^{-1}x| < 2^{-m}} |f(x) - f(y)|^2 d\mu'(x, y) \leq C$$

the constant $C$ is independent of $m$. To show (3.36) let $\varepsilon > 0$ and set

$$|J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)| = |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^\varepsilon |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{1-\varepsilon}$$
by Schwarz inequality
\[
\int \int_{|y^{-1}x| \geq 2^{-m}} |f(x) - f(y)|^2 d\mu'(x, y) \leq 
\int \int_{|y^{-1}x| \geq 2^{-m}} (\int |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^2 |g(z)|^2 dz)(\int |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{2(1-\varepsilon)} dz) d\mu'(x, y) \leq 
\]
\[
C_1 C_2 \|g\|_{L^2}^2 
\]
where
\[
C_1 = \sup_{|y^{-1}x| \geq 2^{-m}} \int |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{2(1-\varepsilon)} dz 
\]
and
\[
C_2 = \sup_x \int \int_{|y^{-1}x| \geq 2^{-m}} |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{2\varepsilon} d\mu'(x, y) 
\]
\[
C_1 = \sup_{|y^{-1}x| \geq 2^{-m}} \int |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^{2(1-\varepsilon)} dz = I + II 
\]
\[
I \leq C \int_{|z^{-1}x| \leq 2^{-m}} (|z^{-1}x|^\alpha Q + |z^{-1}y|^\alpha Q)^{2(1-\varepsilon)} dz 
\]
but we note that if $|z^{-1}x| \leq 2^{-m}$ and $|y^{-1}x| \approx 2^{-m}$, then also $|z^{-1}y| \leq 2^{-m+10}$, and thus
\[
I \leq C \int_{|z^{-1}x| \leq 2^{-m}} |z^{-1}x|^{2(1-\varepsilon)(\alpha - Q)} \leq C 2^{-2m(1-\varepsilon)(\alpha - Q)} 2^{-mQ} 
\]
as long as
\[
2(1 - \varepsilon)(\alpha - Q) + Q > 0 \tag{3.37} 
\]
by the mean value theorem we have
\[
II \leq C 2^{-2m(1-\varepsilon)} \int_{|z^{-1}x| \geq 2^{-m}} |z^{-1}x|^{2(\alpha - Q - 1)(1-\varepsilon)} dz \leq C 2^{-2m(1-\varepsilon)(\alpha - Q)} 2^{-mQ} 
\]
powered that
\[
2(\alpha - Q - 1)(1 - \varepsilon) + Q < 0 \tag{3.38} 
\]
combining (3.37) and (3.38) we get that $\varepsilon$ has to be chosen such that
\[
1 - \frac{Q}{2(Q + 1 - \alpha)} < \varepsilon < 1 - \frac{Q}{2(Q - \alpha)} \tag{3.39} 
\]
The estimates for $C_2$ are similar to $C_1$. Let $E_1 = \{(x, y) : |y^{-1}x| \approx 2^{-m}, |z^{-1}x| \leq 2^{-m}\}$ and $E_2$ the complement of $E_1$ in $|y^{-1}x| \approx 2^{-m}$, then we have
\[
C_2 \leq I' + II' 
\]
40
\[ I' = \int \int_{E_i} |J_\alpha(z^{-1}x) - J_\alpha(z^{-1}y)|^2 d\mu'(x,y) \leq \]
\[ C 2^{m(Q-1)} \int_{|z^{-1}x| \leq 2^{-m}} |z^{-1}x|^2 |2^{(\alpha-Q)\varepsilon} \mu(B(x,2^{-m}) \cap M) d\mu(x) \leq \]
\[ C \int_0^{2^{-m}} r^{2(\alpha-Q)\varepsilon} d(\mu(B(z,r) \cap M)) \leq C 2^{-2m(\alpha-Q)\varepsilon} 2^{-m(Q-1)} \]
as long as \( 2(\alpha - Q)\varepsilon + Q - 1 > 0 \) i.e.,
\[ \varepsilon < \frac{Q - 1}{2(Q - \alpha)} \] (3.40)

The same estimate is satisfied by \( II \) except the condition on \( \varepsilon \) is now
\[ \varepsilon > \frac{Q - 1}{2(Q + 1 - \alpha)} \] (3.41)

combining (3.40) and (3.41) we get that \( \varepsilon \) must be such that
\[ \frac{Q - 1}{2(Q + 1 - \alpha)} < \varepsilon < \frac{Q - 1}{2(Q - \alpha)} \] (3.42)

Now it suffices to note that
\[ C_1 C_2 = C 2^{-2m\beta} \]
which is what we have set to prove provided that \( \varepsilon \) can be chosen so as to satisfy both requirements (3.39) and (3.42). Solving this system of inequalities we see that it is the same as \( \frac{1}{2} < \alpha < \frac{3}{2} \).
\[ \square \]

**3.2.2 The case \( \beta \geq 1 \)**

Let \( k \) be the integer such that \( 0 \leq k < \beta \leq k + 1 \), then we want to prove that the system of functions \( \{f_J = X_J f|_{\mathcal{M}}, |J| \leq k\} \) belongs to \( B_2^p \). Let us note the functions \( f_J \) are well defined, this is because \( X_J f \in S^2_{\alpha-|J|} \) and \( \alpha - |J| > \frac{1}{2} \). Recall that this means that for each net \( \mathcal{F}_m \) of size \( 2^{-m} \), and each set \( Q \in \mathcal{F}_m \) there is a polynomial \( P_Q(t,z) \) in the ambiant space of degree \( N \geq k + 1 \) such that
1. For all \( |J| \leq k \) we have
\[ \sum_{Q \in \mathcal{F}_m} \int_Q |f_J - X_J P_Q|^2 d\mu \leq (c_m 2^{-m(\beta - |J|)})^2 \]
2. For \( k + 1 \leq |J| \leq N \)
\[ \sum_{Q \in \mathcal{F}_m} \sum_{Q' \in \mathcal{F}_{m+1}(Q)} \int_Q |X_J(P_Q - P_{Q'})|^2 d\mu \leq (c_m 2^{-m(\beta - |J|)})^2 \]
and

3. For $|J| \leq N$ we have

$$\sum_{Q \in F_m} \int_Q |X_j P_Q|^2 d\mu \leq c_0$$

the constants $c_m$ satisfy $\sum_{m=0}^{\infty} c_m^2 < \infty$. We start by the proof of 1. For $Q \in F_m$, we let $P_Q$ be the left Taylor polynomial, of degree $N$, given by the formula (2.29) and (2.30) of chapter 2, of the finite sum

$$F_m = \sum_{j=0}^m f_j$$

taken around any fixed point of $Q$, say its center $x_Q$. Let us write those formulas again in this context

$$P_Q(x) = \sum_{|J| \leq N} C_j(x_Q^{-1} x)^J (X_j F_m)(x_Q)$$

denote by $R_Q(x)$ the remainder $F_m - P_Q$, then

$$R_Q(x) = \frac{1}{N!} \sum_{N+1 \leq |J| \leq 2(N+1)} C_j(x_Q^{-1} x)^J \int_0^1 s^{N-1} (1 - s)^N (X_j F_m)(x_Q(s(x_Q^{-1} x))) ds$$

Estimates in 1. follows if we prove (1').

$$\sum_{Q \in F_m} \int_Q |X_j R_Q(x)|^2 d\mu(x) \leq (c_m 2^{-m(\beta-|J|)})^2$$

and

(1'').

$$\sum_{Q \in F_m} \int_Q |X_j(\sum_{j=m+1}^{\infty} f_j(x))|^2 d\mu(x) \leq (c_m 2^{-m(\beta-|J|)})^2$$

Using lemma 12(b) and Minkowski's inequality we have

$$\sum_{Q \in F_m} \int_Q |X_j(\sum_{j=m+1}^{\infty} f_j(x))|^2 d\mu(x) \leq C 2^m (\sum_{j=m+1}^{\infty} ||X_j f_j||)^2 + C 2^m (\sum_{j=m+1}^{\infty} ||\nabla(X_j f_j)||)^2 \leq$$

$$C 2^m (\sum_{j=m+1}^{\infty} 2^{-j(\alpha-|J|)} a_j)^2 + C 2^m (\sum_{j=m+1}^{\infty} 2^{-j(\alpha-|J|)-1} a_j)^2 \leq$$

$$(c_m 2^{-m(\beta-|J|)})^2$$

by Hardy's inequality we have

$$\sum_{m=0}^{\infty} a_m^2 = C \sum_{m=0}^{\infty} 2^{2m(\beta-|J|+\frac{1}{2})} (\sum_{j=m+1}^{\infty} 2^{-j(\alpha-|J|)} a_j)^2 \leq$$

$$C \sum_{m=0}^{\infty} a_m^2 = C ||f||_{L^2}^2$$

42
and this takes care of (1').

Next, \( X_{J}R_{Q}(x) \) is a sum of terms of the type

\[
A(x) = C \sum_{J'} C_{J'} X_{J_{1}}(x^{-1}x)^{J'} \int_{0}^{1} s^{(|J'| - N - 1)}(1 - s)^{|J'| + |J_{1}|} (X_{J_{2}} X_{J_{1}} F_{m})(x_{Q}(s(x_{Q}x)))ds
\]

with \( J = J_{1} + J_{2} \).

\[
|A(x)| \leq C \sum_{J'} 2^{-m(|J'| - |J_{1}|)} \left( \int_{0}^{1} |X_{J_{2}} X_{J_{1}} F_{m}(x_{Q}(s(x_{Q}x)))|^{2}ds \right)^{1/2}
\]

invoking lemma 12(b), it follows that

\[
\sum_{Q \in F_{m}} \int_{Q} |A(x)|^{2}d\mu(x) \leq
\]

\[
C \sum_{J'} 2^{-2m(|J'| - |J_{1}| - \frac{1}{2})} \sum_{Q \in F_{m}} \int_{Q} |X_{J_{2}} X_{J_{1}} F_{m}|^{2}dvol + \sum_{J'} 2^{-2m(|J'| - |J_{1}| + \frac{1}{2})} \sum_{Q \in F_{m}} \int_{Q} |\nabla X_{J_{2}} X_{J_{1}} F_{m}|^{2}dvol
\]

\[
\leq C \sum_{J'} 2^{-2m(|J'| - |J_{1}| - \frac{1}{2})} (\sum_{j=0}^{m} 2^{-j(\alpha - |J'| - |J_{2}|)}a_{j})^{2} + C \sum_{J'} 2^{-2m(|J'| - |J_{1}| + \frac{1}{2})} (\sum_{j=0}^{m} 2^{-j(\alpha - |J'| - |J_{2}| - 1)}a_{j})^{2} \leq
\]

\[
(c_{m}2^{-m(\beta - |J|)})^{2}
\]

by keeping in mind that \(|J'| \geq N + 1\) we can apply Hardy's inequality to conclude, and this takes care of (1').

3. is straightforward we have,

\[
\sum_{Q \in F_{m}} \int_{Q} |X_{J}P_{Q}|^{2}d\mu(x) \leq
\]

\[
C \sum_{Q \in F_{m}} \int_{Q} |X_{J}(f_{0} - P_{Q})|^{2}d\mu + C \sum_{Q \in F_{m}} \int_{Q} |X_{J}f_{0}|^{2}d\mu \leq
\]

\[
C \sum_{Q \in F_{m}} \int_{Q} |X_{J}(f_{0} - P_{Q})|^{2}dvol + C \sum_{Q \in F_{m}} \int_{Q} |\nabla X_{J}(f_{0} - P_{Q})|^{2}dvol +
\]

\[
C \sum_{Q \in F_{m}} \int_{Q} |X_{J}f_{0}|^{2}dvol \leq C \sum_{Q \in F_{m}} \int_{Q} |\nabla X_{J}f_{0}|^{2}dvol \leq c_{0}
\]

Finally we prove 2. For \( Q \in F_{m} \) and \( Q' \in F_{m+1} \), we write the difference \( P_{Q} - P_{Q'} \) like,

\[
P_{Q}(x) - P_{Q'}(x) = \sum_{j=0}^{m} (r_{j}(x_{Q}, x) - r_{j}(x_{Q'}, x)) + r_{m+1}(x_{Q'}, x) - f_{m+1}(x)
\]

(3.43)
where \( r_j(z, x) := f_j(x) - p_j(z, x) \) is the Taylor remainder of the function \( f_j \) taken around the point \( z \). And so,

\[
\sum_{Q \in \mathcal{F}_m} \int_Q |X_j(P_Q - P_{Q'})|^2 d\mu
\]

\[
\leq \sum_{Q \in \mathcal{F}_m} \int_Q \left( \sum_{j=0}^{m} X_j(r_j(x_Q, x) - r_j(x_{Q'}, x)) \right)^2 d\mu + \sum_{Q \in \mathcal{F}_m} \int_Q |X_j r_{m+1}(x_{Q'}, x)|^2 d\mu + \sum_{Q \in \mathcal{F}_m} \int_Q |X_j f_{m+1}|^2 d\mu
\]

(3.44)

The last two terms in the right hand side of (3.44) are dealt with in the same way as before by invoking lemma 12(b), and they are both less than or equal to

\( (c_{m+1} 2^{-(m+1)(\beta - |J'|)})^2 \)

By Taylor’s formula we can write the difference \( r_j(z, x) - r_j(y, x) \) as;

\[
r_j(z, x) - r_j(y, x) = \sum_{1 \leq |J| \leq 2} C_{J_1} (z^{-1}y)^{|J_1|} \int_0^1 s^{|J_1|-1} (X_j^w(r_j(w, x)))|_{w=(z(s(z-x)))} ds
\]

(3.45)

the notation \( X^w \) means that the action is on the \( w \) variable. Writing out the integral representation of \( r_j(w, x) \) and taking the derivatives \( X^w r_j(w, x) \) we get that \( r_j(z, x) - r_j(y, x) \) is a sum of terms of the type

\[
C(z^{-1}y)^{|J_1|} \int_0^1 \int_0^1 s^{|J_1|-1} t^{|J|-N-1} (1-t)^{|J_2|} |X_j r_j(w(t(w^{-1}x)))|_{w=(z(s(z-x)))} ds dt
\]

(3.46)

with \( J_1 = J_3 + J_2 \). Applying \( X_j r_j \) to this typical term we get that \( X_j r_j(z, x) - r_j(y, x) \) is a sum of terms like;

\[
C(z^{-1}y)^{|J_1|} \int_0^1 \int_0^1 s^{|J_1|-1} t^{|J|-N-1} (1-t)^{|J_2|} |X_j r_j X_j (w(t(w^{-1}x)))|_{w=(z(s(z-x)))} ds dt
\]

(3.47)

with \( J' = J_1' + J_2' \) and the expression in the integrand of (3.47) is supposed to be evaluated at \( w = (z(s(z-x))) \). Taking into account

\[ |z^{-1}y| \leq 2^{-m+2} \]

and that

\[ |w^{-1}x| = |(s(z^{-1}y))y^{-1}x| \leq C(s|z^{-1}y| + |y^{-1}x|) \leq 2^{-m+10} \]

also

\[ |X_{J_1'} X_{J_3}((w^{-1}x)^J)| \leq C 2^{-m(|J| - |J_1'| - |J_3|)} \]

if \( |J| - |J_1'| - |J_3| \geq 0 \) which is the case. The terms of the type (3.47) are in absolute value less than or equal to

\[
C 2^{-m(|J_1| + |J|-|J_1'| - |J_3|)} \left( \int_0^1 \int_0^1 |X_j X_j X_j f_j(w(t(w^{-1}x))))|_{w=(z(s(z-x)))} ds dt \right)^{1/2}
\]

(3.48)
summing over \( j \) and square it integrating it over \( Q \), using lemma 12(b) and the estimates on the \( f_j \)'s over \( Q \), and then summing over \( Q \in \mathcal{F}_m \) using Minkowski's inequality we get finally

\[
\sum_{Q \in \mathcal{F}_m} \int_Q \left( \sum_{j=0}^m X_j \left( r_j(x_Q, x) - r_j(x_{Q'}, x) \right) \right)^2 d\mu \leq 
\]

\[
C^2 2^{-m(|J| + |J_1| - |J'_1| - |J_3| + \frac{1}{2})} \left( \sum_{j=0}^m 2^{-j(|J| - |J'_2| - |J_2|)} a_j \right)^2 + 
\]

\[
C^2 2^{-m(|J| + |J_1| - |J'_1| - |J_3| - \frac{1}{2})} \left( \sum_{j=0}^m 2^{-j(|J| - |J'_2| - |J_2| - 1)} a_j \right)^2 \leq 
\]

\[
(c_m 2^{-m(|\beta - |J'|)})^2
\]

and the proof of the restriction theorem is now complete.

\( \square \)
Chapter 4

The Extension Theorem

This chapter is concerned with the extension of functions, or rather systems of functions, defined on a surface \( M \), to the ambient space. It is the converse to the restriction theorem. It was our goal to prove an extension theorem for the spaces \( F^2_\beta \) instead of \( B^2_\beta \), of course that would have been a stronger result. We still do not know whether it is possible or not. Another simple way to ask the question is: are the spaces \( B_\beta \) and \( F^2_\beta \) equivalent? The answer is yes for \( 0 < \beta < 1 \) and would be yes if we can prove an extension to the ambient space for the space \( F^2_\beta \), since the restriction is bounded from the space \( S^2_{\beta+\frac{1}{2}} \) to \( F^2_\beta \). The extension described below is based on two main ingredients, the first objects are polynomials and the second is the Whitney decomposition of open sets in the nonisotropic metric. This technique of extending functions is classical and goes back to Whitney (1930's) who used it to extend Lipschitz functions defined on an arbitrary closed set of \( \mathbb{R}^n \) see [WH] and [ST1] for more on this. Recall that in the classical case (isotropic) the extension using the Whitney decomposition is as follows. First flatten the hypersurface \( M \) by a local coordinate system \((x_1, \ldots, x_n)\) such that \( M = \{x_n = 0\} \). Now we can talk about polynomials in the coordinates \((x_1, \ldots, x_{n-1})\) and use them to define the spaces \( F^2_\beta \). Polynomials \( p \) in \( \mathbb{R}^{n-1} \) have obvious extension to \( \mathbb{R}^n \), just extend them to be constant along the vertical direction (the \( x_n \)-axis). These extended polynomials satisfy the following trivial but important property: let \( B \) be any Euclidean ball of radius \( r > 0 \), and centered on \( M \) then,

\[
\int_B |p(x_1, \cdots, x_{n-1})|^2 dvol \leq Cr \int_{B \cap M} |p(x_1, \cdots, x_{n-1})|^2 d\sigma
\]  

(4.1)

Once we have this, given \( f \in F^2_\beta \) we define its extension on a Whitney cube to be the best polynomial approximation (in the \( L^2(d\sigma) \)-sense) of \( f \) on a corresponding ball on \( M \). With (4.1) and the so-called Markov's inequality (see [JW]) we can prove that the extension is bounded etc.

Of course in the Euclidean case there are other possible ways of extensions for example the reflection method, see [AD] and [ST1] [ST2] for example, is applicable for smooth hypersurfaces, there is also the method of Calderon that one can use when we have a surface that is smooth. The extension of Whitney type is a powerful tool when we don’t have smoothness of the surface. In [JO], one of the most recent (1984) papers dealing with extendibility of
classical Sobolev spaces defined on "almost arbitrary open sets", the so called \((\varepsilon, \delta)\)-domains, this method is used successfully.

Let us return to the nonisotropic setting. The first thing to say is that (1) is false if we replace the Euclidian ball \(B\) by a nonisotropic one and the surface measure by the the measure \(d\mu\), except if the surface is noncharacteristic or the polynomial is actually a constant, in the former case it is true because one can choose a coordinate system \((x_1, \cdots, x_{2n+1})\) such that \(M = \{x_{2n+1} = 0\}\) and for which a transverse vector field say \(X_{2n}\) is given by \(\frac{\partial}{\partial x_{2n+1}}\), and this of course makes (4.1) valid. We don't have a canonical way of extending polynomials to the ambient space and still have (4.1) valid in this coordinate system, the reason being the fact that when we extend by constant along the vertical direction we might exit the balls too quickly especially when the balls are centered at characteristic points. We can use normal coordinates to extend polynomials, but this requires us to fix a base point on \(M\) and this means that the same polynomial has different extensions a property that we certainly do not want. One may seek a universal vector field along which we can perform constant extension. This idea is in fact under investigation and it seems to work at least for \(0 < \alpha < 2\), for the characteristic hyperplane and hence would work for hypersurfaces with isolated characteristic points. Which is an improvement, unfortunately the details of this technique have not been worked out and therefore cannot be included in here, they will appear elsewhere in the future. This technical difficulty led us to seek other methods to deal with the higher derivatives case. The method of Jonsson and Wallin is applicable in our context. This method requires the knowledge of all derivatives even transverse ones to be able to construct extensions, this of course in their setting was necessary, because they deal with complicated \(d\)-sets. Let us state the main theorem.

**Theorem 8** Let \(0 \leq k < \beta \leq k + 1\). There is an extension operator \(E\) taking elements 
\[ f := \{f_J : |J| \leq k\} \] of \(B_\beta^3(M)\) to functions defined on the ambient space such that:

\[ E : B_\beta^3(M) \rightarrow S_\alpha^2 \]

is bounded and such that \(X_J E(f)|_M = f_J\) almost everywhere, where \(\alpha = \beta + \frac{1}{2}\).

After constructing the extension operator \(E\), we plan the proof of the theorem as follows. First, because the difficulty is only technical, we treat the case \(0 < \beta < 1\), in this case constants are enough to use. This case is important because the spaces \(F_\beta^3\) and \(B_\beta^3\) are the same. We start with \(\beta = \frac{1}{2}\), and show that the first derivatives are in \(L^2\) and then use other possible characterizations of the spaces \(S_\alpha^2\).

### 4.1 The Whitney decomposition

The construction of the extension operator is achieved using the Whitney decomposition and a partition of unity. Recall that this cover is obtained by covering \(\Omega\) by balls of radius \(2^{-k}\) and then selecting from this balls that are at distance \(10 \times 2^{-k}\) away from \(M\), for \(k = 0, 1, \cdots\),
integer. This cover is required to meet the condition that the balls do not get pilled up too much, this means that if $B$ and $B'$ are two distinct balls with nonempty intersection then if we shrink them enough they end up disjoint. We summarize all what we need to know about the Whitney decomposition in the following theorem. We refer to [ST1] for the proofs.

**Theorem 9** Let $M$ be a closed subset of $\Omega$, then there are nonisotropic balls $W$ covering $\Omega \setminus M$ with the following properties:

(a) $\Omega \setminus M \subset \bigcup_{B \in W} B$

(b) $\frac{1}{10} r(B) \leq \text{dist}(B, M) \leq 10 r(B)$

(c) $B_* \cap B'_* = \emptyset$, where the notation $B_*$ means the ball with the same center as $B$ but with radius $\frac{1}{10}$ the radius of $B$.

(d) If $B$ and $B'$ are two balls such that $B^* \cap B'^* \neq \emptyset$, then

$$\frac{1}{10} \leq \frac{r(B)}{r(B')} \leq 10$$

(e) Finally No point in $\Omega \setminus M$ belongs to more than 100 balls of $W$

In particular by property (d) we can cover the surface $M$ by sets of the form $B^* \cap M$. Recall also that membership to the spaces $F^2_\alpha$ or $B^2_{\alpha}$ is independent of the covering $\mathcal{F}_k$, and hence we can use $W_k$ to define a grid on $M$, $\mathcal{F}_k := \{B^* \cap M, B \in W_k\}$. In particular for small $\beta$ a possible norm for the spaces $B^2_\beta$ and $F^2_\beta$ is given by:

$$||f||^2_{B^2_\beta} = ||f||^2_{L^2(\mu)} + \sum_{k=0}^{\infty} 2^{2k\beta} \sum_{B \in W_k} \inf_{c} \int_{B^* \cap M} |f - c|^2 d\mu$$

(4.2)

Next we let $\varphi_B$ be a partition of unity subordinate to the cover $W$. The support of $\varphi_B$ is contained in $B^*$, it is identically equal to 1 in $B_*$. The two properties essential in what follows about the $\varphi_B$’s are:

For all $x \in \Omega \setminus M$ we have

$$\sum_{B \in W} \varphi_B(x) = 1$$

(4.3)

and

$$|X_j \varphi_B(x)| \leq C r(B)^{-|J|}$$

(4.4)

### 4.2 Proof of theorem 8

#### 4.2.1 The case $0 < \beta < 1$

Let $f \in F^2_\beta$, for each ball $B \in W$ we denote by $c_B$ the constants in the definition of $F^2_\beta$, such that

$$\sum_{B \in W_k} \int_{B^* \cap M} |f - c_B|^2 d\mu \leq (c_k 2^{-k\beta})^2$$

(4.5)
we can be more precise with the constant \( c_B \) to take it the best constant approximating \( f \) in the \( L^2(d\mu) \)-norm i.e.; its average over the set \( B^{**} \cap M \), but that is not necessary. The extension \( E \) is given by,

\[
E(f)(x) = \sum_{B \in W} \varphi_B(x)c_B \quad \text{for} \quad x \in \Omega \setminus M
\]

(4.6)

The sum defining the extension is locally finite and hence defines a smooth function away from \( M \). Let us check that the resulting function is an element of \( L^2(\Omega) \). Indeed, we have

\[
||E f||^2 \leq \sum_{B \in W} \sum_{B' \in W(B)} \int_B |c_{B'}|^2 = C \sum_{B \in W} |B| \sum_{B' \in W(B)} |c_B|^2
\]

where \( |B| \) denotes the volume of \( B \). But by theorem (2)-e the number of balls in the inner sum is bounded by 100, hence

\[
||E f||^2 \leq C \sum_{B \in W} |B||c_B|^2 = \sum_{B \in W} \frac{|B|}{\mu(B^{**} \cap M)} \int_{B^{**} \cap M} c_B^2 d\mu
\]

\[
\leq C \sum_{k=0}^{\infty} 2^{-k} \sum_{B \in W_k} |f - c_B|^2 d\mu + C||f||_{L^2(d\mu)}^2 \leq C||f||_{L^2}^2 \quad \text{for any} \quad 0 < \beta < 1
\]

Suppose now that \( \beta = \frac{1}{2} \), then we need to check that the first order derivatives \( X(Ef) \) are in \( L^2 \). By (3) we have

\[
\sum_{B \in W} X \varphi_B(x) \equiv 0
\]

(4.7)

and so

\[
||X(Ef)||^2 \leq \sum_{B \in W} r(B)^{-2} |B| \sum_{B' \in W(B)} |c_B - c_{B'}|^2 \leq C \sum_{B \in W} r(B)^{-2} \frac{|B|}{\mu(B^{**} \cap M)} \sum_{B' \in W(B)} \int_{B^{**} \cap M} |c_B - c_{B'}|^2 d\mu
\]

\[
\leq C \sum_{k=0}^{\infty} 2^k \sum_{B \in W_k} \int_{B^{**} \cap M} |f - c_B|^2 d\mu \leq C||f||_{L^2}^2
\]

To prove the estimates needed for the full range \( 0 < \beta < 1 \) we distinguish between two cases. The first case is when \( 0 < \beta < 1/2 \) and the second when \( 1/2 < \beta < 1 \). Instead of proving directly that the extended function is in the Sobolev space \( S^2_{\alpha} \), we may, in the first case, prove that given any tiling \( \mathcal{F}_k \) of the ambient space (not to be confused with the tiling of the surface \( M \) which in the present situation is given to us by the Whitney decomposition), we have

\[
\sum_{B \in \mathcal{F}_k} \inf_{c} \int_B |F(x) - c|^2 dx \leq 2^{-2k(\beta+\frac{1}{2})} c_k^2
\]
for some $c_k$ satisfying
\[ \sum_{k=0}^{\infty} c_k^2 \leq C \|f\|_{F_2}^2 \]
in the second case we have to check that the same holds for the derivatives of first order.

Now let $\mathcal{F}_k$ be a net of mesh of $2^{-k}$ for the ambient space, and let $B \in \mathcal{F}_k$, suppose also that it is contained in a Whitney ball $Q_0$, then by Poincare inequality, we have
\[ \inf_c \int_B |f(x) - c|^2 dx \leq C r(B)^2 \int_B |\nabla f(x)|^2 dx \]
but by (4.7)
\[ \nabla f(x) = \sum_{Q \in W} c_Q \nabla \varphi_Q(x) = \sum_{Q \in W} (c_{Q_0} - c_Q) \nabla \varphi_Q(x) \]
and thus we have,
\[ \inf_c \int_B |f(x) - c|^2 dx \leq C \left( \frac{r(B)}{r(Q_0)} \right)^2 |B| \sum_{Q \in W(Q_0)} |c_Q - c_{Q_0}|^2 \]
and since for $Q \in W(Q_0)$ we have $r(Q) \sim r(Q_0)$, we get that
\[ \inf_c \int_B |f(x) - c|^2 dx \leq C \left( \frac{r(B)}{r(Q_0)} \right)^2 \frac{|B|}{\mu(Q_0 \cap M)} \sum_{Q \in W(Q_0)} \int_{Q_0 \cap M} |f(x) - c_Q|^2 d\mu(x) \quad (4.8) \]
now the balls in $\mathcal{F}_k$ that are contained in Whitney balls are those that are far away from $M$, denote those balls by $\mathcal{F}_k(W)$ i.e.,
\[ \mathcal{F}_k(W) = \{ B \in \mathcal{F}_k : B \subset Q, \text{ for some } Q \in W \} \]
with this we have
\[ \sum_{B \in \mathcal{F}_k} \inf_c \int_B |f(x) - c|^2 dx = \sum_{B \in \mathcal{F}_k(W)} + \sum_{B \notin \mathcal{F}_k} = S_{k,1} + S_{k,2} \]
Let the letter $s$ (instead of $Q$) denote temporarily the homogeneous dimension ($s=2n+2$), using (4.8)
\[ S_{k,1} \leq C \sum_{Q \in W} \sum_{B \in \mathcal{F}_k(Q)} \int_B \left( \frac{r(B)}{r(Q)} \right)^2 \frac{|B|}{\mu(Q_0 \cap M)} \sum_{Q' \in W(Q)} \int_{Q_0 \cap M} |f(x) - c_Q|^2 d\mu(x) \]
\[ \leq C \sum_{m=0}^{k} 2^{-2(k-m)} 2^{-ks} 2^{m(s-1)} \sum_{Q \in W_m} \sum_{Q' \in \mathcal{F}_k(Q)} \sum_{Q' \in W(Q)} \int_{Q_0 \cap M} |f(x) - c_{Q'}|^2 d\mu(x) \]
note also that
\[ \sum_{B \in \mathcal{F}_k(Q)} 1 \leq C 2^{(k-m)s} \]
50
using this we finally get

$$S_{k,1} \leq C 2^{-2k} \sum_{m=0}^{k} 2^m \sum_{Q \in W_m} \int_{Q \cap M} |f(x) - c_Q|^2 d\mu(x) \leq C 2^{-2k(\beta + \frac{1}{2})} c_k^2$$

where

$$c_k^2 = 2^{-2k(\frac{1}{2} - \alpha)} \sum_{m=0}^{k} 2^{2m(\frac{1}{2} - \alpha)} a_m^2$$

and as usual by Hardy’s inequality we conclude that

$$\sum_{k=0}^{\infty} c_k^2 \leq C \sum_{k=0}^{\infty} a_k^2$$

Now we turn to balls $B \not\in \mathcal{F}_k(W)$, pick any cube $Q \in W_{k+1}$ that is contained in $B$ and set

$$c_B := c_Q$$

$$\int_B |F(x) - c_B|^2 dx = \sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} \int_Q |F(x) - c_B|^2 dx$$

$$\int_Q |F(x) - c_B|^2 dx \leq C|Q| \sum_{Q' \in W(Q)} |c_{Q'} - c_B|^2 \leq$$

$$C 2^{-m} \int_{Q \cap M} |f(x) - c_B|^2 d\mu(x) + \sum_{Q' \in W(Q)} C 2^{-m} \int_{Q' \cap M} |f(x) - c_{Q'}|^2 d\mu(x) =$$

$$I(Q) + II(Q)$$

$$\sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} I(Q) \leq C \sum_{m=k+1}^{\infty} 2^{-m} \int_{B \cap M} |f(x) - c_B|^2 d\mu(x) \leq C 2^{-k-1} \sum_{Q \in W_{k+1}} \int_{Q \cap M} |f(x) - c_Q|^2 d\mu(x)$$

and summing over $B \not\in \mathcal{F}_k(W)$ we get

$$\sum_{B \not\in \mathcal{F}_k(W)} \sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} I(Q) \leq C 2^{-k-1} \sum_{Q \in W_{k+1}} \int_{Q \cap M} |f(x) - c_Q|^2 d\mu(x) \leq C 2^{-2(k+1)(\beta + \frac{1}{2})} a_{k+1}^2$$

summing $II(Q)$ over $Q \in W_m(B)$ and then over $m$ and then over $B \in \mathcal{F}_k$ we get

$$\sum_{B \in \mathcal{F}_k} \sum_{m=k+1}^{\infty} \sum_{Q \in W_m(B)} II(Q) \leq C \sum_{m=k+1}^{\infty} 2^{-m} \sum_{Q \in W_m} \int_{Q \cap M} |f(x) - c_Q|^2 d\mu(x)$$

$$\leq C \sum_{m=k+1}^{\infty} 2^{-2m(\beta + \frac{1}{2})} a_m^2 \leq 2^{-2k(\beta + \frac{1}{2})} c_k^2$$
where

\[ c_k^2 = C \sum_{m=0}^{\infty} 2^{-2m(\alpha + \frac{1}{2})} a_m^2 \]

and by the second part of Hardy's inequality we get that

\[ \sum_{k=0}^{\infty} c_k^2 \leq C \sum_{k=0}^{\infty} a_k^2 \]

By the same method we get the estimates needed for the derivatives of first order in the case where \( \frac{1}{2} < \beta < 1 \).

\[ \square \]

4.2.2 The case \( \beta \geq 1 \)

Now we come to the description of the extension operator in the general case. Unfortunately we have to leave the spaces \( F_\beta^2 \) and deal with \( B_\beta^2 \) whose elements are systems of functions. Here we assume that the integer \( N >> 0 \). Let \( 0 \leq k < \beta \leq k + 1 \) and \( f := \{ f_j : |J| \leq k \} \in B_\beta^2 \), recall that this means that for each \( B \in W \) there is a polynomial \( P_B \) of degree \( \leq N \) such that:

(a)

\[ \sum_{B \in \mathcal{F}_m} \int_{B^{**} \cap M} |f_j - X_j P_B|^2 d\mu \leq (c_m 2^{-m(\beta - |J|)}), \quad \text{for all } |J| \leq k \]

(b)

\[ \sum_{B \in \mathcal{F}_m} \sum_{B' \in W(B)} \int_{B^{**} \cap M} |X_j(P_B - P_{B'})|^2 d\mu \leq (c_m 2^{-m(\beta - |J|)}), \quad \text{for } k + 1 \leq |J| \leq N \]

(c)

\[ \sum_{B \in \mathcal{F}_m} \int_{B^{**} \cap M} |X_j P_B|^2 d\mu \leq c_0, \quad \text{for all } |J| \leq N \]

The extension of \( f = \{ f_j \} \) is now given by

\[ F(x) := \mathbf{E}(f)(x) = \sum_{Q \in W} \varphi_Q(x) P_Q(x), \quad \text{for } x \notin \Omega \setminus M \]  

(4.10)

Let \( \alpha = \beta + \frac{1}{2} \). The first thing that needs to be checked is that

\[ ||X_j F|| \leq C (\sum_{m=0}^{\infty} c_m^2)^{1/2} \quad \text{for } |J| \leq \alpha \]  

(4.11)

Let \( x \notin M \), then \( X_j F(x) \) is a sum of two types of terms:

\[ A_{J', J''}(x) = \sum_{Q \in W} X_{J'} \varphi_Q(x) X_{J''} P_Q(x), \quad J' + J'' = J, |J'| \neq 0 \]
and

\[ B_J(x) = \sum_{Q \in W} \varphi_Q(x) X_J P_Q(x) \]

We need the following lemma

**Lemma 13** Let \( P(x) \) be a polynomial and \( Q \) a nonisotropic ball such that, \( \mu(Q^{**} \cap M) \cong r(Q)^{*-1} \) then

\[
\int_Q |P(x)|^2 dx \leq C \sum_J r(Q)^{1+2|J|} \int_{Q^{**} \cap M} |X_J P_Q(x)|^2 d\mu
\]

(4.12)

**Proof:**

It suffices to note that if \( Q \) is unit size then both quantities of (4.12) define norms and they are equivalent. Rescale to get the result.

By (4.7) and the lemma

\[
\|A_{J',J}\|^2 \leq C \sum_{Q \in W} \int_Q |A_{J',J}(x)|^2 dx \leq C \sum_{Q \in W} r(Q)^{-2|J'|} \sum_{Q' \in W(Q)} \int_Q \left| X_{J}(P_Q - P_{Q'}) \right|^2 dx
\]

\[
\leq C \sum_{J} \sum_{Q \in W} r(Q)^{1+2(|J|-|J'|)} \sum_{Q' \in W(Q)} \int_Q \left| X_{J} X_{J'}(P_Q - P_{Q'}) \right|^2 d\mu
\]

it easy to realize that

\[
\sum_J r(Q)^{1+2(|J|-|J'|)} \int_{Q^{**} \cap M} |X_J X_{J'} P|^2 d\mu \leq C \sum_{|J| \geq |J'|} r(Q)^{1+2(|J|-|J'|)} \int_{B^{**} \cap M} |X_J P|^2 d\mu
\]

And hence

\[
\|A_{J',J}\|^2 \leq C \sum_{|J'| \leq |J| \leq k} \sum_{Q \in W} r(Q)^{1+2(|J|-|J'|)} \sum_{Q' \in W(Q)} \int_Q \left| X_{J}(P_Q - P_{Q'}) \right|^2 d\mu
\]

\[
+ C \sum_{|J| \geq k+1} \sum_{Q \in W} r(Q)^{1+2(|J|-|J'|)} \sum_{Q' \in W(Q)} \int_Q \left| X_{J}(P_Q - P_{Q'}) \right|^2 d\mu
\]

by (a) and (b) we get at once

\[
\|A_{J',J}\|^2 \leq \sum_{m=0}^{\infty} 2^{-2m\left(\frac{1}{2}-|J|+\beta\right)} c_m^2 \leq \sum_{m=0}^{\infty} c_m^2
\]

as long as \( |J| \leq \beta + \frac{1}{2} = \alpha \).
Next we turn to $B_J$. This term is estimated in a different way.

$$
||B_J||^2 \leq C \sum_{Q \in W} \int_B |B_J|^2 \, dx \leq C \sum_{Q \in W} \int_B |X_J P_Q|^2 \, dx = \sum_{m=0}^{\infty} \sum_{Q \in W_m} \int_B |X_J P_Q|^2 \, dx
$$

For each $Q \in W_m$, $W_{m-1}(Q)$ denotes the balls in $W_{m-1}$ touching the ball $Q$, the immediate ancestors of $Q$. By theorem 2-d there are at most a finite number of them say less than 100 for each ball $Q$, let us enumerate them keeping in mind that this enumeration depends on $Q$.

$$
W_{m-1}(Q) = \{Q_1^1, Q_1^2, \ldots, Q_1^{N_1}\}, \quad \text{where} \quad N_1 = N_1(Q)
$$

Put $Q_0^{N_0} = Q$, inductively we enumerate the set $W_{m-j}(Q_{j-1}^{N_{j-1}})$,

$$
W_{m-j}(Q_{j-1}^{N_{j-1}}) = \{Q_1^j, Q_2^j, \ldots, Q_j^{N_j}\} \quad \text{for} \quad j = 1, 2, \ldots, m
$$

with this we can write $P_Q$ as a telescopic sum:

$$
P_Q = (P_Q - P_{Q_1^1}) + (P_{Q_1^1} - P_{Q_1^2}) + \cdots + (P_{Q_1^{N_1}} - P_Q)
$$

and

$$
(P_Q - P_{Q_1^1}) + (P_{Q_1^1} - P_{Q_1^2}) + \cdots + (P_{Q_1^{N_1}} - P_{Q_2^1}) + (P_{Q_2^1} - P_{Q_2^2}) + \cdots + (P_{Q_{N_2} - 1} - P_{Q_2^N}) + P_{Q_2^N}
$$

$$
(P_Q - P_{Q_1^1}) + \sum_{i=1}^{N_1} (P_{Q_1^i} - P_{Q_1^{i+1}}) + (P_{Q_1^{N_1}} - P_{Q_2^1}) + \sum_{i=1}^{N_2-1} (P_{Q_2^i} - P_{Q_2^{i+1}}) + \cdots + (P_{Q_{N_{m-1}} - 1} - P_{Q_m})
$$

$$
+ \sum_{i=1}^{N_{m-1}} (P_{Q_m^i} - P_{Q_m^{i+1}}) + P_{Q_m^N}
$$

$$
= \sum_{j=1}^{m} \sum_{i=1}^{N_j} (P_{Q_j^i} - P_{Q_j^{i+1}}) + \sum_{j=0}^{m-1} (P_{Q_j^{N_j}} - P_{Q_j^{N_{j+1}}}) + P_{Q_m^N}
$$

Now use Minkowski’s inequality to get

$$
\left( \sum_{Q \in W_m} \int_B |X_J P_Q|^2 \, dx \right)^{1/2} \leq \sum_{j=1}^{m} \left( \sum_{Q \in W_m} \sum_{l=1}^{N_j} \int_Q |X_J (P_{Q_j^l} - P_{Q_j^{l+1}})|^2 \, dx \right)^{1/2}
$$

$$
+ \sum_{j=0}^{m-1} \left( \sum_{Q \in W_m} \int_Q |X_J (P_{Q_j^{N_j}} - P_{Q_j^{N_{j+1}}})|^2 \, dx \right)^{1/2} + \left( \sum_{Q \in W_m} \int_B |X_J P_{Q_m^N}|^2 \, dx \right)^{1/2}
$$
By the lemma we get

\[ I = \sum_{j=1}^{m} \left( \sum_{Q \in W_m} \sum_{l=1}^{N_j-1} \int_Q \left| X_j(P_{Q_j} - P_{Q_{j+1}}) \right|^2 dx \right)^{1/2} \leq \]

\[ C \sum_{J} 2^{-m(\frac{1}{2}+|J|)} \sum_{j=1}^{m} \left( \sum_{Q \in W_m} \sum_{l=1}^{N_j-1} \int_{Q^* \cap M} \left| X_jX_j(P_{Q_j} - P_{Q_{j+1}}) \right|^2 d\mu \right)^{1/2} \leq \]

\[ C \sum_{|J| \geq |J|} 2^{-m(\frac{1}{2}+|J|)} \sum_{j=1}^{m} \left( \sum_{Q \in W_m} \sum_{l=1}^{N_j-1} \int_{Q^* \cap M} \left| X_j(P_{Q_j} - P_{Q_{j+1}}) \right|^2 d\mu \right)^{1/2} \]

Let us look at the double sum

\[ \sum_{Q \in W_m} \sum_{l=1}^{N_j(Q)-1} \int_{Q^* \cap M} \left| X_j(P_{Q_j} - P_{Q_{j'}}) \right|^2 d\mu \]

Since the balls \( Q_j \) in the second sum are the \( j \)th ancestors of the balls \( Q \in W_m \) the first sum can be replaced by a sum over balls in \( Q \in W_m \) whose projections on \( M' \cap M \) are contained on the projection of balls in \( W_{m-j} \) then followed by a sum over \( W_{m-j} \). There might be repetition in this way but as we said they get repeated only a finite number of times and thus we have

\[ \sum_{Q_{m}} \sum_{l=1}^{N_j(Q)-1} \int_{Q^* \cap M} \left| X_j(P_{Q_j} - P_{Q_{j'}}) \right|^2 d\mu \leq C \sum_{Q \in W_{m-j}} \sum_{l=1}^{N_j(Q)} \int_{Q^* \cap M} \left| X_j(P_{Q} - P_{Q'}) \right|^2 d\mu \]

and therefore we get

\[ I \leq C \sum_{|J| \geq |J|} 2^{-m(\frac{1}{2}+|J|)} \sum_{j=1}^{m} \left( \sum_{Q \in W_m} \sum_{Q' \in W_{m-j}(Q)} \int_{Q^* \cap M} \left| X_j(P_{Q} - P_{Q'}) \right|^2 d\mu \right)^{1/2} \]

By (a) (if \(|\tilde{J}| \leq k\)) and (b) (if \(|\tilde{J}| \geq k + 1\)) we get that the last term is less than or equal to

\[ C \sum_{|J| \geq |J|} 2^{-m(\frac{1}{2}+|J|-|J|)} \sum_{j=1}^{m} 2^{-j(|J|)} c_j \]

which when squared and summed over \( m \), using Hardy's inequality, is less than or equal to

\[ C \sum_{m=0}^{\infty} 2^{-2m(|J|)} c_m^2 \leq C \sum_{m=0}^{\infty} c_m^2 \quad \text{for all} \quad |J| \leq \alpha \]

the other terms are dealt with similarly we

\[ II = \sum_{j=0}^{m-1} \left( \sum_{Q \in W_m} \int_Q \left| X_j(P_{Q_j} - P_{Q_{j+1}}) \right|^2 dx \right)^{1/2} \leq \]

55
Now square it and sum it over \( m \) like before to it less than or equal to
\[
\sum_{m=0}^{\infty} 2^{-2m(\alpha-|J|)}c_m^2
\]
the last term is simpler we use estimate (c) to get it less than or equal to \( c_0 \). Thus we have proved that for all \( |J| \leq \alpha \)
\[
\|X_JF\| \leq C(\sum_{m=0}^{\infty} c_m^2)^{1/2} \leq C\|\{f_J\}\|_{B^3_{\alpha}}
\]
and hence if \( \alpha \) is integer the extension theorem is proved.

Next we treat the case \( \alpha \) noninteger. Let \( l \) be the integer such that \( \beta - \frac{1}{2} < l < \beta + \frac{1}{2} = \alpha \), and set \( \lambda = \alpha - l \), then like the proof in the case of small \( \beta \) we are going to prove that the derivatives \( X_JF \) belongs to \( S^2_{\lambda} \) by showing that
\[
\sum_{B \in \mathcal{F}_m} \inf_c \int_B |X_JF - c|^2 dx \leq 2^{-2m\lambda}c_m^2 \quad |J| \leq l \tag{4.13}
\]
We take as before for each integer \( m \) a grid \( \mathcal{F}_m \) of the ambient space. We prove (13) proving it for each of the terms \( A_{J,J''} \) and \( B_J \). Let \( B \) be a ball that is contained in a Whitney ball \( Q \), then by Poincare inequality we have
\[
\inf_c \int_B |A_{J,J''} - c|^2 dx \leq Cr(B)^2 \int_B |\nabla A_{J,J''}|^2 dx \leq Cr(B)^2 \int_B \sum_{Q' \in W(Q)} |(\nabla X_J\varphi_{Q'}) (X_J\nu_{P_{Q'}})|^2 dx + Cr(B)^2 \int_B \sum_{Q' \in W(Q)} |(X_J\nu_{P_{Q'}}) (\nabla X_J\nu_{P_{Q'}})|^2 dx
\]
\[
= I + II
\]
both terms \( I \) and \( II \) satisfy the same estimates so we treat only one of them. Summing over the balls that are contained in Whitney balls we get
\[
\sum_{B \in \mathcal{F}_m(W)} \inf_c \int_B |A_{J,J''} - c|^2 dx \leq C 2^{-2m} \sum_{j=0}^{m} 2^{2j(|J'|+1)} \sum_{Q \in W} \sum_{B \in \mathcal{F}_m(Q) \cap \mathcal{F}_m(Q')} \int_B |X_J\nu_{P_{Q}} - P_{Q'}|^2 dx
\]
\[
C^{2-2m} \sum_{j=0}^{m} 2^{2j(|J'|+1)} \sum_{Q \in W, Q' \in W(Q)} \int_Q |X_J(P_Q - P_{Q'})|^2 \, dx \\
\leq \leq \\
C^{2-2m} \sum_{|J| \geq |J'|} 2^{2j(|J'|+|J|+\frac{1}{2})} \sum_{Q \in W, Q' \in W(Q)} \int_Q \sum_{Q' \cap W} |X_J(P_Q - P_{Q'})|^2 \, d\mu \\
\leq C^{2-2m} \sum_{j=0}^{m} 2^{2j(|J| - \beta + \frac{1}{2})} \alpha_j^2 \\
\leq 2^{-2m\lambda \alpha_j^2}
\]

where
\[
\sum_{m=0}^{\infty} a_m^2 \leq C \sum_{m=0}^{\infty} 2^{-2m(|J|)} \leq C \sum_{m=0}^{\infty} \alpha_m \text{ for } |J| \leq l
\]

Next we turn to balls that are near \( M \) i.e. \( B \notin F_m(W) \). For these we show that
\[
\sum_{B \notin F_m(W)} \int_B |A_{J,J'}|^2 \, dx \leq 2^{-2m\lambda \alpha_j^2}
\]

\[
\int_B |A_{J,J'}|^2 \, dx \leq C \sum_{j=m}^{\infty} \sum_{Q \in W(B)} \int_Q \sum_{Q' \in W(Q)} (X_J \varphi_{Q'}) (X_{J'}(P_Q - P_{Q'}))^2 \, dx \\
\leq C \sum_{j=m}^{\infty} 2^{|J'|} \sum_{Q \in W(B)} \sum_{Q' \in W(Q)} \int_Q |X_{J'}(P_Q - P_{Q'})|^2 \, d\mu \\
\leq C \sum_{|J| \geq |J'|} 2^{2j(|J| - |J'| - \frac{1}{2})} \sum_{Q \in W(B)} \sum_{Q' \in W(Q)} \int_{Q' \cap W} |X_J(P_Q - P_{Q'})|^2 \, d\mu
\]

After we sum over \( B \notin F_m(W) \) and rearranging the terms we finally get
\[
\sum_{B \notin F_m(W)} \leq 2^{-2m\lambda} (C 2^{2m\lambda} \sum_{j=m}^{\infty} 2^{-2j\lambda \alpha_j^2}) \leq 2^{-2m\lambda \alpha_j^2}
\]

The other term \( B_J \) is estimated in the same manner except that we would have to use the method by which we proved it to be in \( L^2 \), we do not wish to repeat it here.

Thus the extension theorem is proved granted we prove finally that \( (X_J F)|_M = f_J \). Set
\[
G(x) := X_J F(x).
\]
First we should note that since \( |J| \leq k < \beta \) and \( F \in S^2_\alpha (\beta + \frac{1}{2} = \alpha) \) then \( G \in S^2_\lambda \) for some \( \frac{1}{2} < \lambda < 1 \). Set
\[
G(Q) := \frac{1}{|Q|} \int_Q G(x) \, dx
\]

Let \( W_m^* \) be balls of \( W_m \) expanded enough to cover the surface \( M \), and let \( \psi_Q^* \) be a partition of unity subordinate to this cover and finally set
\[
G_m(x) := \sum_{Q \in W_m} \psi_Q^* G(Q)
\]

57
then this function is smooth and hence its restriction to \( M \) is well defined. To prove that the restriction is what we started with it suffices to prove two things:

1. 
\[
\int_M |f_J - G_m|^2 d\mu \to 0 \quad m \to \infty
\]

and

2. 
\[
\int_M |G - G_m|^2 d\mu \to 0 \quad m \to \infty
\]

to prove 1. we have

\[
\int_M |f_J - G_m|^2 d\mu = \int_M \left| \sum_{Q \in W_m} \psi_Q (f_J - G(Q')) |^2 d\mu \leq \sum_{Q \in W_m} \int_{Q \cap M} \sum_{Q' \in W_m(Q)} \psi_{Q'} (f_J - G(Q')) |^2 d\mu
\]

\[
\leq \sum_{Q \in W_m} \sum_{Q' \in W_m(Q)} |f_J - G(Q')|^2 d\mu \leq C \sum_{Q \in W_m} \int_{Q \cap M} |f_J - G(Q)|^2 d\mu
\]

\[
\leq C \sum_{Q \in W_m} \int_{Q \cap M} |f_J - X_J P_Q|^2 d\mu + C \sum_{Q \in W_m} \int_{Q \cap M} |G(Q) - X_J P_Q|^2 d\mu
\]

the first term is by definition less than \( 2^{-2m(2-|J|)} c_m^2 \to 0 \). As for the second term it is easily proved that if \( P \) is a polynomial and \( Q \) any ball then we have

\[
\int_{Q \cap M} |P|^2 d\mu \leq C 2^m \int_Q |P|^2 dvol
\]

(if we use a linear change of variables so as the class of polynomials is preserved and the surface becomes horizontal then the assertion is true by finite dimensionality of polynomials, then rescale to get the general version, it is also wort noticing that the reverse inequality is not true.) And thus we have that the second term is less than

\[
C 2^m \sum_{Q \in W_m} \int_Q |G(Q) - X_J P_Q|^2 dvol \leq C 2^m \sum_{Q \in W_m} \int_Q |G(Q) - G|^2 dvol + C 2^m \sum_{Q \in W_m} \int_Q |G - X_J P_Q|^2 dvol
\]

by (1.) the first term is less than \( c_m^2 2^{-2m(\lambda - \frac{1}{2})} \to 0 \) as \( m \to \infty \)

write \( G - X_J P_Q \) as a sum of terms of the form

\[
c_{J', J''} \sum_{Q' \in W} X_{J'} \varphi_{Q'} X_{J''} (P_{Q'} - P_Q), J' + J'' = J, |J'| \neq 0
\]

plus

\[
\sum_{Q' \in W} \varphi_{Q'} X_J (P_{Q'} - P_Q)
\]

if we use now lemma (11) we get at once

\[
\int_Q |G - X_J P_Q|^2 dvol
\]
\[ \leq C \sum_{J' \cdot J''} \sum_{J} 2^{-2m(|J| - |J'| + \frac{1}{2})} \int_{Q' \cap M} |X_{J''} X_{J'''} (P_{Q} - P_{Q'})|^2 \text{d}\mu + \sum_{J} 2^{-2m(|J| + \frac{1}{2})} \int_{Q' \cap M} |X_{J} X_{J} (P_{Q} - P_{Q'})|^2 \text{d}\mu \]

\[ \leq C \sum_{J' \cdot J'' \cdot |J| \geq |J''|} 2^{-2m(|J| - |J'| + \frac{1}{2})} \int_{Q' \cap M} |X_{j} (P_{Q} - P_{Q'})|^2 \text{d}\mu + \sum_{|J| \geq |J'|} 2^{-2m(|J| - |J'| + \frac{1}{2})} \int_{Q' \cap M} |X_{j} (P_{Q} - P_{Q'})|^2 \text{d}\mu \]

summing over \( Q \in W_m \) and using the definition of \( B_2^2 \) we get the desired result. To prove 2. it suffices to note that from the proof of the restriction theorem that we presented for \( \frac{1}{2} < \alpha < 1 \), we see that if \( Q \) is a Whitney ball and if \( f \in \mathcal{S}_2^2 \), and if we set \( f(Q) := \frac{\int_{Q} |Q| \int_{Q} f \text{dvol}}{\text{vol}(Q)} \), then we ahve

\[ \sum_{Q \in W_m} \int_{Q' \cap M} |f - f(B)|^2 \text{d}\mu \leq C_m 2^{-2m(\lambda - \frac{1}{2})} \]

from which 2. follows immediately.
Chapter 5

Some Generalizations to $p \neq 2$

In this note we shall give indications on how to obtain generalizations to $p \neq 2$ whenever possible. Now, the spaces we are concerned with are $S^p_\alpha := L^p * J_\alpha$. Theorem 1 extends to the $p \neq 2$ case trivially, the limitation there for $p$ and $\alpha$ is that $\alpha > \frac{1}{p}$, and $1 \leq p < \infty$. The same proof can be carried out in this case, the only difference is, whenever we used Schwarz’s inequality we use Hölder’s. The restriction theorem is also true if we adopt the Jonsson and Wallin proof to our situation. Our proof extends to a restriction theorem for the nonisotropic Besov spaces instead; the reason is because we used decomposition of functions that is valid for Besov spaces and not for Sobolev spaces in the case $p \neq 2$. The extension is also true from $B^p_\beta$ spaces on the hypersurface to the ambient space to land in $S^p_\alpha$, $\beta = \beta + \frac{1}{p}$, but now $p$ must be $1 < p \leq \infty$. The only difference and “not obvious” extension to $p \neq 2$ is theorem 2. Recall that we have the embedding

$$S^p_\alpha \rightarrow L^p_{\frac{1}{2}} \quad (locally)$$  

(5.1)

where the spaces on the right hand side of (5.1) are the classical potential spaces.

$$R : S^p_\alpha \rightarrow \Lambda^{p,p}_{\frac{1}{2}-\frac{1}{p}}(M)$$  

(5.2)

provided that $\frac{q}{2} - \frac{1}{p} > 0$; i.e.; $\alpha > \frac{2}{p}$. The spaces on the right hand side of (5.2) are the classical Besov spaces see [ST]; chapter V. In particular (5.2) tells us that the restriction spaces of $S^p_\alpha$ to $M$, is in $L^p(d\sigma)$ if $\alpha > \frac{2}{p}$. Let $M = \{t = 0\}$ be the characteristic hyperplane, and let $\rho = (|z|^4 + t^2)^{\frac{1}{4}}$; then the function $\rho^{-\frac{Q+2}{p}}$, belongs to $S^p_\alpha$ near the origin for all $\alpha < \frac{2}{p}$, but clearly its restriction to $\{t = 0\}$ fails to belong to $L^p(d\sigma)$. Thus, the question remains unsettled only for $\alpha = \frac{2}{p}$ and this amounts to checking whether or not we have boundedness of the operator

$$R \circ J_p : L^p(R^{2n+1}) \rightarrow L^p(M, d\sigma)$$  

(5.3)

We will employ the same weighted methods as in the case of $p = 2$, except here we need two weight functions $w_1(x)$ and $w_2(y)$ on $M$ and $R^{2n+1}$ respectively. So, let $w_1(x) \geq 0$ and
$w_2(y) \geq 0$ be two functions not identically zero on $M$ and $R^{2n+1}$ respectively; and denote by $T$ the operator $R \circ J_{\frac{1}{p}}$. Let $g \in L^p(R^{2n+1})$ having support in, say the unit ball, then for $x \in M$,

$$Tg(x) = \int_{|y| \leq 1} J_{\frac{1}{p}}(y^{-1}x)g(y)dy$$

Set $d\mu(x) = w_1(x)^{-p}d\sigma(x)$ (sorry if it causes confusion this $d\mu$ is not the we've been calling in this thesis) and $d\nu(y) = w_2(y)^{-p}dy$, then it is easily verified that the inequality

$$\|Tg\|_{L^p(d\sigma)} \leq C \|g\|_{L^p(d\nu)}$$

(5.4)

is equivalent to:

$$\int_M (\int_{|y| \leq 1} K(x,y)f(y)d\nu(y))^2 d\mu(x) \leq C \int_{|y| \leq 1} |f(y)|^p d\nu(y).$$

(5.5)

where $K(x,y) = w_2(y)^{p-1}J_{\frac{1}{p}}(y^{-1})w_1(x)$. According to Young's inequality, to have (5) it suffices that we check two estimates:

(a) $\sup_{x \in M} \int_{|y| \leq 1} |K(x,y)|d\nu(y) \leq C$

and

(b) $\sup_{|y| \leq 1} \int_M |K(x,y)|d\mu(x) \leq C.$

in terms of the Lebesgue measure and ordinary surface measure (taking into account the estimate for $J_{\frac{1}{p}}$) (a) and (b) follow from

(a)' $\sup_{x \in M} w_1(x) \int_{|y| \leq 1} |y^{-1}x|^{\frac{2}{p}-Q} w_2(y)^{-1}dy \leq C$

and

(b)' $\sup_{|y| \leq 1} w_2(y)^{p-1} \int_M |y^{-1}x|^{\frac{2}{p}-Q} w_1(x)^{1-p}d\sigma(x) \leq C.$

From our experience with the case $p = 2$, we know what $w_1(x)$ ought to be; $w_1(x) = w(x)^{\varepsilon}$, for some $\varepsilon > 0$ small enough. The problem is to find $w_2(y)$ satisfying (a)' and (b)'. From (b)' we can take $w_2(y)$ to be equal to,

$$(w_2(y))^{-p+1} = \int_M |y^{-1}x|^{\frac{2}{p}-Q} w_1(x)^{1-p}d\sigma(x)$$

(5.6)

and we will be left with (a)'. Let us analyze the integral

$$\int_M |y^{-1}x|^{\frac{2}{p}-Q} w(x)^{\varepsilon(1-p)}d\sigma(x).$$

(5.7)

The factor $|y^{-1}x|^{\frac{2}{p}-Q}$ is at its worst (blowing up) for those $x \in M$ that are near enough to $y$; i.e., within a $d(y, M)$ from $M$. The factor $w(x)^{1-\varepsilon}$ is worst when $x$ is near the characteristic
set $\Gamma$ and the integrand is worst when the singularities are combined, i.e., when $x$ is near $y$ and near the characteristic point that is when $d(y, \Gamma)$ is comparable to $d(y, M)$. The integral in (5.7) is therefore going to be expressed (or estimated) solely in terms of two quantities, $d(y, M)$ the distance of $y$ to $M$ and $d(y, \Gamma)$, the distance of $y$ to $\Gamma$. Let us treat the case $M = \{ t = 0 \}$ first. In this case integral in (5.7) is

$$I = \int_M |y^{-1}x|^{\frac{2}{p} - Q}|x|^{\epsilon(1-p)}d\sigma(x) = I_1 + I_2 + I_3$$

$$I_1 = \int_{|x| \leq |y|^{\frac{1}{2}} |y - x|} |x|^{\epsilon(1-p)}|y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x), \quad I_2 = \int_{\frac{|y|}{2C} \leq |x| \leq C|y|} |x|^{\epsilon(1-p)}|y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x)$$

$$I_3 = \int_{|x| \leq C|y|} |x|^{\epsilon(1-p)}|y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x)$$

We need the following easily proved lemma whose statements are just variants of the triangle inequality.

**Lemma 14** Let $0 \neq x \in M$, $y \in \mathbb{R}^{2n+1}$

- (a) If $|y| < |x|$ and $|y| \cong d(y, M)$ then $|y^{-1}x| \cong |x|$.
- (b) If $|y| \gg |x|$ then $|y^{-1}x| \cong |y|$.
- (c) If $|x| \leq \frac{|y|}{2C}$ then $|y^{-1}x| > \frac{|y|}{2C}$.

If $d(y, M) \ll |y|$ and $x_0 \in M$ is such that $d(y, M) = |x_0^{-1}y|$, then $|x_0| \cong |y|$.

By lemma (b)

$$I_1 \leq C|y|^{\frac{2}{p} - Q} \int_{|x| \leq |y|^{\frac{1}{2}}} |x|^{\epsilon(1-p)}d\sigma(x) \leq C|y|^{\frac{2}{p} + \epsilon(1-p)}$$

$$I_3 = \int_{|x| \leq C|y|} |x|^{\epsilon(1-p)}|y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x)$$

By lemma (a) $|y^{-1}x| \cong |x|$ and thus

$$I_3 \leq C \int_{|x| \geq C|y|} |x|^{-Q + \frac{2}{p} + \epsilon(1-p)}d\sigma(x) \leq C|y|^{-Q + \frac{2}{p} + \epsilon(1-p)}$$

$I_2$ is more subtle and requires a careful analysis

$$I_2 \leq C|y|^{\epsilon(1-p)} \int_{|x| \leq C|y|} |y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x)$$

If $|y| \cong d(y, M)$, we have

$$I_2 \leq C|y|^{\frac{2}{p} + \epsilon(1-p)}$$

If $|y| \gg d(y, M)$, pick a point $x_0 \in M$ such that $|x_0^{-1}y| \cong d(y, M)$ and write

$$\int_{|x| \leq C|y|} |y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x) = \int_{|x_0^{-1}x| \leq d(y, M)} |y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x) + \sum_{k=1}^{\infty} A_k$$

where

$$A_k = \int_{\Delta_k} |y^{-1}x|^{\frac{2}{p} - Q}d\sigma(x)$$

62
where

$$\Delta_k = \{2^{k-1}d(y, M) < |x_0^{-1}x| < 2^k d(y, M)\}$$

$$\int_{|x_0^{-1}x| \leq d(y, M)} |y^{-1}x|^{\frac{2}{p} - Q} d\sigma(x) \leq d(y, M)^{\frac{2}{p} - Q} \sigma(\{ |x_0^{-1}x| \leq d(y, M)\})$$

By lemma 3(b) of chapter 2 we have

$$\sigma(\{ |x_0^{-1}x| \leq d(y, M)\}) \leq C \frac{d(y, M)^{Q-1}}{|x_0|} \quad \text{and} \quad \sigma(\Delta_k) = \frac{(2^k d(y, M))^{Q-1}}{|y|}$$

and by lemma (c)

$$A_k \leq (2^k d(y, M))^{\frac{2}{p} - Q} \sigma(\{ |x_0^{-1}x| \leq 2^k d(y, M)\}) \leq C(2^k d(y, M))^{\frac{2}{p} - Q} \frac{(2^k d(y, M))^{Q-1}}{|y|}$$

and hence,

$$\int_{|x| \leq C|y|} |y^{-1}x|^{\frac{2}{p} - Q} d\sigma(x) \leq C|y|^{-1}d(y, M)^{\frac{2}{p} - 1} \sum_{k=0}^{\infty} (2^k)^{\frac{2}{p} - 1} \leq C|y|^{-1}d(y, M)^{\frac{2}{p} - 1}$$

provided that $p > 2$. Therefore

$$I_2 \leq C|y|^{(1-p)-1}d(y, M)^{\frac{2}{p} - 1}$$

which also contains the estimate for $I_1$ and $I_3$, so,

$$I \leq C|y|^{(1-p)-1}d(y, M)^{\frac{2}{p} - 1} \quad \text{for} \quad p > 2 \quad (5.8)$$

What about $p \leq 2$? If $p = 2$ the integral to estimate is

$$\int_M |y^{-1}x|^{-Q+1} |x|^{-\epsilon} d\sigma(x)$$

Case 1: $|y| \cong d(y, M)$, the same estimate as $I_3$ above yields

$$\int_M |y^{-1}x|^{-Q+1} |x|^{-\epsilon} d\sigma(x) \leq C|y|^{-\epsilon-1}$$

Case 2: $|y| \gg d(y, M)$; break the integral into three parts

$$\int_M |y^{-1}x|^{-Q+1} |x|^{-\epsilon} d\sigma(x) = I_1 + I_2 + I_3$$

$$I_1 = \int_{|x^{-1}x| < d(y, M)} |y^{-1}x|^{-Q+1} |x|^{-\epsilon} d\sigma(x), \quad I_2 = \int_{d(y, M) \leq |x^{-1}x| \leq |y|} |y^{-1}x|^{-Q+1} |x|^{-\epsilon} d\sigma(x)$$

$$I_3 = \int_{|x^{-1}x| > |y|} |y^{-1}x|^{-Q+1} |x|^{-\epsilon} d\sigma(x)$$

63
\[
I_1 \leq C|y|^{-\varepsilon}d(y, M)^{-Q+1}\sigma(\{|x_0^{-1}x| < d(y, M)\}) \leq C|y|^{-\varepsilon-1}
\]

\[
I_2 \leq C|y|^{-\varepsilon}\int_{d(y, M) \leq |x_0^{-1}x| \leq |y|} |y_0^{-1}x|^{-Q+1}d\sigma(x) \leq C|y|^{-\varepsilon}\int_{d(y, M) \leq |x_0^{-1}x| \leq |y|} |x_0^{-1}x|^{-Q+1}d\sigma(x) =
\]

\[
= C|y|^{-\varepsilon}(\int_{d(y, M)} |y|^{-Q+1}d\sigma(\{|x_0^{-1}x| < r\})) \leq C|y|^{-\varepsilon}|y|^{-1} + |y|^{-1}\log(\frac{|y|}{d(y, M)})
\]

\[
I_2 \leq C|y|^{-\varepsilon-1}(1 + \log(\frac{|y|}{d(y, M)}))
\]

\[I_3\] is easy, we notice that \(|y^{-1}x|\) and \(|x|\) are exchangeable and compute that \(I_3 \leq C|y|^{-\varepsilon-1}\) and thus

\[I \leq C|y|^{-\varepsilon-1}(1 + \log(\frac{|y|}{d(y, M)})) \quad (5.9)\]

\(p < 2\): Case 1: \(|y| \equiv d(y, M)\), then, like before

\[
\int_M |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x) = I_1 + I_2
\]

\[
I_1 = \int_{|x| < |y|} |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x) \quad , \quad I_2 = \int_{|x| > |y|} |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x)
\]

\[
I_2 = C\int_{|x| > |y|} |x|^{\frac{1}{p}-Q+\varepsilon(1-p)}d\sigma(x) = C|y|^{(\frac{1}{p}+\varepsilon)(1-p)}
\]

By lemma (a)

\[
I_1 = \int_{|x| < |y|} |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x) \leq C|y|^{\frac{1}{p}-Q} \int_{|x| < |y|} |x|^{\varepsilon(1-p)}d\sigma(x) \leq C|y|^{(\frac{1}{p}+\varepsilon)(1-p)}
\]

Case 2: \(|y| \gg d(y, M)\), we have

\[
\int_M |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x) = I_1 + I_2 + I_3
\]

\[
I_1 = \int_{|x_0^{-1}x| < \frac{|y|}{|\sigma|}} |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x) \quad , \quad I_2 = \int_{|x| < \frac{|y|}{|\sigma|}} |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x)
\]

\[
I_3 = \int_{|x_0^{-1}x| > \frac{|y|}{|\sigma|}, |x| > \frac{|y|}{|\sigma|}} |y^{-1}x|^{\frac{1}{p}-Q} |x|^{\varepsilon(1-p)}d\sigma(x)
\]

integration bu parts gives

\[
I_1 \leq C|y|^{\varepsilon(1-p)} \int_{|x_0^{-1}x| < \frac{|y|}{|\sigma|}} |x_0^{-1}x|^{\frac{1}{p}-Q}d\sigma(x) = C|y|^{\varepsilon(1-p)} \int_{0}^{\frac{|y|}{|\sigma|}} r^{\frac{1}{p}-Q}d(\sigma(\{|x_0^{-1}x| < r\}))
\]

\[
= C|y|^{\varepsilon(1-p)} r^{\frac{1}{p}-Q} \frac{r^{Q-1}}{|x_0|} + C \int_{0}^{\frac{|y|}{|\sigma|}} r^{\frac{1}{p}-Q-1}\sigma(\{|x_0^{-1}x| < r\})dr
\]

64
\[ I_2 \leq C|y|^{\frac{2}{p}-2Q} \int_{|x| \leq |y|} |x|^{(1-p)}d\sigma(x) \leq C|y|^{\left(\frac{2}{p} + \frac{2}{p} + (1-p)\right)} \]

The tail \( I_3 \) is also easy, in this case the quantities \(|x|\) and \(|y^{-1}x|\) are exchangeable, and hence

\[ I_3 \leq C \int_{|x| \geq |y|} |x|^{\frac{2}{p}-2Q +(1-p)}d\sigma(x) = C \int_{|y|}^{\infty} r^\frac{2}{p}-2Q +(1-p)r^{-3} dr \]

\[ \leq C|y|^{\left(\frac{2}{p} + \frac{2}{p} + (1-p)\right)} \]

Hence, if \( p < 2 \) we have a better estimate

\[ \int_M |y^{-1}x|^{\frac{2}{p}-2Q} |x|^{(1-p)}d\sigma(x) \leq C|y|^{\left(\frac{2}{p} + \frac{2}{p} + (1-p)\right)} \quad (5.10) \]

In summary, by (5.6),(5.8),(5.9) and (5.10) \( w_2(y) \) ought to be

\[
 w_2(y) = \begin{cases} 
 C|y|\left(\epsilon - \epsilon \right) d(y,M)^{\frac{2}{p}-\frac{2}{p}-1} & \text{if } \ p > 2 \\
 C|y|^{-\epsilon - 1 + \log\left(\frac{|y|}{d(y,M)}\right)} & \text{if } \ p = 2 \\
 C|y|^{\left(\epsilon + \frac{2}{p}\right)} & \text{if } \ p < 2 
\end{cases}
\]

In order to conclude, we need to check estimate (a)'

\[ \sup_{x \in M} |x|^{\epsilon} \int_{|y| \leq 1} |y^{-1}x|^{\frac{2}{p}-2Q} w_2(y)^{-1} dy \leq C \]

Assume \(|x| \neq 0\) and \( p > 2 \)

\[ \int_{|y| \leq 1} |y^{-1}x|^{\frac{2}{p}-2Q} |y|^{-\left(\epsilon + \frac{1}{p-1}\right)} d(y,M)^{\frac{2}{p}-\frac{2}{p}-1}) dy \leq I + II \]

Let \( E_1 = \{ y : |y| \leq d(y,M), |y| \leq |x| \} \) and \( E_2 = \{ y : |y| \leq d(y,M), |y| > |x| \} \)

\[ I \leq \int_{E_1} |y^{-1}x|^{\frac{2}{p}-2Q} |y|^{-\frac{\epsilon}{2}} dy + \int_{E_2} |y^{-1}x|^{\frac{2}{p}-2Q} |y|^{-\frac{\epsilon}{2}} dy \]

65
By lemma 14(a)

\[ I \leq C |x|^{\frac{2}{p} - Q} \int_{|y| \leq |x|} |y|^{-\epsilon - \frac{1}{p}} dy + \int_{|y| > |x|} |y|^{-\epsilon - Q} dy \leq C |x|^{-\epsilon} \]

Let \( E_k = \{ y : |y| \gg d(y, M) \text{ and } d(y, M) \cong 2^k |x| \} \) and \( F = \{ y : |y| \gg d(y, M) \text{ and } d(y, M) \leq |x| \} \)

\[ II = \sum_{k=0}^{\infty} \int_{E_k} |y^{-1} x|^{\frac{2}{p} - Q} |y|^{-\epsilon - \frac{1}{p} - 1} d(y, M)^{\frac{2-p}{p-1}} dy + \int_{F} |y^{-1} x|^{\frac{2}{p} - Q} |y|^{-\epsilon - \frac{1}{p} - 1} d(y, M)^{\frac{2-p}{p-1}} dy \]

\[ \int_{E_k} |y^{-1} x|^{\frac{2}{p} - Q} |y|^{-\epsilon - \frac{1}{p} - 1} d(y, M)^{\frac{2-p}{p-1}} dy \leq (2^k |x|)^{\frac{2-p}{p-1}} \int_{|y| > 2^k |x|} |y^{-1} x|^{\frac{2}{p} - Q} |y|^{-\epsilon - \frac{1}{p} - 1} dy \]

\[ \leq (2^k |x|)^{\frac{2-p}{p-1}} \int_{|y| > 2^k |x|} |y|^{\frac{2-p}{p-1}} dy = (2^k |x|)^{\frac{2-p}{p-1}} \int_{|y| > 2^k |x|} \frac{dy}{|y|^Q} \leq C (2^k |x|)^{-\epsilon} \]

Summing over \( k \) we get

\[ \sum_{k=0}^{\infty} \int_{E_k} |y^{-1} x|^{\frac{2}{p} - Q} |y|^{-\epsilon - \frac{1}{p} - 1} d(y, M)^{\frac{2-p}{p-1}} dy \leq C |x|^{-\epsilon} \]

The other term of \( II \) is estimated in a similar manner and the case \( p \leq 2 \) is easy to check. The estimates for general \( M \) follow the same ideas, and because of the torture they give, we do not want them to be here.
Bibliography


[ST2] ———— The characterization of functions arising as potentials, I. Bull. Amer. Soc. 69 (1963)