Theoretical Studies of Nuclear Burning on White Dwarfs

by

James Robert Kiger

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Abstract

The nuclear burning on the surface of an accreting white dwarf in a binary star system is modeled using a many-zone representation of the white dwarf’s envelope. This model accounts for both hydrogen and helium burning in the envelope, and utilizes exponentially sized mass zones, fully variable gravity, and arbitrary zone composition. The matter in the envelope is spatially redistributed in every time step by a procedure in which the zone boundaries are redefined according to the mass in each zone. The time dependence of the zones is derived from the stellar structure equations, and is solved by a fourth-order Runge Kutta numerical integration. This is a more sophisticated hydrostatic model than those used by previous authors, but is not too computationally intensive to rule out long time scale studies.

The methods for determining the time dependence of the zones are discussed in detail. Also, the algorithms used to redistribute the mass among the zones and the methods to calculate all quantities of interest are outlined.

We tested our model against several previously published results. The properties of hydrogen flashes and steady hydrogen burning are explored. We found our calculations to be in good agreement with most other results. The dependence of these flashes on accretion rate and white dwarf mass is explored. Also, the transition from flashing behavior to steady burning of the hydrogen is discussed.

As new explorations, we did long-term runs to study the behavior of helium in the envelope while the hydrogen is steadily burning. Also, we allowed the mass accretion rate onto the white dwarf to vary in time. In these cases, we found long time scale oscillations in the accretion rate (both above and below the region of steady burning) produced much shorter time scale phenomena. Also, we determined the response time of the envelope to changes in mass accretion rate. When the oscillations in accretion rate dropped below approximately 10 years the resulting change in luminosity of the star was significantly diminished by this response time.

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Chapter 1

Introduction

Binary systems composed of a normal star and a compact star (white dwarf, neutron star, or black hole) with mass transfer result in luminous sources of X-rays and ultraviolet radiation. As the accreted matter falls down the deep gravitational well of the compact object a significant fraction of the rest mass can be converted to extractable energy, thereby resulting in a highly efficient energy source. Systems in which the compact star is a white dwarf accreting matter at rates of $10^{-10}$–$10^{-8} \text{ M}_\odot/\text{yr}$ form objects known as cataclysmic variables (CV's). These systems include both dwarf novae and classical novae. Outbursts from dwarf novae result from the release of accretion energy and usually produce luminosities of less than $10^{34}$ ergs/s. After accumulating matter for long periods of time (e.g. thousands of years), the accreted matter on the white dwarf may undergo a violent thermonuclear explosion, producing a phenomenon known as a classical nova. Cataclysmic variables and classical novae have been studied for many decades.

More recently, however, a new class of high luminosity ($10^{37}$–$10^{38}$ ergs/s) supersoft sources has been discovered. The most widely accepted model for these objects is that they are binary systems in which the luminosity is provided by relatively stable nuclear burning on an accreting white dwarf — such nuclear burning is 10 times more efficient than the release of gravitational energy that accompanies accretion. This stable nuclear burning on a white dwarf requires much higher accretion rates ($10^{-6}$–$10^{-7} \text{ M}_\odot/\text{yr}$) than those typically associated with CV's. In these supersoft
X-ray systems the donor star, a main sequence or sub-giant star, is overflowing its critical potential lobe (Roche lobe) and is spilling matter onto the companion white dwarf. The mass transfer is driven by thermal timescale "instability" because the donor is more massive than its companion. This accreted matter may either burn steadily, or remain on the surface of the white dwarf until enough mass has built up in an envelope around the dwarf for runaway nuclear burning to take place. These episodes of violent burning, or flashes, recur on a timescale which is dependent on the rate of mass accretion. Low mass accretion rates give rise to highly degenerate, widely spaced flashes (e.g., thousands of years), while high mass accretion rates yield either steady burning in the envelope or a constant buildup of unburned material. The behavior of hydrogen and helium burning in such systems has been studied over the past several decades [9, 16, 8, 2, 3, 5, 10, 15, 11, 6] using both analytic and numerical simulation techniques. In addition, it has been suggested that these supersoft X-ray sources may be the precursors of type Ia supernovae, whose origins remain uncertain.

Supersoft X-ray sources were discovered in the early 1980's and were later determined to form a distinct class of astronomical objects when ROSAT observations identified almost thirty such objects in our galaxy, M31, and the Magellanic clouds [12]. These sources have luminosities on the order of $\sim 10^4 - 10^5 \ L_\odot$ and temperatures of $\sim 10^5 - 10^6 \ K$, and produce photons of much lower energy than "conventional" X-ray sources [15]. A model by van den Heuvel, Bhattacharya, Rappaport and Nomoto [17] postulates that such luminosities and temperatures can be achieved in an accreting binary system with a white dwarf mass of approximately 0.8-1.3 $M_\odot$ and an accretion rate high enough (approximately $10^{-7} \ M_\odot/yr$) to produce steady hydrogen burning in the white dwarf envelope. One of the main aims of the present work was to achieve a better understanding of the long term behavior of this "stable" hydrogen burning and the associated helium burning.

The origin of type Ia supernovae has been a long standing mystery in astrophysics, but these objects are generally believed to result from the explosion of a white dwarf whose mass has grown to exceed the Chandrasekhar limit. This can occur if two white dwarfs coalesce and their combined mass is greater than this limit, or a white dwarf
can accrete matter from a donor star, and if matter is not ejected from the white
dwarf its mass can grow beyond the limit. However, binary systems with very low
accretion rates (< 10^{-9} \ M_\odot/yr) have been believed to experience very violent episodes
of hydrogen burning which eject matter from the white dwarf and lead to classical nova
explosions [16]. More recently it has been suggested that at higher accretion rates
such binary systems may give rise to type Ia supernovae. At accretion rates greater
than approximately 10^{-8} \ M_\odot/yr the accreted hydrogen is either burned steadily or
in non-violent, low-degeneracy flashes which are too weak to expel material from the
white dwarf. In such cases the helium produced from the burned hydrogen builds up
in the envelope of the dwarf. This helium will either experience steady burning on its
own, or undergo a cycle of runaway nuclear burning. If the helium or the products
of helium burning build up enough for the star to exceed the Chandrasekhar limit a
type Ia supernova may result. The other possibility is that the helium will increase
in mass until it burns explosively in a nuclear flash. Undoubtedly, such a flash would
expel matter from the star, resulting in a classical nova, and preventing the dwarf
from reaching the mass necessary to become a supernova. However, if the mass of
helium in the envelope is a significant fraction of the mass of the white dwarf, the
nuclear explosion in the envelope may be powerful enough to ignite the core as well,
resulting in a sub-Chandrasekhar type Ia supernova [4]. One of the major goals of
this research was to deepen our understanding of this helium layer and the associated
helium flashes, especially for white dwarf masses and accretion rates which give rise
to steady hydrogen burning.

Past research on accreting white dwarfs has included both analytic descriptions of
the nuclear burning in the envelope [2, 3] and numerical simulations of the envelope
and its episodes of runaway burning [11, 5, 8, 10, 6, 15]. The numerical models
vary widely in complexity. Some models, e.g. those of Nomoto and Iben [10, 6],
take a highly detailed look at a single, or at most a few flashes. These models
often contain hundreds of zones and account for highly detailed physical phenomena.
Others, notably Paczyński and José, et al. [11, 5], assume a highly simplistic model
of the burning zone in order to study the longer term behavior of the envelope. The
model used by Paczyński was a simple one-zone representation of the entire envelope. He studied only the behavior of bursting hydrogen or bursting helium alone. José, et al.'s improvement on this model added a second zone, so that hydrogen and helium burning could take place simultaneously. In both of these models, the zones were of fixed composition (either all hydrogen or all helium), and the products of the nuclear reactions were immediately removed to either the core or the second zone, in the case of hydrogen burning in José, et al.'s model. The goal of our present research is to develop a more sophisticated numerical hydrostatic model, which is still based on simple enough principles and algorithms so that long-term evolution can be studied without prohibitive amounts of computational time.

Several elements clearly need to be included in an improved long-term numerical model. First, the envelope should be modeled as a stack of a modest number of zones, not just one or two. Also, the zones should be allowed to have arbitrary chemical composition, and both helium burning and hydrogen burning should be allowed to take place in every zone. These steps would give one the ability to study the spatial distribution of the helium and hydrogen in the envelope, and would also show the spatial propagation of a shell flash (e.g. a hydrogen flash begins towards the middle of the envelope and very quickly propagates to the surface). The models by both Paczyński and José, et al. assumed plane-parallel geometry. Along with this, they assumed that the gravitational force on the material in the envelope did not change as the density of the material dropped and the radius increased during a flash or periods of steady burning. This pointed to another clear area for improvement. With several zones, the change in gravitational acceleration experienced by any zone could be simply calculated from the radial expansion of that zone away from the core. Since it has been shown that the thickness of the envelope can increase by up to ten times during a hydrogen flash [6], allowing gravity to vary with radius could have very significant effects.

To these ends, we have developed a hydrostatic model of the accretion envelope which implemented these major improvements and also incorporated several minor refinements over previous models. With this code we are able to divide the envelope
into any number of logarithmically spaced zones, though, in practice usually six are used. The composition of the matter accreted, the accretion rate, and the mass of the white dwarf are all variable. As accretion and nuclear burning take place in the envelope, the chemical compositions of the zones vary. Instead of arbitrarily removing all helium to a lowest zone, we allow the helium produced by hydrogen burning to naturally diffuse through the envelope by redefining the zone boundaries at every time step. Each zone is taken to hold a specific constant fraction of the total envelope mass; so as matter is accreted on the top of the envelope the helium produced by earlier burning is slowly moved to lower and lower zones. The exact mechanics of this "re-zoning" and of the rest of our model are laid out in Chapter 2.

Additionally, our model takes into account the $r^{-2}$ behavior of gravity as the zones swell away from the core; this is in contrast to the plane-parallel atmosphere used in some earlier work. This was most useful in studying the region of parameter space in which the accretion rate was close to the steady burning region. Using this feature we reproduced to well within an order of magnitude Iben's predictions about the behavior of the envelope during a burst and as the accretion rate moves through the steady burning region.

As additional tests of our model, we studied the long term behavior of hydrogen and helium bursting and compared these results to those of Iben, Paczyński, and José, et al.. Also, we explored the boundaries of the steady burning region and compared our results to those of Nomoto [10]. We found our model to agree very well with most previously published results. These results and comparisons are detailed in Chapter 3.

After confirming that our model was producing reasonable results compared to previous work, we moved on to explore new phenomena. In Chapter 4, we describe the long-term behavior of helium during steady hydrogen burning, and the effect of an oscillating mass accretion rate.
Chapter 2

Mechanics of the Model

2.1 Overview of the Computational Model

The main goal of this work was to develop a computational model for the nuclear burning occurring in the envelope of a white dwarf accreting matter from a companion star. As explained in the previous section, both stable and unstable nuclear burning are theoretically well known to occur in such systems. Our aim was to develop a code which could accurately model relatively short-time scale phenomena, such as hydrogen flashes, but which was fast enough to investigate longer term behavior. For instance, one of our goals was to study runaway helium burning, which occurs on time scales several hundred to several thousand times the duration between successive hydrogen flashes. To make such a study feasible demanded a model which could process that number of hydrogen bursts in a reasonable amount of computational time (on the order of one day).

Fundamentally, our code was a multi-zone, hydrostatic model of the envelope surrounding an accreting white dwarf. The model was generalized to deal with white dwarf masses of 0.8–1.3 M\(_\odot\), accretion rates of 10\(^{-11}\)–10\(^{-6}\) M\(_\odot\)/yr, and composition of accreted matter ranging from pure hydrogen to pure helium. The code we developed could calculate the evolution of an arbitrary number of shells, each shell processing hydrogen and helium via nuclear burning and removing all but a constant fraction of the heavier elements produced to the core of the dwarf. To strike a balance between
computational speed and detail in the results, we typically utilized six burning zones. The model allowed each of these zones to have an arbitrary composition (aside from the constant metallicity), and we considered, in each zone, hydrogen burning via the proton-proton chain and the carbon-nitrogen-oxygen cycle, and also helium burning via the triple-alpha reaction.

The boundaries separating the zones were redefined at each time step in the evolution, so that each zone contained a constant fraction of the total mass in the envelope. After attempting several methods of dividing up the total mass into discrete zones, we settled on zones whose mass increased exponentially towards the core of the star. Early on we experimented with several zones of equal mass, but this became unacceptable when we allowed all of the zones to have arbitrary compositions. After only one or two hydrogen shell flashes the uppermost zones in the envelope became so predominately helium that hydrogen burning was all but extinguished, which we deemed unphysical. Since the helium requires a large mass to ignite in a flash, it sits rather inactively for most of the evolution of the star. With several zones of equal mass, it is a very short time after the beginning of the evolution that enough helium has built up, as a product of hydrogen burning, in every zone that the mass fraction of hydrogen is driven to be very small, greatly reducing the energy generated by hydrogen burning. The assumption is that on an actual accreting white dwarf the newly accreted matter mostly lies on the top of the envelope, whereas evenly spaced zones automatically mix all new material with some sizable fraction of what is already in the envelope. One solution to the problem is to have such a large number of equal mass zones that even when enough mass has accumulated for a helium burst, the mass in the top zone is still very small. This would defeat the purpose of our code, though, since it would create such a computational burden that long duration runs would be unrealistic.

Our solution was to forego equally sized zones. Instead we chose zones whose masses decreased exponentially as they approached the surface of the envelope. In this scheme, the uppermost zones are always a small fraction of the total envelope mass, allowing for continued hydrogen bursting, while the lower zones become large
helium reservoirs. After some experimentation we discovered that for six zones a factor of about three in mass between zones yielded very nice results. That is, every zone contained three times as much mass as the zone physically on top of it.

Both the one-zone model of Paczyński [11] and the two-zone model of José [5] utilized parallel-plane approximations, in that they did not allow the gravitational acceleration to vary as the density of the zones decreased and they expanded from the surface of the white dwarf. Our model is capable of allowing gravity to vary with shell expansion. As shown by Iben [6], a hydrogen flash can result in the expansion of the envelope of the dwarf to several times the core radius, significantly decreasing the gravitational force on the outer zones. Under our scheme, fully variable gravity was only really feasible for relatively weak hydrogen flashes and for cases with steady burning. More violent flashes (close to or exceeding the Eddington luminosity) tended to expand the envelope to infinite radius, from which the code would never recover. For such cases the gravity was either fixed, or allowed to decrease only to a specified cutoff level after which it was fixed. However, in the case of such strong bursts we were concerned mainly with the behavior between flashes, and not the specifics of the flashes themselves. Violent flashes are surely hydrodynamic, and perhaps even expel matter from the star, so their detailed behavior was beyond the scope of this project.

The actual mechanics of the model are outlined in the following sections. In each zone several parameters were tracked through every step. Most importantly, the pressure, entropy, temperature, and density fit into the differential equations determining the time dependence of each zone. The energy produced by nuclear burning, the compositions, and the flux into and out of each zone were among the other important quantities followed in each step. From these parameters we also determined the electron degeneracy, the surface luminosity, and the effective surface temperature of the white dwarf.
2.2 The Stellar Structure Equations

The differential equations linking the pressure and entropy of each zone to the temperature and density were derived from the four stellar structure equations. The first of these equations simply relates the differential mass in a layer, $dM$, to the thickness, $dr$, and the density, $\rho$.

$$ \frac{dM}{dr} = 4\pi \rho r^2. \quad (2.1) $$

At every step in the evolution of our model, each shell is in hydrostatic equilibrium. That is, the gravitational force per unit volume is equal to the pressure gradient on the shell. If $g(r)$ is the gravitational acceleration at the radius of the shell,

$$ \frac{dP}{dr} = -g(r)\rho. \quad (2.2a) $$

This can also be written in Lagrangian form with the use of equation 2.1.

$$ \frac{dP}{dM} = -\frac{g(r)}{4\pi r^2}. \quad (2.2b) $$

In our model, the value for $g$ can either be held fixed or allowed to vary as a function of $r$. Fixing $g$ effectively forces the zones to be spatially thin compared to the radius of the white dwarf, while fully variable gravity allows unlimited expansion of the zones, which, as outlined above, is usually undesirable during violent bursting episodes. The third stellar structure equation is a statement of the conservation of energy, relating the net luminosity flowing out of a given shell to the difference of the total nuclear energy produced and the heat added to the zone. Writing the specific entropy as $s$, the temperature as $T$, and the energy per unit time per unit mass due to all of the nuclear reactions as $\epsilon$, we have,

$$ \frac{dL}{dr} = 4\pi \rho r^2 \left( \epsilon - T \frac{ds}{dt} \right). \quad (2.3a) $$
Written in Lagrangian form, this becomes

\[ \frac{dL}{dM} = \epsilon - T \frac{ds}{dt}. \]  \hspace{1cm} (2.3b)

Given that the mean free path for a photon within a star is quite small, the radiative transport of energy throughout the envelope can be treated in the diffusion approximation. The solution to this diffusion problem yields a linear relationship between the energy transported through a shell and the temperature gradient over the thickness of that shell. The solution is given by Kippenhahn [7], and reduces to

\[ \frac{dT}{dr} = \frac{3\kappa \rho L(r)}{64\pi \sigma T^3 r^2}, \]  \hspace{1cm} (2.4a)

or in Lagrangian form

\[ \frac{dT}{dM} = \frac{3\kappa L(r)}{256\pi^2 \sigma T^3 r^4}, \]  \hspace{1cm} (2.4b)

where \( \kappa \) is the radiative opacity of the material in that shell, and \( \sigma \) is Boltzmann's constant.

In the absence of mass accretion, the only explicit time dependence in the four stellar structure equations is the \( dS/dt \) term in equation 2.3a. Due to the mass accretion, the pressure also varies with time. It is convenient to express the total mass of a shell as a column thickness,

\[ d\Sigma = \frac{dM}{4\pi r^2} = \rho dr. \]  \hspace{1cm} (2.5)

Rewriting equation 2.2b,

\[ dP = -\frac{g(r)}{4\pi r^2} dM(r). \]  \hspace{1cm} (2.6)

Integrating this equation outward, through all the overlying mass, gives

\[ P = -\int \frac{g(r)}{4\pi r^2} dM(r). \]  \hspace{1cm} (2.7)

Since the mass is being accreted on the top of and removed throughout the envelope,
the time dependence of $P$ is

$$
\frac{dP}{dt} = - \frac{d}{dt} \left( \int \frac{g(r)}{4\pi r^2} dM(r) \right),
$$

(2.8)

or, using our definition for $\Sigma$,

$$
\frac{dP}{dt} = - \frac{d}{dt} \left( \int g(r) \rho \frac{d\Sigma(r)}{\rho} \right).
$$

(2.9)

This expresses the local time dependence of $P$ as a function of the change in column thickness and gravity for the entire envelope above the point of interest.

Now, we combine the remaining three equations to derive the time dependence of entropy. Equation 2.3b can be rewritten as

$$
\frac{dS}{dt} = \frac{\epsilon}{T} - \frac{1}{T} \left( \frac{dL}{dM} \right).
$$

(2.10)

From equation 2.4b, $dL/dM$ in turn can be rewritten as

$$
\frac{dL}{dM} = \frac{256\pi^2 \sigma}{3} \frac{dT}{dM} = \frac{16\sigma}{3\rho^2} \frac{d\Sigma}{d\rho} = \frac{16\sigma}{3r^2} \frac{dT}{d\Sigma}.
$$

(2.11)

Since $d\rho/d\Sigma = 1/\rho$, the entropy equation becomes

$$
\frac{dS}{dt} = \frac{\epsilon}{T} - \frac{16\sigma}{3T} \left[ \frac{d\kappa}{d\Sigma} \left( \frac{T^3 dT}{\kappa} \right) + \frac{3T^2}{\kappa} \left( \frac{dT}{d\Sigma} \right)^2 + \frac{T^3 d^2 T}{\kappa d\Sigma^2} + \frac{2T^3 dT}{r \rho \kappa d\Sigma} \right].
$$

(2.12)

Notice that since $L = 4\pi r^2 F$,

$$
\frac{dL}{dM} = 4\pi r^2 \frac{dF}{dM} + 8\pi r F \frac{dr}{dM} = \frac{dF}{d\Sigma} + \frac{2F dr}{r d\Sigma} = \frac{dF}{d\Sigma} + \frac{2F}{r \rho}.
$$

(2.13)

So, equation 2.12 can also be written in terms of the flux as

$$
\frac{dS}{dt} = \frac{1}{T} \left( \epsilon - \frac{dF}{d\Sigma} - \frac{2F}{r \rho} \right).
$$

(2.14)

Since the compositions of the zones are allowed to be arbitrary, the opacity can
change from zone to zone, hence I included the \( \frac{d\kappa}{d\xi} \) term in equation 2.12. Paczyński gives the radiative opacity in terms of the hydrogen and carbon fractions (\( X \) and \( Z \) respectively) [11]:

\[
\kappa = 0.2(1 + X) \left[ 1 + 2 \times 10^{26}(0.001 + Z) \frac{\rho}{T^{3.5}} \right].
\] (2.15)

This formula is a simple sum of the electron scattering opacity and the Kramers opacity for free-free transitions [7].

### 2.3 Numerical Integration and Considerations for Discrete Zones

The differential equations derived in the previous section are valid for radially infinitesimal shells of the star’s envelope. In our model, the mass thicknesses of the zones, especially near the core, can actually become rather large. We still use the same differential equations, but this does introduce some subtlety in calculating the derivatives of temperature and opacity with respect to column thickness that appear in equation 2.12.

In our calculation, the temperatures, pressures, densities and entropies are defined at the center of each zone. In order to get an approximate value for the first and second derivatives of temperature with respect to the column thickness, the differences in thicknesses of adjacent zones must be taken into account. For example, if we wish to find \( \frac{dT}{d\Sigma^2} \) and \( \frac{dT}{d\Sigma} \) in a zone with temperature \( T_2 \) and thickness \( \Sigma_2 \), we must also know the temperatures and thicknesses of the adjoining zones. If the zone physically on top of our zone has temperature and thickness \( T_1 \) and \( \Sigma_1 \), and the one immediately below has \( T_3 \) and \( \Sigma_3 \), then the second derivative is approximated by

\[
\frac{dT}{d\Sigma^2} = \frac{2}{\Sigma_2} \left( \frac{T_1 - T_2}{\Sigma_1 + \Sigma_2} - \frac{T_2 - T_3}{\Sigma_2 + \Sigma_3} \right).
\] (2.16)
Immediately, this gives the first derivative.

\[
\frac{dT}{d\Sigma} = \frac{2(T_2 - T_3)}{\Sigma_2 + \Sigma_3} + \frac{d^2 T \Sigma_2}{d\Sigma^2} \frac{T_1 - T_2}{\Sigma_1 + \Sigma_2} + \frac{T_2 - T_3}{\Sigma_2 + \Sigma_3}.
\]  \hspace{1cm} (2.17)

An exactly analogous treatment for opacity yields \( \frac{d\delta}{d\Sigma} \).

Given these expressions for the spatial derivatives of temperature and opacity, and equation 2.12, the time derivative of entropy is determined in each zone based only on the current values of temperature, opacity, density, and mass accretion rate.

In order to accurately determine the time derivative of pressure at the center of each zone the changes in mass in all overlying zones must also be taken into account. To calculate this derivative, the integral in equation 2.9 can be converted into a discrete summation. Since the derivative is sought in the middle of a zone, the summation will include all overlying zones, plus one half of the zone in question. Taking the gravitational acceleration in any zone to be time independent, the discrete form of equation 2.9 gives the pressure in the \( n \)th zone from the top to be

\[
\frac{dP_n}{dt} = \sum_{i=1}^{n-1} \left( g(r_i)\dot{\Sigma}_i \right) + \frac{g(r_n)\dot{\Sigma}_n}{2}
\]  \hspace{1cm} (2.18)

These differential equations are solved by a fourth-order Runge-Kutta numerical integration procedure which calculates the pressure and entropy change through the small time step. The Runge-Kutta procedure used in this model was a highly modified version of the one given by Press [18]. This method requires the time derivatives of pressure and entropy to be calculated four times for each time step, once at each end point, and twice at mid-points. At each of these trial points, nuclear burning rates, opacities, mean molecular weights, and polytropic constants had to be calculated for the test values of pressure and entropy. The integration was performed simultaneously on all the zones, resulting in very good estimates for the evolved values of pressure and entropy.
2.4  Progress of One Step

The model was initiated by specifying an accretion rate for the white dwarf, the mass of the dwarf, and some initial temperature, pressure and density for each zone. The evolution proceeded using an adaptive time step algorithm, which assured that none of the properties of a zone would change by more than a few percent in one step. The time step was based on the changes in that occurred in the previous increment. For each increment in time, several sub-steps were executed to properly compute the changes in each zone. The evolution through a single time increment proceeded in the following manner:

1. The initial temperatures and entropies of all the zones are calculated from the specified initial densities and masses of each zone via a Newton’s method procedure solving the equation of state of the envelope.

2. The time step size is calculated, and new pressures and entropies are calculated for this small change in time. The time dependence of the pressure and entropy are given by the two differential equations (2.9, 2.12) containing the temperature, opacity, derivatives of these with respect to column thickness, and mass accretion rate. Given these differential equations the pressure and entropy are evolved through the time step using a fourth-order Runge-Kutta numerical integrator. In short, this step links the temporal behavior of two of our parameters (pressure and entropy) to the spatial distribution of the other two (temperature and density).

3. The new pressures and entropies having been determined by the numerical integration, corresponding temperatures and densities must be determined by another Newton’s method (from the equation of state). After this step the entropy, density, pressure and temperature are fully self-consistent, but the masses of the zones are not distributed properly, according to the exponential spacing mentioned earlier.
4. Before redistributing the mass among the zones, the changes in composition of each zone must be determined. This change in composition is due to two effects: (i) mass accretion on the top zone, and (ii) nuclear burning in all the zones. Also calculated is the amount of mass that was lost from each zone due to burning of helium. During each time step the metallicity of each zone is held fixed (usually at around 2%), and any carbon produced above this percentage is removed to the white dwarf core.

5. Given that the masses and compositions of all the zones are now determined, the next step is to redefine the zone boundaries, or “re-zone”, so that each zone has the desired fraction of the total envelope mass. This re-zoning is carried out so that each “new” zone is composed of portions of the originally spaced zones. In this way, the compositions and densities of the old zones are used to determine those of the new zones.

6. Since the time step was small, the nuclear burning and the mass accretion were correspondingly small. This means that the change in mass of each zone due to re-zoning will also be small. The density of each zone is known from the re-zoning, and the pressure of each comes immediately from the new masses. Given these, yet another Newton’s method is used to determine the necessary small changes in temperature and entropy arising from the re-zoning procedure.

7. This completes one step in the evolution. The four evolutionary parameters \((P, \rho, T, S)\) are once again entirely consistent, the zones contain the proper relative masses, and we are ready to begin the next time step with another Runge-Kutta integration. At the end of each evolutionary step the other quantities of interest are calculated. These include: the surface temperature and luminosity of the star, the electron degeneracy of every zone, and the rate of buildup of hydrogen in the envelope.
2.5 Nuclear Burning

For this model we considered hydrogen and helium burning in every zone at each step in the evolution. For most of the evolution of a typical low-accretion rate model the helium burning was negligible, but periods of intense helium burning, or "helium flashes", were observed and studied.

Our model included hydrogen burning through the proton-proton, and CNO chains, as well as helium burning through the triple-alpha reaction. Since we were only interested in the amount of energy produced and the amount of material burned by each reaction, it was unnecessary to track every intermediate species. Instead, we used the total energy generation rates for each reaction.

Clayton gives the total energy-generation rate of the PP chain to first order to be [1]:

$$\epsilon_{\text{PP}} = (2.36 \times 10^6)\rho X^2 T_6^{-2/3} \exp(-33.81 T_6^{-1/3}) \text{ erg g}^{-1} \text{ sec}^{-1}, \quad (2.19)$$

and the energy-generation rate of the CNO cycle to be:

$$\epsilon_{\text{CNO}} = (8 \times 10^{27})\rho X X_{CN} T_6^{-2/3} \exp(-152.31 T_6^{-1/3}) \text{ erg g}^{-1} \text{ sec}^{-1}. \quad (2.20)$$

In these equations $T_6 = T/(10^6 \text{ K})$, and $X_{CN} = Z/3$, where $Z$ is the mass fraction of all heavy elements with atomic weight greater than 6 ("metals").

The helium burning obeys a similar expression, also given by Clayton:

$$\epsilon_{3\alpha} = (3.9 \times 10^{11})\rho^2 Y^3 T_8^{-3} \exp(-42.94 T_8^{-1}) \text{ erg g}^{-1} \text{ sec}^{-1}, \quad (2.21)$$

where, $Y$ is the mass fraction of helium and $T_8$ is defined analogously to $T_6$.

These expressions relate some qualitative and semi-quantitative facts about the burning occurring the dwarf's envelope. Both hydrogen reactions vary identically density. They differ in the overall multiplicative factor, and also in the argument to the exponential. As the temperature increases the term in the exponential in both equations becomes less negative, and so the whole exponential grows. This exponential increase is far stronger than the decrease from the $T_6^{-2/3}$ in front. So,
exponentially stronger hydrogen burning should accompany rises in temperature if the density and mass fraction do not vary wildly. In fact, we often see hydrogen flashes accompany an order magnitude rise in the temperature of a zone. Also, as the temperature increases the energy generation via the CNO chain increases far more rapidly than the energy generation via the PP chain. In fact, as the temperature changes from $1 \times 10^7$ K to $2 \times 10^7$ K the ratio of the energy generation rates from these two reactions changes by almost three orders of magnitude. This ratio is plotted over three decades of temperature in Figure 2-1.

## 2.6 Two-Dimensional Newton’s Method

During every time step in the evolution it is necessary to have relations connecting temperature, density, pressure, and entropy. Given any two of these we can calculate, either by direct solution of the equations or through a two-dimensional Newton’s method, all four of these parameters.

Paczyński gives approximate expressions for the total pressure and the internal energy which can be used to deduce the desired relationships [11]. The pressure is composed of three terms. These are the ideal gas pressure, the pressure of completely degenerate (but non-relativistic) electrons, and the radiation pressure.

$$ P = \frac{k}{\mu H} \rho T + K \rho^{5/3} + \frac{a}{3} T^4. \quad (2.22) $$

The first term in this equation is valid only in the non-degenerate region, and the second is valid for the highly degenerate case. The simple sum of these two gives correct results in the cases of extreme degeneracy or non-degeneracy, but is only approximately correct for the partially degenerate case. Similarly, the specific internal energy is given by

$$ u = \frac{3kT}{2\mu H} + \frac{3K \rho^{2/3}}{2} + \frac{aT^4}{\rho}. \quad (2.23) $$

In these equations $k$ is Boltzmann’s constant, $H$ is the atomic mass unit ($1.67 \times 10^{-24}$
Figure 2-1: This demonstrates the strong dependence of the burning rates on temperature. Notice that at temperatures lower than $\sim 2 \times 10^7$ the PP chain dominates the hydrogen burning, while the CNO burning is much greater at higher temperatures. For this plot the mass fraction of hydrogen and metallicity were taken to be 0.98 and 0.02, respectively.
g), and the mean molecular weight is given by [1]:

\[
\mu = \frac{2}{1 + 3X + 0.5Y}
\]  

(2.24)

The constant, \( K \), is dependent on the chemical composition:

\[
K = 3.12 \times 10^{12}(1 + X)^{5/3}.
\]  

(2.25)

From the equation for internal energy we can extract a formula for entropy. From thermodynamics,

\[
Tds = du + Pd(1/\rho) = \frac{\partial u}{\partial \rho} d\rho + \frac{\partial u}{\partial T} dT + \frac{P}{\rho^2} d\rho.
\]  

(2.26)

So, taking the appropriate derivatives of \( u \), we find

\[
Tds = \left( -\frac{aT^4}{\rho^2} - \frac{aH^4}{3\rho^2} - \frac{kT}{\rho \mu H} \right) d\rho + \left( \frac{4aT^3}{\rho} + \frac{3k}{2\mu H} \right) dT.
\]  

(2.27)

This gives the density and temperature derivatives of the specific entropy.

\[
\frac{ds}{d\rho} = -\frac{4aT^3}{3\rho^2} - \frac{k}{\rho \mu H},
\]  

(2.28a)

\[
\frac{ds}{dT} = \frac{4aT^2}{\rho} + \frac{3k}{2\mu H T}.
\]  

(2.28b)

Integrating these equations gives

\[
 s = \frac{4aT^3}{3\rho} - \frac{k}{\mu H} \ln(\rho) + f(T),
\]  

(2.29a)

\[
 s = \frac{4aT^3}{3\rho} + \frac{3k}{2\mu H} \ln(T) + g(\rho),
\]  

(2.29b)

where \( f \) and \( g \) are functions of only \( T \) and \( \rho \), respectively. Combining these two equations yields our equation for entropy:

\[
 s = \frac{4aT^3}{3\rho} + \frac{k}{\mu H} \ln \frac{T^{5/2}}{\rho}.
\]  

(2.30)
It is difficult to invert equations 2.22 and 2.30 into direct expressions for the temperature and density as functions of entropy and pressure. Therefore, these equations can only be directly applied in cases when the pressure and entropy are to be found from the temperature and density. If any other pair of these four parameters is sought, a two-dimensional Newton's method of these equations must be used. For example after the Runge-Kutta integration the new pressures and entropies of all the zones are known, and the corresponding temperatures and densities must be determined with the Newton's method. The actual computation involved in performing such a two-dimensional Newton's method is detailed in the appendix.

2.7 Re-Zoning

The procedure used to redefine the zone boundaries was a straightforward redistribution of mass from the old zones to new zones of proper mass. The goal of this process is to determine the density and compositions of the new zones, so that the entropies and temperatures can be properly determined (the pressures are automatically fixed by the zone masses via equation pressum).

After the forward integration in time, the pressures of the zones are known. Working from the top zone downward, equation 2.18 gives the masses of the zones. As the first step in re-zoning, the new zone masses are calculated with respect to the new total envelope mass based on our logarithmic zoning algorithm. The recombination of the old zones necessary to create these new zones is calculated via the masses in each. This results in new zones which are composed of whole and fractional pieces of the old zones. Knowing the densities and masses of the old zones allows the total densities of the new zones to be calculated. Furthermore, the chemical compositions of all the old zones were determined during the integration step in which the burning rates in each zone were calculated. Using the mass fractions of hydrogen and helium of the old zones, the mass fractions of the new zones are computed. The change in pressure over one integration step is very small, so, in practice, one new zone is usually composed of an old zone, or a large fraction of one, and some tiny portions of
the underlying or overlying zones. The details of the computational procedure used to carry out this rezoning are explained in the appendix.

2.8 Lower Boundary Condition

Of great importance in the calculation is the boundary condition connecting the core of the star to the lowest most zone of accreted matter. Both Paczyński and José, et al. [11, 5] used a zero-flux condition, in which it was assumed that the bottom zone exchanged no heat with the core. In effect, this boundary condition automatically forces the core and the bottom zone to have the same temperature and opacity.

The other possible lower boundary condition is to have the core fixed at a constant temperature and opacity. In this case the core would act as a heat sink or a heat source, depending on whether its temperature is greater than or less than the temperature of the bottom zone.

At first glance, the constant temperature boundary condition seems to be the more physically accurate. Surely, in a white dwarf, the temperature of the core does not fluctuate wildly with the temperature of the atmosphere. However, this boundary condition does raise its own problems. For instance, during periods of strong nuclear burning in the lower zones, the temperature gradient between the core and the bottom zone can be quite large. This results in a very large flux into the core.

The truth may lie somewhere between the two possibilities. For the purposes of this model the choice of boundary conditions did not make a large difference when the burning shell had settled into steady burning or regular flashing. However, the zero flux boundary condition was less problematic and more well behaved, especially when the zones had not yet come into equilibrium immediately after the beginning of an evolution. All of the figures presented later in this thesis were made with the zero-flux condition, unless explicitly noted.
2.9 Other Useful Quantities

In addition to the four evolutionary parameters, there were many instructive quantities we calculated throughout the course of an evolution. Among these were the effective surface temperature of the white dwarf and its luminosity.

Since the temperature of the uppermost zone is calculated at the midpoint of the zone (with respect to mass) the flux out of this zone, and thus out of the star, can be calculated as a function of the opacity, temperature and column thickness of the top zone. The effective surface temperature is defined as the temperature corresponding to a black body of temperature $T_{*}$ which radiates the same luminosity as our star. Thus,

$$T_{\text{eff}}^4 = \frac{L}{4\pi r^2 \sigma} = \frac{F}{\sigma}$$

(2.31)

where $F$ is the flux out of the surface. Rewriting equation 2.4a, we have

$$\frac{dT}{d\Sigma} = \frac{dT}{\rho dr} = \frac{3\kappa L}{64\pi \sigma T^3 r^2} = \frac{3\kappa F}{16\sigma T^3}$$

(2.32)

or,

$$\frac{d(T^4)}{d\Sigma} = \frac{3\kappa F}{4\sigma}.$$  

(2.33)

The thickness of the top zone is always small (due to the exponential zone spacing), and the temperature outside of the star is taken to be zero. Then, the change in temperature from the middle of the top zone to outside the top zone is simply the temperature at the middle of the zone, $T_{\text{top}}$, and the relevant change in column thickness is half of the thickness of the top zone, $\Sigma_{\text{top}}/2$. So,

$$\frac{T_{\text{top}}^4}{\Sigma_{\text{top}}} = \frac{3\kappa F}{8\sigma}.$$  

(2.34)

Applying this to the above equation, we get an expression for the surface temperature in terms only of the temperature, opacity, and thickness of the top zone.

$$T_{\text{eff}} = T_{\text{top}} \left( \frac{8}{3\kappa \Sigma_{\text{top}}} \right)^{1/4}.$$  

(2.35)
In this expression the term multiplying \( T_{\text{top}} \) turns out to be the optical depth of the upper half of the top zone. Given the radius of the top zone, the luminosity of the star follows immediately.

\[
L = 4\pi r^2 \sigma T_{\text{eff}}^4
\]  \hspace{1cm} (2.36)

Another parameter we calculated for every zone was the degeneracy, \( \delta \). One very useful measure of the degeneracy a gas of fermions is the Fermi integral \( \tilde{F}_{1/2} \). The equation of state for a partially degenerate, non-relativistic gas of electrons can be written exactly as [13]

\[
\tilde{F}_{1/2} = \frac{\hbar^3}{4\pi \mu \hbar} \left( \frac{1}{2m_e k} \right)^{3/2} \frac{\rho}{T^{3/2}},
\]  \hspace{1cm} (2.37)

where \( \hbar \) is the atomic mass unit, \( m_e \) is the electron mass, and \( \mu \) is the mean molecular weight. Inserting the values for all the constants, and defining this dimensionless number as our degeneracy

\[
\delta \approx 0.1105 \frac{\rho}{\mu T^{3/2}}.
\]  \hspace{1cm} (2.38)

For \( \delta \ll 1 \), the matter is highly non-degenerate and behaves like an ideal gas, whereas, for \( \delta > 1 \) the gas is highly degenerate.
Chapter 3

Tests of the Model

In order to determine whether our model was producing believable results we tested it against several previously published studies. In particular, we compared the hydrogen and helium flashes produced by our model to those found by Paczyński's one-zone model [11], José, et al. two-zone model [5], and the various models used by Iben [6]. Also we allowed the mass accretion rate to climb through the steady burning region ($\dot{M} \gtrsim 8 \times 10^{-8} M_\odot/yr$), in order to determine the relationships among white dwarf mass, effective surface temperature, and luminosity. These results were compared to similar studies by Iben. In addition, we used our model to determine the lower and upper limits of the steady hydrogen burning region, and compared this to other results.

3.1 Recurrence and Shape of Flashes

One of the most important and most obvious characteristics of envelope burning on a white dwarf is the cycle of hydrogen flashes that occurs at low enough mass accretion rates. This section explores how this flashing cycle depends on mass accretion rate and white dwarf mass. Figure 3-1 is an illustrative example of the basic properties of hydrogen flashes on a white dwarf accreting pure hydrogen. In this figure, gravity in each zone is taken to be constant by fixing the radius of each zone to be the radius of the white dwarf. Since in this case there is insignificant helium burning the mass
of the zones is growing steadily. If there were much helium burning, carbon would be transferred to the core, and the total envelope mass would not increase at the mass accretion rate.

The following figure displays eight useful parameters calculated for the evolution of the envelope. \( L \) and \( T_{\text{eff}} \) are the surface luminosity and effective temperature, measured in \( L_\odot \) and \( K \), respectively. The other six panels display the quantities measured in each of the six zones used in this run. \( T \) is the temperature at the middle of each zone in \( K \), \( \rho \) is the density in \( \text{g/cm}^3 \), and \( M \) is the mass of each zone in units of \( M_\odot \). The mass of each zone is three times larger than the zone on top of it, as required by our re-zoning procedure. The radii (measured in cm) of all the zones are shown to be the same in this case, since the gravity was held fixed for this run. Additionally, the dimensionless parameters \( \delta \), degeneracy, and \( X \), mass fraction of hydrogen, are displayed.

For a large fraction of the cycle, the envelope is in a relatively quiescent non-burning phase. Surface luminosity is low, approximately 10 \( L_\odot \), and the effective temperature is only just above \( 10^5 \) \( K \). Since the mass is re-zoned at every time step, and since the mass of the outermost zones is so small, they are almost always composed entirely of hydrogen. As the hydrogen accretes, though, the fraction of hydrogen, \( X \), in the middle two zones increases. By this point in the evolution of the white dwarf the lowest two zones have basically become large helium reservoirs. Eventually, enough pressure and temperature has built up in enough hydrogen to ignite hydrogen burning. The burning is intense and short lived. In approximately one year the entirety of the hydrogen accreted since the last flash burns to He. The hydrogen fraction in all but the top two zones drops precipitously and the luminosity increases by up to three orders of magnitude. During the actual flash the density of all but the bottommost zone drops and the temperatures of these zones spike up to high values. The majority of hydrogen burning actually occurs in the four outermost zones, but the flux inward also causes small spikes in the density and temperature of the fifth zone, and even causes small dips in the innermost zone. Notice that the electron degeneracy in each zone actually reaches a maximum before the flash commences. The
small jitter in the luminosity and surface temperature results from minor numerical instabilities due to the small size of the outer zone.

Both Paczyński and José, et al. found that hydrogen flashes become more intense and more closely spaced if the accretion rate is held constant and the mass of the white dwarf is increased. Our model also shows this behavior. Figure 3-2 shows hydrogen shell flashes for the same parameters used in Figure 3-1 but with a white dwarf mass of 1.2 $\text{M}_\odot$ — note the difference in the time scale between the two plots. The period between flashes on the 1.2 $\text{M}_\odot$ is reduced to $\sim 650$ years while on the 1.0 $\text{M}_\odot$ the recurrence period is $\sim 2050$ years. Also, the electron degeneracy of the zones during every flash achieves a higher maximum value, implying that the flashes on the heavier core are likely to be more intense as well as more hydrodynamic at the peaks of their outburst.

Furthermore, Paczyński’s and José, et al.’s models predict that an increase in accretion rate leads to a decrease in the time between flashes, and the flashes become less intense. That is, the amount of hydrogen burned in a dwarf with a higher accretion rate is lower than the mass burned on a lower accretion rate dwarf. Therefore, if the accretion rate is increased by a factor of ten, the time between bursts should decrease by more than a factor of ten. The hydrogen will be added ten times faster, but not as much is needed to produce a flash with the low accretion rate model. Figures 3-1, 3-3, and 3-4 clearly demonstrate this trend. All of these runs were made with a 1.0 $\text{M}_\odot$ white dwarf, and the accretion rates were increased from $10^{-9}$ $\text{M}_\odot$/yr to $10^{-8}$ $\text{M}_\odot$/yr to $5 \times 10^{-8}$ $\text{M}_\odot$/yr. These graphs were all generated for segments of time over which the flashing behavior had become more or less regular, and the anomalous behavior due to the initial conditions had died away. For the case of $10^{-9}$ $\text{M}_\odot$/yr accretion rate the period between successive flashes averages out to approximately 2050 years. Increasing the accretion rate by a factor of ten shortens this period to approximately 125 years. Note that in this case more nuclear burning occurs during the non-flashing part of the cycle. This accounts for the higher temperature in each zone, and the more than ten times increase in the non-flashing luminosity. The accretion rate is increased by another factor of two in Figure 3-4, dropping the recurrence period to
Figure 3-1: Several hydrogen shell flashes for a 1.0 $M_\odot$ white dwarf with an accretion rate of $1 \times 10^{-9} M_\odot/\text{yr}$ of pure hydrogen. In this model gravity and radius of the zones are held fixed.
Figure 3-2: Same as figure 3-2 but with a white dwarf mass of 1.2 $M_{\odot}$
about 16 years, almost an eightfold change. The non-flashing luminosity has increased to approximately 3000 ergs/s, while the flashing luminosity has remained nearly the same at about $3 \times 10^4$ ergs/s.

Paczyński and José, et al. did not explore the consequences of variable gravity with their hydrostatic models. However, Iben, Sion, and Starrfield have produced models that take the expansion of the shell, and the corresponding change in gravitational acceleration, into account [6, 14, 15]. With this model he found the effective temperature to experience two spikes during one burst cycle. In the initial phase of the burst cycle the luminosity and temperature of the envelope increase rapidly without a significant increase in shell thickness. The shell then expands greatly, reducing the temperature and holding the luminosity more or less constant. As the runaway nuclear burning dies away the radius contracts, increasing the temperature for a second period, but of longer duration. The white dwarf then settles back into its quiescent state, with temperature and luminosity assuming their low resting values. By allowing the gravitational acceleration to vary we were able to reproduce this phenomenon. Figure 3-5 shows two flashes occurring on a 1.0 $M_\odot$ white dwarf accreting at a rate of $2 \times 10^{-8} M_\odot$/yr. Notice that during a flash the luminosity plateaus at a high value, while the surface temperature has two distinct spikes. During the flash the outermost zone is pushed to a radius of approximately 3.5 times the radius of the core. This expansion forces the density in each zone to become much smaller than it ever becomes with non-variable gravity.

The changes in temperature and luminosity during the burst cycle are shown in more detail in a HR diagram in figure 3-6. This crescent shaped cycle was also reported by Iben. The same type of calculation was repeated for the case when the accretion rate is a factor of two higher ($4 \times 10^{-8} M_\odot$/yr), and the results are shown in figures 3-7 and 3-8. The flashes are much more closely spaced and less violent. The shell only expands to twice the radius of the white dwarf, and the burst fits onto a much smaller portion of the luminosity-temperature plane.

Figures 3-1, 3-3, 3-4, 3-5, and 3-7 show that as the mass accretion rate onto the white dwarf gets lower and lower the degeneracy of the flashes gets larger and
Figure 3-3: Same as figure 3-1 but with an accretion rate of $1 \times 10^{-8} \, M_\odot/yr$. 
Figure 3-4: Same as figure 3-1 but with an accretion rate of $5 \times 10^{-8} \, \text{M}_\odot/\text{yr}$. 
larger. At low enough accretion rates these flashes would correspond to classical nova explosions. These objects have been long studied and eject their envelopes due to the violent hydrodynamic nature of the explosion.

3.2 Steady Burning Region

We also used our model to explore the region of parameter space in which the hydrogen in the envelope is steadily burning, instead of flashing. For accretion rates higher than \( \sim 5 \times 10^{-8} \), the exact values of which depend on \( M_{\text{WD}} \), the envelope will not experience episodes of runaway hydrogen burning, but will burn the hydrogen as fast as it is accreted. This results in a perpetually high surface luminosity.

Figure 3-9 demonstrates the transition from flashing behavior to steady burning. This model had variable gravity, and an accretion rate very near the lower boundary for steady burning. The initial conditions used to begin the run created a transient bursting episode. These bursts were so close together that the time spent in flashing was roughly equal to the time spent not flashing. As the envelope settled into equilibrium this flashing died away and the accreted matter began to burn steadily. The luminosity and surface temperature came to rest at the fairly large values of \( 2.6 \times 10^4 \, L_\odot \) and \( 7.5 \times 10^5 \, \text{K} \).

In addition we explored the dependences of temperature and luminosity in the steady burning region on the accretion rate and the white dwarf mass. By beginning a model with an accretion rate very close to the steady burning limit and gradually increasing the accretion rate in small steps we generated figure 3-10. The tracks in the HR diagram are for three runs with different white dwarf masses. As the accretion rate increased the temperature and luminosity increased roughly along a curve of constant radius before turning over to a curve corresponding to significant radial expansion. As expected, this expansion was accompanied by a slight decrease in luminosity as well as a strong decrease in temperature. At accretion rates beyond the curves shown we began to encounter significant helium burning and even helium flashes. Iben used a steady burning model to generate larger pieces of these same
Figure 3-5: Hydrogen shell flashes for a 1.0 M$_\odot$ white dwarf with an accretion rate of $2 \times 10^{-8}$ M$_\odot$/yr. In this figure the radius and gravity are allowed to vary freely.
Figure 3-6: The relationship between the surface temperature and luminosity through one flash in figure 3-5. Between points represented by a cross there is a time increment of 1/20 year and between points shown with a triangle there is a time increment of 1/200 year. The dashed lines are curves of constant radius, with radius increasing by a factor of 2 between successive lines.
Figure 3-7: Same as figure 3-5, but with accretion rate of $4 \times 10^{-8} \, M_\odot/yr$. 
Figure 3-8: The relationship between the surface temperature and luminosity through one flash in figure 3-7. Between points represented by a cross there is a time increment of 1/20 year and between points shown with a triangle there is a time increment of 1/200 year. The dashed lines are curves of constant radius, with radius increasing by a factor of 1.6 between successive lines.
Figure 3-9: The transition from flashing to steady burning. The accretion rate is $8 \times 10^{-8} \, M_\odot/\text{yr}$ and the white dwarf mass is $1.1 \, M_\odot$. 
curves, and our results match his findings fairly well for accretion rates just above the steady burning limit.
Figure 3-10: The relationship between surface temperature and luminosity for three different white dwarf masses as the accretion rate travels up through the steady burning region. The lowest and leftmost point of each of the three paths is just above the transition from flashing to steady burning. The dashed lines are constant radius curves, with a factor of 1.5 increase in radius between each line.
Chapter 4

New Results

4.1 Oscillating Mass Accretion Rate

We studied the effects on the burning envelope of a temporally oscillating mass accretion rate. Such an oscillating accretion rate could conceivably arise from either an eccentric binary system, or it may simply simulate erratic variations of a characteristic time scale. Observationally, the mass transfer rates in binary systems are known to vary markedly over a whole range of different time scales. Our attempt was to determine the response of the white dwarf envelope at different oscillation frequencies in both the steady burning and flashing regions.

In order to explore the effects of an oscillating mass accretion rate, we used allowed the logarithm of the accretion rate to vary sinusoidally. This admittedly arbitrary scheme forced $\dot{M}$ to increase or decrease by a multiplicative factor of $\alpha$ over a period of $\tau$ years, the mass accretion rate at any time was

$$\dot{M} = \dot{m}_0 e^{\ln(\alpha)\sin(2\pi t/\tau)}.$$  (4.1)

If $t$ is equal to an integral number of periods, $\dot{M}$ is equal to $\alpha\dot{M}_0$, and if $t$ is an odd number of half periods $\dot{M}$ is $(1/\alpha)\dot{M}_0$.

We studied oscillation period ranging from 1-200 years for a variety of intensity factors, $\alpha$. Most interesting to us was the question of whether the luminosity and
temperature of the shell would respond appreciably at the period of oscillation and also whether the oscillating accretion rate would induce variations in the shell behavior at a much different frequency (e.g., would short time scale accretion rate fluctuations cause long time scale changes in the envelope, and vice versa).

Longer oscillation periods become especially interesting when the maximum and minimum values of the accretion rate lie on different sides of the steady burning line. Figure 4-1 shows the behavior of a $1.2 \, M_\odot$ white accreting matter at a median accretion rate of $3 \times 10^{-8} \, M_\odot/yr$. Over the 500 years shown the logarithm of the mass accretion rate was varied sinusoidally with a period of 200 years. At maximum, this variation increased the mass accretion rate to five times the median value, and at minimum it was decreased to a fifth of the median value. Notice that at near the maximum, the accretion rate is within the steady burning region, and the luminosity and effective surface temperature vary relatively smoothly. When the accretion rate decreases, however, flashing behavior begins. As the accretion rate passes the transition from steady burning to bursting, the rate of flashes is high, and their intensity low. Near the minimum accretion rate the bursts are most intense and widely separated. In this particular model, the long time scale variation in accretion rate led to much shorter time scale changes. The bursts themselves are separated by 20 years, at maximum, and the changes in burst frequency and intensity are quite substantial over a similarly small interval of time.

Figure 4-2 shows somewhat similar behavior for a lower accretion rate. In this model the white dwarf mass, oscillation period of mass accretion, and percentage change in accretion rate due to the oscillation were the same as in the previous case. However, this time the median accretion rate was reduced to $1 \times 10^{-8} \, M_\odot/yr$. In this case the rapid bursting behavior occurred near the maximum values of the mass accretion rate. The entirety of this evolution was within the non-steady burning region of parameter space. Near the peak values of the accretion rate the flashes were spaced by only approximately 15 years. As the accretion rate fell, the spacing of the bursts grew. At the minimum accretion rate, the time necessary to build up enough mass for a burst was long enough, and the existing amount of hydrogen was low.
Figure 4-1: The behavior of the envelope for a $1.2 \ M_\odot$ white dwarf with an accretion rate oscillating around $3 \times 10^{-8} \ M_\odot/yr$ by a factor of 5 with a period of 200 years.
enough from the rapid bursting that occurred earlier, that for approximately half of
the 200 year cycle no flashing was evident. During this period of quiescence virtually
no hydrogen is burning, which is markedly different from that of the previous model
in which all the accreted hydrogen was steadily burning. This difference is reflected
in the relative luminosities of the white dwarf during these non-flashing periods. The
model in figure 4-1 has more than 100 times the luminosity of the model in figure 4-2
when both are not flashing. Again, the variations in intensity and spacing of hydrogen
bursts shows that the 200 year oscillation of the mass accretion rate was manifested
in much shorter time scale changes.

In addition to these long time scale oscillations, we also subjected the white dwarf
to much more rapid oscillations and studied the response of the envelope. Figure 4-3
was generated for a 1.2 M\(_{\odot}\) white dwarf experiencing an accretion rate that is varying
with a 10 year cycle and growing and shrinking by a factof 5. In this case, the
median accretion rate is well within the steady burning region (1 \(\times\) 10\(^{-7}\) M\(_{\odot}/\text{yr}\)).
Note that the 10 year period of the accretion rate is directly reflected by a 10 year
oscillation in the luminosity. If the accreted hydrogen were burned instantaneously the
total luminosity would vary exactly as the accretion rate, since the energy generated
by nuclear burning is proportional to the amount of matter burned. If this were the
case, we would observe the luminosity of the white dwarf vary by a factor of 25 from
its minimum to maximum values. We actually find, however, that the luminosity
only varies by a factor of \(~1.3\). This suggests that the envelope in unable to fully
respond on a time scale this short, and that the finite time required to adjust to a
new accretion rate to some extent averages out such short time scale oscillations.

Similar ten-year oscillations, but which cross the boundary between steady burn-
ing and flashing, are shown in figure 4-4. Notice that in this case the luminosity still
exhibits a 10 year cycle, but that the changes in luminosity are much greater than
in the previous model. In fact, the luminosity in this model actually varies by more
than a factor of 25. This suggests that little nuclear burning is taking place for a
significant portion of the 10 year cycle, and then the accreted hydrogen is burned in a
quasi-flash of high luminosity. Lowering the median accretion rate even further, fig-
Figure 4-2: Same as figure 4-1 but with a median accretion rate of $1 \times 10^{-8} \, M_\odot/\text{yr}$
Figure 4-3: A 1.2 $M_\odot$ white dwarf with a median accretion rate of $1 \times 10^{-7} M_\odot$/yr. Over its 10 year period the accretion rate varied by a factor of 5.
Figure 4-4: Same as figure 4-3 but with a median accretion rate of $3 \times 10^{-8} \text{ M}_\odot/\text{yr}$. 
ure 4-5 shows the a white dwarf of the same mass when even the maximum accretion rate \(5 \times 10^{-8} \, M_\odot/\text{yr}\) is within the flashing region. Here, the spikes in luminosity are just hydrogen flashes as seen with constant accretion (e.g. figure 3-5). These flashes occur over a period several times longer than the 10 year period of the accretion rate oscillations. Since there is minimal burning in the non-flashing portions of this evolution, the oscillating accretion rate does not noticeably affect total luminosity. Between flashes the mass is simply building up, and the envelope is unaffected by whether this buildup is steady or varying. Note that there are no discernable changes in the envelope with a period equal to that of the accretion rate.

Finally, we observed the effects of an even faster change in accretion rate. An interval of ten years from the evolution of 1.2 \(M_\odot\) white dwarf with an accretion rate varying at a period of 1 year is shown in figure 4-6. This accretion rate is within the steady burning region. In this case we still observe the envelope to respond at the same frequency as the accretion rate, but the amplitude of the changes is decidedly minimal. The averaging effect seen in the case of 10 year oscillations is even more pronounced in this figure. The surface luminosity only changes by a factor of 1.08 while the accretion rate increases by a factor of 25.

### 4.2 Studies of Long Term Helium Buildup

In addition to studying the steady burning and flashing of hydrogen we explored the long term behavior of the helium in the envelope. As a first test we allowed the white dwarf to accrete pure helium, and observed the runaway nuclear burning of helium that resulted. Figure 4-7 shows a 1.2 \(M_\odot\) white dwarf accreting pure helium at a rate of \(1 \times 10^{-7} \, M_\odot/\text{yr}\). An example of such a system would be accretion from a companion white dwarf, instead of a main sequence star. In this figure the helium accretes for approximately 2800 years before runaway helium burning ensues. The helium flashes associated with this runaway burning closely resemble the hydrogen flashes discussed earlier. Figure 4-8 shows the details of one of these helium flashes. Since these bursts are so violent it was impossible to allow gravity to vary freely in this
Figure 4-5: Same as figure 4-3 but with a median accretion rate of $1 \times 10^{-8} \, M_\odot/\text{yr}$. 
Figure 4-6: Same as figure 4-3, but with an accretion rate oscillating with a period of 1 year.
Figure 4-7: A 1.2 \( M_\odot \) white dwarf accreting pure helium at \( 1 \times 10^{-7} \ M_\odot/\text{yr} \). The gravitational acceleration is held fixed. The energy generation rate of the triple-alpha reaction is shown in the panel labeled \( \epsilon_{\text{He}} \).
Figure 4-8: A closeup of one of the helium flashes in figure 4-7
particular evolution, so both figures were created with the gravitational acceleration held fixed. Notice that the mass in the envelope only grew to \( \sim 6 \times 10^{29} \text{ g} \) \( \sim 3 \times 10^{-4} \text{ } M_{\odot} \) when the helium burst occurred. This is a tiny amount of mass compared to the total mass of the star, and not nearly enough to push the mass of the white dwarf over the Chandrasekhar limit, or to detonate the uncerlying white dwarf.

As another test we allowed a white dwarf to accrete hydrogen at a rate that was within the steady hydrogen burning region. In effect the steady burning of hydrogen allowed helium to effectively accrete steadily into the envelope. Figure 4-9 shows the behavior of such a system over two helium bursts. Due to the hydrogen burning, the temperatures in all the zones, especially the uppermost zones, are significantly higher than in figure 4-7. This increase in temperature induces the helium flash after shorter intervals. In this case the helium flashes are separated by only approximately 1400 years, roughly half the separation of flashes in the pure helium accretion case. This implies that only half the mass that we found in the case for pure He accretion has been accumulated between bursts. Also, the energy generated via the triple alpha reaction is roughly the same in the pure helium and pure hydrogen models, but the energy produced by the hydrogen burning causes the luminosity and effective temperature to be much higher in the pure hydrogen case. Furthermore the luminosity does not have dramatic spikes, but rather slowly dies down after a burst.

Finally, we attempted to study the long term behavior of the white dwarf when the core of the star is held at a constant temperature, the accretion rate of pure hydrogen is within the steady burning region and the gravitational acceleration is free to vary. Figure 4-10 shows \( 8 \times 10^5 \) years of the evolution of a white dwarf with a constant temperature core accreting pure hydrogen. Note that the radius of the envelope has expanded greatly, and the luminosity and surface temperature are dropping steadily. Also, the mass in the envelope has grown to a substantial \( 1/20 \text{ } M_{\odot} \). In carrying out these studies of very long term accretion we discovered a somewhat disconcerting problem with the luminosities predicted by the model. It appears that in runs such as figure 4-10 energy in the envelope is not being properly conserved. We had two methods of testing energy conservation, both of which are plotted in figure 4-11:
Figure 4-9: Same as figure 4-7, but the white dwarf is accreting pure hydrogen and the helium is built up via hydrogen burning. Both the hydrogen and helium energy generation rates are now plotted.
Figure 4-10: A 1.1 $M_\odot$ white dwarf with a core of temperature $3 \times 10^7$ K. The accretion rate is $8 \times 10^{-8} M_\odot$/yr of pure hydrogen. $\epsilon$ is the total energy generation rate of both helium and hydrogen burning.
1. We calculated the total mass per unit time of hydrogen burned in every time step, and multiplied it by \( c^2 \) and the efficiency of nuclear burning (\( \sim 0.007 \) for hydrogen burning). From this, we subtracted the flux from the lowest zone into the core of the star. This gave the total energy available in every time step to add to the luminosity out of the star. In the figure this is labeled as \( E_{\text{AVAL}} \), and it is shown in units of solar luminosities.

2. We also added up the energy generation rates from all three reactions for all the zones and subtracted the flux into the core. This gave the total energy produced by the by nuclear burning which could contribute to the luminosity. This is labeled \( E_{\text{PROD}} \) in the figure, and is also in solar luminosities.

From this figure it is clear that the two different methods of calculating total energy are in quite good agreement. This is a reassuring test of our calculation of the nuclear burning. However, these values lie below the computed luminosity from the stellar photosphere. The reasons for this are still unclear and will be the subject of future work. It may be the case that this discrepancy is due to imperfect calculation of fluxes between zones, caused by the large column thickness of the lowest zones.
Figure 4-11: Accretion onto the constant temperature core of a 1.1 $M_\odot$ white dwarf. $E_{\text{AVAL}}$ and $E_{\text{PROD}}$ are in good agreement, but are approximately a factor of 3.25 smaller than the luminosity.
Appendix A

Newton's Method Computation

At several places per time step within the code it is necessary to extract the temperature and density of a zone from the entropy and pressure. Equations 2.30 and 2.22 give explicit formulas for $s$ and $P$ in terms of $T$ and $\rho$, but are not possible to recast as direct formulas for temperature and density. Thus, it is necessary to perform a two-dimensional Newton's method on these equations.

Given two functions of two variables, $g(x, y)$ and $h(x, y)$, we wish to find the values of $x$ and $y$ which make $g$ and $h$ equal to zero. We begin with an initial guess for $x$ and $y$, which yield values of $g^0$ and $h^0$. Our goal is to find the incremental changes, $\delta x$ and $\delta y$, which approach the roots of $g$ and $h$. Given our initial guesses these changes are given by:

$$
g^0 + \frac{\partial g}{\partial x} \delta x + \frac{\partial g}{\partial y} \delta y = 0, \quad (A.1)$$

$$
h^0 + \frac{\partial h}{\partial x} \delta x + \frac{\partial h}{\partial y} \delta y = 0, \quad (A.2)$$

where $\frac{\partial g}{\partial x}$ symbolizes $\frac{\partial g}{\partial x}$, etc. Multiplying through the first equation by $\frac{\partial h}{\partial y}$ and the second by $\frac{\partial g}{\partial y}$

$$
\frac{\partial h}{\partial y} g^0 + \frac{\partial g}{\partial x} \frac{\partial h}{\partial y} \delta x + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} \delta y = 0, \quad (A.3)$$

$$
\frac{\partial g}{\partial y} h^0 + \frac{\partial g}{\partial y} \frac{\partial h}{\partial x} \delta x + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} \delta y = 0. \quad (A.4)$$
Subtracting and solving for $\delta x$ gives:

$$
\delta x = \frac{\frac{\partial h}{\partial y} g^0 - \frac{\partial g}{\partial y} h^0}{\frac{\partial h}{\partial y} \frac{\partial x}{\partial y} - \frac{\partial g}{\partial y} \frac{\partial h}{\partial y}}.
$$

(A.5)

A similar procedure yields:

$$
\delta y = \frac{\frac{\partial g}{\partial x} h^0 - \frac{\partial h}{\partial x} g^0}{\frac{\partial h}{\partial x} \frac{\partial y}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial h}{\partial x}}.
$$

(A.6)

In the case of our Newton’s method, the two variables we wish to solve for are $\rho$ and $T$, and the functions whose roots we need to find are:

$$
g(T, \rho) = \frac{4aT^3}{3\rho} + \frac{k}{\mu H} \ln \frac{T^{5/2}}{\rho} - s
$$

(A.7)

$$
h(T, \rho) = \frac{a}{3} T^4 + \frac{k}{\mu H} \rho T + K \rho^{5/3} - P,
$$

(A.8)

where the values of $s$ and $P$ are known.

The following is the segment of code actually used to perform this Newton’s method. There were occasionally problems with convergence, and one anomalously large increment would destroy the procedure. As a safeguard against this, we explicitly did not allow any single increment in $T$ or $\rho$ to change the value of that variable by more than a factor of 2.

```c
void newttrho(double tmp[], double rho[], double j[], double mu[], double kpoly[])
{
    /* uses Newton's method to determine the temperature */
    /* and the density from the pressure and the entropy */

    extern int ntot;
    double small=1e-4, g1=1e150, g2=1e150, gif1, gif2, g2f1,
        g2f2, df2, df1, b, drhodt;
    int i;

    /* g1=j[odd]=entropy */
    /* g2=j[even]=pressure */
    /* f1=tmp=temperature */
    /* f2=rho=density */

    for(i=0; i<ntot; ++i) {
        g1=1e150;
```
g2=1e160;
while(fabs(g1/j[2*i+1])>small && fabs(g2/j[2*i])+small) {

    /* the equations for entropy and pressure */
    g1 = (4./3.)*ARAD*pow(tmp[i],3)/rho[i] + (BK/mu[i]/MP)*
        log(pow(tmp[i],1.6)/rho[i]) - j[2*i+1];
    g2 = ARAD*pow(tmp[i],4)/3.+BK*rho[i]*tmp[i]/(mu[i]*MP)+
        kpoly[i]*pow(rho[i],(5./3.))- j[2*i];

    /* calculate the derivatives with respect to temperature and density */
    gif1 = 4.*ARAD*pow(tmp[i],2)/rho[i] + 1.5*BK/(mu[i]*MP*tmp[i]);
    gif2 = -4./3.*ARAD*pow(tmp[i],3)/(rho[i]*rho[i]) - BK/(mu[i]*MP*rho[i]);
    g2f1 = 4.*ARAD*pow(tmp[i],3)/3. + BK*rho[i]/(mu[i]*MP);
    g2f2 = BK*tmp[i]/(mu[i]*MP) + 5.*kpoly[i]*pow(rho[i],(2./3.))/3.;

    /* calculate the increments in temperature and density */
    df2 = (-g1*gf1 + g2*gif1)/(g2f1*gf2 - gif1*g2f2);
    df1 = (g1*gf2 - g2*gif2)/(g2f1*gf2 - gif1*g2f2);

    /* add to the temperature and the density the appropriate amounts */
    /* do not allow the temperature or density to change more than 2x */
    while((rho[i] + df2)/rho[i] < .5 || (rho[i] + df2)/rho[i]>2 ||
        (tmp[i] + df1)/tmp[i] < .5 || (tmp[i] + df1)/tmp[i]>2) {
        df2/=2;
        df1/=2;
    }
    tmp[i] += df1;
    rho[i] += df2;
}
}
Appendix B

Computational Method for Re-Zoning

The algorithm described in this appendix was used to redefine the zone boundaries after every numerical integration step. This re-zoning procedure defines and calculates the densities, pressures and chemical compositions of any number of exponentially sized zones from an arbitrary number of old zones. For the present work the number of zones was never changed within a single run. However, if future incarnations of this model allow the number of zones to vary over the evolution of the envelope, this algorithm will still work.

Define $p$ to be the number of new, exponentially spaced zones, and let each zone have a mass of $\alpha$ times the mass of the zone on top of it. The first step is to determine the mass which should be contained in each of these new zones, given that the total mass of the envelope is known. The mass of the uppermost zone is $M_1$, and the mass of the other zones are determined according to:

\[ M_i = \alpha^{i-1} M_1. \tag{B.1} \]

Using this, the mass of the top zone can be calculated from the total envelope mass, $M_T$.

\[ M_T = M_1 \sum_{i=1}^{p} \alpha^{i-1}. \tag{B.2} \]
This is just a finite geometric series whose sum is equal to

\[ M_T = M_1 \left( \frac{\alpha^p - 1}{\alpha - 1} \right). \]  \hfill (B.3)

Therefore, the mass of any zone is given by:

\[ M_i = M_T \left( \frac{\alpha^i - \alpha^{i-1}}{\alpha^p - 1} \right). \]  \hfill (B.4)

Beginning at the top of the envelope, the new zones are built from the mass in the old zones. Take the masses of the old zones to be \( M_i \) and the number of old zones to be \( q \). The uppermost new zone will be composed of \( k \) entire old zones are some fraction, \( f \), of the \((k+1)\)th zone:

\[ M_1 = \sum_{i=1}^{k} M_i + fM_{k+1}. \]  \hfill (B.5)

When the numbers of old and new zones are the same, as was the case in all the models for this thesis, it is usually the true that the new top zone is actually smaller than the old top zone, since all the mass accreted in a time step adds to the top zone. Under these circumstances only the second term in the above equation is relevant and the new mass is \( M_1 = fM_1 \).

The densities of the old zones, \( \rho_i \) were all known, so the density of the new zone can be determined. If \( \rho'_1 \) is the density of the uppermost new zone, then

\[ \rho'_1 = M_1 \left( \sum_{i=1}^{k} \left( \frac{M_i}{\rho_i} \right) + f \frac{M_{k+1}}{\rho_{k+1}} \right)^{-1}. \]  \hfill (B.6)

Furthermore, the chemical composition of the new zone is determined from the compositions of the old zone. For instance, the mass fraction of hydrogen in the new zone is

\[ X_1^{\text{new}} = \left( \frac{1}{M_1} \right) \left( \sum_{i=1}^{k} X_i^{\text{old}} M_i + f X_{k+1}^{\text{old}} M_{k+1} \right). \]  \hfill (B.7)

The same procedure is carried out for all of the new zones. When computing lower zones, however, the extra fraction of a zone left over must be taken into account. For
instance, if the mass of the second zone equals the rest of the mass from the \((k+1)\)th zone, \(l\) entire zones, and a fraction \(g\) of the \((k+l+2)\)th zone, we have

\[
M_2 = (1 - f)M_{k+1} + \sum_{i=k+1}^{k+l+1} M_i + gM_{k+l+2},
\]

(B.8)

and the density and composition are calculated just like the first zone. Since the masses of the new zones were calculated from the total envelope mass, the \(q\) old zones will exactly fill up the \(p\) new zones.

The pressures on each zone are directly computed from the masses (equation 2.18). Now that \(P\) and \(\rho\) have been computed for the new zones, the corresponding values of \(s\) and \(T\) are computed using a Newton's method on the equation of state, similar to the one described in Appendix A.

The following is the actual C function used to carry out this re-zoning.

```c
void rezone(double ma[], double mb[], double rhoa[], double rhob[], double xa[], double xb[], double ya[], double yb[], int n, int p)
{
    /* remake the zone boundaries:
       ma[], rhoa[], xa[], ya[] are the mass, density, x, y of the old zones
       these are rezoned into mb[], rhob[], xb[], yb[]
    */

double mp[p], mtot=0, mfrac, xt, vt, mx, my;
int i,j,k,l;

    /* zero the new array */
    /*--------------------*/

    for(i=0;i<p;++i) {
        mb[i]=0;
        rhob[i]=0;
    }

    /* calculate the total mass of the shell */
    /*--------------------*/

    for(i=0;i<n;++i) {
        mtot += ma[i];
    }
}
```
for(i=0;i<p;++i)
    mp[i] = mtot*((LOGFACT-1)/pow(LOGFACT,p) - 1)*pow(LOGFACT,i);

j=0;
for(i=0;i<p;++i) {
    vt=0;
    mx=0;
    my=0;
    while((mb[i]+ma[j])<=mp[i] & j<n) {
        mb[i]+=ma[j];
        vt+=ma[j]/rhoa[j];
        mx+=ma[j]*xa[j];
        my+=ma[j]*ya[j];
        ++j;
    }
}

mfrac = mp[i] - mb[i];

if(mfrac/mb[i]>1.e-5) {
    vt+=mfrac/rhoa[j];
    mx+=mfrac*xa[j];
    my+=mfrac*ya[j];
}
/* subtract from the old zone the mass used and complete the mass of */
/* the new zone */

mb[i] += mfrac;
ma[j] -= mfrac;

/* calculate the density and composition of the new zone */

rhob[i] = mb[i]/vt;
xb[i] = mx/mb[i];
yb[i] = my/mb[i];
Bibliography


