Early history of soil–structure interaction

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EARLY HISTORY OF SOIL-STRUCTURE INTERACTION

by

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Hommage à AnestisVeletsos

ABSTRACT

Soil Structure Interaction is an interdisciplinary field of endeavor which lies at the intersection of soil and structural mechanics, soil and structural dynamics, earthquake engineering, geophysics and geomechanics, material science, computational and numerical methods, and diverse other technical disciplines. Its origins trace back to the late 19th century, evolved and matured gradually in the ensuing decades and during the first half of the 20th century, and progressed rapidly in the second half stimulated mainly by the needs of the nuclear power and offshore industries, by the debut of powerful computers and simulation tools such as finite elements, and by the needs for improvements in seismic safety. The pages that follow provide a concise review of some of the leading developments that paved the way for the state of the art as it is known today. Inasmuch as static foundation stiffnesses are also widely used in engineering analyses and code formulas for SSI effects, this work includes a brief survey of such static solutions.

PROLOGUE

As anyone who has tried is surely aware, it is quite difficult to provide a concise definition of Soil-Structure Interaction (SSI) without actually giving a lengthy explanation instead. This predicament is reminiscent of a memorable statement proffered nearly half a century ago by the late US Supreme Court justice Potter Stewart, which when paraphrased in the context of this paper would read:

> *"I shall not today attempt further to define the kinds of material I understand to be embraced within the shorthand description of [soil-structure interaction]; and perhaps I could never succeed in intelligibly doing so. But I know it when I see it".*

Of course, it is eminently clear that the concept of soil-structure interaction refers to static and dynamic phenomena mediated by a compliant soil and a stiffer super-structure, but the discipline encompasses so many different, sometimes tenuously connected aspects that it is difficult indeed to enounce a cogent definition in just a few words. For one, this area of expertise includes the amplification of seismic waves in the soil even before any structure has been erected, so it includes the complex dynamic interactions that arise in soil layers by themselves. Thus, it behooves to begin with a summary of some of the principal problems encompassed by the theory of SSI:

- Response of a soil domain to external dynamic (or even static) sources acting near —or on— the surface. The sources may be concentrated (point loads) or distributed, and they could be harmonic in time or suddenly applied with an arbitrary variation in time (Green's functions, or *fundamental solutions*).

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- Response of the soil to ground-borne vibrations elicited by earthquakes or some other sources, such as fast moving trains, even before any structures lie in their path (*Free field problem*).
- Response of rigid, ideally massless structures to ground-borne waves passing underneath (wave passage or *kinematic interaction*).
- Response of ideally massless footings, foundations, piles or caissons embedded in compliant soils elicited by static, harmonic or transient loads applied directly onto these footings (*static and dynamic stiffnesses*).
- Additional deformation of the soil in the neighborhood of an actual structure caused by feedback of the structure's own inertia (*inertial interaction*).
- Ad hoc numerical methods needed to analyze any of the above problems (finite element models with transmitting boundaries, boundary elements, and so forth).

Thus, it is not surprising that individuals with a wide range of talents and specialties, from mathematicians and scientists to engineers may have participated in the development of this discipline. In the sections that follow we shall succinctly revisit some of the most important early technical developments that led to the state of the art as we know it today, highlighting the pioneers whose work left an indelible mark on the field. This necessitated securing and reading copies of very old papers, a task that was quite difficult not only because of the rarity of old journals and thus of the papers, but also because of the different notation and writing style then in use, not to mention the near universal absence of figures. For example, what many old papers refer to as Poisson's ratio is actually its reciprocal, $m = 1/\nu$. Still, the writer did read *all* of the references, and marveled at what was already accomplished more than a century ago and during the early part of the twentieth century, even if most of it was restricted to linearly elastic systems.

FUNDAMENTAL SOLUTIONS

A fundamental solution is an analytical expression for the response anywhere in a solid elicited by a static or dynamic point source at some arbitrary location. Such expressions can be thought of as influence functions and can be used as tools to construct other, more complex solutions. In the course of the 19th and early part of the 20th century a number of scientists —mostly mathematicians, but also engineering scientists— provided the theoretical framework together with fundamental solutions, or Green's functions, that not only made the posterior development of SSI feasible, but which to this day lie at the heart of the now widely used Boundary Element Method (BEM).

The very first scientists to have addressed the problem of loads on or within an infinite (or semi-infinite) elastic body were the eminent French mathematicians Gabriel Lamé and Benoît Paul Émile Clapeyron, who in the early part of the 19th century addressed the half-space problem with mathematical tools and methods so abstruse and convoluted that they failed to obtain any useful and practical results. Thus, the very first fundamental solution had to await the middle of the 19th century until 1848, when Sir William Thomson —better known as Lord Kelvin—gave expressions for the displacements elicited by concentrated static forces acting at some arbitrary point in an elastic, infinite solid. Very shortly thereafter in 1849, this was followed by the solution of the much more difficult problem of time varying point forces in an infinite medium provided by Sir George Gabriel Stokes, *Lucasian* Professor of Mathematics at Cambridge. The Stokes solution, of which time-harmonic forces and static forces are special cases, constitutes today a cornerstone in the Boundary Element Method and exerts a most profound influence not only on the field of SSI, but also in geophysics, acoustics and other branches of science. A salient characteristic of the Stokes state is that it is one of the few fundamental solutions which are known in closed-form throughout space in both the time and frequency domains. A modern rendition of the Stokes solution in two and three dimensions is found in the well known paper by British mathematicians Eason, Fulton and Sneddon (1956).

In the last quarter of the nineteenth century, another French mathematician, Joseph Valentin Boussinesq published in 1878 a series of short papers in *Comptes Rendus* that sketched a solution method for static, vertical point loads applied onto the surface of an elastic half-space, and also gave a closed form solution for a rigid disk with smooth contact on the surface of a half-space bearing vertical loads, about which more will be said later on. However, it was not until his extensive treatise on the subject saw the light in 1885 that the full details of his method, based on potentials, were revealed. In the interim, the Italian mathematician Valentino Cerruti, Professor of Rational Mechanics and Rector of the University of Rome, published in 1882 a related, massive paper in the journal of the Italian Royal Academy (*Reale Accademia dei Lincei*), to which many modern papers make reference to.

Cerruti's paper is rather general in its approach and makes extensive use of integral theorems in elastostatics known as Betti's principle —similar to those that underlie the modern BEM— and obtains the response in the interior of an arbitrary solid elicited by tractions or displacements prescribed on parts of the external boundaries. Although Cerruti does not use the modern language of continuum mechanics, his paper is concerned with what can clearly be recognized today as a *boundary value problem* involving an elastic body surrounded in part by a *Dirichlet boundary* where displacements are prescribed, and in part by a *Neumann boundary* where tractions are prescribed. Thereafter, he goes on to apply his method to a body of infinite extent limited by a flat surface (i.e. a half-space); yet nowhere does he provide the final equations for the displacement field due to a tangential load famously attributed to him, but leaves this task to the reader—at least in this work. However, his equations do indeed contain the tools necessary to obtain such a solution, and not just for tangential point loads on the surface but for any load distribution, horizontal or vertical. As an example of application of his approach, he does distill from his equations the case of vertical loads and obtains results that are in agreement with Boussinesq's.

Curiously, in the introduction to his memoir Cerruti made a brief allusion to Boussinesq's earlier papers, but seemingly was not very impressed by these, for he commented that they "appear deficient to me in several respects". Still, he did not elaborate further on what those deficiencies might actually have been.

A significant leap forward came in the form of the fundamental solution for a homogeneous half-space subjected to a dynamic load on its surface, which is contained in the celebrated 1904 paper of Sir Horace Lamb, Professor of Mathematics at the University of Adelaide in South Australia. Here, Lamb resorts to a precursor of what constitutes the modern integral transform method to obtain the response to either impulsive (2-D) or suddenly applied (3-D) vertical loads on the surface of an elastic half-space. [Note: the 2-D space has no step-load solution]. However, Lamb lacked in his time the complete set of mathematical tools —not to mention computers— necessary to fully evaluate all of his integrals. Thus, Lamb assessed in some detail only the response in the far field at remote distances from the source. Still, to this day and in his honor, the problem of a dynamic source applied at the surface of an elastic half-space is referred to as *Lamb's problem*.

The next major milestone in fundamental solutions came with the publication in 1936 of Raymond David Mindlin's famous set of closed-form equations for the displacement field elicited by static, vertical and horizontal point loads buried at an arbitrary depth below the surface of an elastic half-space. Legend has it that the brevity of his dissertation at Columbia University in New York raised doubts as to its quality, which were dissipated as soon as departmental headquarters got word that he had managed to solve a very difficult problem that had stumped others before him. Mindlin's publication appeared in the same year that he earned his PhD there, but it was not until 1940 that he was finally appointed Assistant Professor at Columbia. Unbeknownst to him, the Austrian engineer Ernst Melan (1932) had already published four years earlier a solution for the two-dimensional counterpart of buried line loads, a work about which he was alerted to by S. Timoshenko. Mindlin then added in his 1936 paper a comparison with Melan's formulas and confirmed their agreement with his own solution.

About four decades after Lamb, Cagniard (1939) finally managed to evaluate the requisite double integral transforms in Lamb's problem by means of a very ingenious yet arcane contour integration that few understood. Two decades later in turn, de Hoop (1960) succeeded in finding a substantial simplification to Cagniard's procedure in what is now referred to as the Cagniard–de Hoop method. This analytical strategy was also used by Pekeris (1955) and by Chao (1960) to obtain closed-form solutions —i.e. not requiring numerical integrations— for impulsive vertical and horizontal point loads in a half-space, but only when Poisson's ratio is $\nu = 0.25$. Thereafter, Mooney (1974) generalized Pekeris' results for vertical point loads acting on half-spaces with arbitrary Poisson's ratio, but only gave results for the horizontal component of displacement up to Poisson's ratio $\nu = 0.2631$, which is the value at which the false roots of the equation for the speed of Rayleigh waves turn complex. In all of these solutions, displacements are known in closed form only on the surface and on the axis of symmetry below the load, and not at interior points. Concerning impulsive line loads in two dimensions, fully closed-form solutions to Lamb's problem exist as of this writing only for sources on the surface and displacements anywhere in the body, or for buried line loads and displacements on the surface. By contrast, displacements anywhere in the half-space due to dynamic loads contained there can only be obtained by purely numerical means, and the same is true when the loads are harmonic, even when the line loads are applied on the surface, the SH line load case excepting.

The seminal work of Lamb together with its refinements in the ensuing decades in the early and mid- twentieth century provided the exact formulas for the transient response of elastic half-spaces elicited by suddenly applied line and point sources on its surface. It is thus remarkable that comparable closed-form solutions for a single transform, say from the frequency-wavenumber domain to the frequency-space domain, or alternatively, to the wavenumber-time domain, remain lacking. Perhaps the reason lies partly in the difficulty involved in obtaining exact results for a single transform —it is rather curious that the double transform should be easier— but also because the availability of the space-time solution removed much of the motivation to find solutions to such partial transforms. Nonetheless, there is no lack of practical solutions based on numerical approaches. A particularly powerful one is the Thin-Layer Method for the Green's functions of layered media (Kausel 1981, Kausel & Peek, 1982) which now lies at the heart of widely used codes such as PUNCH and SASSI, among many others.

Once these pioneering fundamental solutions became widely known, many other such solutions for distributed loads of various shapes and characteristics followed, including transversely isotropic media, but a review of these is well beyond the scope of this paper. Still, many of the static solutions can be found in the well known reference book by Poulos and Davis (1972), while the dynamic counterpart —including Stokes', Lamb's and Chao's problems—can be found in Kausel (2006).

STATIC SSI

It would appear at first glance that once the solutions for point loads applied onto the elastic half-space were known, distributed loads would readily follow by recourse to convolutions over the loaded surface, but the resulting integrals were rarely tractable when the observation point was taken anywhere within the half-space. Still, Boussinesq (1885) himself considered not only the problem of vertical point loads, but also managed to find solutions for vertical loads uniformly distributed over a finite circular area, although he did so only for the displacements on the surface, on the axis of symmetry, and the average displacement under the load. The latter is a decent approximation to the compliance of a rigid, circular disk. In addition, he also solved the problem of a rigid, circular plate subjected to centered vertical loads, and found both the vertical stiffness K_z as well as the stress distribution σ_z underneath the plate under the assumption that the contact was smooth, i.e. the plate-soil interface was lubricated, not welded. In modern notation, Boussinesq found:

$$K_z = \frac{4Ga}{1-\nu}$$

$$\sigma_z\big|_{z=0} = \frac{P_z}{2\pi a\sqrt{a^2 - r^2}}$$

where P_z is the total vertical load, G is the shear modulus, ν is Poisson's ratio, a is the radius of the plate, and r, z are the radial and vertical coordinates.

Perhaps one of the earliest contributions to foundation engineering may be found in a remarkable paper by Fr. Engesser, who in 1893 wrote *Zur Theorie des Baugrundes* (About the Theory of Soils) in which he discusses the stability and carrying capacity of foundations. Still, quantum leaps in the field had to await the arrival of Austrian engineer Karl Terzaghi, now regarded as the father of soil mechanics. During the 1930's, the Institute of Soil Mechanics at the Technical University of Vienna (*Technische Hochschule Wien*) —founded in 1928 and first directed by Karl Terzaghi— was abuzz with other notable personalities in the then new field, including Arthur Casagrande, Leo Casagrande, Leo Rendulic, Hubert Borowicka, O.K. Fröhlich and Wilhelm Steinbrenner, among others. After Terzaghi, the direction of the institute passed on to Fröhlich, and a dozen years after World War II in 1957, to Borowicka. Thus, it is not surprising to find that a good number of the early, leading advances in soil mechanics and SSI may have emanated from the Austro-German academic circles of which the Vienna institute was preeminent. Among the papers from that world are those by F. Schleicher (1926), K. Marguerre (1931, 1933), E. Melan (1932), W. Steinbrenner (1934), E. Reissner (1936, 1937), H. Borowicka, (1939, 1943a, 1943b), K. Girkmann (1940), G. Schubert (1942), and K. Hruban (1943), just to name a few.

Ferdinand Alois Schleicher, *Privatdozent* at the Technical University Karlsruhe in Germany and later on a civil engineer in the industry, echoed in his own 1926 paper the title of Engesser's, to which he makes reference and pays brief tribute. In this work, Schleicher revisits the Boussinesq problem of vertical loads distributed over the surface of an elastic half-space, and confirms Boussinesq's results for both a disk load and for a rigid disk, referring to these as the plate of zero and infinite bending stiffness, respectively. He then uses these to infer the value of the modulus of sub-grade reaction (*Bodenziffer*), which is useful in the formulation of foundation mechanics problems via distributed Winkler springs, and comments that this coefficient is not a material constant of the soil, but is inversely proportional to the linear dimensions of the supported loads. More significantly, Schleicher then turns his attention to vertical loads uniformly distributed over a rectangular area, and provides what may well turn out to be the first closed-form formulas for such loads. Indeed, he provides explicit expressions for the vertical displacements anywhere on the surface of the half-space, either within or without the loaded area, and observes that the smallest deflection is observed at the four corners and equals one half of the deflection at the center, no matter what aspect ratio the load may have, and whatever Poisson's ratio. In addition, he computes the average deflection under the load, goes on to specialize these for strip loads and square loads, and compares the latter with the disk load.

Less than a decade after Schleicher, Wilhelm Steinbrenner (1934) published a remarkable five-page paper in the short-lived journal *Die Strasse*, in which he considers once more the problem of rectangular loads. In this document he makes the brilliant observation that the vertical stresses anywhere in the soil can readily be inferred from the stresses underneath the symmetry center of the loaded area, and from there to the corners (or vice-versa), a simple calculation which can be obtained by integration of the Boussinesq solution for point loads. Stresses elsewhere then follow by simple superposition of appropriately sized rectangular loads, including negative loads in the case of observation points beyond the edges of the actual load. Without giving any technical details or derivations, he goes on to give his now famous formulas, which are very widely used at present in engineering practice. However, given the obscurity of the journal he chose as venue and had these formulas not been reproduced by Terzaghi in a later edition of his *Erdbaumechanik*, chances are that they might have been overlooked or forgotten, perhaps only to be rediscovered by others in subsequent years. Still, while Steinbrenner's influential paper has been quoted very widely and he received due credit for his insightful contribution, it is highly likely that very few people have actually read it. Thus, it may well belong to the special class of "most widely unread papers".

A decade later and during the dark and fateful years of WW-II, Steinbrenner's colleague H. Borowicka (1943a) published a historical paper where he considered strip footings and circular disks subjected to eccentric vertical loads. He obtained the stress distribution σ_z under both of these types of footings and went on to extract his now widely used rocking stiffnesses K_r. In a nutshell and in modern notation:

Disk:

$$K_r = \frac{8Ga^3}{3(1-\nu)}$$

$$\sigma_z\big|_{z=0} = \frac{3r\cos\theta}{2\pi a^3\sqrt{a^2-r^2}} M_r$$

Strip footing:

$$K_r = \frac{\pi G a^2}{2(1-\nu)}$$

$$\sigma_z\big|_{z=0} = \frac{2x}{\pi a^2\sqrt{a^2-x^2}} M_r$$

where $M_r =$ rocking moment; $\theta =$ azimuth; r = radial distance to some point within the plate; $x =$ abscissa; $a =$ radius or half-width; G = shear modulus; and ν = Poisson's ratio.

In retrospect, it seems peculiar that the first researchers of static SSI should have neglected the plate subjected to the potentially simpler state of torsion, but perhaps this may be because vertical gravity loads were primordial in foundation design while torsional stiffnesses had lesser utility. Thus, it was not until 1944 that Reissner and Sagoci, then both at Massachusetts Institute of Technology, considered the torsional stiffness K_t of a circular plate welded to an elastic half-space. They obtained the following results:

$$K_t = \frac{16 G a^3}{3}$$

$$\tau_{r\theta} = \frac{3}{4\pi}\frac{r}{a^3\sqrt{a^2-r^2}} M_t$$

with $M_t =$ torsional moment; and $\tau_{r\theta}$ = shear stress at plate-soil interface; and the other symbols are as before.

Finally, it was not until the middle of the 20th century that the set of four stiffnesses of a rigid disk was completed, when Mindlin (1949) presented results for a plate subjected to tangential loads, in the horizontal displacement mode now referred to as swaying. Using an unorthodox and most ingenious approach based on the Hertz theory of two bodies in contact, he derived the lateral stiffness K_h of a rigid, circular membrane which is infinitely stiff in its own plane and infinitely flexible in the transverse direction, so no vertical contact stresses are elicited. He found that

$$K_h = \frac{8 G a}{2-\nu}$$

$$\tau_{xz} = \frac{P_x}{2\pi a\sqrt{a^2-r^2}}$$

with τ_{xz} being the contact stress, which is parallel to the applied tangential load P_x.

Comparing the contact stresses in the basic loading cases described, it can be seen that the stress magnification factor $\sqrt{a^2-r^2} \equiv \sqrt{a^2-x^2-y^2}$ is present in all four cases, all exhibit an integrable singularity at the edge of the disk, and none of the contact stresses depend on Poisson's ratio. A large number of authors have discussed the physical significance of these characteristics, so their observations need not be revisited herein.

HISTORICAL OVERSIGHT

It would appear that with Mindlin's contribution one could close the subject of static SSI, but there remains one iniquitous historical oversight that must first be dealt with, to set the record straight. An intense review of the modern geotechnical literature and books on theoretical soil mechanics yielded only scant hints that the problem of uniformly distributed, static loads in the shape of disks and rectangles could by now have been *completely* solved. In other words, it was not at all clear to the writer that displacements and stresses for circular and rectangular loads could be found anywhere in the soil mass by sole recourse to formulas, and not via tables and charts obtained numerically or by means of nomographs such as Newmark's. The standard references listed formulas only for the surface and/or the axis of symmetry, and in most cases, only for a subset of the stresses, usually the vertical component. Attempts by the writer to find the missing components via integral transform methods foundered upon reaching Laplace transforms not listed anywhere in standard tables and which *Matlab* and *Mathematica* were not able to solve either. As it turns out, the details of these two load cases were considered in full in a pair of extraordinary papers of enormous theoretical and historical significance, but which for unfathomable reasons the technical community hardly noticed and thus allowed largely to pass into oblivion. These two forgotten gems are Terazawa (1916) and Love (1929).

Kwan-ichi Terazawa, a distinguished professor and scientist in Japan in the early years of the 20th century chose as venue for his 1916 paper the *Journal of the College of Science of the Imperial University of Tokyo*, a serial publication that may have had only limited circulation outside of Japan. Apparently, the material presented in this paper—in impeccable English, one may add— may have been the subject of a doctoral dissertation of his at Cambridge University in the UK, for a footnote on page 62 indicates that it was written there. Thus, it is not surprising that this massive paper may have been known to some circles in the UK and that copies circulated there. In fact, Love in his 1929 paper does indeed make reference to Terazawa, and this is how it came to our attention. With the kind help of Prof. Emeritus Michio Iguchi of the Science University of Tokyo, we were then able to secure a copy, for it was not available in any of the major libraries in the US.

Unlike Boussinesq's strategy based on potentials, Terazawa got his inspiration from Lamb and formulated the problem of the loaded half-space in terms of integral transforms. By managing to find solutions to *all* of the difficult integrals we alluded to earlier he bested some of the sharpest minds who were to follow him in the decades ahead. Not only did he provide a complete solution to the problem of uniform vertical disk loads, including displacements and stresses at arbitrary interior points, but considered other load distributions, such as a bell load, and observed that the depression (or dishing) left on the surface has an infinite volume even if the work performed by the load is finite. Even more remarkably, he applied onto the surface the load distribution elicited by the vertically loaded, rigid plate, and found the displacements and stresses everywhere. The paper is heavy in mathematics, and employs symbols such as those for the complete elliptic integrals that differ from the current norm, all of which makes reading more difficult. Still, while this paper may turn off the casual reader, it is a must-read for the cognoscenti.

Then there is also the notable 1929 memoir by Love already mentioned. Augustus Edward Hough Love, the *Sedleian* Professor of Natural Philosophy at the University of Oxford and

Fellow of Queen's College, was the most eminent mathematician of the early 20th century to work on the theory of elasticity. His 1929 paper alluded to earlier appears in the bibliography list of R.E. Gibson's 1974 *Rankine Lecture*, but Gibson makes only a brief and non-specific reference to Love's groundbreaking paper. That after Gibson this extraordinary and massive memoir were to have been largely ignored or remain obscure within the technical community is difficult to comprehend. Perhaps the reason may be that the nascent geotechnical world at the time of Love's publication may have been enamored of the engineering solutions emanating from the academic circles spearheaded by Terzaghi, and did not care much for abstract mathematical papers from elasticians elsewhere. Whatever the reasons, the fact is that Love considered in great detail the problem of both disk loads and rectangular loads, and provided complete solutions to both. Thus, Love not only beat Steinbrenner by half a decade, but gave far more detail on the solutions in the interior of the half-space. A simplified, concise rendition of Love's solution for displacements anywhere due to a rectangular load applied at the surface can be found in Becker and Bevis (2004). May we suggest that Geotechnical Engineering owes Love an apology?

A word in closing. When loads other than vertical are considered, especially tangential disk loads, one arrives at additional, difficult-to-solve integrals. While they may not be found in any tables of integrals, and *Mathematica* also fails to solve these, it turns out that a subset of such integrals was provided half a century ago by Eason, Noble and Sneddon (1955). Starting from their solutions, Hanson and Puja, in a remarkable 1997 paper in the *Quarterly of Applied Mathematics*, were able to extend the set of Eason *et al* and found all of the requisite integration formulas. Thus, some 130 years after Boussinesq and Cerruti, the set of analytical tools needed to provide a closed-form solution to distributed static loads, horizontal or vertical, applied onto the surface of an elastic half-space is at long last complete.

DYNAMIC SSI

Properly speaking, the theory of dynamic SSI began in 1936 with a publication by Erich (Eric) Reissner in which he explores the behavior of circular disks on elastic half-spaces subjected to time-harmonic vertical loads. Eric Reissner —born 1913 in Aachen, Germany, and the eldest son of yet another prominent engineering scientist, Prof. Dr. Ing. Hans Reissner— received his doctorate in 1935 from the *Technische Hochschule Berlin-Charlottenburg*, which today corresponds to the Technical University Berlin, or TUB. A few years later in 1938, he earned a second doctorate in mathematics from the Massachusetts Institute of Technology and in 1939 joined the faculty there, where he remained in service until 1970. Thereafter, he moved on to the University of California in San Diego in the capacity of Professor of Applied Mechanics until his passing away in 1996. Besides his pioneering work on SSI, Reissner was also very well known for his contribution to the theory of the *Reissner Plate* and to the *Hellinger-Reissner Variational Principle*, just to name another two of his most conspicuous achievements.

At the time of his dissertation, Reissner was by no means alone in the pursuit of solutions to dynamically loaded foundations. For example, Karl Marguerre penned a couple of remarkable papers in 1931 and 1933 which dealt with harmonically loaded soils and plates, including layered media. Still, while Marguerre gained some interesting engineering insights on the dynamic behavior of such systems, he became ultimately overwhelmed by the sheer complexities of wave propagation and his developments petered out, so it can safely be affirmed that his papers are now mostly of historical and not of practical significance.

In Reissner's 1936 paper, which emanated from his dissertation in Berlin, he assumes the plate to have frictionless contact with the soil. However, he did not actually manage to solve a true mixed boundary value problem, but assumed instead a uniform stress distribution underneath the plate together with the rather coarse assumption that the displacement at the center of the load equals the displacement of the plate. Barely a year later in another path-breaking paper, Reissner (1937) addressed the problem of an elastic half-space excited at the surface by concentrated and distributed torsional sources. He considered point moments, ring moments and distributed torsional disk sources, and assessed the torsional response of massive cylinders under the assumption of contact shearing stresses that increased linearly with distance to the axis. In addition, he also considered the case of a soil stratum of finite depth, a soil layer underlain by an elastic half-space, and even discussed briefly the generalization to soils whose properties change continuously with depth. Despite the simplifications concerning the distribution of contact stresses in either paper, and considering the notable insights he gained without the advantage of computers, especially with regard to *radiation damping* and to the *equivalent mass-spring-damper* analog system, it can be affirmed that Reissner is indeed the grandfather of dynamic SSI.

Just a few years later and in yet another towering achievement, Reissner & Sagoci together with a companion paper by Sagoci in the same 1944 issue of the *Journal of Applied Physics* provided not only the very first rigorous solution ever to a mixed boundary value problem involving a dynamically loaded plate but, together with exact formulas for rigid spheres to which we refer in the ensuing, constitutes one of the very few problems for which closed form solutions are known. Using oblate spheroidal coordinates, they managed to find exact expressions for rigid circular plates loaded in torsion at arbitrarily high frequencies, although they achieved this at the expense of a formulation in terms of admittedly obscure spheroidal wave functions. This choice resulted in formulas that "lacked a simple form", to borrow from the criticisms used in some more modern papers on soil-structure interaction, in which their authors go on to detract somewhat the practical value of the Sagoci solution while endorsing their own one based on integral transforms, or on purely numerical methods. They do this in part because accurate subroutines for spheroidal wave functions are not readily available or easy to come by, and also because the use of oblate and prolate spheroidal coordinates is only applicable to torsional problems, not to mention that the writers did not wish to educate themselves in the use of such functions. We may add that by means of a straightforward stretching of the vertical coordinates, the Reissner-Sagoci solution can readily be generalized to transversely isotropic media (Kausel, 2008).

Some four decades after Reissner & Sagoci, Apsel & Luco (1976) took up once more the use of spheroidal coordinates to provide an exact solution to the torsional response of both prolate and oblate ellipsoidal foundations embedded in an elastic half-space, subjected to a harmonic torque about the vertical axis, and to SH waves propagating along arbitrary directions. Except for the limiting case of a hemispherical foundation, for which exact results can be written, they ultimately used numerical approximations to evaluate the response. Based on their descriptions, it would seem that Apsel & Luco may have been unaware that the special problem of a rigid sphere in a full-space subjected to torsion —which differs trivially from the hemisphere— had already been solved earlier by Chadwick & Trowbridge (1967a) in both the frequency and time domains. An additional exact solution by these same writers (1967b) also exists for a sphere subjected to lateral (or vertical) loads, again both in the frequency and time domain. However, because of the presence of the free surface which serves as a guide for surface waves, the solution for the sphere in a full-space undergoing translation is no longer equivalent to a hemisphere in a half-space, although it probably exhibits similar characteristics. This could be ascertained by comparing the solution for the sphere with the solution for a hemisphere provided by Luco and Wong (1986), and of the four cases evaluated by them the torsional case should provide a perfect match. Now, the expressions for a rigid, massless sphere of radius a contained in a full space with *S* and *P* wave velocities C_S, C_P, mass density ρ, Poisson's ratio ν and subjected to either a dynamic torque $M_z(\omega)$ or a force $P(\omega)$ are so simple that they can be reproduced herein without much ado (the time domain solution is for a unit impulse). Let Ω be the dimensionless frequency, τ the dimensionless time, and β the ratio of S-wave to P-wave velocity, defined by

$$\Omega = \frac{\omega a}{C_S}$$

$$\tau = \frac{C_S\, t}{a} \geq 0$$

$$\beta = \frac{C_S}{C_P} = \sqrt{\frac{1-2\nu}{2-2\nu}}\ ,$$

The rotation and translation of the sphere are then

Torsion

$$\vartheta_z(\omega) = \frac{3}{8\pi G a^3} \frac{1+\mathrm{i}\Omega}{3+3\mathrm{i}\Omega-\Omega^2} M_z$$

$$\vartheta_z(t) = \frac{3}{8\pi\rho C_S\, a^4} \exp\left(-\frac{3}{2}\tau\right)\left[\cos\tfrac{\sqrt{3}}{2}\tau - \tfrac{\sqrt{3}}{3}\sin\tfrac{\sqrt{3}}{2}\tau\right]$$

Translation

$$u_z(\Omega) = \frac{1}{G a}\frac{3}{4\pi}\left[\frac{2+\beta^2-\beta^2\Omega^2+\mathrm{i}\Omega(2+\beta)\beta}{9-\Omega^2\left(1+9\beta+2\beta^2\right)+\mathrm{i}\Omega\left[9(1+\beta)-\Omega^2(1+2\beta)\beta\right]}\right]P$$

$$u_z(t) = \frac{1}{a^2\rho C_S}\frac{3}{4\pi}\left[\frac{1}{\beta(1+2\beta)}\sum_{j=1,3\neq k,l}\frac{2+\beta^2-\beta^2\Omega_j^2+\mathrm{i}\Omega_j(2+\beta)\beta}{\left(\Omega_j-\Omega_k\right)\left(\Omega_j-\Omega_l\right)}\exp\left(\mathrm{i}\Omega_j\tau\right)\right]$$

in which $\Omega_j, j=1,2,3$ are the three roots of the cubic equation

$$\mathrm{i}\,\beta(1+2\beta)\Omega^3 + \left(1+9\beta+2\beta^2\right)\Omega^2 - 9\mathrm{i}\,(1+\beta)\Omega - 9 = 0$$

For Poisson's ratios less than 0.498, $\Omega_2 = -\mathrm{conj}(\Omega_1)$ (i.e. the negative conjugate) while the third root is purely imaginary. Above that threshold all three roots are purely imaginary, of which the first root grows as a^{-1} without bound as the incompressible solid is approached. The response in the time domain for the translational case is qualitatively similar to the impulse response function of a 2-DOF system with supercritical damping. A rigid sphere with arbitrary mass, and displacements anywhere beyond the sphere may also be described explicitly in both the frequency and time domains. In addition, it is a simple matter to find complete solutions to the scattering problems associated with waves impinging onto the sphere.

The most important reason for summarizing the formulas for a sphere in a full space is that together with the Reissner-Sagoci + Apsel-Luco + Chadwick-Throwbidge solutions, these are *exact* formulas, and thus constitute valuable instruments in the arsenal of canonical problems, ready to serve as yardsticks against which the results obtained with approximate numerical methods can —and should— be judged.

The postwar decades of the 50's and 60's saw a rapid expansion of activities in the area of SSI, which led to the publication of a fairly large number of papers dealing with dynamically loaded circular plates resting on half-spaces as well as on strata of finite depth. Some of the most noteworthy among these are the papers by Quinlan (1953), Sung (1953), Bycroft (1956), Warburton (1957), Thomson & Kobori (1963), Awojobi & Grootenhuis (1965), and Gladwell (1968). This was also the time when Barkan's seminal —but by now outdated— book on dynamics of foundations reached publication, of which the English translation from the Russian made its debut in 1962. Another excellent book of this era is due to Richart, Hall and Woods (1970), which continues to be a valuable resource to this day. All of the previously cited papers are based on some kind of approximation, either in the way that the contact stresses are distributed, or in how the integral equations are solved and evaluated. A common characteristic is that they only provide results over a limited range of frequencies, and all but Thomson-Kobori address solely the problem of circular disks. Three of these papers merit further comments.

Bycroft's (1956) formidable paper —in which he spares nothing in heavy mathematical artillery— considers all four modes of vibration and assumes that the stress distribution in the dynamic case can be approximated by the static distribution and determines the plate's compliances by taking a weighted average of the displacements over the loaded area, which is in general an excellent approximation. However, he restricts his analyses to a fairly low frequency range. In addition, he appears to have been unaware of Mindlin's results for he arrives at a somewhat different static stiffness in swaying, namely

$$K_h = \frac{32(1-\nu)}{7-8\nu} G\,a \qquad \text{instead of} \qquad K_h = \frac{8}{2-\nu} G\,a$$

which is larger by a factor $8/7$ when $\nu = 0$, yet identical to Mindlin's when $\nu = 0.5$. This modest difference in horizontal static stiffness may well have resulted from Bycroft's rather inconsistent, simultaneous assumptions of zero rotation of the plate together with zero vertical contact stress, a problem of which he was certainly aware, for he mentioned that much in a brief comment. Some two decades later, Bycroft (1977) picked up the subject once more and provided asymptotic approximations for the compliances in all four modes when the frequencies are high.

Then there is the paper by Thomson & Kobori (1963). Of all works cited above, it is the only one of that era to address the problem of rectangular foundations subjected to vertical loads (but note that Awojobi & Grootenhuis, 1965, also considered 2-D strip footings). Like other papers of this period, they again limited their results to low frequencies, assumed the plate to be smooth, the vertical stresses to be uniform, and rather disappointingly, that the compliance of the plate was the same as the displacement at the center of the rectangle. Thus, these compliances are almost certainly too large.

INTERACTION EFFECTS WITHIN AND NEAR THE STRUCTURE

Up to this point in time, the narration has focused solely on the foundation, but it is the interaction of seismic waves with the structure and its effects on the soil nearby which ultimately matter. As it turns out, the problem of dynamic ground-structure interaction had already been considered as early as 1935 in Japan by Katsutada Sezawa and Kiyoshi Kanai (1935a) who published a truly remarkable, pioneering paper on the subject. In their work, they modeled an idealized structure as a thin cylindrical rod terminated at the base by a hemispherical tip which is embedded in a homogeneous half-space. The latter is subjected to plane, vertically propagating *P* waves which, upon hitting the hemispherical foundation, are partly scattered in all directions and partly transmitted into the rod, which in turn feeds back into the soil and contributes to the scattered field. They had, of course, to make simplifying assumptions in their model, such as the wavelengths of incident waves in the soil being much longer than the width of the structure, and they also ignored diffraction effects on the scattered field elicited by the adjacent free surface. Using analogous methods, they also considered shear beams and bending beams supported by a half-space subjected to vertically propagating S waves. They ultimately concluded that the severity of the motion in the structure was limited by the loss of energy in the form of waves that feed back into the soil, even for an undamped superstructure, so that resonance effects remain limited, and thus, that SSI is beneficial. Contemplating in this day and age those very early results, one can only marvel and wonder how, despite the significant complications entailed by their equations, the researchers managed to evaluate and present plots of amplification functions at a time when computers did not exist.

Half a decade later in 1940, Caltech professor Romeo Raoul Martel offers one of the earliest commentaries in the US on the possible interaction between structures and soils. Although mostly anecdotal in character, his observations cite the results of studies on the 1933 Long Beach earthquake together with observed effects on the Hollywood Storage Building as well as Japanese researches of the 1930's, and opined that damage to buildings on soft soils, deep alluvia, or high elevations can be expected to be more widespread than in buildings resting on firm or level ground. In addition, citing the relative seismic quiescence observed in tunnels, he also speculated on the possible reduction of motion intensity with depth. However, Martel lacked the wherewithal to confirm his predictions, for strong motion instruments were rare in his time, seismic records were few in number and he had little reliable empirical (or theoretical) evidence on which to base his predictions.

The topic was taken up again in 1954 by R.G. Merrit together with earthquake engineering legend, Caltech's Prof. George Housner, who began by observing that horizontal records obtained in basements are similar to records of motions on parking lots nearby, which is evidence that the lateral compliance of the foundation has little or no effect on motions. However, they surmised that rocking could be important and that its telltale signature might conceivably show up in the field records. To demonstrate this effect, they proceeded to idealize the superstructure as a rigid block mounted on a rotational spring whose stiffness was based on the moment of inertia of the foundation about the rocking axis together with the bearing capacity of the soil, and assessed this admittedly simple system with an ad hoc *analog* computer made up of electrical circuits, a tool whose very application would be unthinkable in today's digital world. With this rather crude model, they arrived at the result that any beneficial effects of rocking in reducing base shear would depend strongly on both the earthquake characteristics and the height of the building, yet ultimately they attained inconclusive results, bringing their paper to a close with the observation that "the base shear … will [not be affected] by any degree of foundation compliance that can be expected in standard practice." A few years later, Housner (1957) once more revisits the subject and on the basis of actual strong motion records demonstrates that the Hollywood Storage Building had measurable effects on motions nearby. This elongated building —which Hradilek and Luco (1970) report being mounted on concrete piles— has a short dimension in the North-South direction, and a long dimension in the East West direction. By comparing the motions in the basement in both directions with the corresponding motions recorded outside in the parking lot, Housner observed that waves in the ground that propagated along the long direction suffered significant filtering, yet waves moving in the short direction did not. Thus, he was the first researcher to demonstrate rather conclusively that the phenomenon now referred to as wave passage or *kinematic interaction*, is real and results in decreased effective motions in the vicinity of a relatively rigid structure of sizable proportions. This occurs because the stiff structure cannot accommodate the deformations in the ground elicited by waves shorter in wavelength than the dimensions of the foundation, and thus filters them out.

A decade later, the celebrated engineer and University of Illinois' Prof. Nathan M. Newmark (1969) considered the torsional response of otherwise symmetric structures elicited by waves which pass underneath the foundation, a phenomenon that he referred to as the *Tau Effect*. By this he meant the time delay in excitation to parts of the foundation caused by waves that impinge first on one side of the building and then on the other, i.e. $\tau = L / C_s$, where L is the width of the foundation and C_s is the speed of horizontally propagating shear waves. This phenomenon too is a manifestation of kinematic interaction, and it results in effective seismic motions to the structure that would not exist if the waves propagated vertically and the structure was not embedded. Thereafter, the topic was taken up again many other researchers, including a well known study by Robert Scanlan (1976). A beautifully simple engineering approximation to this phenomenon, which merely requires knowledge of the free-field problem evaluated at the soil-structure interface together with the stiffness functions for the embedded foundation, was given by Iguchi (1982).

During the mid 1960's, Parmelee made initial assessments of SSI effects by means of a very simple structural model with three degrees of freedom, namely the translation and rotation of the base together with the lateral motion of the superstructure. The system was mounted on lateral and rocking springs based on Bycroft's stiffness functions. In the first of his two papers, only static stiffnesses without any damping were used, and only harmonic response functions were obtained, while in the second, he considered the frequency-dependence of the foundation impedances over the limited frequency range for which they were available, and employed synthetic earthquakes with no more than ten terms. Lacking an FFT at the time of this work — Cooley-Tukey's FFT algorithm only saw the light in 1965— and struggling with numerical limitations, this study attained only limited success in evaluating SSI effects.

Shortly thereafter, Sarrazin et al (1972) adopted once again Parmelee's model but accounted also for the height of the center of mass of the foundation above the line of action of the soil springs. More importantly, they used both frequency-dependent impedances and also evaluated the impedances at the coupled soil-structure frequencies so as to be able to work with frequency-independent “springs and dashpots”. They presented extensive sets of graphics with the relative values of rocking, swaying and structural frequencies, and most importantly, the effective damping at the frequency of the coupled mode of vibration as a function of the aspect ratio i.e. of the building height. They found that rocking damping can be very low —just a few percent— but that swaying damping is typically very high, anywhere from 30% to more than 100% of critical value. Thus, they confirm Housner’s contention that in most practical cases swaying is not important. In their study, they also apply random vibration theory to assess the maximum response of the structure to white noise input. By and large, this well-organized investigation concludes that SSI is beneficial in that response amplitudes are decreased in comparison with the response for a fixed based condition. They also point out that hysteretic damping in the soil is very important, especially when the coupled soil-structure frequency is low, i.e. when radiation damping in rocking is low. The results of this compelling study with simple models are most valuable when considered in the context of simplified, code-type design procedures.

Finally, in 1975 and 1977, the eminent Rice University Prof. Anestis Veletsos shows that interacting systems can accurately be modeled via simple systems with modified periods and appropriate levels of damping, which allows applications to design with standard response spectra. As Housner and Sarrazin before him, he observes that the translation of the base is similar to that of the free field, and that rocking is important. Veletsos also concludes that hysteretic damping is essential, that SSI increases damping and reduces deformations in the structure, and that for tall structures the effective damping in SSI may sometimes be less than for fixed-base structures.

CONTEMPORARY ERA

The beginning of the modern era in SSI can be said to have begun some four decades ago with the publication of the profoundly influential papers by Veletsos & Wei (1971) and by Luco & Westmann (1971, 1972), which provided complete rigorous solutions to the problem of circular plates underlain by elastic half-spaces excited dynamically over a broad range of frequencies, and for a wide set of Poisson’s ratios. After these pioneering works made their debut, the rate of progress in the discipline of SSI took a rapid acceleration and diversification, driven mainly by the needs of the nuclear power and offshore industries. In addition to Juan Enrique Luco and Anestis (“Andy”) Veletsos, some of the other principal figures working on dynamic SSI at that time included Harry B. Seed, John Lysmer, Anil Chopra, Izzat M. Idriss, Paul C. Jennings, Jacobo Bielak, Paul Christiano, Hung L. (“Dave”) Wong, Mihailo D. Trifunac, Robert V. Whitman, José M. Roësset, John T. Christian, Milos Novak, W.D. Liam Finn, Hiroshi Tajimi, Takuji Kobori and Ryoichiro Minai, just to name a few heavyweights.

The decades of the mid 1960’s to mid 1970’s were also marked by the triumphal entry of powerful digital computers together with versatile numerical methods —especially finite elements— both of which helped to radically change the research paradigm and shift its emphasis away from purely analytical methods. Thus, instead of continuing to solve highly idealized mathematical problems involving, say, rigid circular disks welded onto perfectly homogeneous half-spaces, it became possible to address irregularly-shaped, flexible foundations embedded in inhomogeneous or layered media, and even account for rather complex effects such as the inelasticity of the soil. This was also the time when sophisticated computer programs such as SHAKE, LUSH, SASSI, and CLASSI entered the scene and —at least in the nuclear power industry— acquired mythical status as the supreme instruments by means of which one could solve nearly any practical SSI problem. As a result, some of the programs began being used in the industry by persons who lacked proper knowledge of the underlying assumptions, i.e. as black boxes, a situation that was clearly undesirable. Then again, as the numerical predictions began to outrun the observable, they stimulated intensive research on experimental methods and laboratory verification, all of which helped to provide a reality-check on the complex numerical models then in vogue.

Peculiarly, although the Finite Element and Boundary Element Methods saw the light at about the same time, the latter took much longer to find widespread use. One of the pioneers in the use of Boundary Elements for foundation mechanics problems was José Dominguez (1978a,b), who first obtained the impedances of rectangular foundations embedded in an elastic half-space.

It was also during this time that the writer was fortunate enough to have worked on the subject of SSI and have published a lengthy seminal paper which received wide attention by bringing into focus some key aspects of SSI and clarifying the source of inconsistencies observed in analyses by alternative methods (Kausel & Roësset 1974). Indeed, many of the SSI models employed up through the early 1970's were relatively "simple" in the sense that they were restricted to systems in which the foundation rested directly onto the surface of a homogeneous half-space, and the seismic motion in the free field was invariant in horizontal planes, e.g. the motions resulted from waves propagating vertically in a laterally homogeneous soil. For such models, the intuitively obvious strategy of prescribing the free-field motion directly underneath the soil "springs" supporting the structure in a formulation in the frequency domain was both sufficient and rigorous. However, when discrete methods of analyses, such as finite elements, started being applied to SSI problems, and especially when embedded structures began to be considered, substantial discrepancies were observed between the results of the numerical analyses and the classical analytical method, which demanded an explanation as to why the differences. This motivated the development by us of the so-called *three-step solution*, which provided the means to accomplish fully consistent comparisons between the results obtained by purely numerical models with finite elements and those by the lumped parameter method based on foundation impedances or "springs" together with seismic motions prescribed underneath these springs. In a nutshell, the three steps in this method are:

- *Kinematic interaction*, which considers the response of the foundation *embedded* in the *actual* soil and subjected to the seismic environment defined in the free field *at the soil-structure interface* before the soil has been excavated.
- *Foundation stiffnesses*, which provide the frequency-dependent impedances for the foundation *embedded* in the *actual* soil medium.
- *Inertial interaction*, where the structure is supported on the impedances determined in step 2, and is subjected at the base to the motions found in step 1.

Thus, the comparisons of the results obtained with both the *direct* approach (i.e. finite elements) and with the *lumped parameter* method were inconsistent —the apples and oranges problem— and disagreed with each other because the wrong stiffnesses and the wrong support motions were used. Details are by now well known and need not be elaborated further herein.

For the record, however, we wish to add that the monikers "kinematic interaction" and "inertial interaction" were originally coined by Prof. Robert V. Whitman at MIT in the months following our paper. To the best of our knowledge, these concepts first appeared in print in our lecture notes in a well-attended course on SSI in Santa Margherita, Italy (Kausel, 1976) and then again in a refereed journal by Kausel, Whitman et al (1978).

From this point on, the discipline broadened enormously and gave rise to myriad papers, to the point that we could not do justice herein with even a cursory review. Examples are:

- The free field problem, especially the amplification of vertically propagating waves by means of practical algorithms which account for inelastic effects, e.g. Schnabel et al (1972).
- Complex material behavior, e.g. Seed & Idriss (1969)
- Consideration of non-horizontally layered soils, undulating or dipping layers, subjected to miscellaneous kinds of seismic waves
- Soils whose properties change continuously with depth, e.g. Vrettos (1991,1999).
- Diffraction of waves by topographic features, such as basins, canyons and depressions, wedges, cliffs or hills, and their effect on structures, e.g. Sánchez-Sesma et al (2000).
- Scattering of waves by foundations embedded in various soil configurations, e.g. Trifunac (1971, 1973), Lee (1988).
- Footings of miscellaneous shapes and with various degrees of embedment, subjected to either forces or arbitrary seismic environments.
- Footings on layered soils, e.g. Luco (1974); Kausel (1974); Gazetas & Roësset, (1976)
- Effect of foundation flexibility, e.g. Savidis & Richter (1979).
- Single piles, pile groups, caissons, and all of these in various kinds of soils, e.g. Kaynia & Kausel (1982).
- Multiple footings, i.e. structure-soil-structure interaction, e.g. Wong & Luco (1986), Lin et al (1987)
- Structural response and effects, e.g. Roësset et al (1972), Bielak (1976)
- Non-linear effects, such as partial lift-off and inelastic soils.
- Poro-elastic effects.
- Transverse isotropy, anisotropy
- Numerical methods, e.g. Karabalis & Beskos (1984), Apsel & Luco (1987), Alarcón & Cano (1989)
- Large scale experimentation on SSI effects

Still, in the next decade or two, a subset of the best contributions by contemporary researchers in SSI will eventually be duly recognized for their lasting value, at which time they too will be the subject of historical surveys yet to be written. In all likelihood, however, such future reviews will have to focus much more narrowly on specific sub-disciplines, given the significant breadth that the subject of SSI has now attained.

EPILOGUE

The writer together with his innumerable colleagues around the world —many of them personally known to him— have been privileged to have contributed to the development of a fascinating technical discipline. We are also grateful for the opportunities that the engineering science community at large as well as the funding agencies afforded us and made our work

possible. Nonetheless, a cause of some concern is the recent proliferation of papers which seemingly rediscover known facts and methods and mistake these for new knowledge. It is not recognition that any in the older generation seeks —as the late pop artist Andy Warhol would have put it, they all had their fifteen minutes of fame— but for the state-of-the-art to make true progress, it behooves for the current and future generations of young researchers and scientist to be aware of what has already been accomplished. It is only by profiting from the predecessors' already accumulated experience, knowledge and wisdom that true progress may accrue in decades to come. Or as the Latin metaphor has it, *nanos gigantum humeris insidentes.*

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