Aircraft Attitude Determination
Using Robust Estimation

by

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Abstract
This thesis investigates the use of robust $H_{\infty}$ estimation for the attitude determination of an aircraft. Specifically, it is shown that acceptable pitch angle estimation may be obtained using merely a rate gyroscope, an airspeed sensor and the aircraft longitudinal equations of motion. No outside navigation aids are assumed. The thesis demonstrates that robust $H_{\infty}$ estimation is a useful tool for such design, yielding significant robustness to aircraft modeling uncertainties and neglected dynamics. For various aircraft operating conditions and gyroscope qualities, the performance of the robust filter is compared to that of a Kalman filter "robustified" by increasing the process noise covariance of its underlying state-space model. Overall, numerical results demonstrate that the robust $H_{\infty}$ filter handles model uncertainties and neglected dynamics better than the robustified Kalman filter.

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Chapter 1

Introduction

In any engineering application involving signals, there is often a noisy component that one wishes to eliminate. Filtering the noise out of a signal has been for many years the focus of much scientific and technical effort. Kalman revolutionized the field when he introduced his filter in the early 1960s, providing a way to design recursive linear least-square estimators.

However, the Kalman filter's optimality is based on four main hypotheses:

- the performance index is least-square errors,

- the plant of interest is described by a linear state-space model,

- one has perfect knowledge of the noise statistics,

- one has perfect knowledge of the plant model.

Over the past few years, ways to deal with the two last restrictions of Kalman filtering have received much attention in the control/estimation community. In [2, 3], Appleby derived steady-state estimators that are robust to both noise and plant model uncertainties. More recently, Mangoubi [15, 16, 17] came up with time-varying estimators also yielding robustness to both noise and plant model uncertainties. Yet, these theoretical results have rarely been applied to practical problems. In [13], robust estimators are obtained for the space shuttle main engine. In [1], Agustin gets very good results for robust estimation of the thrust of the shuttle during reentry.
In this thesis, we investigate the application of robust $H_\infty$ estimators from [15, 16, 17] to the design of an attitude determination system for an aircraft. Specifically, we seek to obtain good pitch angle estimation of the aircraft using merely an airspeed sensor, a rate gyroscope and the aircraft longitudinal dynamic equations of motion. No outside navigation aids are assumed.

The approach in this thesis bases the filter design on the aircraft dynamics, following [10, 14, 20], rather than on the Inertial Navigation System error propagation dynamics, as is usually done (see [18]). As such, whereas traditional navigation filters are conceptually autonomous from the vehicle on which they are installed, the approach in this thesis uses the a priori information provided by the vehicle dynamics to enhance the measurement information provided by the sensors. When designed with conventional Kalman filters, however, such navigation systems are very sensitive to model uncertainties. This thesis investigates ways to reduce or eliminate the sensitivity to plant and noise model uncertainties through the use of the new robust filtering technique from [15, 16, 17]. Because model uncertainties are often unpredictable, we test the robust filter designs with various aircraft models linearized around a wide range of altitude and Mach numbers.

The thesis is organized in the following way. Chapter 2 provides the problem formulation. First, a brief overview of navigation system design is presented. Then the aircraft and sensor models used throughout the thesis are described.

Chapter 3 gives an overview of Kalman filtering. The sensitivity of the filter to mismodeling is analyzed and demonstrated in the context of our navigation problem.

In Chapter 4, a filter robust to noise model uncertainty is presented. This filter does not assume any specific spectral content to the disturbance, but rather seeks to bound the ratio of the error estimation energy to the disturbance energy. The filter is applied to our specific navigation problem in the presence of unmodeled wind gust dynamics.

Chapter 5 presents a filter with robustness to both noise and plant model uncertainties. Here, the robust filter is compared with the Kalman filter and numerical
results are presented for various aircraft operating conditions and sensor model cases.

Finally, in Chapter 6, the conclusion summarizes the results of the thesis and provides suggestions for further work.
Chapter 2

Problem Formulation

2.1 Two different approaches to navigation system designs

Navigation has probably been the most successful application of Kalman filtering since it was first derived.

A widely advocated approach to the navigation filter design is the following: an Inertial Navigation System (INS) provides the vehicle with attitude and position information. Even with the best INS units, attitude/position measurements are contaminated with biases. An external source of attitude/position information, often provided by GPS, is used in a Kalman filter to estimate and correct the bias errors of the inertial system. An example of such a filter is given in Figure 2-1 (from [18]). Note that in this approach, the Kalman filter seeks to estimate the inertial system errors, instead of trying to obtain directly attitude and position information. These errors are then subtracted from the INS measurements in order to reduce or eliminate biases, as eventually biases will grow.

The above approach is viable because the INS and GPS measurement errors have different spectral content. INS typically tracks high frequency components of attitude and position very well, but it is often contaminated with low frequency bias errors. In contrast, the external aiding source is often contaminated with white noise.
disturbances, but it provides a good low frequency tracking of position and attitude. The Kalman filter then blends optimally the high-frequency information of the INS instrument with the low-frequency information of the external aiding source. As such, it operates as a high-pass filter for the INS measurements and as a low-pass filter for the external aiding measurements.

In this approach, the navigation filter is conceptually autonomous from the vehicle on which it is installed, since it is designed to model the dynamics of INS error propagation. Specifically, it does not use the available source of information that is provided by the vehicle dynamic equations of motion. This waste of information is mainly motivated by the fact that the dynamic model of the vehicle may not be accurate enough to provide the filter with valuable information. It may even induce the navigation system into error when the dynamics of the vehicle are not properly modeled.

However, there are some situations where information provided by the vehicle dynamics may be beneficial or even necessary. In particular, this is the case when external aiding instruments are not available. A typical example of this could be a scenario where GPS is the external source of data and it is disabled by jamming. Without the external aiding, the conventional navigation filter will fail to provide an accurate attitude/position determination over the long term. In such a situation, it may be a good idea to use the information provided by the vehicle dynamics.
For this reason, another approach to navigation system design has been proposed [5, 10, 14, 20]: it consists of using a Kalman filter based on the vehicle dynamics for the design of a navigation system. The structure of such a filter is given in Figure 2-2. In this case, the Kalman filter seeks to directly provide an estimate of position and attitude in addition to trying to estimate the errors in the INS. It should also be noted that this filter structure suits well the Kalman filtering philosophy, in that it blends optimally the *a priori* information provided by a dynamic model of the vehicle with the *measurement* information provided by the various sensors.

![Diagram](image)

**Figure 2-2: Alternative approach to navigation filters**

As mentioned earlier, navigation system designers have often been reluctant to rely on vehicle model information, preferring to rely on highly accurate, but expensive INS units. The issue of plant and noise model inaccuracies is definitely a real one in some cases. Our objective is to demonstrate that a new filter design methodology can take into account such uncertainties, and still provide information that can complement an INS.
2.2 Model description

Our problem is the following: we seek to obtain an acceptable estimate of the pitch angle of an aircraft using merely a rate gyroscope, an airspeed sensor and the aircraft dynamics. No outside navigation aids are available to the flight vehicle.

A possible scenario is an autonomous or remotely piloted aircraft flying for reconnaissance during battle. Such aircraft rely on cheap INS units and do not carry outside navigation aids. Even if outside aids are available, they cannot be used too often as jamming is part of the battle scene. Another possibility is an aircraft whose outside navigation aids have failed. Here the navigation system must last as long as possible.

We will use the linear models for a Boeing 747 aircraft, augmented with a rate gyro and an airspeed sensor. It should be noted that the results obtained in this thesis are valid for the case of a small pilotless aircraft as well, since large and small vehicles are often aerodynamically similar.

In this section, we present the plant and sensor models considered.

2.2.1 Aircraft dynamics

An airplane in flight is a very complicated dynamic system. However, assuming that the aircraft can be considered as a rigid body, the dynamic equations simplify considerably. Furthermore, it can be shown that for small perturbations about a nominal operating condition, the lateral and longitudinal dynamics may be decoupled to first order [8]. This simplification allows the lateral and longitudinal states to be separated, each being represented by a four element state vector. We focus on the longitudinal aircraft dynamics.

Figure 2-3 shows the coordinate representation of the plane. The body-axis system (0X_bY_bZ_b) consists of right-handed, orthogonal axes whose origin is fixed at the aircraft center of gravity. Its orientation remains fixed with respect to the aircraft. In this representation, \( V_o \), \( U_o \), \( W_o \), and \( \theta_o \) express, respectively, the nominal total speed, the nominal forward speed, the nominal normal speed, and the nominal pitch
angle for a given operating condition. The pitch angle is measured with respect to an inertial reference, assuming a flat earth.

For small perturbations around nominal, the longitudinal dynamics of the aircraft are given by the following equation [12]:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -Z_{\dot{w}} & 0 \\
0 & -M_{\dot{w}} & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
X_{u} & X_{w} & -W_{o} & -g\cos(\theta_{o}) \\
Z_{u} & Z_{w} & Z_{q} + U_{o} & -g\sin(\theta_{o}) \\
M_{u} & M_{w} & M_{q} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix}
+ 
\begin{bmatrix}
X_{\delta e} & X_{\delta t} \\
Z_{\delta e} & Z_{\delta t} \\
M_{\delta e} & M_{\delta t} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta e \\
\delta t
\end{bmatrix}
$$

(2.1)

In the above equation, the state vector $x = [u \ w \ q \ \theta]^T$ represents perturbations from nominal in forward speed ($ft/s$), normal speed ($ft/s$), pitch rate ($deg/s$) and pitch angle ($deg$) respectively. The command input vector $s = [\delta e \ \delta t]^T$ represents perturbations from nominal in elevator deflection ($deg$) and thrust ($ft/s^2$) respectively. Finally, terms such as $X_u$ or $Z_w$ are dimensional stability derivatives.

These dimensional stability derivatives were obtained in [12] for the Boeing 747
at various Mach numbers and constant altitudes in steady-state rectilinear flight. Equation (2.1) may then readily be inverted and provides us with a continuous-time linear state-space model of the form:

$$\dot{x} = Ax + Bs$$  \hspace{1cm} (2.2)

The $A$ and $B$ matrices are obtained for the various operating conditions listed in Table 2.1, and are explicitly provided in Appendix A. In the following, we will consider that our nominal operating condition is at Mach .7, 40000 ft altitude, while other flights will be considered as plant perturbations from the nominal one. Note that the plant perturbations considered here encompass both Mach number and altitude uncertainties.

<table>
<thead>
<tr>
<th>Mach number</th>
<th>Altitude (kft)</th>
<th>Nominal pitch angle $\theta_o$ (deg)</th>
</tr>
</thead>
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<tr>
<td>.9</td>
<td>40</td>
<td>2.4</td>
</tr>
<tr>
<td>.7</td>
<td>40</td>
<td>7.3</td>
</tr>
<tr>
<td>.5</td>
<td>20</td>
<td>6.8</td>
</tr>
<tr>
<td>.65</td>
<td>sea level</td>
<td>0</td>
</tr>
<tr>
<td>.2</td>
<td>sea level</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Table 2.1: Operating conditions for the Boeing 747

For use with a discrete-time filter, the plane dynamics were discretized with a sampling rate $\Delta t = 0.05s$ using a zero-order hold method.

The next step was to define some process noise associated with the aircraft dynamics. Here, the assumption was that the process noise enters the dynamics through the same channels as the input, defining the discrete-time process noise matrix $G_d$ as 1/10 of the discrete-time input matrix $B_d$. Other process noise matrices may have been used. However, the overall goal of the filter design is to be able cope with various process noise matrices, as this noise may not be properly modeled. In the following, we will indeed show that our designs are able to cope with various wind gust process noises.

Overall, the discrete-time state-space model for the longitudinal dynamics is ex-
pressed by:

\[ x_{k+1} = A_dx_k + B_ds_k + G_d\eta_k \]  

(2.3)

where \( \eta_k \) is a unit covariance white noise.

Our nominal time of simulation was 1000 seconds and we defined the command inputs as follows:

- At \( t = 0 \) s, the command is a step input that gives a steady-state forward velocity increase of 10 ft/s above the reference forward velocity while remaining at a constant altitude for the nominal model (Mach .7/40 kft).

- At \( t = 500 \) s, we reverse this command and obtain a steady-state velocity of 10 ft/s below the reference forward velocity while remaining at a constant altitude for the nominal model (Mach .7/40 kft).

Geometrically (see Figure 2-3), the change in altitude \( h \) around a nominal operating condition is related to the other state variables by the following relation:

\[ \dot{h} = - (W_o + w) \cos(\theta_o + \theta) + (U_o + u) \sin(\theta_o + \theta) \]

Linearizing the above expression and keeping only first order terms in \( u, w \) or \( \theta \), we obtain:

\[ \dot{h} = - w \cos \theta_o + u \sin \theta_o + (U_o \cos \theta_o + W_o \sin \theta_o) \theta \]

\[ = - w \cos \theta_o + u \sin \theta_o + V_o \theta \]

(2.4)

The steady-state change in elevator and throttle required to produce the desired small change in steady-state forward speed while remaining at a constant altitude may be obtained from (2.2) and (2.4) by putting

\[ \dot{u} = \dot{w} = \dot{q} = \dot{\theta} = \dot{h} = 0 \]

We may then solve numerically for the command input vector \( s = [\delta e \ \delta t]^T \) as a function of the desired steady-state forward velocity \( u \). In our case, the command
inputs for the time-simulation may be visualized in Figure 2-4 and the response of the plane to such inputs is given in Figure 2-5.

Figure 2-4: Command inputs considered for the time-simulation

2.2.2 Measurement models

We assume that two states of the plant are measured: the forward velocity is measured with an airspeed sensor, and the pitch rate is measured with a gyroscope. Note that our state of interest, the pitch angle, is not directly measured, as no outside navigation aids are assumed. Rather, it is inferred from the dynamic model of the aircraft.

Airspeed sensor model

An airspeed sensor compares the dynamic and the static pressure around the aircraft, which enables us to obtain a measure of the forward velocity of the aircraft. In this thesis, we will assume that the airspeed sensor provides a measure of the forward speed with a discrete white noise disturbance with standard deviation $\sigma_{ud} = 10$ ft/s.

Gyroscope model

A gyroscope provides the measure of the pitch rate. We consider two main sources of error in this measurement:
Figure 2-5: State response of the plane to the standard command inputs
• The gyro bias

Assuming that the constant bias in the measurement has been initially removed through calibration, we consider only the drift term in the bias. It is modeled by a first-order Markov process with a correlation time constant $\tau$ of 6 hours and a RMS value $\sigma_b$ for the bias stability depending on the gyro quality considered (see Table 2.2).

It thus consists of a continuous-time state-space model [9]:

$$\dot{b} = -\frac{1}{\tau} b + \sqrt{\frac{2\sigma_b^2}{\tau}} \eta_b(t)$$

where $\eta_b(t)$ is a unit intensity continuous-time white noise.

The discretization of such a process leads to:

$$b_{k+1} = a_b b_k + g_b \eta_{bk}$$

with

$$a_b = e^{-\frac{\Delta t}{\tau}} \quad \text{and} \quad g_b = \sigma_b \sqrt{1 - e^{-2\frac{\Delta t}{\tau}}}$$

where $\eta_{bk}$ is a unit variance discrete-time white noise and $\Delta t$ is the sampling interval (see Appendix B).

• The angle random walk

Any gyroscope measurement contains white noise. When integrated, this results in an angle random walk. This means that direct integration of the gyroscope measurement to obtain the pitch angle would result eventually in an unbound error component. By using the plane dynamics, we hope to address this problem and keep a bounded error on the angle estimate.

The value of “angle random walk” provides us with the intensity of white noise $\sigma^2$ for a continuous-time measurement. The variance of the equivalent discrete-
time white noise is then expressed as:

$$\sigma^2_{q_d} = \frac{\sigma^2_q}{\Delta t}$$

where $\Delta t$ is the sampling interval.

In this thesis, two qualities of gyro will be considered and are referred to as “Good” and “Low-cost” in Table 2.2.

<table>
<thead>
<tr>
<th></th>
<th>Good</th>
<th>Low-cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias stability (deg/hr)</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>Angle random walk (deg/sqrt(hr))</td>
<td>.8</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2.2: Gyroscope measurement specifications

Other sources of gyroscope measurement errors such as scale factor error, non-orthogonality error, and misalignment error will be neglected in this thesis. Our hope is to show that filtering design can be made insensitive to the neglected dynamics.

### 2.2.3 Augmented model

Overall, the discrete-time augmented plant consists of the dynamics of the aircraft and those of the gyro bias.

$$
\begin{bmatrix}
  x_{k+1} \\
  b_{k+1}
\end{bmatrix} =
\begin{bmatrix}
  A_d & 0 \\
  0 & a_b
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  b_k
\end{bmatrix}
+ 
\begin{bmatrix}
  B_d \\
  0
\end{bmatrix}
\begin{bmatrix}
  \delta c \\
  \delta t
\end{bmatrix}
+ 
\begin{bmatrix}
  G_d & 0 & 0 & 0 \\
  0 & g_b & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  r_k
\end{bmatrix}
$$

(2.5)

Since we measure the forward velocity and the pitch rate, our measurement equation for this augmented system is:

$$
y_k = 
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_k \\
  b_k
\end{bmatrix}
+ 
\begin{bmatrix}
  0 & 0 & \sigma_u & 0 \\
  0 & 0 & 0 & \sigma_q
\end{bmatrix}
\begin{bmatrix}
  r_k
\end{bmatrix}
$$

(2.6)

where $r_k = [\eta_k \eta_{bk} v_{uk} v_{qk}]^T$ is a unit covariance white noise vector.
This augmented model as well as the sought after filter to estimate the pitch angle may be visualized in Figure 2-6.

Figure 2-6: Augmented plant and filter structure

Our problem is to find a filter that yields an acceptable pitch angle estimate for a wide range of noise and plant model uncertainties.
Chapter 3

Kalman Filtering and Model Uncertainties

3.1 Overview of Kalman filtering

Consider a linear state-space representation of the plant of interest:

\[ x_{k+1} = A_k x_k + G_k r_k \]  \hspace{1cm} (3.1)
\[ y_k = C_k x_k + D_k r_k \]  \hspace{1cm} (3.2)
\[ E(r_k r'_l) = \delta_{kl} I \]  \hspace{1cm} (3.3)

where \( x_k \) denotes the state vector, \( y_k \) denotes the measurement vector, and \( r_k \) is a white noise disturbance normalized to have unit covariance. The matrices \( G_k \) and \( D_k \) define how the disturbance enters the state and the measurement respectively. Any covariance characteristic may be obtained by adjusting the values of matrices \( G_k \) and \( D_k \). Furthermore, the particular case where the process noise and the measurement noise are uncorrelated is obtained when \( G_k D'_k = 0 \).

Given the output of the system, the Kalman filter seeks to obtain the best estimator of the state of the system, with a mean-square error performance index.
Specifically:

\[
\min_{\hat{x}_k} E \left( \| e_k \|^2 \right) \equiv E (e_k^T e_k), \quad k \in [1, K] \quad (3.4)
\]

subject to \ Eqs. (3.1) – (3.3)

with

\[
\tilde{x}_k \equiv x_k - \hat{x}_k \quad \text{and} \quad e_k \equiv M_k \tilde{x}_k
\]

where \( \tilde{x}_k \) is an estimate of \( x_k \) and \( e_k \) is the estimation error sequence weighted by some matrices \( M_k \).

**Recursive solution**

We give here the Kalman filter equations for the case where process noise and measurement noise are uncorrelated \( (G_k D_k^T = 0) \), so as to gain insight on how the filter operates.

It turns out that problem (3.4) has a recursive solution that alternates *update* and *prediction* steps. In the *update* step, the measurement information is optimally incorporated in the current state-estimator. In the *prediction* step, the filter uses the a priori knowledge about the state dynamics so as to propagate the estimator from one sample-time to the next.

Throughout this thesis, we will use the following standard notation: \( \hat{x}_k \) denotes the *a priori* estimate of \( x_k \), i.e. the estimate based on the observations \( y_i \) with \( i < k \), while \( \hat{x}_k^+ \) denotes the *a posteriori* estimate of \( x_k \), i.e. the estimate based on the observations \( y_i \) with \( i \leq k \). The same notation holds for the estimation error \( \tilde{x}_k \) and the error covariance \( P_k \equiv E(\tilde{x}_k \tilde{x}_k^T) \).

The recursion steps are the following:

**Prediction**

\[
\hat{x}_{k+1} = A_k \hat{x}_k^+ \quad (3.5)
\]
\[ P_{k+1} = A_k P_k^+ A_k^T + G_k G_k^T \]  \hspace{1cm} (3.6)

**Kalman gain computation**

\[ K_{k+1} = P_{k+1} C_{k+1}' \left( C_{k+1} P_{k+1} C_{k+1}' + D_{k+1} D_{k+1}' \right)^{-1} \]  \hspace{1cm} (3.7)

**Update**

\[ \hat{x}_{k+1}^+ = \hat{x}_{k+1} + K_{k+1} (y_{k+1} - C_{k+1} \hat{x}_{k+1}) \]  \hspace{1cm} (3.8)

\[ (P_{k+1}^+)^{-1} = P_{k+1}^{-1} + C_{k+1} (D_{k+1} D_{k+1}' \left( D_{k+1} D_{k+1}' \right)^{-1} C_{k+1}' \]  \hspace{1cm} (3.9)

In the prediction step, the filter extrapolates its best estimate at step \( k \) to step \( k + 1 \). Since no measurement is available between steps, the filter has no alternative but to rely solely on the a priori information provided by the state-space dynamics of the state. Hence, in the presence of a zero-mean white noise disturbance, the filter cannot do better than propagate the estimator with the state transition matrix \( A_k \) as is done in equation (3.5).

Equation (3.6) indicates how the uncertainty in the estimation error builds up during the prediction step. Specifically, the error covariance matrix is the sum of two terms. While the first one \( (A_k P_k A_k^T) \) is merely the propagation of the error covariance, the second one \( (G_k G_k^T) \) comes from the process noise and feeds uncertainty into the system.

In the update step, the filter incorporates the measurement information into its estimate through equation (3.8). The a posteriori estimate is given by the a priori estimate plus a corrective term proportional to the innovation term \( (y_{k+1} - C_{k+1} \hat{x}_{k+1}) \), which represents the difference between the measurement and what the filter would have expected for it.

Recalling that the inverse of a covariance matrix may be interpreted as a measure of information, equation (3.9) shows that the a posteriori information about the state is the sum of the a priori information and the measurement information.
In equation (3.7), the Kalman gain is defined as an optimal blending factor between the a priori estimate and the innovation term. Specifically, if the measurement is very accurate (i.e. $D_{k+1}D'_{k+1}$ small) or if the a priori information is very low (i.e. $P_{k+1}$ big), then provided $C_{k+1}$ is invertible, we have that $K_{k+1} \approx C_{k+1}^{-1}$, and equation (3.8) shows that the Kalman filter merely inverts the measurement at each update step. Conversely, if the measurement is very inaccurate (i.e. $D_{k+1}D'_{k+1}$ big), or if the a priori information is very high (i.e. $P_{k+1}$ small), $K_{k+1}$ tends to zero, and the measurement is not taken into account in the update step (3.8).

![Figure 3-1: Kalman filter structure](image)

Overall, the Kalman filter has the structure given in Figure 3-1. Two facts are worth mentioning since they highlight the dependence of the Kalman filter on accurate noise and plant modeling:

- the assumption of a white noise disturbance is crucial to the propagation from $\hat{x}_{k-1}^+$ to $\hat{x}_k$, and then from $\hat{x}_k$ to $\hat{y}_k$. If the noise is colored, and we do not have any knowledge of its correlation structure (so it can not be modeled through some shaping filter), the Kalman filter will lose its optimality.

- during the prediction step, the Kalman filter mimics internally the expected (i.e. mean) behavior of the system. If the a priori knowledge about the plant is erroneous (i.e. if matrices $A_k$ and $C_k$ are wrong), this will induce the filter into error.
Note that the recursive solution to the estimation problem above does not invoke the error weighting matrix sequence $M_k$. This means that the estimator is optimal for each component of the state. In particular, it is not possible to gain estimation accuracy over one specific component of the state at the expense of other ones.

For the sake of future comparison with robust estimators, we give here the filter equations for the one-step predicted case, and without restrictions on $G_kD'_k$ (i.e. process noise and measurement noise may be correlated). The one-step recursion is given by:

$$
\hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{K}_k y_k \\
P_{k+1} = \hat{A}_k P_k \hat{A}_k' + \hat{G}_k \hat{G}_k'
$$

(3.10) \hspace{1cm} (3.11)

with

$$
\hat{A}_k = A_k - K_k C_k \\
\hat{G}_k = G_k - K_k D_k \\
K_k = [G_k D'_k + A_k P_k C'_k][D_k D'_k + C_k P_k C'_k]^{-1}
$$

Although the above equations do not include a perfectly known input sequence $s_k$, they may be easily modified to include it, by performing the following substitutions:

$$
r_k \leftarrow \begin{bmatrix} s_k \\ r_k \end{bmatrix} \\
y_k \leftarrow \begin{bmatrix} s_k \\ y_k \end{bmatrix} \\
C_k \leftarrow \begin{bmatrix} 0 \\ C_k \end{bmatrix} \\
D_k \leftarrow \begin{bmatrix} I & 0 \\ 0 & D_k \end{bmatrix} \\
G_k \leftarrow \begin{bmatrix} B_k & G_k \end{bmatrix}
$$
where $B_k$ describes how the input $s_k$ enters the system. The augmentations to the matrices $C_k$ and $D_k$ are of dimensions corresponding to $s_k$. Conceptually, $s_k$ is considered to be a disturbance, but we measure it exactly.

### Steady-state Kalman filter and frequency domain interpretation

In steady-state, the weighted estimation error sequence $e_k$ is a stationary random process with power spectral density $S_{ee}(j\omega)$. In the following, we denote $G_{er}$ as the transfer function from the disturbance to the weighted estimation error, and $\sigma_i, i = 1, ..., N$ as its singular values. Taking the limit of the criterion of problem (3.4), we obtain:

\[
\lim_{k \to \infty} E(e'_k e_k) = \lim_{k \to \infty} E(\text{trace}(e'_k e_k)) \\
= \text{trace} \left[ \lim_{k \to \infty} E(e'_k e_k) \right] \\
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \text{trace} [S_{ee}(j\omega)] \, d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \text{trace} [G_{er}(j\omega)G'_{er}(j\omega)] \, d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left[ \sum_{i=1}^{N} \sigma_i^2 |G_{er}(j\omega)| \right] \, d\omega \\
\equiv \|G_{er}\|_{H_2}^2
\]

Hence, the steady-state Kalman filter minimizes the $H_2$ norm of the transfer function from the disturbance to the weighted estimation error, i.e. it minimizes the integral of the singular values of $G_{er}$ over all frequencies. In other words, it minimizes the average transmitted energy from the disturbance to the estimation error.

### 3.2 Effect of model uncertainties

Looking carefully at the Kalman filter equations in Section 3.1, we acquired the intuition that the optimality of the Kalman filter was highly dependent on the accuracy of its underlying state-space model. In this section, we confirm this intuition experi-
mentally with the navigation problem of Chapter 2.

Specifically, we design a steady-state Kalman filter around the nominal model (Mach .7/40kft) investigate whether it performs in an acceptable manner for the (Mach .9/40 kft) operating condition.

![Kalman filter - Mach .7/40 kft](image)

![Kalman filter - Mach .9/40 kft](image)

Figure 3-2: Pitch estimation for the Kalman filter - Good gyro case

Figure 3-2 shows the time-domain behavior of the filter for the pitch estimation for both plants. For the nominal plant, the filter does indeed track very well the pitch angle. However, for the perturbed plant, it does a poor job and is induced into error by the wrong plant model. Note that the error in the estimate is particularly large right after the step inputs (at 0 and 500 s). This suggests a strong influence of the command inputs to the error in estimate.

Let us write the augmented dynamics when the Kalman filter is designed around
a nominal plant and applied to a perturbed one:

\[
\begin{bmatrix}
  x_{k+1} \\
  \hat{x}_{k+1}
\end{bmatrix} = \begin{bmatrix}
  A_{per} & 0 \\
  A_{nom}KC_{per} & A_{nom} - A_{nom}K\hat{C}_{nom}
\end{bmatrix} \begin{bmatrix}
  x_k \\
  \hat{x}_k
\end{bmatrix} + \begin{bmatrix}
  B_{per} \\
  B_{nom}
\end{bmatrix} s_k + \begin{bmatrix}
  G_{per} \\
  A_{nom}KD_{per}
\end{bmatrix} d_k
\]

\[
\begin{bmatrix}
  \hat{x}_k \\
  \hat{x}_k
\end{bmatrix} = \begin{bmatrix}
  I_5 & -I_5
\end{bmatrix} \begin{bmatrix}
  x_k \\
  \hat{x}_k
\end{bmatrix}
\]

(3.12)

Here, we consider explicitly the deterministic input sequence \( s_k \) to make its influence clear. \( K \) is the steady-state Kalman gain corresponding to Eq. (3.7). The above state-space model enables us to analyze the transfer function from any input/disturbance to the estimation error. Clearly, it appears that when there is no model uncertainty in the deterministic dynamics of the model (i.e. \( A_{per} = A_{nom}, B_{per} = B_{nom}, C_{per} = C_{nom} \)), the transfer function from the deterministic inputs to the estimation error is zero over all frequencies.

However, when perturbations are considered, Eq. (3.12) shows that the command inputs have a strong influence on the estimation error. In the frequency domain, Figure 3-3 shows that the magnitude of the transfer functions from the command inputs to the pitch estimation error is quite high over a wide range of frequencies.

Overall, this analysis shows that while the Kalman filter is perfectly tuned to a nominal plant, it behaves poorly for a perturbed one. We may want to sacrifice a bit of nominal performance in order to accommodate plant perturbations. This is precisely the aim of robust estimation.
Figure 3-3: Magnitude of the transfer functions from the command inputs to the pitch estimation error for the perturbed Mach .9/40 kft operating condition - Good gyro case
Chapter 4

Minimax Filter: Robustness to Noise Model Uncertainty

With the Kalman filter, we place ourselves in a stochastic setting and make the assumption that the disturbance is white. This means that any non-white stationary dynamics of the noise must have been modeled through some shaping filters, and have been incorporated into an augmented plant. Often, this is not possible for two reasons: the first is that knowledge of the noise statistics could be limited, and the second is that it builds up the state dimension of the Kalman filter, which burdens computations. Also, when the noise is not stationary, it is not even possible to model it through a linear shaping filter.

Hence, a more conservative way to look at the filtering problem is the following: we place ourselves in a deterministic setting and do not assume any prior knowledge about the disturbance. Instead, we seek a filter that provides some deterministic guarantee about the transmitted energy from the disturbance to the estimation error.

4.1 Minimax filter presentation

In this presentation, we assume zero initial conditions to simplify the notation. A full derivation of the minimax filter with arbitralay initial conditions may be found in [15, 16].
Figure 4-1: Input-Output representation of plant estimation problem.

**Problem formulation**

Figure 4-1 shows an input/output representation of a given plant $P$ and an estimator $F$. A disturbance sequence $r$ enters a plant whose output sequence $y$ is in turn fed into a filter, which provides a state estimate sequence $\hat{x}$.

We place ourselves in a finite horizon setting and define the following notation to denote the input and the output sequence:

$$r \equiv [r_0, ..., r_{N-1}] \quad \text{and} \quad y \equiv [y_0, ..., y_{N-1}]$$

The weighted state estimation error is defined as:

$$e \equiv [e_1, ..., e_N]$$
$$e_k \equiv M_k (x_k - \hat{x}_k)$$

Note the time-interval shift for $e$, coming from the fact that in this section, we deal with the *a priori* estimate $\hat{x}_k$ of $x_k$.

The finite-horizon 2-norm of these sequences is given by:

$$\|r\|_2 \equiv \left( \sum_{k=0}^{N-1} r_k^t r_k \right)^{1/2} \quad \text{and} \quad \|e\|_2 \equiv \left( \sum_{k=1}^{N} e_k^t e_k \right)^{1/2}$$

The minimax filter problem formulation assumes a linear state-space representation of the plant of interest:
\[
\begin{bmatrix}
  x_{k+1} \\
e_k \\
y_k \\
\end{bmatrix} = \begin{bmatrix}
  A_k & G_k & 0 \\
M_k & 0 & -M_k \\
C_k & D_k & 0 \\
r_k \\
\end{bmatrix} \begin{bmatrix}
x_k \\
\end{bmatrix}
\]

(4.1)

where no assumptions are made on the disturbance \( r \), except that it is of bounded energy (i.e. finite 2-norm).

The objective is to find a filter that achieves a bound on the ratio of the estimation error energy to the disturbance energy, for any possible disturbance. Mathematically, if we define \( \mathcal{G} \) to be the mapping from the disturbance \( r \) to the estimation error \( e \), this means that we seek a filter that achieves a bound on \( \|\mathcal{G}\|_2 \), the induced 2-norm of \( \mathcal{G} \):

\[
\|\mathcal{G}\|_2 \equiv \sup_{r \neq 0} \frac{\|e\|_2}{\|r\|_2} < \gamma
\]

(4.2)

Note that this criterion is a deterministic one in contrast to that of the Kalman filter.

The above criterion may be arranged as follows:

\[
J = \|e\|_2^2 - \gamma^2 \|r\|_2^2 < 0 \quad \forall r \neq 0
\]

(4.3)

This enables us to define a minimax formulation:

\[
\min_{\hat{x}} \max_r J \text{ subject to Eq.(4.1)} \]

(4.4)

In this game-theoretic formulation, the first player chooses the worst case disturbance and the second one attempts to minimize the estimation error for this worst case disturbance.

**Filter equations**

In [15, 16], it is shown that a solution to the above problem is the following: if the modified Riccati equation
\[ P_{k+1} = (A_k - K_kC_k)H_k(A_k - K_kC_k)' + (G_k - K_kD_k)(G_k - K_kD_k)' \quad k = 0, ..., N - 1 \]
\[ = \tilde{A}_kH_k\tilde{A}_k' + G_k\tilde{G}_k' \tag{4.5} \]

where

\[ H_k^{-1} \equiv P_k^{-1} - \gamma^{-2}M_k'M_k \tag{4.6} \]
\[ K_k = [G_kD_k' + A_kH_kC_k'][D_kD_k' + C_kH_kC_k']^{-1} \tag{4.7} \]

has a solution such that \( H_k \) is positive definite for all \( k \), then a filter that guarantees the bound (4.2) exists. It is then given by the following dynamics:

\[ \hat{x}_{k+1} = (A_k - K_kC_k)\hat{x}_k + K_ky_k \tag{4.8} \]

which have the same form as the one-step predicted Kalman filter (see Eq. 3.10). Note also that the robust estimator dynamics are strictly similar to that of a Kalman filter where the a priori error covariance \( P_k \) is replaced by \( H_k \) at each time step.

**Interpretation**

Looking at equation (4.6), and recalling that, in a stochastic setting, the inverse of the error covariance matrix may be interpreted as a measure of information of the estimator, we see that the robust estimator artificially takes some a priori information out of the system at each time step. Note that the amount of information thrown away is proportional to \( \gamma^{-2} \). When \( \gamma \to \infty \), we have that \( H_k \to P_k \) and we recover the Kalman filter. Conversely, there is a lower bound \( \gamma_{\text{min}} \) on \( \gamma \) such that \( H_k \) is positive definite for all \( k \). This constitutes the minimum achievable bound for the criterion (4.2).

Since the Kalman filter is optimal in the mean-square sense for the nominal system, the robust filter presented above is suboptimal in that sense. But the benefit is that we have a guarantee on the worst case error provided by (4.2). Hence \( \gamma \) may be
thought of as a design parameter trading off mean-square nominal performance with a bound on the estimation error for the worst case performance.

Finally, we note that another design parameter is the estimation error weighting matrix $M_k$. It may be used to weight more heavily the error in some particular component of the state, which will give rise to a different filter. For the sake of comparison, recall that this error weighting matrix has no influence in the Kalman filter case.

We give here the continuous-time counterpart to the discrete-time minimax filter [15, 16]:

\[
\begin{align*}
\dot{x} &= (A - KC)x + Ky \\
K &= (PC' + GD')(DD')^{-1} \\
\dot{P} &= (A - KC)P + P(A - KC)' + \gamma^{-2}PMM'P + (G - KD)(G - KD)' 
\end{align*}
\] (4.9) (4.10) (4.11)

**Prediction and update for the discrete-time minimax filter**

We note that the discrete minimax filter derived above is provided in a one-step predicted form. However, in practical situations, it may be useful to have a distinct prediction and update step equation. Such is the case if we have a different sampling rate for the plant and for the measurements for example.

In this section, we make the assumption that $G_kD_k = 0$ to simplify both the notation and interpretation.

We note that the update equation is a particular case of the one-step equation when $A_k = I$ and $G_k = 0$. In this case, we have that $\hat{x}_k^+ = \check{x}_{k+1}$, and the one-step predicted equation (4.8) collapses to the following update:

\[
\hat{x}_k^+ = \hat{x}_k + K_k u (y_k - C\hat{x}_k) 
\] (4.11)

where $K_k u = H_kC_k(D_kD_k' + C_kH_kC_k)'^{-1}$. Making the reasonable assumption that the update does not depend on matrices $A_k$ and $G_k$ (i.e. it does not depend on the dynamics of the plant), equation (4.8) can be considered to be the update equation
for the minimax filter.

The prediction equation is then obtained by taking the difference between the one-step predicted equation and the update equation (4.11). We obtain:

$$\hat{x}_{k+1} = A_k \hat{x}_k$$  \hspace{1cm} (4.12)

Making a similar analysis for the modified Riccati equation (4.5), it is possible to obtain an update equation for $P_k$:

$$P_k^+ = (I - K_k^u C_k)H_k$$  \hspace{1cm} (4.13)

Using the matrix inversion lemma, the above equation may also be rewritten in an information form:

$$(P_k^+)^{-1} = H_k^{-1} + C_k (D_kD_k')^{-1} C_k'$$  \hspace{1cm} (4.14)

Expanding the one-step modified Riccati equation (4.5) and replacing equation (4.13), we obtain a propagation equation for $P_k$:

$$P_{k+1} = AP_k^+ A' + G_k G_k'$$  \hspace{1cm} (4.15)

Now the missing step between the prediction and update equation for $P_k$ is the following:

$$H_k^{-1} = P_k^{-1} - \gamma^{-2} M_k' M_k$$  \hspace{1cm} (4.16)

which may be interpreted as an information removal step.

Equation (4.14) may be useful if we do not have a measurement for some time step $k$. In such cases, we can just force $(D_kD_k')^{-1}$ to zero (the measurement brings no information), and set $P_k^+ = H_k$ in the 2-step recursion.

Note that equations (4.11), (4.12), (4.14) and (4.15) are strictly similar to the prediction/update equations of the Kalman filter where the extra-step (4.15) appears just prior to update, corresponding to an information removal.

However, be aware that neither $P_k$ nor $P_k^+$ should be interpreted literally as the
actual estimation error covariance in the minimax setting.

Steady-state minimax filter and frequency domain interpretation

The minimax problem formulation may be extended to the infinite-horizon case for a linear time-invariant system. For this case, sequences must be square summable over the infinite horizon, and the finite-horizon 2-norm in problem formulation (4.2) must be replaced by the infinite-horizon one. The steady-state solution is then given by the same equations as the finite-horizon case, with all subscripts 'k' dropped [15, 16].

The following proposition provides the relation between the induced 2-norm and the $H_\infty$ norm in the frequency domain, and enables us to restate the time-domain minimax problem formulation as a frequency-domain one.

Proposition 4.1 Let a stable linear system have a transfer matrix $G(s)$, and let $\mathcal{G}$ denote the linear map it induces from the square summable input sequence to the square summable output sequence. Then, the induced 2-norm of $\mathcal{G}$ coincides with the $H_\infty$ norm of $G(s)$.

\[
\sup_{r \neq 0} \frac{\|e\|_2}{\|r\|_2} = \|G\|_2 = \|G\|_\infty \equiv \sup_{\omega} \sigma_{\text{max}}(G(j\omega)) \quad (4.17)
\]

Proof [4]: Let us denote by $r$ the input sequence and $e = \mathcal{G}r$ its related output sequence. By Parsevaal's identity, we have that:

\[
\sum_{k=1}^{\infty} e_k e'_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |E(j\omega)|^2 d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{+\pi} |G(j\omega)R(j\omega)|^2 d\omega
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi}^{+\pi} \sigma_{\text{max}}(G(j\omega))^2 |R(j\omega)|^2 d\omega
\]

\[
\leq \|G\|_\infty^2 \frac{1}{2\pi} \int_{-\pi}^{+\pi} |R(j\omega)|^2 d\omega
\]

\[
= \|G\|_\infty^2 \sum_{k=0}^{\infty} r_k r'_k
\]

using again Parsevaal's identity. This shows that $\|G\|_2 \leq \|G\|_\infty$. 

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Conversely, suppose that $\gamma < \|G\|_\infty$. It means that for some $\omega_o$, $\sigma_{\text{max}}(G(j\omega_o)) > \gamma$. By continuity, there exists $\eta > 0$ such that $\sigma_{\text{max}}(G(j\omega)) > \gamma$ for all $\omega$ in $[-\omega_o - \eta, -\omega_o + \eta]$ as well as in $[\omega_o - \eta, \omega_o + \eta]$. Pick a noise to be zero outside this frequency range, and such that for each frequency in this range, it coincides with the eigenvector corresponding to the largest eigenvalue of $G'(-j\omega)G(j\omega)$ in these frequencies. The corresponding output $e$ is then given by:

\[
\|e\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |E(j\omega)|^2 d\omega
\]
\[
= \frac{1}{2\pi} \left[ \int_{-\omega_o - \eta}^{-\omega_o + \eta} |E(j\omega)|^2 d\omega + \int_{\omega_o - \eta}^{\omega_o + \eta} |E(j\omega)|^2 d\omega \right]
\]
\[
\geq \frac{1}{2\pi} \left[ \int_{-\omega_o - \eta}^{-\omega_o + \eta} \gamma^2 |R(j\omega)|^2 d\omega + \int_{\omega_o - \eta}^{\omega_o + \eta} \gamma^2 |R(j\omega)|^2 d\omega \right]
\]
\[
= \gamma^2 \|r\|_2^2
\]

Hence $\|G\|_{12} \geq \|G\|_\infty$. □

Proposition 4.1 shows that in steady-state, the minimax filter problem formulation reduces to bounding the $H_\infty$ norm of the transfer function from the disturbance to the weighted estimation error. Recalling that the steady-state Kalman filter minimizes the $H_2$ norm of that transfer function, we see that $\gamma$ may also be considered as a frequency-domain trade-off parameter between $H_2$ and $H_\infty$ performance. The lower achievable bound $\gamma_{\text{min}}$ leads to the so-called $H_\infty$ filter, and for this value of $\gamma$ the bound is indeed met. This filter is then tuned so as to minimize the weighted estimation error for the worst case sinusoidal noise at the worst frequency.

### 4.2 Application of the minimax filter

We apply the minimax filter in continuous-time (for numerical reasons) to our navigation problem of Chapter 2. We illustrate here how the minimax filter may provide robustness to unmodeled noise dynamics.

Taking a continuous-time model of the Boeing 747 at the nominal operating condition Mach 0.7/40 kft, we assume here that the process noise comes from wind gusts.
That is, we consider that the continuous-time dynamics of the aircraft are actually given by:

\[
\begin{bmatrix}
\dot{u} \\
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix} = A
\begin{bmatrix}
{u - u}_w \\
{w - w}_w \\
q \\
\theta
\end{bmatrix} + B
\begin{bmatrix}
\delta e \\
\delta t
\end{bmatrix}
\]

where \( u_w \) and \( w_w \) are zero-mean unit intensity continuous-time white noise modeling wind-gusts. In the above formulation, velocity perturbations \( u \) and \( w \) from Eq. (2.1) have been replaced by \textit{perturbations in airspeed} \((u - u_w)\) and \((w - w_w)\) where \( u_w \) and \( w_w \) are wind-velocity components in body axes. The above state-space dynamics may be rewritten as follows:

\[
\begin{bmatrix}
\dot{u} \\
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix} = A
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} + B
\begin{bmatrix}
\delta e \\
\delta t
\end{bmatrix} + G
\begin{bmatrix}
{u}_w \\
{w}_w
\end{bmatrix}
\]

where \( G \) is a newly defined process noise matrix. Our measurement model is the continuous-time counterpart of the measurement model in Chapter 2.

Several minimax filters were designed with weighting matrix \( M = I \), and various values for the parameter \( \gamma \).

<table>
<thead>
<tr>
<th>Filter</th>
<th>( \gamma )</th>
<th>( |G_{er}|_{\infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kalman filter</td>
<td>( \infty )</td>
<td>2.0507</td>
</tr>
<tr>
<td>Minimax filter</td>
<td>2</td>
<td>1.7203</td>
</tr>
<tr>
<td>Minimax filter</td>
<td>1.4</td>
<td>1.3971</td>
</tr>
<tr>
<td>( H_\infty ) filter</td>
<td>1.348</td>
<td>1.348</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of the bound \( \gamma \) and the actual \( H_\infty \) norm of \( G_{er} \)

Figure 4-2 shows the maximum singular value of the transfer function from the disturbance to the estimation error \( G_{er} \). These plots show clearly that the peak of the maximum singular value over all frequencies decreases with \( \gamma \). It may indeed be
checked in Table 4.1 that $\|G_{er}\|_\infty$ is always bounded from above by $\gamma$ as is guaranteed by the problem formulation 4.2.

An interesting point in Figure 4-2 is that the curve tends to flatten as $\gamma$ decreases towards $\gamma_{\text{min}}$, indicating that $H_2$ performance degrades severely. In a scalar case, this would ultimately lead to an all-pass filter. As a matter of fact, when $\gamma$ tends to $\gamma_{\text{min}}$, it may be seen that in the worst direction, the filter gain finally just inverts the measurement, i.e. it does not filter anything in that direction. The intuitive idea behind this is that if the noise really goes in the worst direction, the filter cannot do a good prediction job. It then has to rely solely on the measurements, which somehow have to be linked to reality.

For this example, we see $\gamma = 1.4$ yields a good compromise between $H_2$ and $H_\infty$ performance.

We make two types of simulation in order to compare the Kalman filter with the minimax filter when $\gamma = 1.4$: the first one assumes a white wind gust noise with unit intensity, as modeled for the filter design. The second one assumes that the
wind gust is the output of some shaping filters. Wind gust models are taken from [6] (see Appendix C) for various altitudes and Mach numbers. We assume here a RMS component of 2 ft/s for both the forward and the vertical gust.

<table>
<thead>
<tr>
<th>Wind gust Type</th>
<th>Kalman Filter</th>
<th>Minimax Filter with $\gamma = 1.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>White wind gust</td>
<td>.0436</td>
<td>.0562</td>
</tr>
<tr>
<td>Shaped wind gust (Mach .7/40 kft)</td>
<td>.1184</td>
<td>.0878</td>
</tr>
<tr>
<td>Shaped wind gust (Mach .9/40 kft)</td>
<td>.1077</td>
<td>.0836</td>
</tr>
<tr>
<td>Shaped wind gust (Mach .5/40 kft)</td>
<td>.1303</td>
<td>.0922</td>
</tr>
<tr>
<td>Shaped wind gust (Mach .5/20 kft)</td>
<td>.1280</td>
<td>.0914</td>
</tr>
</tbody>
</table>

Table 4.2: Steady-state RMS pitch estimation error (deg) - Good gyro case

Table 4.2 compares the steady-state RMS error of the two filters obtained through a covariance analysis. We see that the minimax filter gives up a little bit of nominal performance when wind gusts are white as modeled for the design. This was expected since the Kalman filter is optimal in the mean-square sense for this case. However it is likely that wind gusts will never be exactly white noise. Furthermore, it is difficult to obtain an accurate model for them. We see that the minimax filter offers better performance when we simulate some more realistic wind gusts which are the outputs of some shaping filters.

Hence, in this case, the minimax filter may be considered as an alternate design tool to the traditional Kalman filter, sacrificing a bit of nominal performance, but offering a greater protection against unmodeled noise dynamics.
Chapter 5

Robustness to Noise and Plant Model Uncertainty

In the previous chapter, a filter robust to noise model uncertainty was presented. In case there is no prior knowledge about the statistics of the disturbance, the minimax filter provides some guarantee for the worst case estimation error over all frequencies.

However, in many situations, the most harmful uncertainty source does not come from noise but rather from plant dynamics mismodeling. Indeed, as shown in Section 3.1, the Kalman filter has its own internal representation of the plant it is based on. When this internal representation is wrong, it induces in the filter an error which may be dramatic (see Section 3.2).

This motivates the need for a filter robust to both noise and plant modeling uncertainties, as has been derived by Mangoubi in [15, 16].

5.1 The robust filter

As in the previous chapter, assume zero initial conditions to simplify the notation. The full derivation of the filter with non-zero initial conditions may be found in [15, 16].
Figure 5-1: General representation of robust estimation problem

**Problem formulation**

Figure 5-1 shows an input/output representation of a given plant \( P \) with modeling uncertainties \( \Delta \) and an estimator \( F \). As in the previous chapter, we place ourselves in a finite horizon setting. The sequences \( \epsilon \equiv [\epsilon_1, \ldots, \epsilon_{N-1}] \) and \( \eta \equiv [\eta_1, \ldots, \eta_{N-1}] \) represent the signals connecting the plant and the perturbation. The other sequences and vectors have the same definitions as before. Thus the system is defined by the following linear state-space representation:

\[
\begin{align*}
x_{k+1} &= A_k x_k + G_k d_k \\
\epsilon_k &= S_k x_k + T_k d_k \\
e_k &= M_k (x_k - \hat{x}_k) \\
y_k &= C_k x_k + D_k d_k
\end{align*}
\]

\[
\begin{bmatrix}
x_{k+1} \\
\epsilon_k \\
e_k \\
y_k
\end{bmatrix} =
\begin{bmatrix}
A_k & G_k & 0 \\
S_k & T_k & 0 \\
M_k & 0 & -M_k \\
C_k & D_k & 0
\end{bmatrix}
\begin{bmatrix}
x_k \\
d_k \\
\hat{x}_k
\end{bmatrix}
\]

In this representation \( d_k = [\eta_k \ r_k'] \) is an augmented noise vector comprising the output of the perturbation \( \Delta \) and the noise disturbance.

As in the previous case, we seek a filter that achieves a bound on the ratio of the
estimation error energy to the noise disturbance energy. But this time, we want the
criterion to hold for any perturbation \( \Delta \) of bounded induced 2-norm. This translates
into the following mathematical criterion:

\[
G_{i2} \equiv \sup_{r \neq 0} \frac{\|e\|_2}{\|r\|_2} < \gamma
\]  \hspace{1cm} (5.2)

\( \forall \Delta \) such that \( \|\Delta\|_{i2} \equiv \sup_{\epsilon \neq 0} \frac{\|\eta\|_2}{\|\epsilon\|_2} < 1/\gamma \)

In order to achieve the above criterion, a new criterion is defined where the per-
turbation output \( \eta \) is considered as an additional exogenous disturbance input to the
plant, and the perturbation input \( \epsilon \) is considered as an additional error term. The
new criterion is given by:

\[
\mathcal{J}_1 < \gamma^2 \hspace{0.5cm} \forall \Delta \text{ such that } \|\Delta\|_{i2} < 1/\gamma
\]  \hspace{1cm} (5.3)

where

\[
\mathcal{J}_1 \equiv \sup_{(n,r) \neq 0} \frac{\|e\|_2^2 + \|\epsilon\|_2^2}{\|r\|_2^2 + \|\eta\|_2^2}
\]

In [15], it is shown that the above criterion is indeed sufficient to meet the bound
of (5.2). It may in turn be expressed as:

\[
J_2 = \|e\|_2^2 + \|\epsilon\|_2^2 - \gamma^2(\|d\|_2^2) < 0 \hspace{0.5cm} \forall d = [\eta' \ r']' \neq 0
\]  \hspace{1cm} (5.4)

This enables us to define a game-theoretic formulation of the problem, analogous
to that of the minimax filter:

\[
\min_{\hat{x}} \max_d J_2 \hspace{1cm} (5.5)
\]

subject to \hspace{0.5cm} Eq.(5.1)

However, this time the first player chooses the worst case noise \( \text{and} \) plant dis-
turbance and the second player attempts to minimize both the estimation error \( e \) and
the additional error term \( \epsilon \).
Filter equations

A solution to this robust estimation problem is provided in two stages. In a first stage, a change of variables enables us to to get rid of the sequence \( \epsilon \) in the problem formulation (5.5). The estimation problem then reduces to that of the minimax filter of the previous chapter, which is in turn solved in a second stage.

First stage

Through a "completing the square" argument, it is proven in [15, 16] that

\[
\|\epsilon\|_2^2 - \gamma^2(\|d\|_2^2) = -\gamma^2(\|\bar{d}\|_2^2)
\]

(5.6)

where

\[
\bar{d}_k \equiv Z_k^{1/2} d_k - \gamma^{-2} Z_k^{-1/2} F_k' x_k
\]

(5.7)

if and only if a first modified Riccati equation

\[
X_k = A_k' X_{k+1} A_k + S_k' S_k + \gamma^{-2} F_k Z_k^{-1} F_k'
\]

\[
X_N = 0
\]

(5.8)

where

\[
F_k \equiv S_k' T_k + A_k' X_{k+1} G_k
\]

(5.9)

\[
Z_k \equiv I - \gamma^{-2} (T_k' T_k + G_k' X_{k+1} G_k)
\]

has a solution such that \( X_k \geq 0 \) and \( Z_k > 0 \) for all \( k \).

Second stage

This allows us to redefine \( J_2 \) as:

\[
J_2 = \|\epsilon\|_2^2 - \gamma^2(\|\bar{d}\|_2^2)
\]

(5.10)
Here, we recognize the game-theoretic criterion of the minimax filter (4.3), where the disturbance \( d \) has been replaced by a new disturbance \( \bar{d} \). To apply the solution of Chapter 4, we simply need to express the state-space model (5.1) in terms of the new disturbance \( \bar{d} \). Replacing \( d_k = \gamma^{-2}Z_k^{-1}F'_k x_k + Z_k^{-1/2}\bar{d}_k \) in (5.1) results in:

\[
\begin{align*}
x_{k+1} &= \bar{A}_k x_k + \bar{G}_k \bar{d}_k \\
y_k &= \bar{C}_k x_k + \bar{D}_k \bar{d}_k
\end{align*}
\]  

with

\[
\begin{align*}
\bar{A}_k &= A_k + \gamma^{-2}G_k Z_k^{-1}F'_k \\
\bar{G}_k &= G_k Z_k^{-1/2} \\
\bar{C}_k &= C_k + \gamma^{-2}D_k Z_k^{-1}F'_k \\
\bar{D}_k &= D_k Z_k^{-1/2}
\end{align*}
\]

Then the estimator equations are similar to those of the minimax filter:

\[
\hat{x}_{k+1} = (\bar{A}_k - \bar{K}_k \bar{C}_k)\hat{x}_k + \bar{K}_k y_k
\]

with gain

\[
\bar{K}_k = \left[ \bar{G}_k \bar{D}_k' + \bar{A}_k \bar{H}_k \bar{C}_k' \right] \left[ \bar{D}_k \bar{D}_k' + C_k \bar{H}_k \bar{C}_k' \right]^{-1}
\]

where \( \bar{H}_k \) is given by

\[
\bar{H}_k^{-1} \equiv \bar{P}_k^{-1} - \gamma^{-2}M'_k M_k
\]

and the matrix \( \bar{P}_k \) satisfies a second modified Riccati equation:

\[
\bar{P}_{k+1} = (\bar{A}_k - \bar{K}_k \bar{C}_k)\bar{H}_k (\bar{A}_k - \bar{K}_k \bar{C}_k)' \\
+ (\bar{G}_k - \bar{K}_k \bar{D}_k)(\bar{G}_k - \bar{K}_k \bar{D}_k)'
\]

such that \( \bar{H}_k \) is positive definite for all \( k \).

The design process of the robust filter may thus be summarized as follows: given
a state-space representation in the form of Eq. (5.1),

- select a $\gamma$,
- solve the first Riccati equation (5.8),
- check the conditions $X_k \geq 0$ and $Z_k > 0$,
- select a weighting matrix $M_k$,
- solve the second Riccati equation (5.16),
- check the condition $H_k > 0$.

When dealing with a linear time-invariant plant and a steady-state filter, all subscripts $k$'s may be dropped from the above matrices. The filter then has a frequency-domain interpretation similar to that of the minimax filter, using the fact that the induced 2-norm coincides with the $H_\infty$ norm (see proposition 4.1 in Chapter 4).

Finally, note that the robust filter above is again an extension of the Kalman filter. When no plant perturbation is taken into account, the matrices $S_k$ and $T_k$ are equal to zero. The first Riccati equation therefore becomes superfluous since it yields $X_k = 0$ for all $k$, and we obtain $\bar{A}_k = A_k$, $\bar{G}_k = G_k$, $\bar{C}_k = C_k$, $\bar{D}_k = D_k$ from (5.9) and (5.12). In that case, the robust filter reduces to the minimax filter of Chapter 4. When we further allow $\gamma \to \infty$, the filter becomes a minimax filter with an infinite bound $\gamma$, which is nothing but a Kalman filter (see Chapter 4).

**Parametric uncertainties**

This subsection explains how to fit parametric uncertainties into the state-space formulation of (5.1). Consider a state-space plant with parameter errors [2]:

\[
\begin{align*}
x_{k+1} &= \left( A + \sum_{j=1}^{l} \Delta A_j \delta_j \right) x_k + \left( B + \sum_{j=1}^{l} \Delta B_j \delta_j \right) r_k \\
y_k &= \left( C + \sum_{j=1}^{l} \Delta C_j \delta_j \right) x_k + \left( E + \sum_{j=1}^{l} \Delta E_j \delta_j \right) r_k
\end{align*}
\]  

(5.17)
Each $\delta_j$ represents a parameter error that is normalized as

$$-1 < \delta_j < 1 \quad \forall j = 1, \ldots, l$$

For each $j$, $j = 1, \ldots, l$, we may define the matrix $N_j$ by

$$N_j = \begin{bmatrix} \Delta A_j & \Delta B_j \\ \Delta C_j & \Delta E_j \end{bmatrix} \in \mathcal{R}^{(n_z+n_y) \times (n_z+n_r)}$$

where $n_z$, $n_y$, and $n_r$ are the dimensions of the vectors $x_k$, $y_k$, and $r_k$, respectively. Generally, this matrix will not be of full rank since one parameter rarely affects all of the states and outputs. Hence, using a singular value decomposition, $N_j$ may be decomposed as follows:

$$N_j = \begin{bmatrix} Q_j \\ R_j \end{bmatrix} \begin{bmatrix} S_j & L_j \end{bmatrix} \quad (5.18)$$

where $Q_j \in \mathcal{R}^{n_z \times n_j}$, $R_j \in \mathcal{R}^{n_z \times n_j}$, $S_j \in \mathcal{R}^{n_j \times n_z}$, $L_j \in \mathcal{R}^{n_j \times n_r}$, and $n_j$ is the rank of the matrix $N_j$. Combining Eq. (5.17) and Eq. (5.18), and recalling that $d_k = [\eta'_k \ r'_k]'$, the state-space model of the perturbed system may be rewritten as

$$x_{k+1} = \left[ A + \sum_{j=1}^{l} Q_j \delta_j I_{n_j} S_j \right] x_k + \left[ B + \sum_{j=1}^{l} Q_j \delta_j I_{n_j} L_j \right] r_k$$

$$= A x_k + \left[ Q_1 \ldots Q_l \right] \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_l \end{bmatrix} + B r_k$$

$$= A x_k + Q \eta_k + B r_k$$

$$= A x_k + G d_k \quad \text{with} \quad G = [Q \ B] \quad (5.19)$$

$$y_k = \left[ C + \sum_{j=1}^{l} R_j \delta_j I_{n_j} S_j \right] x_k + \left[ E + \sum_{j=1}^{l} R_j \delta_j I_{n_j} L_j \right] r_k$$
\[
\begin{align*}
= \ C_jx_k + [R_1 \ \cdots \ R_l] \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_l \end{bmatrix} + Er_k \\
= \ Cx_k + R\eta_k + Er_k \\
= \ Cx_k + Dd_k \quad \text{with} \quad D = [R \ \ E]
\end{align*}
\] (5.20)

\[
\epsilon_k = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_l \end{bmatrix} \\
= \begin{bmatrix} S_1 \\ \vdots \\ S_l \end{bmatrix} x_k + \begin{bmatrix} L_1 \\ \vdots \\ L_l \end{bmatrix} r_k \\
= \ Sx_k + Td_k
\] (5.21)

where \( T = \begin{bmatrix} 0 & L_1 \\ \vdots & \vdots \\ 0 & L_l \end{bmatrix} \)

We have created some sequences \( \epsilon \) and \( \eta \) as well as matrices \( S, T, B, \) and \( D \) to fit into the state-space description (5.1).

The relationship between \( \eta \) and \( \epsilon \) is then:

\[
\begin{align*}
\eta &= \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_l \end{bmatrix} \\
= \begin{bmatrix} \delta_1 & \cdots \\ \vdots & \ddots \\ \delta_l \\ \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_l \end{bmatrix} \\
= \Delta \epsilon
\end{align*}
\] (5.22)
The robust filter does not take advantage of the diagonal structure of $\Delta$. Neither does it take into consideration the fact that the parametric uncertainty may be real. Hence it is conservative in the sense that it is robust to model uncertainties that do not occur.

5.2 Application of the robust filter

All our analysis will be for a time-invariant plant with a steady-state filter gain, which is obtained by the same equations as in the finite-horizon case, but with all subscripts $k$ dropped.

In order to apply the robust filter, we consider parametric uncertainties for various models of the aircraft. Specifically, we choose a nominal and a perturbed plant, and define the parametric uncertainty to be a certain percentage of the difference between the two plants. Using the singular value decomposition (5.18), we fit the parametric uncertainty into the state-space formulation (5.1). We then tune the robust filter by adjusting the values of $\gamma$ and $M$ appropriately.

Throughout this chapter, however, we do not use the second modified Riccati equation (5.16) that deals with the noise model uncertainty. Instead, improved results are obtained by using a regular Riccati equation, and thus actually computing a Kalman filter based on the fictitious plant $\bar{A}$, $\bar{C}$, $\bar{C}$, and $\bar{D}$ coming from (5.8) and (5.12). Our interpretation is that even if we consider uncertainties in the aircraft dynamics, we still believe that the disturbance is white, which it actually is in our simulation. Note that when the second modified Riccati equation is not used, the estimation error weighting matrix $M$ does not come into play in the robust filter design.

Two cases are considered depending on the quality of the gyro, which is specified in Table 2.2 of Chapter 2.
Good gyro case

The design parameters for the robust filter design have been selected after a long trial and error procedure so as to obtain the best possible results. We finally came up with values for the design parameters as summarized in the table below:

<table>
<thead>
<tr>
<th>Robust filter design parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal plant</td>
</tr>
<tr>
<td>Perturbed plant</td>
</tr>
<tr>
<td>$dA$</td>
</tr>
<tr>
<td>$dG$</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

In the time domain, our analysis consists of simulating the filters with the various aircraft models considered. For this time-simulation, the nominal command input defined in Figure 2-4 of Chapter 2 is applied.

<table>
<thead>
<tr>
<th>Kalman Filter</th>
<th>Robust Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach .9/40 kft</td>
<td>1.1959</td>
</tr>
<tr>
<td>Mach .7/40 kft (nominal)</td>
<td>.0351</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.3353</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>-4.1867</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.3564</td>
</tr>
<tr>
<td>Mach .23/sea level</td>
<td>.3235</td>
</tr>
</tbody>
</table>

Table 5.1: RMS pitch estimation error (deg) over the 1000 s simulation run - Good gyro case

Table 5.1 shows the RMS pitch estimation error over the 1000 s simulation run for the various operating conditions. The robust filter is clearly suboptimal for the nominal case (Mach .7/40 kft). However, if we consider any other operating condition, the robust filter easily outperforms the Kalman filter. It is interesting to see that the perturbations with the more deleterious effects on the Kalman filter performance are not always the ones that are “farthest” from the nominal operating condition. Here both the Mach .9/40 kft and the Mach .65/sea level operating conditions are extremely sensitive to the influence of the command inputs (elevator and throttle),
leading to huge errors when a nominal Kalman filter is applied. In contrast, the robust filter remains quite insensitive over the whole range of plant perturbations.

Figure 5-2: Pitch estimation for the Kalman and robust filters - Good gyro case

Figure 5-2 compares the Kalman and the robust filter behavior during the time-simulation. Clearly, the Kalman filter provides an excellent estimate for the nominal case. However, the estimate degrades severely for the perturbed plant (Mach .9/40 kft). In contrast, the quality of estimation of the robust filter is nominally less performant, but it is almost indifferent to the plant perturbation. In that case, it may be advantageous to accept a degradation of nominal performance so as to be able to cope with a variety of plant perturbations.
Figure 5-3: Magnitude of the transfer functions from the command inputs to the pitch estimation error - Nominal = Mach .7/40kft; Perturbed = Mach .9/40 kft - Good gyro case
Figure 5-3 shows the magnitude of the transfer function from the inputs to the estimation error for both filters. The Kalman filter/nominal plant case is not plotted since it is zero over all frequencies. The response for the Kalman filter/perturbed plant case is quite high over a wide range of frequencies, indicating that the Kalman filter would not work well for the perturbed plant. For the robust filter, the curves for nominal and perturbed are relatively close to each other, and lower compared to the Kalman filter/perturbed plant curve, indicating better performance for the perturbed case.

Overall, we see that the robust filter provides a good balance between nominal and robust performance. It yields a bearable degradation of nominal performance but keeps that performance over a wide range of operating conditions.

**Low-cost gyro case**

The same analysis is now repeated for the low-cost gyro case. The robust filter is designed with $\gamma = 7$, while all other design parameters are the same as in the good gyro case.

<table>
<thead>
<tr>
<th>Mach .9/40 kft</th>
<th>Kalman Filter</th>
<th>Robust Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach .7/40 kft (nominal)</td>
<td>.0467</td>
<td>.4538</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.6250</td>
<td>.4572</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>5.3015</td>
<td>.4231</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.4860</td>
<td>.4218</td>
</tr>
</tbody>
</table>

Table 5.2: RMS pitch estimation error (deg) over the 1000 s simulation run - Low-cost gyro case

Table 5.2 shows the RMS pitch estimation error over the simulation run for both filters. Even when the gyroscope is of poor quality, the robust filter performs much better than the Kalman filter in the presence of plant perturbations while giving up some nominal performance. However, we note that in comparison with the results in Table 5.1, the steady-state performance of the robust filter is quite dependent on the quality of the sensor. This is not very surprising since the sensor is the only means
by which we refer to the outside world. If the sensor is very poor, and furthermore if the model is very inaccurate, we should not expect good performance. Nevertheless, in 4 out of 5 operating conditions, the error remains below 0.5 degree for the robust filter.

![Graphs showing pitch estimation for Kalman and robust filters](image)

**Figure 5-4:** Pitch estimation for the Kalman and robust filters - Low-cost gyro case

The time-domain responses are plotted in Figure 5-4. We see that the Kalman filter still behaves well when its internal representation of the plant is correct, but it fails to provide a good estimate for the perturbed plant. The robust filter estimate is insensitive to plant perturbations, but it is much noisier than the case of the good gyro (compare with Figure 5-2). It illustrates the fact that robust filter performance is quite dependent on sensor quality. However, even in this case, it may be advantageous
to use a robust filter to protect oneself against model uncertainties.

5.3 Robustness versus performance

In this section, we show that the robust filter design actually consists of a trade-off between robustness and nominal performance. Specifically, we investigate the effect of the parameter $\gamma$ on the robust filter design.

We compare three robust filters with the following design parameters:

<table>
<thead>
<tr>
<th></th>
<th>Robust Filter 1</th>
<th>Robust Filter 2</th>
<th>Robust Filter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal plant</td>
<td>Mach .7/40 kft</td>
<td>Mach .7/40 kft</td>
<td>Mach .7/40 kft</td>
</tr>
<tr>
<td>Perturbed plant</td>
<td>Mach .9/40 kft</td>
<td>Mach .9/40 kft</td>
<td>Mach .9/40 kft</td>
</tr>
<tr>
<td>$dA$</td>
<td>$2 \times (A_{nom} - A_{per})$</td>
<td>$2 \times (A_{nom} - A_{per})$</td>
<td>$2 \times (A_{nom} - A_{per})$</td>
</tr>
<tr>
<td>$dG$</td>
<td>$10 \times (G_{nom} - G_{per})$</td>
<td>$10 \times (G_{nom} - G_{per})$</td>
<td>$10 \times (G_{nom} - G_{per})$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>6.78</td>
<td>8</td>
<td>$10^6$</td>
</tr>
</tbody>
</table>

Table 5.3: RMS pitch estimation error (deg) in the presence of command input - Good gyro case

Table 5.3 compares the RMS pitch estimation error in the presence of a command input for the usual 1000 s simulation run. It is clear that the nominal performance of the filter improves with $\gamma$. Conversely, for any other operating conditions, we see that decreasing the bound $\gamma$ enables us to get more robustness to model uncertainties.

In summary, it appears that decreasing $\gamma$ increases robustness at the expense of nominal performance. This makes intuitive sense because $\gamma$ is an upper bound on the $H_{\infty}$ performance of the filter.
5.4 Robustness and the Kalman filter

The Kalman filter may be overconfident about its own internal representation of the system dynamics. This may have disastrous effects in the case of modeling errors, as we have seen in Section 3.2. A common way to deal with this issue is to overdesign the intensity of process noise, so that the design model appears more erratic to the Kalman filter than it really is.

In this section, we first analyze the effect of overdesigning the process noise covariance in a Kalman filter. We then compare the robust filter with the “overdesigned” Kalman filter.

5.4.1 Analysis

There are two ways to look at the resulting effect of overdesigning the process noise covariance on the Kalman filter.

The first one simply considers the time-varying filter gain calculation:

\[
P_{k+1} = A_k P_k^+ A_k^T + G_k G_k^T
\]
\[
(P_{k+1}^+)^{-1} = P_{k+1}^{-1} + C_{k+1}(D_{k+1} D_{k+1}^T)^{-1} C_{k+1}^T
\]
\[
K_{k+1} = P_{k+1}^+ C_{k+1}(D_{k+1} D_{k+1}^T)^{-1}
\]

Clearly, it appears that increasing \( G_k \) leads to greater values for \( P_{k+1} \), and in turn for \( P_{k+1}^+ \) and \( K_{k+1} \). Hence, a greater gain is obtained, which means that the filter pays more attention to the measurements and less to its internal predicted estimate.

Another way to look at this is in the frequency domain. Let us consider the transfer function from the measurements to the state estimates for the steady-state Kalman filter. Taking the z-transform of \( \hat{x}_{k+1} = (A - K_{\infty} C) \hat{x}_k + K_{\infty} y_k \), we obtain:

\[
\frac{\hat{X}(z)}{Y(z)} = \frac{K_{\infty}}{z I - (A - K_{\infty} C)}
\]

We recognize a first-order low-pass filter. The location of the poles determines the
shape of the filter. Setting the denominator to 0, we get:

\[
z = (A - K_{\infty}C) \\
= A \left[ I - PC'(CPC' + R)^{-1}C \right] \\
= A \left[ I - (AP^+A' + GG')C'(CAP^+A'C' + CGG'C' + R)^{-1}C \right]
\] (5.26)

To acquire some intuition, consider the scalar case. Suppose that \( G \to 0 \) (i.e. almost no process noise), then we will have \( P^+ \to 0 \) for stable dynamics (both the prediction and the update shrink the error covariance), and we obtain from (5.26) that the pole \( z \to A \). This means that the transfer function acquires an increasing low-pass behavior. The filter trusts the model so much that it does not even pay attention to the measurements, and filters everything out.

On the other hand, let us assume that \( G \to \infty \) (i.e. infinite process noise), Eq. (5.26) shows that \( z \to 0 \). This means that the transfer function from the measurement to the state estimate acquires an increasing bandwidth. The Kalman filter does not trust the model, so it must fully rely on the measurements without even filtering them.

A compromise between these two extremes leads to a filter that does not rely too much on its own internal representation of the plant, but which at the same time does a good job of filtering noise out of the measurements. Hence, \( G_k \) may be thought of as a design parameter in the Kalman filter, trading off performance versus robustness. In particular, making \( G_k \) bigger than it actually is provides some robustness to the plant modeling errors for the Kalman filter. In the following, we will refer to such a filter as an “overdesigned Kalman filter”.

For our specific navigation problem, we are interested in estimating the pitch angle, which we do not measure. Instead, we have a measurement of the pitch rate. In the extreme case where we tell the Kalman filter to rely solely on the measurements, we will end up with a filter that will simply integrate the pitch-rate measurement. As such it is completely robust to aircraft model uncertainties! But it is also of no use since integrating the sensor noise will eventually result in an unbounded estimation
error. This is precisely what we wanted to avoid by incorporating our aircraft model in the Kalman filter.

Overall, the performance of an overdesigned Kalman filter should be a compromise between the following two criteria:

- the steady-state behavior of the pitch estimate in the absence of any command input,

- the behavior of the pitch estimate in the presence of command input and how badly it is affected when plant perturbations are considered.

In order to compare the performance of the robust filter designed above with that of an overdesigned Kalman filter, we adopt the following methodology for the design of the latter: instead of taking the process noise matrix $G_d$ to be $1/10$ of the command input matrix $B_d$ in the aircraft state-space model (2.3), we set $G_d = \alpha B_d$, and tune the parameter $\alpha$ so as to obtain approximately the same steady-state performance in the absence of command input as that of the robust filters designed previously. We then compare the respective robustness of the filters by looking at the effect of the command inputs on the pitch estimation error for both filters.

The steady-state performance analysis in the absence of command input is done as follows: given the filter gain $\bar{K}$, and the new state-space model for the filter to be designed around $\bar{A}, \bar{G}, \bar{C}, \bar{D}$, we are able to build the dynamics of the estimator:

$$\hat{x}_{k+1} = (\bar{A} - \bar{K}\bar{C})\hat{x}_k + \bar{K}y_k$$

To obtain the steady-state estimation error covariance in the absence of command input, we build the augmented system:

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ \bar{K}\bar{C} & \bar{A} - \bar{K}\bar{C} \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} G \\ \bar{K}D \end{bmatrix} d_k$$

$$= A_{aug}x_{aug} + G_{aug}d_k$$

(5.27)
\[
\ddot{x}_k = \begin{bmatrix}
I_5 & -I_5
\end{bmatrix}
\begin{bmatrix}
x_k \\
\ddot{x}_k
\end{bmatrix} = C_{aug} x_{kaug}
\]

(5.28)

We are then able to obtain the steady-state estimation error covariance \(P_{\ddot{z}}\):

\[
P_{\ddot{z}} = C_{aug} Y C_{aug}'
\]

where \(Y\) is the solution of the discrete-time Lyapunov equation:

\[
Y = A_{aug} Y A_{aug}' + G_{aug} G_{aug}'
\]

### 5.4.2 Designs and results

Again, we consider the "good" and "low-cost" gyro case separately.

**Good gyro case**

We compare three filters in this section. For the overdesigned Kalman filter, we tune \(\alpha = 4.2\). Two robust filters are designed with the following parameters:

<table>
<thead>
<tr>
<th></th>
<th>Robust Filter 1</th>
<th>Robust Filter 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal plant</td>
<td>Mach .7/ 40 kft</td>
<td>Mach .7/40 kft</td>
</tr>
<tr>
<td>Perturbed plant</td>
<td>Mach .9/ 40 kft</td>
<td>Mach .9/40 kft</td>
</tr>
<tr>
<td>(dA)</td>
<td>(2 \times (A_{nom} - A_{per}))</td>
<td>(4 \times (A_{nom} - A_{per}))</td>
</tr>
<tr>
<td>(dG)</td>
<td>(10 \times (G_{nom} - G_{per}))</td>
<td>(10 \times (G_{nom} - G_{per}))</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>6.78</td>
<td>10.15</td>
</tr>
</tbody>
</table>

The steady-state results in Table 5.4 show clearly that all three filters have approximately the same performance in the absence of command input. Furthermore, it appears that the steady-state performance is almost insensitive to model uncertainties. Our interpretation is that the main source of disturbance comes from the gyroscope noise (which is the same for all aircraft model uncertainties).
<table>
<thead>
<tr>
<th>Mach .9/40 kft</th>
<th>Overdesigned K. F.</th>
<th>Robust F. 1</th>
<th>Robust F. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1075</td>
<td>.0954</td>
<td>.1072</td>
<td></td>
</tr>
<tr>
<td>Mach .7/40 kft (nominal)</td>
<td>.1075</td>
<td>.0953</td>
<td>.1072</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.1075</td>
<td>.0954</td>
<td>.1073</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>.1077</td>
<td>.0954</td>
<td>.1073</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.1075</td>
<td>.0954</td>
<td>.1073</td>
</tr>
</tbody>
</table>

Table 5.4: Steady-state RMS pitch estimation error (deg) in the absence of command input - Good gyro case

More revealing is the behavior of the filters when subjected to some command input, as such inputs will almost always be present. In Table 5.5, we give the RMS estimation error for the entire 1000 s simulation run. As expected, the overdesigned Kalman filter behaves better than both robust filters for the nominal operating condition. But surprising is the fact that the overdesigned Kalman filter handles the nominal perturbation (Mach .9/40kft) as well as first robust filter. It may be seen however, that while giving up some nominal performance, Robust Filter 2 handles the nominal perturbation much better than the other two filters. Overall, we see that both robust filters handle various model perturbations better than the overdesigned Kalman filter. This is especially true when perturbations are significant (e.g. sea level operating conditions).

<table>
<thead>
<tr>
<th>Mach .9/40 kft</th>
<th>Overdesigned K. F.</th>
<th>Robust F. 1</th>
<th>Robust F. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>.3234</td>
<td>.3235</td>
<td>.2195</td>
<td></td>
</tr>
<tr>
<td>Mach .7/40 kft (nominal)</td>
<td>.0768</td>
<td>.1629</td>
<td>.2358</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.1239</td>
<td>.0852</td>
<td>.1391</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>.6221</td>
<td>.0976</td>
<td>.2131</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.3935</td>
<td>.2037</td>
<td>.1346</td>
</tr>
</tbody>
</table>

Table 5.5: RMS pitch estimation error (deg) in the presence of command input - Good gyro case

Figure 5-5 compares the behavior of the overdesigned Kalman filter and that of Robust Filter 2 for the (Mach .9/40 kft) operating condition. It may be seen that the bias of the robust estimator is slightly less than that of the overdesigned Kalman
filter. However, it is not fully eliminated.

Figure 5-5: Pitch estimation for the overdesigned Kalman filter and Robust Filter 2 - Mach .9/40 kft configuration - Good gyro case

Figure 5-6 illustrates the type of improvement we may expect by using Robust Filter 2 when large departures from nominal conditions occur. It may be seen that the robust filter attenuates the bias following the step inputs much better than the overdesigned Kalman filter. In the frequency domain, Figure 5-7 shows that the influence of the command inputs on the estimation error is greatly reduced for the robust filter.
Figure 5-6: Pitch estimation for the overdesigned Kalman filter and Robust Filter 2 - Mach .2/sea level configuration - Good gyro case
Figure 5-7: Magnitude of the transfer functions from the command inputs to the pitch estimation error for the Mach .2/sea level configuration - Good gyro case
Overall we conclude that overdesigning the process noise in a Kalman filter helps to handle plant perturbations. However, when looking at a wide range of model uncertainty, it turns out that a robust filter behaves much better.

**Low-cost gyro case**

In the low-cost gyro case, we compare 2 filters. The first one is an overdesigned Kalman filter where we have tuned $\alpha = 4$. The second one is a robust filter which is designed with $\gamma = 7$ and the same $dA$ and $dG$ parameters as Robust Filter 1 in the preceding section.

The results in Table 5.6 show that we are able to achieve approximately the same performance for both filters in the absence of command input.

<table>
<thead>
<tr>
<th>Mach .9/40 kft</th>
<th>Overdesigned Kalman Filter</th>
<th>Robust Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach .7/40 kft (nominal)</td>
<td>.4150</td>
<td>.4059</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.4151</td>
<td>.4064</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>.4153</td>
<td>.4061</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.4151</td>
<td>.4059</td>
</tr>
</tbody>
</table>

Table 5.6: Steady-state RMS pitch estimation error (deg) in the absence of command input - Low-cost gyro case

Now if we look at the performance in the presence of command input, we see in Figure 5-8 that the robust filter performs better than the overdesigned Kalman filter for the perturbed plant (Mach .9/40 kft). In particular, it enables us to attenuate the bias following some step inputs. This may be checked in the frequency domain in Figure 5-9, where we see that we gain approximately 7 dB at low frequency by using a robust filter instead of an overdesigned Kalman filter for the perturbed plant.

Table 5.7 gives the RMS pitch estimation error in the presence of command input for the standard 1000 s simulation run. We see that the robust filter helps in achieving an overall reasonably good level of performance over a wide range of plant perturbations. However, by comparing these figures with those of Table 5.5, it should
be noted that the performance of both the robust and the overdesigned Kalman filter are strongly affected by the poor quality of the gyroscope.

<table>
<thead>
<tr>
<th></th>
<th>Overdesigned Kalman Filter</th>
<th>Robust Filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach .9/40 kft</td>
<td>.7694</td>
<td>.5091</td>
</tr>
<tr>
<td>Mach .7/40 kft (nominal)</td>
<td>.4046</td>
<td>.4533</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.4234</td>
<td>.4572</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>1.0765</td>
<td>.4231</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.4829</td>
<td>.4218</td>
</tr>
</tbody>
</table>

Table 5.7: RMS pitch estimation error (deg) in the presence of command input - Low-cost gyro case

Figure 5-8: Pitch estimation for the overdesigned Kalman filter and the robust filter for the Mach .9/40 kft configuration - Low-cost gyro case
Figure 5-9: Magnitude of the transfer functions from the command inputs to the pitch estimation error for the Mach .9/40 kft configuration - Low-cost gyro case
Summary

From all these results, we conclude that when sensors are accurate enough, an overdesigned Kalman filter can handle small perturbations in the model. However, for larger model uncertainties, a robust filter will do better.

When sensors are degraded, the robust filter handles plant perturbations better than the overdesigned Kalman filter. However, the quality of estimation remains very dependent on the sensor quality.

Overall, we have shown that a single robust filter design yields acceptable performance for a wide range of linearized aircraft dynamic models.

5.5 **Effect of unmodeled wind gusts**

In this section, the effect of wind gust modelling is investigated. In particular, we show that the robust filters designed in the preceding sections resist process noise model uncertainty well.

Our methodology is the following: simulate the aircraft with a wind gust process noise as is done is Section 4.2. Again, we use the gust models provided in Appendix C and consider a RMS value of 5 ft/s for the wind gust. Outputs of the simulation are then processed through the discrete-time robust filter which provides the pitch estimate.

| Mach .9/40 kft | Usual process noise | .3235 | | Gust (5 ft/s) | .3281 |
| Mach .7/40 kft | .1629 | .1551 |
| Mach .5/20 kft | .0852 | .0926 |
| Mach .65/sea level | .0976 | .1226 |
| Mach .2/sea level | .2037 | .2140 |

Table 5.8: RMS pitch estimation error (deg) in the presence of command input and wind gust for Robust Filter 1 - Good gyro case

Tables 5.8 and 5.9 give the RMS pitch estimation error over the usual 1000 s simulation for the various process noise cases. It is obvious from the results that
<table>
<thead>
<tr>
<th>Mach .9/40 kft</th>
<th>Usual process noise</th>
<th>Gust (5 ft/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach .7/40 kft</td>
<td>.5091</td>
<td>.5085</td>
</tr>
<tr>
<td>Mach .5/20 kft</td>
<td>.4538</td>
<td>.4387</td>
</tr>
<tr>
<td>Mach .65/sea level</td>
<td>.4572</td>
<td>.4676</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.4231</td>
<td>.4970</td>
</tr>
<tr>
<td>Mach .2/sea level</td>
<td>.4218</td>
<td>.4444</td>
</tr>
</tbody>
</table>

Table 5.9: RMS pitch estimation error (deg) in the presence of command input and wind gust for the robust filter - Low-cost gyro case

the unmodeled wind gust process noise does not lower the performance of the robust filters.

Our interpretation is that the main estimation error source is not the unmodeled process noise but rather the mismodeled aircraft dynamics.

These simulations show however that the robust filter obtained in the preceding sections operates well for a wide range of noise as well as plant model uncertainties.
Chapter 6

Conclusion

This thesis applies a recently developed robust $H_{\infty}$ estimation technique to the aircraft attitude determination problem. Specifically, an acceptable pitch angle estimate is obtained using merely a rate gyroscope, an airspeed sensor and the aircraft dynamic equations of motion. Whereas a conventional Kalman filter is shown to be very sensitive to aircraft model uncertainties, robust filtering handles plant and noise model uncertainties better. The next subsection summarizes the thesis content and the last one provides suggestions for further research.

Summary

In Chapter 2, the aircraft and sensor models used throughout the thesis are presented. We give an overview of Kalman filtering in Chapter 3, and we demonstrate its sensitivity to model uncertainty in the context of our navigation problem. In Chapter 4, we present the minimax filter which provides a bound on the transmitted energy from the disturbance to the estimation error, yielding robustness to noise model uncertainties. The influence of the design parameter $\gamma$, trading off $H_{2}$ and $H_{\infty}$ performance, is investigated for our navigation problem. It is further shown that an appropriate tuning of $\gamma$ leads to a filter with robustness to unmodeled wind gust dynamics. In Chapter 5, the minimax filter is generalized to deal with both noise and plant model uncertainties. Numerical results demonstrate that such filters are able to cope with a
wide range of model uncertainties, while still providing an acceptable pitch estimation error. It is further shown that overdesigning the process noise in a regular Kalman filter, while building up some robustness, does not handle model perturbations as well as the robust $H_\infty$ filter. However, results indicate that both the overdesigned Kalman filter and the robust filter performance are quite dependent on the quality of the sensors. Finally, simulations show that the performance of the robust filters does not deteriorate in the presence of unmodeled wind gusts.

Further work

The most direct extension of this work would be to deal with the lateral dynamics of the aircraft, and obtain robust filter designs for both roll and yaw angles. Together with the filter designs presented in this thesis, this would realize a full attitude determination system with robustness to aircraft mismodeling. The next step would be to design a full navigation filter that provides position, velocity and attitude estimates.

Another interesting investigation would be to consider some other robust filtering design methodologies that are less conservative than the $H_\infty$ one. A first possibility is to use the $\mu$ estimator developed in [2, 3] in order to take advantage of the diagonal structure of the parametric uncertainty (see Eq. 5.22). Another avenue would be to consider robust $H_2$ design methodologies [19, 21] that give bounds on the worst case $H_2$ rather than $H_\infty$ cost for a given uncertainty model.

However, it must be kept in mind that worst case designs will always be conservative ones, since actual noise and plant uncertainties are not likely to be as harmful to filter's performance as the worst possible case. These methodologies come from robust control theory where robust stability is the primary issue and must be strictly guaranteed over the entire uncertainty. In contrast, robust estimation is only concerned with robust performance. With this perspective in mind, a new estimation technique that would average the $H_2$ cost over the entire uncertainty would be of interest. Specifically, an “optimal” robust estimation problem would be: given a probabilistic description of model uncertainty, find an estimator that minimizes the averaged $H_2$ cost over the uncertainty. This type of problem has been investigated
from a control point of view by Hagood [11]. An adaptation of this theory to estimation would certainly be an interesting contribution.

Finally, another avenue for further work would be to consider adaptative estimation techniques instead of or in conjunction with robust estimation. When the aircraft model is not accurately known, it may be possible to partially infer it from the sensor measurements. Robust estimation could then deal with the remaining unidentifiable uncertainty.
Appendix A

Boeing 747 Longitudinal Dynamics

The models considered are linearizations of the longitudinal dynamics of the Boeing 747 around various Mach numbers and given constant altitudes [12]. They are given here in continuous-time and were further discretized before implementation with a sampling period of 0.05 s. The four states are: forward velocity (ft/s), normal velocity (ft/s), pitch rate (deg/s), and pitch (deg). The two inputs are: elevator deflection (deg) and thrust (ft/s²).

Mach = .9
Altitude = 40 kft
Pitch angle of linearization = 2.4 deg

\[ A = \]

\[
\begin{array}{ccccc}
-0.0200 & 0.0159 & -0.6366 & -0.5607 \\
-0.0427 & -0.4035 & 15.1645 & -0.0236 \\
0.0034 & -0.1052 & -0.5400 & 0.0002 \\
0 & 0 & 1.0000 & 0 \\
\end{array}
\]
\[ B = \]

\[
\begin{array}{cc}
0.0136 & 1.0000 \\
-0.3266 & -0.0438 \\
-1.2170 & 0.3430 \\
0 & 0 \\
\end{array}
\]

Mach = .7
Altitude = 40 kft
Pitch angle of linearization = 7.3 deg

\[ A = \]

\[
\begin{array}{ccccc}
-0.0019 & 0.0263 & -1.5036 & -0.5566 \\
-0.0701 & -0.2941 & 11.7447 & -0.0718 \\
0.0152 & -0.3563 & -0.3449 & 0.0004 \\
0 & 0 & 1.0000 & 0 \\
\end{array}
\]

\[ B = \]

\[
\begin{array}{cc}
0.0337 & 1.0000 \\
-0.2654 & -0.0439 \\
-0.9686 & 0.3429 \\
0 & 0 \\
\end{array}
\]
Mach = .5
Altitude = 20 kft
Pitch angle of linearization = 6.8 deg

\[
A = \\
\begin{bmatrix}
-0.0025 & 0.0782 & -1.0705 & -0.5572 \\
-0.0690 & -0.4399 & 9.0071 & -0.0675 \\
0.0146 & -0.0943 & -0.4855 & 0.0005 \\
0 & 0 & 1.0000 & 0
\end{bmatrix}
\]

\[
R = \\
\begin{bmatrix}
0.0353 & 1.0000 \\
-0.2997 & -0.0443 \\
-1.0879 & 0.3430 \\
0 & 0
\end{bmatrix}
\]

Mach = .65
Altitude = sea level
Pitch angle of linearization = 0 deg

\[
A = \\
\begin{bmatrix}
-0.0078 & 0.0345 & 0 & -0.5612 \\
-0.1298 & -0.9921 & 12.8234 & 0 \\
-0.0097 & -0.1240 & -1.0925 & 0 \\
0 & 0 & 1.0000 & 0
\end{bmatrix}
\]
B =

\[
\begin{pmatrix}
0 & 1.0000 \\
-0.5826 & -0.0449 \\
-2.0624 & 0.3432 \\
0 & 0
\end{pmatrix}
\]

Mach = .2

Altitude = sea level

Pitch angle of linearization = 8.5 deg

A =

\[
\begin{pmatrix}
-0.0209 & 0.1220 & -0.5701 & -0.5550 \\
-0.2090 & -0.5297 & 3.8343 & -0.0858 \\
0.0096 & -0.0939 & -0.4110 & 0.0012 \\
0 & 0 & 1.0000 & 0
\end{pmatrix}
\]

B =

\[
\begin{pmatrix}
0.0167 & 1.0000 \\
-0.1159 & -0.0452 \\
-0.3764 & 0.3122 \\
0 & 0
\end{pmatrix}
\]
Appendix B

First-order Markov Process

The behavior of a scalar first-order Markov process $x(t)$ is characterized by its root mean squared value $\sigma_x$, and its time constant $\tau$. This process is described in detail in [9]. We give here the propagation equations for the continuous and discrete-time scalar case.

Continuous-time scalar case

$$\dot{x} = -\frac{1}{\tau}x + \sqrt{\frac{2\sigma_x^2}{\tau}}w(t) \quad (B.1)$$

with $w(t)$ is a unit intensity continuous-time white noise.

The variance propagation equation is given by:

$$\dot{E} = -\frac{2}{\tau}E + \frac{2\sigma_x^2}{\tau} \quad (B.2)$$

where $E \equiv E(x^2)$.

From this equation, it may be easily seen that the RMS steady-state value of the process is indeed given by $\sigma_x$. The auto-correlation of such a process is given by:

$$R_{xx}(t) = \sigma_x^2 e^{-\frac{|t|}{\tau}}$$

Hence, the time constant $\tau$ can be interpreted as a measure of how "peaky" the autocorrelation function is.
Discrete-time scalar case

In order to discretize the process, we integrate the continuous-time dynamics (B.1) thanks to the variation of constant formula:

\[ x(t) = e^{-\frac{(t-t_o)}{\tau}} x(t_o) + \int_{t_o}^{t} e^{-\frac{(t-s)}{\tau}} \sqrt{\frac{2\sigma_x^2}{\tau}} w(s) \, ds \]

Taking samples at \( t_k = k\Delta t \), we obtain:

\[ x_{k+1} = e^{-\frac{\Delta t}{\tau}} x_k + \int_{t_k}^{t_{k+1}} e^{-\frac{(t_{k+1}-s)}{\tau}} \sqrt{\frac{2\sigma_x^2}{\tau}} w(s) \, ds \]

The second term is a discrete white noise with variance:

\[
\begin{align*}
& \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} e^{-\frac{(t_{k+1}-s)}{\tau}} \sqrt{\frac{2\sigma_x^2}{\tau}} E[w(s)w(u)] \sqrt{\frac{2\sigma_x^2}{\tau}} e^{-\frac{(u-t_k)}{\tau}} \, ds \, du \\
& = \frac{2\sigma_x^2}{\tau} \int_{t_k}^{t_{k+1}} e^{-\frac{(t_{k+1}-s)}{\tau}} \, ds \\
& = \frac{2\sigma_x^2}{\tau} e^{-\Delta t} \int_{0}^{\Delta t} e^{\frac{s}{\tau}} \, ds \\
& = \sigma_x^2 \left( 1 - e^{-\frac{2\Delta t}{\tau}} \right)
\end{align*}
\]

Hence the propagation equation for the discretized process may be written as:

\[ x_{k+1} = e^{-\frac{\Delta t}{\tau}} x_k + \sigma_x \sqrt{1 - e^{-\frac{2\Delta t}{\tau}}} w_k \quad (B.3) \]

where \( \Delta t \) is the sampling interval and \( w_k \) is a unit variance discrete-time white noise.

Computing the variance propagation equation, we obtain:

\[ E_{k+1} = e^{-\frac{2\Delta t}{\tau}} E_k + \sigma_x^2 \left( 1 - e^{-\frac{2\Delta t}{\tau}} \right) \quad (B.4) \]

where \( E_k \equiv E(x_k^2) \).
Appendix C

Wind Gust Models

Wind gust models from [6] are given here as a function of their RMS value $\sigma$, the aircraft velocity $V$, and the aircraft altitude $h$.

Forward gust velocity $u_w$

$$\frac{u_w(s)}{\eta_u(s)} = \frac{c_1}{sL/V + c_1}$$

where $\eta_u(t)$ is zero-mean white noise with spectral density,

$$Q_u = \frac{2\sigma^2 L}{c_1 V} \Rightarrow RMS(u_w) = \sigma$$

$$c_1 \equiv \frac{(1 + 3\beta/2)^{2/3}}{1 + 3\beta}$$

$$\beta \equiv b/2L$$

$$L \approx L_\infty \frac{h}{h + h_o} \equiv \text{turbulence integral scale}$$

$$L_\infty = 2000 \text{ ft, } \quad h_o = 2500 \text{ ft}$$

$$b = \text{wing span} = 195.68 \text{ ft for the Boeing 747}$$

Vertical gust velocity $w_w$

$$\frac{w_w(s)}{\eta_w(s)} = \frac{\frac{c_3}{(sL/V)^2 + c_2(sL/V) + c_3}}$$
where $\eta_w(t)$ is zero-mean white noise with spectral density,

$$Q_w = \frac{2c_2 \sigma^2 L}{c_3 V}, \quad \Rightarrow \quad \text{RMS}(w_w) = \sigma$$

$$c_2 \equiv \frac{1 + 3\beta}{2\beta^{4/3}}, \quad c_3 \equiv \frac{(1 + \beta)^{2/3}}{\beta^{4/3}}$$
Bibliography


