Constrained Predictive Control for Corporate Policy

by

Thomas Alois Weber

Ing. ECP, Ecole Centrale Paris (1994)
Dipl.-Ing., Rhenish-Westphalian Technical University Aachen (1995)

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Author . . . .
Department of Electrical Engineering and Computer Science
August 29, 1997

Certified by . . . .
Alexandre Megretski
Assistant Professor of Control Theory
Thesis Supervisor

Accepted by . . . .
Richard de Neufville
Chairman, Technology and Policy Program

Accepted by . . . .
Arthur C. Smith
Chairman, Department Committee on Graduate Theses

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Abstract

This Master's thesis uses moving horizon control strategies for deterministic critical resource allocation problems at the level of the firm. Corporate policies, *optimal* with respect to the allocation of critical resources, maximize a specified objective functional.

A new formal approach to so-called *nonsingular* corporate decision making, that aims at finding optimal corporate resource allocation policies over an infinite time-horizon, is presented. For this, constrained predictive control, based on periodic forecasts over a finite time-horizon, is used as a theoretical tool. The methodology is well-adapted to the typically periodic resource allocation decisions, in that it uses moving horizon predictions derived from a model of the firm and its environment. In the nominal case conditions can be given such that the moving horizon optimal cost approximates the infinite horizon theoretically optimal cost, and yields a stabilizing behavior of the system converging towards an optimal equilibrium state.

New results are obtained for dealing with discounted indefinite objective functionals that appear frequently in economic applications. The main theoretical result is that under certain conditions an optimal equilibrium state can be found without solving Bellman's equation or using the Pontryagin Maximum Principle. The critical resource allocation problem can then be formulated equivalently using a positive semi-definite cost functional that is computed explicitly. As an illustration, a simple nonlinear model of a firm in a diffusive market environment is treated in detail.

The methodology is critically reviewed with respect to its practical relevance and potential. An industry example is considered to clarify the main points of the discussion.

Thesis Supervisor: Alexandre Megretski
Title: Assistant Professor of Control Theory

For my Parents.

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Cambridge, Massachusetts
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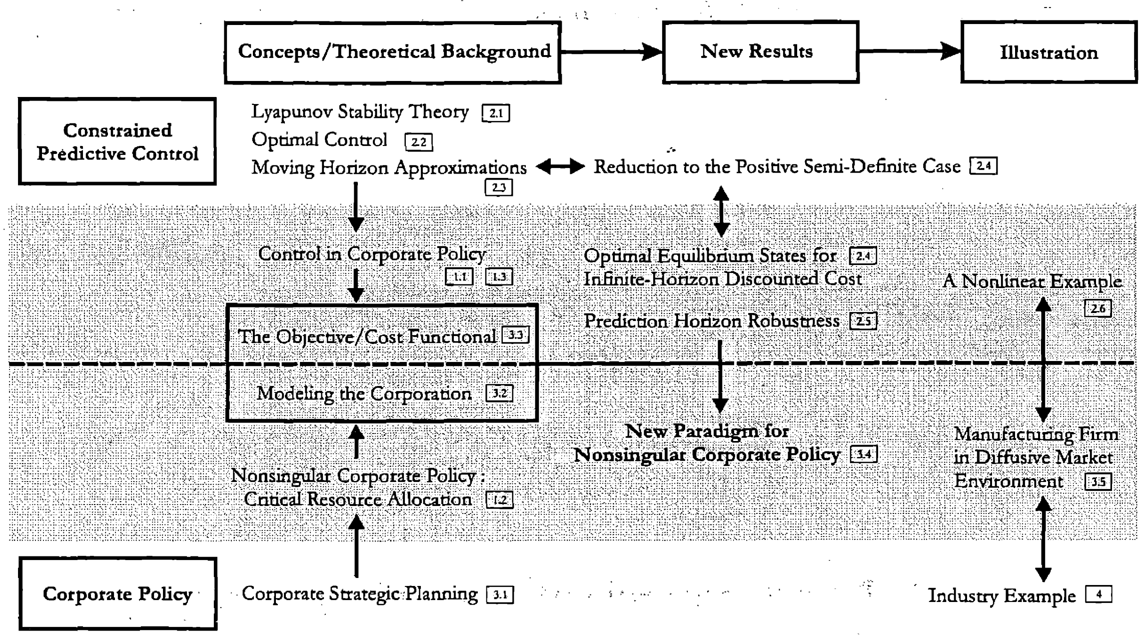
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All human knowledge thus begins with intuitions, proceeds then to concepts, and ends with ideas.
— IMMANUEL KANT

Chapter 1

Introduction

1.1 Control in Corporate Policy

What is the potential for control in corporate policy? This question will appear throughout this thesis. *Control*, is here understood to be a methodology for defining inputs to a system in such a way that it exhibits a certain desired behavior. If a firm is understood as a system, whose evolution can be influenced at least partly by the choice of certain decision variables, then control theory may provide an appropriate framework for thinking about finding good strategies for the selection of inputs to a corporate system, such as to come close to some specified objective. In other words, given a model of the firm that quantifies assumptions about its inner workings and relationships with the external environment, control theory may yield systematic and insightful ways of finding an *optimal corporate policy*, with respect to a chosen objective function.

A difficulty that arises when combining mathematical control theory and real-world management concepts, used by corporate leaders, is that terms on the mathematical side often have a much more precise meaning than related concepts on the management side (cf. Table 1.1 on page 20). On the other hand, there are at times no direct mathematical analoga to concepts that appear to be very well defined in the management environment. In short, it is important when establishing a link between control theory and corporate policy, to identify the key concepts on both sides, and formulate a clear relationship between them, so that derivations make sense from either point of view. Assuming that such a clear correspondence can be established, the above question discussed throughout this work can be framed more precisely as follows:

Question 1 *Can a procedure be defined and conditions given under which control can be used efficiently for corporate policy?*

Thereby we define *corporate policy* as a set of strategic decisions that are central for the intended evolution of a firm, as defined by the appropriate legal and statutory constructs.

It seems clear that it will be impossible to find such a general procedure and conditions that guarantee a potentially efficient use of control for all situations of corporate policy. Since in general decision makers are not rational, strategic choices ill-defined, external events uncertain, and the objectives unclear, mathematical concepts are in most cases too limited to deal fully with Question 1 as formulated above. Moreover, top decision makers in a corporation are often concerned with 'singularities' (in a mathematical sense), i.e., their decisions cause a 'fundamental change' that can hardly be accommodated by modeling assumptions. Such 'singular' decisions include mergers and acquisitions, the introduction of a new product line, or a corporate restructuring program. — It is not this type of decision making that is addressed here, and control theory is likely to be of no particular use for such high-level strategic thinking.

The focus of this work is on corporate policy in a restrictive sense comprising the allocation of resources and choice of strategic variables that are critical to the evolution of the firm[1] and 'non-singular' in a sense. A corporation generally comprises different organizational entities, such as 'divisions' or 'functions' that compete for the allocation of its resources. Resources include funds as well as nonmonetary contributions such as production capacity and central firm-wide services such as training or human resources. Since the available resources are generally limited, their allocation cannot take place independently for the different entities and tradeoffs have to be considered.

The Decision Making Process. Typically, decisions on the allocation of resources and the choice of strategic variables such as the price for a product of the firm, are taken in periodic stages after review of regular internal reports and other data concerning the environment of the firm. Such information describes the past development of the firm with respect to its environment, and may allow predictions of events over a finite *planning horizon.* This horizon, generally extends further into the future as time progresses while decisions are continuously implemented to the best of the knowledge at the last decision stage. The next decision stage then uses a planning horizon of a similar length but extending further into the future than the last planning horizon. In terms of control theory, such a strategy can be described as *moving horizon* as the planning horizon over which an objective is maximized, moves into the future (cf. Figure 1-1). The resource allocation and strategic variable values decided upon at the last decision stage – covering parts of a planning horizon that extended beyond the current decision stage – are discarded and re-determined over a new planning horizon by taking additional information into account that has become available since the last decision stage.

It is important to note that such periodic decision making, that is common to the vast majority of corporate entities, may yield under certain circumstances an inherently unstable resource allocation strategy, even though at each stage a seemingly optimal decision is taken. Indeed 'unstable' behavior may result from nonlinearities in the system model, constraints imposed on inputs and states, or a too short planning horizon. The induced undesired instabilities are often characterized by inefficient cycles, for instance in production capacity. Such cyclical strategies are in most cases costly for the firm, since they usually involve investments in adjusting capacity and human resources. This type of dysfunctional corporate policy is frequently observed in diffusive market environments and has been studied at the behavioral level (cf. [PS93]). It will be discussed further in Section 1.3, together with the advantages of the predictive control methodology.

What is the scope of the decision making process considered? Assuming that the use

[1] Note that for convenience the words corporation and firm will be used interchangeably from this point on.

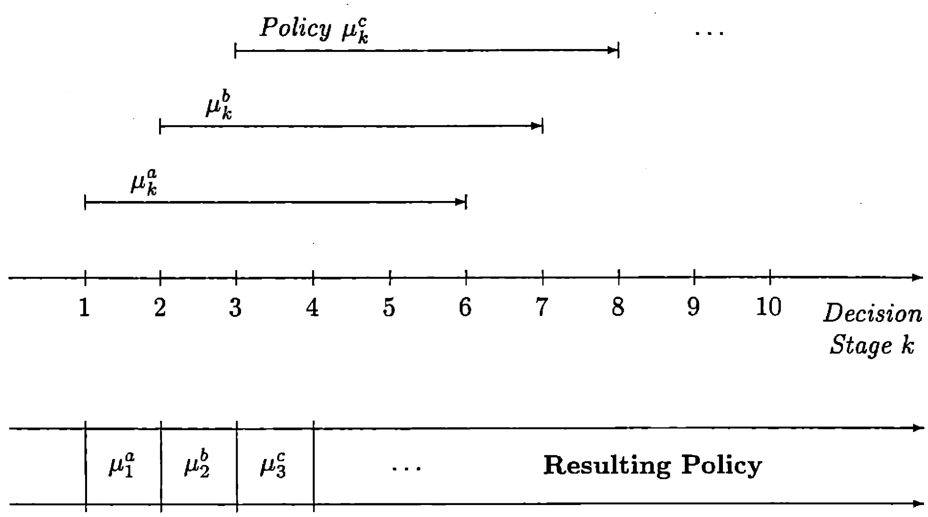

Figure 1-1: Nonsingular Corporate Policy with Moving Planning Horizon.

of control theory is limited to *nonsingular corporate policy*, i.e., corporate policy that does not change the current operations of a firm, Question 1 can be re-stated in terms of a *critical resource allocation problem.* To this end, a *critical resource* of a firm is defined as *any* decision variable that can be influenced *and* is considered critical to achieve a particular corporate objective. Under this definition some critical resources such as the price for a product, may be chosen without *direct* cost considerations by the firm, while for resources in the traditional sense such as production capacity, cost has to be considered immediately. However, in general most critical resources (even price) are limited in the sense that they cannot be chosen arbitrarily.

Question 2 *Can a procedure be defined and conditions given under which control can be used efficiently in critical resource allocation problems, i.e., nonsingular corporate policy?*

Prerequisite for the use of control is a model of the system it should be applied to. Hence, for its use in nonsingular corporate policy a quantitative understanding of the ongoing operations is required. This model has to be well defined mathematically and relate quantifiable *input*, *state*, and *output* to each other. The state of a system thereby completely describes its current status and can be possibly changed by its inputs. A report about the system would generally make statements about its *observable* states and outputs. Some states may not be attained by *any* control input and therefore be not *reachable.*[2] In order to find a control input that is most suited for the goals of the firm, an *objective functional* has to be specified of which a formal definition will be given in Chapter 2.

Question 2 is still not very precise, as it leaves open the choice of the particular control methodology, of which, depending on the situation, they may be many. In particular, the control methodology should be adapted to the natural corporate decision making process

[2] An analysis of such system properties might reveal that it is *impossible* to attain certain goals starting from a given initial configuration (cf. Chapter 2).

for ongoing operations, as described briefly above. Moreover, it will depend on the nature of the considered model (linear/nonlinear, time-variant/invariant), as well as the objective functional (quadratic/non-quadratic, definite/indefinite). These distinctions will become clear later, when formal definitions are advanced. — Since economic systems such as firms are notoriously fraught with uncertainty and hard to identify, it is important to know how the control strategy will behave towards slight changes in the modeling assumptions, or a different choice of parameters. We therefore demand that the control methodology yield *robust* results with respect to uncertainty and modeling assumptions. For nonlinear systems such as the ones typically encountered when modeling economic entities such as firms, theoretical robustness results are hard to obtain, and may depend quite significantly on the particular model considered. In addition to this type of robustness towards *unstructured uncertainty*, a firm naturally operates in a dynamic market environment, where competition can be seen as *structured uncertainty*, since it its actions generally depend on its own behavior. In general, knowledge about a worst-case performance with respect to potential actions of competitors is desirable. However, these issues, involving the use of dynamic game theory, are beyond the scope of this work. The developments here are given for the non-stochastic[3] case in a *monopolistic dynamic market environment.*

Thesis Outline. In the following section different approaches for planning under uncertainty, together with their respective strengths and weaknesses, are reviewed. Then the predictive control methodology, based on a model and moving horizon predictions is introduced and put into context with the reviewed approaches. The main questions that will be addressed in this thesis are formulated here. Chapter 2 introduces the theory of predictive control for both linear and nonlinear system models. The focus will be here on presenting general versions of the available theory. Chapter 3 adapts the developed theory to corporate decision making and presents a nonlinear example with very few decision variables to illustrate the methodology and alert the reader to its potential pitfalls and difficulties. Chapter 4 applies the concept to a practical industry example, namely the Boston Central Artery/Tunnel Project's internal allocation of resources for information technology. Chapter 5 finally discusses the applicability of constrained predictive control for corporate decision makers, draws conclusions and suggests topics for further research.

1.2 Nonsingular Corporate Policy: Critical Resource Allocation

In the following we will discuss a number of formal or semi-formal approaches available for 'nonsingular' corporate policy as introduced in the last chapter. Most of these approaches are partly related to the predictive control methodology that will be presented in more detail in the next section and formally developed in Chapter 2. This review of the literature is non-exhaustive and limited to references that the author found helpful in understanding the main concepts of the presented approaches.

[3]For linear systems and a quadratic objective function stochastic disturbances on the state and/or output of the system can be taken into account by replacing them with their means using the Certainty Equivalence Principle, cf. [ÅW90].

1.2.1 Traditional Capital Budgeting and Resource Allocation

The traditional view of resource allocation resides on microeconomic theory of the firm. The firm is here essentially treated as a 'black box' producing outputs, given appropriate inputs. Its behavior is characterized by a so-called *production function* $F_P : u \mapsto y$ that describes the dependence of the outputs $(y_1, \dots, y_p)' =: y \in \mathbb{R}^p$ of the firm on the allocated resources $(u_1, \dots, u_m)' =: u \in \mathbb{R}^m$. Firms can usually vary the proportions in which they combine different resources, and the *efficient frontier* denotes the hyperplane (possibly only a point) of input combinations that maximizes a scalar objective function $\Pi(y, u)$. All resource combinations on this frontier are considered equally desirable with respect to the objective function Π.

The analysis in this traditional view of resource allocation is usually static (without explicit dependence on time) and limited to states around short and long-run equilibria. Given a production function, and an associated objective function, the traditional resource allocation problem becomes: For a constrained amount of resources, i.e.,

$$u_{i\min} \leq u_i \leq u_{i\max}, \qquad i \in \{1, \dots, m\},$$

determine an admissible (in the sense that it satisfies the above constraints) allocation u^* of resources such that

$$\Pi(F_P(u), u) \longrightarrow \max .$$

Such constrained maximization problems can be generally solved using Lagrange multipliers and do in most cases not pose substantial analytical or computational difficulties. — Perhaps due to its simplicity, the production function approach is fairly well-established. It has been used frequently for steady-state analyses at the level of the firm, an industry, or a national economy. Moreover, statistical methods are available for fitting coefficients of 'standard' production function representations[4] to real-world data.

Traditional capital budgeting is limited by its inability to incorporate uncertainty and dynamic behavior in a straightforward way. The only modeling components in this approach are the expressions of the production and objective functions, so that generally only aggregate statements may be obtained. In addition the methodology is often not appropriate to describe sequential interactions between firms or between market and firm in a monopolistic competition. The approach can be made pseudo-dynamic by applying it repeatedly to different scenarios, but this will generally not yield satisfying dynamic results as no explicit model for the system evolution as a *consequence* of previous decisions is present.

Traditional capital budgeting theory on the other hand focuses on the concepts of maximizing shareholder value. An essential component of the capital budgeting process is the choice among different investment project alternatives, that are typically ranked according

[4]One of the most prominent representations is the *Cobb-Douglas* production function: $F_P(u) = C \prod_{i=1}^{m} u_i^{\alpha_i}$. Using linear regression for the expression $\ln F_P(u) = \ln C + \sum_{i=1}^{m} \alpha_i \ln u_i$, the positive coefficients C and α_i can be readily estimated.

to their *Net Present Value* (NPV),

$$\mathrm{NPV} = \sum_{k=0}^{N} \frac{\mathrm{CF}_k}{(1+r')^k} = \sum_{k=0}^{N} e^{-rk}\mathrm{CF}_k,$$

where the first and the second expression for the NPV are identical for $r = \ln(1+r')$. Although the first form is the one that is used almost exclusively in the management literature to represent discounted cash flows $(\mathrm{CF}_k/(1+r)^k)$, we prefer the second form here for consistency with the continuous-time discounted cost functionals that will be introduced later. The positive constant r is the *discount rate* and describes a required rate of return for the project under evaluation. It usually equals the rate of return of the next best project with equal risk. The NPV of a project is a reliable indicator of its value, in the absence of uncertainty. Including disturbances, one has to consider the expectancy of the NPV, together with the variance of the return on the project. Under a more general setting the project can then be seen simply as an asset that is worth to invest in or not given its return r, its risk σ (i.e., standard-deviation of the return), and its correlation with the 'market portfolio': the collection of all assets available on a financial market. The required rate of return for a project can then be determined under relatively strong simplifying assumptions by the Capital Asset Pricing Model,

$$E\{r\} = r_f + E\{\beta(r_m - r_f)\},$$

where β describes the covariance of the asset with the market portfolio, r_f a risk-free rate of return (given e.g. by government bond rates), and $E\{\beta(r_m - r_f)\}$ an expected *risk premium* over the market return r_m. We will not further develop the substantial theory associated with this approach, and only note that most statements are derived in a stationary, non-dynamic setting, and generally do not consider a dynamic model of the inner workings of a firm. Instead, the market is seen as a collection of numerous actors whose behavior can be described in terms of stochastic objects. — In addition, we remark that this *type* of analysis, which is essentially based on statistical moments such as the mean or variance, is extremely hard to conduct for nonlinear dynamical phenomena, as the underlying probability distributions do not behave nicely from a mathematical point of view. Thus, for nonlinear phenomena, simulation may be the only possible way to estimate the required statistical moments.

1.2.2 Dynamic Optimization

Given an inter-temporal scenario, i.e., a corporate system and objective function(al)[5] that depend explicitly on time, the type of questions that can be addressed become different to the ones in the above static resource allocation problems: Does the system possess a steady-state, that is optimal in the sense that once it has been reached, the value of the objective function cannot be improved? What is an optimal trajectory that leads to such an optimum state, if it exists? Is the optimal policy stable, and if yes, under what conditions?

[5] For dynamical systems, we will refer to an objective functional rather than objective function, since the maximization (or minimization) is carried out with respect to input *functions* rather than numbers as in Section 1.2.1.

A dynamic optimization approach generally requires a model of the firm that describes the evolution of its 'states' $x(t)$, i.e., characteristic dependent variables, as a function of the input $u(t)$ and time t :

$$\dot{x}(t) = f(t, x, u) \tag{1.1}$$

with initial condition

$$x(0) = x_0.$$

The goal is then to find a time and state-dependent resource allocation policy

$$u(t) = \mu(t, x),$$

such that an objective functional $\Pi(u)$ is maximized over a finite or infinite time horizon T. More specifically, the problem is to find u such that

$$\Pi(u) := \int_0^T h(t, x, u)\, dt \longrightarrow \max, \tag{1.2}$$

subject to constraints

$$u \in \Omega(x).$$

Exact definitions of all the sets and functions involved in this general problem description are given in Chapter 2. The positive time horizon T in (1.2) can be finite or infinite.

The above formulation of the dynamic optimization problem is very general and appropriate for a model-based search for optimal policies at the level of the firm. For not too irregular problems, necessary and sufficient conditions for an optimal control-state trajectory are available and can be used for an attempt to find an explicit solution to the dynamic optimization problem. However, due to nonlinearities in the system and the presence of constraints, an analytical treatment is in most cases impossible, so that computer simulation remains the only way to obtain an idea about the nature of the optimal resource allocation for particular initial conditions. A sensitivity analysis can then be performed to find out the dependence of the optimal strategy on the initial conditions. However, this simulation-based study of the system generally does not yield well-founded conclusions, and may be simply impossible to conduct for infinite horizon problems without asymptotically stable steady-state. We note however that there are quite sophisticated approaches in this area using functional approximations such as neural networks.

The predictive control methodology presented later is based significantly on dynamic optimization, as at each decision stage a finite horizon optimal control problem has to be solved on-line to find a control that is to be selected at the next instant. Dynamic optimization *per se* however does not consider stability[6] of the optimal solutions obtained. This will be an essential part of the formal presentation of the predictive control methodology in Chapter 2.

[6]It is clear that for a finite horizon optimization problem the optimal policy will generally be stable in the sense that it is bounded. Stability considerations become of importance as soon as the time horizon is not fixed anymore, and can be considered infinite for all practical purposes.

1.2.3 Decision Analysis

A decision-analytical approach is similar to the dynamic optimization introduced above in that the value of an objective function is to be maximized subject to constraints given by input and state limitations as well as the system's evolution. However, the decision analysis is typically conducted in discrete-time. Uncertainty in the form of a random disturbance w_k is explicitly considered in the state equation

$$x_{k+1} = f_k(x_k, u_k, w_k), \tag{1.3}$$

with initial condition

$$x_0 = a, \tag{1.4}$$

and is characterized by probability distributions $P_k(\cdot \mid x_k, u_k)$. The goal, analogous to the problem presented in Section 1.2.2, is then to maximize the expected value of an objective functional $\Pi(u)$, or equivalently — in accordance with the usual conventions in optimal control theory — to minimize the expected value of a cost functional $J(u) = -\Pi(u)$, i.e.,

$$J(a, u) := \mathop{E}_{w} \left\{ \sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) \right\} \longrightarrow \min, \tag{1.5}$$

subject to (1.3)–(1.4), and

$$u_k \in \Omega_k(x_k). \tag{1.6}$$

The Dynamic Programming Principle, developed to its full potential by Richard Bellman in the 1950s [Bel57], provides a simple algorithm for finding a solution to the above problem, by considering successively 'tail subproblems' of the form,

$$\mathop{E}_{w} \left\{ \sum_{k=i}^{N-1} h_k(x_k, u_k, w_k) \right\} \longrightarrow \min,$$

the optimal policies for which are also optimal for the last time-stretch of the complete problem. More specifically, a dynamic programming algorithm for problem (1.5) can be formulated as follows:

$$\begin{aligned} V_N(x_N) &= 0, \\ V_k(x_k) &= \min_{u_k \in \Omega_k(x_k)} \mathop{E}_{w} \left\{ g_k(x_k, u_k, w_k) + V_{k+1}\left(f_k(x_k, u_k, w_k)\right) \right\}, \end{aligned} \tag{1.7}$$

for all stages $k \in \{0, \ldots, N-1\}$, whereby $V_k(x_k)$ is the optimal cost-to-go from state x_k at time k.

This algorithm describes formally exactly what is the basis of *Decision Analysis*, namely the finding of optimal policies based on an N-stage decision tree, in which each branch can be reached from an earlier branch with a certain transition probability. The problem is then to find for a given initial state the path with highest expected payoff, or equivalently, the minimum expected cost, given that disturbances may occur along the way. With such disturbances the resulting policy is in effect a feedback policy, since the disturbance has an influence on the actual state that is being reached from a particular state (even though one might have intended to go to another state), so that at the next time step the policy has to consider this fact (cf. Figure 1-2).

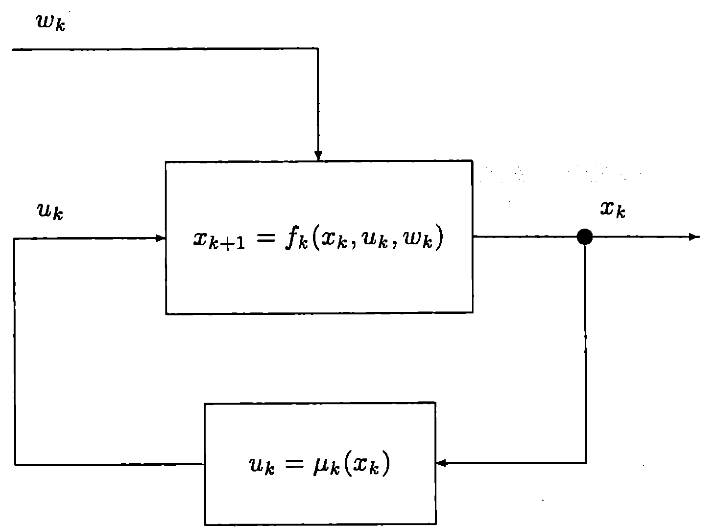

Figure 1-2: State Feedback Policy under Uncertainty.

1.3 The Predictive Control Methodology

The model-based predictive control methodology will now be presented in more detail and related to the approaches presented in the previous section. As already mentioned earlier, predictive control employs a moving horizon, in the sense that at each stage one finds an optimal limited lookahead[7] policy based on a system model and then implements only the first of the computed control moves, only to repeat this optimization at the next stage. In mathematical terms, the optimal policy at time $i \geq 0$ is obtained by solving

$$\hat{\mathcal{P}}(i,a): \quad \hat{J}_i := \mathop{E}_{w}\left\{\sum_{k=i}^{i+M-1} h_k(x_k, u_k, w_k)\right\} \longrightarrow \min, \tag{1.8}$$

subject to (1.3)–(1.4), (1.6), considering the endpoint constraint

$$x_{i+M} = x^e, \tag{1.9}$$

with positive horizon length M and a set of admissible states $\hat{X}_i$ that will be defined properly in Chapter 2. By obtaining solutions for the above problem at each time i, one implicitly determines a *moving horizon policy* $\hat{\pi}$ by the collection of first steps of the computed policies $\{\hat{\mu}_k^i, \ldots, \hat{\mu}_{k+M-1}^i\}$, i.e.,

$$\hat{\pi} = \{\hat{\mu}_0, \hat{\mu}_1, \hat{\mu}_2, \ldots\} := \{\hat{\mu}_0^0, \hat{\mu}_1^1, \hat{\mu}_2^2, \ldots\},$$

[7]over a finite prediction horizon

so that $u_k = \hat{\mu}_k(x_k),\ k \geq 0$ defines an infinite-horizon state-feedback law (cf. Figure 1-1).

In the case of a linear time-invariant system,

$$x_{k+1} = Ax_k + Bu_k + w_k$$

and quadratic cost,

$$h_k(x_k, u_k, w_k) = x_k' R x_k + u_k' Q u_k,$$

with positive definite matrices R and Q of appropriate dimensions, the methodology is well-known under the name of Model Predictive Control (MPC) and has been used extensively in practice, particularly for relatively slow varying dynamical systems such as those encountered in chemical process control. As a special case of general moving horizon predictive control, MPC requires at each time step the on-line solution of a finite horizon quadratic programming problem. Only the first of the computed optimal control moves is usually implemented and at the next time-step the procedure is repeated. Main advantage of the MPC methodology over other control approaches such as linear-quadratic Gaussian (LQG) control, is that input constraints can be handled in a quite elegant manner. For chemical processes, the quadratic objective function is appropriate to penalize state deviations from a 'reference trajectory' as well as to keep control action small. Considering state deviations $x - \bar{x}$ from a reference trajectory $\bar{x}$ in the cost (1.3) with $Q = \rho I$, and $R = I$, the cost that is to be minimized at time i becomes

$$\hat{J}_i = \sum_{k=i}^{i+M-1} \left(\|x_k - \bar{x}_k\|_2^2 + \|u_k\|_2^2 \right). \tag{1.10}$$

A typical MPC configuration is depicted in Figure 1-3.

The quadratic cost functional (1.3) is in many cases not appropriate for managerial resource allocation decisions. Also, reference trajectories are usually not available for the states of a corporate system, so that MPC in its traditional form is not very helpful in finding an optimal corporate policy. Moreover, corporate systems are often nonlinear, so that a formulation of the moving horizon control methodology more general than MPC is needed (cf. Section 1.3.2).

1.3.1 Interface between Decision Tool and Corporate Reality

The interesting feature of the predictive control methodology is that it is adapted to the corporate decision making process based on periodic revisions as described in Section 1.1. Moreover, at each stage, standard techniques from dynamic optimization can be applied to determine the next control move.

In order to be useful as a tool for decision making the dynamic model of the firm and the objective functional have to be closely linked to the real concerns of decision makers. We will discuss the choice of an appropriate objective functional in more detail in Section 3.3. Here we only remark that for many applications it is useful to consider expected

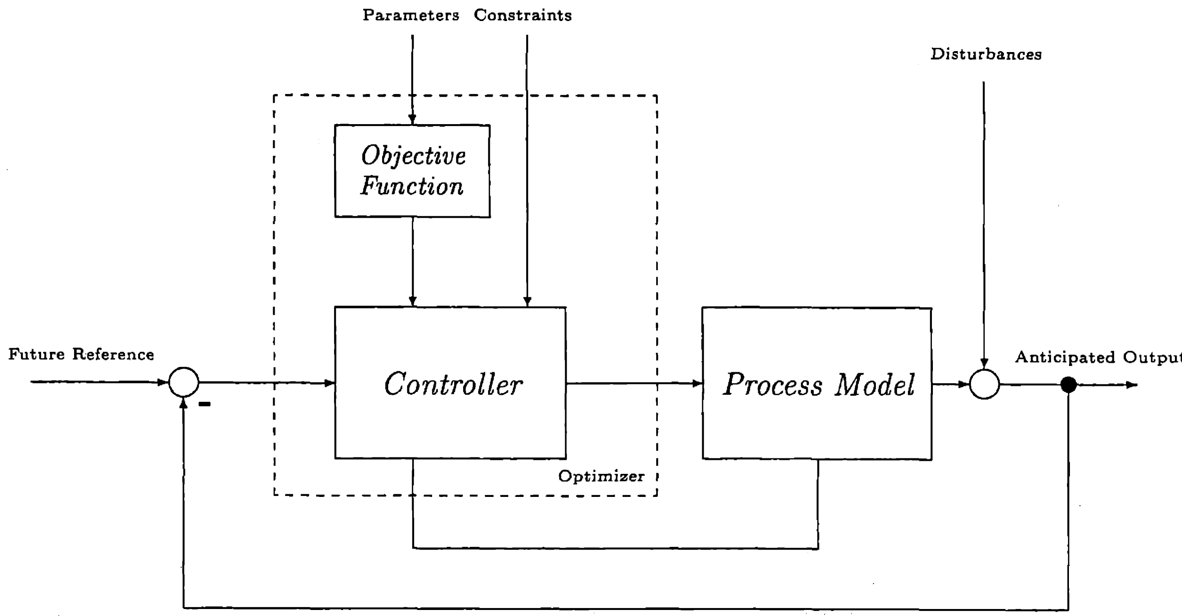

Figure 1-3: Typical MPC Configuration.

discounted profits,

$$\hat{\Pi}_i = -\underset{w}{E}\{\sum_{k=i}^{i+M-1} e^{-rk} h(x_k, u_k, w_k)\},$$

where $h(x_k, u_k)$ according to our convention to consider minimization rather than maximization problems, describes a momentary *cost*. Thus, in the absence of uncertainty the problem $\hat{\mathcal{P}}(i,a)$ becomes

$$\hat{\mathcal{P}}(i,a) \qquad \hat{J}_i = \sum_{k=i}^{i+M-1} e^{-rk} h(x_k, u_k) \longrightarrow \min, \tag{1.11}$$

subject to (1.3)–(1.4), (1.6), and possibly with endpoint constraint (1.9). An important difference to the linear-quadratic case discussed before is that the cost kernel h in (1.11) is not necessarily bounded from below, i.e., h may be sign-*indefinite*.[8] This is not desirable for optimization and we will introduce in Chapter 2 a method that is useful in certain cases to transform the problem (1.11) into an equivalent form with positive semi-definite cost functional.

Below we have summarized some of the terminology in management with respect to critical resource allocation, together with their respective counterpart in control theory.

[8]In other words, there may be no finite constant that when added to h would *a priori* prevent it from changing sign.

Firm	—	System
Planning Horizon	—	Prediction Horizon
Implementation Horizon	—	Control Horizon
Development	—	System Evolution, Trajectory
Item	—	State Variable
Resource, Decision Variable	—	Input Variable, Control
Objective/Value	—	Objective Functional = −(Cost Functional)
Resource Allocation	—	Control
Best Strategy	—	Optimal Control
Resource Constraints	—	Control/Input Constraints
Status/State	—	State
Status Report	—	Observation
Requirements	—	State Constraints
External Factors	—	Boundary Conditions/Unmodeled Dynamics
Uncertainty	—	Noise, Disturbance
Decision Maker	—	Controller
Decision Making under Uncertainty	—	Robust Control

Table 1.1: Management Terminology and Correspondences in Control Theory.

Clearly these terms are not equivalent, but are here used in a similar manner for both contexts. They will be used in parallel for developments in this Master's thesis.

1.3.2 A Moving Horizon Approach

Based on the remarks leading to the formulation of the moving horizon optimal control problem with discounted cost (1.11), we will now formulate more specific questions that will be investigated in this thesis:

Question 3 *Can a procedure be defined and conditions given that allow an application of the moving horizon control methodology to approximate an optimal infinite horizon resource allocation? In particular, can statements be made about the robustness of this procedure with respect to the choice of the planning horizon?*

A quite general answer to the first part of this question, i.e., conditions under which

$$\hat{\pi} = \{\hat{\mu}_k\}_{k=0}^{\infty} \longrightarrow \pi = \{\mu_k\}_{k=0}^{\infty}, \quad \text{as } M \to \infty,$$

where π is the optimal policy for the corresponding infinite horizon optimal control problem

$$\mathcal{P}(i,a): \qquad J_i = \sum_{k=i}^{\infty} e^{-rk} h(x_k, u_k) \longrightarrow \min, \tag{1.12}$$

has been provided by Keerthi and Gilbert [KG88]. Their result assumes however a positive semi-definite cost functional and the existence of an optimal equilibrium state x^e, i.e., a state that once it has been reached cannot be improved upon. In addition, no constructive statement about the necessary length of the planning horizon M to come within a specified 'neighborhood' of the optimal infinite horizon cost $V(i, a)$ has been made. We will examine part of Question 3 for an example of practical relevance in Chapter 2.

Regarding the underlying optimal control problem of finding the best resource allocation strategy for a given dynamical system, the following question is of considerable practical relevance:

Question 4 *Are there conditions under which there exists an optimal state for a corporate critical resource allocation problem? Can this state be reached under a stabilizing optimal corporate policy?*

A formal development of constrained moving horizon predictive control is given in the next chapter. Basic concepts of corporate strategy in relation to nonsingular corporate policy will be introduced later in Chapter 3, when the relation between control theory and managerial practice will be established. Modeling approaches and a practical application will be discussed at that point.

1.4 Notes and Sources

Resource Allocation and Optimization. For an overview of the traditional approaches to resource allocation and capital budgeting, see [EL88] and [Bie88], [BM91] respectively. Dynamic optimization in management has been considered by [KS91] and [CH87]. For a introductory treatment of decision trees and 'dynamic strategic planning' based on decision analysis, see [dN90]. A more advanced book on dynamic programming and optimal control is [Ber95], which provides many practical examples.

MPC. The theoretical discussion of MPC (although the principle was known and used in practice before) has been initiated largely by Clarke's Generalized Predictive Control [CMT87], which was later extended by Soeterboek as well as De Vries and Verbruggen providing a more general (quadratic) criterion function [Soe92] for the multivariable case [VV93]. For an overview, see [Mor93]. Although these approaches had nice practical applications with a relatively easy input/output constraint handling, stability could not be guaranteed. Rossiter and Kouvaritakis introduced first ideas for stable model predictive controllers under constraints in the SISO and MIMO[9] case [RK93b], [RK93a]. Parallel to these results in an input/output setting, formulations in state-space have been proposed by Morari and Zheng [Mor93], [ZM94], and a constrained multivariable stable MPC implementation in the state space was given by Heise and Maciejowski [HM94]. A unified treatment of the different stabilizing approaches to MPC can be found in [Web95]. Robustness results for constrained MPC, in particular with respect to plant model uncertainty, have been provided recently using linear matrix inequalities [KBM96].

Constrained Predictive Control. Moving horizon approximations to constrained infinite horizon optimal control problems in a general nonlinear setting have been investigated in discrete time by Keerthi and Gilbert [KG86b], [KG88]. Based on their results,

[9] SISO: Single Input — Single Output; MIMO: Multiple Input — Multiple Output.

continuous-time nonlinear moving horizon controllers with quadratic cost have been introduced by Mayne and Michalska [MM90], who also provided some robustness results [MM93]. The use of moving horizon control as a potential tool for managers is new. To the author's knowledge, no moving horizon predictive controllers for discounted indefinite cost functionals have been considered so far in the literature.

I have had my results for a long time, but I do not yet know how I am to arrive at them.
— CARL FRIEDRICH GAUSS

Chapter 2

Constrained Predictive Control

Predictive control makes explicit use of a model to forecast the evolution of a dynamical system, and determine, based on these forecasts and the minimization of a cost functional, inputs to the system. This control methodology is *predictive* in the sense that in the presence of uncertainty, the forecasts are based on an estimator (or predictor) that attempts to minimize the error of the forecasts in some given metric. Here though, we will focus mainly on using a methodology analogous to predictive control for the *nominal* case, i.e., in the absence of explicit uncertainty. In this case, the above predictions degenerate to a simulation, since the evolution of the system is perfectly described by its model when disturbances are not present. We will however still term this approach predictive, as it establishes baseline results for more sophisticated developments that take into account structured or unstructured uncertainty.[1] Nevertheless, even though uncertainty is excluded from the main discussions, some statements about the robustness towards disturbances and modeling errors may be derived, e.g., by using Lyapunov stability theory.

This chapter introduces the theory on constrained predictive controllers relevant for its potential use as a tool for resource allocation decisions in nonsingular corporate policy. *First*, some fundamental definitions and theorems on dynamical systems and their stability properties will be given for both the continuous-time and the discrete-time case. In fact, since corporate policy can generally be considered as not very time-critical from a computational viewpoint — data updates occur typically weekly or daily — a discrete-time analysis seems fully sufficient. However, some developments and analyses can be conducted elegantly for continuous-time systems, where the underlying concepts often appear more natural, due to smoothness assumptions on the functions involved. Thus, both approaches will be presented in a complementary manner and used each where convenient. *Second*, we will introduce the basic constrained infinite horizon (IH) and moving horizon (MH) optimal control problems (OCPs), that describe the objective maximization problem and its real-world formulation. *Third*, conditions for MH approximations of the solution to the IH OCP are given, which is fundamental for the usefulness of the predictive control methodology. — The chapter concludes with two lemmas by the author, that have been developed for dealing with managerial resource allocation problems, characterized by a discounted indefinite objective functional. In addition we will provide some robustness considerations of the quality of the approximation with respect to the length of the time-horizon. An application

[1] In the well-known case of a linear system and quadratic cost functional (cf. MPC in Section 1.3), such baseline discussions are in many instances sufficient, for a *Certainty Equivalence Principle* holds, cf. [ÅW90].

of the predictive control methodology to marketing expenditure and price selection in a diffusive market environment, will be given at that point.

2.1 Basic Concepts

In this section we will review important concepts in control theory for both the continuous-time and discrete-time case that will be used later to summarize results on predictive controllers.

Definition 1 (System) *A system Σ is a relation between signals s_i, $1 \leq i \leq r$, of the form $\Sigma = \{(s_1, s_2, \ldots, s_r)\}$. A signal s is thereby a real-valued mathematical function $s : \mathcal{T} \to \mathbb{R}$, $t \mapsto s(t)$, defined over a time-set $\mathcal{T} \in \{\mathbb{Z}, \mathbb{R}\}$. The system Σ specifies the set of admissible signal r-tuple.*

Typically, the first m components of an element of a system Σ are called *input*, while the last $p = r - m$ components denote its *output*.[2] In the remainder of this work, we will only consider systems with inputs $(u_1, \ldots, u_m)' =: u$ and outputs $(y_1, \ldots, y_p)' =: y$, that can be represented in one of the following *state space* forms:

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{2.1}$$
$$y(t) = g(t, x(t), u(t)), \tag{2.2}$$

if $\mathcal{T} = \mathbb{R}$, corresponding to an ordinary differential equation (ODE) description, appropriate for *continuous-time* systems; or,

$$x_{k+1} = f_k(x_k, u_k), \tag{2.3}$$
$$y_k = g_k(x_k, u_k), \tag{2.4}$$

if $\mathcal{T} = \mathbb{Z}$, corresponding to a finite difference equation (FDE) representation, appropriate for *discrete-time* systems.

The functions $f : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $g : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ are called *system function* and *output function* respectively. $\mathcal{X} \subseteq \mathbb{R}^n$ denotes the *state space*, $\mathcal{Y} \subseteq \mathbb{R}^p$ the *output space*, and $\mathcal{U} \subseteq \mathbb{R}^m$ the *control space*.

REMARK In this Master's thesis it is sufficient to consider systems where $g \equiv x$, i.e., where the output is equal to the state variables, since we are here only concerned with models where the states are accessible and relevant to the decision maker. This assumption simplifies the necessary analysis, seems however appropriate for many management applications, where it is satisfied naturally by the problem formulation.

[2] In principle, a system need have neither inputs nor outputs.

2.1.1 Continuous-Time

Given a measurable control input $u : [t_0, t_1] \to \mathcal{U}$, a function $x : [t_0, t_1] \to \mathbb{R}^n$ is called a *solution* to the ODE (2.1) with initial condition $x(t_0) = x_0$, if $f(\cdot, x(\cdot), u(\cdot))$ is locally integrable (i.e., measurable and essentially bounded) on the system trajectory $\{(x(t), u(t)) \mid t \in [t_0, t_1]\}$, and

$$x(t) = x_0 + \int_{t_0}^{t} f(\theta, x(\theta), u(\theta))\, d\theta, \tag{2.5}$$

$\forall\, t \in [t_0, t_1]$. A *maximal* solution of (2.1) is a solution that cannot be extended beyond a time $t_\infty \in \mathcal{T} \cup \{\infty\}$, the so-called *escape time*. For convenience, when referring to a system's evolution for all 'future' times $t \geq t_0$ for some given $t_0 < t_\infty$, a maximal solution of the system equation (2.1) on $[t_0, t_\infty)$ is meant. Sufficient conditions for existence and uniqueness of such a solution on a finite time interval are provided by the following theorem.

Theorem 1 (Global Existence and Uniqueness for Continuous-Time Systems) *Suppose for a given input $u : [t_0, t_1] \to \mathcal{U}$, that $\bar{f}(t, x) := f(t, x, u(t))$ is piecewise continuous in t and satisfies the following global Lipschitz condition,*

$$\begin{aligned} \|\bar{f}(t, x) - \bar{f}(t, y)\| &\leq L\|x - y\|, \\ \|\bar{f}(t, x_0)\| &\leq M, \end{aligned}$$

$\forall\, x, y \in \mathcal{X}$, $\forall\, t \in [t_0, t_1]$, and some constants $L, M \geq 0$. Then, the state equation (2.1) with the initial condition $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$ for the given input u.

Proof. See [Kha92], p. 81.

REMARK

1. For any admissible[3] control $u(t)$, the function $\bar{f}(t, x) := f(t, x, u(t))$ in Theorem 1 describes a new system with no control inputs,

$$\dot{x} = \bar{f}(t, x). \tag{2.6}$$

   Such systems without explicit dependence on the input u are called *autonomous*. In the following we consider mainly such systems without explicit dependence on u, since they arise naturally, once the input has been chosen or in the case of state feedback, when $u = \mu(t, x)$.

2. In the continuous-time case it will be always assumed that the system function satisfies the conditions in Theorem 1, which is sufficient for the developments in this Master's thesis. Hence, all continuous-time state space representations that will be introduced later on, are guaranteed to possess a unique solution over some nontrivial time interval.

To characterize important properties of the systems that will be examined later, several fundamental definitions are now provided. These definitions will be given for the continuous-time case. Formulations of analogous discrete-time concepts can be obtained in most cases by simply replacing derivatives by finite differences, and adjusting the notation accordingly (cf. Section 2.1.2).

[3] In general we mean by an *admissible* control one that satisfies additional constraints such as $u \in \Omega(x)$, cf. Section 2.1.2.

Definition 2 (Dynamic Equilibrium) *A dynamical system of the form (2.1) is said to be in (dynamic) equilibrium at x^e from time t_0 on, if for some input $u^e(t)$:*

$$0 = f(t, x^e, u^e(t)), \tag{2.7}$$

$\forall t \geq t_0$. *For a given equilibrium point x^e at time t_0, we will refer to an input $u(t)$ satisfying (2.7) as corresponding equilibrium control $u^e(t)$.*

Let $E(t)$ denote the set of all equilibrium points at time t. It is clear that $E(t) \subset E(t+\tau)$, for all $\tau \geq 0$. The usual time-invariant definition for equilibrium points is a special case of Definition 2 for systems of the form

$$\dot{x} = f(x, u). \tag{2.8}$$

Tuples (x^e, u^e) satisfying a condition analogous to (2.9),

$$0 = f(x^e, u^e), \tag{2.9}$$

$\forall t \geq t_0$, composed of an equilibrium point $x^e \in E$ and a corresponding (constant) equilibrium control u^e, are called state-control equilibrium tuple. Let us now consider stability properties of a feedback system, i.e., a system where the input u is a function of the state x: $u = \mu(x)$.[4]

Definition 3 (Feedback Stability) *An equilibrium state x^e of the time-invariant system (2.8) with $u = \mu(x)$ is*

(i) stable, if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|\bar{x} - x^e\| < \delta \Rightarrow \|T_t^f(\bar{x}) - x^e\| < \epsilon,$$

$\forall t \geq 0$, where $T_t^f(\bar{x})$ is the so-called differential flow,[5] i.e., the solution of the initial value problem

$$\dot{z} = f(z, \mu(z)), \qquad z(0) = \bar{x},$$

at time t.

(ii) asymptotically stable, if it is stable and there is $\delta > 0$ such that

$$\|\bar{x} - x^e\| < \delta \Rightarrow \lim_{t\to\infty} T_t^f(\bar{x}) = x^e.$$

(iii) globally asymptotically stable, if $\forall \bar{x} \in \mathcal{X}$:

$$\lim_{t\to\infty} T_t^f(\bar{x}) = x^e.$$

(iv) unstable, if it is not stable.

---

[4] Stability properties for *static* equilibria can be obtained by substituting a constant feedback law, i.e., $\mu(x) \equiv u^e = const.$

[5] The notion of flow for dynamical systems depends essentially on a semi-order property of the time-set $\mathcal{T}$ and is available in discrete time as well (cf. Section 2.1.2). Such a flow possesses a characteristic semi-group property, since $\left(T_{t_1}^f \circ T_{t_2}^f\right)(\bar{x}) = T_{t_1+t_2}^f(\bar{x})$, $\forall \bar{x} \in \mathcal{X}$, $\forall t_1, t_2 \geq 0$.

REMARK For time-varying autonomous systems (cf. remark after Theorem 1), a point x^e will be called *uniformly stable*, *uniformly asymptotically stable*, etc., if the above conditions hold for $x = x(t_0)$ without dependence on the initial time t_0. Exact definitions of uniform stability for time-varying discrete-time systems will be given in the next section (cf. Definition 9).

To analyze equilibrium points with respect to the above stability properties, it is useful to consider functions describing essential aspects of the system behavior. Some useful theorems based on functions of the Lyapunov type are given below. These statements play a crucial role when establishing stability results for moving horizon predictive controllers in Section 2.3.

Definition 4 (Lyapunov Function) *Consider the autonomous system*

$$\dot{x} = f(t, x) \tag{2.10}$$

on an open set $X \subset \mathcal{X}$. A function $V : (t_0, t_1) \times X \to \mathbb{R}$ is called Lyapunov function, if $V(t, x(t))$ is monotonically decreasing for $t \in (a, b) \subset (t_0, t_1)$ whenever $x(\cdot)$ is a solution of (2.10) on (a, b) such that the trajectory $\{x(t) | \, t \in (a, b)\} \subset X$.

REMARK

1. Lyapunov functions as defined above are particularly useful on an *invariant* subset of the state space, i.e., a set $\tilde{X} \subset \mathcal{X}$ such that $T_t^f(x) \in \tilde{X}$, $\forall x \in \tilde{X}$, $\forall t \geq 0$. In this case the corresponding Lyapunov function (if it exists) will be negative semi-definite for an entire state trajectory $\{x(t) | \, t \geq t_0\}$ starting at $x(t_0) \in \tilde{X}$.

2. For continuous system functions f and continuously differentiable functions V, proving that V is a Lyapunov function on an invariant set $\tilde{X}$ can be accomplished by showing that
$$\dot{V}(t, x(t)) = \nabla_t V + \nabla_x V \cdot f(t, x(t)) \leq 0, \tag{2.11}$$
$\forall t \geq t_0$, whenever $x(t_0) \in \tilde{X}$ and $x(t) = T_{t-t_0}^f(x(t_0))$.

In order to characterize stability properties, e.g., through Lyapunov stability criteria given below, it is convenient to consider a special class of scalar functions, the so-called class $\mathcal{K}$ functions.

Definition 5 (Class $\mathcal{K}$) *A continuous function $\varphi : [0, R) \to \mathbb{R}_+$, $\rho \mapsto \varphi(\rho)$, is said to belong to class $\mathcal{K}$, if it is strictly monotonically increasing on $[0, R)$ and $\varphi(0) = 0$. In the case that $R = \infty$ and $\varphi(\rho) \to \infty$ as $\rho \to \infty$, it is said to belong to class $\mathcal{K}_\infty$.*

Theorem 2 (Lyapunov Stability Criteria) *Let $x^e \in \mathcal{X}$ be an equilibrium point of the respective dynamical systems considered below and a ball $B_\rho(x^e) := \{x \in \mathcal{X} | \, \|x - x^e\| < \rho\}$ around x^e with some radius $\rho > 0$ be given.*

(i) Assume $V : B_\rho(x^e) \to \mathbb{R}$ is a continuously differentiable function, and f in the time-invariant autonomous system equation

$$\dot{x} = f(x), \tag{2.12}$$

a continuous function such that

$$\nabla_x V \cdot f(x) \leq 0, \tag{2.13}$$

$\forall\, x \in B_\rho(x^e)$. Then, if x^e is an isolated local minimum of V, it is stable. Moreover, if the inequality (2.13) is strict $\forall\, x \in B_\rho \setminus \{x^e\}$, then x^e is an asymptotically stable equilibrium point.

(ii) If $V : [t_0, \infty) \times B_\rho(x^e) \to \mathbb{R}$ is a Lyapunov function for the autonomous system (2.10) such that

$$\varphi_1(\|x - x^e\|) \leq V(t,x) \leq \varphi_2(\|x - x^e\|), \tag{2.14}$$

$$\dot{V}(t, x(t)) = \nabla_t V + \nabla_x V \cdot f(t, x(t)) \leq -\varphi_3(\|x - x^e\|), \tag{2.15}$$

$\forall\, t \geq 0$, $\forall\, x \in B_\rho$, where $\varphi_i(\cdot)$ are class $\mathcal{K}$ functions defined on $[0, \rho)$, then the point x^e is uniformly asymptotically stable.

(iii) If the conditions in (ii) hold for $\rho = \infty$, i.e., $B_\rho = \mathcal{X}$ and φ_1, φ_2 are class $\mathcal{K}_\infty$, then the point x^e is globally asymptotically stable.

Proof. (i): [Per96], pp. 131–132; (ii),(iii): [Kha92], pp. 169–172.

Other fundamental control theoretic concepts besides stability, needed here, are the notions of *controllability* and *observability*. A system is controllable if it can be steered from any state to any other state in finite time. A system is observable, if when its output is recorded over some finite time interval, its initial state can be determined. We will now give more precise definitions of these notions.

Definition 6 (Controllability) *The system (2.1) is (pointwise) controllable, if for any initial condition $x(t_0) = x_0 \in \mathcal{X}$ there is an admissible control $u(t)$ that steers the system to a given state $x_1 \in \mathcal{X}$ in finite time, i.e., $\forall\, x_0, x_1 \in \mathcal{X}$ there exist $0 < \tau < \infty$ and $u_0^1 : [t_0, t_0 + \tau] \to \mathcal{U}$ such that*

$$T_\tau^{\bar{f}}(x_0) = x_1,$$

where $\bar{f}(t,x) := f(t, x, u_0^1(t))$.

Remark A system is said to be *reachable*, if all its states can be reached from a particular state (say, the origin) in finite time, i.e., if it is controllable from the origin.[6]

[6] A controllable system is reachable from *any* state.

Definition 7 (Observability) *The system (2.1) is (pointwise) observable, if knowledge of its input $u(t)$ and output $y(t)$ over some time interval $(t_0, t_0+\tau)$, of finite length $0 < \tau < \infty$ suffices to determine its initial state $x(t_0)$.*

REMARK In the developments that follow we will need particulary the controllability and reachability property, as we will be interested in systems that can be steered from any state to a particular 'optimal' state. — The observability property will play a lesser role here, since we restrict ourselves to managerial systems where the states are in most cases readily available and can therefore be considered as output (cf. remark after Definition 1). In this case of *state feedback*, the observability property is trivially satisfied.

2.1.2 Discrete-Time

The discrete-time notions introduced here complement the above treatment of continuous-time systems, in the sense that state and control constraints are now incorporated explicitly. In addition, the stability definitions and corresponding theorems will be formulated for time-varying systems. The reader will appreciate the analogies between discrete-time and continuous-time approach and either will be used when appropriate for the analysis of examples in later sections.

Consider the discrete-time system (2.3) for $k \geq i \in \mathcal{T} = \mathbb{Z}$:

$$x_{k+1} = f_k(x_k, u_k), \tag{2.16}$$

with initial condition

$$x_i = a, \tag{2.17}$$

such that

$$(i, a) \in \Xi := \{(i, a) | \, i \in \mathcal{T}, a \in \mathcal{X}_i\}. \tag{2.18}$$

As in continuous time, the collection of maps $f_k : \mathcal{X}_k \times \mathcal{U}_k \to \mathcal{X}_{k+1}$ is called *system function*, whereby $\mathcal{X}_k$ and $\mathcal{U}_k$ are subsets of the *state space* $\mathcal{X} \subseteq \mathbb{R}^n$ and the *control space* $\mathcal{U} \subseteq \mathbb{R}^m$ respectively. Ξ denotes the set of all admissible initial conditions. Note also that the evolution of system (2.16) is constrained to states x_k in the *state constraint sets* $\mathcal{X}_k$ and control inputs u_k in the *control constraint sets* $\Omega_k(x_k) \subset \mathcal{U}_k$ such that in a more convenient notation

$$(x_k, u_k) \in \mathcal{Z}_k(x_k) := \mathcal{X}_k \times \Omega_k(x_k), \tag{2.19}$$

$\forall \, k \geq i$.

Given an initial time $i \in \mathcal{T}$ let a *policy* (or control law) for system (2.16) consist of a sequence of functions

$$\pi^i := \{\mu_k^i\}_{k \geq i} \tag{2.20}$$

where $\mu_k^i : \mathcal{X}_k \to \mathcal{U}_k, \; \mu_k^i(x_k) \mapsto u_k$. Such a policy is called *admissible*, if

$$\mu_k^i(x_k) \in \Omega(x_k), \tag{2.21}$$

$\forall x_k \in \mathcal{X}_k$, $\forall k \geq i$. Applying an admissible policy π^i to the system (2.16), one obtains an autonomous feedback system,

$$x_{k+1} = \bar{f}_k(x_k), \tag{2.22}$$

with $\bar{f}_k(x_k) := f_k(x_k, \mu_k^i(x_k))$, completely analogous to (2.6) in continuous time. This deterministic state feedback system is illustrated in Figure 2-1, where for convenience $\bar{f}$ has been written again f.

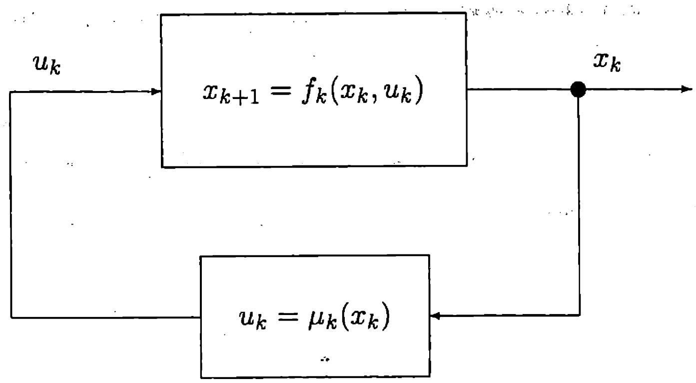

Figure 2-1: State Feedback.

Definition 8 (Equilibrium) *A state $x^e \in \mathcal{X}$ is called an equilibrium of the autonomous system*

$$x_{k+1} = f_k(x_k), \tag{2.23}$$

if there exists $i \in \mathcal{T}$ such that $(i, x^e) \in \Xi$ and

$$T_k^f(i, x^e) = x^e, \tag{2.24}$$

$\forall k \geq 0$.

Remark This definition of an equilibrium for autonomous time-varying discrete-time systems corresponds essentially to Definition 2 in continuous time, of which it is a mere reformulation.

In the next section we will see that in general it is the goal to find policies that are *optimal* in the sense that they minimize a particular cost functional. In effect, the policy $\mu_k^i(x_k)$ describes the input for the system for each possible state state at time k. The system with such an implemented state feedback policy becomes autonomous. The following theorem states some sufficient conditions for equilibria of such autonomous feedback systems.

Definition 9 (Uniform Stability for Time-Varying Systems) *An equilibrium state x^e of the autonomous system (2.23) is*

(i) uniformly stable (US), if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|a - x^e\| < \delta \Rightarrow \|T_k^f(i,a) - x^e\| < \epsilon,$$

$\forall (i,a) \in \Xi,\ \forall k \geq 0.$

(ii) uniformly asymptotically stable (UAS), if it is US and there is $\delta > 0$ such that

$$\|a - x^e\| < \delta \Rightarrow \lim_{k\to\infty} T_k^f(i,a) = x^e,$$

$\forall (i,a) \in \Xi.$

(iii) globally uniformly asymptotically stable (GUAS), if it is US and

$$\lim_{k\to\infty} T_k^f(i,a) = x^e,$$

$\forall (i,a) \in \Xi.$

(iv) exponentially stable (ES), if there exist constants $\sigma, C > 0$ such that

$$\|T_k^f(a) - x^e\| < Ce^{-\sigma k}$$

$\forall (i,a) \in \Xi,\ \forall k \geq i.$

The definition for Lyapunov functions $V(k, x_k)$ in discrete time is completely analogous to the continuous-time version (cf. [Wil70], p. 172), when replacing t by k and the monotonicity condition (2.11) by

$$\Delta V(k, x_k) = V(k+1, x_{k+1}) - V(k, x_k) \leq 0. \tag{2.25}$$

Sufficient conditions for the stability of an equilibrium point x^e are given by the following theorem (as formulated in [KG88]).

Theorem 3 (Asymptotic Stability) *Let x^e be an equilibrium point of the autonomous system (2.23) and $B_\rho(x^e) := \{x \in \mathcal{X} \mid \|x - x^e\| < \rho\}$ be a ball centered at x^e with some radius $\rho > 0$. Suppose that there are functions $V : \Xi \to \mathbb{R}$ and $\varphi_1, \varphi_2, \varphi_2 \in \mathcal{K}_\infty$, a constant $\delta > 0$ and a positive integer $L > 0$ such that the following conditions are satisfied:*

$$\varphi_1(\|a - x^e\|) \leq V(i,a) \leq \varphi_2(\|a - x^e\|), \tag{2.26}$$

$$\varphi_3(\|a - x^e\|) \leq V(i,a) - V(i+L, T_L^f(i,a)), \tag{2.27}$$

$$0 \leq V(i,a) - V(i+1, T_1^f(i,a)), \tag{2.28}$$

$\forall (i,a) \in \Xi$ *for which $a \in B_\rho(x^e)$. Then*

(i) the equilibrium point x^e is UAS.

(ii) if $\rho = \infty$, x^e is GUAS.

Proof. (Outline) The proof for this theorem has not been provided in [KG88]. However, it follows immediately from a discrete-time version of the Lyapunov stability criteria given in Theorem 2, after noting that conditions (2.27)–(2.28) imply

$$\begin{aligned} 0 &\leq V(i+1, T_1^f(i,a)) - V(i+2, T_2^f(i,a)), \\ 0 &\leq V(i+2, T_2^f(i,a)) - V(i+3, T_3^f(i,a)), \\ &\vdots \\ 0 &\leq V(i+L-1, T_{L-1}^f(i,a)) - V(i+L, T_L^f(i,a)). \end{aligned}$$

Subtraction of these inequalities from (2.27) yields,

$$V(t+1, T_1^f(i,a)) - V(i,a) \leq -\varphi_3(\|a - x^e\|),$$

the direct discrete-time analogon to condition (2.15) in Theorem 2, so that the result follows immediately from there by discretization. □

The notions of controllability and observability carry over directly (by change of notation only) to discrete-time systems. To prove the stability of feedback systems obtained by the solution of OCPs that will be introduced in the next section, Keerthi and Gilbert [KG88] introduced properties **C** and **O** that generalize the corresponding strong requirements of uniform complete controllability and observability for linear systems that are well-known [Kai80] and will not be discussed here. Property **C** allows to determine an upper bound on the size of the state-control tuple when steering the system to a given (x^e, u^e), while property **O** gives a lower bound on the size of the output-control tuple as a function the size of the initial state deviation from a given state x^e.

Definition 10 (Property C) *The system (2.16) has property* **C***, if there exists $N_c > 0$, and a $\mathcal{K}_\infty$ function φ_c such that: $\forall\,(i,a) \in \Xi$ there exists a sequence $\{(x_k, u_k)\}_{k \geq i}$ such that*

$$\sum_{k=i}^{i+N_c-1} \|(x_k, u_k) - (x^e, u^e)\| \leq \varphi_c(\|a - x^e\|), \tag{2.29}$$

and $(x_k, u_k) = (x^e, u^e)\ \forall\, k \geq i + N_c$.

Definition 11 (Property O) *The system (2.3)-(2.4) has property* **O***, if there is $N_o > 0$, and a $\mathcal{K}_\infty$ function φ_o such that: $\forall\,(i,a) \in \Xi$ there exists a sequence $\{(x_k, u_k, y_k)\}_{k \geq i}$ such that*

$$\sum_{k=i}^{i+N_o-1} \|(y_k, u_k) - (g_k(x^e, u^e), u^e)\| \geq \varphi_o(\|a - x^e\|). \tag{2.30}$$

REMARK

1. The above properties **C** and **O** are defined with respect to a (constant) equilibrium state-control tuple, whose existence is assumed.
2. Property **O** is trivially satisfied for systems with state feedback, i.e., when $g_k(x_k, u_k) = x_k$, for all k.

2.2 Finite and Infinite Horizon Optimal Control

The input u to the general system (2.1) can often be selected so as to optimize the behavior of the system over time $t \in [0, T]$ with respect to a certain cost functional, that serves as criterion of the relative preference between different control inputs for a given initial state of the system. Goal is then to find a policy $\pi^* = \{\mu^*(t, \cdot)\}_{t\in[0,T]}$ and thus a feedback law $u^*(t) = \mu^*(t, x)$ that minimizes the cost functional. Below we summarize two fundamental results for this class of problems and give a brief overview of numerical methods for efficiently solving such OCPs. This will lay ground for Section 2.3, where we will discuss moving horizon approximations of infinite horizon OCPs, and present several potentially useful results for management applications.

2.2.1 Fundamental Theorems

Let us consider here the basic problem of finding an optimal policy $u^*(t) = \mu^*(t, x)$ that minimizes a cost functional

$$J(u) := l(x(T)) + \int_0^T h(\theta, x(\theta), u(\theta))\, d\theta, \tag{2.31}$$

where $h : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is called the *instantaneous cost* (or cost kernel) and $l : \mathcal{X} \to \mathbb{R}$ the *terminal cost*. For continuous-time systems the finite horizon OCP can be written as follows:

$$l(x(T)) + \int_0^T h(\theta, x(\theta), u(\theta))\, d\theta \longrightarrow \min \tag{2.32}$$

subject to

$$\begin{aligned} \dot{x} &= f(t, x, u), && (2.33)\\ u &\in \Omega(x), && (2.34)\\ x(0) &: \text{given}, \quad x(T) : \text{possibly fixed}. && (2.35) \end{aligned}$$

The corresponding infinite horizon formulation is obtained for $l = 0$ and $T \to \infty$ in (2.32), and will be discussed below under additional assumptions. To simplify the analytical treatment and the presentation of the results, we will impose quite strong regularity assumptions, and in most cases the reader may expect to find in the literature theorems with similar statements under weaker hypotheses. In particular we require here that the cost kernel h be continuous, and the terminal cost function l as well as the system function f be continuously differentiable. Furthermore, the constraint set $\Omega(x)$ is assumed to be *biconvex*,[7] that is $\Omega(x)$ is convex for any x, and $\forall\, x_1, x_2 \in \mathcal{X}$:

$$u_1 \in \Omega(x_1), u_2 \in \Omega(x_2) \Rightarrow \vartheta u_1 + (1 - \vartheta)u_2 \in \Omega(\vartheta x_1 + (1 - \vartheta)x_2) \quad \forall\, \vartheta \in (0, 1).$$

We furthermore assume that the OCP (2.32) subject to (2.33)–(2.35) possesses a solution, in the sense that there exists an admissible state-control trajectory $\{(x^*, u^*) \,|\, t \in [0, T]\}$ that satisfies the constraints and minimizes the above cost. A sufficient condition for such a trajectory to be optimal is given by the following theorem.

[7] An example for such a biconvex set is $\Omega(x) = \{u \in \mathcal{U} \,|\, \omega(x, u) \geq 0\}$, where $\omega : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is a concave function in x and u.

Theorem 4 (Hamilton-Jacoby-Bellman (HJB) Equation) *Assume that the function $V(t,x)$ is continuously differentiable in t and x and is a solution to the HJB equation*[8]

$$-\nabla_t V(t,x) = \min_{u\in\Omega(x)} \{h(t,x,u) + \nabla_x V(t,x)\cdot f(t,x,u)\}, \tag{2.36}$$

$$V(T,x) = l(x), \tag{2.37}$$

$\forall x \in \mathcal{X},\ \forall t \in [0,T]$. *Suppose also that $u^* = \mu^*(t,x)$ attains the minimum in (2.36) $\forall t \in [0,T]$, $\forall x \in \mathcal{X}$. Then under the assumptions of Theorem 1, $V(t,x)$ is the unique solution of the HJB equation and*

$$V(t,x) = \int_t^T h(\theta, x^*(\theta), \mu^*(\theta))\, d\theta \tag{2.38}$$

is the optimal cost-to-go, where $x^*(t) = T_t^f(x)$ when the optimal control u^* is applied.

Proof. See [Ber95], pp. 93–94.

REMARK

1. In most interesting cases analytical solutions to the partial differential equation (2.36) cannot be obtained. Even numerically, the HJB equation is not easy to solve, so that its use is often reduced to a mere verification tool.
2. Continuous differentiability of the value function V is a strong requirement that for many applications is not satisfied. Nevertheless, the HJB equation remains of use, if more general notions of derivatives (such as subdifferentials) are used, which is subject of so-called *Nonsmooth Analysis* initiated by Frank H. Clarke [Cla83], [Cla89], which is however beyond the scope of this Master's thesis.

Necessary conditions that have to be satisfied by an optimal state-control trajectory hold under considerably weaker conditions, and are generally subsumed under Pontryagin's Maximum Principle (PMP). For this introduce the Hamiltonian function $H : \mathcal{T}\times\mathcal{X}\times\mathcal{U}\times\mathbb{R}^n \to \mathbb{R}$,

$$H(t,x,u,p) = h(t,x,u) + p'\cdot f(t,x,u). \tag{2.39}$$

Theorem 5 (Pontryagin Maximum Principle) *Let $\{(x^*(t), u^*(t)) \mid t \in [0,T]\}$ be an optimal state-control trajectory for the OCP (2.32), with $x^*(0) = x(0)$, given. In addition, let $p(t)$ be the solution of the adjoint equation*

$$\dot{p}(t) = -\nabla_x H(t, x^*(t), u^*(t), p(t)), \qquad p(T) = 0, \tag{2.40}$$

[8] For convenience and to minimize the reader's distraction, we will write here and in what follows "min" instead of the more precise "inf" in the minimization of cost functionals. At times the actual minimum may not exist and we will mean by "min" the infimum in these cases.

with the transversality condition

$$p(T) = \nabla l(x^*(T)). \tag{2.41}$$

Then,

$$u^*(t) = \arg \min_{u \in \Omega(x)} H(t, x^*(t), u, p(t)), \tag{2.42}$$

$\forall t \in [0, t]$.

Proof. See [PBGM62], pp. 99–108.

REMARK In practice, using the necessary condition (2.42), one can express u^* as a function of x^* and p and obtain a $2n$-dimensional *two-point boundary problem*, with split boundary conditions

$$x^*(0) = x(0), \quad p(T) = \nabla l(x^*(T)).$$

Such problems are typically very hard to solve analytically. Numerical 'shooting' algorithms do exist for obtaining candidates for solutions. However, it is generally not easily possible to find *globally* optimal solutions (cf. also Section 2.2.2). — If $x(T) = (x_1(T), \dots, x_n(T))'$ is fixed only for some components $i \in I \subseteq \{1, \dots, n\}$, then only the adjoint boundary conditions

$$p_j(T) = \frac{\partial l(x^*(T))}{\partial x_j}$$

for $j \in \{1, \dots, n\} \setminus I$ hold.

Infinite Horizon OCPs. The infinite horizon version of problem (2.32) is as expected:

$$\int_0^\infty h(\theta, x(\theta), u(\theta))\, d\theta \longrightarrow \min, \tag{2.43}$$

subject to

$$\dot{x} = f(t, x, u), \tag{2.44}$$

$$u \in \Omega(x), \tag{2.45}$$

$$x(0) \ : \ \text{given}. \tag{2.46}$$

A rigorous treatment of optimality conditions for the above problem is quite involved, and poses a number of technical difficulties. A sufficient condition such as the HJB equation in the finite horizon case is in many instances not available. In fact, due to the integration over an infinite time-interval, the cost may become unbounded and/or stability issues have to be discussed.[9]

A formulation of the PMP for the infinite horizon OCP can be found in [CH87], together with a discussion of weaker forms of optimality, appropriate for diverging cost. We will here only motivate the Bellman inequality that results from the following observations: Assume

[9] For reasonable problem formulations the *optimal* IH cost should stay bounded.

that for any admissible state x in the above problem, an optimal cost-to-go,

$$V(t,x) = \int_t^\infty h(\theta, x^*(\theta), u^*(\theta))\, d\theta = \min_{u\in\Omega(x^u)} \int_0^\infty h(\theta, x^u(\theta), u(\theta))\, d\theta$$

does exist. Then, by varying the time slightly from t to $t+\delta$ with $\delta > 0$, we observe that

$$V(t, x(t)) - V(t+\delta, x(t+\delta)) \leq \int_t^{t+\delta} h(\theta, x^u(\theta), u(\theta))\, d\theta,$$

where x^u is the state trajectory induced by the control u and as $\delta \to 0^+$ one obtains

$$-\left(\nabla_t V(t,x) + \nabla_x V(t,x) \cdot f(t,x,u)\right) \leq h(t,x,u),$$

which comparing it with (2.36) corresponds to an 'HJB inequality':

$$0 \leq h(t,x,u) + \nabla_t V(t,x) + \nabla_x V(t,x) \cdot f(t,x,u). \tag{2.47}$$

Taking the minimum with respect to all admissible controls, together with some additional analytical considerations, would result in the HJB equation (2.36), that here as before, need not have a solution.

In the special case of a time-invariant system and cost, the well-known *Bellman inequality* results:

$$0 \leq h(x,u) + \nabla_x V(t,x) \cdot f(x,u), \tag{2.48}$$

yielding the *Bellman equation*,

$$0 = \min_{u\in\Omega(x)} \{h(x,u) + \nabla_x V(t,x) \cdot f(x,u)\}. \tag{2.49}$$

with some additional considerations in a more rigorous development.

Let us now introduce the concept of an 'optimal equilibrium state' of an IH OCP of the type (2.43). The existence of such a state (or more precisely: admissible state-control tuple) is essential for the stability results in Section 2.3.

Definition 12 (Optimal Equilibrium State) *Consider the* IH OCP *defined by (2.43), subject to (2.44)–(2.46). If there exists an equilibrium state-control tuple* (x^e, u^e) *and a continuously differentiable function* $\psi : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$ *such that*

$$(x^e, u^e) = \arg \min_{(x,u)\in\mathcal{X}\times\Omega(x)} \{h(t,x,u) + \nabla_t \psi(t,x) + \nabla_x \psi(t,x) \cdot f(t,x,u)\} \tag{2.50}$$

$\forall\, t \geq 0$, *then* (x^e, u^e) *is called* ***optimal*** *(equilibrium) state-control tuple.*

REMARK

1. The OCP will not be changed through ψ, since[10]

$$\int_0^\infty \left(h(\theta,x,u) + \frac{d}{d\theta}\psi(\theta,x(\theta)) \right) d\theta = \psi(\infty,x(\infty)) - \psi(0,x(0)) + \int_0^\infty h(\theta,x,u)\, d\theta.$$

2. If one knows the optimal cost-to-go $V(t,x)$ for the system, $\psi(t,x) = V(t,x) + const.$ satisfies the requirements of Definition 12. If x^e is a known optimal equilibrium, one can set at least $\psi(t,x^e) = V(t,x^e) + const.$

3. From Definition 12, it is not clear how to find a function ψ, which is apparently a solution to the HJB inequality (2.47). For the special case of a time-invariant system and discounted cost (cf. remark below), Lemma 1 in Section 2.4 will provide some sufficient conditions for an equilibrium state-control tuple (x^e, u^e) to be optimal.

REMARK For our developments in Section 2.3.2, we will note here that the case of a time-invariant system (2.8) and discounted cost kernel (with discount factor $r > 0$),

$$e^{-rt}h(x,u),$$

the problem can still be considered as time-invariant. For this consider the value function $V(t,x)$ that is here assumed to exist, and note that for any admissible x :[11]

$$\begin{aligned} V(t,x) &= \min_{u\in\Omega(x^u)} \int_t^\infty e^{-r\theta} h(x^u(\theta),u(\theta))\, d\theta \\ &= e^{-rt} \min_{u\in\Omega(x^u)} \int_0^\infty e^{-r\theta} h(x^u(t+\theta),u(t+\theta))\, d\theta \qquad (2.51) \\ &= e^{-rt} \min_{u\in\Omega(x^u)} \int_0^\infty e^{-r\theta} h(x^u(\theta),u(\theta))\, d\theta, \qquad (2.52) \end{aligned}$$

and thus

$$V(t,x) = e^{-rt}V(0,x). \qquad (2.53)$$

With this, the HJB equation becomes (after substituting $V(x)$ for $V(0,x)$):

$$rV(x) = \min_{u\in\Omega(x)} \{h(x,u) + \nabla V(x)\cdot f(x,u)\}, \qquad (2.54)$$

which we will refer to as Bellman's equation for discounted cost.

[10]Here and later we will assume that ψ is 'bounded at infinity', i.e., that $\lim_{t\to\infty} \psi(t,x(t)) < \infty$ for any admissible state trajectory.

[11]The time-invariance of the system function is used from (2.51) to (2.52).

2.2.2 Numerical Methods

Since for moving horizon controllers a finite horizon optimal control problem has to be solved on-line at every decision stage, we will give a brief overview about computational methods (based on [Sch96]) and in particular briefly describe the algorithm used by the software package employed. A simple classification of algorithms for solving OCPs is depicted in Figure 2-2. There are direct and indirect methods. — Indirect methods usually use necessary conditions of optimality such as the PMP (Theorem 5), which involves the solution of two-point boundary problems, using e.g. so-called shooting algorithms [RS72].

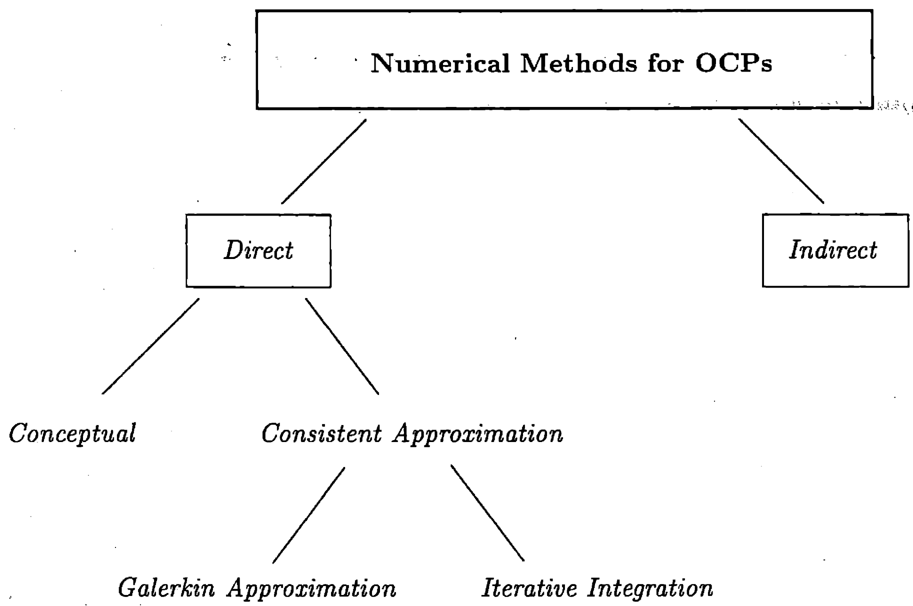

Figure 2-2: Numerical Methods for Solving OCPs.

Main drawback of these methods is that they may provide local solutions instead of global ones and need close initial guesses. Hence, they often exhibit a low degree of robustness with respect to the initial guess. Moreover, since the conditions employed by the algorithms are generally not sufficient for optimality, one may obtain solutions that are *not* minimizers of the OCP. — Direct methods try to minimize the objective functional directly. Within the class of direct methods, *conceptual* algorithms are based on finite dimensional optimization and functional approximation (e.g., Euler's method for piecewise constant control representation, cf. references in [Sch96], p. 9). In this Master's thesis, we will use a numerical software package[12] based on successive approximation via iterative integration, where a succession of finite dimensional, discrete-time OCPs are solved that are more and more accurate representations of the original continuous-time OCP.

[12]RIOTS: Recursive Integration Optimal Trajectory Solver by A. Schwartz [Sch96], a toolbox for Matlab. Matlab is a registered trademark of The Mathworks, Inc.

2.3 Moving Horizon Control Approximations

General conditions for stability of MH controllers and their approximating behavior towards IH optimal controllers have been given by Keerthi and Gilbert [KG88] and the first part of this section will essentially summarize their results. These discrete-time developments have been later formulated for nonlinear continuous-time systems with quadratic cost functionals by Mayne and Michalska [MM90]. The discrete-time results hold for quite a general class of time-varying nonlinear systems with positive semi-definite cost functional. In order to guarantee stability of an optimal MH policy, the existence of an optimal state-control tuple (x^e, u^e) of the underlying IH optimization problem (cf. Definition 13) is required.

Critical resource allocation problems such as the one presented in Section 2.6, frequently involve cost functionals that do *not* satisfy the standard assumptions immediately. In fact, a natural formulation of the cost functional often yields a *sign-indefinite* cost kernel that is not bounded from below *a priori*,[13] and the existence of an optimal state-control tuple is not obvious in many cases. Thus, more general results are needed to deal with typical control problems in nonsingular corporate policy making.

In Section 2.4 we will present a useful lemma for reducing problems with indefinite discounted cost in continuous-time to a formulation with positive semi-definite cost, analogous to the ones considered by Keerthi and Gilbert in discrete-time. Unfortunately an analogous formulation of the lemma in discrete-time requires stricter assumptions that diminishes the practical relevance of this statement. Thus, we have chosen to translate the discrete-time results to the continuous-time case for time-invariant systems and discounted cost functionals under the (strong) assumption that an optimal cost-to-go solving Bellman's equation (2.49) exists. — It is interesting to note at this point that to use the lemma the special structure of the problem is sufficient, and one does not need to solve Bellman's equation, nor is a use of the PMP for IH OCPs required.

2.3.1 Keerthi and Gilbert's Results

Consider the constrained discrete-time system (2.3) for $k \geq i \in \mathcal{T} = \mathbb{Z}_+$:

$$x_{k+1} = f_k(x_k, u_k), \tag{2.55}$$

with initial condition

$$x_i = a, \tag{2.56}$$

and definition of f_k for $k \geq i \geq 0$ as in Section 2.1.2.

- Define the constrained **infinite horizon OCP** $\mathcal{P}(i, a)$ in the following way:

$$\mathcal{P}(i,a): \quad J_i := \sum_{k=i}^{\infty} h_k(x_k, u_k) \longrightarrow \min, \tag{2.57}$$

[13] This means that there is no finite constant K such that $\forall\,(t,x,u) \in \mathcal{T} \times \mathcal{X} \times \mathcal{U}:\; h(x,u) \geq K$, without taking the system equation into account.

subject to (2.55)–(2.56) and

$$(x_k, u_k) \in \mathcal{Z}_k(x_k) = \mathcal{X}_k \times \Omega_k(x_k). \tag{2.58}$$

The set of admissible states for $\mathcal{P}(i, a)$ is

$$X_i := \{a \in \mathcal{X}_i \,|\, \exists \textbf{ admissible } \{(x_k, u_k)\}_{k=i}^{\infty} : \mathcal{P}(i, a) \textbf{ yields } J_i < \infty\}. \tag{2.59}$$

- The corresponding constrained **moving horizon OCP** $\hat{\mathcal{P}}(i, a)$ can then be formulated as follows:

$$\hat{\mathcal{P}}(i, a) : \quad \hat{J}_i := \sum_{k=i}^{i+M-1} h_k(x_k, u_k) \longrightarrow \min, \tag{2.60}$$

subject to (2.55)–(2.56), (2.58), and the endpoint constraint

$$x_{i+M} = x^e, \tag{2.61}$$

with positive horizon length M and the set of admissible states

$$\hat{X}_i := \{a \in \mathcal{X}_i \,|\, \exists \textbf{ admissible } \{(x_k, u_k)\}_{k=i}^{i+M} \textbf{ for } \hat{\mathcal{P}}(i, a)\}. \tag{2.62}$$

We will call $V(i, a)$ the optimal cost-to-go for the IH problem $\mathcal{P}(i, a)$, and $\hat{V}(i, a)$ the optimal cost-to-go for the MH problem. The optimal MH policy $\{\hat{\mu}_k^*\}_{k \geq i}$ is thereby given by

$$\{\hat{\mu}_i^*, \hat{\mu}_{i+1}^*, \hat{\mu}_{i+2}^*, \ldots\} = \{\hat{\mu}_i^{i+M}, \hat{\mu}_{i+1}^{i+1+M}, \hat{\mu}_{i+2}^{i+2+M}, \ldots\},$$

where $\hat{\mu}_j^{j+M}$ is the optimal finite horizon policy at time j, over prediction horizon M. The optimal MH policy is nothing else than a concatenation of the first steps of the different optimal finite horizon policies.

Definition 13 (Optimal State-Control Tuple) *Let* $(i, a) \in \Xi$ *and consider the* IH OCP $\mathcal{P}(i, a)$. *If there exists an equilibrium state-control tuple* (x^e, u^e) *and a continuous function* $\psi : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$ *such that*

$$(x^e, u^e) = \arg \min_{(x,u) \in \mathcal{Z}_k(x)} \{h_k(x, u) + \psi(k+1, f_k(x, u)) - \psi(k, x)\},$$

$\forall\, k \geq i$, *then* (x^e, u^e) *is called an* ***optimal*** *(equilibrium) state-control tuple for* $\mathcal{P}(i, a)$.

Remark The function ψ in the above definition does not change the optimization problem $\mathcal{P}(i, a)$, since for any $(i, a) \in \Xi$:

$$\sum_{k=i}^{\infty} \{h_k(x_k, u_k) + \psi(k+1, f_k(x_k, u_k)) - \psi(k, x_k)\} = J_i - \psi(i, a),$$

and minimization over $J_i - \psi(i, a)$ instead of J_i yields the same result.

Assume that the system (2.55) possesses an equilibrium state-control tuple (x^e, u^e) such that $\forall\, k \geq 0$:

$$(x^e, u^e) \in \mathcal{Z}_k(x^e), \quad f_k(x^e, u^e) = x^e, \quad h_k(x^e, u^e) = 0. \tag{2.63}$$

In addition, consider the following assumptions:[14]

(A1) There is an (admissible) optimal equilibrium state-control tuple (x^e, u^e).

(A2) $\mathcal{Z}_k$ is closed, and the functions $f_k : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $g_k : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ are continuous and $h_k : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is lower semi-continuous $\forall\, k \geq 0$.

(A3) There are $\mathcal{K}_\infty$ functions φ_1 and φ_2 such that

$$\varphi_1(\|(x,u) - (x^e, u^e)\|) \overset{\text{(A3.1)}}{\leq} h_k(x,u) \overset{\text{(A3.2)}}{\leq} \varphi_2(\|(x,u) - (x^e,u^e)\|), \tag{2.64}$$

$\forall\,(x,u) \in \mathcal{X} \times \mathcal{U},\ \forall\, k \geq 0.$

REMARK Assumption (A3.1) is automatically satisfied if h_k is continuous $\forall\, k$ and $h_k \geq h$ for some continuous function $h : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$. To see that, consider

$$\varphi_1(\rho) = \max\{\rho, \max\{h(x,u) \mid \|(x,u)\| \leq \rho\}\},$$

so that condition (A3.1) is clearly satisfied.

Theorem 6 (Existence of Solutions) *Consider the problems $\mathcal{P}(i,a)$ and $\hat{\mathcal{P}}(i,a)$ for a in X_i and $\hat{X}_i$ respectively, and suppose that the system (2.55) has property* **C** *with control horizon N_c. Furthermore, suppose that assumptions (A1), (A2), and (A3.1) hold true. Then*

(i) $(x^e, u^e) \in \operatorname{int}\{X_i\}$ and the IH problem $\mathcal{P}(i,a)$ has a solution.

(ii) if $M \geq N_c$, it is $(x^e, u^e) \in \operatorname{int}\{X_i\}$ and the MH problem $\hat{\mathcal{P}}(i,a)$ has a solution.

Proof. See [KG88], pp. 280–281; [KG86a].

Theorem 7 (Stability of Optimal Feedback Law) *Assume that the system (2.55) has property* **C** *and satisfies (A1)-(A3). Let $F_k(x_k) = f_k(x_k, \mu_k^*(x_k))$, where $u_k^* = \mu_k^*(x_k)$ describes an optimal feedback law for $\mathcal{P}(i,a)$, and define $\hat{F}_k$ with $\hat{u}_k^* = \hat{\mu}_k^*(x_k)$ for $\hat{\mathcal{P}}(i,a)$ accordingly. Then for all $(i,a) \in \Xi$:*

(i) $\lim_{k\to\infty} T_k^F(i,a) = x^e$. The state $x = x^e$ is the only equilibrium of the system $x_k = F(x_k)$ and is UAS.

(ii) $\lim_{k\to\infty} T_k^{\hat{F}}(i,a) = x^e$. The state $x = x^e$ is the only equilibrium of the system $x_k = \hat{F}(x_k)$ and is UAS.

(iii) in the unconstrained case, i.e., when $\mathcal{Z}_k = \mathbb{R}^n \times \mathbb{R}^m$, the equilibrium state x^e in statements (i)-(ii) is GUAS.

Proof. See [KG88], pp. 282–284.

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[14] Keerthi and Gilbert assumed in their paper [KG88] that the optimal state-control tuple is at the origin. This is however unrealistic for typical management applications, so that we will treat the general case for nonzero (x^e, u^e) in order to avoid involved changes of variables later.

Theorem 8 (Moving Horizon Approximation) *If the hypotheses of Theorem 7 are satisfied, then for any $(i,a) \in \Xi$ and $\epsilon > 0$ there exist $L = L(i,a,\epsilon) > 0$ such that*

$$M \geq L \quad \Rightarrow \quad \left(a \in \hat{X}_i \;\; \textbf{and} \;\; \hat{V}(i,a) \leq V(i,a) + \epsilon \right).$$

In addition, there exist $\rho > 0$, and given $\epsilon > 0$ there is a positive integer $L = L(\epsilon) \geq N_c$ such that

$$\hat{V}(i,a) \leq V(i,a) + \epsilon,$$

$\forall M \geq L$, $\forall a \in B_\rho(x^e)$.

Proof. See [KG88], pp. 285–286.

Remark From the above theorems it is not clear how to choose the length of the time horizon (as a function of the initial state) such as to make ensure that the MH optimal cost stays within an ϵ-neighborhood of the optimal IH cost. For a practical application of the MH control methodology such statements are essential however, and we will make an effort to determine appropriate bounds for the necessary length of the prediction horizon when discussing robustness issues in Sections 2.5 and 2.6.

2.3.2 Continuous-Time Considerations

We will now derive semi-formally analogous results to the discrete-time case for discounted time-invariant continuous-time IH OCPs that typically occur in the context of critical resource allocation problems.

Let us consider the IH OCP (2.43), subject to (2.8), (2.45)–(2.46), for a time-invariant system and discounted cost kernel (2.66), as introduced in the remark on page 37. Assume that an optimal cost-to-go,

$$V(t,x) = e^{-rt}V(0,x) =: e^{-rt}V(x), \tag{2.65}$$

exists that solves Bellman's equation (2.49). Suppose also that the system possesses the following properties $\bar{\mathbf{C}}$ and $\bar{\mathbf{O}}$ with respect to the positive semi-definite cost kernel

$$e^{-rt}h(x,u). \tag{2.66}$$

To define properties $\bar{\mathbf{C}}$ and $\bar{\mathbf{O}}$, we will assume that there exists an admissible (constant) optimal equilibrium state-control tuple (x^e, u^e), for which:

$$h(x^e, u^e) = 0.$$

Definition 14 (Property $\bar{\mathbf{C}}$) *The system (2.8) has property $\bar{\mathbf{C}}$ with respect to (2.66), if there exists $T_c > 0$ and a $\mathcal{K}_\infty$ function φ_c such that: $\forall \bar{x} \in \mathcal{X}$ there exists a state-control trajectory $\{(x^u(t), u(t)) \,|\, 0 \leq t \leq T_c\}$ such that $x^u(0) = \bar{x}$ and*

$$\int_0^{T_c} e^{-r\theta}h(x^u(\theta), u(\theta))\, d\theta \leq \varphi_c(\|\bar{x} - x^e\|), \tag{2.67}$$

and $(x^u(t), u(t)) = (x^e, u^e)$, $\forall t \geq T_c$.

Definition 15 (Property $\bar{\mathbf{O}}$ for State Feedback) *The system (2.8) has property $\bar{\mathbf{O}}$ with respect to (2.66), if there exists $T_o > 0$, and a $\mathcal{K}_\infty$ function φ_o such that: any admissible state-control trajectory $\{(x^u(t), u(t)) \mid 0 \leq t \leq T_o\}$ satisfies*

$$\int_0^{T_o} e^{-r\theta} h(x^u(\theta), u(\theta))\, d\theta \geq \varphi_c(\|\bar{x} - x^e\|). \tag{2.68}$$

Remark Properties **C** and **O** (cf. Definitions 10 and 11) relate to properties $\bar{\mathbf{C}}$ and $\bar{\mathbf{O}}$ introduced here as they, together with the boundedness assumptions (A3.1) and (A3.2) on h_k, imply similar properties in discrete-time. In fact, properties $\bar{\mathbf{C}}$ and $\bar{\mathbf{O}}$ have been introduced *with respect to* h as a shortcut to such conclusions and thus to simplify the continuous-time developments. The reader may expect that the results below will also hold under assumptions analogous to the discrete-time case that may then be easier to verify.

For the above IH OCP, let $V_T(t,x)$ be the optimal cost-to-go of the corresponding finite horizon OCP, where instead of (2.43) one considers

$$\int_0^T e^{-r\theta} h(x,u)\, d\theta \longrightarrow \min, \tag{2.69}$$

subject to the additional endpoint constraint

$$x(T) = x^e. \tag{2.70}$$

Furthermore, let $\hat{V}(t,x)$ be the optimal MH cost-to-go, i.e.,

$$\hat{V}(t,x) = \sum_{k \geq \lfloor t/\tau \rfloor} \min_{u \in \Omega(x^u)} \int_{\max\{t,k\tau\}}^{(k+1)\tau} e^{-r\theta} h(x^u(\theta), u(\theta))\, d\theta, \tag{2.71}$$

where $0 < \tau < T$ is the *control horizon*, over which the optimal finite horizon policy (computed at 'decision stage' $k\tau$) is implemented. Expression (2.71) means that at time $t \geq 0$, the optimal IH policy is pieced together by solving a finite horizon OCP at each instant $k\tau$ over the interval[15]

$$I_k := [k\tau, k\tau + T], \tag{2.72}$$

$\forall\, k \geq \lfloor t/\tau \rfloor$, subject to the initial and endpoint constraints

$$\begin{aligned} x(t) = \bar{x} \quad &: \quad \text{given}, && (2.73)\\ x\left((k\tau)^-\right) \quad &= \quad x\left((k\tau)^+\right), && (2.74)\\ x\left(k\tau + T\right) \quad &= \quad x^e. && (2.75) \end{aligned}$$

Properties $\bar{\mathbf{C}}$ and $\bar{\mathbf{O}}$ imply that for $T \geq \bar{T} := \max\{T_c, T_o\} + \tau$:

$$\varphi_o(\|x - x^e\|) \leq V(t,x) \leq \hat{V}(t,x) \leq V_T(t,x) \leq \varphi_c(\|x - x^e\|), \tag{2.76}$$

[15] cf. Figure 1-1 on page 11

$\forall x \in \mathcal{X}$. Thereby is $V_T(t, x)$ for $t \in [k\tau, (k+1)\tau]$ the optimal cost of the finite horizon OCP

$$\int_{I_k} e^{-r\theta} h(x, u)\, d\theta \longrightarrow \min, \tag{2.77}$$

subject to (2.73)–(2.75). It follows from (2.65) and (2.76) that

$$\dot{V}(t, x) = -re^{-rt}V(x) \leq -r\varphi_o(\|x - x^e\|),$$

so that with Theorem 2 the optimal equilibrium point x^e is globally asymptotically stable under the IH optimal control law $u^* = \mu^*(x)$, i.e., for any initial state $x(0) =: \bar{x}$ one has

$$\lim_{t\to\infty} T_t^F(\bar{x}) = x^e,$$

where $F(x) := f(x, \mu^*(x))$ denotes the IH optimal feedback system function.

We will now show the asymptotic stability of the MH optimal controller. Since the problem can be considered as time-invariant (cf. remark on page 37), it is enough to consider the problem over the time interval $[0, T]$.

For a given initial state $\bar{x}$, take a particular admissible state-control trajectory $\{(\tilde{x}(t), \tilde{u}(t)) \mid 0 \leq t \leq T\}$ such that property $\bar{\mathbf{C}}$ is satisfied. In particular one has then

$$x(t) = x^e,$$

$\forall t \in [\bar{T}, T]$, where we have assumed as above that $T \geq \bar{T}$. Using property $\bar{\mathbf{O}}$, we can write

$$\varphi_o(\|\bar{x} - x^e\|) \leq V_T(0, \bar{x}) \leq \int_0^{\bar{T}} e^{-r\theta} h(\tilde{x}(\theta), \tilde{u}(\theta))\, d\theta \leq \varphi_c(\|\bar{x} - x^e\|),$$

and

$$\varphi_o(\|\bar{x} - x^e\|) \leq V_T(\tau, \bar{x}) \leq e^{-r\tau} \int_0^{\bar{T}} e^{-r\theta} h(\tilde{x}(\theta), \tilde{u}(\theta))\, d\theta \leq \varphi_c(\|\bar{x} - x^e\|), \tag{2.78}$$

Thus for

$$\tau > \ln\left(\frac{\varphi_c(\|\bar{x} - x^e\|)}{\varphi_o(\|\bar{x} - x^e\|)}\right)$$

we obtain

$$V_T(\tau, \bar{x}) \leq V_T(0, \bar{x}).$$

If in addition there is a positive constant $K = K(\bar{x})$ such that

$$0 \leq \ln\left(\frac{\varphi_c(\|x - x^e\|)}{\varphi_o(\|x - x^e\|)}\right) \leq K(\bar{x}),$$

$\forall x \in B_{\|\bar{x}\|}(x^e)$, then for $\tau > K$:

$$x(0) = \bar{x} \;\Rightarrow\; x(\tau) \in \{x \mid \|x\| < \bar{x}\} \cup \{x^e\}. \tag{2.79}$$

Moreover, by extending both the prediction horizon T and the control horizon τ, the distance $\|x(\tau) - x^e\|$ can be made arbitrarily small, since[16]

$$V_T(t,x) \longrightarrow V(t,x), \qquad \text{as } T \to \infty.$$

Thus the sequence $\left\{\hat{V}(k\tau, \bar{x})\right\}_{k\geq 0}$ is monotonically decreasing. Since the set $B_{\|\bar{x}\|}(x^e)$ is bounded and by (2.79) invariant, any trajectory starting in $B_{\|\bar{x}\|}(x^e)$ must have at least one accumulation point there.[17] However, because of (2.70) and (2.79) there can be no other accumulation point than x^e and thus the MH optimal state trajectory converges asymptotically towards the optimal equilibrium state x^e for sufficiently large T and τ.

2.4 A New Result for Discounted Indefinite Cost

Optimal control problems with discounted cost play an important role in finding value-maximizing strategies for an economic system such as a firm. A theoretical treatment of OCPs for managerial systems often considers an infinite horizon formulation of the cost functional, since as operations continue indefinitely, appropriate terminal conditions are not available. — In practice, even though the overall problem is IH, decisions are taken periodically, to the best of current knowledge over a generally finite prediction horizon. In the previous section we have provided conditions under which the optimal IH policy can be approximated by an optimal control law, based on periodic predictions over a moving horizon of finite length. As emphasized, these results hinge on the fact that the cost kernel is positive semi-definite and attains its only minimum at an optimal state-control tuple. In economic problems such hypotheses are often not satisfied as the cost kernel may not be bounded from below (cf. example in Section 2.6). In this case, additional work is required to find a suitable 'potential' function ψ (cf. Definition 12 on page 36) such that

$$h + \frac{d\psi}{dt} \geq 0,$$

and zero for an equilibrium state-control tuple (x^e, u^e). This is generally not an easy task, since ψ has to satisfy the HJB inequality (2.47). — Once an appropriate ψ has been found and several other assumptions are satisfied, stability of the moving horizon policy in discrete time[18] is achieved by imposing at each stage a terminal condition that would steer the system to an optimal equilibrium state at the end of the finite horizon, if the optimal finite horizon control at that stage would be implemented over the whole length of the horizon.

In summary, the question of reducing a given IH OCP with discounted cost and indefinite cost kernel to an equivalent problem with the cost kernel minimal at an optimal equilibrium state is of great importance for statements about stability and convergence of an MH policy, typically employed by practical decision makers.

[16]This is clear from the fact that V_T is monotonically decreasing in T for $T \geq T_c$ (by property $\bar{\mathbf{C}}$) and bounded from tightly below by $\hat{V}$.

[17]If for a trajectory $S := \{x(t) \,|\, t \geq 0\}$ there is a sequence $\{x(t_k)\}_{k\geq 0} \subset S$ with $\lim_{k\to\infty} t_k = \infty$ such that $\lim_{k\to\infty} x(t_k) = y$, then y is called an *accumulation point* of S.

[18]Stability in continuous-time does *not* hold by analogy. In fact, it is very hard to formulate sufficient conditions that are not too limiting for our purposes and imply the existence of an optimal cost. In [MM90], a restriction to quadratic cost functionals was imposed.

Let us now consider the problem of finding an optimal state-control tuple in continuous time. We will present a sufficient condition for such a problem reduction together with candidates for optimal equilibrium states, that is particularly useful for problems that only penalize a single control variable. A discrete-time version of this result will be provided afterwards.

Continuous Time. Consider the time-invariant system

$$\dot{x}_1 = f_1(x, u_1), \tag{2.80}$$
$$\dot{x}_2 = f_2(x, u_2), \tag{2.81}$$

where $f_1 : X \times \mathbb{R}^{m_1} \to \mathbb{R}^{n_1}$, $f_2 : X \times \mathbb{R}^{m_2} \to \mathbb{R}^{n_2}$ are continuous functions satisfying the conditions of Theorem 1, n_1, m_1, n_2, m_2 positive integers and $X := X_1 \times X_2 \subset \mathbb{R}^{n_1+n_2}$ is a closed nonempty set of admissible states $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, for $t \geq 0$.

Consider also the IH OCP (2.43) with discounted cost functional,

$$\int_0^\infty e^{-rt} h(x, u_2)\, dt \longrightarrow \min, \tag{2.82}$$

subject to (2.80)–(2.81), and

$$(u_1, u_2) \in \Omega_1(x_1) \times \Omega_2(x_2) \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \tag{2.83}$$

where $h : X \times \mathbb{R}^{m_2} \to \mathbb{R}$ is a piecewise continuous function (the cost kernel).

The following assumptions are required for our result:

($\bar{\mathbf{B}}$1) For any given admissible x and $v_2 := \dot{x}_2$, the control u_2 in (2.81) is uniquely determined, i.e., there is a function $\zeta : X \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_2}$ such that

$$v_2 = f_2(x, \zeta(x, v_2)),$$

$\forall x \in X$, $\forall v_2 \in \mathbb{R}^{n_2}$.

($\bar{\mathbf{B}}$2) The mapping $\phi : X_2 \times \mathbb{R}^{n_2} \to \mathbb{R}$,

$$(x_2, v_2) \mapsto \min_{\xi_1 \in X_1} \{h(\begin{bmatrix} \xi_1 \\ x_2 \end{bmatrix}, \zeta(\begin{bmatrix} \xi_1 \\ x_2 \end{bmatrix}, v_2))\} =: \phi(x_2, v_2)$$

exists,[19] and is of the form

$$\phi(x_2, v_2) = q(x_2) + \nabla\psi(x_2) \cdot v_2 \tag{2.84}$$

$\forall x_2 \in X_2$, $\forall v_2 \in \mathbb{R}^{n_2}$, whereby $q, \psi : X_2 \to \mathbb{R}$ are piecewise continuous functions such that for a given constant $r > 0$ the function

$$q(x_2) + r\psi(x_2)$$

[19] i.e., the minimum in the expression for ϕ exists and is finite $\forall\, (x_2, v_2) \in X_2 \times \mathbb{R}^{n_2}$

is bounded from below on X_2, where ψ is assumed to be piecewise continuously differentiable. Let $x^o = \begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix}$ be determined by

$$x_2^o = \arg\min_{x_2 \in X_2} \{q(x_2) + r\psi(x_2)\}, \tag{2.85}$$

$$x_1^o = \arg\min_{\xi_1 \in X_1} \{h(\begin{bmatrix} \xi_1 \\ x_2^o \end{bmatrix}, \zeta(\begin{bmatrix} \xi_1 \\ x_2^o \end{bmatrix}, 0))\}. \tag{2.86}$$

($\bar{\mathbf{B}}$**3**) There exists a constant (equilibrium) control $(u_1^o, u_2^o) \in \Omega_1(x_1^o) \times \Omega_2(x_2^o)$ satisfying

$$0 = f_1(x^o, u_1^o), \tag{2.87}$$

$$u_2^o = \zeta(x^o, 0). \tag{2.88}$$

Lemma 1 *Consider the OCP (2.82), subject to (2.80)–(2.83), and assume that the conditions ($\bar{B}$1)–($\bar{B}$3) are satisfied. Then, (x^o, u^o) is an optimal (equilibrium) state-control tuple.*

Proof. For any $x \in X$, $v_2 \in \mathbb{R}^{n_2}$ we define

$$\Delta(x, v_2) := h(x, \zeta(x, v_2)) - \phi(x_2, v_2).$$

It is clear that under assumptions ($\bar{B}$1)–($\bar{B}$2) the function $\Delta : X \times \mathbb{R}^{n_2} \to \mathbb{R}$ is well-defined and positive semi-definite, since for any (x, v_2) the RHS of the last equation,

$$h(x, \zeta(x, v_2)) - \min_{\xi_1 \in X_1} \{h(\begin{bmatrix} \xi_1 \\ x_2 \end{bmatrix}, \zeta(\begin{bmatrix} \xi_1 \\ x_2 \end{bmatrix}, v_2))\} \geq 0.$$

In addition $\Delta = 0$, whenever (x, v_2) is an element of a hyperplane $P \subset X \times \mathbb{R}^{n_2}$ defined by

$$P: \; h(x, \zeta(x, v_2)) = \phi(x_2, v_2).$$

Using representation (2.84) for $\phi(x_2, v_2)$, the cost functional (2.82) can be written in the form

$$\begin{aligned} \int_0^\infty e^{-rt} h(x, v_2)\, dt &= \int_0^\infty \left[e^{-rt} \left(q(x_2) + r\psi(x_2) + \Delta(x, v_2) \right) + \frac{d}{dt} \left(e^{-rt} \psi(x_2) \right) \right] dt, \\ &= -\psi(0) + \int_0^\infty e^{-rt} \left(q(x_2) + r\psi(x_2) + \Delta(x, v_2) \right) dt, \\ &\geq \frac{m}{r} - \psi(0), \end{aligned} \tag{2.89}$$

where

$$m := \min_{x_2 \in X_2} \{q(x_2) + r\psi(x_2)\}.$$

The lower bound (2.89) for the cost is attained for any initial equilibrium states $x(0) =: \bar{x}$ such that

$$\bar{x}_2 = \arg\min_{x_2 \in X_2} \{q(x_2) + r\psi(x_2)\},$$

$$\textbf{and} \qquad (\bar{x}, 0) \in P.$$

The state $\bar{x}$ is an equilibrium if and only if

$$\begin{aligned} 0 &= f_1(\bar{x}, \bar{u}_1), \\ 0 &= f_2(\bar{x}, \bar{u}_2). \end{aligned}$$

This is the case if admissible controls $(\bar{u}_1, \bar{u}_2) \in \Omega_1(\bar{x}_2) \times \Omega_2(\bar{x}_2)$ can be chosen accordingly, which completes the proof. ■

REMARK This lemma is in general useful *only* if $n_2 = 1$, since then for a given $\phi(x_2, v_2)$, affine in v_2, the function $\psi(x_2)$ in (2.84) can always be determined. In the case $n_2 > 1$ this is still possible, if there is a function $\Psi : X_2 \to \mathbb{R}^{n_2}$ such that

$$\phi(x_2, v_2) = q(x_2) + \Psi(x_2) \cdot v_2,$$

and

$$\frac{\partial \Psi_i}{\partial x_j} = \frac{\partial \Psi_j}{\partial x_i},$$

$\forall\, i, j \in \{1, \ldots, n_2\}$, on an open and simply connected domain $\tilde{X}_2 \subset X_2$ of dimension n_2, since then the vector-field $\Psi = (\Psi_1, \ldots, \Psi_{n_2})'$ is *irrotational* there. In this case a scalar potential $\psi : \tilde{X}_2 \to \mathbb{R}^{n_2}$ can be computed as follows:

$$\psi(x_2) = \int_{\bar{x}_2}^{x_2} \Psi(\xi) \cdot d(\xi_1, \ldots, \xi_{n_2}),$$

where $\bar{x}_2 \in \tilde{X}_2$ is an arbitrary, fixed reference point.[20]

Lemma 2 *Under the assumptions of Lemma 1, the OCP (2.82) subject to (2.80)–(2.81), is equivalent to*

$$\int_0^\infty e^{-rt}\tilde{h}(x, v_2)\, dt \longrightarrow \min, \tag{2.90}$$

subject to

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1), \\ \dot{x}_2 &= v_2, \end{aligned} \tag{2.91}$$

and

$$u_1 \in \Omega_1(y), \qquad \zeta(x, v_2) \in \Omega_2(x_2), \tag{2.92}$$

$\forall\, x \in X_1 \times X_2,\ \forall\, t \geq 0,$ *whereby*

$$\tilde{h}(x, v_2) := \Big(q(x_2) + r\psi(x_2) - m\Big) + \Big(h(x, \zeta(x, v_2)) - \phi(x_2, v_2)\Big) \geq 0. \tag{2.93}$$

Proof. See remark below.

REMARK The above assertion is in fact a direct consequence or just a restatement of the

[20] The details can be found in standard textbooks on Advanced Calculus (see e.g. [Heu95]). For $n_2 = 2$ the above result follows from Green's Theorem and for $n_2 = 3$ from Stoke's Theorem.

proof to Lemma 1. It has been mainly formulated for its practical value and will be referred to later. In particular, we note that any optimal equilibria are the same for the two OCPs and can be therefore also determined by (2.85)–(2.86), subject to (2.87) and (2.92) for $v_2 = 0$.

Discrete Time. The above results carry (under somewhat stronger assumptions) over to the discrete-time case. To show this consider the system

$$x_{k+1}^1 = f_1(x_k, u_k^1), \tag{2.94}$$
$$x_{k+1}^2 = f_2(x_k, u_k^2), \tag{2.95}$$

where $f_1 : X \times \mathbb{R}^{m_1} \to \mathbb{R}^{n_1}$ and $f_2 : X \times \mathbb{R}^{m_2} \to \mathbb{R}^{n_2}$ are continuous functions, and $x_k := \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} \in X := X_1 \times X_2$ with X and n_1, m_1, n_2, m_2 as above.

Introduce the time-invariant discounted IH OCP,

$$\sum_{k=0}^{\infty} e^{-rk} h(x_k, u_k) \longrightarrow \min, \tag{2.96}$$

subject to (2.94)–(2.95), and

$$(u_k^1, u_k^2) \in \Omega_1(x_k^1) \times \Omega_2(x_k^2). \tag{2.97}$$

The following assumptions (B1)–(B3) are analogous to the hypotheses ($\bar{\text{B}}$1)–($\bar{\text{B}}$3) in continuous time.

(B1) For any given admissible x_k and x_{k+1}, the control u_k^2 in (2.95) is uniquely determined $\forall\, k$, i.e., there is a well-defined function $\zeta : X \times X_2 \to \mathbb{R}^{m_2}$ such that

$$u_k^2 = \zeta(x_k, x_{k+1}^2),$$

$\forall\, (x_k, x_{k+1}^2) \in X \times X_2$.

(B2) The mapping $\phi : X_2 \times X_2 \to \mathbb{R}$,

$$(x_k^2, x_{k+1}^2) \mapsto \min_{\xi \in X_1} \{h_k(\begin{bmatrix} \xi \\ x_k^2 \end{bmatrix}, \zeta(\begin{bmatrix} \xi \\ x_k^2 \end{bmatrix}, x_{k+1}^2))\} =: \phi(x_k^2, x_{k+1}^2)$$

exists and is of the form

$$\phi(x_k^2, x_{k+1}^2) = q(x_k^2) + \psi(x_{k+1}^2) - \psi(x_k^2) \tag{2.98}$$

$\forall\, (x_k^2, x_{k+1}^2) \in X_2 \times X_2$, $\forall\, k \geq 0$, whereby q, $\psi : X_2 \to \mathbb{R}$ are piecewise continuous such that for a given constant $r > 0$ the function

$$q(x_k^2) - \left(1 + e^{-r}\right) \psi(x_{k+1}^2)$$

is bounded from below, i.e.,

$$m_q := \min_{\xi^2 \in X_2} q(\xi^2), \qquad m_\psi := \min_{\xi^2 \in X_2} \psi(\xi^2)$$

both exist and are finite. In addition, assume that there is a state $x_{opt} = \begin{bmatrix} x^1_{opt} \\ x^2_{opt} \end{bmatrix}$ that satisfies

$$\begin{aligned} x^2_{opt} &= \arg\min_{\xi^2 \in X_2} q(\xi^2) \ \textbf{and} \ x^2_{opt} = \arg\min_{\xi^2 \in X_2} \psi(\xi^2), && (2.99) \\ x^1_{opt} &= \arg\min_{\xi^1 \in X_1} \{h(\begin{bmatrix} \xi^1 \\ x^2_{opt} \end{bmatrix}, \zeta(\begin{bmatrix} \xi^1 \\ x^2_{opt} \end{bmatrix}, 0))\}. && (2.100) \end{aligned}$$

(B3) There exists an (equilibrium) control $(u^1_{opt}, u^2_{opt}) \in \Omega_1(x^1_{opt}) \times \Omega_2(x^2_{opt})$ satisfying

$$\begin{aligned} 0 &= f_1(x_{opt}, u^1_{opt}), && (2.101) \\ u^2_{opt} &= \zeta(x_{opt}, 0). && (2.102) \end{aligned}$$

Lemma 3 *Consider the OCP (2.96), subject to (2.94)–(2.95), and assume that the conditions (B1)–(B3) are satisfied. Then, (x_{opt}, u_{opt}) is an optimal (equilibrium) state-control tuple.*

Proof. For any $x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \in X$, $y \in X_2$ define

$$\Delta(x,y) := h(x, \zeta(x,y)) - \phi(x^2, y).$$

The function $\Delta : X \times X_2 \to \mathbb{R}$ is positive semi-definite, since $\forall\,(x,y) \in X \times X_2$:

$$\Delta(x,y) = h(x,\zeta(x,y)) - \min_{\xi \in X_1}\{h(\begin{bmatrix} \xi \\ x^2 \end{bmatrix}, \zeta(\begin{bmatrix} \xi \\ x^2 \end{bmatrix}, y))\} \geq 0.$$

In addition $\Delta(x,y) = 0$, whenever $(x,y) \in P$ with $P \subset X \times X_2$ a hyperplane defined by

$$P: \quad h(x,\zeta(x,y)) = \phi(x^2, y).$$

As a consequence, we have for that any $(x_k, x^2_{k+1}) \in X \times X_2$:

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-rk} h(x_k, u_k) &= \sum_{k=0}^{\infty} e^{-rk} \left(q(x^2_k) + \psi(x^2_{k+1}) - \psi(x^2_k) + \Delta(x_k, x^2_{k+1}) \right) \\ &= \sum_{k=0}^{\infty} e^{-rk} \Big\{ \left(q(x^2_k) + \left(1 - e^{-r}\right) \psi(x^2_{k+1}) + \Delta(x_k, x^2_{k+1}) \right) \\ &\qquad + \left(e^{-r}\psi(x^2_{k+1}) - \psi(x^2_k) \right) \Big\} \\ &= -\psi(x^2_0) + \sum_{k=0}^{\infty} e^{-rk} \left(q(x^2_k) + \left(1 - e^{-r}\right) \psi(x^2_{k+1}) + \Delta(x_k, x^2_{k+1}) \right) \\ &\geq m_\psi - \psi(x^2_0) + \frac{m_q}{1 - e^{-r}}. && (2.103) \end{aligned}$$

The lower bound in (2.103) is achieved for any initial equilibrium state $\bar{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} \in X$ for which

$$\bar{x}^2 = \arg\min_{\xi^2 \in X} q(\xi^2) = \arg\min_{\xi^2 \in X_2} \psi(\xi^2)$$
$$\textbf{and} \quad (\bar{x}, \bar{x}^2) \in P.$$

The state $\bar{x}$ is an (admissible) equilibrium, if and only if

$$\zeta(\bar{x}, \bar{x}^2) \in \Omega(\bar{x}^2), \quad \textbf{and}$$
$$\exists\, \bar{u}^1 \in \Omega(\bar{x}^1) \textbf{ such that } \bar{x}^1 = f_1(\bar{x}, \bar{u}^1),$$

which completes the proof. ■

REMARK

1. For $\phi(x_k^2, x_{k+1}^2) = \phi'(x_k^2) + \phi''(x_{k+1}^2)$ the functions ψ and q are simply determined by

$$\psi = \phi'', \qquad q = \phi' + \phi''.$$

2. Note that the assumed representation (2.98) for ϕ is quite restrictive. In particular, it does generally *not* allow a simple discretization of (2.84), the form of ϕ in the continuous-time case. This limits the utility of Lemma 3 considerably, so that we will not formulate an (apparent) analogon to Lemma 2 here.

2.5 Robustness Issues

Most of the stability results in Section 2.3 were obtained without specific consideration of how long the prediction horizon has to be so that the optimal moving horizon cost-to-go is guaranteed to stay in a neighborhood of the optimal infinite horizon cost-to-go. Such statements are important however for the practical applicability of the theory. In particular this yields guidelines for the decision maker of how to structure the decision making process, and over which prediction horizon a policy has to be devised so as to avoid the costly unstable and cyclical decision making, that is being observed frequently in practice, cf. [PS93].

We will limit our discussion to time-invariant systems and discounted cost in the continuous-time case, that we have treated in Section 2.3.2. Generically speaking for IH OCPs with discounted cost kernel lower bounds for time horizons are quite easy to obtain, since the optimal cost-to-go functions decay exponentially (cf. (2.65) on page 42). Figure 2-3 illustrates this point.

Thus, knowing that the state is within some δ-neighborhood of the optimal equilibrium state x^e implies bounds on the cost functional. On the other hand, knowledge about bounds on the cost-functional such as relations (2.76) or (2.78), can be used to conclude that the state has to be within a neighborhood of x^e after a certain time τ. More specifically, if there exist $\mathcal{K}_\infty$ functions φ_1, φ_2 such that for an optimal cost-to-go $V(t, x) = e^{-rt}V(x)$:

$$\varphi_1(\delta) \leq V(t, x) \leq \varphi_2(\delta),$$

$\forall\, \delta > 0$, then for any *fixed* $\delta > 0$ one can conclude that

$$V(t+\tau, x) \leq \varphi_1(\delta),$$

and

$$\|x(t+\tau) - x^e\| \leq \delta,$$

for some large enough τ (cf. Figure 2-3). In the case of the MH OCP with discounted cost in Section 2.3.2 we found that

$$\tau \geq \max_{x \in B_{\|\bar{x}\|}(x^e)} \ln\left(\frac{\varphi_c(\|x - x^e\|)}{\varphi_o(\|x - x^e\|)}\right) =: K(\bar{x}) \tag{2.104}$$

is sufficient for a given initial state $\bar{x}$, if the prediction horizon T satisfies

$$T \geq \max\{T_c, T_o\} + \tau. \tag{2.105}$$

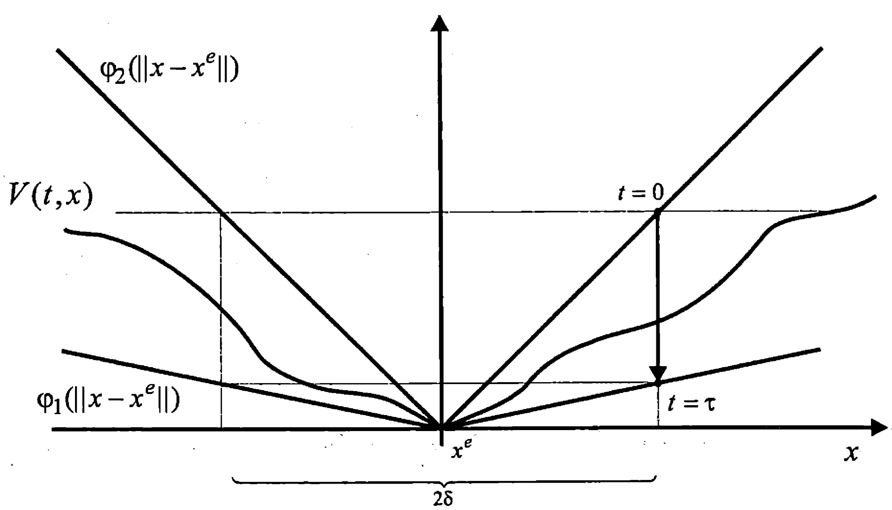

Figure 2-3: Time Horizon Robustness.

2.6 A Nonlinear Example

The following nonlinear ODE state space model describes a firm in a diffusive market environment that can chose its marketing expenditures $c_1 u_1$ and the price for its product u_2 :

$$\begin{aligned} \dot{x}_1 &= -\alpha_1 x_1 + u_1, & (2.106)\\ \dot{x}_2 &= D(x, u_2) - \beta x_2, & (2.107) \end{aligned}$$

where the demand D for the product is given by

$$D(x, u_2) := (x_{2\max} - \gamma u_2 - x_2)(\alpha_2 x_1 + \alpha_3 x_2). \tag{2.108}$$

The 'marketing effect' x_1 is modeled to follow the control

$$u_1 \in \Omega_1 = [0, \infty), \tag{2.109}$$

via a first-order lowpass with characteristic time $1/\alpha_1$ and to impact linearly on $\dot{x}_2$, the rate of change of the 'products in use'. More specifically, $x_2(t)$ represents the installed base at time t, which can never exceed the market potential $x_{2\max}$. The products are assumed to have a finite 'average' life-time $(1/\beta)$, so that for a vanishing demand the installed base decays to zero exponentially at the rate $\beta \geq 0$. In addition, the demand cannot be negative and the choice of the price is therefore confined to

$$u_2 \in \Omega_2(x_2) = [0, (x_{2\max} - x_2)/\gamma]. \tag{2.110}$$

The optimal control problem of maximizing at time $t \geq 0$ discounted profits Π_t, or equivalently minimizing discounted cost $J_t := -\Pi_t$ can therefore be written in the form

$$J_t = \int_t^\infty e^{-r\theta} \Big(c_1 u_1 - u_2 D(x, u_2) \Big) \, d\theta \longrightarrow \min, \tag{2.111}$$

subject to (2.106)–(2.110), for all admissible initial states $(x_1(t), x_2(t)) =: (\bar{x}_1, \bar{x}_2)$ given by

$$(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2 := [0, \infty) \times [0, x_{2\max}]. \tag{2.112}$$

All the parameters $\alpha_i, c_1, \gamma, r, x_{2\max}$ are given positive constants, whereby we will set $\gamma = 1$ henceforth. This choice of γ can be made without loss of generality, since the system equations (2.106)–(2.107) for $\gamma \neq 1$ can be reduced to the case $\gamma = 1$ by considering $\tilde{x}_2 := x_2/\gamma$ and $\tilde{x}_{2\max} := x_{2\max}/\gamma$ instead of x_2 and $x_{2\max}$ in equation (2.107).

Below, we will find the only optimal equilibrium state using Lemma 1. Then complete controllability will be established in the whole part of the state space that can contain candidates for optimal equilibrium states. It will be shown that this subset C can be reached from almost any other state in the state space and estimates for the time to steer the system to an optimal state will be provided. Such estimates can be made independent of initial and final state of a trajectory on any connected compact subset $\bar{Y}$ of C.

Optimal Equilibrium State. Determining an optimal equilibrium state — in the sense that once such a state has been reached, it cannot be improved upon — by using PMP or Bellman's equation is not an easy task. Instead we try to apply Lemma 1. From equation (2.107) one obtains

$$u_2 = -\frac{\dot{x}_2 + \beta x_2}{\alpha_2 x_1 + \alpha_3 x_2} + x_{2\max} - x_2,$$

$\forall$ admissible $(x_1, x_2) \neq 0$. Defining $v_2 := \dot{x}_2$ we have that the function $\zeta : ([\epsilon_1, \infty) \times [\epsilon_2, x_{2\max}]) \times \mathbb{R} \to \mathbb{R}$,

$$\zeta(x, v_2) = -\frac{v_2 + \beta x_2}{\alpha_2 x_1 + \alpha_3 x_2} + x_{2\max} - x_2, \tag{2.113}$$

is well-defined for any fixed, small enough $\epsilon_1, \epsilon_2 \geq 0$ such that $\epsilon_1 + \epsilon_2 > 0$.

The cost J_t in (2.111) can be re-written using (2.106) as follows:

$$\begin{aligned} J_t &= \int_t^\infty e^{-r\theta} \left(c_1(\dot{x}_1 + \alpha_1 x_1) - u_2 D(x, u_2)\right) d\theta \\ &= \int_t^\infty \left[e^{-r\theta} \left(c_1(r + \alpha_1)x_1 - u_2 D(x, u_2)\right) + c_1 \frac{d}{d\theta}\left(e^{-r\theta} x_1\right)\right] d\theta \\ &= \int_t^\infty e^{-r\theta} h(x, u_2)\, d\theta - c_1 e^{-rt} \bar{x}_1, \end{aligned} \tag{2.114}$$

where in (2.114) we have introduced a cost kernel

$$h(x, u_2) := \tilde{c}_1 x_1 - u_2 D(x, u_2), \tag{2.115}$$

with $\tilde{c}_1 := c_1(r + \alpha_1)$ and $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Substituting the expression for $u_2 = \zeta(x, v_2)$ into the cost kernel $h(x, u_2)$ yields

$$h(x, \zeta(x, v_2)) = \tilde{c}_1 x_1 + \frac{(v_2 + \beta x_2)^2}{\alpha_2 x_1 + \alpha_3 x_2} - (x_{2\max} - x_2)(v_2 + \beta x_2),$$

which can be minimized with respect to $x_1 \in X_1$ so as to obtain

$$\min_{x_1 \in X_1} \{h(x, \zeta(x, v_2))\} = \phi(x_2, v_2),$$

where

$$\begin{aligned} \phi(x_2, v_2) &= 2\sqrt{\frac{\tilde{c}_1}{\alpha_2}}(v_2 + \beta x_2) - \frac{\alpha_3 \tilde{c}_1}{\alpha_2} x_2 - (x_{2\max} - x_2)(v_2 + \beta x_2) \\ &= (k_1 x_2 + \beta x_2^2) + (k_2 + x_2) v_2, \end{aligned}$$

and the constants k_1, k_2 given by

$$\begin{aligned} k_1 &= 2\sqrt{\frac{\tilde{c}_1}{\alpha_2}}\beta - \alpha_3 \frac{\tilde{c}_1}{\alpha_2} - \beta x_{2\max}, \\ k_2 &= 2\sqrt{\frac{\tilde{c}_1}{\alpha_2}} - x_{2\max}. \end{aligned}$$

The function is $\phi : X \to \mathbb{R}$ is well-defined by the above relations, and in particular we have that ϕ is of the form

$$\phi(x_2, v_2) = (k_1 x_2 + \beta x_2^2) + \frac{d}{dx_2}\left(k_2 x_2 + \frac{x_2^2}{2}\right) v_2, \tag{2.116}$$

and $(k_1 + rk_2)x_2 + (\beta + \frac{r}{2})x_2^2$ bounded from below on X_2, so that all assumptions of Lemma 1 are satisfied. Therefore

$$J_t \geq \left(\frac{m}{r} - (c_1\bar{x}_1 + k_2\bar{x}_2 + \frac{\bar{x}_2^2}{2})\right) e^{-rt}, \tag{2.117}$$

$\forall t \geq 0$, where

$$m = \min_{x_2 \in X_2} \{(k_1 + rk_2)x_2 + (\beta + \frac{r}{2})x_2^2\}, \tag{2.118}$$

and the only candidate for an optimal equilibrium state $x^o = \begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix}$ is determined by[21]

$$\begin{aligned} x_2^o &= \arg\min_{x_2 \in X_2} \{(k_1 + rk_2)x_2 + (\beta + \frac{r}{2})x_2^2\} = \left[-\frac{k_1 + rk_2}{2\beta + r}\right]_{X_2} \\ &= \left[\frac{(\beta + r)(x_{2\max} - 2\sqrt{\frac{c_1(r+\alpha_1)}{\alpha_2}}) + c_1(r + \alpha_1)\frac{\alpha_3}{\alpha_2}}{2\beta + r}\right]_{X_2}, \end{aligned} \tag{2.119}$$

and

$$\begin{aligned} x_1^o &= \arg\min_{x_1 \in X_1} \{h(\begin{bmatrix} x_1 \\ x_2^o \end{bmatrix}, \zeta(\begin{bmatrix} x_1 \\ x_2^o \end{bmatrix}, 0))\} \\ &= \left[x_2^o \left(\frac{\beta}{\sqrt{\alpha_2 c_1 (r + \alpha_1)}} - \frac{\alpha_3}{\alpha_2}\right)\right]_{X_1}. \end{aligned} \tag{2.120}$$

These candidates are indeed equilibria, if there are $u_1 \in \Omega_1$ and $u_2 \in \Omega_2(x_2^o) = [0, x_{2\max} - x_2^o]$ such that

$$\begin{aligned} u_1 &= \alpha_1 x_1^o, \\ u_2 &= \begin{cases} \zeta(x^o, 0) & \text{if } x_2^o \neq 0, \\ \text{free} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.121}$$

This is the case if and only if

$$\zeta(x^o, 0) = -\frac{\beta x_2^o}{\alpha_2 x_1^o + \alpha_3 x_2^o} + x_{2\max} - x_2^o \geq 0. \tag{2.122}$$

[21] For convenience we introduce a projection function $[\cdot]_I : \mathbb{R} \to \mathbb{R}$ for an interval I as follows:

$$[\xi]_I = \begin{cases} \inf I & \text{if } \xi \leq \inf I \\ \sup I & \text{if } \xi \geq \sup I \\ \xi & \text{otherwise.} \end{cases}$$

Controllability. The system (2.106)-(2.107) is controllable on an open set $\tilde{X} \subset X_1 \times X_2$, if for any $x^i, x^f \in \tilde{X}$ there exist a finite constant $\tau = \tau(x^i, x^f) \geq 0$ and an admissible control $u(t) \in \Omega_1 \times \Omega_2(x_2(t))$, $0 \leq t \leq \tau$ such that it evolves from the initial state $x(0)$ to the final state $x(\tau) = x^f$ when u is applied. — The system is uniformly controllable if the constant τ does not depend on x^i, x^f.

Consider now our system on the interior of the set of all admissible states, $\tilde{X} := \text{int}\{X_1 \times X_2\}$. At each point $x \in \tilde{X}$ the possible directions, in which the state can evolve, lie inside a cone that is described by a set of candidate vectors for 'extremal directions',

$$\left\{ \begin{bmatrix} v_{1\,\min} \\ v_{2\,\min} \end{bmatrix}, \begin{bmatrix} v_{1\,\max} \\ v_{2\,\min} \end{bmatrix}, \begin{bmatrix} v_{1\,\min} \\ v_{2\,\max} \end{bmatrix}, \begin{bmatrix} v_{1\,\max} \\ v_{2\,\max} \end{bmatrix} \right\} \Bigg|_{x=(x_1,x_2)'},$$

where we define $v_{j\,\max} := \sup_{u_j \in \Omega_j} \{\dot{x}_j\}$ and $v_{j\,\min} := \inf_{u_j \in \Omega_j} \{\dot{x}_j\}$ for $j = 1, 2$. In the case of system (2.106)-(2.107) one obtains

$$\begin{aligned} v_{1\,\min} &= -\alpha_1 x_1, & v_{2\,\min} &= -\beta x_2, \\ v_{1\,\max} &= +\infty, & v_{2\,\max} &= -\beta x_2 + (x_{2\,\max} - x_2)(\alpha_2 x_1 + \alpha_3 x_2), \end{aligned}$$

$\forall\,(x_1, x_2) \in \tilde{X}$.

From this it becomes clear that it is always possible to find a control that steers the system from x^i to x^f in finite time, whenever

$$x^i, x^f \in \{x \in \tilde{X} \,|\, v_{1\,\min} < 0 < v_{1\,\max} \textbf{ and } v_{2\,\min} < 0 < v_{2\,\max}\} =: C(\tilde{X}),$$

since then there is an admissible control that can move the system in any desired direction[22]. More specifically, one obtains that in the example

$$C(\tilde{X}) = \{x \in \tilde{X} \,|\, x_1 > \frac{x_2}{\alpha_2}\left(\frac{\beta}{x_{2\,\max} - x_2} - \alpha_3\right)\}. \tag{2.123}$$

Trajectories starting at states x^i anywhere in $\tilde{X}$ can reach final states x^f in

$$\begin{aligned} R(\tilde{X}; x^i) &= C(\tilde{X}) \cup \{x \in \tilde{X} \,|\, \bar{x}_1(t) \geq x_1,\ x_2^i > x_2, \textbf{ and} \\ &\qquad \bar{x}(t) \textbf{ solution of } \left\{ \begin{array}{l} \dot{\bar{x}}_1 = v_{1\,\min} \\ \dot{\bar{x}}_2 = v_{2\,\max} \end{array}, \bar{x}(0) = x^i \right\}, t \geq 0\}. \end{aligned} \tag{2.124}$$

Other reachable and controllable states may lie on $\partial R(\tilde{X}; x^i)$ and $\partial C(\tilde{X})$ respectively, but these marginal cases will not be treated here, as such states may in general not be reached in finite time, and are of no particular importance for our further discussion. Hence all the statements below will be given 'modulo' the boundary of the set in consideration. — *Uniform* controllability can be established for any compact subset $\bar{Y} \subset C(\tilde{X})$, and τ is then the maximum time needed to connect two points in $\bar{Y}$. This time must be finite, since otherwise some states in $C(\tilde{X})$ could not be reached from $\bar{Y}$.

[22]Note that $C(\tilde{X})$ is simply connected, because $\tilde{X}$ is.

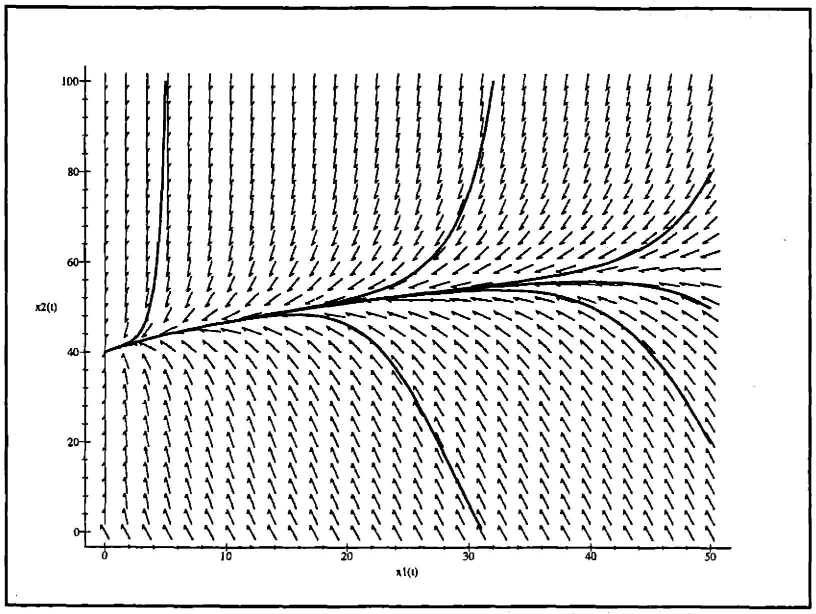

Figure 2-4: Phase Diagram for $\beta < \alpha_3 x_{2\,\max}$.

We note at this point that the set $C(\tilde{X})$ or even its closure $\bar{C}(\tilde{X})$ does in general *not* contain all controllable states. However, it turns out that it contains all controllable states as $\alpha_1 \to 0^+$. Consider Figures 2-4 and 2-5 that show the phaseportrait of the system

$$\begin{cases} \dot{x}_1 = v_{1\,\min} \\ \dot{x}_2 = v_{2\,\max} \end{cases}$$

as used in the definition of $R(\tilde{X}; x^i)$ in (2.124).

The set $\bar{C}(\tilde{X})$ contains necessarily all nonzero candidates for optimal equilibrium states as determined by equations (2.119)–(2.121), since condition (2.122) corresponds to the expression of $C(\tilde{X})$ in equation (2.123).

Let us now estimate a time constant $\tau_u(\bar{Y})$ for uniform controllability as a function of the size of the set $\bar{Y} \subset C(\tilde{X})$. We assume that the target state x^f is in $C(\tilde{X})$, while the initial state $x^i \in \tilde{X}$.

Clearly, an upper bound for $\tau_u(\bar{Y})$ can be obtained when computing the time needed to go from one state x^i to another state x^f (both in $\bar{Y}$) using some *particular* control strategy. We will distinguish two cases, namely $x_2^f \leq x_2^i$ and $x_2^f > x_2^i$:

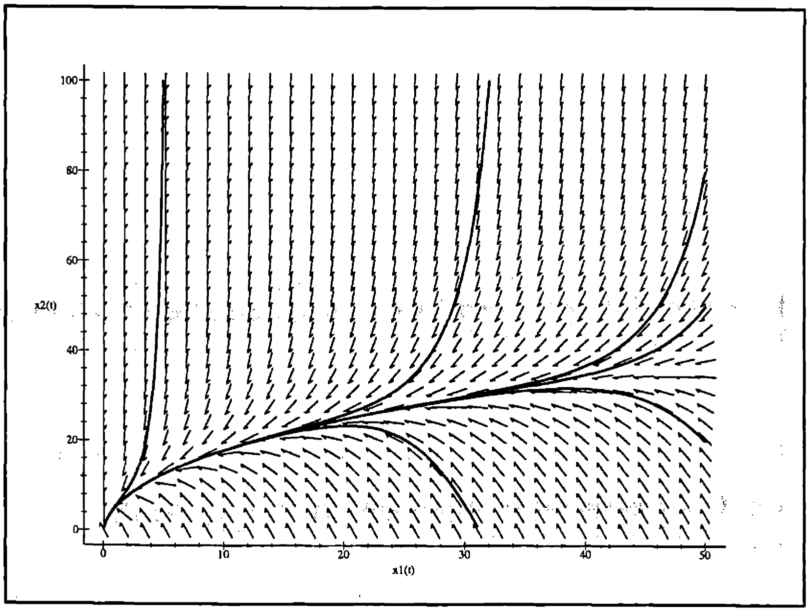

Figure 2-5: Phase Diagram for $\beta \geq \alpha_3 x_{2\,\max}$.

(i) $\boxed{x_2^f \leq x_2^i}$ Considering the phase diagrams (Figures 2-4 and 2-5),[23] strategy should be to first adjust x_2 from x_2^i to x_2^f while holding x_1 constant, and then to move x_1 from x_1^i to x_1^f while holding x_2 constant. This is always possible since $v_{2\,\min} = -\beta x_2 < 0$ is independent of x_1 and $v_{2\,\max} > 0$ in $C(\tilde{X})$. An appropriate control input is then:

$$u_1(t) = \begin{cases} (x_1^f - x_1^i)\delta(t - \frac{1}{\beta}\ln\left(\frac{x_2^i}{x_2^f}\right)), & \text{if } x_1^f \geq x_1^i, \\ \alpha_1 x_1^f \varepsilon(-t + \frac{1}{\beta}\ln\left(\frac{x_2^i}{x_2^f}\right)) & \text{if } x_1^f < x_1^i, \end{cases} \tag{2.125}$$

$$\begin{aligned} u_2(t) &= \left(x_{2\,\max} - x_2^i e^{-\beta t}\right)\varepsilon(-t + \frac{1}{\beta}\ln\left(\frac{x_2^i}{x_2^f}\right)) \\ &\quad + \left(\frac{\beta x_2^f}{\alpha_2 x_1(t) + \alpha_3 x_2^f} - x_{2\,\max} + x_2^f\right)\varepsilon(t - \frac{1}{\beta}\ln\left(\frac{x_2^i}{x_2^f}\right)), \end{aligned} \tag{2.126}$$

for $0 \leq t \leq \tau_1(x^i, x^f) := \lambda(\frac{1}{\alpha_1}\ln\left(\frac{x_1^i}{x_1^f}\right)) + \frac{1}{\beta}\ln\left(\frac{x_2^i}{x_2^f}\right)$.

(ii) $\boxed{x_2^f > x_2^i}$ The strategy is here reversed. One first adjusts x_1, then x_2. This always works, since in this case both initial and final state are in the controllable region

[23]Maple V Release 4 has been used to generate the phase portraits, with parameter configurations $(\alpha_1, \alpha_2, \alpha_3, r, c_1, x_{2\,\max}) = (1, .05, .1, .1, .05, 100)$ and $\beta \in \{6, 11\}$. Maple is a registered trademark of Waterloo Maple, Inc.

$C(\tilde{X})$. A possible control input is given by:

$$u_1(t) = \begin{cases} (x_1^f - x_1^i)\delta(t) + \alpha_1 x_1^f \varepsilon(t), & \text{if } x_1^f \geq x_1^i, \\ \alpha_1 x_1^f \varepsilon(t - \frac{1}{\alpha_1}\ln\left(\frac{x_1^i}{x_1^f}\right)) & \text{if } x_1^f < x_1^i, \end{cases} \tag{2.127}$$

$$u_2(t) = \left(\frac{\beta x_2^f}{\alpha_2 x_1(t) + \alpha_3 x_2^f} - x_{2\max} + x_2^f\right)\varepsilon(t - \frac{1}{\alpha_1}\ln\left(\frac{x_1^i}{x_1^f}\right) - \tau_2'), \tag{2.128}$$

for $0 \leq t \leq \tau_2(x^i, x^f) := \tau_2'(x^i, x^f) + \lambda(\frac{1}{\alpha_1}\ln\left(\frac{x_1^i}{x_1^f}\right))$, where[24]

$$\tau_2'(x^i, x^f) = \frac{2}{\sqrt{K(x_1^f)}}\left[\operatorname{arctanh}\left(\frac{2x_2^f\alpha_3 + \alpha_2 x_1^f + \beta - \alpha_3 x_{2\max}}{\sqrt{K(x_1^f)}}\right) - \operatorname{arctanh}\left(\frac{2x_2^i\alpha_3 + \alpha_2 x_1^f + \beta - \alpha_3 x_{2\max}}{\sqrt{K(x_1^f)}}\right)\right],$$

and

$$K(x_1^f) = 4\alpha_3\alpha_2\, x_{2\max}\, x_1^f + (\alpha_3 x_{2\max} - \alpha_2 x_1^f - \beta)^2 > 0.$$

An estimate, i.e., upper bound, for $\tau(x^i, x^f)$ is then

$$\hat{\tau}(x^i, x^f) = \max\{\tau_1, \tau_2\} = \lambda(\frac{1}{\alpha_1}\ln\left(\frac{x_1^f}{x_1^f}\right)) + \max\{\frac{1}{\beta}\ln\left(\frac{x_2^i}{x_2^f}\right), \tau_2'(x^i, x^f)\}. \tag{2.129}$$

This estimate can be made independent of x^i and x^f by considering the following compact subset $\bar{Y}$ of $C(\tilde{X})$ (cf. Figure 2-6):

$$\begin{aligned} \bar{Y} = \{x \in \tilde{X} \,|\, 0 < \chi_{1\min} \leq \; & x_1 \; \leq \chi_{1\max} < \infty, \\ 0 < \chi_{2\min} \leq \; & x_2 \; \leq \frac{\alpha_3 x_{2\max} - \beta - \alpha_2\chi_1 + (1-\epsilon)\sqrt{K}}{2\alpha_3}\}, \end{aligned} \tag{2.130}$$

for given positive constants $\chi_{1\min}$, $\chi_{1\max}$, $\chi_{2\min}$, and ϵ. For the set $\bar{Y}$ to be nonempty, one must have $\chi_{1\min} \leq \chi_{1\max}$, and for any $x_1 \in [\chi_{1\min}, \chi_{1\max}]$ that

$$0 < \epsilon < 1 - \frac{|2\alpha_3\chi_{2\min} + \alpha_2 x_1 + \beta - \alpha_3 x_{2\max}|}{\sqrt{K(x_1)}}.$$

---

[24]From $\dot{x}_2 = v_{2\max}$ one obtains for $x_1 = x_1^i$ that $\tau_2' = \int_{x_2^i}^{x_2^f} \frac{dx_2}{(x_{2\max} - x_2)(\alpha_2 x_1^i + \alpha_3 x_2) - \beta x_2}$. In addition, for constants a, b, c with $a > 0$, $c \geq 0$ we have $\int_{\xi_1}^{\xi_2} \frac{d\xi}{-a\xi^2 + b\xi + c} = \frac{2}{\sqrt{4ac+b^2}}\operatorname{arctanh}\left(\frac{2a\xi - b}{\sqrt{4ac+b^2}}\right)$ $\forall [\xi_1, \xi_2] \subset (\frac{b - \sqrt{4ac+b^2}}{2a}, \frac{b + \sqrt{4ac+b^2}}{2a})$. In this particular case the constants are given as $a = \alpha_3$, $b = \alpha_3\, x_{2\max} - \alpha_2 x_1^f - \beta$, and $c = \alpha_2\, x_{2\max}\, x_1^f$.

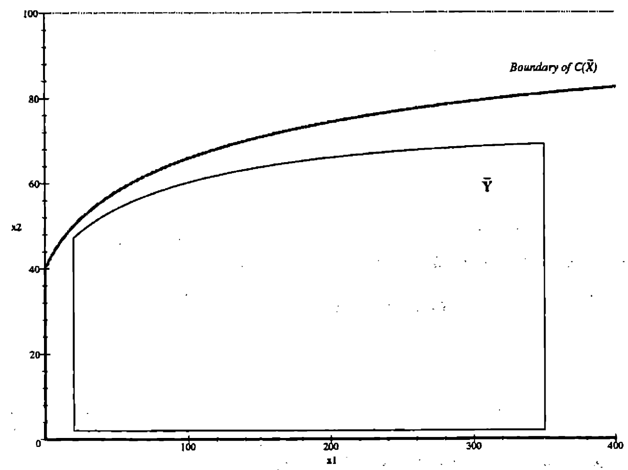

Figure 2-6: The Region $\bar{Y} \subset C(\tilde{X})$.

This can be guaranteed by choosing $\epsilon > 0$ such that

$$\epsilon < 1 - \max\left\{ \frac{|2\alpha_3\chi_{2\min} - \alpha_3 x_{2\max} + \beta|}{\sqrt{K(0)}}, \frac{|2\alpha_3\chi_{2\min} + \alpha_2 x_1 + \beta - \alpha_3 x_{2\max}|}{\sqrt{K(\chi_{1\max})}} \right\}. \quad (2.131)$$

From the above it is clear that an estimated upper bound $\hat{\tau}_u$ for the minimum time needed to steer the system (2.106)–(2.107) from one state to another, cannot become smaller as the diameter of $\bar{Y}$ increases. Thus, to ensure that a number of extremal points

$$\{(\bar{x}_1^1, \bar{x}_2^1)', (\bar{x}_1^2, \bar{x}_2^2)', \ldots, (\bar{x}_1^N, \bar{x}_2^N)'\} \subset C(\tilde{X})$$

be included in $\bar{Y}$, while its diameter stays relatively small, one can choose the above constants as follows[25]:

$$\begin{aligned} \chi_{1\min} &= \min\{\bar{x}_1^1, \ldots, \bar{x}_1^N\}, \qquad & \chi_{2\min} &= \min\{\bar{x}_2^1, \ldots, \bar{x}_2^N\}, \\ \chi_{1\max} &= \max\{\bar{x}_1^1, \ldots, \bar{x}_1^N\}, \end{aligned}$$

In addition to condition (2.131) the constant ϵ has to satisfy

$$0 < \epsilon < 1 - \frac{|2\alpha_3\chi_2 + \alpha_2\chi_{1\max} + \beta - \alpha_3 x_{2\max}|}{\sqrt{K(\chi_1)}},$$

where

$$(\chi_1, \chi_2) := \arg \max_{(\bar{\xi}_1, \bar{\xi}_2) \in \{\bar{x}^1, \ldots, \bar{x}^N\}} \bar{\xi}_2.$$

[25] One could also choose the convex hull of the extremal points. However the set $\bar{Y}$ as specified here is simpler and appears very natural for the given problem.

With this We can finally give an upper bound for τ_u on $\bar{Y}$:

$$\boxed{\hat{\tau}_u(\bar{Y}) = \frac{1}{\alpha_1}\ln\left(\frac{\chi_{1\,\max}}{\chi_{1\,\min}}\right) + \max\{\frac{1}{\beta}\ln\left(\frac{\chi_{2\,\max}}{\chi_{2\,\min}}\right), \bar{\tau}_2'\}} \tag{2.132}$$

where

$$\chi_{2\,\max} := \frac{\alpha_3 x_{2\,\max} - \beta - \alpha_2\chi_{1\,\max} + (1-\epsilon)\sqrt{K(\chi_{1\,\max})}}{2\alpha_3},$$

and

$$\bar{\tau}_2' = \frac{4\ \text{arctanh}\,(1-\epsilon)}{\sqrt{K(\chi_{1\,\min})}}. \tag{2.133}$$

Numerical Example. Consider the case, when $\alpha_1 = 1$, $\alpha_2 = .05$, $\alpha_3 = .1$, $\beta = .6$, $r = .1$, $c_1 = .05$, and $x_{2\,\max} = 100$. The (only) optimal equilibrium state $x^o \in X_1 \times X_2 = [0,\infty) \times [0, x_{2\,\max}]$ is then given by equations (2.119)–(2.120), and thus

$$\begin{aligned} x_2^o &= \left[\frac{(\beta + r)(x_{2\,\max} - 2\sqrt{\frac{c_1(r+\alpha_1)}{\alpha_2}}) + c_1(r+\alpha_1)\frac{\alpha_3}{\alpha_2}}{2\beta + r}\right]_{X_2} &= 52.8,\\ x_1^o &= \left[x_2^o\left(\frac{\beta}{\sqrt{\alpha_2 c_1(r+\alpha_1)}} - \frac{\alpha_3}{\alpha_2}\right)\right]_{X_1} &= 498.5, \end{aligned}$$

with optimal equilibrium controls $(u_1^o, u_2^o) \in [0,\infty) \times [0, x_{2\,\max} - x_2^o]$:

$$\begin{aligned} u_1^o &= \alpha_1 x_1^o = 498.5,\\ u_2^o &= \zeta(x^o, 0) = 46.15. \end{aligned}$$

Furthermore we have, using Lemmas 1 and 2, that

$$\begin{aligned} \tilde{h}(x, v_2) &= \Big(q(x_2) + r\psi(x_2) - m\Big) + \Big(h(x, \zeta(x, v_2)) - \phi(x_2, v_2)\Big)\\ &= \Big(-68.64\ x_2 + .65\ x_2^2 + 1812.18\Big)\\ &\quad + \bigg(.055\ x_1 - \bigg(-\frac{.6\ x_2 + v_2}{.05\ x_1 + .1\ x_2} + 100 - x_2\bigg)(.6\ x_2 + v_2)\\ &\quad - \big(-58.85\ x_2 + .6\ x_2^2 + (-97.9 + x_2)v_2\big)\bigg) \qquad (2.134)\\ &= \frac{(.6\ x_2 + v_2)^2}{.05\ x_1 + .1\ x_2} - 2.1\ v_2 + .65\ x_2^2 - 69.79\ x_2 + .055\ x_1 + 1812.18\\ &\geq 0, \end{aligned}$$

$\forall\,(x, v_2) \in X \times \mathbb{R}$. — A contourplot of $\tilde{h}$ is given in Figure 2-7.

Let $\bar{Y}$ be defined by $\chi_{1\,\min} = x_1^o/1000$, $\chi_{1\,\max} = 2\ x_1^o$, $\chi_{2\,\min} = x_2^o/100$. Then

$$\epsilon := .1231 < 1 - \max\left\{\frac{|-9.29|}{\sqrt{112.36}}, \frac{|40.56|}{\sqrt{3534.87}}\right\}$$

satisfies the condition (2.131), and the time to reach points in $\bar{Y}$ for can be bounded from

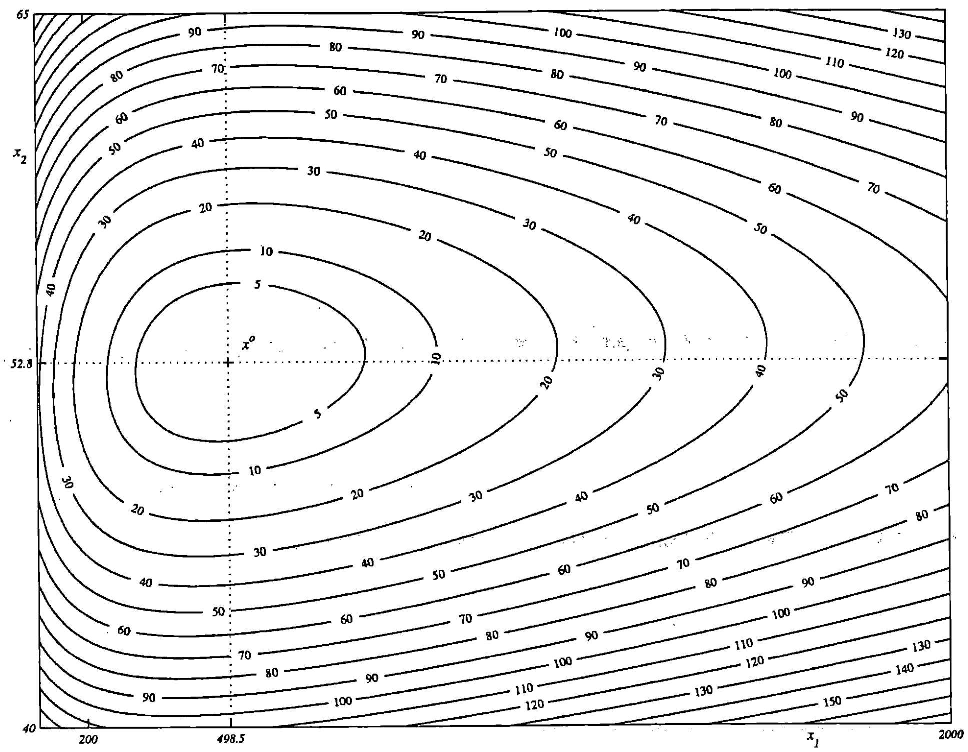

Figure 2-7: Contourplot of the Positive Semi-Definite Cost Kernel $\tilde{h}(x, 0)$ in (2.134).

above, using (2.132)–(2.133), by

$$\begin{aligned}\hat{\tau}_u(\bar{Y}) &= \ln(200) + \max\left\{\frac{1}{.6}\ln\left(\frac{58.42}{.528}\right), \frac{4\ \mathrm{arctanh}(1 - .1231)}{\sqrt{.4985}}\right\} \\ &= 5.298 + \max\{7.844, 7.717\} \;=\; 13.142.\end{aligned}$$

Stability Issues. The strong assumptions under which we have proven (semi-formally) asymptotic convergence in Section 2.3.2 are quite hard to verify in practice, even for small-scale nonlinear systems such as the one given by (2.106)–(2.107). To illustrate this fact, we will outline a proof that the positive semi-definite cost kernel $\tilde{h}$ obtained above is bounded from below by a $\mathcal{K}_\infty$ function. For this, note that

$$\begin{aligned}\tilde{h}(x, v_2) &= \Big(q(x_2) + r\psi(x_2) - m\Big) + \Big(h(x, \zeta(x, v_2)) - \phi(x_2, v_2)\Big) \\ &= \left(\beta + \frac{r}{2}\right)(x_2 - x_2^o)^2 + \Big(h(x, \zeta(x, v_2)) - \phi(x_2, v_2)\Big) \qquad (2.135) \\ &\geq 0.\end{aligned}$$

In particular, if there is $\epsilon > 0$ such that

$$0 \leq \tilde{h}(x, v_2) \leq \epsilon,$$

for some $(x, v_2) \in X \times \mathbb{R}$, then one can conclude that necessarily

$$|x_2 - x_2^o| \leq \epsilon,$$

and (x_1, v_2) lie inside a bounded volume,

$$L(\epsilon): \quad \Big\{(x, v_2) \in X \times \mathbb{R} \,\Big|\, h(x, \zeta(x, v_2)) - \phi(x_2, v_2) \leq \epsilon \text{ and } |x_2 - x_2^o| < \epsilon \text{ and } v_2 \geq -\beta(x_2^o + \epsilon)\Big\}. \tag{2.136}$$

One can think of the small volume $L(\epsilon)$ as being an ϵ-neighborhood of the the line-piece

$$\left\{(x, v_2) \,\middle|\, h(\begin{bmatrix} x_1 \\ x_2^o \end{bmatrix}, \zeta(\begin{bmatrix} x_1 \\ x_2^o \end{bmatrix}, v_2)) = \phi(x_2^o, v_2) \text{ and } x_1 \geq 0 \text{ and } v_2 \geq -\beta(x_2^o + \epsilon)\right\} \subset P,$$

where P is the plane defined in the proof of Lemma 1 on page 50. Under the assumption that

$$x_2 \to x_2^o \quad \Rightarrow \quad v_2 \to 0 \tag{2.137}$$

the diameter of the volume $L(\epsilon)$ does converge to zero as $\epsilon \to 0^+$, i.e.,

$$\lim_{\epsilon \to 0^+} \operatorname{diam} L(\epsilon) = \lim_{\epsilon \to 0^+} \sup_{a,b \in L(\epsilon)} \|a - b\| = 0. \tag{2.138}$$

The supplementary condition (2.137) is justified, if v_2 is assumed to be smooth, and thus implicitly that u_1 is bounded. Thus the second term in (2.135) can be bounded by a $\mathcal{K}_\infty$ function, which completes the proof-outline.

To get a $\mathcal{K}_\infty$ function that can serve as an *upper* bound for $\tilde{h}$ is considerably more involved and beyond our scope here. Verifying properties $\bar{\mathbf{C}}$ and $\bar{\mathbf{O}}$ may at times be even more difficult. One possbility to verify property $\bar{\mathbf{C}}$ is to compute the cost for a particular control strategy from a given initial state $\bar{x}$ to the optimal equilibrium state x^o (as a function of $\bar{x}$ and x^o). On pages 58–59 we have given a suitable control implementation that can be used to steer the system to any reachable state.[26]

Choice of the Planning Horizon. To find lower bounds for the control horizon τ and the prediction horizon T that guarantee stability of the corresponding optimal MH policy (cf. Section 2.5) has proved quite difficult, as discussed above. However, it is possible to state that the prediction horizon T should be no less than the shortest time needed to steer the system from a given initial state to the optimal equilibrium state using admissible control inputs. For the bounded region $\bar{Y}$ defined in (2.130), the time-estimate $\hat{\tau}_u(\bar{Y})$ in (2.132) is therefore a good indicator. For our numerical example, we conclude that

$$T \geq \hat{\tau}_u(\bar{Y}) = 13.142$$

is likely to be a good choice. Indeed, simulation using the Riots software (cf. [Sch96]) in-

[26]Numerical/symbolic computations performed by the author using Maple proved to be very complex and had to be performed for every quadrant in Figure 2-7 separately. However no simple enough results could be obtained appropriate for an inclusion in this Master's thesis.

dicates that we obtain instability for $(\tau, T) = (2, 10)$ and stability for $(\tau, T) = (5, 15)$ (cf. Figure 2-8).

Simulation. Simulation results[27] for $T = 15$ and $\tau = 5$ for $t \in [0, 20]$ are shown in Figure 2-8. We have thereby used a discretization grid with 300 spline break-points and a maximum of 100 iterations per finite horizon optimization. As a reference the optimal finite horizon policy over the entire time interval [0,20] (with fixed terminal state x^o and initial state $x(0) = (0, 5)')$ is depicted in Figure 2-9 on the next page.[28]

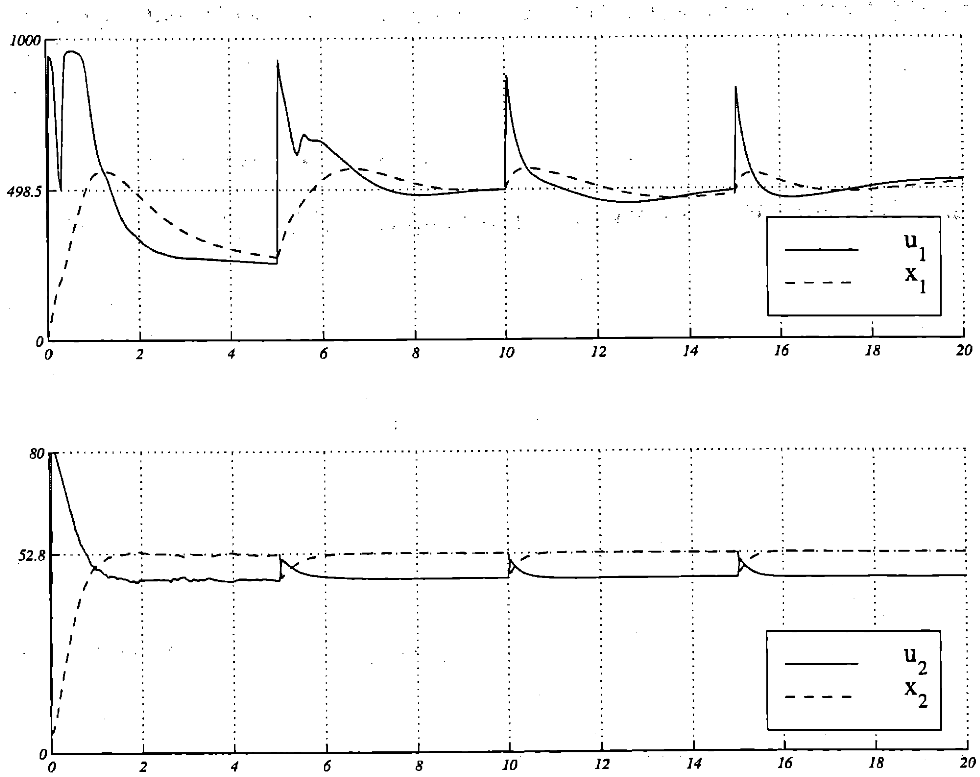

Figure 2-8: Optimal MH Control and State Trajectories for $(\tau, T) = (5, 15)$.

2.7 Notes and Sources

The Basics. A good introduction to the theory of nonlinear continuous-time systems is provided by [Kha92], and [Per96]. Stability issues for discrete-time systems are discussed in [Wil70].

Optimal Control. The classical reference for the treatment of necessary optimality conditions to finite-horizon OCPs remains [PBGM62]. An elementary discussion of the main

[27] For the initial state $x(0) = (0, 5)'$, the MH optimal cost over the time interval $[0, 20]$ was obtained to: -11,480.18.

[28] The "end-effects" in the finite horizon optimal trajectories are normal. An infinite horizon optimal state trajectory would naturally converge asymptotically towards the optimal equilibrium state x^o.

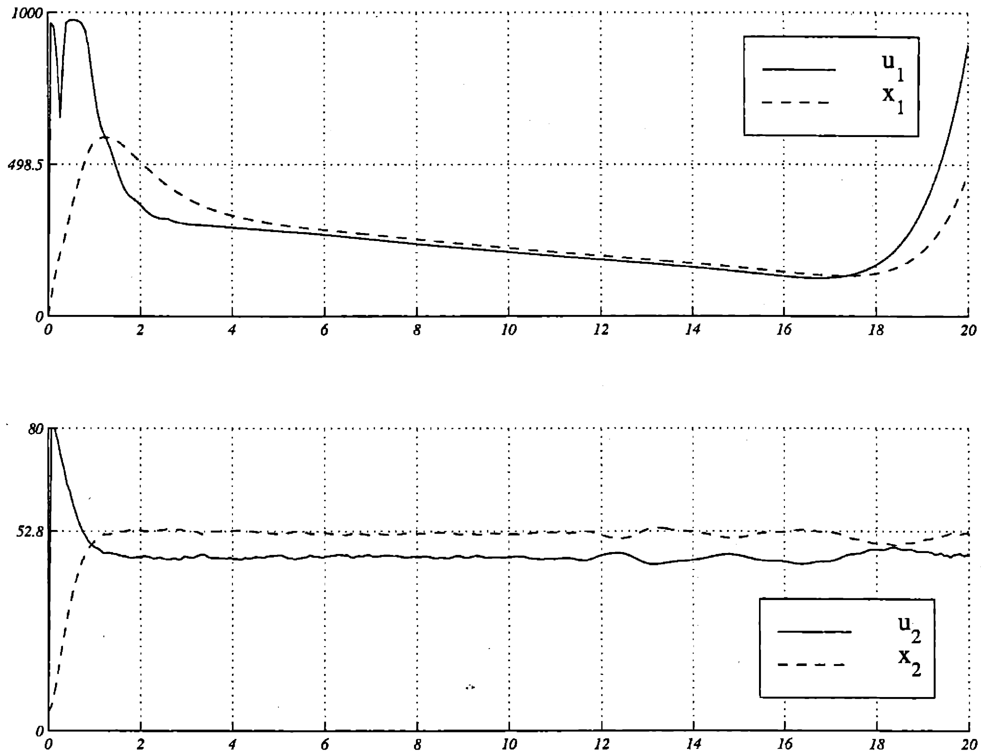

Figure 2-9: Optimal Finite Horizon Control and State Trajectories for $(\tau, T) = (20, 20)$.

practical issues, together with a brief introduction into optimal control theory, can be found in [Ber95]. For an advanced treatment of the HJB equation without the strong smoothness assumptions made here, the reader is referred to Clarke's books on nonsmooth analysis [Cla83], [Cla89], and to [Vin88] for a discussion of the relationship between HJB equation and the PMP. A comprehensive treatment of infinite horizon OCPs with special emphasis on economic problems is given in [CH87]. Shooting algorithms for numerically solving two-point boundary problems arising from the use of the PMP are discussed in [RS72], while direct solution methods based on Runge-Kutta iterative integration schemes are developed in [Sch96]. The software package Riots based on the latter developments by Schwartz has been used for numerical computations in this Master's thesis (cf. Section 2.2.2).

Constrained Predictive Control. Keerthi and Gilbert's results [KG88] have been discussed at length in the text. Their developments are in fact slightly more general, including a variable control horizon and output feedback. For linear systems and quadratic costs stability results for moving horizon controllers have been obtained earlier by [KP78], [KBK83]. Continuous-time results for nonlinear systems and quadratic costs have been established by Mayne and Michalska in [MM90], with nondifferentiable optimal cost-to-go function in [MM91], and under consideration of robustness issues in [MM93].

Diffusion Modeling.[29] The logistic diffusion model for consumer durables has been introduced by Frank M. Bass in the late 1960s [Bas69]. The diffusion models considered in

[29] cf. also the notes on information technology diffusion on page 91

the literature are usually one-dimensional and allow for a variety of parameters that can be influenced to change the particular growth characteristic. Good summary reviews are provided by [MMB90] and [Par94]. The diffusion model adopted here has been motivated by the system dynamics model in [PS93], although no formal modeling in the ODE state space form is known to the author. An analysis of a simpler one-dimensional model (equation (2.107) for $\alpha_2 = 0$, $\alpha_3 = \gamma = 1$) developed in [VC73] for optimal fish harvest is discussed in [CH87], pp. 39–42). To the knowledge of the author, the optimal equilibrium states for the diffusion model (2.106)–(2.107) have been provided here for the first time and no MH control schemes have been applied to this system before.

Imagination is more important than knowledge.
— ALBERT EINSTEIN

Chapter 3

Predictive Control for Corporate Policy

After having laid the theoretical foundation for the moving horizon predictive controller in the last chapter, we will now examine the applicability of this control methodology to nonsingular corporate policy as introduced in Section 1.1. To do this, we will firstly position nonsingular corporate policy (or critical resource allocation) within the area of corporate strategic planning as a whole. Then, constrained predictive control will be examined in detail for the use as a decision tool in critical resource allocation. For this we will focus on each of its elements, namely objective functional, modeling of the activity in question, and moving horizon control strategy. In the last section we will illustrate the methodology on a theoretical level for a manufacturing firm in a diffusive market environment. In the next chapter we will then indicate how the methodology may be of use in a real-world situation, and discuss the Boston Central Artery/Tunnel Project and its internal allocation of information technology resources.

3.1 Corporate Strategy and Critical Resource Allocation: The Context

In this section we will deal with Corporate Strategy only as far as it is indispensable for a general understanding of our critical resource allocation methodology and its potential practical relevance. The presentation is mainly based on Hax and Majluf's approach to corporate strategic planning, and for a more detailed treatment of the issues the reader is referred to their book and the references there. We will occasionally point to other sources in the substantial management literature available on the topic, where necessary.

Corporate Strategic Planning. Under corporate strategy we will understand here a coherent set of (strategic) decisions seeking to attain long-term superior financial performance of the firm. The strategic planning process takes place at several levels of the firm: the corporation or firm itself, the business unit level, and possibly the functional level. At each of these levels, strategic planning is based on both an environmental scan and an internal scrutiny. On the basis of the data sets resulting from these analyses, together with an overall mission statement, one attempts to identify and prioritize key strategic goals and a set of actions to achieve those goals. The last step in this process, usually conducted at the corporate level and then at the functional or business level, is the approval of a *resource*

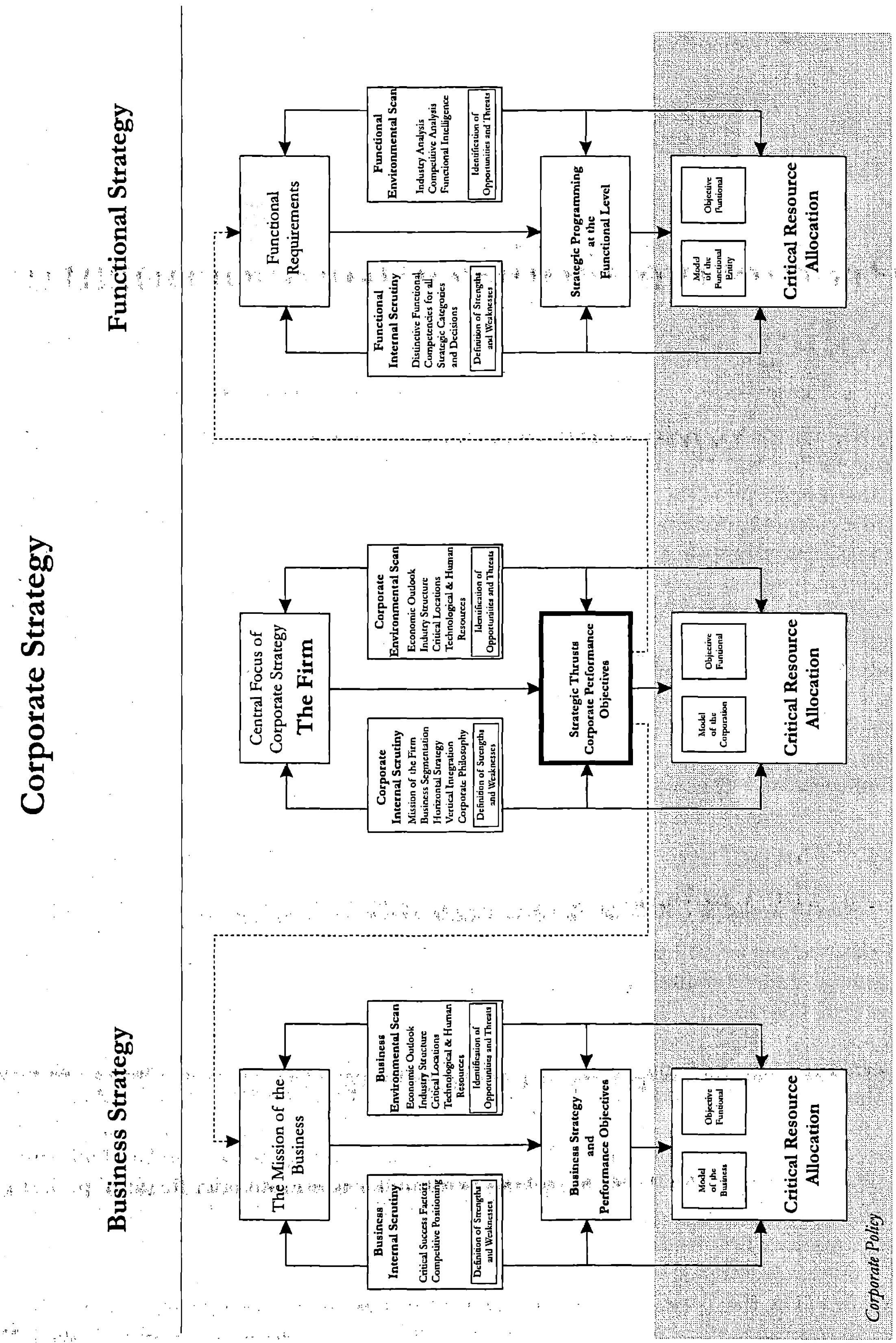

Figure 3-1: A Corporate Strategic Planning Process (adapted from [HM96]).

allocation plan, that ultimately defines what strategic actions will take place and how the necessary means for an implementation of the different measures will be rationed over time (cf. Figure 3-2). Since the outcome of intermediate steps of the strategic plan is usually not certain, such a resource allocation plan or corporate policy should be flexible. In fact, as a function of the current state of the firm and its environment, rather than a fixed rule for setting the decision variables in a pre-determined manner, corporate policy can deal efficiently with uncertainty. In this manner, strategic goals and performance objectives may be approximated even in the the presence of unexpected events.

The different elements of strategic planning, namely environmental scan, internal scrutiny, strategy, and corporate policy, will now be discussed in turn (cf. Figure 3-1).

Environmental Scan. The boundary conditions of the firm are investigated, as determined by the general economic outlook, the industry structure relevant for the firm's activities. Such an environmental scan also comprises an analysis of critical locations for production factors of the firm, as well as technological and human resources.

- Corporate level: After having fixed a time-frame for the analysis, one generally performs a geographical segmentation, so that the scan is conducted for each segment separately, if possible. Then an analysis of economic factors[1] (such as GNP growth, CPI, unemployment rate, population growth, disposable income, etc.) is performed with respect to their past development, together with projections for their future development. The next important step for an environmental scan at the corporate level is to identify and analyze primary industry sectors, critical for the firm's product-service offerings. From there one can devise alternative planning scenarios (or different sets of boundary conditions) that can be used as different parameter configurations for the modeling explained in Section 3.2.

- Business level: A structural industry analysis in the spirit of Porter [Por80] is in most cases appropriate. This type of analysis is standard and will not be detailed any further here. Complementary types of industry profiles such as strategic group and financial statement analyses are available (cf. [HM96], pp. 97–117).

- Functional level: The benchmarking of similar functions (such as manufacturing, service, finance, procurement, marketing, etc.) at the industry level is important for defining boundary conditions for a model of the firm at the level of a function.[2]

The outcome of an environmental scan is ideally a summary of opportunities and threats, as well as a fair characterization of the current state of relevant parameters for the firm's further development and its investment portfolio. The variables described here will be generally beyond the direct influence of the firm and can be considered as quasi-fixed[3] in the short and medium term when devising the first steps of a corporate strategy.

Internal Scrutiny. This structured process is intended to identify the potential strengths and weaknesses of the firm with respect to its competitors, and in particular to determine its competitive positioning with respect to the rest of the industry.

[1]GNP: Gross National Product; CPI: Consumer Price Index.
[2]Details can be found in [HM96], pp. 323–346.
[3]This does *not* mean time-invariant.

- Corporate level: The focus is here to identify opportunities to gain 'corporate competitive advantage' that is transferable to different business units.

- Business level: Porter's value chain framework (cf. [Por85]) is a useful tool to assess the competitive positioning of the firm and evaluate the business.

- Functional level: Each function has to meet the requirements imposed by the rest of the firm. Overall strengths and weaknesses as well as opportunities for gaining competitive advantage at the functional level should be identified here.

The outcome of the internal scrutiny is a set of strategic goals and performance objectives, that form the basis for a strategy development.

Strategy. After having identified the strengths and weaknesses of the firm and defined its opportunities and threats, it is time to devise a coherent set of actions aimed at superior financial performance of the firm. At each level the terminology is slightly different, but the content quite similar as performance objectives are defined, together with a broad action scheme to achieve these objectives. Such an action scheme is called *corporate policy.* We will focus here, as explained in Section 1.1 on a subset of corporate policies, namely those 'nonsingular' action schemes concerned with the ongoing allocation of critical resources (see below).

Singular vs. Nonsingular Corporate Policy. The distinction between singular and nonsingular corporate policy is arbitrary in a sense, since one might argue in both directions, that *corporate policy can be neither truly singular nor truly nonsingular.* We will use this terminology however, to distinguish budgeting for one-time/irregular events and what we term a critical resource allocation, which is the provision of a stream of decision variable settings designed to achieve strategic goals based on a model and expressed by an objective functional. Both corporate modeling and objective functional will be explained in detail in the sections that follow, together with the application of a moving horizon control strategy to approximate the infinite-horizon optimal objective.

3.2 Modeling the Firm

There are different methodologies that have established themselves for the modeling of socio-economic systems such as firms (or corporations). We will consider here three specific methods in more detail: causal loop diagrams, stock-and-flow modeling using System Dynamics software and mathematical modeling using ODE or FDE representations. Each method will be illustrated here with an ongoing example of a manufacturing firm in a diffusive market environment that builds upon the nonlinear example considered at the end of Chapter 2.

Causal Loop Diagrams. In this most elementary form of formalizing the understanding of cause-and-effect mechanisms in a firm, causal relationships between two variables A and $B(A, \ldots)$ are visualized by an arrow linking the two items,

$$A \longrightarrow B,$$

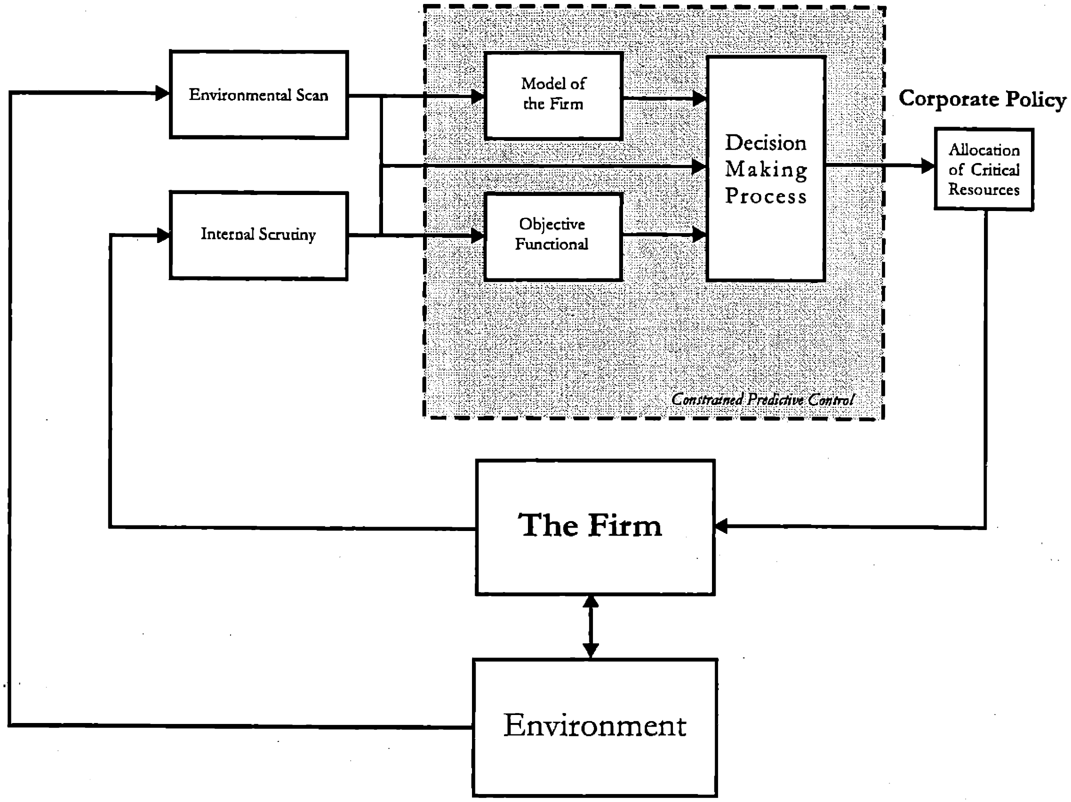

Figure 3-2: Nonsingular Corporate Policy Making.

and it is usually noted on the arrow if the influence of A on B is positive or negative, corresponding to the sign of $\partial B/\partial A$ on the relevant domain for A. If the the above sign is not unique on the domain considered, it is omitted (or replaced by a questionmark).

To illustrate the use of causal loop diagrams (CLDs), consider now a firm in a diffusive monopolistic market environment, analogous to our nonlinear example at the end of the last chapter. A possible CLD when considering only the two decision variables marketing expenditure and price is depicted in Figure 3-3.

Stocks and Flows: System Dynamics. Although causal loop diagrams provide a good intuitive understanding of causal relationships, it is generally not appropriate for simulation or optimization purposes, since it does not contain quantitative elements. By defining state variables as stocks and their time-derivatives as flows, one can visualize an ODE state space model in a so-called stock-and-flow diagram (SFD).[4] To illustrate this technique that is frequently used in management, and for which specialized software does exist,[5] consider an extended version of the diffusive market model, where the additional decision variables production and capacity on order are included. This extension of the model allows to consider a manufacturing firm that can chose to divide its critical resources between marketing,

[4]For the use in a software package, these equations are discretized yielding an FDE representation.
[5]Here we have used Vensim by Ventana Systems, Inc. (cf. Figure 3-3), and iThink by High Performance Systems, Inc. (cf. Figure 3-4).

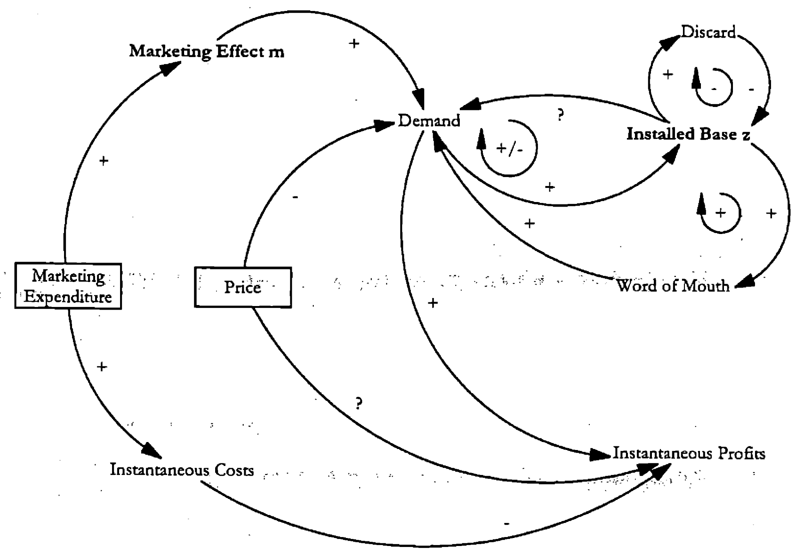

Figure 3-3: Causal Loop Diagram for Example in Section 2.6 with $(x_1, x_2) = (m, z)$.

capacity expansion, maintaining stock, and actual production. In addition, the firm can chose a nonnegative price for the product in such a way that the demand stays positive.

Description of the Model. A typical and efficient modeling approach at the level of the firm splits the overall model into different sectors that can be built and tested independently before they are linked together. In this case we have split up the firm into Marketing, Production, and Finance sectors, that influence each other and all interact with a Market sector, that can be considered as the environment of the firm. The description of this model corresponds to the more explicit ODE formulation given below and exact formulas (cf. equations (3.1)–(3.4)) can be found there. Let us consider the Marketing sector first. The activity of this sector is measured in some form of 'marketing effect' m that without ongoing expenditures diminishes asymptotically to zero. Also, an expenditure $c_1 u_1$ generates a delayed marketing effect that is modeled here via a first-order lowpass. In the Production sector, one has the choice to build up capacity to a level u_2 that is delivered with a delay (modeled in the same manner as in the Marketing sector), and producing u_4 into the stock. Storing items in the stock s incurs a storage fee $c_5 s$. However, a positive stock is necessary to be able to sell the product in the (monopolistic) market environment. The Market sector is modeled using a diffusion model, exactly as in Section 2.6. The demand D thereby depends on the (non-negative) price u_3 chosen for the product, the current marketing effect m and the installed base z, i.e., the number of users that is currently using the product. This number z is assumed to be bounded from above by a 'market potential' $z_{\max}$ that cannot be exceeded. The financial results of the firm, i.e., its cumulative discounted profits, depend on the price charged for the product times the demand, minus the cost incurred through the use of production factors, cf. relation (3.17) on page 77.

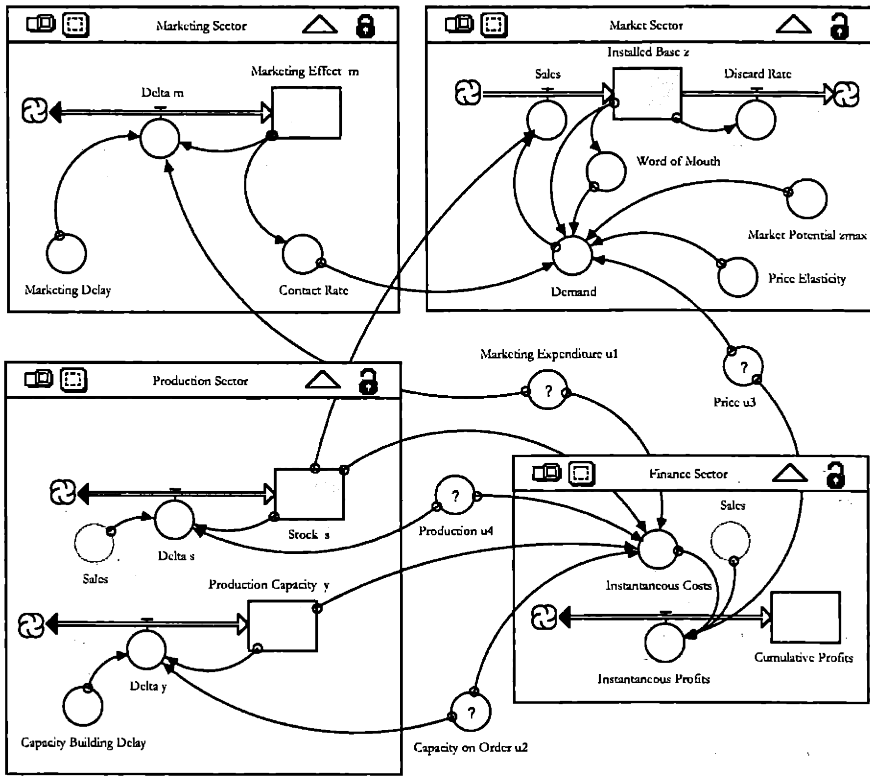

Figure 3-4: Stock-and-Flow Diagram of a Manufacturing Firm in a Diffusive Market.

Mathematical Modeling. A mathematically more explicit, equivalent form of expressing the model of the manufacturing firm introduced above, is a ODE state space representation as in (2.1) on page 24. Using the ODE representation, the above model can be written as follows:

$$\dot{m} = -\alpha_1 m + u_1, \tag{3.1}$$

$$\dot{y} = -\alpha_2 (y - u_2), \tag{3.2}$$

$$\dot{z} = -\beta z + \varepsilon(s) D(x, u_3), \tag{3.3}$$

$$\dot{s} = -\varepsilon(s) D(x, u_3) + u_4, \tag{3.4}$$

where the *demand* D is given by

$$D(x, u_3) \equiv (z_{max} - \gamma u_3 - z)(\alpha_3 m + \alpha_4 z), \tag{3.5}$$

with the parameters $0 < \alpha_i,\ \gamma < \infty,\ 0 \leq \beta < \infty$.

The (linear) constraints on state $x = (m, y, z, s)'$ and control $u = (u_1, u_2, u_3, u_4)'$ for $t \geq 0$

are

$$0 \leq u_1(t) < \infty, \quad (3.6)$$
$$0 \leq u_2(t) < \infty, \quad (3.7)$$
$$0 \leq u_3(t) \leq \frac{z_{\max} - z(t)}{\gamma}, \quad (3.8)$$
$$0 \leq u_4(t) \leq y(t), \quad (3.9)$$

with initial conditions (at $t = 0$) :

$$0 \leq m(0) < \infty, \quad (3.10)$$
$$0 \leq y(0) < \infty, \quad (3.11)$$
$$0 \leq z(0) \leq z_{\max} < \infty, \quad (3.12)$$
$$0 \leq s(0) < \infty. \quad (3.13)$$

The properties of this model and its analogies to the example in Section 2.6 will be examined below in Section 3.5.

3.3 The Objective Functional

The choice of an appropriate objective functional plays a crucial role in a systematic control-based approach to critical resource allocation. The scalar objective functional has to express the tradeoffs between different goals pursued by the firm on an ordered set such as the real numbers, so that clear preferences between outcomes of different strategies can be established.

For firms, a common objective is the maximization of some form of 'value'. Definitions of *value* depend however on the perspective taken. — At the level of an individual business, we can consider the 'added' value of a player in a vertical chain that extends from suppliers through the firm to buyers of products and services of the firm. This added value amounts to the difference between the value created by all players in the chain and the value created by all players except the one in question [BHS96]. Competitors are not excluded from this definition and the essential imperative at the business level is to achieve a 'favorable asymmetry' yielding a positive added value. — At the level of the firm, the assessment of value is somewhat different. Its added value can be understood as the amount by which the value of corporate output exceeds the value of all its inputs [DK90]. Ultimately the total value of a publicly traded corporation is equal to its market value, i.e., the number of outstanding shares times the price per share. However, for a successful strategy at the corporate level, an appropriate objective function for value maximization has to capture variables that can be influenced by strategic decisions. For an external valuation of the firm the following variables are often of some importance for analysts:

$$\text{M/B} = (\textit{Market Value})/(\textit{Book Value});$$
$$\textit{Spread} = (\textit{Return on Equity}) - (\textit{Cost of Equity Capital}) =: \text{ROE} - k_E;$$
$$\textit{Growth} = (\textit{Profits Reinvested})/\textit{Equity}.$$

Under certain assumptions, there are relations between these variables that can be stated explicitly. The so-called 'Profitability Matrix' summarizes these relations in the case of a constant growth assumption (cf. [HM96], pp. 295–298). Such performance measures are however hard to use in a decision making context, since they are generally beyond the direct influence of the decision maker. Here we resort to a much simpler objective functional, namely discounted costs.

Discounted Profits. As outlined before, a first approach that is adopted here for a value-based strategy is to maximize *discounted profits* over a finite prediction horizon. In other words, we will consider here the problem of minimizing a time-dependent cost, generated by a dynamical system. Since the evolution of the system, and therefore the generated cost depends essentially on the selected control input, the fundamental question faced by decision makers is how to select the 'optimal', i.e., cost-minimizing input to the system under consideration. As discussed in Chapter 1, the focus here is on systems relevant to a managerial decision maker such as a corporation or firm. Assuming in nonsingular corporate policy that the manager's goal is to maximize some form of 'value' or objective functional in his critical resource allocation problem, we find that given a dynamic model of the firm, this can be expressed equivalently as a standard problem in optimal control, namely the dynamic minimization of a to cost functional, equal to minus the original objective functional (cf. page 16).

Henceforth, we will assume that the length in time of the project that requires the periodic resource allocation decisions is essentially infinite.[6] The fundamental problem becomes then:

$$\int_0^{\infty} e^{-r\theta} h(x, u)\, d\theta \longrightarrow \min, \tag{3.14}$$

subject to

$$\begin{aligned} \dot{x} &= f(x, u), && (3.15) \\ u &\in \Omega(x), && (3.16) \end{aligned}$$

In full generality, the system function f and the cost kernel h could be time-varying and more general constraints might be imposed on the control input u. However, this exceeds the scope of this thesis and is subject to future research.

In practice however, the manager can only optimize over a finite planning horizon, and is therefore limited to try to approximate the infinite horizon optimal decisions,which may yield an unstable overall policy. This phenomenon is generally not well understood by decision makers and we conjecture that this may explain inefficient, cyclical resource allocation decisions, portrayed and analyzed on an empirical basis e.g. by Paich and Sterman [PS93]. It is therefore of some importance to provide statements about the necessary length of the planning horizon, or more specifically about the quality of approximation to the optimal infinite horizon objective, by the moving horizon optimal policy. This discussion will be conducted here mainly for the nominal case, i.e., in the absence of uncertainty.[7]

[6]This seems reasonable, since many projects are *ongoing*, i.e., without definite ending. In addition, the truncation error is small, if the length of the project and/or the discount factor are sufficiently large.

[7]In the presence of uncertainty, one can resort to the *expected value* of the above measures, if a statistical model of the disturbance is available. If a model of the behavior of the environment, as embodied say, by an opponent, is available, then dynamic game theory may provide a good theoretical framework to find good 'worst-case strategies.' This is however beyond the scope of this Master's thesis.

3.4 A New Paradigm for Critical Resource Allocation

Let us now consider the question, under what conditions the moving horizon predictive control strategy is applicable to problems in critical resource allocation at the level of the firm and in how far this can be considered as a new paradigm.

The core of the methodology is really nothing else than the formalization of the decision making process in Figure 3-2 on page 71. The elements needed for the formalized approach are then:

1. *Model of the Firm*;
2. *Objective Functional*;
3. *Optimal Equilibrium State*;
4. *Planning (Prediction) Horizon T and Implementation (Control) Horizon τ of Appropriate Lengths.*

We have discussed how to obtain the *first two* elements in this list over the past sections. The usual approaches stop at this point, or may at most provide some computerized methods for a brute-force optimization of the model. Since we are interested in a *stabilizing* infinite horizon optimal policy that is adapted to a periodic decision making process over a finite prediction horizon, more elements are needed.

The *third* element the search for an optimal equilibrium state, i.e., a state of the system that, once it has been reached, cannot be improved upon. To find this state in the case of a discounted objective functional, Lemma 1 can be used under certain conditions. The explicit knowledge of such a state is very valuable information for the decision maker, as it *translates the abstract problem of maximizing some functional into a concrete problem of arriving at clearly defined state of the system.* — The fundamental problem in management is that overall performance measures (such as the ones given in Section 3.3 on page 74) are not related in an obvious way to the underlying dynamical system. Knowing an optimal equilibrium state is an important step forward, even though it may not be clear *a priori* how to arrive there in the "cheapest" possible way. Rewriting a sign-indefinite problem such as the example in Section 2.6, using Lemma 2 might in addition yield important insights on the structure of the cost, i.e., it may highlight the variables that *actually* matter given the dynamics of the system.

Element *four* of the constrained predictive control methodology is the determination of appropriate planning and implementation horizons, or in control terminology (cf. Table 1.1 on page 20) prediction and control horizons. This information can be expected to have a profound impact on the decision making process, as it defines what should be minimal time-spans for reporting and predicting the evolution of the model in order to yield a stabilizing policy, at least in the nominal case.[8]

See Section 5.2 for a summary and discussion of the implications for the decision maker.

[8]Certain robustness statements can be made directly, since the Lyapunov stability results can be expected to hold also for slightly perturbed systems. However as noted before, a direct treatment of structured or unstructured uncertainty is beyond the scope of this Master's thesis.

3.5 A Model of a Manufacturing Firm in a Diffusive Market

We will now show that with some additional considerations the results derived in Section 2.6 can be used for more complex models. This is true in particular when just the linear part part of model (2.106)–(2.107) is augmented. To illustrate the point, consider the (nonlinear) model (3.1)–(3.4) of a manufacturing firm, describing its evolution in a diffusive market environment subject to the constraints (3.6)–(3.13), cf. also Figure 3-4. The goal of the firm at time $t \geq 0$ is to maximize discounted profits Π_t, or equivalently, to minimize a (generalized) *cost* $J_t := -\Pi_t$,

$$J_t := \int_t^\infty e^{-r\theta} \Big(c_1 u_1 + c_2 \lambda(u_2 - y) + c_3 y + c_4 u_4 + c_5 s - u_3 \varepsilon(s) D(x, u_3) \Big) \, d\theta, \qquad (3.17)$$

where $r > 0$ is a *discount factor* and $c_i \geq 0$ are cost coefficients.

As explained above, the firm can influence its 'marketing effect' m and capacity expansion $\lambda(u_2 - y)$ by choosing *marketing expenditure* $c_1 u_1$ and *capacity-on-order* u_2 appropriately, provided that all other constraints are satisfied. Furthermore, the firm has to determine the current production level u_4 (not exceeding capacity y), and the price u_3 it charges for the product in a monopolistic, diffusive market environment. This diffusive environment is characterized by an *installed base* z, i.e., the number of products currently in use. The installed base is bounded from above by $z_{\max}$ (market saturation). However, since the product is often assumed to have a finite 'average' lifetime ($\beta > 0$), this installed base can shrink over time when current *sales* $\varepsilon(s)D(x, u_3)$ are lower than the number of discarded products βz. The difference between cumulative production and cumulative sales is the (non-negative) stock s.

Let us now make several observations about this model, before we determine its optimal equilibrium state-control tuple using Lemma 1 in analogy to the example in Section 2.6.

Observation 1 *For $u_1, u_2, u_4, y, s < \infty$ the cost functional J_t is unbounded from below, if the system evolution described by equations (3.1)–(3.4) is* not *considered.*

Proof. Let $u_1, u_2, u_4, y, s \leq M < \infty$. Then, for any fixed $0 < z < z_{\max}$, $0 < u_3 < \frac{z_{\max} - z}{\gamma}$, the integrand $h(x, u) \leq M \sum_{i=1}^5 c_i - u_3 \varepsilon(s) D(x, u_3)$. Thus, we have for any $s > 0$:

$$h(x, u) \leq M \sum_{i=1}^{5} c_i - (z_{\max} - \gamma u_3 - z)\alpha_3 m \longrightarrow -\infty, \quad \text{as } m \to \infty.$$

In particular, if $m(t)$ and constants $t_0, C > 0$ are chosen such that

$$(z_{\max} - \gamma u_3 - z)\alpha_3 m(t) \geq C e^{rt},$$

$\forall\, t \geq t_0$, then

$$J_t \leq M \frac{e^{-rt}}{r} \sum_{i=1}^{5} c_i - C \int_t^\infty d\theta = -\infty,$$

$\forall\, t \geq t_0$, which completes the proof. ∎

REMARK This observation shows that the cost functional J_t is *not* sign-definite.

Observation 2 *The cost functional J_t is bounded from below if the system equations (3.1)–(3.4) are taken into account.*

Proof. Clearly,

$$J_t \geq -\int_t^\infty e^{-r\theta} u_3 \varepsilon(s) D(x, u_3)\, d\theta,$$

and $0 \leq z(t) \leq z_{\max}$ for all $t \geq 0$. From equation (3.3), it follows that for any $r > 0$:

$$\varepsilon(s) D(x, u_3) \leq \dot{z} + \beta z + \frac{r}{2}\left(z_{\max} - z\right),$$

which, using (3.8) and the fact that $z \in [0, z_{\max}]$, yields

$$\gamma u_3 \varepsilon(s) D(x, u_3) \leq \left(\dot{z} + \beta z\right)\left(z_{\max} - z\right) + \frac{r}{2}\left(z_{\max} - z\right)^2.$$

For the RHS of the last inequality we obtain

$$-\frac{1}{2}\frac{d}{dt}\left(z_{\max} - z\right)^2 + \frac{r}{2}\left(z_{\max} - z\right)^2 + \beta z\left(z_{\max} - z\right) \leq -\frac{1}{2} e^{rt} \frac{d}{dt}\left\{e^{-rt}\left(z_{\max} - z\right)^2\right\} + \beta \frac{z_{\max}^2}{4},$$

where we have used that $\beta z\left(z_{\max} - z\right) \leq \beta z_{\max}^2/4$ for all z. Hence,

$$-e^{-rt} u_3 \varepsilon(s) D(x, u_3) \geq \frac{1}{2\gamma}\frac{d}{dt}\left\{e^{-rt}\left(\left(z_{\max} - z\right)^2 + \frac{\beta z_{\max}^2}{2r}\right)\right\},$$

and consequently,

$$J_t \geq -\frac{e^{-rt}}{2\gamma}\left(\left(z_{\max} - z\right)^2 + \frac{\beta z_{\max}^2}{2r}\right) \geq -\frac{z_{\max}^2}{2\gamma}\left(1 + \frac{\beta}{2r}\right),$$

$\forall\, t \geq 0$, completing the proof. ■

REMARK

1. The lower bound on J_t in Observation 2 can be improved by noting that

$$\gamma u_3 \varepsilon(s) D(x, u_3) \leq \left(\dot{z} + \beta z\right)\left(z_{\max} - z\right) + \frac{r}{2}\left(z_{\max} - z\right)^2 - \frac{r}{2}\left(z_{\max} - z\right)^2.$$

   Defining

$$\bar{z} := \frac{(r + \beta) z_{\max}}{r + 2\beta} = \arg\max_z \left\{\beta z\left(z_{\max} - z\right) - \frac{r}{2}\left(z_{\max} - z\right)^2\right\} \in [0, z_{\max}],$$

   one can conclude that

$$\begin{aligned} -e^{-rt} u_3 \varepsilon(s) D(x, u_3) &\geq \frac{1}{2\gamma}\left(\frac{d}{dt}\left\{e^{-rt}\left(z_{\max} - z\right)^2\right\} + \beta \bar{z}\left(z_{\max} - \bar{z}\right) - \frac{r}{2}\left(z_{\max} - \bar{z}\right)^2\right) \\ &= \frac{1}{2\gamma}\frac{d}{dt}\left\{e^{-rt}\left(\left(z_{\max} - z\right)^2 + \frac{\beta^2 z_{\max}^2}{2r(r + 2\beta)}\right)\right\}, \end{aligned}$$

and
$$J_t \geq -\frac{z_{\max}^2}{2\gamma}\left(1+\frac{\beta^2}{2r(r+2\beta)}\right), \quad \text{for all } t \geq 0. \tag{3.18}$$

2. An alternative proof can be given by directly referring to our example in Section 2.6:

   From the system equations, together with the restrictions on the initial conditions (3.10)–(3.13), it follows that
$$y(t), s(t) \geq 0,$$
   $\forall t \geq 0$. Thus one obtains for the cost functional J_t in (3.17) :
$$J_t \geq \int_t^\infty e^{-r\theta}\left(c_1u_1 - u_3D(x,u_3)\right)\, d\theta. \tag{3.19}$$

   The RHS in the last inequality corresponds to the cost functional (2.111) on page 53. And indeed if $s > 0$ the system equations and constraints for m and z map identically to equations (2.106)–(2.107) in Section 2.6, so that Observation 2 follows directly from (2.117). If on the other hand $\varepsilon(s) < 1$ then the functional can only become larger and Observation 2 still holds.

Let us now determine the optimal state-equilibrium tuple (x^o, u^o). For this, note first that
$$\begin{aligned} J_t &\geq \int_t^\infty e^{-r\theta}\left(c_1(\alpha_1+r)m + c_3y + c_4\left(\dot{s}+\varepsilon(s)D(x,u_3)\right) + c_5s - u_3\varepsilon(s)D(x,u_3)\right)\, d\theta \\ &= \int_t^\infty \left[e^{-r\theta}\left(\tilde{c}_1m + c_3y + \tilde{c}_4s - (u_3-c_4)\varepsilon(s)D(x,u_3)\right) + \frac{d}{d\theta}\left(e^{-r\theta}\left(c_1m+c_4s\right)\right)\right] d\theta \\ &= -e^{-rt}(c_1m(t)+c_4s(t)) + \int_t^\infty e^{-r\theta}\left(\tilde{c}_1m + c_3y + \tilde{c}_4s - u_3D(x,u_3)\right)\, d\theta, \end{aligned} \tag{3.20}$$

where we have used the abbreviations
$$\begin{aligned} \tilde{c}_1 &:= c_1(\alpha_1+r), \\ \tilde{c}_1 &:= c_4r + c_5. \end{aligned}$$

From the system equations (3.1)–(3.4) it is clear that $x^e = 0$ is an equilibrium state with associated (admissible) equilibrium control $u^e = (0,0,u_3^e,0)'$ ($u_3^e \in [0, z_{\max}/\gamma]$). Thus, for an *optimal* state-control tuple (x^o, u^o) the cost kernel h cannot be positive, i.e., for any parameter configuration in the above model
$$0 \geq h(x^e,u^e) \geq h(x^o,u^o)$$
must hold. Thus, if
$$c_3y^o \leq \tilde{c}_1m^o - (u_3^o - c_4)D(x^o,u_3^o),$$
one can compute the optimal equilibrium state-control tuple analogous to Section 2.6 by considering
$$\tilde{u}_3 := u_3 - c_3 - \tilde{c}_4$$

instead of u_3.[9] Hence, the relevant portion of the lower bound of the cost functional in (3.20) can again be bounded from below by

$$\int_t^\infty e^{-r\theta}\left(\tilde{c}_1 m + -\tilde{u}_3(\tilde{z}_{\max} - \gamma\tilde{u}_3)(\alpha_2 m + \alpha_3 z)\right) d\theta,$$

with[10]

$$\tilde{z}_{\max} := z_{\max} - \gamma(c_3 + \tilde{c}_4).$$

The new constraint set for $\tilde{u}_3$ is given by

$$\tilde{\Omega}_3(x) = [-(c_3 + \tilde{c}_4), (\tilde{z}_{\max} - z)/\gamma].$$

This bound can be achieved, which follows simply from using the results found in Section 2.6 for the non-restrictive[11] case when $\gamma = 1$. Equations (2.119)–(2.120) yield

$$\begin{aligned} z^o &= \left[\frac{(\beta + r)((\tilde{z}_{\max} - \gamma c_4) - 2\sqrt{\frac{\tilde{c}_1}{\alpha_2}}) + \tilde{c}_1\frac{\alpha_3}{\alpha_2}}{2\beta + r}\right]_{[0,\tilde{z}_{\max}]}, \\ m^o &= \left[x_2^o\left(\frac{\beta}{\sqrt{\alpha_2\tilde{c}_1}} - \frac{\alpha_3}{\alpha_2}\right)\right]_{[0,\infty)}. \end{aligned}$$

From the equilibrium condition $\dot{x} = 0$ and the binding constraint (3.9), one can conclude that

$$\begin{aligned} y^o &= (\tilde{z}_{\max} - \tilde{u}_3^o - z^o)(\alpha_2 m^o + \alpha_3 z^o), \\ s^o &= 0^+, \end{aligned}$$

with associated equilibrium controls

$$\begin{aligned} u_1^o &= \alpha_1 m^o, \\ u_2^o &= y^o, \\ u_3^o &= -\frac{\beta z^o}{\alpha_2 m^o + \alpha_3 z^o} + z_{\max} - z^o, \\ u_4^o &= y^o, \end{aligned}$$

where we have re-substituted u_3 for $\tilde{u}_3$. In the case when

$$c_3 y^o \geq \tilde{c}_1 m^o - (u_3^o - c_4)D(x^o, u_3^o),$$

it follows from (3.5) that $x^o = 0$.

REMARK

1. We have shown that indeed for the seemingly more complex model (3.1)–(3.4) essentially the same considerations as for (2.106)–(2.107) in Section 2.6 hold. This seems

[9] Note that because the constraints for y^o are binding in this case, we have actually $y^o = D(x^o, u_3^o)$.

[10] It has been assumed here that $z_{\max} - \gamma(c_3 + \tilde{c}_4) \geq 0$. If this is not satisfied, the optimal equilibrium state is zero.

[11] This has been explained on after relation (2.112) on page 53.

plausible *a posteriori*, since the pairs of model equations (3.1),(3.3) and (3.2),(3.4) are almost completely decoupled.

2. The role of the stock s is interesting. In the nominal case discussed here, it is apparently optimal not to keep any stock, or in other words to have production feeding the demand directly. In the presence of uncertainty however, one may conjecture that role of the stock becomes essential to compensate for demand uncertainty (sales can only take place if there is sufficient stock). To prove or disprove this conjecture could be subject to further research.

3.6 Notes and Sources

Corporate Strategy. Hax and Majluf's [HM96] approach to corporate strategic planning provides a general framework, integrating a variety of thought models, both factor-driven and market-driven, into an overall process to derive a coherent set of strategic decisions. A market-driven approach [Por80], [Por85], [Por91] thereby postulates that competitive advantage is derived from an appropriate competitive positioning in a given industry structure, whereas a factor-driven strategic planning considers the proper resources of the firm as the main source of competitive advantage [Pet91]. In the corporate strategic planning framework adopted here, both views are present in environmental scan and internal scrutiny respectively. Furthermore, from a corporate perspective it is desirable to identify so-called 'core competencies' [PH90], i.e., transferable sources of competitive advantage at the level of a corporation, rather than a mere business unit. — All these considerations allow the establishment of a model of the firm and an objective functional for critical resource allocation.

Modeling the Firm. The System Dynamics modeling approach was introduced by Jay W. Forrester at MIT in the 1960s [For61]. The methodology has been refined since and is now widely used in practice. The available literature is very fragmented and there is a need for a good unfied treatment of the different modeling approaches. [Ric96] is an excellent collection of recent articles in the field, and [Ric92] provides a good practical introduction based on a specific easy-to-use software package. Mathematical modeling based on simulation of differential equations including the use of optimal control techniques has been provided recently by [Blo96].

Objective Functions. From an economic viewpoint, value is the return that exceeds the cost of capital. Value can be derived only in the state of disequilibrium of (idealized) market forces of supply and demand. Thus, the quest for value from the perspective of the firm is the search for intertemporal opportunities to create such a 'favorable disequilibrium'. A firm's strategic goal of creating value translates then to the search of investment opportunities (or projects) that yield a positive NPV over a significant time horizon (cf. Section 1.2.1, p. 14). The discounted profits approach adopted here is directly derived from the quest for a maximal NPV. For a more detailed discussion of value from the perspective of the firm, see [Ste91] and [MKM94]. — We note here that economic preferences can be at times nicely expressed in terms of (possibly multiobjective) *utility* functions. Utility functions can be introduced on an axiomatic basis (see [dN90] for an introductory review with practical examples). — In an uncertain environment, matters are more complicated (cf. footnote 7 on page 75). Investments under uncertainty using stochastic modeling are considered in [DP94], while [BO95] provides a comprehensive introduction to dynamic non-cooperative game theory. See [BHS96] for references in cooperative game theory that ultimately is more adapted to the business environment, where coalition structures are common practice.

The New Paradigm. The application of a moving horizon predictive control methodology to critical resource allocation problems is new. The term *nonsingular* corporate policy to denote an infinite horizon critical resource allocation strategy has been coined here.

Manufacturing Firm in a Diffusive Market Environment. The ODE model presented here is a natural extension of the one in Section 2.6 and has been developed by the author, considering many useful suggestions by Professor Alexandre Megretski (MIT).

The source of all great mathematics is the special case, the concrete example.
— Paul Richard Halmos

Chapter 4

A Real-World Example

To illustrate the application of the methodology outlined in the previous chapters, a real-world example is provided. Due to the theoretical complexity of our approach, the level of abstraction and simplification here is still considerable and our main goal is certainly *not* to provide a complete solution to the problems of one particular firm, but to demonstrate the critical resource allocation strategy developed earlier, as well as to discuss its applicability and current limitations.

First, we will briefly describe the Boston Central Artery/Tunnel Project, its scope and organization. Here we will pose the problem, namely how to allocate resources for the internal diffusion of information technology (IT) in an optimal manner. *Second*, the IT functional strategy at the level of the firm with respect to the corporate strategic planning framework presented in Chapter 3 is briefly discussed. *Third*, a model and corresponding estimates for its parameters are introduced, before in the the last section we that may be obtained using the constrained predictive control methodology are critically reviewed.

4.1 Introduction

The Boston Central Artery/Tunnel (CA/T) project is a multibillion dollar undertaking and at present one of the largest infrastructure efforts in the United States. The core of the CA/T project is the transformation of the Interstate I-93 through Boston and an extension of I-90 beneath Boston Harbor and on to Logan Airport. The latter extension, including the new Ted Williams Tunnel, was opened in December 1995 to commercial vehicles. At present, major works are being conducted on an eight-lane underground section of I-93 replacing its elevated portion that has been dividing the city over the past decades.

The CA/T project is conducted by the Massachusetts Highway Department (MHD) in cooperation with a joint venture (B/PB) of two construction companies, namely Bechtel and Parsons Brinckerhoff. Project completion is scheduled for the end of the year 2004, and cumulative expenses are currently estimated to exceed US$7 billion. The overall organization of the project is illustrated in Figure 4-1 and the geographical scope of the project together with the current construction contract packages can be seen in Figure 4-2.

The Problem. We will focus here on the internal information technology strategy of the CA/T. The project possesses a dedicated IT section that has been recently subordinated to

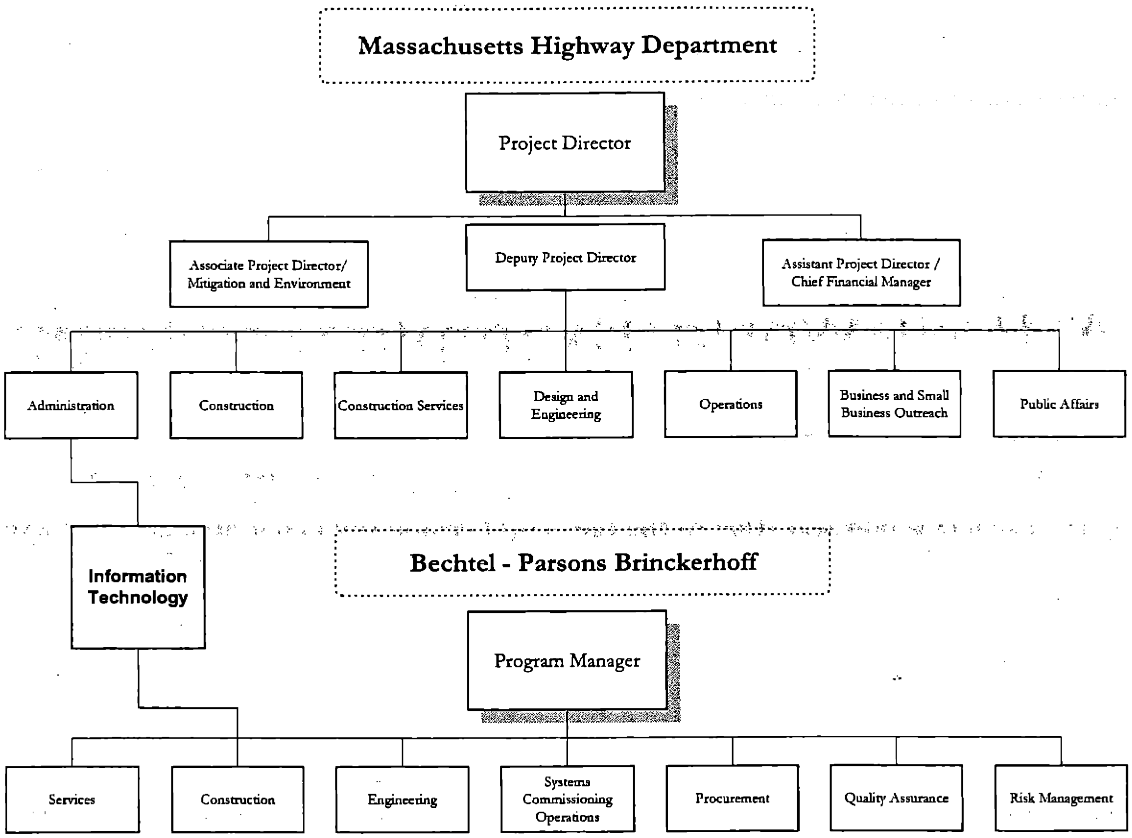

Figure 4-1: The CA/T Project Management System (Source: CA/T).

the construction department of the B/PB joint venture, but also reports to the client, MHD. In the light of quite substantial investments in information technology in the past as well as the CA/T project's still quite low levels of office automation and workflow integration, the question is *how to capitalize on the "returns" of the past IT investments, and how to possibly determine "optimal" spending levels for furthering IT diffusion, so as to maximize the overall benefit of the project.*

Here already it should be clearly stated that a theoretically completely satisfying solution to the above problem cannot be given, as the returns on IT investment are in itself highly controversial, and still a largely open area for research (cf. [BH96]). Also it should be noted that the CA/T project's internal documentation about spending on and benefits from IT is rather fragmented, so that we are forced to make quite substantial assumptions that are themselves open to discussion.

4.2 Functional Strategy

The IT department represents a major "function" at the level of the CA/T project (or "the firm," to be consistent with the developments in earlier chapters). It centrally allocates all information technology resources for the firm, including hardware, software, connection lines, training, as well as on-line help. These resources can be seen as *critical*, since the management of the different construction contracts and the monitoring of the work in progress are largely information-based and therefore depend essentially on the quality of the provided information systems. — To illustrate our approach, let us now apply the corporate strategic planning framework outlined in Chapter 3 (cf. Figure 3-1 on page 68) with respect to its functional component. We first define the functional requirements for information technology, and then briefly outline functional internal scrutiny as well as functional environmental scan to explain the "strategic programming" that has been developed with regard to IT diffusion at the level of the firm.

Functional Requirements. Information technology enables the aggregation and disaggregation of information-based work. A major part of the activities in the field offices managing the different contracts is information-based and can thus be substantially enhanced through the use of IT. Investments in IT resources are required to achieve measurable benefits in the medium term. Such measurable benefits include a reduction in man-hours at the level of the field offices, as well as increased quality and efficiency of the work-output. The work-output can thereby be summarized as the (possibly weighted) number of standard workflows successfully completed (cf. Table 4.1), while the work-input is seen here simply as the number of required man-hours.

Functional Environmental Scan. A good indicator for benchmarking CA/T project's workflow efficiency at an industry level is Bechtel's own experience with the use of a state-of-the-art proprietary project management system (Infoworks) in some of its other construction projects. The software is reported to significantly enhance workflows and to result in substantial savings and its potential use in the CA/T project is currently under evaluation.[1]

Functional Internal Scrutiny. The different construction contracts for the CA/T project are managed by field offices in which the so-called resident engineer represents the B/PB joint venture, and serves as a designated authorized representative of MHD for assigned contracts and/or projects. One field office can generally manage different contracts and/or can have certain employees, such as the scheduling engineer, working on different contracts at the same time. The field office monitors the work performed by its assigned contractor and authorizes it by issuing work-orders. It also handles contract modifications and the contractors' bi-weekly payment applications.

The current degree of automation in a typical field office of the CA/T project is quite low and varies from contract to contract. At present, only communications via email, deficiency reports at the level of the field office, and sometimes schedule/cost updates at the level of the contractor are fully automated. Most other significant workflows (cf. Table 4.1), such as the processing of contract modifications, are — though not fully automated — enhanced through the use of computers.

Strategic Programming at the Functional Level. Global priorities for the IT section are set by the MHD project director and the program manager of the B/PB joint venture. On an operational level, the IT section is subordinated to the B/PB construction department and reports in regular intervals to the MHD administration. The annual budget for IT in the CA/T project is fixed and in the order of magnitude of US$1 M. This budget is at times augmented to cover extraordinary expenses.

[1] No quantification of the exact savings obtained was available to the author.

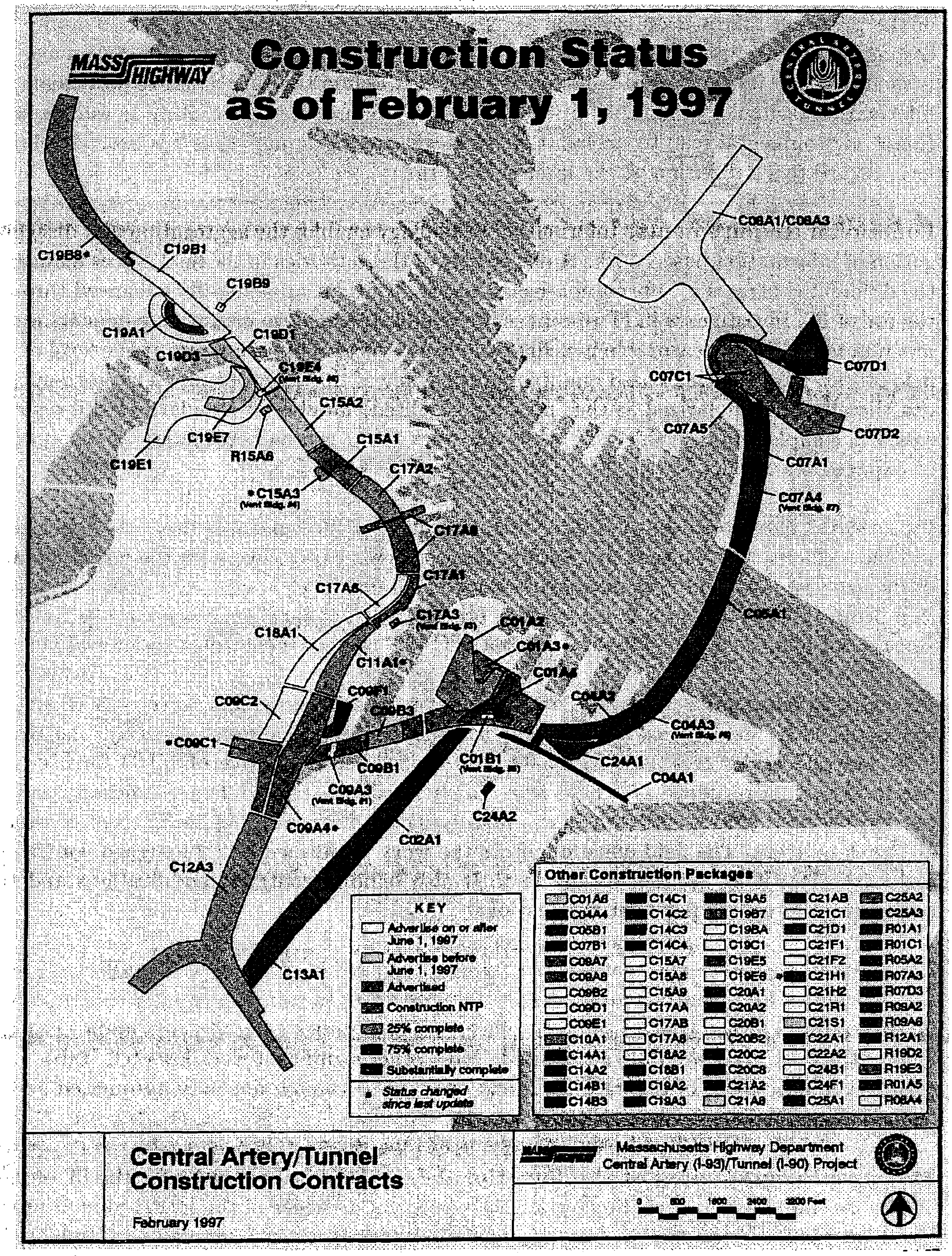

Figure 4-2: Geographical Scope of the Boston CA/T Project (Source: CA/T).

Document/Workflow	Originator	Automated Y/N
Contract Modification	Resident Engineer	N
Correspondence Letter	Contractor/Resident Engineer	N
Deficiency Report	Resident Engineer	Y (internally)
Design Update	Section Design Consultant (SDC)	N
Internal Email	MHD/(B/PB)/SDC	Y
Progress Payment	Contractor/(B/PB)	N
Request for Information	Contractor/Resident Engineer	N
Schedule/Cost Update	Contractor	Y
Test Result	Laboratory	Y (quasi)

Table 4.1: Typical Field Office Workflows (Source: CA/T).

4.3 Modeling IT Diffusion at the Level of the Firm

We will adopt here a highly simplified, though plausible model of IT diffusion, based on logistic growth, analogous to the one presented in Section 2.6. In fact, logistic growth has been used in the 1960s by [Cho67] to describe the adoption of computers and can been fitted to statistical data. A widely used model for the diffusion of consumer durables developed by Bass [Bas69] is also based on logistic growth.

Causal Loop Diagram. Consider now the Causal Loop Diagram in Figure 4-3, which shows a simple model of how to think of IT diffusion at the level of the firm. The model is driven by expenditure in IT, which results in IT infrastructure. In the presence of a more sophisticated IT infrastructure the demand for IT can be expected to rise, which then increases the "level of informatization," i.e., the percentage of workflows (or more specifically: man-hours) that are "informatized." If left alone, without 'inflow' generated by IT expenditure, the level of informatization z is assumed to slowly drop as the machines become obsolete. On the other hand, a high level of informatization in one part of the project produces a technology "spillover effect" which increases the overall demand for IT. The demand for IT is dampened by the required benefit from its usage, i.e., say the required reduction in man-hours achieved through its adoption. The overall benefits are then the cumulative difference of the achieved man-hour savings and the expenditure made in IT.[2]

IT Diffusion. The diffusion of IT or level of informatization is measured here in terms of a "penetration percentage," i.e., the percentage of activities that can be regarded as fully enhanced through the usage of IT. More specifically, it can be understood as the fraction of man-hours spent while working on an integrated IT system. Clearly, the maximum level of informatization that can be expected under this definition is less than 100 percent, since not all workflows can be fully informatized. A more formal model based on an ODE

[2]This "tayloristic" view of IT benefits as exclusively being measurable in terms of man-hour reductions is of course highly simplified. However, since in critical resource allocation decisions different strategies have to be evaluated on an ordered set for their relative merits, we will express all measurable savings in 'equivalent' man-hours.

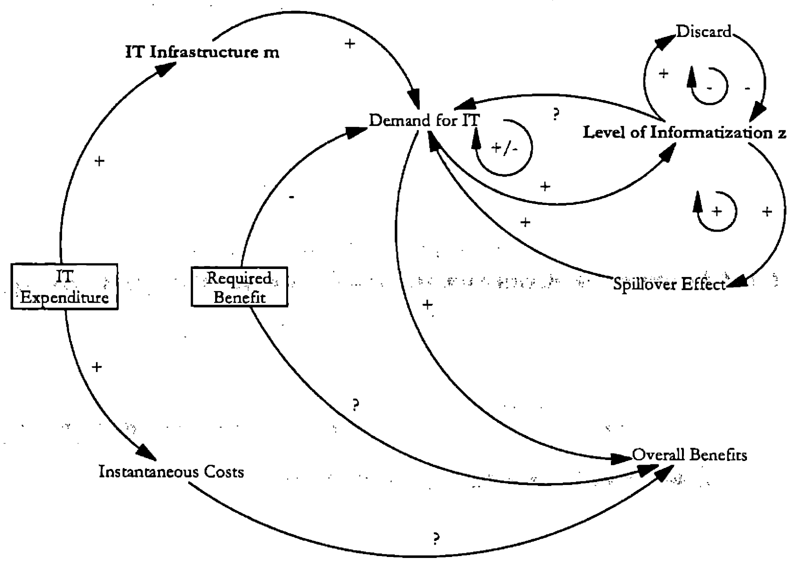

Figure 4-3: A Highly Simplified Model of IT Diffusion.

representation has been developed here as follows:

$$\begin{aligned} \dot{m} &= -\alpha_1 m + u_1, \\ \dot{z} &= c_2 D(x, u_2) - \beta z, \end{aligned}$$

where

$$D(x, u_2) := (z_{\max} - \gamma u_2 - z)(\alpha_2 m + \alpha_3 z), \qquad [\$/\text{month}]$$

and

m	:	IT Infrastructure	[$]
z	:	Level of Informatization (LoI)	[% LoI]
u_1	:	IT Expenditures	[$/month]
c_1	:	IT Investment/(Man-Hour Savings·(% Informatization))	[$/((man-hours/month) · (% LoI))]
c_2	:	1/(Cost of Informatization)	[(% LoI)/$]
u_2	:	Man-hour Savings	[man-hours/month]
$\bar{\tau}$	:	Time before Savings Realized	[months]
D	:	Demand for IT	[(% LoI)/month]

The coefficients in the above ODE model are defined as follows:

α_1	:	1/(IT Installation Delay)	[1/month]
α_2	:	Sensitivity towards new Installation of IT	[1/((%) · month)]
α_3	:	Spillover Effect	[$/ ((%)^2 · month)]
β	:	Informatization Discard Rate	[1/month]
γ	:	IT Demand Elasticity towards Required Benefit	[(%)/(man-hours/month)]
r	:	Discount Rate	[1/month]

Objective Functional. As explained above, the objective functional[3] summarizes the overall benefits obtained from the investment in IT. We will adopt here the following cost functional:

$$J_t = \int_t^{\infty} e^{-r\theta} \left(u_1 - e^{-r\bar{\tau}} c_1 c_2 u_2 D(x, u_2) \right) d\theta. \quad [\$]$$

The instantaneous benefits $u_2 D(x, u_2)$ are discounted by the supplementary factor $e^{-r\bar{\tau}}$, whereby the constant $\bar{\tau}$ represents an estimate for the average time needed to realize these benefits. The structure of the ODE model and the objective functional correspond to the nonlinear theoretical example given in Section 2.6. A direct correspondence can be established when the above cost functional is written in the form

$$J_t = c_1 c_2 e^{-r\bar{\tau}} \int_t^{\infty} e^{-r\theta} \left(\tilde{c}_1 u_1 - u_2 D(x, u_2) \right) d\theta,$$

where

$$\tilde{c}_1 := \frac{e^{r\bar{\tau}}}{c_1 c_2}. \qquad \text{[man-hours/month]}$$

Thus we can use the results obtained there and get optimal levels for the information system spending, once appropriate values for the model parameters have been found. To use these results directly one has to substitute

$$(\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{z}_{\max}) := (c_2\alpha_2, c_2\alpha_3, z_{\max}/\gamma)$$

in equations (2.119)–(2.120) on page 55. After having re-substituted the original parameters back one obtains:

$$z^o = \left[\gamma \frac{(\beta + r)(z_{\max}/\gamma - 2\sqrt{\frac{\tilde{c}_1 (r+\alpha_1)}{c_2\alpha_2}}) + \tilde{c}_1 (r + \alpha_1)\frac{\alpha_3}{\alpha_2}}{2\beta + r} \right]_{[0, z_{\max}]},$$

$$m^o = \left[z^o \left(\frac{\beta}{\sqrt{\tilde{c}_1 (r + \alpha_1) c_2 \alpha_2}} - \frac{\alpha_3}{\alpha_2} \right) \right]_{[0,\infty)}.$$

One of the major drawbacks of high abstraction levels during the modeling phase is that the modeling parameters become highly aggregated constants that are not readily available in practice and therefore have to be estimated.

Parameter Estimates. To obtain rough estimates for possible parameter values, the author visited two field managing contracts for the CA/T project (C17A2 and C09C1). The current and potential informatization status, i.e., the actual time fraction spent using IT and an estimated maximal time fraction is listed for a typical field office in Table 4.2. We have thereby determined the current level of informatization as the weighted average time spent using IT, where as weights we have chosen the relative cost of the field office activity in question. From this we obtain for the maximum level of informatization $z_{\max} = 59\%$, while the current level is estimated to about $z = 41\%$.

[3] Recall that usually (Objective Functional) = −(Cost Functional) to obtain a minimization instead of a maximization problem, cf. discussion on page 16.

The monthly discount rate has been chosen to $r = 15\%/12 = .0125$, consistent with the conclusion in [BH96] that IT capital can generally be expected to generate returns quite similar to other production factors. The determination of the other model parameters is considerably more difficult since their definition is complicated and the available data is very limited. We will simply give a possible value for each parameter. The discard rate for IT resources is extremely low as used computers are transferred across the project according to current needs. Hence we assume an average lifetime until the end of the project in 2004, which results in $\beta = 1/(7 \cdot 12) = .0119$. The IT installation delay is approximately 3 months and thus $\alpha_1 = 1/3$. We estimate that the savings should be realized in about two years and therefore set $\tau = 24$. To estimate c_1 we assume that about US$500 have to be invested to achieve labor savings of one man-hour per month. This estimated is based on the approximate cost for a database system that allows the office engineer to save about 12 hours per month on payment applications ($c_1 = 6000/12 = 500$). The parameters γ, α_2, and α_3 could not be estimated reliably. However the results of the model are quite robust with respect to their choice, and we have assumed as an example: $\gamma = 2.5/10 = .25$, $\alpha_2 = 1$, $\alpha_3 = 5000/(3 \cdot (2.5)^2) = 266$. With these parameter values, one obtains an optimal steady-state level of informatization $z^o = 29.5\%$ that is considerably *lower* than the current level. — However we point out that is model is extremely simplified and considers essentially only the substitution of labor, so that these results have to be considered with care.

Job (# persons)	Actual/Maximal Time for Usage of IT [%]	Relative Man-Hour Cost [%]
Resident Engineer (RE)	15/50	12
Assistant RE	45/80	8
Scheduling Engineer	90/95	10
Office Engineer (OE)	60/80	8
Assistant OE	75/85	6
Claims & Changes	90/95	6
Lead Field Engineer	40/60	10
Field Engineer (4)	15/35	25
Clerk/Typist	85/95	5
TOTAL:	41/59	100

Table 4.2: Informatization Status for Typical Field Office (Source: Own Estimates).

4.4 Results

In order to value IT, one has to determine the gains in profitability, productivity and consumer surplus that could be achieved through its use.[4] Our estimates have been based purely on potential productivity gains and internal diffusion phenomena that are hard or impossible to estimate reliably. Our model (whose structure has been chosen for convenience

[4] These dimensions for the assessment of the value of information technology have been introduced by Hitt and Brynjolfsson [HB96].

close to the one discussed in Section 2.6) proved too aggregate to yield directly usable results. The indication that the optimal steady-state level of informatization is below the current one can have several causes: (1) insufficient quality estimation model parameter estimation; (2) over-simplification in model; (3) the current informatization is indeed too high. The likely causes here are that the model structure is too simple to describe the underlying phenomenon, and that appropriate parameter values could not be obtained with any degree of certainty.

We therefore conclude that the constrained predictive control methodology for nonlinear systems seems at present too limited to deal with more than "small-scale" phenomena for which models with reliable parameter estimates are available. On the other hand, we note that for the "real-world problem" presented here no satisying theoretical models do exist, so that one cannot expect an answer to come from a methodology that in itself is based on a sound understanding of the system.

4.5 Notes and Sources

Selection of the Example. It is difficult to find a real-world example that satisfies the quite restrictive modeling assumptions that we have made in the course of the theoretical developments in Chapter 2. A straightforward application of the diffusion model would have been the calibration of marketing expenses and price levels for a newly launched product. Unfortunately, the author did not have access to such such data. — Nevertheless, we believe that the internal diffusion of information technology on such a large scale and centralized manner as in the CA/T project, represents in principle a valid illustration of the theory, if the highly aggregated model structure is accepted. The author wishes to thank Professor Feniosky Peña-Mora (MIT) for his support and invaluable help in providing access to the data for this case-study.

Value of Information Technology. Assessing the value of IT is still a controversial subject in the literature [BH96]. The production function approach (cf. page 13) is widely used for statistical analyses at an industry level; see [HB96] for an excellent recent study. Unfortunately, this type of analysis is not very helpful for a given particular firm, since the production function approach as discussed in Section 1.2.1 treats a firm essentially as a "black box," with no regard for the different workflows involved. A conceptual process-oriented[5] framework that distinguishes between informational, automational, and transformational value of IT, has been given recently in [MGK96]. Though, such methodologies rarely provide more than categories and hardly help to actually quantify the business-value derived from the use of information technology.

IT Diffusion. As already mentioned in the text, IT diffusion has been modeled using logistic growth in the 1960s by Gregory C. Chow [Cho67]. Logistic growth is still widely used to explain technological diffusion and diffusion of consumer durables (cf. discussion of diffusion modeling in Section 2.7 on page 65). The modeling of diffusion is thereby particularly important during the startup phase of a new product, or in so-called "experimental markets," cf. [PS93]. The competition of different technologies can be modeled using two

[5] An excellent generic overview of processes in a manufacturing firm is given in [Toz96], pp. 106–107.

coupled logistic growth models, yielding a Lotka-Volterra predator-prey model [PU96]. An application of our methodology to such a case of multi-mode technological interaction would require interesting modifications of the model and is subject to further research.

A great truth is a truth whose opposite is also a great truth.
— THOMAS MANN

Chapter 5

Conclusion and Further Research

5.1 The Questions — The Answers

To conclude, the questions posed in Sections 1.1 and 1.3 will be addressed in reverse order, progressing from the more specific to the general.

Question 4 *Are there conditions under which there exists an optimal equilibrium state for a corporate critical resource allocation problem? Can this state be reached under a stabilizing optimal corporate policy?*

We have characterized critical resource allocation at the level of the firm as an optimal control problem. To formalize the problem, the decision maker needs: (1) a dynamic model of the firm that quantifies assumptions about its inner workings and relationship with the environment; (2) an objective functional that measures the performance of a particular resource allocation policy. A resource allocation policy is thereby a set of rules that guides the decision maker in his choice of decision variables or resources. Optimal control theory formalizes the search for an *optimal* critical resource allocation policy.

Optimal Equilibrium State. An optimal equilibrium state is a state of the system[1] that once it has been reached cannot be improved upon. Finding such a steady-state of the system (if it exists), provides an associated optimal admissible resource allocation policy so that the system remains in equilibrium there (cf. Definition 12 on page 36). For a class of time-invariant nonlinear systems and discounted (possibly indefinite) cost functionals, a new explicit procedure for finding optimal equilibrium states is provided by Lemma 1 (cf. page 47) that does *not* require the burdensome use of Bellman's inequality or the Pontryagin Maximum Principle.

Asymptotic Convergence and Stability. The moving horizon control methodology uses predictions over a finite time horizon to regularly update the current policy that is being implemented to the best of the knowledge at the last decision stage (cf. Figure 1-1 on page 11). A stabilizing moving horizon optimal controller requires the existence of an optimal state-control tuple discussed in the last paragraph. If this state can be reached in finite time using admissible controls (cf. Definition 6 on page 28), then conditions for asymptotic convergence to this optimal state for a general class of nonlinear discrete-time

[1]This state not just depends on the system, but also on the objective functional.

systems have been given by Keerthi and Gilbert (cf. Section 2.3.1). In our semi-formal discussion in Section 2.3.2 we point out, that only under certain (stronger) assumptions analogous results hold for the continuous-time case.

Robustness. In the presence of uncertainty it is important to know how the theoretical results vary under the influence of disturbances. In all developments in this Master's thesis we have restricted ourselves to the nominal case in the absence of uncertainty. This establishes baseline results for a more sophisticated application of constrained predictive control to (nonsingular) decision making under uncertainty. Still, since the stability results obtained here are based on what is called Lyapunov stability theory, statements can be made with respect to slight variations in the initial assumptions. We have assumed that there are no modeling errors, and we have limited our robustness discussions to the choice of the time horizon (cf. Section 2.5). For linear systems and quadratic cost functionals robustness results have been obtained elsewhere (cf. notes in Section 2.7). However realistic corporate models are nonlinear and relevant cost functionals are typically nonquadratic.

Question 3 *Can a procedure be defined and conditions given that allow an application of the moving horizon control methodology to approximate an optimal infinite horizon resource allocation? In particular, can statements be made about the robustness of this procedure with respect to the choice of the planning horizon?*

Moving Horizon Control Methodology. The relevance of moving horizon control for decision makers has been pointed out in Section 1.3 and Section 3.4. The informal procedure of how to arrive at a suitable system model and how to choose an objective functional is detailed in Sections 3.2 and 3.3. A formal development can be found in Section 2.3. The conditions for a successful application of the method, among which the existence of an optimal state-control tuple that is both an equilibrium and a minimizer of the cost kernel, were discussed in the above answer to Question 4. The theoretically optimal cost that is attained by applying an infinite horizon optimal policy can be approximated with the moving horizon control methodology. We have thereby shown in a concrete example (cf. Section 2.6) that although the *moving* horizon optimal cost is "close" to the optimal *infinite* horizon cost, the moving horizon optimal policy possibly does not uniformly approximate the infinite horizon optimal policy.

Planning Horizon. The lower bounds for planning and implementation horizon in the continuous-time case were given in Section 2.5. For practical purposes these lower bounds may be too large, and this potentially presents a methodological drawback. Certainly the bounds have to be improved, but 'long' required prediction and implementation horizons, indicate an inherently unstable decision making process. In other words, the market dynamics may require longer time-horizons for a moving horizon policy to be stabilizing, and thus the decision maker might have to re-evaluate the riskiness of his business.

Question 2 *Can a procedure be defined and conditions given under which control can be used efficiently in critical resource allocation problems, i.e., nonsingular corporate policy?*

Only a limited answer to this question was produced by the analysis. The moving horizon methodology was applied to small-scale systems only. These systems although having the advantage of accessibility, are usually hard or impossible to fit to reality, and require very aggregated parameters (cf. real-world example in Chapter 4). In addition there is no software currently available that allows the integration of modeling and optimal control problems, as required for the use by decision makers.

Other related procedures, such as decision analysis discussed in Section 1.2.3, possess already a large basis of theory and applications and are at present more sophisticated than constrained predictive control. The methodology developed in this thesis has the potential to be integrated with one of these more established methodologies.

Question 1 *Can a procedure be defined and conditions given under which control can be used efficiently for corporate policy?*

To find good corporate policies in the general and possibly "singular" case (as explained on page 10), formal models and dynamic optimization cannot be expected to yield any reasonable answers. Moreover, such methodologies may cloud the view and intuition of a good decision maker. — However, we believe that in the subset of critical resource allocation problems or *nonsingular corporate policy* formal methods may play a significant role in complementing and interpreting the information and assumptions available on the environment and the inner workings of the firm.

5.2 Implications for the Decision Maker

Let us distinguish three main implications of our developments for a typical decision maker, before we give comments on the discussed examples.

1. The new paradigm for critical resource allocation induced by the application of the constrained predictive control methodology has been explained in Section 3.4. Our main conclusions were that the required finding of an optimal equilibrium state for the critical resource allocation problem is in fact beneficial to the decision maker, since it translates the abstract problem of maximizing some objective functional into a concrete problem of arriving at a clearly defined state of the system. Still there may be many strategies to get to the optimal state, and there remains the problem of selecting the lowest cost strategy.

2. In the case of an *indefinite objective functional*, the decision maker might also profit from a reformulation using Lemma 2 on page 48 as the problem is transformed into an equivalent form with sign-definite functional that may simplify the problem. Indeed, the new cost functional may reflect more "what really matters" and help to gain insights into the structure of the problem.

3. One of our main concerns was the *stability of nonsingular corporate policies.* The methodology proposed here guarantees stability in the nominal case, under assumptions that may be extremely hard to verify. Further work is needed to develop conditions that can be easily tested in practice and provide the decision maker with good lower bounds on prediction and implementation horizon. — Although beyond the scope of this analysis, the methodology could possibly be applied to cases of structured uncertainty in a competitive market environment, and so helping decision makers avoid the costly boom-bust cycles that are frequently observed in practice.

The Examples. The examples discussed were based on a nonlinear diffusion model of the market. The model is based on a logistic growth assumption that is widely used to explain adoption phenomena for consumer goods or technology. The cost functional, namely discounted profits is applicable to a large class of economic problems. The treatment was limited here to the deterministic case in a market without competition, and interesting extensions can be expected from there. The special structure of our examples satisfied the assumptions of the lemmas developed here so that equivalent formulations of the problems with positive semi-definite cost functional could be obtained.

5.3 Research Directions

Constrained Predictive Control. The area of nonlinear robust control is largely open for research. Managerial decision making could benefit from exploration of robustness results with respect to errors in the system model in combination with a systematic strategy for continually updating a nonlinear system model given an incoming stream of new information. There is also scope for the application of dynamic game theory and potentially stochastic methods to accommodate explicitly for uncertainty. In addition further research would need to include an extensive examination of the practical limitations of the selection scale, that is large, medium or small-scale optimization.

Corporate Policy. An exploration of "modularization" of standard model components and associated might be useful to the building and development of test models. Finally there is an ongoing need for research into the hoary question of just what, other than financial capital, is of "value" to a firm.

Table of Symbols

B	Ball in $\mathbb{R}^n$
C	Set of Controllable States
F_P	Production Function
f, F	System Functions
g	Output Function
h	Cost Kernel
H	Hamiltonian Function
I	Interval, Index Set
J	Cost Functional
l	Terminal Cost Function
p	Adjoint Variable
P	Hyperplane
r	Discount Factor
R	Set of Reachable States
T, T_t^f	Time Horizon, Flow
u	Control Variable
V	Lyapunov Function; Optimal Cost-to-Go
x	State Variable
X	State Constraint Set
y	Output Variable
ε	Step Function : $\varepsilon(\xi) := \begin{cases} 1, & \text{if } \xi \geq 0 \\ 0, & \text{otherwise} \end{cases}$
λ	Positive Identity : $\lambda(\xi) := \xi\,\varepsilon(\xi)$
φ	$\mathcal{K}_\infty$ Function
ψ	Potential
Ψ	Vectorfield
π	Policy
τ	Control Horizon
Ξ	Set of Admissible Initial States
Ω	Control Constraint Set
$\mathcal{P}$	Optimal Control Problem
$\mathcal{U}$	Control Space
$\mathcal{X}$	State Space
$\mathcal{Y}$	Output Space
$\mathbb{R}$	Set of Real Numbers
$\mathbb{Z}$	Set of Integer Numbers

Bibliography

[ÅW90] K.J. Åstrœm and B. Wittenmark. *Computer Controlled Systems.* Prentice Hall, Englewood Cliffs, New Jersey, 1990.

[Bas69] F.M. Bass. A New Product Growth Model for Consumer Durables. *Management Science*, 15(1):215–227, 1969.

[Bel57] R.E. Bellman. *Dynamic Programming.* Princeton Univ. Press, Princeton, New Jersey, 1957.

[Ber95] D.P. Bertsekas. *Dynamic Programming and Optimal Control, Vol. I.* Athena-Scientific, Belmont, Massachusetts, 1995.

[BH96] E. Brynjolfsson and L. Hitt. Paradox Lost? Firm-Level Evidence on the Returns to Information Systems Spending. *Management Science*, 42(4):541–558, 1996.

[BHS96] A.M. Brandenburger and Jr. H.W. Stuart. Value-Based Business Strategy. *Journal of Economics and Management Strategy*, 5(1):5–24, 1996.

[Bie88] H. Bierman. *The Capital Budgeting Decision: Economic Analysis of Investment Projects.* Macmillan, New York, 1988.

[Blo96] M.W.J. Blok. *Dynamic Models of the Firm.* Lecture Notes in Economics and Mathematical Systems. Springer, Berlin, 1996.

[BM91] R.A. Brealy and S.C. Myers. *Principles of Corporate Finance.* McGraw-Hill, New York, 1991.

[BO95] T. Basar and G.J. Olsder. *Dynamic Noncooperative Game Theory.* Academic Press, London, 1995.

[CH87] D.A. Carlson and A. Haurie. *Infinite Horizon Optimal Control.* Lecture Notes in Economics and Mathematical Systems. Springer, Berlin, 1987.

[Cho67] G.C. Chow. Technological Change and the Demand for Computers. *American Economic Review*, 57(5):1117–1130, 1967.

[Cla83] F.H. Clarke. *Optimization and Nonsmooth Analysis.* CMS Ser. of Monogr. and Adv. Texts. Wiley Interscience, New York, 1983.

[Cla89] F.H. Clarke. *Methods of Dynamic and Nonsmooth Optimization.* CBMS-NSF. Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1989.

[CMT87] D.W. Clarke, C. Mohtadi, and P.S. Tuffs. Generalized Predictive Control, Part I & II. *Automatica*, 23(2):137–160, 1987.

[DK90] E. Davis and J. Kay. Assessing Corporate Performance. *Business Strategy Review*, 2:1–16, 1990.

[dN90] R. de Neufville. *Applied Systems Analysis: Engineering Planning and Technology Management.* McGraw-Hill, New York, 1990.

[DP94] A.K. Dixit and R.S. Pindyck. *Investment under Uncertainty.* Princeton Univ. Press, Princeton, New Jersey, 1994.

[EL88] R.D. Eckert and R.H. Leftwich. *The Price System and Resource Allocation.* The Dryden Press, Chicago, 1988.

[For61] J.W. Forrester. *Industrial Dynamics.* MIT Press, Cambridge, Massachusetts, 1961.

[HB96] L. Hitt and E. Brynjolfsson. Productivity, Business Profitability and Consumer Surplus: Three Different Measures of Information Technology Value. *MIS Quaterly*, 20(2):121–142, 1996.

[Heu95] H. Heuser. *Lehrbuch der Analysis, Teil 2.* Teubner, Stuttgart, 1995.

[HM94] S.A. Heise and J.M. Maciejowski. Stable Multivariable Model Based Predictive Control. In *Proc. of the 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, Florida, December 1994.

[HM96] A.C. Hax and N.S. Majluf. *The Strategy Concept and Process.* Prentice Hall, Englewood Cliffs, New Jersey, 1996.

[Kai80] T. Kailath. *Linear System Theory.* Prentice Hall, New York, 1980.

[KBK83] W.H. Kwon, A.M. Bruckstein, and T. Kailath. Stabilizing State-Feedback Design via the Moving Horizon Method. *Intern. J. of Contr.*, 37:631–643, 1983.

[KBM96] M. Kothare, V. Balakrishnan, and M. Morari. Robust Constrained Model Predictive Control Using Linear Matrix Inequalities. *Automatica*, 32(10):1361–1379, October 1996.

[KG86a] S.S. Keerthi and E.G. Gilbert. An Existence Theorem for Discrete-Time Infinite Horizon Optimal Control Problems. *IEEE Trans. on Autom. Contr.*, AC-30:907–909, 1986.

[KG86b] S.S. Keerthi and E.G. Gilbert. Moving-Horizon Approximations for a General Class of Optimal Nonlinear Infinite-Horizon Discrete-Time Systems. In *Proc. 20th Ann. Conf. on Info. Sci. and Syst.*, pages 301–306, Princeton Univ., 1986.

[KG88] S.S. Keerthi and E.G. Gilbert. Optimal Infinite-Horizon Feedback Laws for a General Class of Constrained Discrete-Time Systems: Stability and Moving Horizon Approximations. *Journal of Optimization Theory and Applications*, 57(2):265–293, May 1988.

[Kha92] H.K. Khalil. *Nonlinear Systems.* Macmillan, New York, 1992.

[KP78] W.H. Kwon and A.E. Pearson. On Feedback Stabilization of Time-Varying Discrete Linear Systems. *IEEE Trans. on Autom. Contr.*, 23:479–481, 1978.

[KS91] M.I. Kamien and N.L. Schwartz. *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management.* North-Holland, Amsterdam, 1991.

[MGK96] J.G. Mooney, V. Gurbaxani, and K.L. Kraemer. A Process-Oriented Framework for Assessing the Business Value of Information Technology. *The Database for Advances in Information Technology*, 27(7):814–824, 1996.

[MKM94] J.M. McTaggart, P.M. Kontes, and M.C. Mankins. *The Value Imperative: Managing for Superior Shareholder Returns.* The Free Press, 1994.

[MM90] D.Q. Mayne and H. Michalska. Receding Horizon Control of Nonlinear Systems. *IEEE Trans. on Autom. Contr.*, 35(7):814–824, July 1990.

[MM91] H. Michalska and D.Q. Mayne. Receding Horizon Control of Nonlinear Systems without Differentiability of the Optimal Value Function. *Systems Control Lett.*, 16(2):123–130, 1991.

[MM93] H. Michalska and D.Q. Mayne. Robust Receding Horizon Control of Constrained Nonlinear Systems. *IEEE Trans. on Autom. Contr.*, 38(11):1623–1633, November 1993.

[MMB90] V. Mahajan, E. Muller, and F.M. Bass. New Product Diffusion Models in Marketing: A Review and Directions for Research. *Journal of Marketing*, 54(1):1–26, 1990.

[Mor93] M. Morari. Model Predictive Control: Multivariable Control Technique of Choice in the 1990s? In *Proc. Workshop on Advances in MBPC*, pages 22–37, Oxford, UK, 1993.

[Par94] P.M. Parker. Aggregate Diffusion Forecasting Models in Marketing: A Critical Review. *International Journal of Forecasting*, 10:353–380, 1994.

[PBGM62] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko. *The Mathematical Theory of Optimal Processes.* Wiley Interscience, New York, 1962.

[Per96] L. Perko. *Differential Equations and Dynamical Systems.* Springer, New York, 1996.

[Pet91] M.A. Peteraf. The Cornerstones of Competitive Advantage: A Resource-Based View. *Strategic Management Journal*, 14:179–191, 1991.

[PH90] C.K. Prahalad and G. Hamel. The Core Competence of the Corporation. *Harvard Business Review*, pages 71–90, May 1990.

[Por80] M.E. Porter. *Competitive Strategy: Techniques for Analyzing Industries and Competitors.* The Free Press, New York, 1980.

[Por85] M.E. Porter. *Competitive Advantage: Creating and Sustaining Superior Performance.* The Free Press, New York, 1985.

[Por91] M.E. Porter. Towards a Dynamic Theory of Strategy. *Strategic Management Journal*, 12:95–117, 1991.

[PS93] M. Paich and J. Sterman. Boom, Bust, and Failures to Learn in Experimental Markets. *Management Science*, 39(12):1439–1458, 1993.

[PU96] C.W.I. Pistorius and J.M. Utterback. A Lotka-Volterra Model for Multi-Mode Technological Interaction: Modeling Competition, Symbiosis and Predator-Prey Modes. Working Paper 3929, MIT Intern. Center for Research on the Management of Technology, Cambridge, Massachusetts, March 1996.

[Ric92] B. Richmond. *Introduction to Systems Thinking and iThink.* High Performance Systems, Inc., Hanover, New Hampshire, 1992.

[Ric96] G.P. Richardson. *Modelling for Management : Simulation in Support of Systems Thinking, Vol. I+II.* Aldershot, Hants, England, 1996.

[RK93a] J.A. Rossiter and B. Kouvaritakis. Advances in Generalized and Constrained Predictive Control. In *Proc. Workshop on Advances in MBPC*, pages 303–317, Oxford, UK, 1993.

[RK93b] J.A. Rossiter and B. Kouvaritakis. Constrained Stable Generalised Predictive Control. *IEE Proc.-D*, 140(4):243–254, July 1993.

[RS72] S.M. Roberts and J.S. Shipman. *Two-Point Boundary Value Problems: Shooting Methods.* Elsevier, New York, 1972.

[Sch96] A.L. Schwartz. *Theory and Implementation of Numerical Methods Based on Runge-Kutta Integration for Solving Optimal Control Problems.* Phd thesis, Univ. of California at Berkeley, 1996.

[Soe92] R. Soeterboek. *Predictive Control: A Unified Approach.* Prentice Hall, New York, 1992.

[Ste91] G.B. Stewart. *The Quest for Value: A Guide for Senior Management.* Harper Business, 1991.

[Toz96] E.E. Tozer. *Strategic IS/IT Planning.* Butterworth Heinemann, 1996.

[VC73] T.L. Vincent and E.M. Cliff. An Optimal Policy for a Fish Harvest. *Journal of Optimization Theory and Applications*, 12:485–496, 1973.

[Vin88] R.B. Vinter. New Results on the Relationship between Dynamic Programming and the Maximum Principle. *Math. Contr. Sign. Syst.*, 1:97–105, 1988.

[VV93] R.A.J. De Vries and H.B. Verbruggen. Multivariable Unified Predictive Control. In *Proc. Workshop on Advances in MBPC*, pages 84–102, Oxford, UK, 1993.

[Web95] T. A. Weber. New Model Predictive Control Techniques for Stabilizing Systems Under Constraints. Diploma thesis, Rhenish-Westphalian Technical Univ. Aachen, Imperial College London, May 1995.

[Wil70] J.L. Willems. *Stability Theory of Dynamical Systems.* Wiley Interscience, New York, 1970.

[ZM94] A. Zheng and M. Morari. Global Stabilization of Linear Discrete-Time Systems with Bounded Controls: A Model Predictive Control Approach. In *Proc. ACC*, pages 2847–2851, Baltimore, 1994.