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A DIRECT APPROACH TO COMPENSATOR DESIGN FOR DISTRIBUTED PARAMETER SYSTEMS

by

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ABSTRACT

We present a direct approach to finite-order compensator design for distributed parameter systems, i.e., one that is not based on reduced order modelling. Instead, we use a parametrization around an initial compensator which displays both controller order and closed-loop stability in a convenient way. The main result is an existence theorem which holds for a wide class of linear time-invariant systems (parabolic, delay, damped hyperbolic). The most important assumptions are: bounded inputs and outputs, finitely many unstable modes, completeness of eigenvectors. An example is included, to illustrate the feasibility of our method for purposes of design.

1. Introduction

In the context of systems described by linear partial differential equations or functional differential equations, the problem of stabilization by feedback gains some challenging features that are not present in the finite-dimensional situation. For instance, it is no longer easy to establish necessary and sufficient conditions for the existence of a finite-dimensional compensator that will produce a closed-loop system with a prescribed stability margin. It is an important practical problem to find at least sufficient conditions which will hold for a wide class of interesting systems, since implementation of state feedback ([1], [2]) or of controllers of infinite order ([3], [4], [5]) is often not possible. The most popular approach consists of replacing the infinite-dimensional system by a finite-dimensional "reduced order model", and applying standard techniques to obtain a finite-dimensional compensator for this model. The pertinent question is, of course, how we can be sure that the compensator will also stabilize the original, infinite-dimensional system. It has been shown by examples that, under unfavorable circumstances, the interaction of the controller with the unmodelled part of the system (sometimes termed "spillover") may be such as to de-stabilize the closed-loop system as a whole [6]. Existence results for finite-dimensional compensators have been established recently on the basis of a "zero spillover" assumption ([5], [7], [8]), but this assumption is severely restrictive. Also, existence results can be based on a suitable concept of 'closeness' of the reduced-order model and the actual system. This approach is taken in [9], where the results are still limited in nature. At this point, it should be emphasized that a concept of 'closeness' is also crucial in any study of parameter uncertainty. This aspect is, as well as order reduction, inherent in many discussions of modelling. For the sake of theoretical clarity, we shall keep these two issues apart. In the present paper, we shall assume that the infinite-dimensional system to be controlled is known precisely, and we shall construct a finite-dimensional compensator under this assumption. It is expected that this result can then be used in a further study of what can be done under conditions of parameter uncertainty.

Our approach is not based on reduced-order modelling, and therefore we call it a "direct approach". The core of our method is a certain parametrization of compensators for a given system, which displays both the stability properties of the closed-loop system and the order of the compensator in a convenient way. We shall try to explain the basic idea in Section 2. In Section 3, the set-up is described in a more rigorous fashion. The main result, which establishes the existence of finite-dimensional compensators for a wide class of time-invariant linear systems (including parabolic systems, delay systems, and damped hyperbolic systems), will be given in Section 4. The method of proof is constructive and can be turned into an actual design method, as will be shown by an example in Section 5. Some final remarks follow in Section 6.

2. Heuristics

The purpose of this section is to describe the main idea behind the development in the rest of the paper, without entering into technical details. A rigorous set-up will be described in the next section; here, we just want to give a heuristic discussion.

So let us consider a linear system in its standard state-space form

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.1}$$

where we assume that the pair (A,B) is stabilizable and the pair (C,A) is detectable. We can then choose F such that A+BF is stable and G such that A+GC is stable, and the standard full-order compensator (see, for instance [10]) is then formed by

$$\begin{cases} \hat{x}'(t) = (A+GC)\hat{x}(t) - Gy(t) + Bu(t) \\ u(t) = F\hat{x}(t) \end{cases} \tag{2.2}$$

In the finite-dimensional situation, it is well-known that the closed-loop system obtained by combining (2.1) and (2.2) is described by a system matrix whose eigenvalues are those of A+BF and A+GC taken together ([10,§ 5.2]). Let us examine the compensator (2.2) a little more closely. We can re-write the compensator equations as

$$\begin{cases} \hat{x}'(t) = (A+BF+GC)\hat{x}(t) - Gy(t) \\ u(t) = F\hat{x}(t) \end{cases} \tag{2.3}$$

and hence the compensator transfer matrix is

$$\phi_c(s) = -F(sI-A-BF-GC)^{-1}G. \tag{2.4}$$

Now, there is no reason why (2.3) should represent a minimal realization of this transfer function. If it is not, then the compensator order can be reduced. Even if the McMillan degree of ϕ_c coincides with the order of the system (2.3), there may be transfer matrices with considerable lower McMillan degree that are close enough to ϕ_c to guarantee that they as well will stabilize (2.1). In order to find such transfer matrices, one possible strategy would be to take ϕ_c and to change it a little bit by turning near-cancellations into actual cancellations, thereby decreasing the order of its minimal realization.

The question is, of course, under what conditions we can be sure that such a procedure will lead to a finite-dimensional compensator, if the original system (2.1) is infinite-dimensional. To get at least a partial answer to this, let us return to the state-space setting. The realization (2.3) is non-minimal if the pair (A+BF+GC,G) is not reachable or the pair (F,A+BF+GC) is not observable. We shall concentrate on the reachable set of the pair (A+BF+GC,G), which is of course the same as the reachable set of the pair (A+BF,G). This set is characterized as the smallest subspace V such that $(A+BF)V \subset V$ and $\text{im } G \subset V$. The basic idea which underlies the present paper is the observation that, by manipulation of G alone, we can implement a strategy of slightly perturbing the compensator transfer matrix to decrease its McMillan degree. Even if the original im G is not contained in any (A+BF)-invariant subspace of interesting dimension, it may very well be true that close to G there is a $\tilde{G}$ such that im $\tilde{G}$ does fit into a low-dimensional (A+BF)-invariant subspace. Then the reachable set of the pair $(A+BF+\tilde{G}C,\tilde{G})$ will also be low-dimensional, say equal to k, and it will be possible to construct a compensator of order k based on F and $\tilde{G}$. The stability of the closed-loop system will then depend on A+BF and $A+\tilde{G}C$. We didn't change A+BF, so there is no problem for that part, and it follows from the theorem on continuity of eigenvalues that the stability of $A+\tilde{G}C$ follows from that of A+GC if $\tilde{G}$ is close enough to G. (Actually, we shall use another theorem below, which gives us a ball around G where stability of $A+\tilde{G}C$ is guaranteed: see Lemma 4.3.)

It can also be seen directly from the differential equations (2.2) that a reduction of compensator order is possible if there is a nontrivial subspace V with $(A+BF)V \subset V$ and im $G \subset V$. For this purpose, re-write (2.2) as

$$\begin{cases} \hat{x}'(t) = (A+BF)\hat{x}(t) + G(C\hat{x}(t)-y(t)) \\ u(t) = F\hat{x}(t) \end{cases} \tag{2.5}$$

The equation for $\hat{x}(t)$ is seen to be given by the evolution operator A+BF together with a driving input which enters through G. Since the stabilization action of the compensator should take place for any initial value of $x(\cdot)$, we may as well suppose that $\hat{x}(0) = 0$. Then it is clear that $x(t)$ will be in V for all time. Consequently, no larger state space than V is necessary for $\hat{x}$.

As a third possible interpretation, consider the following matrix argument. Again, if V is a subspace such that $(A+BF)V \subset V$ and im $G \subset V$, then we obviously have the following matrix representations for A+BF and G, with respect to a suitable basis.

$$A+BF = \begin{pmatrix} A_{11}+B_1F_1 & A_{12}+B_1F_2 \\ 0 & A_{22}+B_2F_2 \end{pmatrix} , \qquad G = \begin{pmatrix} G_1 \\ 0 \end{pmatrix} . \tag{2.6}$$

As is easily established from (2.1) and (2.2), the equation describing the closed-loop system is

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} = A_e \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} , \quad A_e = \begin{pmatrix} A & BF \\ -GC & A+BF+GC \end{pmatrix} \tag{2.7}$$

Using the special forms in (2.6) to describe the compensator dynamics, we see that the evolution operator A_e in (2.7) can be given as a three-by-three block matrix:

$$A_e = \begin{pmatrix} A & BF_1 & BF_2 \\ -G_1C & A_{11}+B_1F_1+G_1C_1 & A_{12}+B_1F_2+G_1C_2 \\ 0 & 0 & A_{22}+B_2F_2 \end{pmatrix} \tag{2.8}$$

It is evident from this representation that if A_e is stable, then the two-by-two left upper block in A_e must also be stable. This means that we are able to build a stabilizing compensator (of order dim $\mathcal{V}$) based on G_1, F_1, and $A_{11}+B_1F_1+G_1C_1$. Technically speaking, this is perhaps the cleanest way to describe the situation, and we shall use basically this approach in the rigorous development of later sections.

In summary, the proposed method is the following. We start by selecting a full-order compensator that stabilizes the original system. Then, we parametrize a set of nearby compensators on the basis of the 'injection mapping' G. This parametrization is not necessarily complete, but the stability of the resulting closed-loop systems is easily monitored, and, in particular, there is a ball around the original injection mapping where stability is guaranteed. Moreover, the points in the parameter space where the compensator order is reduced to a given number k are easily spotted, because they correspond to the k-dimensional invariant subspaces of A+BF, which are, at least theoretically speaking, known. So this parametrization allows us to do an effective search for low-order stabilizing compensators. In the infinite-dimensional case, we expect that it will be possible to prove the existence of a finite-dimensional stabilizing compensator if there are finite-dimensional (A+BF)-invariant subspaces arbitrarily close to any given subspace, i.e., if we have completeness of eigenvectors. No further essential restrictions will be required. We shall now proceed to make this precise. It should be emphasized that the procedure we have sketched is meant for theoretical purposes; several alterations may be made to advantage, when a similar method is to be used for practical design purposes. This will be illustrated in the example of Section 5.

3. Assumptions and Preliminaries

We shall consider systems of the form

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & x(t) \in \mathcal{X},\ u(t) \in \mathcal{U} \\ y(t) = Cx(t), & y(t) \in \mathcal{Y} \end{cases} \tag{3.1}$$

under the following basic assumptions:

(A1) A is the generator of a strongly continuous semigroup $T(\cdot)$ of bounded linear operators on the Banach space $\mathcal{X}$.

(A2) B is a bounded linear mapping from the finite-dimensional input space $\mathcal{U}$ into $\mathcal{X}$.

(A3) C is a bounded linear mapping from $\mathcal{X}$ into the finite-dimensional output space $\mathcal{Y}$.

For the general theory of semigroups, we refer to [11]. The condition (A2) requires that the control enters the system in a 'distributed' way, i.e., as a forcing term, rather than via the boundary conditions. The condition (A3) excludes, for instance, taking point observations on an L_2-space. We make these boundedness assumptions here for simplicity.

Following [12, p.181], we shall say that the spectrum of an operator is <u>discrete</u> if it consists only of isolated eigenvalues with finite multiplicities. We shall make the following assumption because it is convenient, and also because it covers the commonly encountered cases.

(A4) The spectrum of A is discrete.

As a measure of stability, we shall use the <u>growth constant.</u> This constant is obtained for every semigroup T(t) (from now on, we shall use the term 'semigroup' as a synonym for 'strongly continuous semigroup of bounded linear operators on a Banach space') by the following formula [11, p.306]:

$$\omega_0 := \inf_{t\in[0,\infty)} \frac{1}{t}\log||T(t)|| = \lim_{t\to\infty} \frac{1}{t}\log||T(t)|| < \infty. \qquad (3.2)$$

The semigroup is said to be <u>asymptotically stable</u> if its growth constant is negative, and the absolute value of the growth constant is then also called the <u>stability margin</u>. Obtaining a reasonable stability margin is a primary purpose of feedback control, and we shall suppose that a desired minimum degree of stability has been specified by a growth constant $\omega<0$ which will be fixed from now on. A semigroup will be called simply <u>stable</u> if its growth constant is smaller than or equal to ω. We shall assume that there are only finitely many unstable or nearly unstable modes:

(A5) There exists $\delta>0$ such that the half-plane $\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\,\lambda>\omega-\delta\}$ contains only finitely many eigenvalues of A.

Under this assumption, we can draw a simple closed curve enclosing precisely those eigenvalues of A that have real parts larger than ω. From this, we obtain a decomposition of the state space X as in [12, p.178]. We shall write $X=X_u \oplus X_s$ where X_u is called the <u>unstable modal subspace</u> and X_s is the <u>stable modal subspace</u>. Correspondingly, the following notation will be used with respect to this decomposition:

$$A = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} \qquad B = \begin{pmatrix} B_u \\ B_s \end{pmatrix}, \qquad C = (C_u \; C_s) \quad . \qquad (3.3)$$

As in the finite-dimensional case, we shall need assumptions on the stabilizability of the pair (A,B) and the detectability of the pair (C,A). In the present context, these are most easily expressed in the following way.

(A6) The pair (A_u, B_u) is controllable.

(A7) The pair (C_u, A_u) is observable.

Note that both pairs involve only operators between finite-dimensional spaces, so that we can rely on the familiar finite-dimensional concepts.

Next, we need an assumption of a somewhat more technical nature. Let $\delta>0$ satisfy the condition of (A5). Then it is clear that one can also do a decomposition of $\mathcal{X}$ with respect to the eigenvalues of A that have real parts larger than $\omega-\delta$ (rather than ω). Let $A_s^{\omega-\delta}$ denote the operator that is obtained in this way, similarly to A_s. It has been shown in [2, App. 2] that $A_s^{\omega-\delta}$ generates a semigroup. We shall assume the following.

(A8) The growth constant of the semigroup generated by $A_s^{\omega-\delta}$ is smaller than ω.

We know, of course, that the eigenvalues of $A_s^{\omega-\delta}$ all have real parts smaller than or equal to $\omega-\delta$, but counter examples ([11, p.665], [13]) show that this in itself does not guarantee that the growth constant of the semigroup will be bounded by $\omega-\delta$ or by ω. One solution, then, is to introduce a "spectrum determined growth assumption" like (A8). This solution has been proposed in [2], where it has also been argued that the assumption holds for various important classes of semigroups.

For an alternative, we should consider our ultimate purposes. To the system (3.1), we want to add a finite-dimensional <u>compensator</u> of the form

$$\begin{cases} w'(t) = A_c w(t) + G_c y(t) & w(t) \in \mathcal{W}, \quad \dim \mathcal{W} < \infty \\ u(t) = F_c w(t) + Ky(t) & \end{cases} \tag{3.4}$$

Doing so, we obtain a closed-loop system which looks like

$$\frac{d}{dt}\begin{pmatrix}x\\w\end{pmatrix}(t) = A_e\begin{pmatrix}x\\w\end{pmatrix}(t) \tag{3.5}$$

where the closed-loop system mapping A_e is given by

$$A_e = \begin{pmatrix}A+BKC & BF_c\\ G_cC & A_c\end{pmatrix}. \tag{3.6}$$

This operator generates a semigroup on $X \oplus W$, since it is a bounded perturbation of

$$\tilde{A}_e = \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix} \tag{3.7}$$

[11, p. 389]. For our purposes, it will be easily sufficient if we know the following:

(A8)' For any choice of the matrices K, F_c, G_c, and A_c in (3.6), the growth constant of the semigroup generated by A_e is equal to $\sup\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A_e)\}$.

We shall primarily use (A8), because this assumption is probably in most cases more directly verifiable (see [2]). However, in some instances it may be easy to check that (A8)' is true, and then (A8) can be dispensed with. In engineering contexts, (A8)' is often assumed without mentioning.

For our final assumption, we point out that we shall call any nonzero vector in the range of the eigenprojection associated with a given eigenvalue [12, p.181] an eigenvector, so this includes 'generalized eigenvectors'. A set of elements of X is called complete (in X) if the finite linear combinations of these elements form a dense set in X. We assume the following.

(A9) The eigenvectors of A form a complete set in X.

Completeness of eigenvectors is a common property for diffusion operators, delay operators, and wave operators as well; see, for instance, [14, p.325], [15, pp.465-470], [16, pp.278-289], [17], [18], and [19, p.250]. Under the stated assumptions, it will be shown below (Lemma 4.5) that there exists a feedback mapping $F: X \to U$ such that the spectrum of A+BF is discrete and contained in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \leq \omega\}$, and such that the eigenvectors of A+BF form a complete set in X. We could use this statement to replace both (A6) and (A9), but since these assumptions are stated directly in terms of A, we prefer to use then, rather than an indirect (be it weaker) expression.

For easy reference, we shall state here the following lemma, which will be used repeatedly. The proof presents no basic difficulties and will be omitted.

Lemma 3.1. Suppose that A_{11} and A_{22} are generators of semigroups on the Banach spaces X_1 and X_2, respectively, with growth constants ω_1, and ω_2. Suppose also that $A_{21}: X_1 \to X_2$ is a bounded linear mapping. Then the operator on $X_1 \oplus X_2$ defined by

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \tag{3.8}$$

generates a semigroup whose growth constant equals $\max(\omega_1, \omega_2)$.

4. Existence Result

Our aim in this section is to prove the following result.

Theorem 4.1. Consider the system (3.1), and suppose that the assumptions (A1)-(A8) hold for some given growth constant ω. Then there exists a compensator of finite order such that the evolution of the controlled system is described by a strongly continuous semigroup with growth constant smaller than or equal to ω.

For convenience, we shall break up the proof of this theorem into four separate lemmas.

Lemma 4.2. Consider the system (3.1) under the assumptions (A1)-(A3). Let ω be a given growth constant, and suppose that there exist a finite-dimensional subspace $\mathcal{V} \subset D(A)$ and linear mappings $F: \mathcal{V} \to \mathcal{U}$ and $G: \mathcal{Y} \to X$ with the following properties:

$$\text{im } G \subset \mathcal{V} \tag{4.1}$$

the semigroup generated by A+GC has growth constant

$$\omega_1 \leq \omega \tag{4.2}$$

$$(A+BF)x \in \mathcal{V} \text{ for all } x \in \mathcal{V} \tag{4.3}$$

the (finite-dimensional) semigroup generated by $A+BF|_{\mathcal{V}}$ has growth constant $\omega_2 \leq \omega$. (4.4)

Then there exists a compensator of the form (3.4), which has (finite) order equal to dim $\mathcal{V}$, and which is such that the evolution of the controlled system is described by a semigroup with growth constant $\max(\omega_1, \omega_2) \leq \omega$.

Proof. Introduce a new linear space $\mathcal{W}$ isomorphic to $\mathcal{V}$, and let $R:\mathcal{V} \to \mathcal{W}$ be the mapping that provides the isomorphism. Define a compensator of the form (3.4) by setting $K=0$, $F_c = FR^{-1}$, $G_c = -RG$, and $A_c = R(A+BF+GC)R^{-1}$. (Note that it follows from (4.1) and (4.3) that G_c and A_c are well-defined, even though R is not defined on all of $\mathcal{X}$.) We can write the following differential equation for the controlled system:

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}(t) = A_e \begin{pmatrix} x \\ w \end{pmatrix}(t) \tag{4.5}$$

with the extended system mapping A_e given by

$$A_e = \begin{pmatrix} A & BF_c \\ G_cC & A_c \end{pmatrix} = \begin{pmatrix} A & BFR^{-1} \\ -RGC & R(A+BF+GC)R^{-1} \end{pmatrix} . \tag{4.6}$$

Consider the following subspace of the extended state space $\mathcal{X}_e := \mathcal{X} \oplus \mathcal{W}$:

$$\mathcal{M} := \{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \mathcal{V} \}. \tag{4.7}$$

There is an obvious isomorphism between $\mathcal{V}$ and $\mathcal{M}$, given by

$$Tx = \begin{pmatrix} x \\ Rx \end{pmatrix} , \quad x \in \mathcal{V} . \tag{4.8}$$

The space $\mathcal{X}_e$ can also be decomposed as $\mathcal{X} \oplus \mathcal{M}$, rather than as $\mathcal{X} \oplus \mathcal{W}$. Written with respect to this decomposition, A_e will have the form

$$\tilde{A}_e := HA_eH^{-1} , \tag{4.9}$$

where the isomorphism $H: \mathcal{X} \oplus \mathcal{W} \to \mathcal{X} \oplus \mathcal{M}$ is defined by

$$H = \begin{pmatrix} I & -R^{-1} \\ 0 & TR^{-1} \end{pmatrix} \tag{4.10}$$

By straightforward computation, we find that

$$\tilde{A}_e = \begin{pmatrix} A+GC & 0 \\ -TGC & T(A+BF)T^{-1} \end{pmatrix} . \tag{4.11}$$

Noting that $T(A+BF)T^{-1}$ is similar to $A+BF|_{\mathcal{V}}$, we now immediately get the result by an application of Lemma 3.1.

<u>Lemma 4.3</u>: Consider a pair of mappings (C,A) under the assumptions (A1), (A3), (A4), (A5), (A7), and (A8). Then we can find a linear mapping $G:\mathcal{Y} \to \mathcal{X}$ and a constant $\eta>0$ such that, for every $\tilde{G}$: $\mathcal{Y} \to \mathcal{X}$ satisfying $||G-\tilde{G}|| < \eta$, the semigroup generated by $A+\tilde{G}C$ is stable.

<u>Proof</u>. We shall use the same modal decomposition that has been used to formulate (A8), and we shall further decompose the 'unstable' parts $A_u^{\omega-\delta}$ and $C_u^{\omega-\delta}$ (cf. (3.3)) in order to display the unobservable subspace of this pair. The final result of these operations is a decomposition of the form

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}, \qquad C = (C_1 \;\; C_2 \;\; 0) \tag{4.12}$$

where $\mathrm{Re}\,\lambda \leq \omega-\delta$ for $\lambda \in \sigma(A_{11})$, the pair (C_2, A_{22}) is observable, and $\omega-\delta < \mathrm{Re}\,\lambda \leq \omega$ for $\lambda \in \sigma(A_{33})$. (The last inequality follows from (A7).) By the observability of the pair (C_2, A_{22}), there exists a G_2 such that all eigenvalues of $A_{22} + G_2C_2$ have real parts smaller than ω. Define G by

$$G = \begin{pmatrix} 0 \\ G_2 \\ 0 \end{pmatrix} . \tag{4.13}$$

In general, for $G = (\tilde{G}_1^t\ \tilde{G}_2^t\ \tilde{G}_3^t)^t$, we get

$$A + \tilde{G}C = \begin{pmatrix} A_{11}+\tilde{G}_1C_1 & \tilde{G}_1C_2 & 0 \\ \tilde{G}_2C_1 & A_{22}+\tilde{G}_2C_2 & 0 \\ \tilde{G}_3C_1 & A_{32}+\tilde{G}_3C_2 & A_{33} \end{pmatrix} \tag{4.14}$$

If $\tilde{G}=G$, it follows from our construction, from Lemma 3.1, and from assumption (A8), that the two-by-two left upper block in (4.14) generates a semigroup whose growth constant is smaller than ω. By the general result on bounded perturbation of semigroups (see, for instance, [20, p.38]), this entails that the same block will also generate a stable semigroup if $||\tilde{G}-G||$ is small enough. Since the eigenvalues of A_{33} all have real parts smaller than or equal to ω, this means again by Lemma 3.1, that the semigroup generated by $A+\tilde{G}C$ is stable as well.

Lemma 4.4. Consider the system (3.1) under the assumptions (A1)-(A3). Let $G:\mathcal{Y}\to\mathcal{X}$ by a given injection mapping, and suppose that there exists $F:\mathcal{X}\to\mathcal{U}$ such that the eigenvectors of A+BF are complete in $\mathcal{X}$. Then, for any $\eta>0$, there exist a finite-dimensional subspace $\mathcal{V}\subset D(A)$ and a mapping $\tilde{G}:\mathcal{Y}\to\mathcal{X}$ such that

$$||\tilde{G}-G|| < \eta \tag{4.15}$$

$$(A+BF)x \in \mathcal{V} \text{ for all } x \in \mathcal{V} \tag{4.16}$$

$$\text{im } \tilde{G} \subset \mathcal{V} \tag{4.17}$$

Proof. Pick some orthonormal basis $\{y_1,\ldots,y_p\}$ of $\mathcal{Y}$, and write $g_i:=Gy_i$. Let $\eta>0$ be given. For every $i=1,\ldots,p$, there exists a finite set $\{x_{i1},\ldots,x_{iN(i)}\}$ of generalized eigenvectors of A+BF such that

$$||g_i - \sum_{j=1}^{N(i)} \alpha_{ij} x_{ij}|| < \eta \tag{4.18}$$

for suitable constants α_{ij} $(i=1,\ldots,p;\ j=1,\ldots,N(i))$. To every (i,j) there exist a $\lambda_{ij} \in \mathbb{C}$ and an $n_{ij} \in \mathbb{N}$ such that

$$(\lambda_{ij} - (A+BF))^{n_{ij}} x_{ij} = 0 . \tag{4.19}$$

Now define $\tilde{G}: \mathcal{Y} \to \mathcal{X}$ by $\tilde{G} y_i = \tilde{g}_i$ $(i=1,\ldots,p)$, where

$$\tilde{g}_i := \sum_{j=1}^{N(i)} \alpha_{ij} x_{ij} \tag{4.20}$$

and let $\mathcal{V}$ be the subspace defined by

$$\mathcal{V} := \operatorname{span}\{(\lambda_{ij} - (A+BF))^k x_{ij} \,|\, i=1,\ldots,p;\ j=1,\ldots,N(i);\ k=0,\ldots,n_{ij}-1\} \tag{4.21}$$

Then $\tilde{G}$ and $\mathcal{V}$ satisfy the requirements.

Lemma 4.5: Consider a pair of operators (A,B), and suppose that the assumptions (A1), (A2), (A4), (A5), (A6), and (A9) hold. Then there exists a bounded linear mapping $F: \mathcal{X} \to \mathcal{U}$ such that the spectrum of $A+BF$ is discrete, all eigenvalues of $A+BF$ have real parts smaller than or equal to ω, and the eigenvectors of $A+BF$ are complete in $\mathcal{X}$.

Proof. Doing a modal decomposition with respect to the eigenvalues of A in $\{\lambda \in \mathbb{C} \,|\, \operatorname{Re} \lambda > \omega\}$, we obtain a direct sum representation $\mathcal{X} = \mathcal{X}_u \oplus \mathcal{X}_s$, and corresponding block representations for A and B:

$$A = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix} , \qquad B = \begin{pmatrix} B_u \\ B_s \end{pmatrix} . \tag{4.22}$$

By (A6), we can choose F_u such that the eigenvalues of $A_u + B_uF_u$ are in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \le \omega\}$ and such that they are distinct from the eigenvalues of A_s. Define F by

$$F = (F_u \quad 0). \tag{4.23}$$

Then the spectrum of A+BF will consist of the eigenvalues of A_s together with those of $A_u+B_uF_u$. Because the two sets are separated, there is a corresponding modal decomposition, which we shall indicate by $X = X_s \oplus X_n$ ('n' for 'new'). Hence, every vector $x \in X$ can be written as $x = x_s + x_n$ with $x_s \in X_s$ and $x_n \in X_n$. By (A9), x_s can be approximated by linear combinations of eigenvectors of A in X_s, which are, as a consequence of the special form of F, also eigenvectors of A+BF. Because X_n is a finite-dimensional (A+BF)-invariant subspace, x_n is equal to some linear combination of eigenvectors of A+BF. We conclude that x can be approximated by linear combinations of eigenvectors of A+BF. Thus, the eigenvectors of A+BF are complete in X.

Proof (of Thm. 4.1). Choose G as in Lemma 4.3 and F as in Lemma 4.5. Let $\eta>0$ be the constant from Lemma 4.3, and use Lemma 4.4 to obtain $\tilde{G}: Y \to X$ and $V \subset D(A)$ satisfying (4.15-17). Finally, apply Lemma 4.2 to the subspace V and the mappings F and $\tilde{G}$.

The proof of the theorem is constructive, and therefore it suggests a design method. Depending on the particular type of equation one has at hand, one may vary the actual form of this method in order to avoid unnecessary work. In the next section, we shall illustrate this by an example.

5. Design Example

Consider the following system, which is of the 'delay' type:

$$\begin{cases} x_1'(t) = -\frac{\pi}{2}\, x_1(t-1) + x_2(t) \\ x_2'(t) = u(t) \end{cases} \tag{5.1}$$

$$y(t) = x_1(t). \tag{5.2}$$

To write these equations in the standard form (3.1), we use the following set-up (cf. [21]). Let $M_2(-1,0)$ denote the product space $\mathbb{R} \times L_2(-1,0)$, and let $H^1(-1,0)$ be the set of functions on $[-1,0]$ whose distributional derivative is in $L_2(-1,0)$ [22, p.44]. By Sobolev's lemma [22, p.97], the mappings $\phi \mapsto \phi(-1)$ and $\phi \mapsto \phi(0)$ are well-defined and continuous functions on $H^1(-1,0)$. For the equation (5.1), the state space will be

$$\mathcal{X} := M_2(-1,0) \oplus \mathbb{R}\ . \tag{5.3}$$

The elements of this linear space will be written as column vectors with two components, where the first component is in $M_2(-1,0)$ and will be written as a row vector (ϕ_0,ϕ) with $\phi_0 \in \mathbb{R}$ and $\phi \in L_2(-1,0)$, and the second component is in $\mathbb{R}$. The operator A is defined by

$$D(A) := \left\{ \begin{pmatrix} (\phi_0,\phi) \\ \alpha \end{pmatrix} \middle|\ \phi_0 \in \mathbb{R}, \phi \in H^1(-1,0),\ \alpha \in \mathbb{R},\ \phi(0) = \phi_0 \right\} \tag{5.4}$$

$$A \begin{pmatrix} (\phi_0,\phi) \\ \alpha \end{pmatrix} := \begin{pmatrix} (-\frac{1}{2}\pi\phi(-1)+\alpha,\ \phi') \\ 0 \end{pmatrix} \tag{5.5}$$

The input space $\mathcal{U}$ and the output space $\mathcal{Y}$ are both equal to $\mathbb{R}$, and the mappings B and C are given by

$$B\alpha = \begin{pmatrix} (0,0) \\ \alpha \end{pmatrix} \tag{5.6}$$

$$C \begin{pmatrix} (\phi_0, \phi) \\ \alpha \end{pmatrix} = \phi_0 \ . \tag{5.7}$$

We shall also use the complexifications of these spaces and operators, without change of notation.

It follows from the results of [23] (see also [21]) that the operator A generates a semigroup on $\mathcal{X}$. It is seen immediately that the operators B and C are bounded. The spectrum of A is discrete, and the eigenvalues are precisely the roots of the characteristic equation

$$\det \begin{pmatrix} \lambda + \frac{\pi}{2} e^{-\lambda} & -1 \\ 0 & \lambda \end{pmatrix} = 0 \tag{5.8}$$

[18; Prop. 4.2]. The characteristics function

$$\Delta_A(\lambda) := \lambda(\lambda + \frac{\pi}{2} e^{-\lambda}) \tag{5.9}$$

has roots at 0, $\pm \frac{1}{2} \pi i$, and at infinitely many other points in the complex plane which are given approximately by

$$\lambda_k \cong -\log(4k+1) \pm \frac{\pi}{2} (4k+1) i \qquad (k \in \mathbb{N}) \ . \tag{5.10}$$

Rules for deriving such formulas are given in [30]. All roots are simple. We see that there are only finitely many eigenvalues of A to the right of any vertical line in the complex plane, as is true in general for delay equations [24; p.114]. The stabilizability of the pair (A,B) and the detectability of the pair (C,A) can be verified conveniently using the generalization of the Hautus test ([25], [26]) that was given in [3].

Because

$$\operatorname{rank}\begin{pmatrix} \lambda+\frac{\pi}{2}e^{-\lambda} & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} = 2 \qquad \text{for all } \lambda\in\mathbb{C}\ , \tag{5.11}$$

the pair (A,B) is stabilizable no matter how the desired growth constant ω is chosen. Likewise, detectability of the pair (C,A) also holds for any ω because

$$\operatorname{rank}\begin{pmatrix} \lambda+\frac{\pi}{2}\,e^{-\lambda} & 1 \\ 0 & \lambda \\ 1 & 0 \end{pmatrix} = 2 \qquad \text{for all } \lambda\in\mathbb{C}\ . \tag{5.12}$$

Adding a compensator of the form (3.4) to the system (5.1-2) will lead to a closed-loop system which still has the basic form of a delay equation:

$$x'(t) = A_1x(t-1) + A_0x(t)\ . \tag{5.13}$$

Consequently, the closed-loop semigroup will be compact for $t>1$ ([31]; see also [24]), and this is sufficient to guarantee that its growth constant is determined by the spectrum of its generator [11, p.467]. So we can use assumption (A8)' instead of (A8). Finally, the completeness of the eigenvectors of A follows from [17; Cor. 5.5].

We have verified that all assumptions of Section 3 are satisfied for any choice of the desired growth constant ω. Hence, it follows from Thm. 4.1 that any degree of stability can be obtained by adding a finite-dimensional compensator to the system (3.1). Let us design such a compensator to obtain a stability margin of 1; so we set $\omega=-1$.

<u>First Step</u>. By the stabilizability of the pair (A,B), there exists an F such that the eigenvalues of A at 0 and $\pm\frac{\pi}{2}i$ are shifted to new eigenvalues at -1 and $-1\pm\frac{\pi}{2}i$ for A+BF. If μ is an eigenvalue of A+BF, it is easily

verified that the corresponding eigenvector is given by

$$\phi = \begin{pmatrix} (\phi_0, \phi) \\ \alpha \end{pmatrix}, \qquad \phi(\theta) = e^{\mu\theta}\phi_0 (\theta \in [-1,0]), \quad \alpha = (\mu + \frac{\pi}{2} e^{-\mu})\phi_0. \tag{5.14}$$

The eigenvector will be normalized such that $C\psi = 1$ if we put $\phi_0 = 1$. In that case, we also have

$$F\psi = \mu(\mu + \frac{\pi}{2} e^{-\mu}) = \Delta_A(\mu) . \tag{5.15}$$

<u>Second Step</u>. The matrices of A_u and C_u with respect to the basis

$$\begin{pmatrix} (1, \cos\frac{\pi}{2}\theta) \\ 0 \end{pmatrix} \quad \begin{pmatrix} (0, \sin\frac{\pi}{2}\theta) \\ 0 \end{pmatrix} \quad \begin{pmatrix} (\frac{2}{\pi}, \frac{2}{\pi}) \\ 1 \end{pmatrix} \tag{5.16}$$

of X_u are given by

$$A_u = \begin{pmatrix} 0 & \frac{1}{2}\pi & 0 \\ -\frac{1}{2}\pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C_u = \begin{pmatrix} 1 & 0 & \frac{2}{\pi} \end{pmatrix} \tag{5.17}$$

A straightforward pole placement procedure leads to the conclusion that $A+GC$ will have new eigenvalues at $-\frac{1}{2}\pi$ (double) and $-\pi$ if we take

$$G = -\pi \begin{pmatrix} (2, \cos\frac{\pi}{2}\theta + 2\sin\frac{\pi}{2}\theta + 1) \\ \frac{1}{2}\pi \end{pmatrix} \tag{5.18}$$

<u>Third Step</u>. Although it is possible, in principle, to compute η such that $A+\tilde{G}C$ will be stable for each $\tilde{G}$ with $||\tilde{G}-G|| < \eta$, it does not seem attractive

to perform the actual computations and, moreover, the bound we obtain may be unnecessarily conservative. Rather, we shall proceed in an algorithmic way. Let us select

$$\tilde{G} = 2.08 \begin{pmatrix} (1,\ e^{-\theta}\cos\frac{\pi}{2}\theta) \\ -1 \end{pmatrix} - 9.08 \begin{pmatrix} (0,\ e^{-\theta}\sin\frac{\pi}{2}\theta) \\ \frac{1}{2}\pi(1-e) \end{pmatrix} - 8.36 \begin{pmatrix} (1,\ e^{-\theta}) \\ -1+\frac{1}{2}\pi e \end{pmatrix} \tag{5.19}$$

which is obtained by orthogonally projecting G into the subspace spanned by the eigenvectors of A+BF corresponding to the eigenvalues at -1 and $-1 \pm \frac{1}{2}\pi i$. A convenient way to compute the eigenvalues of $A+\tilde{G}C$ is provided by the Weinstein-Aronzain theory [12, p.244], from which it follows that these eigenvalues can be found as the zeros of

$$\Delta_{A+\tilde{G}C}(\lambda) := \Delta_A(\lambda)(1-C(\lambda-A)^{-1}G) . \tag{5.20}$$

If $\tilde{G}$ maps into a subspace spanned by finitely many eigenvectors of A+BF, so that

$$G = \sum_{k=1}^{m} \gamma_k \begin{pmatrix} (1,\ e^{\mu_k\theta}) \\ \mu_k + \frac{\pi}{2} e^{-\mu_k} \end{pmatrix} \tag{5.21}$$

then we have the more explicit formula

$$\Delta_{A+\tilde{G}C}(\lambda) = \Delta_A(\lambda) - \sum_{k=1}^{m} \gamma_k (\lambda-\mu)^{-1}(\Delta_A(\lambda) - \Delta_A(\mu)). \tag{5.22}$$

Using this, we can employ a simple Newton method to compute the eigenvalues of $A+\tilde{G}C$, where $\tilde{G}$ is given by (5.19). Initial guesses are provided by (5.10) and by the assigned values $-\frac{\pi}{2}$ and $-\pi$. The results are given in Table I.

We see that this trial is easily successful, and so we shall base our design on F, $\tilde{G}$, and the subspace V spanned by the eigenvectors of A+BF associated with the eigenvalues at -1 and $-1 \pm \frac{\pi}{2} i$.

Fourth Step. Written in a somewhat sloppy way (with omission of isomorphisms), our compensator is given by

$$w'(t) = (A+BF+\tilde{G}C)w(t) - \tilde{G}y(t) \tag{5.23}$$

$$u(t) = Fw(t) \tag{5.24}$$

where the state space of w(t) is the three-dimensional subspace of $M_2(-1,0) \oplus \mathbb{R}$ that is spanned by the vectors

$$w_1 = \begin{pmatrix} (1,\ e^{-\theta} \cos \frac{\pi}{2}\theta) \\ -1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} (0, e^{-\theta} \sin \frac{\pi}{2}\theta) \\ \frac{1}{2}\pi(1-e) \end{pmatrix} \quad w_3 = \begin{pmatrix} (1,\ e^{-\theta}) \\ -1 + \frac{1}{2}\pi e \end{pmatrix}. \tag{5.25}$$

The coordinates of $\tilde{G}$ with respect to this basis are given by (5.19). The matrices of A+BF and C are easily found to be

$$A+BF = \begin{pmatrix} -1 & \frac{1}{2}\pi & 0 \\ -\frac{1}{2}\pi & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = (1 \quad 0 \quad 1) \quad . \tag{5.26}$$

Finally, we can use (5.15) to calculate $Fw_1 = 5.24$, $Fw_2 = 1.13$, and $Fw_3 = -3.27$. We finally arrive at the following compensator equations:

$$w'(t) = \begin{pmatrix} 1.08 & 1.57 & 2.08 \\ -10.65 & -1 & -9.08 \\ -8.63 & 0 & -9.36 \end{pmatrix} w(t) + \begin{pmatrix} -2.08 \\ 9.08 \\ 8.36 \end{pmatrix} y(t) \tag{5.27}$$

$$u(t) = (5.24 \quad 1.13 \quad -3.27)\,w(t). \tag{5.28}$$

The eigenvalues of the closed-loop system are given by -1, $-1 \pm \frac{\pi}{2} i$, and the eigenvalues of $A+\tilde{G}C$ as listed in Table I. Consequently, the closed-loop growth constant is exactly equal to -1.

In conclusion, we can say that the computational work needed to obtain the finite-dimensional compensator has been quite moderate: nothing was needed that goes beyond the power of hand-held calculators. Also, note that it has not been necessary to compute the modal projection. The method could be implemented as an interative procedure, with the third step as the iteration step. The iteration consists of projecting G into a series of trial (A+BF)-invariant subspaces of increasing dimension. In this interpretation, Thm. 4.1 can be viewed as a convergence result, guaranteeing that the procedure will terminate after a finite number of steps. Finally, we note that the compensator we obtain is in the standard finite-dimensional form, unlike the compensators obtained from algebraic methods (see, for instance, [27]), which in general contain delay elements.

6. Final Remarks

Although we have worked an example to show that the method presented here is in principle feasible as a design procedure, the main emphasis of this paper has been on establishing the existence result on finite-dimensional compensators for a wide class of infinite-dimensional systems. There are many other design considerations, besides the stability margin, that have to be taken into account in any practical situation, such as robustness properties and sensitivity reduction. Fortunately, the method we have employed leaves a great deal of freedom, and in particular the selection of the initial F and G is expected to be helpful in obtaining good closed-loop properties. We did not really scrutinize our method to arrive at as low as possible controller orders; here, too, further research promises to be fruitful. The parameterization on the basis of the injection mapping G is particularly suited for situations in which we have few outputs and many inputs; in the reverse situation, one should work with a parametrization on the basis of the feedback mapping F and with subspaces of finite codimension. It has been shown in [28] that ideas very similar to the ones presented here will lead to finite-dimensional compensators that solve tracking and regulation problems for distributed parameter systems. It is, of course, of interest to extend our results to situations in which we have unbounded control and sensing; results in this direction have been reported recently in [29].

roots of $\Delta_{A+GC}(\lambda)$	roots of $\Delta_{A+\tilde{G}C}(\lambda)$
-1.571 (double)	$-1.491 \pm 0.288\ i$
-3.142	-3.401
$-1.604 \pm 7.647\ i$	$-1.609 \pm 7.854\ i$
$-2.198 \pm 13.98\ i$	$-2.197 \pm 14.14\ i$
$-2.567 \pm 20.29\ i$	$-2.565 \pm 20.42\ i$
$-2.835 \pm 26.60\ i$	$-2.833 \pm 26.70\ i$
$-3.046 \pm 32.89\ i$	$-3.045 \pm 32.99\ i$
$-3.220 \pm 39.19\ i$	$-3.219 \pm 39.27\ i$
$-3.368 \pm 45.48\ i$	$-3.367 \pm 45.55\ i$
$-3.497 \pm 51.77\ i$	$-3.497 \pm 51.84\ i$
$-3.612 \pm 58.06\ i$	$-3.611 \pm 58.12\ i$

TABLE I. EFFECTS OF PERTURBATION OF G.

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