Three Essays on Economic Dynamics

by

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ABSTRACT

This thesis consists of three essays:

In Chapter 2, I construct a dynamic general equilibrium model with capital, multiple equilibria, and stochastic phase switching. Under the model, recessions coming after long booms are likely to be faster, as are booms coming after long recessions.

In Chapter 3, co-authored with Michael Kremer, we create a model of an open-access, renewable, storable natural resource in which expectations of extinction may be self-fulfilling. We examine potential policy implications.

In Chapter 4 "On the Effect of Changing Activity on HIV Prevalence", co-authored with Michael Kremer, we investigate the effect of a public-health externality in epidemiology and find that, with a simple model, some members of the population may reduce the long-run prevalence of HIV by increasing their rate of change of sexual partners. Calibrations of the model indicate that these effects may be significant in relatively low prevalence populations.

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Chapter 1

Introduction


Chapters 2 and 3 both concern themselves with how externalities may generate multiple equilibria in dynamic, forward-looking, rational models. Chapter 4 is unrelated, and focuses on the effects of a public-health externality on HIV/AIDS epidemiology.

Chapter 2 presents a general equilibrium model of an economy in which aggregate labor has a positive externality on production by individual capital and labor. This produces multiple rational expectations equilibria, which persist when constant hazard phase switching sunspots are introduced. Because capital adjusts slowly, the model implies that recessions coming after long booms should be faster in their initial phase than recessions coming after short booms. The same applies to booms coming after recessions.

Chapter 3 investigates the effects of expectations on equilibria in markets for storable goods produced from open-access renewable natural resources. Existing models of open-access resources are applicable to non-storable resources, such as
fish. Many open-access resources, however, are used to produce storable goods. Elephants, rhinos, and tigers are three prominent examples. Anticipated future scarcity of these resources will increase current prices, and current poaching. This implies that, for given initial conditions, there may be rational expectations equilibria leading both to extinction and to survival. Governments may be able to eliminate extinction equilibria by promising to implement tough anti-poaching measures if the population falls below a threshold. Alternatively, they, or private agents, may be able to eliminate extinction equilibria by accumulating a sufficient stockpile of the storable good, and threatening to use it to manipulate the price of the good, much as central banks use currency reserves to defend an exchange rate. A version of this chapter has been published as NBER Working Paper 5674 (Kremer & Mercom 1996).

Chapter 4 is an analysis of the effects on HIV prevalence of changing sexual activity. Reductions in the rate of partner change by those with low sexual activity increase the average probability of HIV infection in the remaining pool of available partners. This increases prevalence among people with high activity, and since high activity people disproportionately influence the spread of HIV, may increase long-run prevalence in the population as a whole. Calculations using a preferred mixing model and survey data on sexual activity indicate that in low prevalence populations, most people have low enough activity that further reductions would increase the endemic steady state prevalence. To the extent that these results prove robust in more realistic models, they support the case for targeting public health messages urging reduced sexual activity to high activity people, and emphasizing condom use, rather than abstinence.
Chapter 2

Why Recessions After Longer Booms are Swifter: Dynamic Multiple Equilibria With Uncertainty

2.1 Introduction

In this paper I present a dynamic model in which an externality in aggregate labor affects individual Cobb-Douglas production. With perfect foresight, the model has multiple equilibria according to which level of aggregate labor the economy coordinates on. When there is a sunspot which causes random changes in coordination with constant hazard, the economy switches between high and low activity phases in a way which is rationally anticipated by consumers. I relate the sunspot to imperfect information about the state of the economy. The system with coordination uncertainty, thus, still has multiple equilibria, but it is ergodic.

The model has a number of implications for the structure of changes in economic
activity. First, since the phases are persistent, serial correlation should be expected in economic indicators.

Second, the model has the property that real interest rates should be lower in a recession than in a boom, given the same level of capital.

Third, recessions after a long boom should be faster initially than those after a relatively short boom.

Fourth, it may be Pareto improving for the government to publish less precise economic information during a recession than during a boom.

There is substantial evidence that the existence of multiple equilibria, or “phase switching” is consonant with US business cycles as in, for example, Hamilton (1989), Diebold, Rudebusch & Sichel (1993), and Filardo (1994). Combined with this, there is a wealth of micro-founded theory to suggest that non-convex production is easily possible, and some empirical evidence to suggest that externalities may actually be quite large (Caballero & Lyons 1992). Finally, the large theoretical body of literature on multiple equilibrium models shows that they are easy to construct using plausible parameter values (As in the JET symposium on Growth, Fluctuations, and Sunspots: e.g. (Benhabib & Perli 1994, Benhabib & Rustichini 1994, Benhabib & Farmer 1994, Gali 1994)).

There is a substantial literature concerning models (at least since Diamond (1982), Kiyotaki (1988), and Cooper & John (1988)), in which increasing returns, or externalities lead to multiple equilibria.

Excluding models which generate cycles, or other, more complex, limit sets\(^1\), there are two different kinds of multiple equilibria: those arising from a locally indeterminate steady state (so that the stable manifold has dimension greater than one), and those arising from global indeterminacy between two distinct steady states which may or may may not be locally determinate.

Models with local indeterminacy, but one steady state have been extensively

\(^1\)Such as Diamond & Fudenberg (1987), a perfect foresight search model exhibiting stable cycles.
analyzed. A sunspot may cause "endogenous" fluctuations around the single steady state, which have been used to explain business cycles in a way which doesn't depend on technology shocks, as in Farmer & Guo (1994), or Woodford (1992).

I focus on global indeterminacy. Many models of this sort have been presented but they all have one or more of the following properties: perfect foresight, partial equilibrium, or myopic agents.

Almost all models assume perfect foresight to find solutions. This makes it difficult usefully to discuss changes in phase, as such changes must either be perfectly anticipated, or totally unexpected. Few would suggest that such phenomena as business cycles are perfectly anticipated. Assuming that changes are unexpected instantly renders one vulnerable to the Lucas Critique if one is trying to discuss changes which a rational consumer might reasonably expect to be recurrent.

Many models which do deal with uncertainty\(^2\) only do so in the context of partial equilibrium, and the analysis is facilitated by being able to hold many variables fixed.

The globally indeterminate models which do have switches in equilibrium, achieve this either by means of making agents myopic (Chamley 1996), or by giving capital no role in production (Murphy, Schleifer & Vishny 1989), which is tantamount to myopia. The model in this paper is similar to both of these in that the multiple equilibria are driven by a labor externality. What is different is that agents are forward looking, and capital affects production. This means that anticipation of a change in equilibrium may cause immediate changes in consumption now, and that there are lags in behavior, as capital takes a while to catch up with a new locally steady state after a change of equilibrium.

In this model, phases where economic activity is high are Pareto-superior to those where activity is low. If the government can affect the sunspot process in a way to keep the economy in a good state, or to kick the economy out a bad state, then policy will have long term consequences for welfare, and it is likely that some

\(^{2}\)e.g. (Obstfeld 1986), a model of currency crises, and (Kremer & Morcom 1996), a model of extinction of species used to produce storable goods.
kinds of active, Keynesian, fiscal policy would be effective in this kind of model.

This paper is organized as follows. In Section 2.2, I set up the economy, and solve the system for the case of perfect foresight. In Section 2.3, I introduce sunspot coordination uncertainty, and examine a plausible sub-class of possible equilibria. In Section 2.4, I conclude, and discuss some implications of the model.

2.2 Perfect Foresight, Perfect Information

2.2.1 The Economy

Consumers have utility separable in consumption and labor. Their utility is CRRA with respect to consumption, and the choice of labor is from one of two values\(^3\). Production is, effectively, Cobb-Douglas. The results in this section, and the subsequent sections, are stable\(^4\), so that they do not depend crucially on the specific functional forms adopted here, which considerably simplify the mechanics of the proofs.

There is only one good in the economy, which is both a capital and a consumption good. It does not depreciate, and it may be invested, or eaten at rate \(c\). The total individual capital stock is \(k\). Since consumers are insatiable, and the marginal product of capital in production is always positive, capital will never be thrown away, and so the rate of change of the capital stock must just be total production, minus what is consumed, or:

\[
\dot{k} = y - c, \tag{2.1}
\]

where \(y\) is production.

**Definition 2.1 (Consumers)** There is a continuum of identical consumers, each with capital \(k(t)\) and consumption \(c(t)\), supplying labour \(l(t) \in \{l_0, l_1\}\), where \(\Delta l = \]

\(^3\)The restriction to a two point labor supply is for mathematical convenience only. In Appendix A.3, I include a development of the theory for a continuous labor choice, CRRA disutility of labor, and Cobb-Douglas production; the results are, essentially, unchanged.

\(^4\)"Stable" in the sense that the structure of the system, and the existence of multiple equilibria, will not be changed by small perturbations of utility or production.
\( l_1 - l_0 > 0 \). Consumers have instantaneous utility:

\[
u(c_i, l_i) = \frac{c_i^{1-b}}{1-b} - U_i, \quad \text{for } i = 0 \text{ or } 1,
\]

(2.2)

\( \Delta U = U_1 - U_0 > 0 \). Consumers choose \( c \) and \( l \) to maximize:

\[
W = \int_0^\infty e^{-rs} u(c, l) \, ds,
\]

(2.3)

subject to production and budget constraints.

CRRA utility in consumption means that consumers have a constant intertemporal elasticity of substitution of \( 1/b \). They discount at a constant rate, \( r \), and utility is time-additive.

**Definition 2.2 (Production)** Each consumer has the same production technology. If capital is \( k \), labor \( l \), and aggregate labor is \( \Lambda \), the technology produces the good at rate:

\[
y = E(\Lambda)lk^a = Lk^a,
\]

(2.4)

where \( 0 < a < 1 \) is a constant, and \( L = E(\Lambda)l \) is "effective labor". Define \( E_i = E(l_i) \), and \( L_i = E_i l_i \). I assume that \( E \) is increasing in \( l \), so that \( E_0 < E_1 \), so that labor has positive externalities.

Note that, apart from the externality, production is equivalent to Cobb-Douglas with share of capital in production of \( a \), but since labor can only have two values, one can drop the power of \( (1 - a) \) by rescaling the labor values. In Appendix A.3, I analyze the system with a continuous choice of \( l \) and true Cobb-Douglas production, and similar results to those with the two-point labor-supply hold.

The fact that \( E \) is increasing in \( l \) means that it is more productive to work hard when everyone else is. The existence of multiple equilibria hinges on this. It will turn out that, for suitable parameter values, there are two possible classes of equilibrium, one in which everyone provides labor \( l_1 \), and the other in which everyone chooses labor \( l_0 \).
There are two choice variables, $c$, and $l$, and one state variable, $k$. This is, reasonably, the smallest number that can capture the effects this paper discusses. Without capital, the model is only trivially, if at all, dynamic, as today’s decisions cannot affect tomorrow’s. Missing out labour, an externality in capital cannot easily create multiple steady states and multiple equilibria given the same initial conditions. This is because capital is not a “jump” variable. An externality in consumption is not particularly plausible intuitively.

Note that this model has no growth or change in technology. The population is constant, and the production function is static. Dynamics in this model should be interpreted in terms of business cycles in a static, non-growing economy. It is easily possible, with perfect foresight, to rework the model in an economy with population growth or exogenous technical change. The equilibria would then be “balanced growth paths” as in, for example, Benhabib & Perli (1994). Introducing growth substantially complicates the model when there is uncertainty, though, so I have chosen to use a static model for clarity of exposition.

### 2.2.2 Equilibrium

I assume that each consumer is too small relative to the whole economy for his or her labor choice to affect aggregate labour; the consumer will choose consumption and labour, taking aggregate labour as given. There are, thus, two different dynamic regimes, depending on whether aggregate labor is high or low. For consistency, consumers must prefer to work hard when everyone else is, and must prefer to shirk when everyone else is\(^5\). Situations where consumers prefer a different level of labor from the aggregate cannot be equilibria, as aggregate labor must be the same in equilibrium as the labor supplied by all the identical consumers. Capital and labor must be in a certain range for this to be true, as summarized in:

\(^5\)This is the requirement that there are Nash Equilibria in which everyone chooses the same amount of labor.
Theorem 2.3 (Dynamics) (See Figure 2.1) With perfect foresight and perfect information, and if aggregate labor is $l_1$ and $(k,c) \in \mathcal{R}_1$ given by

$$\mathcal{R}_1 = \left\{ (k,c) : \quad c^b \leq \frac{\Delta l}{\Delta U} E_1 k^a \right\}, \quad (2.5)$$

then consumers are locally maximizing utility so long as they choose labor $l_1$, and

$$\begin{align*}
\dot{c} &= \frac{c}{b} \left( aL_1 k^{a-1} - r \right) \\
\dot{k} &= L_1 k^a - c. \quad (2.6)
\end{align*}$$

If aggregate labor is $l_0$, the dynamics at a local maximum of utility are as in equations (2.6), changing $L_1$ for $L_0$, but $(k,c) \in \mathcal{R}_0$, given by

$$\mathcal{R}_0 = \left\{ (k,c) : \quad c^b \geq \frac{\Delta l}{\Delta U} E_0 k^a \right\}, \quad (2.7)$$

Proof: Appendix A.1

The meaning of the regions $\mathcal{R}_i$ is as follows: if aggregate labor is high, production is high for a given level of labor and capital. If capital is too low, though, the extra production gained from working hard is not high enough to offset the utility lost by so doing, and it is not worth it for the consumer to work hard. In that case, it is optimal for the consumer to shirk. If, on the other hand, aggregate labor is low, then it is worth working hard if capital is high enough. This may be seen from the following: $c^{-b}$ is the shadow price of capital (the "Lagrange Multiplier" in the Hamiltonian) in units of instantaneous utility. The change in the rate of production of capital is $\Delta l E_1 k^a$, so that the utility value of changing labor is $c^{-b} \Delta l E_1 k^a$. For the consumer to prefer not to switch labor, this must be less than the change in disutility of labor from switching.

Figure 2.1 shows the situation of Theorem 2.3. The upper diagram shows the high aggregate labor ($l_1$) dynamics. The unshaded region is $\mathcal{R}_1$. The solid lines are where the rate of change of capital and consumption are, respectively, zero. The
Figure 2.1: Dynamics and Allowed Regions $R_i$
arrows show the direction of motion of the economy in the corresponding region. The lower diagram shows the dynamics for low aggregate labor ($l_0$). Here the unshaded region is $R_0$. These diagrams correspond to $a = 0.35$, $b = 0.5$, $r = 0.1$, $l_0 = 1$, $l_1 = 2$, $E_0 = 1$, $E_1 = 2$, and $\Delta U = 0.8$

In equilibrium, because everyone is the same, they must all be choosing to supply the same amount of labor. This means that there will be two dynamic regimes, one in which everyone chooses labor $l_0$, and another in which everyone chooses labor $l_1$.

In any perfect foresight equilibrium, the path of consumption must be continuous$^6$. There are many perfect foresight equilibria in which aggregate labor changes at preordained times. In between each change, the system must follow the dynamics of Theorem 2.3. The actual path chosen will be such that there is no jump in consumption to the next phase. Almost all of these will, in general, be paths which would eventually go outside the allowed region, $R_i$, or which would diverge asymptotically if the system stayed in one phase for long enough. With perfect foresight, since everyone knows precisely what will happen, the transition times must be preordained so that this is not a problem. With stochastic phase switching, since no one knows in advance how long a phase will last, this is no longer true. When I consider stochastic phase switching in the next section, in order to avoid the possibility of bubbles and divergent paths or the economy straying outside $R_i$, I restrict the equilibrium paths to those which are sustainable in the following sense:

**Definition 2.4 (Sustainable Equilibrium)** An equilibrium in phase $i$ is sustainable if and only if, following the dynamics of Theorem 2.3,

1. The economy stays in the region $R_i$ for all $t \geq 0$,

2. Consumption and capital are such that utility, $W$ is bounded$^7$ on the equilibrium path.

---

$^6$If it were not, consumers would be better off by changing consumption near the discontinuity so as to smooth consumption, since they have concave utility of consumption.

$^7$This condition is just the familiar transversality, or Inada, condition.
Restricting the model to sustainable equilibria forces the equilibrium path to be in the stable manifold of a bounded limit set of the dynamics in each phase, and selects only those paths which stay in the allowed region, \( \mathcal{R}_i \). The only possible limit sets in this economy are fixed points with one dimensional stable manifolds. The sustainable equilibrium paths must, therefore, be on a saddle path leading to a fixed point.

**Theorem 2.5 (Steady States)** If, and only if,

\[
\frac{b-a}{l_1^{1-a}E_1^{\frac{b-1}{1-a}}} \leq \frac{\Delta l}{\Delta U} \left( \frac{a}{r} \right)^{\frac{a(1-b)}{1-a}} \leq l_0^{\frac{b-a}{1-a}} E_0^{\frac{b-1}{1-a}}, \quad \text{then:} \tag{2.8}
\]

There are 2 saddle-path stable steady states \( S_0 \) and \( S_1 \), where \( S_i = (K_i, C_i) \), for labor \( l_i \), and

\[
K_i = \left( \frac{aL_i}{r} \right)^{\frac{1}{1-a}}, \quad C_i = \frac{r}{a} K_i. \tag{2.9}
\]

Clearly, \( K_1 > K_0 \) and \( C_1 > C_0 \), since \( L_1 > L_0 \)

The saddle paths are defined by the curves \( c = \Gamma_i(k) \). Both \( \Gamma_i \) are upward sloping. See Figure 2.2.

A necessary, though not sufficient, condition that there be two steady states is that either \( b < a \), \( b < 1 \), or both.

**Proof:** Appendix A.1

\( \diamond \)

Figure 2.2 shows the steady states \( S_i \) and the saddle paths \( \Gamma_i(k) \). The unshaded region is the intersection of \( \mathcal{R}_1 \) and \( \mathcal{R}_0 \). In this case \( (a = 0.35, b = 0.5, r = 0.1, \) other parameters as for Figure 2.1), the steady states and the saddle-paths all lie inside both regions \( \mathcal{R}_i \). In a wide range of numerical simulations, so long as the system satisfied the inequalities (2.8) so that the steady states were in the allowed regions, the saddle paths were also in the allowed regions for values of capital between \( K_0 \) and \( K_1 \). I have, however, been unable to prove that this is always the case.
Figure 2.2: Steady-States and Saddle Paths of the Perfect Foresight System
The necessary condition is an implication of the “if and only if” requirement. If \( b \) were very high, and thus the intertemporal elasticity of substitution were very low, then it may be that, at the high “steady state”, consumers would rather shirk for extra consumption now, ignoring the long-run effects this would have on the capital stock. Something similar happens in reverse at the low “steady state”.

Additional restrictions must be imposed on \( a \) and \( b \) before I may identify the \( l_1 \) phase with a boom, and the \( l_0 \) phase with a recession. The 1 saddle path is not always above the 0 saddle path. If the intertemporal elasticity of substitution is very high (or, in other words, if \( b \) is very low), people react to the onset of a recession by eating as much as they can early and, thereby, choosing a very steep and rapid decline to the recession steady state. In a boom, the opposite happens, and people start with very low consumption and it grows rapidly to the boom steady state. If \( b < a \), the boom equilibrium consumption path is below the equilibrium consumption in a recession, for the same level of capital. The relative positions of the equilibrium paths are summarized in the following:

**Theorem 2.6 (Saddle-Path Ranking)** The relative positions of the saddle paths, \( \Gamma_i \), depend on the relationship between \( a \) and \( b \):

1. If \( a > b \), then \( \Gamma_0(k) > \Gamma_1(k) \) for all \( k \in [K_0, K_1] \).
2. If \( a < b \), then \( \Gamma_0(k) < \Gamma_1(k) \) for all \( k \in [K_0, K_1] \).
3. If \( a = b \), \( \Gamma_0(k) = \Gamma_1(k) \) for all \( k \in [K_0, K_1] \): the saddle paths are coincident, and are the line \( c = rk/a \).

**Proof:** Appendix A.1

\( \Diamond \)

Deaton (1992) discusses estimates of the intertemporal elasticity of substitution. A number of studies find \( b \) to be from 0.25 to 0.75, depending on the sample and
the level of aggregation. A reasonable estimate of $a$ would be of the order of $0.35^8$. Without trying to calibrate my model, these estimates suggest that it is not unreasonable to assume that $a \leq b \leq 1$, which I shall assume for the rest of the paper. In fact $a$ and $b$ appear to be close enough, that one could probably assume their equality without going too far wrong, especially if there is a lot of uncertainty about the phase (see next section). In the diagrams and numerical solutions of this paper, I take $a = 0.35$, $b = 0.5$, for cases where $a < b$, and $a = 0.35$, $b = 0.2$ for cases where $b < a$.

Restricted to sustainable equilibria, it is impossible for the economy to change phase with perfect foresight, unless $a = b$. The saddle paths, $\Gamma_1$ never cross, so that any phase transition would have to have have discontinuous consumption, which is ruled out by perfect foresight and rational expectations. The class of equilibria I consider with perfect foresight is, thus very small.

**Theorem 2.7 (Perfect Foresight Sustainable Equilibrium)** A **Boom, or Phase 1 sustainable perfect foresight equilibrium, $E_1(k_0)$, is a path of capital, consumption, and labor $(k(t), c(t), l_1)$ such that:**

1. $k(0) = k_0$, the initial level of capital.
2. $c(t) = \Gamma_1(k(t))$ \(\forall t\), and $(k_0, \Gamma_1(k_0)) \in R_1$.
3. $\dot{k} = L_1 k(t)^a - c(t)$ \(\forall t\).
4. If $a = b$, then the dynamics are exactly soluble, and

$$k(t)^{1-a} = \frac{aL_1}{r} + \left(k_0^{1-a} - \frac{aL_1}{r}\right)e^{-\frac{r}{s}(1-a)t}, \quad (2.10)$$

and $c(t) = rk(t)/a$.

A **Recession**, or **Phase 2 s.p.f. equilibrium, $E_0(k_0)$, is analogously defined, by changing ones to zeros in an obvious way. (See Figure 2.3)\(^8\)**

---

\(^8\)Prescott (1986) gives the share of labor in production, $1 - a$, as 64%.
Proof: All except the last part are easy corollaries of the previous results. For the last part, if \( a = b \), the saddle is the trajectory \( c = rk/a \). Thus \( \dot{k} = Lk^a - rk/a \), which is a separable, first-order ODE, which may be easily solved to get the expression above.

\[
\diamond
\]

Note that an equilibrium is determined entirely by the initial capital \( k_0 \), and aggregate labor.

Figure 2.3 shows equilibrium paths for the three possible cases. The top diagram is where \( a < b \): the boom path, \( \Gamma_1 \), is everywhere above the recession path, \( \Gamma_0 \). In the middle diagram, \( a = b \): the boom and recession equilibrium paths are coincident. The bottom diagram shows the case in which \( a > b \): the boom equilibrium path, \( \Gamma_1 \) is below the recession path, \( \Gamma_0 \), so that for a given level of capital, consumption in “boom” is lower than consumption in a “recession”.

In Figure 2.3, I only show the saddle-paths for capital between \( K_0 \) and \( K_1 \). As we shall see in the next section, when there is phase switching, the economy must end up with capital in the uncertainty analog of \( (K_0, K_1) \) after a long enough time. If, therefore, the economy has any history, initial capital must be in \( [K_0, K_1] \). Once capital is in \( [K_0, K_1] \), it must stay there for ever.

It is possible that, for some values of parameters, the equilibria will not exist for all values of capital between \( K_0 \) and \( K_1 \), as the saddle-paths they depend on may move outside \( \mathcal{R}_i \). This doesn’t matter with perfect foresight: since the phase never changes once it has been initially set, the possible non-existence of another equilibrium can’t affect an existing equilibrium if all agents know that the current equilibrium will persist with probability one. When I discuss anticipated stochastic phase shifts in the next section, it matters a lot. For there to be a phase transition from boom to recession, or vice versa, there must be a saddle path for the system to change to at the current level of capital. For ease of exposition, I restrict attention to cases where the saddle paths do exist over all relevant ranges, so that the economy

23
Figure 2.3: Perfect Foresight Equilibria

- a less than b

- a equal to b

- a greater than b
cannot get stuck in a permanent boom or recession.

As noted above, though, since it appears from numerical simulations that, whenever the steady states exist, the saddle-paths lie entirely in the regions $R_i$, this caveat may be a lesser restriction than one might have thought.

2.3 Coordination Uncertainty: Phase Switching

In this section, I relax the assumption of perfect foresight, and consider a class of equilibria in which there are rationally anticipated, stochastic changes in phase.

The expectation of the change of phase will change the equilibrium path from what it was in the perfect foresight case but will preserve the 2 phase structure discussed in the previous section. In this context, the equilibrium path can, and will with probability one asymptotically, be discontinuous. Because the discontinuity is driven by a random event, it cannot be anticipated completely. In this case, therefore, there are sustainable equilibria from which changes in phase are possible, in contrast to the perfect foresight case.

The model I introduced in the last section has identical agents. If all the information available is public, each and every agent knows how every other agent would react to the information: since they are identical, their reactions must all be the same. If an agent received information saying that the economy had changed endogenously, she would know that the information was false because, if she hadn’t changed anything, how could anyone else have done? Because of this, the only possible way to change phase in such a model is by means of a sunspot, where some agreed upon exogenous coordination device, such as the direction of the wind on a particular day, causes everyone to change at the same time.

The coordination I introduce in this section is a sunspot. I wish to identify the sunspot with reactions to public information about aggregate economic activity, so that perceived low economic activity can induce recession. Of course, if that is the agreed sunspot, then that will work as well as any other. On the other hand, any
random exogenous event would work just as well. I prefer to think of the sunspot
I introduce as a reduced form version of the reactions of heterogenous agents, who
don’t know each others’ preferences, to economic information, as I discuss in the
next section.

Suppose there are two distinct phases, determined by aggregate labor, as with
perfect foresight. Suppose that, when the economy is in phase 0 with labor \( l_0 \), there
is a constant hazard \( p_0 \) that a sunspot will occur, and everyone will switch labor to
\( l_1 \) and move the economy to phase 1. Likewise, when in phase 1, there is a constant
hazard \( p_1 \) that a sunspot will change the economy to phase 0\(^9\). Define the transition
hazard vector \( p = (p_0, p_1) \).

The results of this section are based on the following intuition: suppose that the
probability of changing to the other phase is small. Then the path followed ought
to be close to the equilibrium path followed under perfect foresight. That means
that, when there is a change, the system will jump to a determinate saddle path for
the new “equilibrium”. Given that agents know what the probability of a change is,
and where the system will go if there is a change, they can react appropriately so
as to maximise their utility; if they know a recession is likely, they may save rather
than consume as much as they would have done assuming that the boom would go
on for ever. By positing a determinate path for the system in one phase, I write the
dynamics for the undetermined one in terms of the other phase’s equilibrium path,
and the transition hazard. This yields a simultaneous system of dynamics for the
phases which may be solved by iteratively integrating the equations numerically.

**Definition 2.8 (Equilibrium Paths)** Let the equilibrium paths in each phase be
defined by \( c = \Theta_i(k, p) \). For now, assume their existence; I prove their existence
below, by construction. Clearly, \( \Theta_i(k, 0) = \Gamma_i(k) \), the perfect foresight paths, since
\( p = 0 \) corresponds to perfect foresight.

\(^9\)The time between switches in phase will be exponentially distributed, with phase \( i \) lasting an
expected time \( 1/p_i \).
I shall, in general, drop \( p \) from the notation, where \( p \) is constant, in order to make the proofs easier to read.

Consumers' utility depends on the phase the economy is in, and the level of capital. Because utility is time-additive, actions taken in one phase only affect utility in the other phase to the extent that they change the level of capital. This implies that the only way the other phase's equilibrium path can enter the decisions of the consumers is as a function of the level of capital the economy will jump to after the change in phase.

The consumers know that if they are in Phase \( i \), when there is a change of phase to Phase \( j \), aggregate labor will jump to \( l_j \), and consumption will jump to \( \Theta_j(k) \). Using a Bellman Equation approach, I find a differential equation for the path of consumption and capital before the change, and the "saddle-path" solution to this will turn out be the path \( \Theta_i \). For \( p \) "small", the equation for the evolution of \( c \) is close to the perfect foresight equation. Uncertainty introduces an extra term into the dynamics of consumption. If, at current consumption, consumers expect future consumption to be lower in the event of a switch to the other phase, so that \( c > \Theta \), then the slope of the consumption path will be steeper than it would have been with perfect foresight. This means that the equilibrium saddle path with uncertainty must have lower consumption for any value of capital than with perfect foresight. This is just "consumption smoothing" in equilibrium: consumers expect a transition to a less productive phase in the future, so they eat less and invest more now. If, on the other hand, consumers expect consumption to rise in the event of a change of phase, the opposite happens: the consumption path is less steep than it would have been under perfect foresight, and the equilibrium saddle path moves higher.

**Theorem 2.9 (Dynamics Under Sunspot Uncertainty)** *If the system is in Phase \( i \), and there is a hazard \( p_i \) that the system will change phase to Phase \( j \) at consumption level \( \Theta_j(k) \), and labor \( l_j \), then \((k, c) \in R_i\), and the path in Phase \( i \)
satisfies the following differential equations:

\[
\begin{align*}
\dot{c} &= \frac{c}{b} \left[ a L k^{a-1} - r \right] + \frac{c \theta_i}{b} \left[ \frac{c^b}{\Theta_j(k)^b} - 1 \right] \\
\dot{k} &= L k^a - c.
\end{align*}
\] (2.11)

**Proof:** Appendix A.2

\[\square\]

Thus, if the equilibrium path in phase \( j \), \( \Theta_j \) is known, we may solve the differential equations (2.11) to find the equilibrium path in Phase \( i \).

I restrict the class of equilibria considered, by only considering sustainable equilibria, as defined in the last section. This means that the equilibrium path has to be bounded, and has to stay in \( \mathcal{R}_i \), no matter how long the phase should turn out to last. Just as in the perfect foresight case, this means that the sustainable paths must be saddle paths.

**Definition 2.10 (Phase \( i \) Sustainable Response)** If the consumer in Phase 1 knows that she will switch to the path \( \Phi(k) \) in Phase 0, then her only sustainable equilibrium path in Phase 1 is \( B_1[\Phi] \), a saddle path solution to the dynamic equations (2.11). The sustainable response in Phase 0 to a path is defined analogously, and is denoted by \( B_0[\Phi] \).

These sustainable response functions are analogous to best response functions in Game Theory, except that the response is to an agent's own actions in the future, and not to another player's actions now.

The next theorem shows that such sustainable responses do exist, and satisfy reasonable properties.

In this, and the remaining theorems of this section, I assume that \( p \) is small to ensure that there are no bifurcations in the system so that the solutions to the dynamical equations are uniformly continuous in the parameters and the other phase path. This is a sufficient condition for the purposes of this paper, but is by no means
necessary. In fact, solutions exist and have similar properties for large values of \( p \), as I show by numerical methods at the end of this section.

I exclude the case where \( a = b \) for the rest of the section. By Theorem 2.6, this implies that the Phase 0 and Phase 1 saddle paths are coincident with perfect foresight. They must also be coincident with uncertainty. This is because, in that case, there is no “jump” when a phase transition happens, and so the dynamics, within a particular phase, will be exactly the same as in the perfect foresight case.

**Theorem 2.11 (Properties of Sustainable Response Functions)** For \( p \) small enough, if \( \Gamma_0 \leq \Phi \leq \Gamma_1 \) and \( a < b \) (\( \Gamma_1 \leq \Phi \leq \Gamma_0 \) if \( a > b \)), and if \( \Phi \) is upward sloping and has bounded and continuous derivative, then

1. The dynamical system defined by:

\[
\dot{c} = \frac{c}{b} \left[ aL_i k^{a-1} - r \right] + \frac{cp_i}{b} \left[ \frac{c^b}{\Phi_j(k)^b} - 1 \right] \\
\dot{k} = L_i k^a - c
\]

(2.12)

has a unique steady state \( S_1(\Phi) \), and a unique saddle path \( c = B_i(\Phi)(k) \). \( S_1 \) is increasing in \( \Phi \), while \( S_0 \) is decreasing in \( \Phi \) for \( a < b \). For \( a > b \), \( S_1 \) is decreasing in \( \Phi \), while \( S_0 \) is increasing in \( \Phi \) (See Figure 2.4)

2. \( B_i(\Phi) \) is between \( \Gamma_0 \) and \( \Gamma_1 \), is an increasing function of capital, \( k \), is continuously differentiable, and has bounded derivative, with the bound independent of \( \Phi \).

3. \( B_i \) is an increasing function of \( \Phi \) in that, if \( \Phi(k) > \Phi'(k) \) \( \forall k \), then \( B_i(\Phi) > B_i(\Phi') \) everywhere.

4. \( B_1(\Gamma_1) = \Gamma_1 \), and \( B_0(\Gamma_0) = \Gamma_0 \)

**Proof:** Appendix A.2

\( \diamond \)
Figure 2.4: Locally Steady-States and the Saddle Path with Imperfect Foresight

\[
\begin{align*}
\mathcal{S}_1(c) & \quad \text{(old cost function)} \\
\mathcal{S}_1(c) & \quad \text{(new cost function)} \\
\bar{c}(k) & \quad \text{(capital) } \\
\bar{k}(c) & \quad \text{(output) } \\
\end{align*}
\]
Figure 2.4 illustrates the sustainable response function, and its properties. The presence of uncertainty rotates the line where \( \dot{c} = 0 \) around the point where the perfect foresight \( \dot{c} = 0 \) line intersects \( \Phi \). This causes the "steady state" to move up the line where \( \dot{k} = 0 \). \( B_1(\Phi) \) is the saddle-path to this new point, and lies below \( \Gamma_1 \), but above \( \Phi \).

Notice that the steady state \( S_t(\Phi) \) is not steady in a global sense. While the system will stay there so long as the phase doesn't change, as soon as there is a change in phase, the system starts to move again. I shall call the \( S_t(\Phi) \) "Locally Steady States" (LSS) to reflect this.

In a manner analogous to Nash Equilibrium, equilibrium phase paths \( \Theta_i \) must satisfy \( \Theta_1 = B_1(\Theta_2) \), and \( \Theta_2 = B_2(\Theta_1) \), so that each phase is a sustainable response to the other.

That such paths exist\(^{10}\) is shown in the next theorem.

**Theorem 2.12 (Existence of Sunspot Equilibrium Paths)** For \( p \) small enough, and if \( a < b \) equilibrium paths \( \Gamma_1 > \Theta_1 > \Theta_0 > \Gamma_0 \) exist, and are continuously differentiable. (See Figure 2.5). If \( a > b \), then equilibria also exist, but \( \Gamma_1 < \Theta_1 < \Theta_0 < \Gamma_0 \).

**Proof:** Appendix A.2

Figure 2.5 shows the sunspot equilibrium paths, \( \Theta_i \) for \( p_0 = p_1 = 0.2 \), and all other parameters as in Figure 2.1. For comparison, the figure also shows the perfect foresight equilibrium paths, \( \Gamma_i \). As implied by Theorem 2.11, the equilibrium paths with uncertainty lie between the perfect foresight paths. The figure only shows the paths between the locally steady-state values of capital. As mentioned in the previous section, after a long enough period of time, the system must be within that range of capital. In the example shown in the figure, the entirety of the \( \Theta_i \)

---

\(^{10}\)I have, as yet, been unable to prove uniqueness, even within the very restricted class of constant transition hazard equilibria.
Figure 2.5: Imperfect Foresight Equilibrium Paths $\Theta_i$
lie inside \( R_i \), so the equilibrium is consistently defined. All the examples I discuss have this property, which, in the light of simulations, doesn’t appear to be too restrictive. When there is a phase transition, consumption jumps from one \( \Theta_i \) to the other. For this to be consistent, they must both be defined over all ranges of capital the economy may be in. If this were not the case and, say, the boom path didn’t exist near the recession LSS level of capital, the economy could get stuck in a permanent recession. While not economically unpalatable (deep recessions can, in reality, be very hard to end), this would mean that a globally constant transition hazard would not be appropriate. Since this is the main device ensuring that the model is tractable, I rule it out for the purposes of the paper.

Note that, as above, the fact that \( p \) is small is sufficient, but not necessary to guarantee the existence of an equilibrium.

In practice, in numerical simulations, even for large values of \( p \), the sequence of functions used in the proof appears to converge smoothly, and quite rapidly, so this does not appear to be a problem. As \( p \) increases, the equilibrium paths move closer together.

I am now in a position formally to define the imperfect foresight equilibria:

**Definition 2.13 (Imperfect Foresight Sustainable Equilibria)** An imperfect foresight sustainable equilibrium is uniquely defined, given \( k_0 \), initial capital, an ordered, countable set of transition times at which the economy changes phase\(^{11}\), \( \omega = \{\tau_n\} \), transition hazards \( p \), and an initial choice of phase, \( \lambda = l_0 \) or \( l_1 \). Such an equilibrium is a path \((k(t), c(t), l(t))\) such that:

1. The economy alternates in phase, starting in phase \( \lambda \), with \( k(0) = k_0 \), and switching to the other phase at each \( \tau_n \).

2. While in phase \( i \):

\[
l(t) = l_i, \tag{2.13}
\]

\(^{11}\)The transition times are random. The complete state space is the set of all ordered transition times together with an initial choice of phase
\[ c(t) = \Theta_1(k(t), p), \text{ and} \]
\[ \dot{k}(t) = L_1 k(t)^a - c(t). \]

An equilibrium, thus, depends both on history, via initial capital, \( k_0 \), and on expectations, via the transition hazards, \( p \), and the initial phase. Since the transition times are random, though, after a sufficiently long period of time, looking at the system will tell one nothing about which point it started from. In other words, the economy is ergodic.

Figure 2.6 shows two simulated equilibrium paths of consumption for particular transition time realizations. The top panel shows an equilibrium when \( p_0 = p_1 = 0.2 \). This corresponds to an average phase length of five years. The bottom panel shows an equilibrium for \( p_0 = p_1 = 1 \), for an expected phase length of one year. Both simulations have share of capital in production, \( a = 0.35 \), inverse elasticity of intertemporal substitution, \( b = 0.5 \), and discount rate \( r = 0.1 \).

I have selected sustainable equilibria only in order to get a manageable set of equilibria. There are many more equilibria. Because the system will not stay in any one phase for ever, even an asymptotically divergent path won’t be a problem if the system is unlikely enough to stay in that phase for long. In this way, bubbles which last a random length of time should be completely reasonable as equilibria: if the transition hazard to a non-bubble path is high enough, then expected utility will be finite on the bubble path.

### 2.4 Implications and Conclusion

In this section, I discuss a number of implications of the model.

Since the equilibrium interest rate is just the marginal product of capital, \( a L_1 k^{a-1} \), for the same level of capital, the interest rate will be higher in a boom than a recession, because effective labor, \( L_1 \) will be higher. In a boom, the interest rate starts high and, as \( k \) increase, falls gradually to \( r + p_1 (1 - \Theta_1(k)^b/\Theta_0(k)^b) \). This will be
Figure 2.6: Simulated Paths of Consumption for Transition Hazards $p_i = 0.2$, and $p_i = 1$
less than (resp. greater than) \( r \) at the boom locally steady state if \( a < b \) (resp. \( a > b \)). With perfect foresight, the steady-state interest rate would be exactly \( r \), but with uncertainty, consumers know that the phase will change to recession some time, when the interest rate will fall. In a recession, the interest rate starts low, and gradually rises to \( r + p_1(1 - \Theta_0(k)^b/\Theta_1(k)^b) \). This is above \( r \) at the recession locally steady-state.

Thus, under the model, one would expect to see real interest rates rise at the onset of a boom, and fall at the onset of a recession.

Both recessions and booms should start vigorously (i.e. the time rate of change of consumption and capital should be relatively high), and then tend to slow down as the duration of the phase increases. This is because, as a phase progresses, the economy approaches a locally steady-state, and the rate of approach slows as it gets nearer along the stable manifold.

A related implication of this model is that the rate of change of capital and consumption will be history dependent in a particular way: after a long boom, if there is a change in phase, the subsequent recession will faster in its initial stages than after a long boom. This is because after a long boom, the economy will be closer to the boom locally steady state than it will be after a short boom, starting from the same level of initial capital in both cases. If the economy is closer to the boom steady state, it must be further away from the recession steady state, and phases are faster the further away they are from the phase steady state. A similarly, after a long recession, the subsequent boom should be faster initially than a boom after a short recession.

This should be a very general property of most globally indeterminate models with a non-jumping state variable such as capital. In such models, the locally steady states of the economy are separate, and the economy will move towards a locally steady state faster the further away from the steady state it is. The longer the economy is in one phase, the further it gets from the other phases' steady states,
and the faster will be the movement back when the phase changes again.

Consider the model of this paper with a set equal to \( b \). In that case, the evolution of \( \kappa = k^{1-a} \) follows from equation (2.10)

\[
\kappa(t) = A_1 + (\kappa_0 - A_1)e^{-zt}, \text{ in a boom, and}\]
\[
\kappa(t) = A_0 + (\kappa_0 - A_0)e^{-zt}, \text{ in a recession.}\]

In these equations, \( x = r(1 - a)/a, \kappa_0 \) is the initial capital, and \( A_i = aL_i/r \). If the economy starts in a boom at time 0 and capital \( k_0 \), changes to a recession at time \( t_1 \), and changes back to a boom again at time \( t_1 + t_0 \), then at time \( t_1 \), \( \kappa \) will have the value:

\[
\kappa(t_1) = A_1 + \frac{(\kappa_0 - A_1)}{\tau_1},
\]

and by the time the economy has switched back to a boom again,

\[
\kappa(t_1 + t_2) = A_0 + \left( A_1 + \frac{1}{\tau_1} \right) - A_0 \frac{1}{\tau_2},
\]

where \( \tau_i = e^{zt_i} \).

The time rate of change of \( \kappa \) after a period \( t \) in Phase \( i \) will be:

\[
\dot{\kappa} = -x(\kappa_0 - A_i)e^{-zt}.
\]

At the beginning of the initial boom, this is

\[
\dot{\kappa} = -x(\kappa_0 - A_1),
\]

and at the beginning of the boom starting at time \( t_1 + t_2 \), the rate of change of \( \kappa \) is:

\[
\dot{\kappa}' = -x \left( A_0 + \left( A_1 + \frac{1}{\tau_1} \right) \frac{1}{\tau_2} - A_1 \right).
\]

The difference between the rates of change at the beginning of the second boom and the first boom is then \( \Delta \dot{\kappa} = \dot{\kappa}' - \dot{\kappa} \), and

\[
\Delta_{\text{brb}} \dot{\kappa} = x(A_1 - A_0) \left[ \left( 1 - \frac{1}{\tau_2} \right) - \frac{A_1 - \kappa_0}{A_1 - A_0} \left( 1 - \frac{1}{\tau_1 \tau_2} \right) \right].
\]
The subscript \(brb\) stands for "boom, recession, boom". The quantities inside the square brackets of this equation are, in principle, measurable. \(\tau\) may be calculated from \(t\) with estimates for \(r\) and \(a\). The expression \((a_1 - \kappa_0)/(A_1 - A_0)\) is a measure of the distance the economy is from the maximum level of capital possible, and should be related to a measure of the output gap.

Equation (2.21) says that there should be a positive relationship between the quantity in the square bracket, denoted by \(T_{brb}\), and the change in the rate of change of \(\kappa\).

A similar calculation for a recession for \(t_1\) followed by a boom for \(t_2\), followed by another recession, shows that, in this case,

\[
\Delta_{rbr}\kappa = -x(A_1 - A_0) \left[ \left(1 - \frac{1}{\tau_2}\right) - \frac{\kappa_0 - A_0}{A_1 - A_0} \left(1 - \frac{1}{\tau_1\tau_2}\right) \right].
\]  
(2.22)

Here, \(\Delta_{rbr}\kappa\) should depend negatively on \(T_{brb}\), the quantity in the square brackets.

In a model with local fluctuations, \(\Delta\kappa\) should be uncorrelated with \(T\). The effects of any local correlation between measures of economic activity should be substantially lessened by the fact that the measurements are separated by a time \(t_1 + t_2\); so long as the local correlation dies out quickly enough, measurements spaced further apart should have less correlation. In any case, local fluctuations with simple autocorrelated shocks would give the same relationship between \(\Delta\kappa\) in \(brb\) transitions as well as \(rbr\) transitions in a single equilibrium model.

A test of this implication would have to address many issues:

First, how should one define booms and recessions? The model in this paper is static, whereas most real economies have a positive growth trend. Should recessions be defined according to the normal meaning of the word, \(i.e.\) periods of negative real growth, or should they be periods when growth is below trend? Should one assume a constant trend, or should it be allowed to change over time? This will make a large difference to the results of any test.

Second, what should one use to instrument for \(\kappa\)? In the model of this paper, \(\kappa = k^{1-a}\), where \(k\) is capital. Measurement of capital would be very sensitive to the
depreciation rate. It would be better to use consumption which should be positively correlated with capital, but consumption is very volatile.

Third, what would one use as a measure of \((\kappa_0 - A_1)/(A_1 - A_0)\), the "output gap"?

Fourth, in any one country there would probably not be data over enough phase transitions to give a test of any great power. How should one aggregate data from different countries to get more data points?

Fifth, most of the data available are post-war. Governments in that period have been at times very active in Keynesian demand management. One would expect this to add further noise to the picture, and noise which will probably be correlated with the cyclic behavior one would be trying to measure.

Sixth, the model in this paper is in continuous time; booms and recessions may begin or end at any time. Most aggregate economic statistics are published regularly, but infrequently.

As such, a reasonably careful econometric test of this is well beyond the scope of this paper.

The coordination mechanism in the model is, formally, a sunspot. I believe that this should be seen as a reduced form of a situation with heterogeneous agents who don’t know each others’ preferences or actions, other than through published, aggregated, economic information. If this is the case, then the sunspot could plausibly be identified with publication of economic data, such as GDP, inflation, employment, etc.

Matsui & Matsuyama (1995) present a model in which agents repeatedly play a 2x2 game with two Nash Equilibria. Dynamics are introduced by requiring that agents commit to a course of action for some time; their opportunities to change action arrive stochastically. The result is model in which the economy may move from one of the Nash Equilibria to another. Because all information is local, the agents' current decisions depend only on the history of the economy's path: there can
be no jumps in economic activity. Chamley (1996) has a model where heterogeneous agents only know about each others' actions as a result of the economic outcome of their actions, but where jumps are possible. I would like to think of my model as a reduced form of a game combining these models.

Consider a two player game where agents alternate choices of high or low activity, and their actions are fixed for the next period while the other player chooses her action. The only source of information to each player is a public measurement of the last player's action, which has noise. The players do not know enough about each others’ preferences or reactions to be able to reason about what the other player would have done in the previous round: they must take the signal as their only source of information. They must react only to the public signal about the last period's activity. Assume that it is better for both players to coordinate their actions.

The game will be played in the following way: if a player sees a signal that the other player played low in the previous period, it will be worth her while to play low in the current period. If she sees a high signal, it will be worth playing high. So long as the public information about the economy signals its true state, that state will continue. If there is an erroneous signal, however, the player whose turn it is to move will switch activity from low to high, or vice versa. In this way, switches in activity between high and low are generated. Imagine, now, that there are lots of agents and, at the aggregate level, one can't see who moves when. The switches will still occur in a similar way, but they will look just like sunspot changes, driven by public economic information. Note, however, that the public information is not a sunspot in the true sense: the public information signal carries true information above and beyond that of a pure coordination mechanism.

In this way, it should be possible that an anomalous figure for GDP growth could, rationally, convince agents that the economy is in recession. That belief will then become self-fulfilling, and there will be a recession, until another statistic is
published showing, again falsely, that the recession has ended. The more likely statistics are to be wrong, the more likely they are to cause a change of phase.

This interpretation means that the government may manipulate the precision of public information and cause real effects on the economy. If the coordination mechanism were just a sunspot, then agents could choose another one arbitrarily upon which to coordinate, but if they actually need the information for its own sake, they cannot do this. So long as private provision of the public information is costly enough\(^\text{12}\), the economy must do the best it can with the figures the government gives it.

From this follows that it may be in the government’s—and the society’s—best interest that the precision of economic statistics be controlled (I do not consider the possibility that information be falsified). The intuition for this is quite simple. If information is imprecise, then a switch in phase is more likely\(^\text{13}\). In a recession, this is good: it increases the chance that it will end. In a boom, it is bad, for the same reason. Thus, if the government has the power to affect the precision of the information, it will want to have more error when the economy is in a bad state, in the hope that the economy will change state more quickly.

This could potentially be done by firing all the government statisticians at the beginning of a recession, and rehiring them at the start of a boom. Interestingly, but anecdotally, in Britain during the recession of the early Eighties, Mrs. Thatcher substantially cut the budget of the Central Statistical Office, and changed the way unemployment was calculated several times.

This model also raises the possibility that government fiscal intervention could have real effects. If the government owned a significant proportion of the productive capacity of the economy, so that its labor supply decisions had market power, then it may be possible for the government to effect an end to recessions, and at least

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\(^\text{12}\) Few organizations other than governments have the wherewithal to measure national accounting figures, even in principle.

\(^\text{13}\) This is only the case if all available information is similarly imprecise. Otherwise agents would just focus on the indicators known to be more reliable.
raise consumers expectations of the chance that they will end, either of which would be beneficial.

If, during a recession, the government chooses \( l_1 \), then, it may boost aggregate labor enough that it is worth other agents choosing \( l_1 \) as well. With an economic indicators interpretation of the sunspots, this would not work for sure, but would increase the probability that indicators would signal a boom, even if people knew what the government was trying to do. In order to finance this, the government would, presumably, borrow, and repay the money during a subsequent boom. In this way, government spending will not crowd out private spending, so long as it is during a recession. In a boom, increased spending by government would have no effect, as aggregate private labor would already be \( l_1 \).
Chapter 3

Elephants

3.1 Introduction

Twenty-nine percent of threatened birds worldwide and more than half the threatened mammals in Australasia and the Americas are subject to over-harvesting (Goombridge 1992). Most models of open-access resources assume that the good is non-storable (Clark 1976, Gordon 1954, Schaefer 1957). While this may be a reasonable assumption for fish, it is inappropriate for many other species threatened by over-harvesting, as illustrated in Table 3.1. Although 30 percent of threatened mammals are hunted for presumably non-storable meat, 20 percent are hunted for fur or hides, which are presumably storable, and approximately 10 percent are threatened by the live trade (Goombridge 1992).

African elephants are a prime example of a resource which is technologically difficult to protect as private property, and is used to produce a storable good. From 1981 to 1989, Africa’s elephant population fell from approximately 1.2 million to just over 600,000 (Barbier, Burgess, Swanson & Pearce 1990). Dealers in Hong Kong stockpiled large amounts of ivory (New York Times Magazine 1990). As the elephant population decreased, the constant-dollar price of uncarved elephant tusks rose from $7 a pound in 1969 to $52 per pound in 1978, and $66 a pound in 1989.
Table 3.1: Some Species Used for Storable Goods, or by Collectors


<table>
<thead>
<tr>
<th>Bears</th>
<th>Lizards</th>
<th>Medicinal Plants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Giant Panda</td>
<td>Horned Lizard</td>
<td>species of Dioscorea</td>
</tr>
<tr>
<td>Asiatic Black Bear</td>
<td>L. A. Spectacled Caiman</td>
<td>species of Ephedra</td>
</tr>
<tr>
<td>Grizzly Bear</td>
<td>Caiman crocodilus</td>
<td>Dioscorea deltoidea</td>
</tr>
<tr>
<td>S. A. Spectacled Bear</td>
<td>Tegus Lizard</td>
<td>Rawolfia serpentina</td>
</tr>
<tr>
<td>Malayan Sun Bear</td>
<td>Monitor Lizard</td>
<td>Curcuma spp.</td>
</tr>
<tr>
<td>Himalayan Sloth Bear</td>
<td>Varanus niloticus</td>
<td>Parkia rosburghi</td>
</tr>
<tr>
<td>Cats</td>
<td>V. exanthematicus</td>
<td>Voacanga gradifolia</td>
</tr>
<tr>
<td>Tiger</td>
<td>V. salvator</td>
<td>Orthosiphon aristasus</td>
</tr>
<tr>
<td>Cheetah</td>
<td>V. bengalensis</td>
<td>species of Aconitum</td>
</tr>
<tr>
<td>Lynx felis</td>
<td>V. flavescens</td>
<td>Trees</td>
</tr>
<tr>
<td>Lynx canadensis</td>
<td>Snakes</td>
<td>Astronium urundeva</td>
</tr>
<tr>
<td>Ocelot</td>
<td>Python reticulatus</td>
<td>Aspidosperma polyneuron</td>
</tr>
<tr>
<td>Little spotted cat</td>
<td>P. molurus</td>
<td>Ilex paraguaiensis</td>
</tr>
<tr>
<td>Margay</td>
<td>P. curtus</td>
<td>Didymopanax morotoni</td>
</tr>
<tr>
<td>Geoffroy’s Cat</td>
<td>P. sebae</td>
<td>Araucaria hunsteinii</td>
</tr>
<tr>
<td>Leopard Cat</td>
<td>Bunectes spp.</td>
<td>Zeyhera tuberculose</td>
</tr>
<tr>
<td>Other Mammals</td>
<td>Boa Constrictor</td>
<td>Cordia milleni</td>
</tr>
<tr>
<td>Black Rhino</td>
<td>Rat snake</td>
<td>Atriplex repanda</td>
</tr>
<tr>
<td>Amur Leopard</td>
<td>Dog-faced Water Snake</td>
<td>Cupressus atlantica</td>
</tr>
<tr>
<td>Caucasian Leopard</td>
<td>Sea snakes</td>
<td>Cupressus dupreziana</td>
</tr>
<tr>
<td>Markhor Goat</td>
<td>Butterflies</td>
<td>Diospyros hemitales</td>
</tr>
<tr>
<td>Saiga Antelope</td>
<td>Schaus Swallowtail</td>
<td>Aniba duckei</td>
</tr>
<tr>
<td>Cape Fur Bull Seal</td>
<td>Homerus Swallowtail</td>
<td>Ocotea porosa</td>
</tr>
<tr>
<td>Sea Otter</td>
<td>Birdwing</td>
<td>Bertheletia excelsa</td>
</tr>
<tr>
<td>African Elephant</td>
<td>Ornithoptera alezandrea</td>
<td>Dipterix alata</td>
</tr>
<tr>
<td>Chimpanzees</td>
<td>Orchids</td>
<td>Abies guatemalensis</td>
</tr>
<tr>
<td>Toads</td>
<td>Dendrobium aphyllum</td>
<td>Tectona hamiltoniana</td>
</tr>
<tr>
<td>Colorado River Toad</td>
<td>D. bellatulum</td>
<td>Mahogany</td>
</tr>
<tr>
<td>Turtles</td>
<td>D. chrysotoxum</td>
<td>Teak</td>
</tr>
<tr>
<td>Hawshill Sea Turtle</td>
<td>D. farneri</td>
<td>Other Plants</td>
</tr>
<tr>
<td>Egyptian Tortoise</td>
<td>D. scabrilingue</td>
<td>Himalayan Yew</td>
</tr>
<tr>
<td>American Box Turtle</td>
<td>D. sensile</td>
<td>Green Pitcher Plant</td>
</tr>
<tr>
<td>Birds</td>
<td>D. thyrsiflorum</td>
<td>Sm. Begonia</td>
</tr>
<tr>
<td>Red and Blue Lorry</td>
<td>D. unicum</td>
<td>Chisos Mt Hedgehog Cactus</td>
</tr>
<tr>
<td>Parrots</td>
<td>Rattan</td>
<td>Key Tree Cactus</td>
</tr>
<tr>
<td>Quetzal</td>
<td>Calamus caesius</td>
<td>Nellie Cory Cactus</td>
</tr>
<tr>
<td>Roseate Spoonbill</td>
<td>C. manan</td>
<td></td>
</tr>
<tr>
<td>Macaws Ara spp.</td>
<td>C. optimus</td>
<td></td>
</tr>
<tr>
<td>Hyacinth Macaw</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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(Simmons & Kreuteo 1989). The higher prices increased incentives for poaching.

Recently, governments have toughened enforcement efforts with a ban on the ivory trade, shooting of poachers on sight, strengthened measures against corruption, and the highly publicized destruction of confiscated ivory\(^1\). This crackdown on poaching has been accompanied by decreases in the price of elephant tusks (Bonner 1993). Since these policy changes reduce short-run ivory supply as well as demand, it is not clear that the fall in price would have been predicted under a static model, and indeed most economists did not predict this decline. However, the fall in price is consistent with the dynamic model set forth in this paper, under which improved anti-poaching enforcement may increase long-run ivory supply by allowing the elephant population to recover.

Under the model, anticipated future scarcity of storable resources leads to higher current prices, and therefore to more intensive current exploitation. For example, elephant poaching leads to expected future shortages of ivory, and thus raises future ivory prices. Since ivory is a storable good, current ivory prices therefore rise, and this creates incentives for more poaching today. Because poaching creates its own incentives, there may be multiple rational expectations paths of ivory prices and the elephant population for a range of initial populations.

In order to gain intuition for why there may be multiple rational expectations equilibria, it is useful to consider the following two period example, for which we thank Marty Weitzman. Suppose that each year there is a breeding season during which population grows by an amount \(B(x)\) given an initial population of \(x\). Following the breeding season, an amount \(h\) is harvested. Denote the elephant population at the beginning of the harvest season in year one as \(x_0\). Then the population at the end of the harvest in year one will be \(x_0 - h_1\), and the population at the end of the harvest in year two will be \(x_0 - h_1 + B(x_0 - h_1) - h_2\). To keep the model as simple as possible, we assume that the world ends after two years. See Table 3.2

\(^1\)In September 1988, Kenya's president ordered that poachers be shot on sight, and in April 1989 Richard Leakey took over Kenya's wildlife department.
Table 3.2: Time Line for Two Period Example

<table>
<thead>
<tr>
<th>Time</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial (year 1)</td>
<td>$x_0$</td>
</tr>
<tr>
<td>After harvest, $h_1$, in year 1</td>
<td>$x_0 - h_1$</td>
</tr>
<tr>
<td>After breeding in year 2</td>
<td>$x_0 - h_1 + B(x_0 - h_1)$</td>
</tr>
<tr>
<td>After harvest, $h_2$, in year 2 (end of world)</td>
<td>$x_0 - h_1 + B(x_0 - h_1) - h_2$</td>
</tr>
</tbody>
</table>

Let $c$ denote the cost of harvesting an animal, and denote the amount of the good demanded at a price of $p$ as $D(p)$. Assume $D' < 0$ and $D(\infty) = 0$. The interest rate, which is assumed to be the only cost of storage, is denoted $r$.

There will be an equilibrium in which the animal is hunted to extinction in year 1 if the initial population is less than enough to satisfy demand during the first year at a price of $c$, plus demand during the second year at a price of $(1 + r)c$. Algebraically, this can be written as: $x_0 < D(c) + D((1 + r)c)$.

There will be an equilibrium in which the animal does not go extinct if the initial population, minus the amount required to satisfy first-year demand at price $c$, plus the births in the breeding season, is more than enough to satisfy second period demand at price $c$. This will be the case if $x_0 - D(c) + B(x_0 - D(c)) > D(c)$.

If both conditions hold, then there will be multiple equilibria. In one, the animal survives. In the other, the price is high enough that the population is eliminated in the first period, and the breeding that would have satisfied second-period demand never takes place. There will be multiple equilibria if the initial stock is such that:

$$D((1 + r)c) + D(c) > x_0 > 2D(c) - B(x_0 - D(c)).$$

Note that as the interest rate increases, there will be an extinction equilibrium for a diminishing range of initial population levels. For sufficiently high interest rates, there will only be a single equilibrium path of population for any initial stock, just as in non-storable fisheries models.

Note that the example above implicitly assumes that the good is destroyed when it is consumed. It thus applies to goods such as rhino horn, which is consumed in
traditional Asian medicines. We will call such goods *storable* and distinguish them from *durable* goods, which are not used up when they are consumed\(^2\). In an earlier version of this paper, we showed that there could be multiple equilibria in a two-period model of durable goods. This paper models storable, but not durable, goods, but we believe that, except where noted, the results would be qualitatively similar for durable goods.

In the remainder of the paper we use a continuous time, infinite-horizon model, which allows us to solve for steady-state population and prices, and to examine cases in which extinction is not immediate following a shift in expectations, or the path of population and prices is stochastic.

The model carries several policy implications. It suggests that even if the population level is steady, so that standard models would predict the continued survival of the species, the species could still be vulnerable to a switch to an extinction equilibrium. One way to eliminate the extinction equilibrium would be to increase the population of the animal by providing additional habitat. This is, however, likely to be expensive.

If governments have credibility, they may be able to eliminate the extinction equilibrium, and coordinate on the high population equilibrium, merely by promising to implement tough anti-poaching measures if the population falls below a threshold. This suggests that laws which provide little protection to non-endangered species, and practically unlimited protection to endangered species may be justified in some cases.

Finally governments or conservation organizations may be able to eliminate the extinction equilibria by building sufficient stockpiles of the storable good, and threatening to sell the stockpile if the animal becomes endangered or the price rises beyond a threshold. This is somewhat analogous to central banks using foreign exchange reserves to defend an exchange rate (Obstfeld 1986, Obstfeld 1994). Stockpiles

\(^2\)Ivory is often considered an example of a durable good.
could be built either by deliberately harvesting animals, or by storing confiscated contraband taken from poachers, rather than either destroying or selling it.

A number of other papers find multiple equilibria in models of open-access resources with small numbers of players (Lancaster 1980, Haurie & Pohjohla 1987, Levhari & Mirman 1973, Reinganum & Stokey 1985, Benhabib & Radner 1992). In these models, each player prefers to grab resources immediately if others are going to do so but to leave resources in place, where they will grow more quickly, if others will not consume them immediately. Tornell & Velasco (1992) introduce the possibility of storage into this type of model.

The effects examined in the previous models are unlikely to lead to multiple equilibria if there are many potential poachers, each of whom assumes that his or her actions have only an infinitesimal effect on future resource stocks, and on the actions chosen by other players. This paper argues there may nonetheless be multiple equilibria for open-access renewable resources used in the production of storable goods, because if others poach, the animal will become scarce, and this will increase the price of the good, making poaching more attractive. Because poaching transforms an open-access renewable resource into a private exhaustible resource, this paper can be seen as helping unify the Gordon-Schaefer analysis of open-access renewable resources with the analysis of Hotelling (1931) of optimal extraction of private non-renewable resources.

The remainder of the paper is organized as follows. Section 3.2 presents the standard Gordon-Schaefer fisheries model, in which storage is impossible. Section 3.3 shows how the model can be adapted to allow for storage, and classifies the possible equilibria. Section 3.4 discusses equilibria in which people believe there is some probability that the economy will coordinate on extinction and some probability the economy will coordinate on survival. Section 3.5 concludes with a discussion of policy implications.
3.2 The Standard Gordon-Schaefer Model With No Storage

In the standard Gordon-Schaefer model, as set forth by Clark (1976),

\[
\frac{dx}{dt} = B(x) - h, \tag{3.1}
\]

where \( x \) denotes the population, \( h \) is the harvesting rate, and \( B \), the net-births function, is the rate of population increase in the absence of harvesting\(^3\). \( B(0) = 0 \), since if the population is extinct, no more animals can be born. We will measure the population in units of carrying capacity, so \( B(1) = 0 \), and \( B(x) \) is strictly negative for \( x > 1 \). \( B \) is strictly positive if population is positive and less than 1. This implies that, without harvesting, the unique stable steady state for the population is 1.

The rate of harvest will depend on the demand and the marginal cost faced by poachers. The marginal cost of poaching, \( c \), is a decreasing function of the population \( x \), so that \( c = c(x) \), with \( c'(x) < 0 \). We assume that \( c'(x) \) is bounded and that there is a maximum poaching marginal cost of \( c_m \), so that \( c(0) = c_m \).

Given price, \( p \), consumer demand is \( D(p) \), where \( D \) is continuous, decreasing in \( p \), and zero at and above a maximum price \( p_m \). We will restrict ourselves to the case in which \( p_m > c_m \), so that some poaching will be profitable, no matter how small the population. This condition is necessary for extinction to be a stable steady-state.

Since the good is open-access, and storage is assumed to be impossible, its price must be equal to the marginal poaching cost. Algebraically, \( p = c(x) \). The harvest must be exactly equal to consumer demand, so \( h = D(c(x)) \). The evolution of the system in which storage is impossible is thus described by:

\[
\frac{dx}{dt} = B(x) - D(c(x)) \equiv F(x). \tag{3.2}
\]

We assume that \( B \), \( D \), and \( c \) are differentiable. Since \( B(0) = 0 \), and \( p_m > c_m \), \( D(c(0)) > 0 \), so that \( F(0) < 0 \), as illustrated in Figure 3.2. Thus, zero is a stable

\(^3\)This is often taken to be the logistic function \( B(x) = x(1 - x) \)
steady state of equation 3.2. $F(1) < 0$ since $B(1) = 0$, and $D(c(1)) > 0$. We will consider the case in which $F$ is positive at some point in $(0, 1)$, so that extinction is not inevitable. Assuming that $F$ is single-peaked\(^4\), there will generically be points $X_S$, and $X_U$ so that $F$ is negative and increasing on $(0, X_U)$, positive on $(X_U, X_S)$, and negative and decreasing on $(X_S, 1]$. Hence, if population is between 0 and $X_U$, it will become extinct, whereas if it starts above $X_U$, it will tend to the high steady state, $X_S$. Thus, if storage is impossible, there will be multiple steady states, but a unique equilibrium given initial population.

### 3.3 Equilibria with Storage

This section introduces the possibility of storage into a Gordon-Schaefer type model. We assume that storage is competitive, that there is no intertemporal substitution in demand for the good, and that the cost of storage is an interest cost, with rate $r$.

We will look for rational expectations equilibria, or paths of population, stores, and price in which poachers, consumers, and storers are behaving rationally at all times. This section considers perfect foresight equilibria, in which the path is deterministic; Section 3.4 considers equilibria in which the path is stochastic. The steady states of the model with storage are the same as those in the model without storage, as we show below\(^5\). Indeed, the stable steady states of the last section comprise the entire stable limit set of the system with storage (i.e. there are no cycles or chaotic attractors).

We analyze the fairly general model introduced in the last section with two stable steady states, one at zero and the other at $X_S$. In fact, the propositions of this section can be easily generalized to cover much more general models in which there are many stable steady states, or extinction is not stable.

\(^4\)For most of the sequel, we don't strictly need $F$ to be single peaked, but this requirement simplifies the analysis and the notation, and is not too restrictive.

\(^5\)We will make a distinction between a steady state, which is a stationary value of population and stores, and an equilibrium.
Figure 3.1: Dynamics of the Gordon-Schaefer Model with No Storage
Our strategy for finding equilibria is as follows. Simple accounting arithmetic and the absence of arbitrage opportunities in poaching and storage yield local equilibrium conditions on the possible equilibrium paths. Because there may or may not be storage or poaching, it turns out that there are three possible different dynamic regimes: no storage, storage, and no poaching. Using the local equilibrium conditions, we derive differential equations for the equilibrium paths in each regime. The steady states give terminal or boundary conditions which allow us completely to determine the equilibrium paths, which we represent using phase diagrams in population-stores space. The steady states provide a terminal condition that allows us completely to characterize the equilibrium paths.

3.3.1 Local Equilibrium And Feasibility Conditions

The local equilibrium conditions are determined by the absence of arbitrage opportunities for both poachers and storers of the good:

The Storage Condition

The possibility of storage introduces constraints on the path of prices. As in Hotelling (1931), in order to rule out arbitrage,

$$\frac{dp}{dt} = rp, \text{ if } s > 0,$$

(3.3)

where $s$ denotes the amount of the good that is stored. If the price were rising less quickly, people would sell their stores, and if the price were rising more quickly, people would hold on to their stores, or poach more. This “storage condition” is slack when stores are zero. In this case, $dp/dt \leq rp$, because otherwise people would find it profitable to hold stores.

The Poaching Condition

Because poaching is competitive, if there is poaching at all, the price of the good must be equal to the marginal cost of poaching another unit of the good, which is
$c(x)$, if the population is $x$. Thus the "poaching condition" is that:

$$p = c(x), \text{ if there is poaching.} \quad (3.4)$$

This condition is slack if there is no poaching, in which case $p \leq c(x)$.

Note that, in addition to the local equilibrium conditions above, there are some feasibility conditions:

"Conservation of Elephants"

At all times, the increase in stores plus the increase in population must equal the net births minus the amount consumed, or

$$s + \dot{x} = B(x) - D(p). \quad (3.5)$$

Note that, as mentioned earlier, we assume that the good is destroyed when it is consumed. Note also that animals which die naturally cannot be turned into the storable good.$^6$

Finally, both population, $x$, and stores, $s$, must be non-negative at all times.

The above conditions imply that, once on an equilibrium path, population, stores, and price, must be a continuous function of time. This is because, with perfect foresight, jumps would be anticipated and arbitrated. See Appendix B.1, Proposition B.1 for a more formal proof. As we discuss below, there may be an initial jump to get to the equilibrium path.

These conditions must be satisfied at all points on a rational expectations equilibrium path. There are four conceivable dynamic regimes for the system, depending on which of the storage and poaching conditions (equations 3.3 and 3.4) are binding at any time, but only three of these potential regimes are actually possible:

---

$^6$We write the conservation condition as an equality. Because the price is positive, no one would throw the good away voluntarily.
No Storage Regime

Stores are zero, but there is poaching. The zero profit condition for poaching implies that $p = c(x)$. The storage condition restricts the rate at which the price can rise and not induce storage ($\dot{p} \leq rp$). Because the price is inversely related to the population, it is possible to translate this condition that prices may not rise too fast into a condition that the population may not fall too fast: differentiating $p = c(x)$, the condition that no one wants to hold positive stores becomes

$$\frac{dx}{dt} \geq r\frac{c(x)}{c'(x)}. \quad (3.6)$$

In the No Storage Regime, the dynamics are the same as Section 3.2, the model with no storage:

$$\dot{x} = B(x) - D(c(x)) \quad (3.7)$$
$$s = 0$$
$$p = c(x)$$

Storage Regime

Stores are positive and there is poaching, so $dp/dt = rp$, and $p = c(x)$. Here, the exponential path of the price translates into a differential equation for population: differentiating $p = c(x)$ gives the same expression, but with equality, that we had for the No Storage regime (3.6). Given the path of population and, hence, price and consumption, the dynamics of stores are determined by "conservation of elephants" (3.5), and we can express all the local equilibrium dynamics in terms of the population, $x$:

$$\dot{x} = r\frac{c(x)}{c'(x)} \quad (3.8)$$
$$\dot{s} = B(x) - D(c(x)) - \dot{x}$$
$$\dot{p} = rc(x).$$
No Poaching Regime

Stores are positive, but there is no poaching. Without poaching, the rate of change of population is just the net birth rate. All demand is being satisfied from stores, so stores must be falling at the same rate as demand. For stores to be positive, price must be rising exponentially at rate $r$. The dynamics can thus be summarized by:

$$
\begin{align*}
\dot{x} &= B(x) \\
\dot{s} &= -D(p) \\
\dot{p} &= rp.
\end{align*}
$$

Note that since there is no poaching, it is not possible to substitute $c(x)$ for $p$.

No Storage, No Poaching

This is impossible if population is positive, since it would imply that there is no consumption, so the price must be $p_m$, but $p_m$ is greater than $c_m$, which is the maximum marginal cost of poaching, so there would have to be poaching, which contradicts the assumption that there was no poaching.

To be in steady state, stores must be zero because, when they are positive, price must be rising exponentially. This means that there are only two stable steady states: extinction, in which population and stores are zero, and what we will call the "high steady state", in which population is $X_S$, stores are zero, and price is $c(X_S)$. If stores are zero, then the system must be in the No Storage regime, and will thus have the same steady states as the model with no storage in Section 3.2, i.e. $x = 0$ or $X_S$.

3.3.2 Dynamics Within The Storage And No Storage Regimes

We shall begin by looking at the two regimes in which there is poaching: No Storage and Storage.
Equilibrium Paths in the No Storage Regime

For the system to be in the no storage regime in equilibrium, people must not want to hold positive stores, so price must not be rising faster than \( rp \). Since the price is determined by the population, \( p = c(x) \), storage implies that the population cannot fall too fast. Specifically, from equations (3.6) and (3.7),

\[
B(x) - D(c(x)) \geq r \frac{c(x)}{c'(x)}.  \tag{3.10}
\]

As is clear from Figure 3.3.2, for small enough \( r \), equation (3.10) will hold if and only if \( x \in [X_U^*, X_S^*] \), where \( X_S^* \) and \( X_U^* \) are the two critical points at which the storage condition is just binding, i.e. \( B - D = rc/c' \). Moreover, \( 0 < X_U^* < X_U < X_S < X_S^* \).

If the system starts with population in \((X_U, X_S^*)\) and no stores, then it is an equilibrium to follow the No Storage Regime dynamics to \( X_S \), the stable steady state. If the system starts with no stores and a population of exactly \( X_U \), the unstable steady state, the system will stay there. Here, as elsewhere, for the sake of clarity, we shall not discuss measure zero cases like this in any detail.

If the system starts with no stores and with population in \([X_U^*, X_U]\), then the No Storage dynamics will eventually take population to a point less than \( X_U^* \). At some point, therefore, the system must leave the No Storage Regime and enter the Storage Regime. We discuss this after we have found the equilibrium paths in the Storage Regime.

Equilibrium Paths in the Storage Regime

The dynamics of population are determined by the price, which is rising exponentially. The dynamics of stores are determined by "conservation of elephants": what is harvested and not consumed must be stored. We may rewrite equation (3.8) as a differential equation for the trajectory of stores, \( s \), in terms of \( x \):

\[
\frac{ds}{dx} = \frac{c'(x)}{rc(x)} \left\{ B(x) - D(c(x)) - r \frac{c(x)}{c'(x)} \right\}.  \tag{3.11}
\]

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Figure 3.2: The Storage Condition in the No Storage Regime

Storage Condition Solved when $\dot{x} = F(x) > 0$ implies $x \in [x_u^*, x_s^*]$
$dx/dt$ is still just $rc(x)/c'(x)$, which is strictly negative, and bounded above.

Equation (3.11) implies that rational expectations trajectories in population-stores space must have stores decreasing with population, $x$, if $x < X_U^*$, or $x > X_S^*$. Stores must be an increasing function of population if $x \in (X_U^*, X_S^*)$. There is a maximum of stores at $X_U^*$, and a minimum at $X_S^*$. To see the intuition for this, note that if population is very high or very low, population would tend to fall rapidly without stores, and as may be seen from Figure 3.3.2, it would fall rapidly enough that price would be rising faster than rate $r$. In order to prevent population from falling too rapidly, part of demand must be satisfied out of stores, and so stores must decrease with time. $X_U^*$ and $X_S^*$ are the points at which, in the absence of stores, the population would fall just fast enough that price would rise at rate $r$. Between $X_U^*$ and $X_S^*$, the price would rise more slowly than rate $r$ with no storage. For an equilibrium with stores, therefore, more than current demand must be being harvested and stores must increase to make the population fall fast enough so that price rises at exactly rate $r$.

Equation (3.11) is the differential equation for the trajectories of equilibria in population-stores space. The equilibria are now to be determined by boundary conditions. One possibility is that stores run out while population is still positive, and the system enters the No Storage Regime. The only place at which this can possibly happen is where population is exactly $X_S^*$. To see why, consider the following: to be in the No Storage Regime, $x \in [X_U^*, X_S^*]$. Because population, stores and price are continuous in equilibrium, the system must leave the Storage regime at the same point at which it enters the No Storage regime. As explained above, stores are decreasing as a function of $x$, so strictly increasing as a function of time ($x$ is falling) if $x \in (X_U^*, X_S^*)$, and at a maximum at $x = X_U^*$. But stores have to run out at the point of transition from the Storage to No Storage Regime, so stores must have been falling, (or at least not increasing or at a maximum) immediately before the transition. The only point remaining at which stores could run out is, therefore,
The other possible boundary condition is that population becomes extinct before stores run out. Since \( x \) is decreasing at a rate which is bounded below while stores are positive, the population must become extinct in finite time if stores do not run out. After that, stores will be consumed until they reach zero as well. It turns out that the quantity of stores remaining when the population becomes extinct is uniquely determined in a rational expectations equilibrium. To see this, note that the price charged for the last unit of stores must be \( p_m \), the maximum price people are willing to pay for the good, or a storer would profit by waiting momentarily to sell his or her stock. The zero profit condition in poaching implies that the price when the population becomes extinct must be \( c(0) = c_m \). Price is rising exponentially while stores are positive, so we can calculate the amount, \( U(p) \), consumed from the time when price is \( p \) until price reaches \( p_m \):

\[
U(p) = \int_0^{1 \ln\left(\frac{p_m}{p}\right)} D\left(pe^{rt}\right) dt.
\]

The amount of stores remaining at the moment of extinction must, therefore, be \( U(c_m) \).

We have shown that there can only be two equilibrium paths in the Storage Regime (See Figure 3.3.2)

1. **High Steady State Storage Equilibrium** In this equilibrium, population starts at \( x > X^*_n \). The system evolves until stores run out when population is \( X^*_n \), and then enters the No Storage Regime. The equations \( p = c(x) \), and \( \frac{dp}{dt} = rp \) determine the path of population and price. Stores are given by \( s = s_+(x) \), where:

\[
s_+(x) = \int_{X^*_n}^{x} \frac{c'(q)}{rc(q)} \left\{ B(q) - D(c(q)) - r\frac{c(q)}{c'(q)} \right\} dq.
\]

2. **Extinction Storage Equilibrium** In this equilibrium, population becomes extinct, and at that moment, stores = \( U(c_m) \). The equations \( p = c(x) \) and
\[ \frac{dp}{dt} = rp \] determine the path of population and price. Stores are given by
\[ s = s_e(x), \]
where:
\[ s_e(x) = U(c_m) + \int_0^x \frac{c'(q)}{rc(q)} \left\{ B(q) - D(c(q)) - r \frac{c(q)}{c'(q)} \right\} dq. \quad (3.14) \]

For this to be an equilibrium, stores must stay positive at all times along this path. If stores would have to become negative at some point in the future, this path is not an equilibrium. If \( s_e(x) \) is ever negative, we define \( X_{\text{max}} \) to be the smallest positive root of \( s_e(x) \). If there is none such, we say that \( X_{\text{max}} = \infty \).

To be an equilibrium, the starting population must be less than \( X_{\text{max}} \).

\( s_e(x) \) and \( s_+(x) \) are parallel. Both have a minimum at \( X_5^* \). It is clear from Figure 3.3.2 that \( X_{\text{max}} \) is finite if and only if \( s_e(x) \) lies below \( s_+(x) \). If \( X_{\text{max}} \) is finite, it must lie between \( X_U^* \) and \( X_5^* \).

Transitions Between Storage and No Storage Regimes

We now examine under which circumstances an equilibrium path can move from the No Storage to the Storage Regime. If the initial population is small enough, an equilibrium path can move to the Storage regime and, thence, to extinction. It may have to do this: if \( X_U^* > 0 \) and the system starts in the No Storage regime with population less than \( X_U \), then the system must eventually move to the Storage regime because if it didn’t, the population would fall fast enough to violate the storage condition once it had fallen past \( X_U^* \). On an equilibrium path, the system must move to the Storage regime before that point is reached. The system may also move to the Storage regime when it doesn’t strictly have to. By continuity of stores, the system must make the transition from the No Storage to the Storage regime where \( s_e(x) = 0 \), i.e. at \( X_{\text{max}} \). If the path in the No Storage regime crosses \( X_{\text{max}} \), then the system can move to the Storage regime path \( s_e \) leading to extinction. At such a transition, the rates of change of population, stores, and price will jump, but

\[ \text{The transition cannot happen at } X_5^*, \text{ because the population would be falling there in the No Storage regime, so the system could never reach that point.} \]
Figure 3.3: Storage Regime Equilibrium Paths

$X_{max} = \infty$ if $s_e \geq s_+$

$s_e(x)$

U($c_m$)

$s_e(x)$

U($c_m$)

$s(x)$
the storage and poaching conditions are not violated, because the levels will not jump.

We may thus define two sets of points on equilibrium paths in the Storage and No Storage regimes, $A_e$, the set of points leading to extinction, and $A_+$, the set of points leading to the high steady state, as illustrated in Figure 3.3.2.

The system must end up on one of these paths, $A_+$ or $A_e$. Given arbitrary initial values of population and stores $(x_0, s_0)$, there can either be an initial cull, or there can be an interlude when there is no poaching, as discussed below.

3.3.3 Moving To Equilibrium

If the initial population and stores are not on one of the equilibria identified above, then one of two things will happen. If the initial point in population-stores space is below the equilibrium paths above, then the system may jump instantaneously to one of the equilibrium paths via a cull. If the initial point is above an equilibrium, demand may be satisfied from stores with no poaching for a while until the path meets $A_e$ or $A_+$.

Culling

If the system starts above an equilibrium path, there may be an instantaneous harvest, which we shall call a “cull”. In this case, the price starts high enough that it is above the marginal cost implied by the initial population, $c(x_0)$, and there will be instantaneous poaching up to the point at which the price is equal to the marginal cost\(^8\). Although continuity of price, population, and stores is required by rationality, such a jump is allowed if it is unanticipated, or at the “beginning of time”, as it is in this case. We will make a distinction between “initial” values of population and

\(^8\)If the marginal cost of poaching became sufficiently great as the instantaneous rate of poaching became great enough, the harvest would take place over time, rather than instantaneously. Structurally, though, there is little real difference in the two approaches: the rational expectations equilibria are determined by the boundary conditions (where people anticipate the system must end up), and these are essentially the same in both cases.
Figure 3.4: The Storage and No Storage Regime Equilibrium Sets $A_+$ and $A_e$

1. $X_{max} = \infty$
2. $X_u < X_{max} < X_s$
3. $X_{max} < X_u$

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stores and "starting" values, which are the values just after the initial cull. When we need to indicate this, we will write \((x_0, s_0)\) for initial population and stores, and \((x(0), s(0))\) to denote starting (i.e. at time 0 on the equilibrium path) values.

In a cull, live elephants are killed and turned into dead elephants one-to-one. This means that, in population-stores space, the system moves up a downward sloping diagonal, and the total quantity of elephants, dead or alive, is conserved. We call this quantity \(Q = x + s\). For a cull to be rational, it must take the system to a point on one of the equilibrium paths we identified above, \(A_e\) or \(A_+\).

To get to the high steady state equilibrium path by culling, initial population and stores must lie below the line \(s = s_+(x)\), and \(x_0 > X_S^*\).

To get to the path leading to extinction, if \(X_{\text{max}}\) is infinite, the initial point must lie below \(s = s_e(x)\). If \(X_{\text{max}}\) is finite, points below \(s_e(x)\) can also cull to the equilibrium, but there may also be other points from which this is feasible. In particular, if the curve \(s = s_e(x)\) has a tangent of gradient -1, then, as illustrated in Figure 3.3.3 points above the curve, but below the tangent can also reach the extinction equilibrium by culling. A quick look at equation (3.11) shows that the points at which \(s_e(x)\) has gradient -1 are \(X_U\) and \(X_S\), but only the tangent at \(X_U\) lies above the curve. The value of \(Q\) at this tangency, \(Q_{\text{max}}\), is the maximum value \(Q\) may have so that the extinction equilibrium may be reached via culling. If \(X_{\text{max}} < X_U\) this tangency doesn't exist, and only points below \(s_e\) can cull to the extinction equilibrium.

**No Poaching**

If there are sufficient initial stores, there will be equilibria in which the starting price is below \(c(x)\), and there is no poaching for a time while demand is satisfied out of stores. Eventually poaching must resume, at a point on \(A_e\) or \(A_+\). While there is no poaching, population will be rising, and stores falling as they are consumed. Price is rising exponentially, at rate \(r\). In population-stores space, trajectories with
Figure 3.5: Definition of $Q_{\text{max}}$

\[ X_{\text{max}} = 0 \text{ if } S_e \geq S_+ \]

\[ S_e(x) \]

\[ U(c_m) \]

\[ S_i(x) \]

\[ X_u \]

\[ X_{\text{max}} \]

\[ X_s \]

\[ S_e(k) \]

\[ S_i(k) \]

\[ 0 \]

\[ \text{Population} \]

\[ \text{Stores} \]
no poaching must be downward sloping and population must be increasing so long as population is less than one.

When poaching resumes at a point on one of the $A_i$ paths, price, population, and stores are all determined. Given the end point, there is a unique, downward sloping no poaching trajectory leading to it. In order for no poaching to be rational, and for an initial point to end up on one of the $A_i$, the initial point must lie on one of these trajectories (Figure 3.3.3). To get to the path leading to the high steady state, the initial point must lie to the right of the boundary of the set of points on trajectories leading to $A_+$, which we denote $L_+$, and above the curve $s = s_+(x)$. To get to the path leading to extinction, the initial point must lie to the left of the boundary of the set of points on trajectories leading to points on $A_e$, which we denote $L_e$. We include a more formal treatment of this in the Appendix B.1, Proposition B.3.

We have now found all the possible equilibria of the model with storage. As illustrated in Figure 3.3.3, population-stores space may be divided into at most three regions depending on whether there exist equilibria leading to extinction, the high steady state, or both. In the first region, there is no equilibrium path leading to extinction. This will be the case if the initial population and stores are high enough, so that killing and storing enough to get to extinction would mean that stores would have to be held long enough that the storers would lose money. In the second region, there is no equilibrium path leading to the high steady state. This is the case if population and stores are low enough that, even if poaching were temporarily to cease and demand were to be satisfied from stores until they should run out, the population cannot recover enough to guarantee species survival. The third region is where there are multiple possible equilibria, some to extinction, and some to the high steady state. In this deterministic, perfect foresight model, which equilibrium is chosen is determined by exogenously formed, self-fulfilling expectations.

Depending on parameter values, some of these regions may be empty. It is possible that there will be no region in which survival is assured. If $X_{max}$ is infinite,
Figure 3.6: Equilibria with No Poaching
Figure 3.7: Coexistence of Equilibria
any point can get to the extinction set $A_e$, either through a cull if it lies below $s_e$, or by an interlude with no poaching if it lies above $s_e$.

If, on the other hand, $X_{\text{max}}$ is small enough (less than $X_U$), then there will be no region of multiple equilibria, and the fate of the system will be entirely determined by its initial point, and not by expectations.

Note that, if there is an initial no poaching interlude, the population will be rising to start with even if the eventual fate of the system is extinction. There will often be over-shooting with No Poaching equilibria, and one should not, therefore, become complacent if elephant populations are increasing.

It turns out that $X_{\text{max}}$ and $Q_{\text{max}}$ are both decreasing in $r$, the storage cost. For proofs, see Appendix B.1, Proposition B.2. This should not come as a surprise. $Q_{\text{max}}$ tells us the largest population can be and still reach extinction via culling and a storage equilibrium path. The larger the population, the longer stores have to be held before extinction. This is clearly going to be less desirable with higher storage costs. Increasing the storage cost thus always reduces the region of phase space from which extinction is possible. Governments could increase storage costs by threatening prosecution of anybody found to be storing the good. The international ban on ivory trade may have had this effect.

For sufficiently large $r$, $X_{\text{max}}$ will be less than $X_U$, and there will be no region of multiple equilibria at all; the ultimate fate of the species is the same as in the model with no storage possible, given the same initial conditions. In this sense, our model converges to the standard Gordon-Schaefer model as storage cost rises.

If $Q_{\text{max}} > X_S$, then even starting from the high steady state with no stores, the population will be vulnerable to coordination on the extinction equilibrium. This highlights another possible policy response to limit the possibility of extinction: the government or private conservation organizations may increase the size of habitat available to the species. Increasing the habitat, while leaving demand unchanged, will increase the steady state population, $X_S$, more than proportionally. At the
same time, \( Q_{\text{max}} \) will fall. We show, in Appendix B.1, Proposition B.4 that, for sufficient habitat, \( X_S \) will be above \( Q_{\text{max}} \), and the species will then be safe from speculative attacks leading to extinction when it is in the high steady state.

### 3.4 Non-Deterministic Equilibria

So far, we have focused on perfect foresight equilibria, in which all agents believe that the economy will follow a deterministic path. This section considers a broader class of rational expectations equilibria in which agents may attach positive probability to a number of future possible paths of the economy. One reason to consider this broader class of equilibria is that the perfect foresight equilibrium concept has the uncomfortable property that there may be a path from \( A \) to \( B \), and from \( B \) to \( C \), but not from \( A \) to \( C \). To see this, note that if \( Q_{\text{max}} \) is greater than \( X_S \), then for sufficient initial population, the only equilibrium will lead to the high steady state. For a system that starts in the high steady state, however, an extinction storage equilibrium would also be possible.

Note also that the concept of no poaching regimes is also much more relevant when stochastic paths are admissible, since in order to have an equilibrium with no poaching, there must be stores, and the only way stores can be generated within the model is through a storage equilibrium. However, within the limited class of perfect foresight equilibria, people must assign zero weight to the possibility that there might be a switch from an storage regime to a no poaching regime.

While we have not fully categorized the extremely broad class of equilibria with stochastic rational expectations paths, we have been able to describe a subclass of equilibria, which we conjecture illustrates some more general aspects of behavior.

We consider equilibria in which agents believe there is a constant hazard that a sunspot will appear and that, when this happens, the economy will switch to the extinction storage equilibrium, with no possibility of any other switches\(^9\).

\(^9\)In some cases, extinction is instantaneous after the switch, so there is no way that agents could
Thus all agents know that the economy will switch to the extinction equilibrium eventually with probability one, but they are unsure when. We divide time into two parts: before the sunspot (B.S.), and after the sunspot (A.S.). The equilibrium behavior A.S. is simple: it is just the extinction equilibrium we found in the last section. In this section, we look for equilibria B.S. in which the population does not become extinct.

We first derive a stochastic analogue of the storage condition. We then show that there are equilibria with a small switching hazard in which the behavior is similar to that seen in Section 3.3: the B.S. steady state population is \( X_S \), and no stores are held in this steady state. There are also equilibria with a higher switching hazard in which positive stores are held in B.S. steady-state equilibrium in anticipation of a switch to the extinction equilibrium. There cannot be equilibria with a hazard rate above a certain threshold, because in this case extinction would become so likely that it would become certain and the system would have to jump immediately to the extinction equilibrium.

We also show that while the species can survive a series of small increases in the hazard rate of switching by building up stores after each increase, it might not be able to sustain the same increase in the hazard rate if it took place in a single jump, because the required increase in storage would be so great as to drive the species into extinction.

In this section, it is mathematically more convenient to work with the total of stores and population, \( Q \), rather than stores, \( s \). Since \( Q = s + x \), working with \( (x, Q) \) is equivalent to working with \( (x, s) \).

For the sake of clarity and brevity, we relegate all proofs to Appendix B.2, where we derive the results of this section more formally.
3.4.1 B.S. Equilibrium Conditions

Let the B.S. state be \((x, Q, p)\), and the A.S. state be \((x_e, Q, p_e)\). Note that, since the switch happens instantaneously by culling, \(Q\) doesn’t change when the sunspot appears. If there are positive stores, the expected profit from storage must be zero, so that, if we denote by \(\pi\) the hazard rate that the sunspot will appear:

\[
\dot{p} + (p_e - p)\pi = rp, \text{ if } s > 0.
\]  

(3.15)

If stores are zero, it must be because expected profits from storage are not positive, and so

\[
\dot{p} + (p_e - p)\pi \leq rp, \text{ if } s = 0.
\]  

(3.16)

Equations (3.15) and (3.16) are just generalizations of the storage condition when the price change is stochastic, and not necessarily continuous.

We shall consider equilibria in which, once the system is in the extinction equilibrium, there is no possibility of further change. \((x_e, Q)\) must, therefore, lie on the extinction storage equilibrium path derived in Section 3.3. In some cases, there may be more than one point on this path to which the system could jump. We will consider equilibria in which the system jumps to the lowest possible population on this path. Thus the population after the switch is a function of \(Q\): \(x_e(Q)\) is the smallest population such that \(s_e(x_e(Q)) + x_e(Q) = Q\). This is illustrated in Figure 3.4.1. Because harvesting cannot increase the population, for people to believe in the possibility of a switch to the extinction equilibrium we must have \(x \geq x_e(Q)\). We will also only formally consider cases in which \(Q_{\text{max}}\) is bigger than \(X_S\). The system must, therefore, be in the region where \(Q_{\text{max}} \geq Q \geq x > x_e(Q)\).

By assumption, the system jumps to the extinction equilibrium path if there is a sunspot. We may, therefore, determine the A.S price as a function of \(Q\), the total stores plus population at the time of the sunspot. \(Q\) is conserved during the switch. \(p_e\) is a decreasing function of \(Q\) (as we prove in Appendix B.2, Proposition B.5), and it is continuous on \([0, Q_{\text{max}}]\). When \(Q \leq U(c_m)\), the system jumps straight to
Figure 3.8: Switching to the Extinction Equilibrium

Q > Q_{max}, no call to X_e(Q) is possible

Q = x + S_e(x)

X = Q

X = X_{max}

STOCKS < 0 for Q < K

Q, population

Q

stores + population

X_e(Q), no call possible

X_e(Q)

U(c_m)

X_u

X_c

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extinction, and \( p_e = U^{-1}(Q) \). When \( Q > U(c_m) \), the population jumps to \( x_e(Q) \), and \( p_e = c(x_e(Q)) \).

We first consider the system dynamics when there are positive stores, so that equation (3.15) holds. By an argument analogous to that used in Section 3.3, while stores are positive (i.e. \( Q \geq x \)), the system evolves before the sunspot according to the differential equations:

\[
\begin{align*}
\dot{Q} &= B(x) - D(c(x)) \equiv F(x) \tag{3.17} \\
\dot{x} &= \frac{1}{c'(x)} [(r + \pi)c(x) - \pi p_e(Q)].
\end{align*}
\]

### 3.4.2 B.S. Steady States

It is rather easy to solve for the steady states of equation (3.17). The first equation tells us that, for total stores and population to be constant, population must be either \( X_S \), or zero (we ignore \( X_U \)), just as was the case in the deterministic case in the last section. We are interested in the steady state at \( X_S \). Given the population \( X_S \), the B.S. price will be \( c(X_S) \). To determine the steady state level of stores, note that the more stores, the lower the A.S. price will be, and so the less profitable it will be to speculate on the sunspot’s appearance. There will thus be a unique level of stores plus population, \( Q_S \), that satisfies the storage condition with equality. The second equation of (3.17) allows us to solve for this level of stores in terms of the interest rate, the sunspot hazard, and the characteristics of the extinction equilibrium after the sunspot:

\[
Q_S = p_e^{-1} \left( \frac{r + \pi}{\pi} c(x_S) \right). \tag{3.18}
\]

Because stores must be positive, this is only a feasible B.S. steady state if \( Q_S \geq X_S \). This will be the case for \( \pi \geq \pi_l \), where

\[
\pi_l = \frac{r c(X_S)}{p_e(X_S) - c(X_S)}. \tag{3.19}
\]

If \( \pi < \pi_l \), the storage condition cannot be satisfied with equality at \( X_S \), and there must be zero stores in steady state.
Because it must be possible to reach the A.S. equilibrium path via culling, it is also the case that we must have $Q_S \leq s_e(X_U) + X_U$. If $Q_{\text{max}}$ is finite, the right hand side of equation (3.19) is just $Q_{\text{max}}$. Even if $Q_{\text{max}}$ is infinite, this equation still holds, as if stores are too large at $X_S$, one cannot cull to a point on the extinction equilibrium. For this to hold, $\pi$ must be below $\pi_h$, where

$$\pi_h = \frac{rc(X_S)}{c(X_U) - c(X_S)}.$$  \hfill (3.20)

In summary, then, if $\pi < \pi_l$, then there is no B.S. steady state equilibrium with positive stores. In this case, $X_S$ is a steady state with no stores, just as it was in the perfect-foresight case of Section 3.3: the sunspot probability is low enough that the costs of holding positive stores outweigh the expected profit when the sunspot happens.

If $\pi_l < \pi < \pi_h$, then there is a B.S. steady state with positive stores, $(X_S, Q_S)$. The possibility of the sunspot causes agents to hold positive stores in anticipation, so the more likely the sunspot’s occurrence, the higher the stores held in anticipation of it (see Appendix B.2, proposition B.6).

If $\pi > \pi_h$, there exists no B.S. steady state at $X_S$. This is because if the sunspot probability is high enough, extinction becomes self-fulfilling even before the sunspot happens, and there is no steady state equilibrium before the sunspot apart from extinction.

### 3.4.3 B.S. Equilibrium Dynamics

We now summarize the main features of the B.S. equilibrium dynamics. Details of this are in Appendix B.2, Propositions B.6 - B.11. Figure 3.4.3 illustrates a possible phase diagram in $(x, Q)$ space.

Steady states for population $X_U$ are always totally unstable, so we ignore them.

If $\pi < \pi_l$, then the dynamics are basically the same as for the perfect foresight case. Along the equilibrium path, there will be some level of positive stores $s^*_+(x)$ when population is above a critical value $X^*_S$. In this case, population decreases
Figure 3.9: The Equilibrium Stable Manifold Path
towards \( X_S \) over time, and stores are falling and run out at \( X_S^0 \), the stochastic analogue of \( X_S^* \), where the storage condition equation (3.16) is just binding. The system continues to the B.S. steady state \( x = X_S, s = 0 \) in the No Storage regime exactly as in Section 3.3. As we show in Appendix B.2, proposition B.8, \( X_S^\pi \) is decreasing with \( \pi \). Obviously \( X_S^0 = X_S^* \), and \( X_S^{\pi_l} = X_S \).

If \( \pi_l < \pi < \pi_h \), the B.S. equilibrium path is the saddle path of the fixed point \( (X_S, Q_S) \). This saddle path rises with increasing \( \pi \). This means, as is not surprising, that if the probability of a sunspot is higher, then higher stores will be held for all population levels, not just at the steady state. We illustrate the dynamics in Figure 3.4.3.

### 3.4.4 Comparative Statics and Unanticipated Changes in the Transition Hazard

Now we consider unanticipated changes in \( \pi \), the transition hazard. If \( \pi \) increases\(^{10}\), the B.S. equilibrium path moves up, and there will be a cull to get to the new equilibrium path from the old one, so long as the increase in \( \pi \) is not too big. If the increase in \( \pi \) is too big the required cull will be so large that the new equilibrium path cannot be reached, and the system will switch immediately to the extinction equilibrium. Note that a large increase in \( \pi \) need not necessarily lead to extinction if it happens gradually, in a series of small, unanticipated steps. If the increase in \( \pi \) is slow enough, then the equilibrium is sustainable up to the point at which the equilibrium ceases to exist altogether (i.e. \( \pi_h \)).

Thus, if a policy maker knew that there had to be a shift in expectations towards a higher probability of extinction, and somehow had control over the timing of that shift, it would be best to make the shift gradual, rather than rapid. This hints that if policy makers have access to continuously changing information about the state of the population, it might be best for them to release this information on a regular basis.

\(^{10}\)If \( \pi \) decreases, of course, the price would have to fall to get back to the stable manifold, and poaching would stop for a while in a non-deterministic version of a no poaching equilibrium.
Figure 3.10: B.S. Equilibrium Paths as \( \pi \) Increases
basis, rather than simply trying to cover up bad news about the availability of the resource and hope that the situation repairs itself before people find out. This can only be conjectured, however, because in the model, there is no uncertainty about the population, only about what other agents are thinking.

We conjecture that if there was a chance of switching to a no poaching equilibrium at any point on the A.S. trajectory, the rate of growth of prices in the extinction equilibria would have to be higher. Similarly, the possibility of switching back to an extinction equilibrium from a no poaching equilibrium would mean that the rate of growth of prices would have to be lower in the no poaching equilibrium. Note that the possibility of switching to a no poaching equilibrium makes it harder to have a storage equilibrium, just as increasing storage costs would.

3.5 Conclusions and Policy Implications

This paper has argued that there may be multiple rational expectation paths of population and prices for open-access resources used in the production of storable goods. Expectation of future poaching will increase future prices, and this will increase current prices, thus rationalizing the initial increase in poaching. Note that this argument does not apply to non-storable goods, such as fish, because the price of fish depends only on current supply and demand, and not expectations of prices. It also does not apply to privately held goods, such as oil, since anticipation of higher prices will lead people to postpone extracting the resource.

It is becoming cost-effective for people to assert property rights to elephants in a few areas of Africa (Simmons & Kreuteo 1989). Most elephants, however, continue to live in open-access areas, and only a fraction of the elephant population can profitably be protected as private property\textsuperscript{11}. If elephants can only be supported as private property above a certain price, then there may be one equilibrium in which\textsuperscript{11}

\textsuperscript{11}It is expensive to protect elephants as private property, since they naturally range over huge territories and ordinary fences cannot contain them (Bonner 1993).
they are a plentiful open-access resource at a low price, and another equilibrium in which they are a scarce private resource at a high price.

The analysis carries several policy implications. First, it indicates that in order to assure the survival of a species, it may be necessary to preserve a large enough herd not only to allow the species to survive at current equilibrium poaching levels, but also to prevent an equilibrium with a higher level of poaching. If \( Q_{\text{max}} > X_S \), then the population may appear safe, but may in fact be vulnerable to a switch in equilibrium. One way to rule out the extinction equilibrium is to increase the habitat for the animal, so that the steady-state population becomes greater than \( Q_{\text{max}} \).

It may be possible for governments and international organizations to avoid the extinction equilibrium if they can commit to drastic measures to prevent extinction. This could keep prices down and reduce the incentive to poach. If a government or international organization could credibly announce that it would spend a large amount on elephant protection if the herd fell below a certain critical size, it might never actually have to spend the money, whereas if the same government spent a moderate amount on elephant protection each year, the herd might become extinct. The model thus suggests a rationale for conservation laws that extend little protection to a species until it is declared endangered, and then provide extensive protection without regard to cost\(^\text{12}\).

While conservationists and governments may wish to coordinate on low-poaching equilibria, people who hold stores will prefer to coordinate on a high-poaching equilibrium, in which the species becomes extinct. In fact, although game officials in Zimbabwe removed the horns of some rhinos in order to protect them from poaching, poachers killed the rhinos anyway. The New York Times (1994), quotes a wildlife official as explaining their behavior by saying “If Zimbabwe is to lose its entire rhino

\(^{12}\)On the other hand, the model suggests that if the government plans to impose such strong anti-poaching enforcement that the long-run harvest will decline, it should announce these regulations before they are imposed, because this may lead to a rush to poach.
population, such news would increase the values of stockpiles internationally.\footnote{It is also possible that the poachers killed the rhinos to obtain the stumps of their horns, or to make rhino poaching easier in the future.}

Note that if there were a "George Soros" of elephants who had sufficient resources, or were not subject to credit constraints, he or she could use his or her market power to coordinate on the extinction equilibrium simply by offering to buy enough of the good at a high enough price. A speculator who already owned some of the good would make substantial profits by inducing coordination on the extinction equilibrium, so this equilibrium may be more likely in the absence of government intervention, assuming the parameters are such that both extinction and survival equilibria exist.

The model also suggests that it may be possible to eliminate the extinction equilibrium by accumulating a sufficient stockpile of the storable good, and threatening to release it onto the market if the animal goes extinct or becomes sufficiently endangered\footnote{Note that this policy is more likely to be time consistent than policies which promise to spend arbitrary amounts of resources to preserve an animal. If the animal is already going extinct, there is no reason not to sell the stockpile.}. As illustrated in Figure 3.3.3, if $Q_{\text{max}}$ is finite, but greater than the high steady state, $X_{S}$, then an extinction equilibrium will exist in steady state if the government does not stockpile stores. If the government or a conservation organization holds stores greater than the boundary of the region where the extinction equilibria cease to exist, and credibly promises to release them onto the market if the population falls below a threshold, the extinction equilibrium will be eliminated. The organization holding stores would have to pay the interest costs on the stores, and this would entail a financial loss, but the price might be worth paying if the organization valued conservation, and the stores eliminated the extinction equilibrium.

If $Q_{\text{max}}$ is infinite, stores cannot eliminate the extinction equilibrium, but they can extend the range of the survival equilibrium. For example, suppose that the initial stock is $X_{S}$, and the initial stores are zero, but that there is an exogenous
shock to population, for example due to disease. If there are no stores, then the species will be driven to extinction if the population dips below \( X^*_U \). However, if there are sufficient stores, there will be a no poaching equilibrium in which demand is satisfied by stores and the population can recover.

Bergstrom (1990) has suggested that confiscated contraband should be sold onto the market. This analysis suggests that confiscated supplies of goods such as rhino horn should be held and released onto the market only if it appears that the market is coordinating on the extinction equilibrium. For example, a rule might be adopted that confiscated rhino horn would be sold only if the rhino population dipped below a certain level, or the price rose above a certain level.

Stores could be built up not only by confiscating contraband, but also by harvesting. Sick animals could be harvested, and animals could be harvested during periods when population is temporarily above its steady state level, due, for example, to a run of good weather.

Building up stores will reduce the population, but only temporarily. Once the target stockpile has been accumulated, harvesting to build up the stockpiles can be discontinued, and the live population will return to the same level as in the absence of stockpiling. The presence of the stockpiles, however, will permanently eliminate or reduce the chance of a switch to the extinction equilibrium. If \( Q_{max} \) were less than \( X_S \) it is particularly important to build up stockpiles gradually, so as to prevent the population from falling below \( Q_{max} \), and thus creating an opportunity for coordination on the extinction equilibrium.

Many conservationists oppose selling confiscated ivory on the market, for fear that it would legitimize the ivory trade. Building stores achieves the same goal of depressing prices, but without the disadvantage of legitimizing the ivory trade. Stores could potentially be held until scientists develop cheap and reliable ways of marking or identifying “legitimately” sold animal products so they can be distinguished from illegitimate products.
While stockpiles may help promote conservation of animals which are killed for goods which are storable but not durable, such as rhino horn, this analysis does not strictly apply to durable goods, i.e. those which are not destroyed by those who gain utility from them. Elephant ivory is often considered an example of such a durable good. The government has no reason to wait before selling confiscated durable goods, since in any case, private agents will store any durable goods sold on the market. In practice, however, there are presumably few completely durable goods. Even ivory is not perfectly durable, since it depreciates, and uncarved ivory is not perfectly substitutable for carved ivory, due to changing styles and demand for personalized ivory seals.

In the perfect foresight model of Section 3.3, no private stores were held by speculators in the high steady-state. However, if the price is stochastic, either due to sunspot coordination, as in Section 3.4, or to exogenous shocks, such as weather or disease, then speculators may hold stores, and government stores may crowd these out. In the example considered in Section 3.4, government stores would crowd out private stores one for one, until the government accumulated greater stores than would be held by private agents. Any further accumulation by the government would reduce the range of equilibria in which agents could anticipate extinction.

Finally, it is worth noting that this model reinforces the case for international coordination of conservation policy by suggesting that, if one country reduces the price of ivory by protecting its elephants, this reduces the incentive to poach in other countries. In conventional models of non-storable resources, increased anti-poaching efforts in one country will initially drive up the price of the good, encouraging extra poaching in other countries. Under this model, increased anti-poaching efforts in one country may reduce poaching in other countries, both in the short run and in the long run.
Chapter 4

On The Effect of Changing Activity On HIV Prevalence

4.1 Introduction

Increases in the frequency of partner change by low-activity people may reduce long-run HIV prevalence, and even lead to elimination of the disease. To see the intuition, suppose that nine partners per year were required for HIV to be endemic in a homogeneous population. Consider a population in which a small minority had ten partners a year, and the majority had no partners at all. In this population, the disease would persist among the active minority. If the inactive majority decided to have one partner each over their lifetimes, and if the groups mixed homogeneously, then on average the high-activity people might have five partnerships a year with the low-activity people and five a year with other high-activity people. In this case, the disease would not be sustained, because half the new infections would occur among low-activity people who would not infect others.

Calculations using survey data on sexual activity and simple epidemiological models suggest that the example above is more than a theoretical curiosity. We develop expressions for the cut-off level of sexual activity below which increases in
activity will reduce steady-state prevalence. More than 80 percent of the population in a comprehensive study of sexual activity in Britain had low enough activity that reductions in activity would increase steady-state prevalence in a standard susceptible-infected (SI) epidemiological model with preferred mixing. Simulations using this simple model suggest that if everybody who had one partner every five years reduced their frequency of partner change by five percent, steady-state prevalence would increase by seven percent under homogeneous mixing.

There are several important caveats. First, reductions in activity by low-activity people are only likely to increase steady-state prevalence in populations that initially have low steady-state prevalence. The counterintuitive effects discussed in this paper are thus likely to be relevant for heterosexuals in developed countries, but not for homosexuals, heterosexuals in the highest-prevalence areas of Africa, or IV drug users. Second, increases in activity by low-activity people will increase prevalence temporarily. Third, the conclusions in this paper may be weakened if reductions in activity by low-activity people make it harder for high-activity people to find partners, and they consequently reduce their own activity. In general, since the model abstracts from important features of the epidemic (for example, our model does not allow concurrent partnerships or different sexes), the results should be considered provisional.

To the extent that the results of this paper prove robust in more complex models, though, they reinforce arguments for targeting public health messages urging reductions in the frequency of partner change to high-activity people. Under plausible behavioral assumptions, they also suggest that messages aimed at low-activity people should stress condom use, rather than abstinence.

Whereas traditional public health analysis focuses on examining the effect of public health interventions on prevalence, economic analysis focuses on externalities from sexual behavior. Economists use the term "externality" to describe a situation in which individuals do not bear the full costs, or receive the full benefits of their
actions, and typically argue that governments should usually not intervene in situations in which there are no externalities, but should discourage actions, such as polluting, which create negative externalities. The idea is that individuals should be free to decide which risks they wish to assume, but that society has a right to intervene to the extent that these actions effect others. Given that people will have different philosophical views on the issue, we report both the cut-off rates of partner change below which people will create negative externalities by reducing their sexual activity, and the cut-off rate of partner change below which people will increase total prevalence by reducing their sexual activity.

A large literature examines the dynamics of sexually transmitted diseases under a variety of mixing patterns, and considers the effect of changes in activity (Anderson & May 1991, Anderson, May, Boilly, Garnett & Rowley 1991, Gupta, Anderson & May 1989, Hethcote 1993, forthcoming, Hethcote & Yorke 1984, Kaplan 1990, Kaplan, Cramton & Paltiel 1990, May & Anderson 1987, Over & Piot 1992). Both this paper and Kremer (1996b) follow independent work by Whitaker & Rentin (1992). Their under-appreciated paper shows that in a two group example with homogenous mixing, an increase in activity by the low-activity group may reduce steady-state prevalence. Whitaker & Rentin (1992), however, explicitly disclaim empirical relevance of their model for AIDS. Our analysis differs from Whitaker & Rentin (1992) by analytically approximating the cutoff level of activity below which these counterintuitive effects arise with an arbitrary distribution of activity, and using empirical data to show that these effects are likely to be important in low-prevalence populations, but not in high-prevalence populations. Moreover, unlike Whitaker & Rentin (1992), this analysis demonstrates that equal absolute increases in activity by all members of the population can reduce prevalence. Finally, we show that the results are robust under preferred mixing.

Kremer (1996a) analyzes the externalities from sexual activity using the economic concept of asymmetric information. This paper presents the results in purely
epidemiological terms. It also further develops the theory and mathematical methods of Kremer (1996b) using a different approach to determine the cut-off levels of activity, and extending the theory to models with preferred mixing. Kremer (1996a) considers how the argument in this paper is modified if high-activity people change their activity in response to changes in activity by low-activity people.

The paper is organized as follows: In Section 4.2, we define the model we use, and cite some results concerning its stability, and the endemic steady state. In Section 4.3, we solve for the cut-off levels of activity below which increasing activity reduces prevalence among other members of the population, or prevalence in the population as a whole. In Section 4.4, we examine the special case of homogenous mixing. In Section 4.5, we calculate the cut-off values of activity using data on the frequency of partner change among heterosexuals in the UK. In Section 4.6, we discuss policy implications, and possible future research.

4.2 The SI Model With Preferred Matching

We use a simplified version of the preferred mixing SI model as presented by Jacquez, Simon, Koopman, Sattenspiel & Perry (1988). In this model, people are born and die according to a Poisson process with parameter \( \delta \), independent of whether or not they are infected. There are \( N \) groups of people, classified by the number of sexual partners they have per year, \( i_k \), where \( k \in \{0, \ldots, N - 1\} \), and each group represents a proportion \( \alpha_k \) of the total population. The mean sexual activity per year is \( \mu = \sum_{k=0}^{N-1} \alpha_k i_k \), and the variance of sexual activity is \( \sigma^2 = \sum_{k=0}^{N-1} \alpha_k (i_k - \mu)^2 \).

The prevalence of HIV in each group is \( y_k \). The overall, or average, prevalence, \( \bar{y} \), and the activity weighted prevalence, or pool-risk, \( \lambda \), are defined by:

\[
\bar{y} = \sum_{k=0}^{N-1} \alpha_k y_k, \quad \lambda = \sum_k w_k y_k, \quad \text{where} \quad w_k = \frac{\alpha_k i_k}{\mu}.
\]  

(4.1)

The pool-risk, \( \lambda \), is the probability that a partner picked at random will be
infected; one is more likely to pick a partner from a higher activity group, since they are in the pool more often.

People match randomly with probability $1 - \gamma$, and match within their own group with probability $\gamma$. When $\gamma = 0$, the model reduces to homogeneous mixing. If $\gamma = 1$, there is restricted mixing (people select partners only from their own activity group).

Those born are uninfected, and enter activity groups pro rata to existing group size. This ensures that the relative sizes of the groups, $\{\alpha_k\}$, are constant, which simplifies the analysis. If an infected person matches with an uninfected person, the probability of transmission is $\beta$.

The dynamics of HIV infection are, thus described by:

$$\dot{y}_k = -\delta y_k + \gamma \beta i_k y_k (1 - y_k) + (1 - \gamma) \beta i_k \lambda (1 - y_k). \quad (4.2)$$

We normalize time to measure it in dimensionless units by using a new variable $T = \delta t$. We also define “normalized activities” for each group, $r_k = \frac{\beta}{\delta} i_k$. The normalized activities have mean $\mu_r = \frac{\beta}{\delta} \mu$, and variance $\frac{\beta^2}{\delta^2} \sigma^2$. The weights, $w_k$, used to define $\lambda$ are unchanged, as is $\lambda$. After these substitutions, and where dots now refer to differentiation with respect to dimensionless time,

$$\dot{y}_k = -y_k + \gamma r_k y_k (1 - y_k) + (1 - \gamma) r_k \lambda (1 - y_k). \quad (4.3)$$

The first term of equation (4.3) is the effect of infected people dying and being replaced at the same rate by uninfected people (because we have normalized time, this rate is unity). The second term is the rate of infection from matches within the group, with probability $\gamma$. The third term is the rate of infection from random matches, with probability $1 - \gamma$.

The dynamics and stability of such models are relatively well understood. There is either a stable endemic state, or the disease dies out, depending on the system parameters. Formally, Jacquez et al. (1988) show in Appendix A that
1. There is a compact, convex invariant set, \( C \), in the state space. The state space is the set of group prevalences, \( \{ (y_0, \ldots, y_{N-1}) : 0 \leq y_k \leq 1 \ \forall k \} \), with dynamics given by equation (4.3).

2. \( 0 \), the no-disease state is always a steady state.

3. The threshold function \( G \) is defined as:

\[
G(\gamma, \{ r_k \}, \{ \alpha_k \}) = (1 - \gamma) \left( \mu_r + \frac{\sigma_r^2}{\mu_r} \right) + \gamma \max_k \{ r_k \},
\]

where \( \mu_r \) and \( \sigma_r^2 \) are the mean and variance of the normalized activity levels \( \{ r_k \} \).

If \( G < 1 \), then \( 0 \) is a unique globally asymptotically stable steady state. If \( G > 1 \), then \( 0 \) is unstable, and the endemic steady state, \( Y = (Y_0, \ldots, Y_{N-1}) \), exists and is locally asymptotically stable, where

\[
Y_k = \frac{\left( (1 + r_k(\Lambda(1 - \gamma) + \gamma))^2 - 4\gamma r_k \right)^{1/2} - r_k(\Lambda(1 - \gamma) - \gamma) - 1}{2\gamma r_k}
\]

if \( 0 < \gamma < 1 \), and

\[
Y_k = \frac{\beta_i k \Lambda}{1 + \beta i_k \Lambda}, \quad \text{if } \gamma = 0.
\]

We use capital letters to refer to all quantities at the endemic steady state. Thus \( Y = (Y_0, \ldots, Y_{N-1}) \) are the group prevalences, \( \bar{Y} \) is the average prevalence, and \( \Lambda \) is the pool-risk in steady state. Note that prevalence in any group in endemic steady state is only directly a function of the group's own normalized activity, and the pool-risk, \( \Lambda \), given \( \gamma \).

In the rest of the paper we will examine how changes in the normalized activity levels, \( \{ r_k \} \), and, thus, the real activity levels \( \{ i_k \} \) affect the endemic steady state. It is possible, though, that changes in activity levels could wipe out the disease entirely in steady state. We consider this possibility numerically in more detail in Section 4.5.
4.3 The Effects of Changing Activity on Prevalence

In this section, we consider the effects on steady state pool-risk, $\Lambda$, and prevalence, $\bar{Y}$, of changing the activity level of one of the groups. To this end, we shall always assume that $0 \leq \gamma < 1$, so there is at least some cross-group mixing. We also assume that $G > 1$, so that the endemic steady state exists and is stable.

If people from groups with group prevalence less than $\Lambda$ increase their activity then, in the short-run, $\Lambda$ will fall as the chance of meeting someone from a low activity group, who is less likely to be infected, increases. Under certain conditions, this effect persists in the steady state, as well, and $\Lambda$ and $\bar{Y}$ may fall. In this section, we examine such effects in the model of Section 2. There will be two quantities of interest.

First, we define $j_e$ to be the real number of partners below which an increase in activity causes a reduction in steady state pool-risk, $\Lambda$. The equivalent normalized activity level is $s_e = \beta j_e / \delta$. This definition is equivalent to saying that, for $r_0 < s_e$, $d\Lambda / dr_0 < 0$. As we show below, $j_e$ is always positive, so that groups with low enough activity could always reduce steady state pool-risk by increasing activity a bit. The subscript "e" stands for externality; increases in activity by such groups benefit the rest of the population, by reducing the risk they face from a randomly drawn partner, even though they may increase prevalence among the group increasing its activity.

Second, we define $j_t$ to be the cut-off level of real activity below which an increase in activity by a small group leads to a reduction in the long-term prevalence, $\bar{Y}$. The equivalent normalized quantity is $s_t = \beta j_t / \delta$. This definition is equivalent to saying that, for $r_0 < s_t$, $d\bar{Y} / dr_0 < 0$. We derive necessary and sufficient conditions that $j_e$ should be positive, and we show that $j_t \leq j_e$, if it is positive, since if $\bar{Y}$ is lower, $\Lambda$ must be too.

---

1If $\gamma = 1$, a person who increases activity will increase their own chance of infection, and will not affect anyone else's chance of infection outside his or her own group.
First, we prove a technical Lemma, giving expressions for some derivatives of expressions which will prove useful for the results of the rest of the section.

**Lemma 4.1** Considering \( Y_k \) as a function of \( r_k \) and \( \Lambda \), the following are true for \( \Lambda, Y_k > 0 \) and for all \( k \):

\[
\frac{\partial Y_k}{\partial \Lambda} = \frac{(1 - \gamma)(1 - Y_k)Y_k}{(1 - \gamma)\Lambda + \gamma Y_k^2} < \frac{Y_k}{\Lambda}, \quad (4.7)
\]

\[
0 < \sum_k w_k \left( 1 - \frac{\partial Y_k}{\partial \Lambda} \right) \quad (4.8)
\]

\[
r_k \frac{\partial Y_k}{\partial r_k} = \frac{(1 - Y_k)(\Lambda + \gamma(Y_k - \Lambda))Y_k}{(1 - \gamma)\Lambda + \gamma Y_k^2}, \quad \text{and} \quad (4.9)
\]

\[
\frac{\partial Y_k}{\partial r_k} = \frac{(1 - Y_k)^2(\Lambda + \gamma(Y_k - \Lambda))^2}{(1 - \gamma)\Lambda + \gamma Y_k^2}. \quad (4.10)
\]

**Proof:** At the steady state, \( Y_k = r_k(1-Y_k)(\gamma Y_k + (1-\gamma)\Lambda) \). For equation (4.7), differentiate this identity with respect to \( \Lambda \):

\[
\frac{\partial Y_k}{\partial \Lambda} = -\frac{Y_k}{1-Y_k} \frac{\partial Y_k}{\partial \Lambda} + \left[ \gamma \frac{\partial Y_k}{\partial \Lambda} + (1 - \gamma) \right] \frac{Y_k}{\gamma Y_k + (1 - \gamma)\Lambda}.
\]

Rearranging yields the equality in (4.7). For the inequality, note that \( 1 - Y_k < 1 \), and \( (1 - \gamma)\Lambda + \gamma Y_k^2 > (1 - \gamma)\Lambda \), and the result follows.

For (4.8), the inequality from (4.7) gives \( \sum w_k \frac{\partial Y_k}{\partial \Lambda} < \frac{1}{\Lambda} \sum w_k Y_k = 1 \), by definition of \( \Lambda \).

For (4.9), differentiate \( Y_k = r_k(1-Y_k)(\gamma Y_k + (1-\gamma)\Lambda) \) with respect to \( r_k \), multiply by \( r_k \), and rearrange.

For (4.10), note that \( r_k = Y_k/[(1 - Y_k)(\gamma Y_k + (1 - \gamma)\Lambda)] \).

\( \diamond \)

Without loss of generality, we shall consider the effects of changing the normalized activity of the zero group. The cut-off level \( s_e \) below which increases in normalized activity reduce steady state pool risk, \( \Lambda \), must be such that \( d\Lambda/dr_\theta = 0 \) when the zero group has normalized activity \( s_e \).
Proposition 4.2

\[
\frac{d\Lambda}{dr_0} = \frac{\alpha_0}{\mu_r} \frac{Y_0 - \Lambda + r_0 \frac{\partial Y_0}{\partial r_0}}{\sum w_k \left( 1 - \frac{\partial Y_k}{\partial \Lambda} \right)} \tag{4.11}
\]

\[
= -\frac{\alpha_0}{\mu_r} \frac{(Y_0^2(\Lambda - \gamma) - 2(1 - \gamma)\Lambda Y_0 + (1 - \gamma)\Lambda^2)}{(1 - \gamma)\Lambda + \gamma Y_0} \sum w_k \left( 1 - \frac{\partial Y_k}{\partial \Lambda} \right). \tag{4.12}
\]

Proof: Combining equation (4.4) and the definition of \( \lambda \), which applies just as well to \( \Lambda \), steady-state pool-risk, we find:

\[
\Lambda = \sum_{k=0}^{N-1} w_k Y_k(\gamma, r_k, \Lambda) = \Psi(\gamma, \{r_k\}, \Lambda).
\]

Differentiating this with respect to \( r_0 \),

\[
\frac{d\Lambda}{dr_0} = \frac{d\Psi}{dr_0} = \left( \frac{\partial}{\partial r_0} + \frac{d\Lambda}{dr_0} \frac{\partial}{\partial \Lambda} \right) \Psi.
\]

From the definition of \( w_k \) (equation (4.1)),

\[
\frac{\partial w_k}{\partial r_0} = \frac{\partial}{\partial r_0} \left( \frac{r_k \alpha_k}{\mu_r} \right) = \frac{\alpha_k}{\mu_r} \delta_{0,k} - \frac{\alpha_0}{\mu_r} w_k,
\]

where \( \delta_{0,k} \) is 1 when \( k = 0 \), and zero otherwise. Then:

\[
\frac{d\Lambda}{dr_0} = \frac{\alpha_0}{\mu_r} \left( Y_0 + r_0 \frac{\partial Y_0}{\partial r_0} - \Lambda \right) + \frac{d\Lambda}{dr_0} \left( \sum_{k=0}^{N-1} w_k \frac{\partial Y_k}{\partial \Lambda} \right).
\]

Rearranging yields equation (4.11). Substituting for \( r_0 \partial Y_0/\partial r_0 \) from Lemma 4.1, equation (4.9) gives (4.12).

◊

The intuition behind equation (4.11) is as follows: the first two terms in the numerator are the net short-term effect of the change in \( r_0 \) on the pool-risk, \( \Lambda \): if the zero group has lower prevalence than \( \Lambda \), increasing normalized activity will cause \( \Lambda \) to fall. The third term is the long run effect on \( \Lambda \) of the increase in the zero group's prevalence from its own increased normalized activity. The denominator shows how the effect is magnified by the change in prevalence caused in the rest of
the population by the change in zero group behavior. \( s_e \) is the normalized activity level for which \( \frac{dA}{dr_0} = 0 \), when \( r_0 = s_e \).

Consider (4.12). The denominator is always positive, by Lemma 4.1, so the sign is determined by the numerator, which is a quadratic in \( Y_0 \). We may prove the following:

**Proposition 4.3** \( \frac{dA}{dr_0} \) is strictly negative for \( Y_0 \in [0,Y_e) \), zero at \( Y_0 = Y_e \), and strictly positive for \( Y_0 \in (Y_e,1] \), where \( 0 < Y_e < \Lambda \), and

\[
Y_e = \frac{\Lambda(1 - \gamma)}{\Lambda - \gamma} \left( 1 - \sqrt{\frac{1 - \Lambda}{1 - \gamma}} \right). \tag{4.13}
\]

**Proof:** Consider the numerator of equation (4.12):

\[-Y_0^2(\Lambda - \gamma) + 2(1 - \gamma)\Lambda Y_0 - (1 - \gamma)\Lambda^2.

Its value at \( Y_0 = 0 \) is \( -(1 - \gamma)\Lambda^2 \), which is strictly negative. Its value at \( Y_0 = \Lambda \) is \( (1 - \gamma)\Lambda^2 \), which is strictly positive. There must, therefore, be at least one root between 0 and \( \Lambda \). In fact, there is always exactly one root in \([0, \Lambda]\).

If \( \Lambda = \gamma \), the numerator is linear, and has only one root at \( Y_0 = \Lambda/2 \).

If \( \Lambda \neq \gamma \), then the numerator has exactly two roots, \( Y^\pm \) given by:

\[Y^\pm = \frac{\Lambda(1 - \gamma)}{\Lambda - \gamma} \left( 1 \pm \sqrt{\frac{1 - \Lambda}{1 - \gamma}} \right) .\]

\( Y^+ \) is always greater than one, and so cannot lie in \([0, \Lambda]\). There is, thus, only one root in \([0, \Lambda]\), which is \( Y^- = Y_e \)

\[\diamondsuit\]

Now, in steady state,

\[r_k = \frac{Y_k}{(1 - Y_k)(\gamma Y_k + (1 - \gamma)\Lambda)},\]

which is increasing in \( Y_k \) for fixed \( \Lambda \). This means that the root of \( dA/dr_3 \), \( Y_e \) corresponds to a normalized activity level, \( s_e \), so that:

\[s_e = \frac{Y_e}{(1 - Y_e)(\gamma Y_e + (1 - \gamma)\Lambda)}.

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This normalized activity level \( s_e \) is related to the real activity level \( j_e \) as \( s_e = \beta j_e / \delta \). Substituting the value of \( Y_e \) from Proposition 4.3 gives the following:

**Theorem 4.4** If a group has an activity of less than \( j_e \) partners per year, then a small increase in activity by that group will reduce steady-state pool-risk, \( \Lambda \), where

\[
    j_e = \frac{\delta}{\beta(\gamma + (1 - \gamma)\Lambda)^2} \left( \gamma - (1 - \gamma)\Lambda + \sqrt{\frac{1 - \gamma}{1 - \Lambda}}(\Lambda - (1 - \Lambda)\gamma) \right),
\]

and:

\[
    0 < j_e < \frac{\delta}{\beta(1 - \Lambda)}.
\]

**Proof:** As discussed above. The bounds on \( j_e \) come from \( 0 < Y_e < \Lambda \).

\( \Diamond \)

We now turn our attention to how average prevalence, \( \bar{Y} \), in the population as a whole is affected by changes in the activity of one group. It will be helpful, first, to define:

\[
    H = \frac{1}{\mu_r} \frac{\sum k \alpha_k \frac{\partial Y_k}{\partial \Lambda}}{\left( 1 - \sum w_k \frac{\partial Y_k}{\partial \Lambda} \right)}.
\]

Note that, by inequality (4.8) in Lemma 4.1, \( H > 0 \).

**Proposition 4.5**

\[
    \frac{d\bar{Y}}{dr_0} = \alpha_0 \left( \frac{\partial Y_0}{\partial r_0} + \frac{1}{\mu_r} \left( \frac{\sum k \alpha_k \frac{\partial Y_k}{\partial \Lambda} (Y_0 + r_0 \frac{\partial Y_0}{\partial r_0} - \Lambda)}{\sum k w_k \left( 1 - \frac{\partial Y_k}{\partial \Lambda} \right)} \right) \right)
    \]

\[
    = \alpha_0 \left( 1 - Y_0 \right)^2 \left( \Lambda(1 - \gamma) + \gamma Y_0 \right)^2 - H (Y_0^2 (\Lambda - \gamma) - 2(1 - \gamma)\Lambda Y_0 + (1 - \gamma)\Lambda^2) \right) \]

\[
    (1 - \gamma)\Lambda + \gamma Y_0^2.
\]

**Proof:**

\[
    \frac{d}{dr_0} \bar{Y} = \sum_k \alpha_k \frac{dY_k}{dr_0} = \sum_k \alpha_k \left( \frac{\partial Y_k}{\partial \Lambda} \frac{d\Lambda}{dr_0} + \frac{\partial Y_k}{\partial r_0} \right), \text{ and substitute for } \frac{d\Lambda}{dr_0} \text{ from (4.12)}.
\]

This gives the first equation of the Proposition.

The second comes from substituting for the partial derivatives of \( Y_0 \) from Lemma 4.1 and rearranging.

\( \Diamond \)

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When we calculated \( \frac{dA}{dr_0} \), it turned out that \( \alpha_0 \) only explicitly appeared as a factor, or in the denominator. This meant that the cut-off for externalities, \( Y_e \), was not affected by the size of the group whose activity was being changed. This is not the case with \( \frac{dY}{dr_0} \), so the cut-off will be a complicated function not just of activity and pool-risk, but of group size as well. We are actually more interested in the effects a change in activity by an individual has on population prevalence. This is equivalent to the case where \( \alpha_0 \) is very small. Because \( \alpha_0 \) is very small, and \( Y_k \) for \( k > 0 \) is not a function of \( r_0 \), \( H \) will not vary much as \( r_0 \), and hence \( Y_0 \), changes. Thus we may assume that \( H \) is approximately constant as \( r_0 \) and, hence, \( Y_0 \), is varied.

**Theorem 4.6** If, and only if, \( H > 1 - \gamma \), then there exists \( Y_t \in (0, Y_e) \) such that a small group with group prevalence below \( Y_t \) will decrease steady-state average prevalence, \( \bar{Y} \) by increasing group activity.

**Proof:** Consider the numerator of (4.17):

\[
(1 - Y_0)^2(\Lambda(1 - \gamma) + \gamma Y_0)^2 - H \left( Y_0^2(\Lambda - \gamma) - 2(1 - \gamma)\Lambda Y_0 + (1 - \gamma)\Lambda^2 \right). \tag{4.18}
\]

\( Y_t \) exists if and only if this expression is negative for \( Y_0 < Y_t \), and zero at \( Y_t \). It is continuous in \( Y_0 \). Its value at zero is \( \Lambda^2(1 - \gamma)^2 - H\Lambda^2(1 - \gamma) \). This is strictly negative if and only if \( H > 1 - \gamma \).

Now, consider its value at \( Y_e \). \( Y_e \) is a root of the coefficient of \( H \), so the value of the whole expression at \( Y_e \) is \( (1 - Y_e)^2(\Lambda(1 - \gamma) + \gamma Y_0)^2 \), which is always strictly positive.

If \( H > 1 - \gamma \), by the intermediate value property, there must, exist \( Y_t \) so that the numerator is zero at \( Y_t \), and strictly negative for \( 0 \leq Y_0 < Y_e \). In this range, \( d\bar{Y}/dr_0 \) is negative, and the theorem follows.

\( \diamond \)

The activity level \( j_t \) corresponding to \( Y_t \) is the threshold below which increases in activity reduce steady state prevalence. It seems clear that once \( Y_0 \) is above \( Y_t \),

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increases in activity would lead to increases in overall steady state prevalence, but we have not been able to prove that the expression (4.18) has no more roots between \( Y_t \) and \( Y_e \), so it is conceivable that there could be an interval above \( Y_0 = Y_t \) where \( \frac{dY}{dr_0} \) is negative. We have not been able to find any cases where this happens, and we would not expect to. The stronger statement, that the sign of \( \frac{dY}{dr_0} \) is increasing in \( Y_0 \) on \([0, \Lambda]\) is true in the case of homogeneous mixing (see Section 4). This is sufficient to prove that \( \frac{dY}{dr_0} \) has a maximum of one root in \([0, \Lambda]\). That this is true for \( \gamma = 0 \) implies, by continuity, that it is true for \( \gamma \) small but positive.

Quartic equations are soluble in closed form. We could, therefore, obtain an exact closed-form solution for \( Y_t \), where it exists, in terms of \( H, \Lambda, \) and \( \gamma \). Since \( s_t \), and hence \( j_t \) is related to \( Y_t \) by a quadratic in \( Y_e \), we could then find \( j_t \) explicitly. In general, though, \( \frac{dY}{dr_0} \) has no rational roots, and so any such expression would be too complicated to be useful. In the case where \( \gamma = 0 \), we are able to factorize expression (4.18) relatively easily.

### 4.4 Homogeneous Matching: Some Specialized Results

We now specialize to the case of homogenous mixing, in which \( \gamma = 0 \). In this case, we obtain stronger results about the existence and uniqueness of \( j_t \), the activity level below which long run average prevalence is reduced by increases in activity. We also obtain a simple closed form expression for \( j_t \) in terms of the activity weighted prevalence, \( \Lambda \), and the system parameters.

It is well known that if \( \gamma = 0 \), the endemic steady state will exist and be locally asymptotically stable if and only if \( \mu_r + \sigma_r^2/\mu_r > 1 \). Substituting \( \gamma = 0 \) into the expressions in Section 3 implies that

\[
Y_e = 1 - \sqrt{1 - \Lambda}, \quad \text{and} \quad j_e = \frac{\delta}{\beta \Lambda} \left( \frac{1}{\sqrt{1 - \Lambda}} - 1 \right). \tag{4.19}
\]

The expression for \( \frac{dY}{dr_0} \) factorizes when \( \gamma = 0 \), so that we can solve for \( Y_t \) and

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hence \( j_t \) in closed form. We can also prove that \( Y_t \) is the unique point at which 
\[
\frac{d\tilde{Y}}{dr_0} = 0,
\]
if any such point exists:

**Proposition 4.7** If there is homogeneous matching, so that \( \gamma = 0 \), and if \( 0 < \Lambda < 1 \), then:

\[
\frac{d\tilde{Y}}{dr_0} = -\alpha_0 \left\{ (Y_0 - 1)^2(H - \Lambda) - H(1 - \Lambda) \right\}. \tag{4.21}
\]

If \( H \geq 1 \), then \( \exists Y_t \) such that \( 0 < Y_t < Y_e < \Lambda < 1 \), and such that \( \frac{d\tilde{Y}}{dr_0} \) is strictly negative on \([0,Y_t]\), zero at \( Y_t \), and strictly positive on \((Y_t,1]\), where

\[
Y_t = 1 - \sqrt{\frac{H(1 - \Lambda)}{H - \Lambda}}. \tag{4.22}
\]

If \( H < 1 \), then \( \frac{d\tilde{Y}}{dr_0} \) is strictly positive on \([0,1]\).

**Proof:** Substitute \( \gamma = 0 \) in the expression of Proposition 4.5 and simplify for equation (4.21).

\((Y_0 - 1)^2\) is decreasing in \( Y_0 \) for \( 0 < Y_0 < 1 \) and \( H > \Lambda \), so that (4.21) is increasing in \( Y_0 \). At \( Y_0 = 0 \), it has value \(-\alpha_0\Lambda(H - 1)\), which is negative if and only if \( H > 1 \).

When \( \gamma = 0 \), \( Y_e = 1 - \sqrt{1 - \Lambda} \) so that, at \( Y_0 = Y_e \), \( \frac{d\tilde{Y}}{dr_0} \) has value \( \alpha_0\Lambda(1 - \Lambda) \), which is strictly positive, since \( 0 < \Lambda < 1 \).

\( \frac{d\tilde{Y}}{dr_0} \) has, therefore, exactly one root between 0 and \( Y_e \). This is \( Y_t \). Solving the quadratic yields (4.22). Note that, if \( H > 1 \), it must also be true that \( H > \Lambda \).

If \( H < 1 \), \( \frac{d\tilde{Y}}{dr_0} \) is still monotone on \( Y_0 \in [0,1] \). As above, the value at \( Y_0 = 0 \) is positive. At \( Y_0 = 1 \), the value is \( \alpha_0H(1 - \Lambda) \), which is also positive. \( \frac{d\tilde{Y}}{dr_0} \) must be strictly positive on \([0,1]\).

\( \Diamond \)

If \( H > 1 \), then we may use the relation that 
\( r_k = Y_k / \left[ (1 - Y_k)\Lambda \right] \) in steady state to find the value of \( r_0, s_t \), which corresponds to \( Y_t \). This gives a level of real activity, \( j_t \), such that increases in activity by individuals with activity less than \( j_t \) reduce steady state endemic prevalence, \( \tilde{Y} \), where:

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Theorem 4.8 If and only if $H > 1$, there exists a positive real activity level, $j_t \in (0, j_e)$, below which increases in activity lead to a fall in steady state average prevalence, $\bar{Y}$.

$$j_t = \frac{\delta}{\beta \Lambda} \left( \frac{\sqrt{1 - \Lambda/H}}{\sqrt{1 - \Lambda}} - 1 \right).$$

(4.23)

Proof: From Proposition 4.7. $r_0 = Y_0 / [(1 - Y_0) \Lambda]$. Substitute $Y_t$ for $Y_0$, and the result follows.

In the case where $\gamma = 0$, we may find an expression for $H$ in terms of quantities which are, at least in principle, observable. First:

Lemma 4.9 If we define $\sigma_Y^2 = \sum \alpha_k Y_k^2 - \bar{Y}^2$ to be the variance in prevalence amongst the groups, then the endemic steady state group prevalences satisfy:

$$\bar{Y} = \mu \beta \Lambda (1 - \Lambda),$$

(4.24)

$$\mu_r \Lambda^2 = \sigma_Y^2 + \bar{Y}^2 + \Lambda \mu_r \sum_k w_k Y_k^2.$$  

(4.25)

Proof: Since $Y_k = \frac{\Lambda r_k}{1 + \Lambda r_k}$, $Y_k + r_k Y_k \Lambda = r_k \Lambda$. Multiplying by $\alpha_k$ and summing gives equation 4.24. Multiplying by $\alpha_k Y_k$ and summing gives equation 4.25

◊

Lemma 4.10

$$H = \frac{\Lambda (\bar{Y} - \bar{Y}^2 - \sigma_Y^2)}{\mu_r \Lambda^2 - \sigma_Y^2 - \bar{Y}^2}.$$  

(4.26)

$H > 1$, and so $j_t > 0$, if and only if:

$$\sigma_Y^2 > \bar{Y} \left[ \frac{\Lambda^2}{(1 - \Lambda)^2} - \bar{Y} \right].$$  

(4.27)

Proof: When $\gamma = 0$, $\partial Y_k / \partial \Lambda = (1 - Y_k) Y_k / \Lambda$, from Lemma 4.1. Then $\sum \alpha_k \partial Y_k / \partial \Lambda = (\bar{Y} + \bar{Y}^2 - \sigma_Y^2)$. $\sum w_k (1 - \partial Y_k / \partial \Lambda) = \sum w_k Y_k^2 / \Lambda$. Substitute these into the definition for $H$, and:

$$H = \frac{\Lambda (\bar{Y} - \bar{Y}^2 - \sigma_Y^2)}{\Lambda \mu_r \sum w_k Y_k^2}.$$  

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Using the second equation of Lemma 4.9, the expression for \( H \) results.

The bound on \( \sigma_Y^2 \) comes from using the first equation of Lemma 4.9 to replace \( \mu_r \Lambda \) with \( \bar{Y}/(1-\Lambda) \).

\( \Diamond \)

Substituting this expression for \( H \) in the formula for \( j_t \) of equation (4.23):

**Theorem 4.11** If and only if:

\[
\sigma_Y^2 > \bar{Y} \left[ \frac{\Lambda^2}{(1-\Lambda)^2} - \bar{Y} \right],
\]

(4.28)

then \( j_t \) is positive, and is given by:

\[
j_t = \frac{\delta \mu (1-\Lambda)}{\beta \bar{Y}} \left\{ \sqrt{\frac{\bar{Y}(1-2\Lambda)}{(1-\Lambda)^2 (\bar{Y}(1-\bar{Y}) - \sigma_Y^2)}} - 1 \right\}.
\]

(4.29)

\( \Diamond \)

Note that because this theorem holds for \( \gamma = 0 \), \( j_t \) must also exist and be positive for \( \gamma \) in some neighborhood of \( \gamma = 0 \) by continuity, as mentioned in Section 4.3.

In the homogeneous population case, this reduces to the formula of Kremer (1996b):

\[
j_{thom} = i \frac{\sqrt{(1-Y)(1-2Y)} - (1-Y)^2}{Y(1-Y)}.
\]

(4.30)

We are interested in whether the counter-intuitive result that increases in activity may reduce prevalence is likely to hold in reality. Equation (4.28) means, roughly, that, the lower the prevalence and the larger the variance in prevalence (and hence activity), the more likely such effects are. On the other hand, even if prevalence is very low, if the variance is very high, so that the pool risk is large enough, \( j_t \) will not be positive. More formally,

**Proposition 4.12**

1. If \( \bar{Y} > 1/2 \), then increases in activity always increase steady state prevalence.
2. One may construct a population with arbitrarily low steady state prevalence in which increases in activity by people of any activity level increase overall prevalence.

Proof:

1. Since $Y_k$ is in $[0, 1]$, the variance by any measure is less than $1/4$. $\bar{Y}$ is always less than $\Lambda$, thus

$$H < 1 \iff \frac{1}{4} \bar{Y} \left( \frac{\bar{Y}^2}{(1 - \bar{Y})^2} - \bar{Y} \right) < 1.$$  \hspace{1cm} (4.31)

The inequality holds if $\bar{Y} > 1/2$.

2. Consider a general population. If we take a very small group and increase their activity, we can make that group's prevalence arbitrarily close to 1. As we increase activity, $\Lambda$ grows, and we may make this as close to 1 as we like. Thus we can make the RHS of equation (4.27) as big as we want and, since the LHS is always less than $1/4$, we can always find a population where $H < 1$, no matter how low the prevalence.

\[ \diamond \]

4.5 Simulations Calibrated to Data From U.K. Heterosexuals

In this section, we apply the model to data on rates of partner change taken from The National Survey of Sexual Attitudes and Lifestyles (NSSAL), a comprehensive survey of sexual behavior in Britain encompassing 18,000 people. It found that the mean number of heterosexual partners in the last five years was 1.98$^2$ and the variance was 19.03 (Johnson, Wadsworth, Wellings & Field 1993). Table 4.1 shows

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$^2$We confine attention to the exclusively heterosexual population, because the sample size was too small to make reliable inferences about homosexuals.
the distribution of numbers or heterosexual partners in the last five years. In the simulations below, we assume that the annual rate of partner change is one-fifth the number of heterosexual partners over the last five years.

Table 4.1: Numbers of Partners per Year over the Last Five Years

<table>
<thead>
<tr>
<th>ptrs/yr</th>
<th>%age</th>
<th>ptrs/yr</th>
<th>%age</th>
<th>ptrs/yr</th>
<th>%age</th>
<th>ptrs/yr</th>
<th>%age</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.95</td>
<td>2.2</td>
<td>0.218</td>
<td>4.4</td>
<td>0.033</td>
<td>8</td>
<td>0.061</td>
</tr>
<tr>
<td>0.2</td>
<td>62.56</td>
<td>2.4</td>
<td>0.261</td>
<td>4.6</td>
<td>0.010</td>
<td>9</td>
<td>0.010</td>
</tr>
<tr>
<td>0.4</td>
<td>10.56</td>
<td>2.6</td>
<td>0.069</td>
<td>4.8</td>
<td>0.023</td>
<td>10</td>
<td>0.037</td>
</tr>
<tr>
<td>0.6</td>
<td>6.39</td>
<td>2.8</td>
<td>0.033</td>
<td>5</td>
<td>0.061</td>
<td>12</td>
<td>0.026</td>
</tr>
<tr>
<td>0.8</td>
<td>3.44</td>
<td>3</td>
<td>0.347</td>
<td>5.2</td>
<td>0.008</td>
<td>13</td>
<td>0.003</td>
</tr>
<tr>
<td>1</td>
<td>2.05</td>
<td>3.2</td>
<td>0.026</td>
<td>5.4</td>
<td>0.008</td>
<td>14</td>
<td>0.018</td>
</tr>
<tr>
<td>1.2</td>
<td>1.64</td>
<td>3.4</td>
<td>0.063</td>
<td>6</td>
<td>0.149</td>
<td>15</td>
<td>0.015</td>
</tr>
<tr>
<td>1.4</td>
<td>0.90</td>
<td>3.6</td>
<td>0.031</td>
<td>6.4</td>
<td>0.005</td>
<td>16</td>
<td>0.003</td>
</tr>
<tr>
<td>1.6</td>
<td>0.70</td>
<td>3.8</td>
<td>0.027</td>
<td>7</td>
<td>0.027</td>
<td>18</td>
<td>0.003</td>
</tr>
<tr>
<td>1.8</td>
<td>0.24</td>
<td>4</td>
<td>0.263</td>
<td>7.6</td>
<td>0.005</td>
<td>20</td>
<td>0.006</td>
</tr>
<tr>
<td>2</td>
<td>0.83</td>
<td>4.2</td>
<td>0.003</td>
<td>7.8</td>
<td>0.005</td>
<td>100</td>
<td>0.002</td>
</tr>
</tbody>
</table>

In this section, we use data from the heterosexual population, because the sample was too small to make reliable inferences about the homosexual population. Our model, though, is a single-sex model. The results in this section cannot, therefore, strictly apply to a heterosexual population. This section is designed to illustrate the point that the cut-off values $j_e$ and $j_f$ may be above the activity levels of a significant proportion of the population. Since the properties in the single-sex model giving rise to the effects we discuss in this paper would be qualitatively similar to those of a two sex model—the source of the externalities is unchanged—we believe that the qualitative results of this section should at least be indicative of what one would be

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3The data in the NSSAL study were weighted before analysis to correct for differing responses in different geographical regions, and for differential probability of selection of individuals living in households of different sizes. Details of this may be found on pages 54–55 of Johnson et al. (1993).

4There is no unproblematic way of moving from the theoretical concept of the rate of partner change to empirical observations of the number of partners per period, since people may have several partners simultaneously, and may reestablish old partnerships (Pan-American Health Organization 1992). We assume that those people who have one partner over a five year period have an average of 0.2 partners per year. If the people listed as having one partner every five years actually changed partners less frequently, as seems plausible, the variance of sexual activity would be even larger. This would exacerbate some of the counterintuitive effects discussed in this paper.
likely to find using a two-sex model. A full theoretical and simulated analysis of a two-sex model is, though, beyond the scope of this paper.

For the simulations, the data from Table 4.1 were aggregated into 18 activity groups\(^6\) in order to simplify the calculations. Taking the NSSAL partner change rates as given, we then calculate \(\{N_k\}\), \(\bar{Y}\), and \(\Lambda\) as a function of \(\beta/\delta\) and \(\gamma\) by numerically solving the equations for the steady states of equation (4.2). In this section, we use real time, not normalized time. Inverting this relation allows us to express \(Y_k\), \(\beta/\delta\), and \(\Lambda\) as functions of \(\bar{Y}\) and \(\gamma\), the overall prevalence, and the degree of preferential partner mixing. We then calculate \(j_e\) and \(j_t\) using the methods of Section 3 and relate them to the NSSAL distribution to get an idea of what percentage of the population is likely to increase prevalence or pool-risk by reducing activity under various assumptions about long-run endemic prevalence and matching patterns.

The results are summarized in Table 4.2 and Figures 4.1 and 4.2. The first entry in each cell of Table 4.2 shows \(j_e\), the number of partners below which reductions in the number of partners will cause an increase in steady state pool-risk, \(\Lambda\), for particular values of \(\bar{Y}\), steady-state prevalence, and \(\gamma\), the degree of assortativeness in mixing. The second entry in each cell of Table 4.2 shows \(j_t\), the cutoff number of partners below which reductions in the number of partners will increase steady-state prevalence in the population as a whole. The figures in parentheses show the proportion of the population with less than the cutoff frequency of partner change in the NSSAL sample. Thus, for example, if the distribution of rates of partner change were as given in the NSSAL sample, \(\gamma\) was 0.5, and the transmission rate were such that steady-state prevalence was 1/2 of 1 percent, then the 97 percent of the population with less than 1.8 partners per year would increase prevalence among others, generating negative externalities, by reducing their rate of partner change. The 88 percent of the population with less than 0.67 partners per year would

\(^6\)Groups as for Table 4.1 up to 1.6 partners per year, then groups: 1.8–2.2 pts. per year, 2.4–3.8, 4–5.4, 6–6.4, 7–7.8, 8–9, 10–13, 14–15, 16–20, and above 20.
Table 4.2: $j_e$ and $j_t$ by Steady State Prevalence and Assortativeness in Mixing

($j_e$) in first entry, $j_t$ in second entry, percentage of population below cut-off value in parentheses)

<table>
<thead>
<tr>
<th>$\gamma = 0$</th>
<th>$\gamma = 0.25$</th>
<th>$\gamma = 0.5$</th>
<th>$\gamma = 0.75$</th>
<th>$\gamma = 0.95$</th>
<th>$\lim_{\gamma \to 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = .005$</td>
<td>0.91 (92%)</td>
<td>1.43 (96%)</td>
<td>1.80 (97%)</td>
<td>2.08 (98%)</td>
<td>2.28 (98%)</td>
</tr>
<tr>
<td></td>
<td>0.79 (88%)</td>
<td>0.78 (88%)</td>
<td>0.68 (88%)</td>
<td>0.59 (82%)</td>
<td>0.67 (88%)</td>
</tr>
<tr>
<td>$Y = .01$</td>
<td>0.83 (92%)</td>
<td>1.16 (94%)</td>
<td>1.38 (96%)</td>
<td>1.54 (96%)</td>
<td>1.61 (97%)</td>
</tr>
<tr>
<td></td>
<td>0.68 (88%)</td>
<td>0.61 (88%)</td>
<td>0.51 (82%)</td>
<td>0.43 (82%)</td>
<td>0.45 (82%)</td>
</tr>
<tr>
<td>$Y = .02$</td>
<td>0.74 (88%)</td>
<td>0.93 (92%)</td>
<td>1.05 (94%)</td>
<td>1.11 (94%)</td>
<td>1.12 (94%)</td>
</tr>
<tr>
<td></td>
<td>0.54 (82%)</td>
<td>0.44 (82%)</td>
<td>0.34 (72%)</td>
<td>0.28 (72%)</td>
<td>0.37 (72%)</td>
</tr>
<tr>
<td>$Y = .05$</td>
<td>0.59 (82%)</td>
<td>0.66 (88%)</td>
<td>0.70 (88%)</td>
<td>0.71 (88%)</td>
<td>0.67 (88%)</td>
</tr>
<tr>
<td></td>
<td>0.33 (72%)</td>
<td>0.22 (72%)</td>
<td>0.15 (9%)</td>
<td>0.11 (9%)</td>
<td>0.13 (9%)</td>
</tr>
<tr>
<td>$Y = .1$</td>
<td>0.47 (82%)</td>
<td>0.50 (82%)</td>
<td>0.50 (82%)</td>
<td>0.48 (82%)</td>
<td>0.43 (82%)</td>
</tr>
<tr>
<td></td>
<td>0.13 (9%)</td>
<td>0.05 (9%)</td>
<td>0.007 (9%)</td>
<td>0.001 (9%)</td>
<td>0.04 (9%)</td>
</tr>
<tr>
<td>$Y = .2$</td>
<td>0.35 (72%)</td>
<td>0.35 (72%)</td>
<td>0.34 (72%)</td>
<td>0.31 (72%)</td>
<td>0.26 (72%)</td>
</tr>
<tr>
<td></td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>$Y = .3$</td>
<td>0.28 (72%)</td>
<td>0.28 (72%)</td>
<td>0.26 (72%)</td>
<td>0.24 (72%)</td>
<td>0.20 (9%)</td>
</tr>
<tr>
<td></td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>$Y = .5$</td>
<td>0.20 (9%)</td>
<td>0.19 (9%)</td>
<td>0.17 (9%)</td>
<td>0.16 (9%)</td>
<td>0.14 (9%)</td>
</tr>
<tr>
<td></td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
</tbody>
</table>

increase steady-state prevalence by reducing their rate of partner change. Thus, even assuming a steady state prevalence among UK heterosexuals of 1% (which is likely to be a high estimate), more then eighty percent of the population have low enough sexual activity that they would reduce the overall prevalence in the long run by increasing activity. More than ninety percent would produce negative externalities for the rest of the population by decreasing their activity.

Somewhat surprisingly, preferred mixing does not necessarily mitigate the counterintuitive effects discussed in this paper, and in some cases can even exacerbate them. In low-prevalence populations, the proportion of the population that generates negative externalities by reducing sexual activity actually increases with $\gamma$, the degree of preferred mixing (Figure 4.2). The proportion of the population that increases total steady-state prevalence by reducing sexual activity does not change monotonically with $\gamma$, the proportion of sexual activity which is within groups (Figure 4.2). For all values of $\gamma$ in the table, though, $j_t$ is greater than the number of partners of
Figure 4.1: $j_t$ as a Function of $\gamma$
Figure 4.2: $j_e$ as a Function of $\gamma$

Degree of Preferred Mixing, $\gamma$

Level of activity below which increases in activity decrease endemic pool-risk

$Y=0.5\%$

$Y=1\%$

$Y=5\%$

$Y=10\%$

$Y=20\%$
80% of the population, assuming steady-state prevalence is less than 1%\(^6\).

To understand the intuition for why \(j_e\) and \(j_t\) may increase with \(\gamma\), note that as \(\gamma\) rises, prevalence falls among low-activity people. Thus when low-activity people match in the general pool they cause greater reductions in weighted average prevalence. Moreover, as \(\gamma\) rises, prevalence rises less steeply in the number of partners for those with few partners. Thus low-activity people will increase their own probability of infection by a smaller amount if they increase their activity. These effects may cause increases in \(\gamma\) to increase \(j_e\) and \(j_t\). On the other hand, as \(\gamma\) rises, people are less and less likely to match with the general pool, and this effect will cause \(j_t\) to fall with \(\gamma\).

Making the assumption that rates of partner change are similar, but transmission risk is higher, in other societies and situations, we may extrapolate to higher prevalence populations. In sub-Saharan Africa as a whole, prevalence among adults is approximately 2.5 percent and in Thailand it is 2 percent (World Health Organization 1994), so a substantial portion of the population may generate negative externalities and increase steady-state prevalence by reducing activity in these populations. However, in the highest risk areas of Africa, where prevalence among adults is as high as one third, low-activity people probably will not increase total steady-state prevalence by having fewer partners. The situation is probably the same amongst homosexuals in large urban areas of developed countries. Similarly, prevalence among IV-drug users is high enough that reductions in needle-sharing by infrequent users are unlikely to increase total prevalence\(^7\).

If low activity groups increase their activity, while the mean activity will increase, the variance of activity will decrease, at least for moderate increases in activity. A look at the threshold function, \(G\), in equation (4.4) suggests that, under some

\(^6\)In these simulations, \(j_t\) is higher for high or low \(\gamma\) than for moderate \(\gamma\). This is not true for a general distribution of rates of partner change, however. In simulations using data from (Anderson & May 1991), \(j_t\) was often higher for moderate than for extreme values of \(\gamma\).

\(^7\)A sample of IV drug users in Thailand showed that 43 percent were HIV positive (World Health Organization 1994). In Argentina, Brazil, and Uruguay, HIV prevalence among IV drug users is more than 50 percent in some communities (Pan-American Health Organization 1992).
circumstances if $\gamma$ is small enough, the reduction in variance caused by an increase in activity could actually be enough completely to eradicate the disease by causing $G$ to fall below 1 and the endemic steady state to become unstable. For the NSSAL distribution, $G$ is minimized if everyone with activity less than 0.76 partners per year raises their activity to that level. With this new distribution of activity, the most $\beta/\delta$ can be and have $G < 1$ is 0.65 if matching is random ($\gamma = 0$). This maximal value of $\beta/\delta$ falls very rapidly in $\gamma$ to 0.01 when $\gamma$ approaches 1. With random matching, $\beta/\delta$ must be about 0.56 to give 1/2% prevalence, and 0.62 to give 1% prevalence. This calibrated level of $\beta/\delta$ falls more slowly with assortative matching. This means that, while $\beta/\delta$ is low enough that the disease would be wiped out if matching were random, and people increased activity as discussed above, the result is extremely sensitive to $\gamma$. In fact, for the case where long run prevalence is 1/2%, $\gamma$ could be no more than 0.003 for an activity increase to wipe out the disease. This suggests that increased activity would never wipe out the disease, as any realistic model would have rather less than perfectly homogeneous matching.

4.5.1 Dynamics

This paper mostly focuses on comparing steady states. Even if increases in rates of partner change cause decreases in long run prevalence, they must, in the short run, increase prevalence. However, it is also useful to examine the time-path of prevalence in response to reductions in the frequency of partner change.

The transition period required before prevalence increases in response to a reduction in activity is fairly long under the simple SI model, but much shorter under more realistic models. Table 4.3 shows the dynamics of prevalence in response to reduced activity under the basic SI model, a more realistic model with AIDS-induced mortality, and a still more realistic model in which infectiousness is higher during

---

8The death rate from causes other than AIDS is assumed to be 0.03, and the death rate from AIDS is 0.1. The model requires a higher transmission rate to match any given steady-state prevalence. It also requires that a higher proportion of the population be born into highly active groups in order to match the observed proportion of high-activity groups in the population. It does
Table 4.3: Dynamics Under Different Models

In all three models prevalence is set at 0.005, and the group with 1 partner per 5 years reduces its activity to 0.8 partners per 5 years.

<table>
<thead>
<tr>
<th>Model</th>
<th>Basic SI</th>
<th>AIDS Mortality</th>
<th>Varying $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Years Until Incidence = Initial Incidence</td>
<td>5.4</td>
<td>1.4</td>
<td>0.14</td>
</tr>
<tr>
<td>% increase in long run prevalence</td>
<td>30.9</td>
<td>16.5</td>
<td>15.0</td>
</tr>
<tr>
<td>Years until prevalence is halfway to new steady state</td>
<td>185</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>% initial change in incidence</td>
<td>-0.81</td>
<td>-0.74</td>
<td>-0.74</td>
</tr>
</tbody>
</table>

Note: The simulation with varying $\beta$ requires three times as many state variables as groups. In order to keep the number of state variables manageable, the simulations in this table therefore aggregated everyone with more than 8 partners per year into a single group. The number of partners in this group was chosen not to be the average in the group, but to keep $\mu + \sigma^2/\mu$ in the simulation equal to $\mu + \sigma^2/\mu$ in the original data.

the first few months of infection before the immune system has responded\(^9\). The dynamics are much faster in the more realistic models because the people infected immediately after the change in activity are not likely to continue to infect others for long. The simulations examine the impact of a reduction in activity to 0.16 partners per year by all people with 0.2 partner per year. In all three simulations, initial prevalence is one half of one percent. The first row of Table 4.3 shows that incidence returns to its original level after 5.4 years in the basic model, after 1.4 years in a model that incorporates the effect of the disease on mortality, and after only two months if infectiousness depends on the stage of infection.\(^{10}\) steady-state prevalence increases in response to the reduction in activity under all three models, but the effect is greatest under the basic model. The third row shows the number of years not allow for the effect of AIDS-induced mortality on the birth rate, as would be appropriate in studies of aggregate population dynamics in some high prevalence African countries.

\(^9\)Jacquez, Simon, Koopman & Longini (1994) find that transmission rates are high in the first few months after infection, before the immune system has responded, and again in the final stage of the disease when AIDS has developed and the immune system has been overwhelmed. Transition probabilities between stages of the disease in the simulations reported here are taken from (Jacquez et al. 1994), and converted to Poisson hazard rates. The hazard rates for progression into the next stage are 0.970 and 0.119 in the first and second stage respectively. The death rate in the final stage of infection is 0.53. The transmission probability is set at 0.01 in the final stage, and 0.001 in the second stage. In the first stage, it is calibrated to match the desired prevalence, given $\gamma$.

\(^{10}\)Incidence is the flow of new infections. Prevalence, the stock of infected people as a fraction of the population, takes approximately twice as long to return to its original level.
Table 4.4: Months Until Incidence Rises to its Original Level in Response to a Reduction in Activity

Top entry in each cell assumes a reduction from 0.4 partners per year to 0.3 partners per year. Bottom entry in each cell assumes a reduction from 0.2 partners per year to 0.1 partners per year. Dashes indicate that \( j_t \) is less than the number of partners. Infectiousness is assumed to vary with the stage of infection as in (Jacquez et al. 1994).

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( \gamma = 0 )</th>
<th>( \gamma = 0.5 )</th>
<th>( \gamma = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>4.2</td>
<td>10.4</td>
<td>23.8</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>5.2</td>
<td>9.6</td>
</tr>
<tr>
<td>0.01</td>
<td>4.7</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>2.2</td>
<td>6.6</td>
<td>12.7</td>
</tr>
<tr>
<td>0.02</td>
<td>7.3</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3.2</td>
<td>9.1</td>
<td>18.7</td>
</tr>
<tr>
<td>0.05</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>6.0</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

required for prevalence to fall halfway from its initial level to its new steady-state level\(^{11}\). The last row shows the initial percentage change in incidence in response to the reduction in activity. Note that in all three models, the initial reduction in incidence is negligible compared to the steady-state increase.

Although the dynamics are affected most strongly by AIDS-induced mortality and by varying infectiousness with the stage of the disease, Table 4.4 shows the dynamics are slower the greater are \( \gamma \), the degree of assortativeness in mixing, \( Y \), the steady-state prevalence, and \( i \), the number of partners in the group changing its activity. Both \( j_e \) and \( j_t \) seem to be lower under models that allow for mortality effects of the disease and for infectiousness to vary with the stage of infection.

Note that everyone with activity less than \( j_e \), the cut-off for creating negative externalities in the long-run by reducing activity, will also create negative externalities in the short-run by reducing activity. In addition, all those with activity between \( Y_e \) and \( \Lambda \) will create negative externalities in the short-run, but not the

\(^{11}\)In the model in which infectiousness varies with the stage of infection, prevalence initially declines in response to a reduction in activity and then overshoots its steady-state value, before declining to a new steady-state value above its original level. The time until 50% of the steady-state change is attained therefore underemphasizes the costs of reductions in activity.
long-run, by reducing activity.

4.6 Policy Implications and Directions for Future Research

This paper has argued that, in low-prevalence populations, reductions in the frequency of partner change by low-activity people may create negative externalities and increase the long-run prevalence of AIDS. Given the limitations of the data, the simplifying assumptions of the model, and the mismatch between a homosexual model and heterosexual data, the results should be considered preliminary. However, to the extent that these results are confirmed in future research, they reinforce arguments that public health messages urging reduced activity should be targeted to high-activity people, and should emphasize condom use, rather than abstinence.\footnote{We are not suggesting that public health officials encourage people to have more partners, since anyone who followed such advice would face a higher probability of infection, and people have an expectation that public health officials will inform them about how to protect themselves from health risks.}

Public health messages can be targeted both through their content, and the choice of advertising media. For example, the "Get high, Get stupid, Get AIDS." campaign warning people about the links between substance abuse, unprotected sex, and AIDS may have targeted high-activity people more than the mass mailing of AIDS-prevention literature to all U.S. households in the early days of the epidemic.

Before adopting these conclusions, however, more research is needed. For example, further work is necessary to see if these results are robust when differences between the sexes, the age-structure of mixing, the process of partnership-formation and dissolution, more general mixing patterns, time-varying infectiousness, and mortality effects of the disease are explicitly modeled.

This paper has examined the consequences of changes in the frequency of partner change by low-activity people, holding constant the number of partners of others. In fact, since reductions in the number of partners by low-activity people increase
prevalence in the pool of available partners, they may lead to further reductions in activity. Kremer (1996a) explores a model in which people choose an activity level depending on prevalence in the pool of available partners, and shows that there may be multiple equilibria, as in Akerlof's model of the market for lemons (Akerlof 1970).

Although allowing the number of partners to change in response to prevalence alters the predicted spread of the epidemic, it does not change the direction of externalities from reductions in activity. Each person's welfare (as defined in Economics) depends on the pool-risk, because it is this probability of infection which determines the trade-off between having a desired number of partners and probability of infection. If low-activity people reduce their activity and thus increase the pool-risk, this reduces the welfare of high-activity people, even if it induces high-activity people to have fewer partners, and thus reduces their probability of infection.

Reductions in activity by low-activity people could also directly cause high-activity people to reduce their activity by making it harder for them to find additional partners. This paper implicitly assumed zero search costs, and used the example of a bar in which one could always find a partner. One could also imagine a "dating" model, in which people went on dates and decided whether or not to have sex, and there was a limit of one date per day. If the low-activity people decided to have sex on fewer dates, the high-activity people would automatically also have sex on fewer dates. To the extent that this "dating" model is correct, and reductions in activity by low-activity people cause high-activity people to reduce activity rather than to match with each other, increases in activity by low-activity people will be less likely to reduce steady-state prevalence. However, while the date model may be a good model for low-activity people, we believe that it is not likely to be a good model for the tail of the distribution with extremely high activity, which disproportionately influences the spread of the disease. This group is not likely to continue dating without sex, but instead to seek other sexual partners.
If high-activity people respond to a potential partner's abstinence by seeking a new partner, but respond to a potential partner's preference for condom use by agreeing to use a condom, then public health messages directed to low-activity people urging abstinence could actually increase prevalence, but messages urging condom use would reduce prevalence.
Bibliography


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New York Times (1994), 'Zimbabwe's rhino efforts faulted'.


Appendix A

Appendices for Chapter 2
A.1 Proofs of Theorems in Section 2.2

In this appendix, I solve the Consumer’s Problem using Hamiltonian methods, and prove sufficient conditions for the first-order solution to be a maximum of consumer utility.

A.1.1 Proof of Theorem 2.3

The present value Hamiltonian for this problem is:

\[ H = \frac{c^{1-b}}{1-b} - U + m(E(\Lambda)lk^a - c). \]

The first order conditions are then:

\[ \frac{\partial H}{\partial c} = c^{-b} - m = 0 \]
\[ \dot{m} = m(r - aE(\Lambda)lk^{a-1}) \]
\[ \dot{k} = E(\Lambda)lk^a - c. \]

For sufficient second-order conditions that this path be a maximum, I use Arrow’s Theorem (see, for example, Kamien & Schwartz (1991)). This says that if the choice variable first-order conditions solved to give \( c \) and \( l \) as functions of \( m \) and \( k \) maximise the Hamiltonian for all \( (m, k) \) and, if the maximised Hamiltonian is concave in \( k \) for all \( m \), then the first order conditions identify a maximum of the objective function.

First, check that the choice of \( c \) given by \( c = m^{-1/b} \) maximizes the Hamiltonian. \( \frac{\partial^2 H}{\partial c^2} = -bc^{-(1+b)} \), so that \( c \) maximizes \( H \) if and only if \( b > 0 \). In order for the consumer to be risk-averse, I have already restricted \( b \) to be greater than zero, so this is satisfied.

Second, check that a choice of \( l = l_1 \) maximizes \( H \) when aggregate labor is \( l_1 \), and similarly for \( l_0 \). If aggregate labor is high, \( E = E_1 \). If a consumer chooses labor \( l_1 \), her Hamiltonian is:

\[ H_1(l_1) = \frac{c^{1-b}}{1-b} - U_1 + m(E_1l_1k^a - c). \]
If she chooses labor \( l_0 \), her Hamiltonian is:

\[
H_1(l_0) = \frac{c^{1-b}}{1-b} - U_0 + m(E_1 l_0 k^a - c).
\]

For \( l_1 \) to be the optimal choice, \( H_1(l_1) \geq H_1(l_0) \), or:

\[
\Delta U \leq mE_1 \Delta l k^a.
\]

Since \( m = c^{-b} \), it is optimal for the consumer to pick \( l_1 \) if everyone else is so long as:

\[
\frac{\Delta U}{\Delta l} \leq c^{-b} k^a E_1.
\]

The equivalent condition that \( l_0 \) is optimal if everyone else has picked \( l_0 \) is that:

\[
\frac{\Delta U}{\Delta l} \leq c^{-b} k^a E_0.
\]

Third, the maximized Hamiltonian must be concave in \( k \) for all \( m \). Differentiating \( H \),

\[
\frac{\partial^2 H}{\partial k^2} = a(a - 1)mE(\Lambda)lk^{a-2}.
\]

This is negative if and only if \( 0 < a < 1 \), or if the share of capital in production is between 0% and 100%.

Eliminating \( m \) to get dynamic equations in terms of the choice variables \( c \) and \( l \), and the state variables \( k \) and \( L \).

\[\begin{align*}
\dot{c} &= \frac{c}{b}(AE(\Lambda)lk^{a-1} - r) \\
\dot{k} &= E(\Lambda)lk^a - c.
\end{align*}\]

\( \diamond \)

### A.1.2 Proof of Theorem 2.5

If there are to be two steady states, then they must be at \( S_i \), as specified in the theorem. \( \dot{c} = 0 \) where \( k = (aL_i/r)^{1/(1-a)} \). \( \dot{k} = 0 \) where \( c = L_i k^a \), or where \( c/k = L_i k^{a-1} = r/a \).
For the steady states to be allowed, it must be that \((K_i, C_i) \in \mathcal{R}_i\). This is true for the high labor phase if and only if:

\[
C_i^0 \Delta U \leq \Delta l E_1 K_1^a \\
(L_1 K_1)^b \Delta U \leq \Delta l E_1 K_1^a
\]

On substituting for \(K_1\), and noting that \(L_1 = l_1 E_1\), the lower half of the if and only if inequality of Theorem 2.5 results. The upper half comes from the same treatment of the condition that \((K_0, C_0) \in \mathcal{R}_0\)

\(\dot{c}\) is *decreasing* in \(k\), and is zero on the line \(k = K_i\). \(\dot{k}\) is *decreasing* in \(c\), and is zero on the upward sloping line where \(c = L_i k^a\). The resulting steady state is saddle-stable for each \(i\).

The necessary condition in the theorem comes from the fact that \(E_0 < E_1\) and \(l_0 < l_1\). The if and only if condition can only be fulfilled if one of the exponents \((b - a)/(1 - a)\) or \((b - 1)/(1 - a)\) is negative (or both of them). By assumption, \(0 < a < 1\), so either \(b < a\), or \(b < 1\).

\[\Diamond\]

### A.1.3 Proof of Theorem 2.7

Along the line where \(c = r K/a\), the gradient of a trajectory of the dynamics is \(\dot{c}/\dot{k}\).

\[
\dot{c} = \frac{r k}{a b} (a L k^{a-1} - r) \\
\dot{k} = L_i k^a - \frac{r K}{a}.
\]

Thus,

\[
\frac{\dot{c}}{\dot{k}} = \frac{r}{b}.
\]

If \(a = b\), this lies along the line so the line is the same as the saddle-paths in both cases (The line passes through \((K_i, C_i)\).)
If \( a < b \), then between \( K_0 \) and \( K_1 \), \( c \) and \( \hat{k} \) are both positive in Phase 1. This means that the trajectories cross the line \( c = rK/a \) from above. Once below this line, trajectories cannot cross back, so the saddle path must be above the line \( c = rK/a \). Similarly, the saddle path for the 0 Phase must lie below the line \( c = rK/a \).

If \( a > b \), the situation is reversed.

◊

A.2 Proofs of Theorems in Section 2.3

A.2.1 Proof of Theorem 2.9

Define the value function, \( J^i(k) \), for a particular phase, by

\[
J^i(k)_{c,l} = E_t \left[ \int_t^{\infty} u(c_t, l_t)e^{-r(s-t)}ds \mid \text{phase } i \text{ at time } t \right].
\]

It is a "present-value" value-function, in that it is discounted from time \( t \), not from time 0.

Suppose the system is in phase \( i \) to begin with, but in each time interval \( \delta t \), it has a probability \( p_i \delta t \) of switching to phase \( j \). Then:

\[
J^i = \max_{c,l} \left( \int_t^{t+\delta t} u e^{-r(s-t)}ds + E \left[ \int_{t+\delta t}^{\infty} u(c_t, l_t)e^{-r(s-t-\delta t)-r\delta t}ds \right] \right)
\]

\[
= \max_{c,l} \left( J^i + \delta t \left( u - (p_i + r)J^i + \hat{k}_i J^i + J^i J^j \right) + o(\delta t) \right)
\]

Since \( \hat{k} \) is known, I substitute, and get a control variable constraint, and a Bellman equation:

\[
u_c = J^i_k
\]

\[(p_i + r) J^i - p_i J^j = u + J^i_k (Elk^a - c).\]

The system must be in the same region \( R_i \) as for the perfect foresight system in order that consumers should want to choose their individual labor to be the same.
as aggregate labor. To see why this is so, note that the expression on the right hand side of the Bellman equation is the same as that in the Hamiltonian. The relative benefits of changing $l$ are, therefore, the same in both cases.

Differentiate the Bellman equation with respect to $t$:

$$
\dot{k} \left[ (p_i + r) J_k^i - p_1 J_k^j \right] = u_c \dot{c} + u_{cc} \dot{c}k + u_c Elak^{a-1} \dot{k} - u_c \dot{c}
$$

$$
\Rightarrow
$$

$$(p_i + r) u_c^{(i)} - p_1 u_c^{(j)} = u_{cc}^{(i)} \dot{c}^{(i)} + u_c^{(i)} Elak^{a-1}.
$$

$c^{(j)}$ is the consumption path in phase $j$. By assumption, we know this already: it is just $\Theta_j(k)$. We may, thus, write the dynamics for phase $i$ in terms of the dynamics for phase $j$ as stated in the Theorem.

$\diamond$

### A.2.2 Proof of Theorem 2.11

If $p$ is small, then the phase diagram of the dynamics in $i$ expecting a change to $\Phi$ will be close to that of the perfect foresight system, so that $S_i$ and $B_i(\Phi)$ will exist, as required.

Consider the phase diagram in more detail. The line where $\dot{k}$ is unchanged. The line where $\dot{c} = 0$ is as shown in Figure 2.4. For Phase 1, the steady state moves to the right, and the saddle path is below the perfect foresight saddle path. For Phase 2, the steady state moves to the left, and the new saddle path is above the perfect foresight path. Increasing $\Phi$ in phase 1 moves the saddle path up, and the steady state to the left, and does the same in phase 0.

The problem with finding a bound on the derivative of $B_i(\Phi)$ is that, near the locally steady state, $\dot{k}$ is arbitrarily close to zero, which could cause the slope of the $(k, c)$ trajectory to diverge. Consider a small number, $\epsilon$. Near the locally steady state, the gradient of the saddle path will be bounded, and the bound will depend on $p$. Outside the $\epsilon$-neighbourhood of the LSS, the saddle-path will be bounded.
away from the line where \( \dot{k} = 0 \), and so its gradient will be bounded. Thus \( \mathcal{B}(\Phi) \) is bounded globally, and independently of \( \Phi \).

\( \Diamond \)

### A.2.3 Proof of Theorem 2.12

Consider the following sequence of functions:

\[
F_i^0 = \Gamma_0; \\
F_i^N = \mathcal{B}_0(F_i^{N-1}) \\
F_i^N = \mathcal{B}_1(F_i^{N-1}), \quad \forall N \geq 1.
\]

Then both \( F_i^N \) are, by Theorem 2.11, increasing sequences of functions with bounded derivatives, bounded above by \( \Gamma_1 \). They must both, therefore, converge (since the set of bounded, continuous functions with bounded derivative on a compact interval is a compact set), to a function with bounded derivative in the image of \( \mathcal{B}_i \). Define \( \Theta_i = \lim_{N \to \infty} F_i \), and everything works as claimed.

\( \Diamond \)

### A.3 Continuous Labor Supply: Some Results

In this appendix, I state some results for what happens when there is a continuous choice of labor, and Cobb-Douglas production. The results are broadly the same as with the 2-point labor supply model I develop in the text, but the dependence of labor on capital and equilibrium consumption complicates the equations significantly, though adds little.

With instantaneous utility and production:

\[
\begin{align*}
    u(c, l) &= \frac{c^{1-b} - l^{1-d}}{1-b} \\
    y &= E(\Lambda)l^{1-a}k^a,
\end{align*}
\]

The dynamics turn out to be:

\[
l^{1-d} = c^{-b}(1-a)E(\Lambda)l^{1-a}k^a
\]
\[
\dot{c} = \frac{c}{b} (a E(\Lambda) l^{1-a} k^{a-1} - r) \\
\dot{k} = E(\Lambda) l^{1-a} k^{a} - c.
\]

The second two of these are the same as for the 2-point labor model, apart from extra exponents and factors of \(1 - a\). The first equation is a control variable constraint, which takes the place of the regions \(R_i\). The control constraint in principle determines \(l\) in terms of equilibrium consumption, and capital.

It is possible that the externality, \(E\), is such that \(l\) is not uniquely determined by \(c\) and \(k\). In that case, the solution for \(l\) will have 2 branches, a high labor branch, \(l_1(c, k)\), defined over a region \((k, c) \in R_1\), and a low labor branch \(l_0(k, c)\) defined for \((k, c) \in R_0\). If we define \(L_i = E(l_i) l_i^{1-a}\), then the equations become:

\[
\dot{c} = \frac{c}{b} (a L_i(k, c) k^{a-1} - r) \\
\dot{k} = L_i(k, c) k^{a} - c.
\]

These are exactly the same as the 2-point labor model, except that \(L_i\) is now a function of \(k\) and \(c\), and not a constant.

This equation system has all the same structural properties as we had before: multiple steady states, \(S_i\), and upward sloping saddle-paths \(\Gamma_i\). Even the saddle-path ranking Theorem 2.6 holds, but if \(a = b\), while the saddle-paths are coincident, they are not necessarily the line \(c = r k / a\).

With uncertainty, things are, again, almost unchanged. The control variable constraint is unchanged, and so the equations are identical, except \(L_i\) is now a function of \(k\) and \(c\). The sustainable response functions are defined in the same way, and have most of the same properties. I have not proved the existence of equilibria in this case, as the proof that the image sets of the \(B_i\) are compact is harder. I have no reason, however, to suspect that it is any different in an important way.
Appendix B

Appendices for Chapter 3
B.1 Proofs for Section 3.3

Proposition B.1 The path of population, \( x \), stores, \( s \), and price, \( p \), is continuous on an equilibrium path.

Proof: Together, the storage and poaching conditions imply that the equilibrium price path must be continuous in time. A jump up in price would violate the storage condition, and a jump down in price would imply an instantaneous infinite growth rate of the population, which is impossible.

While there is poaching, \( p = c(x) \), which is continuous and monotonic, so population, \( x \), must be continuous. In the no poaching regime, population develops as equation 3.9, so is continuous. Population cannot jump suddenly across regime changes, either, as that would require a jump in price so there is an instantaneous harvest. Population is thus continuous. Stores are differentiable within regimes, and so are continuous. For there to be a jump in stores across regimes, there would have to be an instantaneous harvest, which would require a jump in price, which is impossible. Hence stores are continuous.

\diamond

Proposition B.2 1. The maximum initial value of population plus stores the system may have and still get to the storage equilibrium path \( s_e(x) \) is \( Q_{\text{max}} \), where

\[ Q_{\text{max}} = \max \{ X_{\text{max}}, s_e(x_U) + X_U \} \]  \hspace{1cm} (B.1)

2. \( Q_{\text{max}} \) is decreasing in storage cost, \( r \).

Proof:

1. \( Q_{\text{max}} \) must either be \( X_{\text{max}} \), or the point lying on the \( s = 0 \) axis and the tangent to \( s_e(x) \) of gradient -1. These tangencies occur at \( X_U \) or \( X_S \) (\( F(x) = 0 \) in equation 3.11 gives \( ds/dx = -1 \)). \( s_e(x) \) is concave at \( X_S \), so \( Q_{\text{max}} \) cannot be associated with \( X_S \).
2. If $Q_{max} = X_{max}$, then $s_e(Q_{max}) = 0$. Differentiating with respect to $r$,

$$s'_e(Q_{max}) \frac{\partial Q_{max}}{\partial r} + \frac{\partial s_e}{\partial r}(Q_{max}) = 0. \quad (B.2)$$

$s'_e(x)$ is just equation 3.11, and $\frac{\partial s}{\partial r}$ is:

$$\frac{\partial s_e}{\partial r}(x) = \frac{\partial}{\partial r} \left\{ \int_{X_S^*}^{x} \frac{c'(u)}{rc(u)} \left( F(u) - \frac{rc(u)}{c'(u)} \right) du \right\} \quad (B.3)$$

$$= -\frac{1}{r} \left( s_e(x) + x - X_S^* \right) \frac{\partial X_S^*}{\partial r} \frac{c'(u)}{rc(u)} \left( F(u) - \frac{rc(u)}{c'(u)} \right) \bigg|_{u = X_S^*}$$

$$= -\frac{1}{r} \left( s_e(x) + x - X_S^* \right) \leq 0.$$

since $x > X_S^*$, and $s_e(x)$ must be non-negative. The second term is zero, because at $X_S^*$, the storage condition is satisfied with equality, and that is precisely what is in the parentheses. Because $s_e(x)$ must be strictly decreasing at $X_{max}$,

$$\frac{\partial Q_{max}}{\partial r} = -\frac{\partial s_e}{\partial r}(X_{max}) \frac{1}{s'_e(X_{max})} \leq 0 \quad (B.4)$$

If $Q_{max} = s(X_U) + X_U$, then, because $X_U$ is independent of $r$, the result follows in the same way, but then the equivalent of equation B.3 holds because the second term vanishes because $X_U$ is independent of $r$.

\diamond

Proposition B.3 If initial population and stores are $(x_0, s_0)$ then if, and only if

$$(x_0, s_0) \in \bigcup_{t=0}^{\infty} P\Phi^{-1}(A_i) = E_i. \quad (B.5)$$

where $P$ is the projection operator $P(x, s, p) = (x, s)$, $i = +$ or $e$, and $\phi$ is the time evolution operator mapping $(x(0), s(0), p(0))$ to $(x(t), s(t), p(t))$, there is a starting price $p_0$ and poaching resumption time $t_p$ so that $\phi_i(x_0, s_0, p_0)$ is a no poaching equilibrium leading to the point $(x, s_i(x), c(x))$ at time $t_p$ for some $x$. These equilibria are not, in general, unique. There may be equilibria leading to $A_e$ and $A_+$. There may also be cases where the equilibrium passes through $PA_e$ or $PA_+$ on its way to

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another point. If there are multiple equilibria from the same point \((x_0, s_0)\), then the one with the lower starting price must have a steeper trajectory in \((s, x)\) space, since stores will be consumed faster with a lower price.

In other words, there is a no poaching equilibrium ultimately leading to the steady state \(X_S\) if and only if \(L_+(x_0) < s_0\) and \(s_0 > s_+(x_0)\), where \(L_+\) is the left boundary of the set \(E_+\) defined in equation B.5. Likewise, there is a no poaching equilibrium leading to extinction if and only if \(L_e(x_0) < s_0\), and \(s_0 > s_e(x_0)\). See Figure 3.3.3. \(L_i\) are downward sloping, \(L_e\) and \(L_+\) will be the same line if \(X_{\text{max}} \leq X_U\).

**Proof:** By Figure 3.3.3. \(L_i\) are downward sloping, because they are possible no poaching paths, and so stores are decreasing, while population is increasing.

\(\diamondsuit\)

**Proposition B.4** If the habitat available to the population is increased sufficiently, it is always possible to make \(X_S > Q_{\text{max}}\).

**Proof:** Denote the available habitat by \(K\), and the total population, in real units by \(\phi\). Thus \(x = \phi/K\). We assume that demand, measured in real units, is independent of habitat, and that the poaching marginal cost, \(c\), is a function only of population relative to habitat. Thus \(c(x) = c(\phi/K)\). The dynamics of the population in real units without storage will be:

\[
\dot{\phi} = KB(\phi/K) - D(c(\phi/K)),
\]

which implies that

\[
\dot{x} = B(x) - \frac{1}{K}D(c(x)).
\]

The steady states \(X_S\), and \(X_U\) will be functions of habitat, \(K\), and are such that the RHS of equation B.7 is zero. Differentiating, we find that

\[
\frac{\partial X_S}{\partial K} > 0, \text{ and } \frac{\partial X_U}{\partial K} < 0.
\]

This means that increasing the habitat more than proportionally increases the population in the high steady state. This is not unsurprising, given that demand has not changed.

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In the region where \( Q_{\text{max}} \) is close to \( X_S \), \( Q_{\text{max}} = X_U + s_e(X_U) \). We may write this as:

\[
Q_{\text{max}} = X_U + \frac{U}{K} + \int_0^{X_U} \left( B(q) - \frac{D(c(q))}{K} - \frac{rc(q)}{c'(q)} \right) \frac{c'(q)}{rc(q)} dq. \tag{B.9}
\]

When we differentiate this expression with respect to \( K \),

\[
\frac{\partial Q_{\text{max}}}{\partial K} = \frac{\partial X_U}{\partial K} - \frac{U}{K^2} + \int_0^{X_U} \frac{D(c(q))c'(q)}{K^2rc(q)} dq - 1 < -1. \tag{B.10}
\]

Thus \( Q_{\text{max}} \) is falling with \( K \) at a rate bounded away from zero. \( X_S \) is rising with \( K \). It must be, therefore, that we can find \( K \) large enough that \( Q_{\text{max}} < X_S \).
B.2 Proofs for Section 3.4

Proposition B.5 \( p_e \) is a decreasing function of \( Q \), continuous on \([0, Q_{\text{max}}]\). For \( Q \in [0, U(c_m)] \), \( p_e(Q) = U^{-1}(Q) \). For \( Q \in [U(c_m), Q_{\text{max}}] \), \( p_e(Q) = c(x_e(Q)) \).

Proof: If \( Q > U(c_m) \), the system jumps to \( x_e(Q) \), and the price is then \( c(x_e(Q)) \). If \( Q < U(c_m) \), the price will be \( U^{-1}(Q) \). \( x_e(Q) \) is decreasing in \( Q \), and continuous as required. \( U^{-1} \) is decreasing in \( Q \). \( U^{-1}(U(c_m)) = c_m = c(0) = c(x_e(U(c_m))), \) so \( p_e \) is continuous at \( U(c_m) \).

◊

Proposition B.6 \( dQ/dt = 0 \) when \( x = X_U \) or \( X_S \). The line where \( dx/dt = 0 \), \( Q_0(x) \), is increasing in \( x \), and increasing in \( p \). For \( p \) in a suitable region (see below), there are two steady states, \( (X_U, Q_U) \), and \( (X_S, Q_S) \), and

\[
Q_i = p_e^{-1} \left( \frac{r + \pi}{\pi} c(X_i) \right), \text{ where } i \text{ is } U \text{ or } S. \tag{B.11}
\]

Both \( Q_i \) are increasing with \( p \). The line \( x = x_e(Q) \) is a trajectory of the system. See Figure 3.4.1.

Proof: Since \( dQ/dt = F(x) \), \( dQ/dt = 0 \) iff \( x = X_S \) or \( X_U \). From equation 3.17, the line where \( dx/dt = 0 \) satisfies:

\[
(r + \pi)c(x) = \pi p_e(Q_0(x)). \tag{B.12}
\]

Differentiating with respect to \( x \),

\[
(r + \pi)c'(x) = \pi p_e'(Q_0(x)) \frac{\partial Q_0}{\partial x}. \tag{B.13}
\]

\( c' \) and \( p_e' \) are both negative, so \( Q_0 \) must be increasing with \( x \), for given \( p \). Differentiating the same equation with respect to \( p \),

\[
c(x) = p_e(Q_0) + \pi p_e'(Q_0) \frac{\partial Q_0}{\partial \pi}. \tag{B.14}
\]

So that, on rearranging,

\[
\frac{\partial Q_0}{\partial \pi} = \frac{c(x) - p_e(Q_0)}{\pi p_e'}. \tag{B.15}
\]
When the system jumps, population cannot rise, so price cannot fall. Hence, \( c(x) < p_e \), and equation B.15 is positive. Steady states are where \( dQ/dt = dx/dt = 0 \). Substituting \( X_S \) or \( X_U \) into equation B.12 quickly yields equation B.11

Proposition B.7 \((X_U, Q_U)\) is totally unstable; there may or may not be oscillatory behavior. \((X_S, Q_S)\) is hyperbolic for all \( \pi \). The stable manifold is thus a line, upward sloping, and passing through \((X_S, Q_S)\). See Figure 3.4.3

Proof: Consider \( x \) and \( Q \) near the steady states, \((x, Q) = (X_i + \xi, Q_i + \theta)\), where \( \xi \) and \( \theta \) are small. Using Taylor’s theorem on equation 3.17 (and assuming that we’re allowed to), to first order,

\[
\begin{pmatrix}
\dot{\xi} \\
\dot{\theta}
\end{pmatrix} = \begin{pmatrix}
\pi & -p'(Q_i) \\
F'(X_i) & 0
\end{pmatrix} \begin{pmatrix}
\xi \\
\theta
\end{pmatrix}.
\]

(B.16)

The eigen-values of this linearized system are:

\[
\lambda_{i}^{\pm} = \frac{1}{2} \left[ \pi + \sqrt{(r + \pi)^2 - \frac{4\pi F'(X_i)p'(Q_i)}{c'(X_i)}} \right].
\]

(B.17)

\( p' \) and \( c' \) are always negative. \( F'(X_U) > 0 \), and \( F'(X_S) < 0 \). Thus at \((X_S, Q_S)\) the discriminant is strictly larger than \((r + \pi)^2\), and so one eigen-value is strictly positive, the other is strictly negative. This means that, locally, there exist 1 dimensional stable and unstable manifolds for this fixed point. At \((X_U, Q_U)\), the discriminant is less than \((r + \pi)^2\), and may be negative. Both eigen-values have, therefore, strictly positive real parts, and the steady state is totally unstable.

Proposition B.8 \(X_S^0\) is decreasing with \( \pi \). \(X_S^0 = X_S^* > X_S\). There always exists \( \pi_l \) at which \( X_S^{\pi_l} = X_S \), so that for \( 0 < \pi < \pi_l, X_S^\pi > X_S \), and for \( \pi > \pi_l, X_S^\pi < X_S \).

\[
\pi_l = \frac{rc(X_S)}{p_e(X_S) - c(X_S)}.
\]

(B.18)
Proof: $X^*_S$ is the point at which:

$$F(X^*_S) = \frac{1}{c'(X^*_S)} [(r + \pi)c(X^*_S) - \pi p_e(X^*_S)].$$  \hspace{1cm} (B.19)

When $\pi = 0$, this is the same as the relation defining $X^*_S$. Let $A(x, \pi) = [(r + \pi)c(x) - \pi p_e(x)]/c'(x)$. Since the line $A(x, \pi)$ for constant $\pi$ crosses $F(x)$ from below, $\partial A(X^*_S)/\partial x > F'(X^*_S)$. Hence,

$$\frac{\partial X^*_S}{\partial \pi} \frac{\partial A}{\partial x} + \frac{\partial A}{\partial \pi} = \frac{\partial X^*_S}{\partial \pi} F' \hspace{1cm} (B.20)$$

$$\Rightarrow$$

$$\frac{\partial X^*_S}{\partial \pi} = \frac{\partial A}{\partial \pi} \frac{\partial A}{\partial x} < 0$$

If $X^*_S = X_S$, then $F(X_S) = 0$, so $\pi_l$, if it exists, satisfies $(r + \pi)c(X_S) = \pi p_e(X_S)$.

Because $p_e > c$, a solution does exist.

◊

Proposition B.9: If stores run out, they must do so at $X^*_S$. This is only possible if $X^*_S > X_S$, in which case it is a minimum of stores. If $X^*_S < X_S$ it is a maximum.

Proof: The rate of change of stores goes from $-tc$ to 0 as $x$ falls across $X^*_S$.

But if $X^*_S < X_S$, then $A(X_S, \pi) > 0$, and so $x$ is increasing, not decreasing. Thus stores are at a maximum, as stated.

◊

Proposition B.10: For given parameters, there is only one equilibrium path in the storage regime. If $\pi < \pi_l$, there is a path $Q = \delta^*_Q(x) + x$ where stores run out at $X^*_S > X_S$ and the system reverts to the no storage regime. If $p \geq \pi_l$, the equilibrium path is the stable manifold of the fixed point $(X_S, Q_S)$.

Proof: If $\pi < \pi_l$, then stores may run out at $X^*_S$, and we get exactly the same equilibrium structure as in Section 3.3. The system cannot go to extinction before the switch, because that path would be the one the system would switch to. In that case, assuming a $\pi$ hazard of switching is meaningless. The system cannot follow
the stable manifold of \((X_S,Q_S)\). Why not? Proposition B.6 proves that \(\pi p_e(Q_S) = (r + \pi)c(X_S)\). If \(\pi < \pi_I\) then \((r + \pi)c(X_S) < \pi p_e(X_S)\), so we must have \(X_S > Q_S\). This would mean that, if the system were on the stable manifold, it would have to tend to a point with strictly negative stores, which is not allowed. Thus it is not rational ever to be on the stable manifold. If \(\pi > \pi_I\), then the opposite happens: stores may not run out at \(X_S^S\), but the system may move along the stable manifold in equilibrium.

\(\diamondsuit\)

**Proposition B.11** If \(\pi > \pi_h\), then the system must be in the extinction equilibrium, where \(\pi_h\) is the hazard rate at which \(Q_S = Q_{\text{max}}\), or

\[
\pi_h = \frac{rc(X_S)}{c(X_U) - c(X_S)}.
\]  

(B.21)

**Proof:** \(Q_S\) is increasing in \(\pi\) and, once past \(\pi_I\), \((X_S,Q_S)\) is the only stable steady state before the switch to extinction. As discussed above, \(Q \leq Q_{\text{max}}\) for all points in equilibrium before the switch. Thus if \(Q > Q_{\text{max}}\), the stable manifold to \((X_S,Q_S)\) cannot be an equilibrium. If there is a \(\pi\) at which \(Q_S = Q_{\text{max}}\), then it satisfies: \(\pi p_e(Q_{\text{max}}) = (r + \pi)c(X_S)\). But \(p_e(Q_{\text{max}})\) is just \(c(X_U)\). Solving this for \(\pi\), such a \(\pi_h\) does exist, and is as claimed.

\(\diamondsuit\)

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