Dynamic Models Of Asset Returns And Trading

by

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Abstract
This thesis develops three dynamic models of asset returns and trading. The first model studies the effects of market closures (e.g. daily closures) on the return generating process and trading strategies over the trading day. The second model studies the effects of futures trading on the underlying behavior of spot prices and quantities. The role of futures as risk management tools is emphasized. The third model studies the behavior of returns and open interest in futures markets where investors trade to both hedge and speculate on private information. The informational role of futures is emphasized.

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To My Parents
Acknowledgments

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Introduction

This thesis develops three dynamic models of asset returns and trading. Chapter One (joint with Jiang Wang) models periodic closures of a competitive stock market in which investors trade for both allocational and informational reasons. During periods of market open, investors trade continuously and during periods of market close, investors hold on to their closing positions from previous trading periods. It is shown that periodic closures can generate rich patterns in the time variation of trading activities and stock returns, including those consistent with empirical findings: (1) the mean and volatility of instantaneous excess share returns on the stock can exhibit U-shaped patterns over the trading periods, (2) trading activities are higher around the close and the open, (3) open-to-open returns are more volatile than close-to-close returns, (4) returns over trading periods can be higher than returns over non-trading periods, (5) returns over trading periods are more volatile than returns over non-trading periods. It is also shown that periodic market closures can lead to prices that are more informative about the stock's future payoffs.

Chapter Two develops an equilibrium model of a competitive futures market to analyze the effects of futures expiration on the behavior of underlying spot prices and holdings (e.g. spot return volatility and storage). Futures are not redundant securities in this economy and cannot be priced by arbitrage. Instead, spot and futures prices are simultaneously determined. The introduction of futures has allocational effects. This model generates a number of testable implications regarding spot return volatility and spot holdings (storage or inventory). It is shown that spot return volatility and holdings depend on the time-to-maturity of the futures in a nonlinear fashion. Additionally, it is shown that endogenizing the spot market: price and trades generates time-to-maturity patterns in open interest not possible with an exogenously specified spot price process.

Chapter Three develops an equilibrium model of a competitive futures market in which investors trade to both hedge and speculate on private information. Private information about future payoffs to holding spot positions generates an adverse selection cost to trading in futures. In this model, the adverse selection cost to trading varies across the life of a given contract and across contracts with different time-to-maturity at a given time. This variation across time and across contracts produces a number of patterns in the return volatility and open interest of contracts. For instance, the return volatility and open interest of a futures may initially rise with time and then fall as it expires. Adverse selection can also lead to an increase in the proportion of open interest in the nearby contract (nearest to maturity) to open interest across all contracts traded.
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Chapter 1

Trading And Returns Under Periodic Market Closures (Joint w/ Jiang Wang)

In this paper, we model periodic closures of a competitive stock market and analyze the time variation in equilibrium returns and trading activities. We consider a continuous-time stock market where the exogenous information flow is homogeneous over time and investors consume continuously, but the market is closed periodically. Investors earn income from traded securities as well as non-traded sources, and have private information about stock payoffs. When the stock market opens, investors trade the stock to hedge the risk of their non-traded income and to speculate on future stock payoffs using their private information. When the market closes, the investors can no longer trade the stock and have to hold on to their closing positions from the previous trading period. In anticipation of and following periods of market closures, investors trade differently at different points in time over the trading periods. The resulting equilibrium, including the return distribution of the stock and investors' trading behavior, exhibits periodic variations over the trading periods. The purpose of our analysis is to increase our understanding of the time-varying patterns in security returns and trading activities that are associated with periodic market closures. Examples include the intraday and intra-week variations in stock returns, return volatility and trading volume.

Stock market closures impact the economy in two ways: it precludes investors from trading in the market, and it prevents investors from learning about the economy by observing market prices and trading activities. The lack of trading during future market close gives rise to time variation in investors' trading strategies, especially in their hedging demand. The inability to adjust stock positions increases the risk of holding the stock over closures. Consequently, investors actively hedge the risk in their non-traded income before the market closes, but cut down their hedging positions at the closing point. The anticipated decrease in investors' hedging demand at the close tends to make the stock price decrease over time as the stock becomes a less attractive hedging vehicle and tends to make the stock less sensitive to exogenous shocks (to investors' non-traded income). The lack of market prices as a source of information gives rise to time variation in the information asymmetry among investors. The information asymmetry increases during the closure, and often decreases as trading continues during market open. The
decrease in information asymmetry during market open tends to make the stock price increase as a smaller premium is demanded on the stock and tends to make the stock more sensitive to exogenous shocks (to stock payoffs). The actual time variation in the stock price depends on the interaction between these two effects: the effect of time-varying hedging demand and the effect of time-varying information asymmetry. In addition, the information accumulation during market closures and the discrete position changes at the closing points give rise to abnormally high trading activities at the open and close, respectively.

In the absence of private information, investors trade only for hedging reasons. The effect of time-varying hedging demand determines the time variation in stock price, which now has a higher mean and variance at the market open than at the market close. Consequently, the return volatility decreases over the trading periods. The mean return can also be decreasing. The expected return over the non-trading periods is higher than the expected return over the trading periods. The open-to-open returns are more volatile than close-to-close returns.

In the presence of private information, investors trade for both hedging and speculative reasons. The time variation in investors' hedging demand and in information asymmetry jointly determine the time variation in stock price, which can exhibit rich patterns. For some parameter values, the effect of time-varying hedging demand dominates around market open and the effect of time-varying information asymmetry dominates around market close. In this case, both the mean return and return volatility are U-shaped during the trading periods, higher around the open and close and lower in the middle. The decrease in information asymmetry during the trading periods leads to increasing prices; hence, the return over trading periods is higher than the return over non-trading periods. Furthermore, trading reveals the investors' private information to the market; hence, returns over the trading periods tend to be more volatile than returns over the non-trading periods.

The interaction between investors' hedging activity and the market's information flow also gives rise to other interesting phenomena. As investors adjust their hedging positions in anticipation of future market closures, the endogenous information flow in the economy also changes. In particular, market closures reduce the level of hedging trade, especially at the close, due to the risk of carrying positions overnight. The decrease in hedging trade then makes market prices more informative about investors' private information since price movements are now more likely due to speculative trade than due to hedging trade. In fact, periodically closing the market can make the market prices more informative about future stock payoffs.

There is an extensive and growing literature on the empirical patterns of stock returns and trading activities over trading periods. These patterns include

(1) intraday returns and volatility are U-shaped\(^1\)

(2) intraday trading volume is U-shaped\(^2\)

(3) open-to-open returns are more volatile than close-to-close returns\(^3\)

(4) week-end returns are lower than week-day returns\(^4\)

(5) returns over trading periods are more volatile than returns over non-trading periods.\(^5\)

These patterns are often robust with respect to different market micro-structures (such as NYSE, NASDAQ, the interbank market of currencies).

The empirical findings have generated strong interest in developing theoretical models to understand them [e.g., Admati and Pfleiderer (1988, 1989), Foster and Viswanathan (1990), Brock and Kleidon (1992), Slezak (1994), Spiegel and Subrahmanyam (1995)]. Many of these models take a single period with multiple rounds of trading as a trading day. The initial point is interpreted as the open and the end point is interpreted as the close. Without modeling the actual opening and closing of the market, these models do not capture the dynamic nature of market opening and closing, and the results often depend on particular assumptions about the exogenous information flow (such as the revelation of true asset value at the terminal date). Two exceptions are Brock and Kleidon (1992) and Slezak (1994). Slezak (1994) models a stock market of differently informed traders with a single closure. The closure is modeled as a single date with more information arrival than the surrounding dates. Brock and Kleidon (1992) directly model periodic market closures. Unfortunately, their analysis is partial equilibrium, and hence cannot speak to the intraday patterns in returns and trading activities.

Our model differs from the existing models in several aspects. First, in our model, investors trade for both allocational and informational reasons. This stands in contrast to the noisy rational expectations models in the literature in which investors' allocational trade is exogenously specified. The fact that investors optimally choose the timing and size of their allocational trade is important in understanding the equilibrium return and trading patterns.\(^6\) Second, we model the periodic closure and opening of the market in a dynamic equilibrium framework. Third, in contrast to many of the existing models, we assume a competitive market to avoid market microstructure issues. By analyzing periodic market closures, we focus on those effects on equilibrium returns and trading activities that are solely associated with closures. We show that merely periodic market closures can qualitatively generate all the empirical patterns mentioned above.

---

\(^2\)See, e.g., Jain and Joh (1988).


\(^6\)Admati and Pfleiderer (1988) allow some of the liquidity traders to time their trade of given sizes and show that this can change the nature of the equilibrium. But they do not provide a complete justification for the behavior of liquidity traders. For example, trade sizes are exogenously specified and not all investors can time their trade.
A feature of our model is that market closures intrinsically affect the return generating process. This differs from the model of Admati and Pfleiderer (1988), in which investors can concentrate in trading, leading to concentration in price movements. Their model itself, however, does not determine when trade and price volatility concentrate. The U-shaped volatility pattern results only as one of many possible equilibria. Another feature of our model is that the interaction between repeated market open and close determines the return patterns. In particular, what happens around the market close is related to future re-opening, and what happens around the open is related to closures in the past and future. This is different from the model of Slezak (1994) with a single closure, in which the open is not affected by future close and the close is not affected by past open. Our model is closely related to the models of Campbell and Kyle (1993) (under homogeneous information) and Wang (1993, 1994) (under heterogeneous information).

The paper proceeds as follows. The model is described in Section 1 and the equilibrium notion is defined in Section 2. In Section 3, we analyze the equilibrium and the time variations in stock returns and trading activity under symmetric information. In Section 4, we examine the effect of information asymmetry on stock returns and trading activity. Section 5 concludes. All proofs are in the appendix.

1. The Model

We consider an economy of a single good defined on a continuous time-horizon $[0, \infty)$. The underlying uncertainty of the economy is characterized by an $n$-dimensional standard Wiener process $\mathbf{w}_t$, $t \in [0, \infty)$. The economy consists of two classes of identical investors denoted by $i = 1, 2$ with population weight $\omega$ and $1 - \omega$, respectively, where $\omega \in [0, 1]$. Since investors within the same class are identical, for convenience we also refer to any investor in class-$i$ as investor $i$, $i = 1, 2$. The economy is further defined below.

A. Investment Opportunities

Investors can invest in publicly traded securities or in private investment opportunities. Publicly traded securities include a risky stock and a risk-free money-market account. Private investment opportunities are linear production technologies available only to individual investors. The payoffs of these investments are as follows:

(a) Each share of the stock pays a cumulative dividend $D_t$ where

$$dD_t = G_t dt + b_d dw_t \quad (D_0 = 0) \tag{1a}$$

$$dG_t = -a_G G_t dt + b_G dw_t \quad (G_0 = g_0). \tag{1b}$$

(b) The money-market account pays a positive, constant rate of return $r$. 

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(c) Investor i's private technology yields a cumulative excess rate of return $q_{i,t}$ ($i = 1, 2$) where

$$dq_{i,t} = Y_{i,t}dt + b_q dw_t \quad (q_{i,0} = 0) \quad (2a)$$

$$dY_{i,t} = -a_Y Y_{i,t}dt + b_i dw_t \quad (Y_{i,0} = y_{i,0}). \quad (2b)$$

Here, $a_q$, $a_Y$ are positive constants, and $b_D$, $b_A$, $b_q$, $b_i$ ($i = 1, 2$) are constant matrices of proper order. Since the system $\{G_t, Y_{1,t}, Y_{2,t}\}$ follows a Gaussian Markov process, it completely determines the distribution of future payoffs on the stock and investors' private technologies.$^7$

Investors can continuously invest in their private technologies. They can also trade the securities in a competitive securities market when the market opens.$^8$ The money market opens continuously but the stock market opens only periodically. Thus, investors can adjust their money-market account at any time, but they can trade the stock only when the stock market is open. The dividend and interest payments are received continuously as they accrue. $P_t$ denotes the share price of the stock (in units of current consumption) when the stock market is open.

---

Figure 1-1: Time Line

Figure 1 shows the time sequence of events. Here, $[t_k, n_k]$ is the $k$-th (stock) trading period, and $(n_k, t_{k+1})$ the $k$-th non-trading period, where $k = 0, 1, 2, \cdots, t_k = k(T+N)$, $n_k = t_k + T$, $T$ is the length of trading periods and $N$ the length of non-trading periods. For convenience, we refer to the periods when the stock market is open and close as "day" and "night", respectively. When $N = 0$, the economy reduces to the continuous-trading case, and when $T = 0$, it reduces to the discrete-trading case.

B. Information Distribution

---

$^7$The returns on investors' private technologies are assumed to be perfectly positively correlated. There is no loss of generality in assuming this, and our results stay qualitatively the same if partial correlation is allowed.

$^8$The competitive assumption can be justified by allowing many investors within each class of investors. Given that the investors within each class are identical, we can further construct, for each class of investors, a representative investor, who has the same information, preferences, investment opportunities as each individual investor, but the total endowments of all investors in the class. The risk-tolerance of the representative investor is the sum of the risk-tolerance of individual investors in the class [see, e.g., Rubinstein (1974)]. The economy then can be viewed as consisting of the two representative investors, interacting competitively in the securities market.
For $t \geq 0$, let $\mathcal{I}_t \equiv \{P_s, D_s, G_s, Y_{1,s}, Y_{2,s}, q_{1,s}, q_{2,s} : 0 \leq s \leq t\}$ denote the full information set about the economy and $\mathcal{I}_{i,t}$ the information set of investor $i$, $i = 1, 2$. In general, $\mathcal{I}_{i,t} \subseteq \mathcal{I}_t$. In this paper, we consider two specific cases of information distribution among the investors, the case of "symmetric information" and the case of "asymmetric information":

(a) symmetric information - $\mathcal{I}_{1,t} = \mathcal{I}_{2,t} = \mathcal{I}_t$

(b) asymmetric information - $\mathcal{I}_{1,t} = \mathcal{I}_t$ and $\mathcal{I}_{2,t} = \mathcal{I}_{2,0} \otimes \{P_s, D_s, Y_{2,s}, q_{2,s} : 0 \leq s \leq t\}$

where $\mathcal{I}_{2,0}$ denotes class-2 investors’ prior information (on $G_0$ and $Y_{1,0}$).  

In both cases, all investors observe realized dividends and market prices of the stock. Each investor also observes the expected and realized return of his own private technology. In the case of symmetric information, all investors have perfect information about the economy. In the case of asymmetric information, class-1 investors have perfect information about the economy and class-2 investors only have publicly available information on the stock and information on their own private investment opportunities. In this case, $\mathcal{I}_{2,t} \subseteq \mathcal{I}_{1,t}$ and class-1 investors have superior information than class-2 investors.

C. Endowments, Policies and Preferences

Each investor is endowed with one share of the stock. He chooses investment and consumption policies to maximize the expected utility over his life-time consumption. For investor $i$ ($i = 1, 2$), let $\{c_{i,t} : t \in [0, \infty)\}$ and $\{y_{i,t} : t \in [0, \infty)\}$ be his consumption policy and production policy, respectively, and $\{\theta_{i,t} : t \in T\}$ his trading policy in the stock, defined only when the stock market is open. In short, we use $\{c_{i,t}, y_{i,t}, \theta_{i,t}\}$ to denote investor $i$'s policy, which must be adapted to his information $\mathcal{I}_{i,t}$. We further restrict investors' trading and production policies to predictable, square-integrable processes with respect to the return processes of the stock and production technologies [see, e.g., Harrison and Pliska (1981)] and the consumption policy to integrable processes.

Footnotes:

9For convenience, it is assumed that class-1 investors observe class-2 investors' technology shock $Y_{2,t}$ in the case of asymmetric information. Alternatively, one can assume that class-1 investors only have private information about $G_t$ and their own technology shock $Y_{1,t}$, but not about class-2 investors' technology shock. Note, however, that in this case class-1 investors can perfectly infer class-2 investors' technology shock from the market price in a linear equilibrium whenever the stock market is open. The only difference is that during the non-trading periods, class-1 investors now have to filter about $Y_{2,t}$ instead of directly observing it as assumed in the paper. We do not expect the results to be very different in these two cases and adopt the simpler information structure here for easy exposition.

10In general cases of symmetric information, investors can have imperfect information about the economy, as long as their information are the same. The perfect information assumption is not necessary for our analysis and is adopted here only for ease in exposition. In general cases of asymmetric information, class-1 investors' information needs not be superior to class-2 investors' information, and solving the equilibrium becomes more difficult. See, e.g., He and Wang (1995).

11The notion of integrability here refers to integrability over any finite period $[0, t]$, $\forall \ t \geq 0$.  

13
For tractability, we assume that all investors maximize the expected utility of the following form:

\[
E\left[-\int_t^\infty e^{-\rho(s-t)-\gamma \alpha_i s} ds \left| I_{i,t}\right.\right], \quad i = 1, 2
\]

(3)

where \(\rho\) and \(\gamma\) (both positive) are the time discount coefficient and the relative risk-aversion coefficient, respectively.

\textbf{D. Distributional Assumptions}

We further assume that \(w_t = [w_{D,t}, w_{G,t}, w_{1,t}, w_{2,t}, w_{q,t}]',\) i.e., the Wiener process \(w_t\) is 5-dimensional, and

\[
b_D = \sigma_D [1, 0, 0, 0, 0], \quad b_G = \sigma_G [0, 1, 0, 0, 0] \\
b_1 = \sigma_1 [0, 0, \kappa_-, \kappa_+, 0], \quad b_2 = \sigma_2 [0, 0, \kappa_+, \kappa_-, 0], \quad b_q = \sigma_q [\kappa_{Dq}, 0, 0, 0, \sqrt{1 - \kappa_{Dq}^2}]
\]

where \(\kappa_\pm = \frac{1}{2} (\sqrt{1 + \kappa_{12}} \pm \sqrt{1 - \kappa_{12}}), \kappa_{Dq} \in (-1, 1)\) and \(\kappa_{12} \in [-1, 1].\) The above specification about the underlying shocks to the economy has simple interpretations. For example, \(w_{D,t}\) fully characterizes instantaneous shocks to the dividend \(D_t\) and \(\sigma_D\) gives its instantaneous volatility. The above form of the \(b\)'s specifies a particular correlation structure among the shocks to different variables. In particular, \(\kappa_{Dq}\) is the correlation between dividends and returns on investors' private technologies, and \(\kappa_{12}\) is the correlation between the expected returns on the two classes of investors' private technologies. To fix ideas, we restrict \(\kappa_{Dq}\) to be positive in the remainder of the paper. We use this correlation structure to simplify exposition and our results do not depend on this particular choice (provided that \(\kappa_{Dq} \neq 0).\)

In the model defined above, there are two reasons to trade the stock. First, an investor's portfolio of risky investments consists of holdings of the stock and investments in his private technology. Since returns on private technologies and the stock are correlated, an investor's stock holding depends not only on the expected return on the stock itself but also on the expected return on his private technology. In particular, investors use the stock to hedge the risk from their private investments. As investors adjust their private investments in response to changes in the expected returns, they also revise their hedging positions in the stock, which generates the allocational trading in the market. Second, some investors have private information about future stock payoffs. Thus, they take speculative positions based on their private information in anticipation of future profits. This generates the informational trading in the market.

For easy reference, Table 1.1 lists all the model parameters. We also introduce some notation. Let \(E_{i,t} \equiv E[\cdot | I_{i,t}],\) \(i = 1, 2,\) and \(E_t \equiv E[\cdot | I_t].\) Define \(dQ_t \equiv dP_t + dD_t - \tau P_t dt\) to be the instantaneous excess return on one share of stock when the market is open, \(e \equiv E[dQ_t]/dt, \sigma_Q^2 \equiv E[(dQ_t)^2]/dt\) its first two unconditional moments, and \(e_{i,t} \equiv E_{i,t}[dQ_t]/dt, \sigma_{Q,i}^2 \equiv E_{i,t}[(dQ_t)^2]/dt\) its first two conditional moments given investor \(i\)'s information. In order
to relate to the empirical findings, we define the simple share return on the stock from \( s \) to \( t \) to be

\[
R_{s,t} = P_t - P_s + \int_s^t dD_r dr.
\]

where \( s, t \in \mathcal{T} \) and \( t > s \). The simple share returns from open to open, close to close, open to close and close to open are then given by \( R^{oo} \equiv R_{t_k,t_{k+1}}, \ R^{cc} \equiv R_{n_k,n_{k+1}}, \ R^{co} \equiv R_{t_{k+1},n_{k-1}}, \ R^{oc} \equiv R_{n_{k-1},t_{k+1}} \), respectively. Also, define

\[
F_t \equiv \mathbb{E}_t \left[ \int_t^\infty e^{-r(s-t)} dD_s ds \right] = \frac{1}{r + a_G} G_t.
\]

\( F_t \) gives the expected future dividends on the stock discounted at the risk-free rate and \( dF_t = -\frac{a_G}{r + a_G} G_t dt + b_P dw_t \) where \( b_P = \frac{1}{r + a_G} b_G \). In addition, let \( \mathcal{T}_k \equiv [t_k, n_k] \) denote the \( k \)-th trading period (day), \( N_k \equiv (n_k, t_{k+1}) \) the \( k \)-th non-trading period (night), \( \mathcal{T} \equiv \bigcup_k [t_k, n_k] \) all the trading periods, and \( \mathcal{N} \equiv \bigcup_k (n_k, t_{k+1}) \) all the non-trading periods.

**Table 1.1: Summary of Model Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population weight of type-1 investors</td>
<td>( \omega )</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>( r )</td>
</tr>
<tr>
<td>Volatility of dividend</td>
<td>( \sigma_D )</td>
</tr>
<tr>
<td>Mean-reversion coefficient of expected dividend rate</td>
<td>( a_G )</td>
</tr>
<tr>
<td>Volatility of expected dividend rate</td>
<td>( \sigma_G )</td>
</tr>
<tr>
<td>Volatility of returns on private technologies</td>
<td>( \sigma_q )</td>
</tr>
<tr>
<td>Mean-reversion coefficient of expected returns on private technologies</td>
<td>( a_V )</td>
</tr>
<tr>
<td>Volatility of expected returns on investor 1’s private technology</td>
<td>( \sigma_1 )</td>
</tr>
<tr>
<td>Volatility of expected returns on investor 2’s private technology</td>
<td>( \sigma_2 )</td>
</tr>
<tr>
<td>Correlation of shocks to dividends and returns on private technologies</td>
<td>( \kappa_{Dq} )</td>
</tr>
<tr>
<td>Correlation of shocks to expected returns on private technologies</td>
<td>( \kappa_{12} )</td>
</tr>
<tr>
<td>Length of trading periods</td>
<td>( T )</td>
</tr>
<tr>
<td>Length of non-trading periods</td>
<td>( N )</td>
</tr>
</tbody>
</table>

Bold-face letters denote matrices (or vectors). For a set of elements \( e_1, e_2, \ldots, e_m \) (of proper order), let \( \text{diag}\{e_1, e_2, \ldots, e_m\} \) denote a diagonal matrix, stack \( \{e_1, e_2, \ldots, e_m\} \) the column matrix, and \( \{e_1, e_2, \ldots, e_m\} \) the row matrix with the given elements. For a matrix \( a, a_{ij} \) denotes its \((ij)\) element, \( a_i \) its \( i \)-th row, and \( \text{tr}(a) \) its trace. \( 1_{ij}^{(m,n)} \) denotes the index matrix of order \((m \times n)\) with its \((ij)\) element being one and other elements being zero, and \( i^{(m)} \) denotes the identity matrix of order \( m \). Also, for any two random variables given by \( dX_j = \mu_j dt + b_{x_j} dw_t \), \( j = 1, 2 \), \( \sigma_{x_j x_{j'}} = b_{x_j} b_{x_{j'}} \), \( j, j' = 1, 2 \), denotes the instantaneous covariation of the two variables [see Karatzas and Shreve (1987) for a discussion of covariation processes].
2. Definition of Equilibrium

An equilibrium of the above economy is defined by a stock price process \( \{P_t : t \in T\} \), such that investors follow policies \( \{c_{i,t}, y_{i,t}, \theta_{i,t}\} \), \( i = 1, 2 \), to maximize their expected utilities and the stock market clears. The equilibrium stock price and investors' optimal policies can in general be expressed as a function of time and a set of variables that fully characterizes the state of the economy. Specifically, \( P_t = P(\bullet ; t) \) (for \( t \in T \)) and \( \{c_{i,t}, y_{i,t}, \theta_{i,t}\} = \{c_i(\bullet ; t), y_i(\bullet ; t), \theta_i(\bullet ; t)\} \), \( i = 1, 2 \), where \( \bullet \) denotes the relevant state variables. We call \( P(\bullet ; t) \) the price function and \( \{c_i(\bullet ; t), y_i(\bullet ; t), \theta_i(\bullet ; t)\} \) the policy functions of investor \( i \).

The price function and policy functions are time dependent since the structure of the economy changes over time. The day-time and the night-time are different since the stock market opens only during the day. Different points in time during a day are also different since the trading opportunities remaining in the day are different. The time variation is, however, periodic: the day and the night repeat themselves over time. Thus, we only consider periodic equilibria of the economy, in which the price function and investors' policy functions exhibit periodicity in time.

**Definition 1** In the economy defined above, a periodic equilibrium is defined by the stock price function \( P(\bullet ; t) \) and investors' policy functions \( \{c_i(\bullet ; t), y_i(\bullet ; t), \theta_i(\bullet ; t)\} \), \( i = 1, 2 \), such that (i) the policies maximize investors' expected utility, (ii) stock market clears, and (iii) the price function and the investors' policy functions are periodic in \( t \) with periodicity \( T + N \).

In particular, the stock price function has the same functional form in terms of the relevant state variables for every day:

\[
P(\bullet ; \tau) = P(\bullet ; t_k + \tau)
\]

where \( k = 0, 1, \ldots \) and \( \tau \in [0, T] \). Actual values of the stock price can be different from day to day since values of the state variables can be different.

Given a stock price process \( P_t \), we now state the investors' budget constraints and their optimization problem. Investor \( i \)'s financial wealth at \( t \) consists of two parts: balance in the money-market account denoted by \( W_{i,t}^0 \) and position in the stock \( \theta_{i,t} \). During the day, i.e., \( t \in T_k \) (\( k = 0, 1, \ldots \)), the stock market is open and his stock position can be marked to the market. Investor \( i \)'s total financial wealth is given by the market value of his stock position and money market account: \( W_{i,t} = W_{i,t}^0 + \theta_{i,t} P_t \). He can freely adjust his stock and money-market positions to finance consumption and private investments. His budget constraint is specified by the dynamics of his total financial wealth \( W_{i,t} \). During the night, i.e., \( t \in N_k \), the stock market is closed and investor \( i \) holds a fixed number of stock shares \( \theta_{i,n_k} \), his closing position from the previous day. The stock positions cannot be marked to the market due to the market closure. Investor \( i \) now has to finance his consumption and private investments through his money-market account. His budget constraint is specified by the dynamics of his money-market account \( W_{i,t}^0 \). His overnight stock holding \( \theta_{i,n_k} \) now becomes a state variable.
Given the budget constraints, the investors’ optimization problem is expressed as follows. For \( i = 1, 2 \) and \( k = 0, 1, \cdots \),

\[
\begin{align*}
t \in T_k : \quad J_i \left( W_{i,t}, \bullet; t \right) & \equiv \sup_{\{c_i, \theta_i, t_i\}} E_{i,t} \left[ - \int_t^{n} e^{-\rho(s-t) - \gamma c_i,s} ds + J_{i,t}^{*} \right] \\
& \text{s.t.} \quad dW_{i,t} = (rW_{i,t} - c_{i,t}) \ dt + \theta_{i,t} dQ_t + y_{i,t} dq_{i,t} \tag{5a}
\end{align*}
\]

\[
\begin{align*}
t \in N_k : \quad J_i^{*} \left( W_{i,t}, \theta_{i,n_k}, \bullet; t \right) & \equiv \sup_{\{c_i, \theta_i, t_i\}} E_{i,t} \left[ - \int_t^{t_k+1} e^{-\rho(s-t) - \gamma c_i,s} ds + J_{i,t_k+1}^{*} \right] \\
& \text{s.t.} \quad dW_{i,t} = \left( rW_{i,t} - c_{i,t} \right) \ dt + \theta_{i,n_k} dD_t + y_{i,t} dq_{i,t} \tag{5b}
\end{align*}
\]

where \( J_i \left( W_{i,t}, \bullet; t \right) \) and \( J_i^{*} \left( W_{i,t}, \theta_{i,n_k}, \bullet; t \right) \) are, respectively, investor \( i \)'s value function during the day and night, and \( \bullet \) denotes the relevant state variables that characterize the return processes on the stock and his private technology given his information.

At the close and the open, the optimality of investors’ policies yields the following boundary conditions:

\[
\begin{align*}
J_i \left( W_{i,n_k}, \bullet; n_k \right) & = E_{i,n_k} \left[ J_i^{*} \left( W_{i,n_k} - \theta_{i,n_k} P_{n_k}, \theta_{i,n_k}, \bullet; n_k \right) \right] \\
J_i^{*} \left( W_{i,t_k}, \theta_{i,n_k}, \bullet; t_k+1 \right) & = E_{i,t_k} \left[ J_i \left( W_{i,t_k} + \theta_{i,n_k} P_{n_k}, \bullet; t_k+1 \right) \right] \tag{6a}
\end{align*}
\]

where \( k = 0, 1, \cdots \), \( i = 1, 2 \), \( t^- \) and \( t^+ \) denote, respectively, the time right before and after \( t \). Furthermore, if the stock price is a periodic function of the underlying state variables, the optimization problem of individual investors also exhibits periodicity. Thus, we require

\[
J_i \left( \cdot, \cdot; t_k \right) = J_i \left( \cdot, \cdot; t_{k+1} \right), \quad i = 1, 2 \tag{7}
\]

i.e., investor \( i \)'s value function, as a function of his wealth and the state variables, is periodic in time with periodicity \( T + N \). Equations (6-7) give the boundary conditions we need to solve the optimization problem given a periodic price function.

In equilibrium, the stock market clears. The market clearing condition requires that:

\[
\omega \theta_{1,t} + (1-\omega) \theta_{2,t} = 1, \quad t \in T. \tag{8}
\]

Thus, a periodic equilibrium of the economy is given by a periodic price function (4) such that investors adopt optimal policies given by the solution to (5), (6-7), and the market clears as (8) requires.

The state of the economy is determined by investors’ wealth and their information on current and future investment opportunities, \( I_{1,t} \) and \( I_{2,t} \). Without loss of generality, we assume that
the investors' information at $t$ can be characterized by a set of state variables $\Psi_t$, which is a sufficient-statistic set for $I_{1,t}$ and $I_{2,t}$. In the case of symmetric information, $\Psi_t = \{G_t, Y_{1,t}, Y_{2,t}\}$. In the case of asymmetric information, $\Psi_t$ includes additional variables. Since class-2 investors now do not observe the true value of $G_t$, they rely on their expectations about future stock dividends to form their policies. Thus, $\Psi_t = \{G_t, Y_{1,t}, Y_{2,t}, \tilde{G}_t, \tilde{Y}_{1,t}, \ldots\}$ where $\tilde{G}_t \equiv E_{2,t}[G_t]$ and $\tilde{Y}_t \equiv E_{2,t}[Y_{1,t}]$ are, respectively, class-2 investors' conditional expectations of $G_t$ and $Y_{1,t}$.

Given the constant risk-free rate and constant absolute risk aversion in preferences, investors' demand of risky investments only depends on the distribution of future payoffs, independent of their wealth [see, e.g., Merton (1969)]. We thus seek an equilibrium in which the stock price is independent of investors' wealth, i.e., $P_t = P(\Psi_t; t)$. Furthermore, we restrict ourselves to the linear equilibrium in which the price function is linear in $\Psi_t$.

**Definition 2** In the economy defined above, a linear, periodic equilibrium is a periodic equilibrium in which the stock price has the form: $P_t = L(\Psi_t; t)$, $t \in \mathcal{T}$, where $L(\cdot; t)$ denotes a linear function, which is time dependent with periodicity $T+N$, and $\Psi_t$ denotes a sufficient statistic set of investors' information at $t$, $I_{1,t}$ and $I_{2,t}$.

Further specification of $\Psi_t$ in the case of asymmetric information is given in Section 4.

### 3. The Case of Symmetric Information

We first analyze the equilibrium in the case of symmetric information when $I_{i,t} = I_t$ $\forall$ $t \geq 0$, $i = 1, 2$, and all investors have perfect information about the economy. This case serves three purposes: (1) to understand some basic properties of the model; (2) to derive the trading and return patterns under periodic market closures when investors trade only for allocational reasons; (3) to illustrate the solution of a linear, periodic equilibrium.

#### 3.1. The General Solution

In the case of symmetric information, $G_t$, $Y_{1,t}$ and $Y_{2,t}$ fully characterize future payoffs of the stock and private technologies, hence $\Psi_t = \{G_t, Y_{1,t}, Y_{2,t}\}$. Equation (1) implies that the stock's future dividend flow consists of two additive components: a riskless component determined by $G_t$ and a risky component independent of $G_t$ with zero mean.\footnote{From (1), $D_t - D_s = \int_s^t G_r dr + \int_s^t b_r dw_r$, and $G_t = G_s e^{-a(t-s)} + \int_s^t e^{-a(r-s)} b_r dw_r$, where $t \geq s \geq 0$. Thus, $D_t - D_s = G_s \int_s^t e^{-a(r-s)} dr + \left[\int_s^t \int_s^t e^{-a(r'-s)} b_r dw_r + \int_s^t b_r dw_r\right]$. The first term gives the deterministic component in future dividends and the second term gives the risky component, which has zero mean conditional on the information at $s$.} The present value of the riskless component is simply $F_t$. No arbitrage requires that in a linear, periodic equilibrium, the stock
price must have the following form:

\[ P_t = F_t - \lambda X_t, \quad t \in \mathcal{T} \]  

(9)

where \( X_t = [1, Y_{1,t}, Y_{2,t}]' \) and \( \lambda = [\lambda_0, \lambda_1, \lambda_2] \) is time-dependent with periodicity \( T+N \). Let 
\( a_x = \text{diag}(0, -a_Y, -a_Y) \) and 
\( b_x = \text{stack}(0, b_1, b_2) \). Then, 

\[ dX_t = a_x X_t dt + b_x dw_t. \]

Given the price function in (9), the excess share return on the stock is 

\[ dQ_t = a_Q X_t dt + b_Q dw_t \]

where \( a_Q = \lambda \left( r(t^{(3)} + a_x) - \lambda \right) \) and \( b_Q = b_D + b_F - \lambda b_X \). Its first two unconditional moments are given by 
\( e = a_Q^{(0)} = r \lambda_0 - \lambda_0 \) and 
\( \sigma_Q^2 = b_Q b_Q' \). The conditional expected excess share return on the stock \( e_t = a_Q X_t \), which depends only on \( X_t \).

Since \( X_t \) follows an Ornstein-Uhlenbeck process, the investors’ investment opportunities (including their private technologies) are fully specified by \( X_t \). Consequently, their optimal policies and value functions only depend on their wealth, \( X_t \) and \( t \). For \( i = 1, 2 \), define 
\( a_{i,q} = \mathbb{I}_{1,i+1}, \]
\( a_{i,w} = \text{stack}(a_Q, a_{i,q}) \) and 
\( b_w = \text{stack}(b_Q, b_Q) \). We have the following result on investors’ optimal policies:

**Lemma 1** When \( \mathcal{I}_{i,t} = \mathcal{I}_t \) \( \forall \ t \geq 0, \ i = 1, 2 \), and the stock price has the form in (9), investor \( i \)'s optimal policies and his value function have the form

\[
\begin{align*}
\left( \begin{array}{c}
\theta_{i,t} \\
y_{i,t} \\
h_{i,y}
\end{array} \right) &= \left( \begin{array}{c}
h_{i,\theta} \\
h_{i,y}
\end{array} \right) X_t \\
\alpha_{i,t} &= -\frac{1}{\gamma} \ln r + r W_{i,t} + \frac{1}{2\gamma} X_t' v_i X_t \\
J_{i,t} &= -\exp \left\{ -\rho t - r \gamma W_{i,t} - \frac{1}{2} X_t' v_i X_t \right\} \tag{10}
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
y_{i,t} \\
h_{i,y}
\end{array} \right) &= \frac{1}{\gamma} \sigma_{w}^{-1} \left( a_{i,q} X_t - r \gamma \sigma_{dq} \theta_{i,nk} \right) \\
\alpha_{i,t} &= -\frac{1}{\gamma} \ln r + r W^*_{i,t} + \frac{1}{2\gamma} \left( X_t' v_i^* X_t - 2 \theta_{i,nk} u_{i,0}^* X_t - \theta_{i,nk}^2 u_{i,0}^* \right) \\
J^*_{i,t} &= -\exp \left\{ -\rho t - r \gamma W^*_{i,t} - \frac{1}{2} \left( X_t' v_i^* X_t - 2 \theta_{i,nk} u_{i,0}^* X_t - \theta_{i,nk}^2 u_{i,0}^* \right) \right\}
\end{align*}
\]

where \( k = 0, 1, \cdots, \) 
\( W_{i,t} = W_{i,t}^0 + \theta_{i,nk} F_t \), and 

\[
\begin{align*}
\left( \begin{array}{c}
h_{i,\theta} \\
h_{i,y}
\end{array} \right) &= \left\{ \begin{array}{c}
\frac{1}{\gamma} \sigma_w^{-1} \left( a_{i,w} - \sigma_{wx} v_i \right), \\
\frac{1}{\gamma} \sigma_{w}^{-1} \left( r \gamma \lambda - u_{i,0}^* \right)
\end{array} \right\}, \quad t \in [t_k, n_k] \\
\left( \begin{array}{c}
h_{i,\theta} \\
h_{i,y}
\end{array} \right) &= \left\{ \begin{array}{c}
\frac{1}{\gamma} \sigma_{w}^{-1} \left( r \gamma \lambda - u_{i,0}^* \right) \\
\frac{1}{\gamma} \sigma_{w}^{-1} a_{i,q} - \frac{1}{\gamma} \left( r \gamma \sigma_{dq} (r \gamma \lambda - u_{i,0}^*) \right)
\end{array} \right\}, \quad t = n_k
\end{align*}
\]
Here, \( v_i, v_i^* \) are (3 \times 3) symmetric matrices and \( u_i^*, u_{i,0}^* \) are, respectively, (3 \times 1) and (1 \times 1) matrices satisfying

\[
t \in [t_k, n_k]: \quad \dot{v}_i = v_i \sigma_{xx} v_i - (a_i, w - \sigma_{wx} v_i) \sigma_{i,ww}^{-1} (a_i, w - \sigma_{wx} v_i) + rv_i - (a_i, v_i + v_i a_x') - [\tilde{\sigma} + \text{tr}(\sigma_{xx} v_i)] I_{11}^{(3,3)}
\]

\[
t = n_k : \quad v_i(n_k) = v_i^*(n_k^+) + \frac{1}{u_{i,0}^*(n_k^+)} \left[ r\gamma \lambda(n_k) - u^*(n_k^+) \right]' \left[ r\gamma \lambda(n_k) - u^*(n_k^+) \right]
\]

\[
t \in (n_k, t_{k+1}) : \quad \begin{cases} 
\dot{v}_i^* = v_i \sigma_{xx} v_i + rv_i^* - (a_i, v_i^* + v_i^* a_x') - a_i, q^{-1} a_i, q \\
- [\tilde{\sigma} + \text{tr}(\sigma_{xx} v_i^*)] I_{11}^{(3,3)} 
\end{cases}
\]

\[
(11)
\]

\[
t = t_{k+1} : \quad \begin{cases} 
v_i^*(t_{k+1}^-) = v_i(t_{k+1}) \\
u_i^*(t_{k+1}^-) = r\gamma \lambda(t_{k+1}) \\
u_{i,0}^*(t_{k+1}^-) = 0
\end{cases}
\]

and

\[
v_i(t_k) = v(t_{k+1}) \quad (12)
\]

where \( \tilde{\sigma} = 2(\rho + r \ln r - r) \).

Given \( \lambda \), the above lemma expresses investor \( i \)'s policies as simple functions of the matrices \( v_i \) and \( v_i^* \), \( u_i^*, u_{i,0}^* \), defined by (11-12) on \( T_k \) and \( N_k \), respectively.

In order to show that a linear, periodic equilibrium exists, we need to show that there exists a periodic \( \lambda \) such that the stock market clears:

\[
\omega h_{1,\theta} + (1 - \omega) h_{2,\theta} = I_{11}^{(1,3)}, \quad t \in T.
\]

From the definition of \( h_{i,\theta} \) in Lemma 1 (\( i = 1, 2 \)), we have

\[
t \in [t_k, n_k] : \quad \dot{\lambda} = \lambda (ri - a_i) - (\sigma_{ww})^{-1} \{ r\gamma 2 I_{11}^{(1,3)} - (\sigma_{ww}^{-1})_{12} [\omega a_{1, q} + (1 - \omega) a_{2, q}] \\
+ (\sigma_{ww}^{-1} \sigma_{wx} [\omega v_1 + (1 - \omega) v_2]) ; \}
\]

\[
t = n_k : \quad \lambda = \frac{1}{r\gamma} \left[ \frac{\omega}{u_{i,0}^*(n_k^+)} + \frac{1 - \omega}{u_{i,0}^*(n_k^+)} \right]^{-1} \left\{ [I_{11}^{(1,3)} + \left[ \omega u_i^*(n_k^+) + (1 - \omega) u_{i,0}^*(n_k^+) \right] \left[ u_i^*(n_k^+) + u_{i,0}^*(n_k^+) \right]^{-1} \right\}.
\]

Periodicity requires that

\[
\lambda(t_k) = \lambda(t_{k+1}). \quad (14)
\]
Solving a linear, periodic equilibrium reduces to solving the system (11) and (13) with boundary conditions (12) and (14). (11) and (13) define a system of first-order ordinary differential equations (ODE) fully specifying the undetermined matrices $v_i$, $v_i^*$, $u_i^*$, $u_{i,0}^*$ and $\lambda$, $i = 1, 2$. In particular, given any $v_i(t_{k+1})$ and $\lambda(t_{k+1})$, solving the system backwards gives $v_i(t_k)$ and $\lambda(t_k)$ (as well as their values from $n_k$ to $t_k$). This is the familiar initial-value problem (which seeks the solution of an ODE given its value at a fixed point in time). Our problem, however, has the boundary condition (12) and (14) which require that $v_i(t_k)$ and $\lambda(t_k)$ equal $v_i(t_{k+1})$ and $\lambda(t_{k+1})$, respectively. Solving such a problem is known as the two-point boundary-value problem, which seeks the solution of an ODE, the values of which at two given points in time satisfy a given relation. We have the following lemma concerning the existence of a solution to the given system:

**Lemma 2** The system defined by (11) and (13) with boundary conditions (12) and (14) has a solution.

The actual values of the time-dependent coefficients can only be solved numerically in general. The method for numerical solutions are discussed in more detail in Appendix D. Theorem 1 summarizes the above discussions:

**Theorem 1** When $I^i = I_t \forall t \geq 0$, $i = 1, 2$, and for $\omega$ close to one, a linear, periodic equilibrium exists in which the stock price has the form in (9) and the investors' policies have the form in Lemma 1.

We now consider the general properties of the equilibrium. The stock price consists of two components: the expected value of future payoffs $F_t$ and the risk discount $\lambda X_t = \lambda_0 + \sum_i \lambda_i Y_{i,t}$, $t \in T$. The dependence of the price on $Y_{i,t}$, $i = 1, 2$, arises from the (positive) correlation between the payoffs of the stock and investors' private technologies. When $Y_{i,t}$ is high, the expected return on investor $i$’s private technology is high and he wants to invest more in the technology. At the same time, he also wants to decrease his stock position to reduce the risk of his overall portfolio. In equilibrium, the price has to adjust downward and the expected return on the stock increases to induce other investors to hold more stocks.

Investors' optimal investment policies, $\theta_{i,t}$ and $y_{i,t}$, are both linear functions of $X_t$. During the night, the stock market is closed and investors hold their closing stock positions from the previous day, $\theta_{i,n_k}$ ($i = 1, 2$), for $t \in (n_k, t_{k+1})$. Investor $i$’s investment in his private technology and consumption not only depend on $X_t$ but also depends on $\theta_{i,n_k}$. During market open, investor $i$’s stock position can be expressed as follows:

$$\theta_{i,t} = h_{i,0}^{(0)} + h_{i,0}^{(1)} Y_{1,t} + h_{i,0}^{(2)} Y_{2,t}, \quad i = 1, 2.$$

---

13 In the current case of a periodic solution, values of the solution at (any) two points apart by the length of periodicity are equal.
As an example, we consider investor 1’s stock position $\theta_{1,t}$, which consists of three components. The first component $h_{1,0}^{(0)}$ gives his unconditional stock position. Under perfect symmetry between the two classes of investors (when $\sigma_1 = \sigma_2$ and $\omega = 1/2$), $h_{1,0}^{(0)}$ is simply 1, the stock share per capita. The second component $h_{1,0}^{(1)} Y_{1,t}$ arises from investor 1’s hedging demand. As his private technology changes with $Y_{1,t}$, he adjusts his stock position to hedge the risk of his private investments. The third component, $h_{1,0}^{(2)} Y_{2,t}$, arises from his market-making activity. As $Y_{2,t}$ changes, class-2 investors’ private technology changes and they adjust their stock positions for their hedging needs. In accommodating the hedging needs of class-2 investors, investor 1 takes the opposite position, which leads to the third component in his stock holding. Here, $h_{1,0}^{(1)}$ and $h_{1,0}^{(2)}$ characterize, respectively, the intensity of investor 1’s hedging and market-making activities in equilibrium.\footnote{The above interpretation of an investor’s stock holding is an over-simplification for ease in exposition. The economic forces behind each of the components (defined as part of the holding associated with a particular state variable) are more complicated. Take investor 1’s hedging component as an example. It originates from his hedging needs in response to technology shock $Y_{1,t}$. His hedging trade then changes the expected return on the stock, which in turn affects his stock holding. Moreover, since the expected return on the stock now becomes stochastic and correlated with the realized returns, investor 1 also uses the stock to hedge changes in future expected returns on the stock as well as his technology. The net dependence of his stock holding on $Y_{1,t}$ has contributions from all these channels. See Merton (1971, 1990) for general discussions of investors’ optimal asset demands when expected returns are stochastic.}

Lemma 1 shows that $h_t$ is discontinuous at $n_k$ ($k = 0, 1, \cdots$), the end point of each trading day. Since the state variables follow continuous processes, the jump in $h_t$ implies that investors change their stock positions (and their private investments) discretely right at the market close. This is not surprising given the discrete regime change at the close. The lack of trading opportunities when the market closes changes the risk of carrying stock positions overnight. Consequently, investors want to hold different hedging and market making positions overnight than during the day. In a continuous-time setting like the current one, investors can always maintain their active positions throughout the day and wait until the last instant to establish their closing positions. The stock price, however, remains continuous at the closing point as required by no-arbitrage [see, e.g., Huang (1985)].

### 3.2. Two Special Cases

Two special cases can help us to develop some intuition about the economy. To simplify our analysis, we let $\sigma_1 = \sigma_2 = \sigma_Y$ and $\omega = 1/2$ in this subsection. Thus, there is perfect symmetry between the two classes of investors.

#### A. Homogeneous Investors

When $\kappa_{12} = 1$, $Y_{1,t} = Y_{2,t} = Y_t$, $\forall \ t \geq 0$ (assuming that $Y_{1,0} = Y_{2,0}$) and the two classes of investors face the same investment opportunities, both public and private. In this case, all investors are identical, the market is effectively complete, and the equilibrium is simple [see,
Proposition 1 When $\sigma_1 = \sigma_2 = \sigma_\gamma$ and $\kappa_{12} = 1$, the economy has a unique linear, periodic equilibrium. The equilibrium stock price is

$$P_t = F_t - (\bar{\lambda}_0 + \bar{\lambda}Y_t), \quad t \in T$$

where $\bar{\lambda}_0 = \gamma(\bar{\sigma}_D^2 + \sigma_\gamma^2 + \bar{\lambda}^2 \sigma_\gamma^2)$, $\bar{\lambda} = \frac{\sigma_D \kappa_D \kappa_{q}}{\sigma_q (r + a_\gamma + \sigma_\gamma^2)}$, $v = \frac{1}{2\sigma_q^2} \left[-(r + 2a_\gamma) + \sqrt{(r + 2a_\gamma)^2 + 4\sigma_\gamma^2 / \sigma_q^2}\right]$ and $\bar{\sigma}_D^2 = \sigma_D^2 (1 - \kappa_{Dq}^2)$.

Clearly, $\bar{\lambda}_0 \geq 0$ and $\bar{\lambda} \geq 0$ (assuming $\kappa_{Dq} \geq 0$). The equilibrium price function is now independent of $T$ and $N$. Being homogeneous, investors always hold equal shares of the market (day or night) and there is no trading. The market open and close have no effect on investors' real investment and consumption decisions. Thus, the stock price function is independent of how the market opens and closes.

When $\sigma_\gamma = 0$, $P_t = F_t - \bar{\lambda}_00$ where $\bar{\lambda}_00 = \gamma(\bar{\sigma}_D^2 + \sigma_\gamma^2)$. This is the case with constant investment opportunities. The expected excess return on the private technologies (same for all investors) is zero, and the expected excess share return on the stock is $e_t = r\bar{\lambda}_0$. At zero expected excess returns, the private technologies attract negative investments because investors use them to hedge the risk of their stock investments. As a result, the total risk of the stock becomes $\bar{\sigma}_D^2 + \sigma_\gamma^2$ instead of $\sigma_D^2 + \sigma_\gamma^2$. $\bar{\lambda}_00$ gives the risk discount on the stock which is proportional to the aggregate risk-aversion and the total risk of the stock.

When $\sigma_\gamma > 0$, investors face stochastic private investment opportunities. When the expected excess return on private technologies is high (low), they invest more (less) in it. They would like to change their stock positions accordingly. This, however, is not achievable given that all investors are identical. As a result, the stock price has to adjust and an additional premium is demanded on the stock in equilibrium. In fact, the unconditional expected excess share return on the stock, $r\bar{\lambda}_0$, is now greater than $r\bar{\lambda}_00$, its value under constant investment opportunities. The difference, $\gamma\bar{\lambda}^2\sigma_\gamma^2$, is the extra premium associated with stochastic private investment opportunities, which increases with the risk-aversion, the absolute value of $\kappa_{Dq}$ and $\sigma_\gamma^2$. The conditional expectation of excess share returns on the stock is $e_t = r\bar{\lambda}_0 + (r + a_\gamma)\bar{\lambda}Y_t$. It depends positively on the expected excess return on private technologies $Y_t$.

B. Heterogeneous Investors with Permanent Market Closure

When investors are heterogeneous (i.e., $\kappa_{12} \neq 1$), they do use the stock market to hedge the risk of their private investments. Here, we only consider the special case when $\kappa_{12} = -1$ and $N = \infty$. In this case, $Y_{1,t} = -Y_{2,t}$. Since there is perfect symmetry between the two classes of investors, shocks to individual investors' private technologies exactly cancel at the aggregate level and there is no aggregate uncertainty about the investors' private investment opportunities. $N = \infty$ implies that the stock market opens during $[0, T]$ and then closes permanently thereafter. We
have the following result:

**Proposition 2** When $\omega = 1/2$, $\sigma_1 = \sigma_2 = \sigma_T$, $\kappa_{12} = -1$ and $N = \infty$, the economy has a unique linear equilibrium. The equilibrium stock price during market open is $P_t = F_t - \lambda_0$ where $\lambda_0 = \tilde{\lambda}_{00} + (\tilde{\lambda}_0 - \tilde{\lambda}_{00})e^{-r(T-t)}$. The investors' stock holdings are

$$\theta_{i,t} = \begin{cases} 
1 - |\bar{h}|Y_{i,t}, & t \in [0, T) \\
1 - (\bar{\lambda}/\tilde{\lambda}_0)Y_{i,T}, & t = T \\
\theta_{i,T}, & t > T 
\end{cases}$$

where $\bar{h} = -\kappa_Pq/(r\sigma_T)$. Furthermore, $0 < \bar{\lambda}/\tilde{\lambda}_0 < |\bar{h}|$.

Facing different private technologies, investors trade in the stock market to hedge the risk of their private investments. Since the risk exposures of the two classes of investors are perfectly negatively correlated, so are their hedging needs. Consequently, their hedging trade does not move the stock price.\(^{15}\)

Far away from the closure point, i.e., $t \ll T$ (assuming $T \gg 1/r$), the economy is close to the situation of continuous trading. The risk discount $\lambda_0 \approx \tilde{\lambda}_{00}$, its value in the absence of changes in private technologies. In this case, investors are able to mutually insure the hedgable risk from their private investments completely. After the market closes, the economy enters artauxky. The lack of trading after the closure makes the investors bear the risk of their private investments. As in the case of homogeneous investors, shadow prices of the stock now co-varies with each investor's private investment opportunity given the correlation of their payoffs. This increase in volatility of the (shadow) price makes the stock more risky as perceived by the investors when the market closes. In particular, the risk discount reaches $\tilde{\lambda}_0$ at the close, its value under no trading.\(^{16}\) As $t$ approaches $T$ (from left), $\lambda_0$ approaches $\tilde{\lambda}_0$ from $\tilde{\lambda}_{00}$.

As implied by Lemma 1, there is a discrete change in investors' stock positions at $T$. Before the market close, they trade in the same fashion as in the case of continuous trading. In addition to their fair share of the market (one share per capita), they take on opposite positions to mutually hedge the risk from their private investments. Their trading behavior is not affected at all as the closure time approaches. At the point of market closure, they dis-continuously revise their hedging positions in the stock. In particular, Proposition 2 explicitly shows that investors cut their hedging positions. As mentioned above, the lack of trading makes the stock more risky as perceived by the investors when the market closes. Thus, they discretely reduce their hedging positions at the close.

\(^{15}\)This result is obtained under the assumption that shocks to $Y_{i,t}$ are uncorrelated with the shocks to the stock's cash flows. In the more general case where they are correlated, there is an additional hedging premium since now stock provides a vehicle to hedge the idiosyncratic technological shocks.

\(^{16}\)When $Y_{i,T} = 0$, for example, $\theta_{i,T} = 1$, and (from Proposition 1) investor $i$'s shadow price of the stock is $P^*_{i,t} = F_t - \tilde{\lambda}_0 - \tilde{\lambda}Y_{i,t}$ for $t > T$. At the closure point, the market price equals their shadow prices, which is $P_T = F_T - \tilde{\lambda}_0$. Thus, $P_T = F_T - \tilde{\lambda}_0$. For arbitrary value of $Y_{i,T}$, the intuition is the same.
3.3. Trading and Return Patterns under Periodic Market Closures

We now analyze the equilibrium with periodic market closures under symmetric information. The intuition from the special cases suggests the following picture. Market closures give rise to time variation in investors' hedging demand during market open. After the market open but away from the close, investors can trade continuously and are more aggressive in taking on hedging positions as their private investments change. At the close, however, the investors discretely reduce their hedging positions as they perceive higher risk in holding overnight positions. As a result, $\lambda_0$ takes a higher value at the close than at the open, and $\lambda$ takes a lower value at the close than at the open.

In the following discussions, we use numerical examples to illustrate the predictions of the model. We choose specific parameter values in the numerical illustrations that are "reasonable". For example, the daily interest rate is set at 0.1% and the daily volatility of share returns at 0.05. We also choose the risk-aversion parameter to yield a daily excess share return of 0.002.\footnote{Since we use share returns in this paper, one should be careful in relating the return numbers here to the observed numbers on the rate of return (which is the return on one dollar instead of one share). We have not tried to calibrate the model to actual data, although it is possible [see, e.g., Campbell and Kyle (1993)].}

For the remaining degrees of freedom, the choice for the parameter values is quite arbitrary when the qualitative features of the results are not sensitive to the particular values. Given that the equilibrium is periodic, in all our illustrations, we only present the patterns during a single trading period. The opening and closing points are, respectively, $\tau = 0$ and $\tau = T$.

In Figure 1-2(a) and (b), we plot the intraday patterns of $\lambda_0$ and $\lambda$, respectively. Clearly, $\lambda_0$ increases during the trading period, indicating a decreasing price over the day. The lower price at the close reflects the increase in risk to hold the stock overnight. $\lambda$ decreases during the day. Hence, the price is least sensitive at the close to the investors' technological shocks as they halt their allocational trading.

The investors' intraday holdings of the stock and the private investments are similar to those in the case of permanent market closure considered in Section 4.2. Figure 1-2(c) shows that during the day, investors are active in taking stock positions in response to shocks to their private investment opportunities and $|h|$ is high. At the end of the day, they drastically reduce their hedging positions as the discrete drop in $|h|$ indicates. The value of $h$ maintains more or less at the same level after the opening since the effect of closure is less significant. As the closing point approaches, investors start to adjust their positions. However, they do not necessarily adjust the positions in the same direction as their closing trades. This is partially related to how the price adjusts to approach its value at the close.

From the intraday stock price, we can calculate the mean and the variance of instantaneous stock returns. Figure 1-3(a) shows that the mean excess share return $e$ is decreasing during the day. Since the variability in the underlying state variables is constant over time, the intraday variability of the stock returns is determined purely by $\lambda$, the sensitivity of price to aggregate
Figure 1-2: Intraday coefficients of the price function, $\lambda_0$ and $\lambda$, and sensitivity of stock positions to technological shocks, $h$, under symmetric information. The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.5$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $a_c = .85$, $\sigma_c = 0.08$, $\sigma_q = 0.5$, $a_r = 0.25$, $a_1 = a_2 = 7$, $\kappa_{Dq} = 0.5$, $\kappa_{12} = 0$.

given that $\lambda$ decreases during the day as shown in Figure 1-2(b), the instantaneous price volatility is decreasing in the day as Figure 1-3(b) shows.

Figure 1-3: The mean and standard deviation of instantaneous excess share returns on the stock under periodic market closure and symmetric information, plotted against time over a trading day. The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.5$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $a_c = .85$, $\sigma_c = 0.08$, $\sigma_q = 0.5$, $a_r = 0.25$, $a_1 = a_2 = 7$, $\kappa_{Dq} = 0.5$, $\kappa_{12} = 0$.

We now briefly discuss how the equilibrium change as we change different parameters. For a wide range of parameter values we have explored, the qualitative feature of the equilibrium is the same as shown in Figure 1-2. The quantitative features can change with parameter values.

A parameter of particular interest is $a_r$. As we increase $a_r$, shocks to investors’ private technologies become more transitory. We then expect that hedging becomes less costly and there is more hedging trade. Indeed, Figure 1-4(b) shows that the level of $\lambda$ decreases with $a_r$ and the stock price becomes less sensitive to technology shocks. Figure 1-4(c) further shows that the level of hedging trade (measured by $|h|$) increases with $a_r$. Increasing hedging benefits as well as decreasing price volatility both reduce the risk discount on the stock, $\lambda_0$, as $a_r$ increases. It is important to note how the time variation in $|h|$ and $\lambda$ change with $a_r$. For more transitory technological shocks, investors can better foresee future needs to reduce their hedging positions.
Given that they cannot trade after the market closes, the reduction in their hedging demand at the close is larger. This is illustrated by both the larger drop in $|h|$ at the close and larger decrease in $\lambda$ during the day as $a_\gamma$ increases.

\[ \lambda_0 - \hat{\lambda}_{00} \quad \lambda \quad h \]

**Figure 1-4:** Intraday coefficients of the price function, $\lambda_0$ and $\lambda$, and sensitivity of stock positions to technological shocks, $h$, for different values of $a_\gamma$ under periodic market closure and symmetric information. The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.5$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $a_G = 0.85$, $\sigma_G = 0.08$, $\sigma_a = 0.5$, $\sigma_1 = \sigma_2 = 7$, $\kappa_{DG} = 0.5$, $\kappa_{12} = 0$.

Other parameters also change the quantitative feature of the equilibrium. For example, when $a_G$ decreases, fluctuations in dividend growth become more persistent. The current stock price then becomes more sensitive to changes in current dividend growth. As a result, the stock price becomes more volatile and the stock as a hedging vehicle becomes riskier. Thus, the level of hedging decreases and so does $\lambda$. The parameter $\sigma_G$ has a similar effect. For brevity, these results are not shown here.

Given the abrupt reduction in investors’ stock positions at the close, the trading activity at the close is abnormally high. Also, the information accumulation during the night gives rise to large trading activity at the open. Thus abnormally high trading activities at the open and the close are the simple outcomes of periodic market closures in our model.

We now consider the discrete returns. Given that $\lambda_0$ is a periodic function of time on $T$, the expected simple return over a calendar day must be zero. Since the stock price is lower at the close than at the open, i.e., $\lambda_0(0) < \lambda_0(T)$, the expected return over the day is lower than the expected return over the night.

**Proposition 3** $E[R^{co}] = E[R^{cc}] = 0$. For $\lambda_0(0) < \lambda_0(T)$, $E[R^{cc}] \leq 0 \leq E[R^{co}]$.

$\lambda(0) > \lambda(T)$ implies that the stock price is more sensitive to the aggregate technological shocks at the open than at the close. Thus the open-to-open returns are more volatile than the close-to-close returns. Also, since the stock price linearly depends on the underlying state variables, which are mutually independent, the open-to-close returns and the open-to-close returns have the same volatility. Note that there is continuous trading from open to close while there is no trading from close to open. This result suggests that under symmetric information, the
volatility of discrete simple returns over a given period may not depend on whether or not there is trading within the period per se.

**Proposition 4** For $\lambda(0) > \lambda(T)$, $\text{Var}[R^\infty] > \text{Var}[R^c]$. For $T = N$, $\text{Var}[R^c] = \text{Var}[R^{co}]$.

Table 1.2 reports the mean and variances of discrete simple returns for the given set of parameter values. Clearly, the mean return over the closing period is positive while the mean return over the opening period is negative, and the return volatility is the same over closing periods as over the opening periods, both contradicting the empirical findings.

**Table 1.2: Unconditional Moments of Discrete Returns under Symmetric Information**

<table>
<thead>
<tr>
<th>Returns</th>
<th>Mean (×10⁻⁴)</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^{oo}$</td>
<td>0.00</td>
<td>0.0424</td>
</tr>
<tr>
<td>$R^{cc}$</td>
<td>0.00</td>
<td>0.0421</td>
</tr>
<tr>
<td>$R^{oc}$</td>
<td>-0.12</td>
<td>0.0142</td>
</tr>
<tr>
<td>$R^{co}$</td>
<td>0.12</td>
<td>0.0142</td>
</tr>
</tbody>
</table>

The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.5$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $\alpha_D = 0.85$, $\sigma_G = 0.08$, $\sigma_Q = 0.5$, $\alpha_Y = 0.25$, $\sigma_1 = \sigma_2 = 7$, $\kappa_{DQ} = 0.5$, $\kappa_{12} = 0$.

In summary, in the case of symmetric information, the volatility of the stock returns decreases over the trading periods. High trading activity occurs at the open and close. The open-to-open return is more volatile than the close-to-close return. The stock price decreases during the trading periods, giving higher return over the non-trading periods than over the trading periods. The return volatility is the same for the periods of market open and the periods of market closure. The existence of asymmetric information changes these results as the next section shows.

4. The Case of Asymmetric Information

We now consider the equilibrium in the case of asymmetric information, in which $I_{1,t} = I_t$ and $I_{2,t} = I_{2,0} \otimes \{P_s, D_s, Y_{2,s}, q_s : 0 \leq s \leq t\}$.\(^{18}\) In this case, class-1 investors, observing the true dividend growth and technological shocks to both classes of investors, have full information about the economy. Class-2 investors, observing their own technological shocks and realized dividends and prices of the stock, have only partial information about the economy. Thus, in this section, we also refer to class-1 investors as the informed investors and class-2 investors as the uninformed investor. The informed investors now trade the stock for both allocational and

\(^{18}\)Since we only consider the periodic equilibrium of the model, we assume that class-2 investors' prior information $I_{2,0}$ is consistent with what the periodic equilibrium implies.
informational reasons. They trade not only to meet their own or other investors’ hedging needs, but also to speculate on their private information. The uninformed investors, however, trade only for allocational reasons, either to meet their own hedging needs or to accommodate class-1 investors’ hedging needs.

We first present the general solution to equilibrium under asymmetric information. We then analyze the return and trading patterns during the trading periods and contrast them with those under symmetric information. We find that under asymmetric information, periodic market closures can generate U-shape patterns in the mean and variance of the stock return during the trading periods. The mean and variance of the stock return can be higher over the trading periods than over the non-trading periods. Furthermore, with periodic market closures, the market prices of the stock can be more informative about its future payoffs than when the market is continuously open.

4.1. The General Solution

In the current model, $G_t$, $Y_{1,t}$ and $Y_{2,t}$ fully determine future asset payoffs. Given that the uninformed investors do not directly observe $G_t$ and $Y_{1,t}$, they have to form expectations about them. Let $\Psi_t = [G_t, Y_{1,t}, Y_{2,t}, \tilde{G}_t, \tilde{Y}_{1,t}]'$ denote the set of conditional expectations of these variables by both the informed and uninformed investors. Similar to the case of symmetric information, we consider a linear, periodic equilibrium. In particular, we seek an equilibrium of the following form

$$P_t = L(\Psi_t; t) = \lambda_G G_t + \lambda_{\tilde{G}} \tilde{G}_t - \lambda_0 - \lambda_Y Y_{1,t} - \lambda_{Y2} Y_{2,t} - \lambda_{Y1} \tilde{Y}_{1,t}, \quad t \in T$$

(15)

where the $\lambda$’s are time dependent with periodicity $T+N$. Let $X_t = [1, Y_{1,t}, Y_{2,t}, \tilde{G}_t, \tilde{Y}_{1,t}]'$ and $X_{i,t} = E_{i,t}[X_t]$ (i.e., $X_{1,t} = X_t$ and $X_{2,t} = [1, \tilde{Y}_{1,t}, Y_{2,t}]'$). The stock price function in (15) can then be re-expressed as follows:

$$P_t = F_t - \lambda X_t, \quad t \in T$$

(16)

where $\lambda = [\lambda_0, \lambda_1, \lambda_2, -\frac{1}{r+\alpha G} - \lambda_0, 0]$ (defined on $T$) is time-dependent with periodicity $T+N$.\(^{19}\)

In what follows, we first show that in an equilibrium of the above form, $\Psi_t$ is indeed a sufficient-statistic set of investors’ information on future asset payoffs. We then compute the uninformed investors’ conditional expectations, and derive investors’ optimal policies given the price function (16). Finally, we discuss the existence of a linear, periodic equilibrium in the form of (16).

---

\(^{19}\)Equation (15) can be rewritten as $P_t = (\lambda_G + \lambda_{\tilde{G}}) G_t - \lambda_0 - (\lambda_{Y1} - \lambda_{\tilde{Y}1}) Y_{1,t} - \lambda_{Y2} Y_{2,t} - \lambda_{\tilde{G}} (G_t - \tilde{G}_t) - \lambda_{\tilde{Y}1} (Y_{1,t} - \tilde{Y}_{1,t})$. Since $P_t \subseteq 2_2, t$ from (15), $\lambda_G G_t - \lambda_{Y1} Y_{1,t} = \lambda_G G_t - \lambda_{Y1} \tilde{Y}_{1,t}$ for $t \in T$, i.e., $\lambda_G (G_t - \tilde{G}_t) = \lambda_{Y1} (Y_{1,t} - \tilde{Y}_{1,t})$. Thus, term $(Y_{1,t} - \tilde{Y}_{1,t})$ can be eliminated from the price function. The same arbitrage argument as in the case of symmetric information further requires that $\lambda_G + \lambda_{\tilde{G}} = \frac{1}{r+\alpha G}$. Thus, we obtain the form in (16) with $\lambda_1 = \lambda_{Y1} - \lambda_{\tilde{Y}1}$ and $\lambda_2 = \lambda_{Y2}$. 

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Let $Z_t = [G_t, Y_{t,t}]'$ be the vector of state variables that the uninformed investors do not observe and $\tilde{Z}_t = E_{2,t}[Z_t]$, $\omega_t = E_{2,t} \left[ (Z_t - \tilde{Z}_t)(Z_t - \tilde{Z}_t)' \right]$ their conditional mean and variance of $Z_t$, respectively. Then,

$$dZ_t = a_z Z_t dt + b_z dw_t$$

where $a_z = \text{diag}(a_G, a_Y)$ and $b_z = \text{stack}(b_G, b_Y)$. For class-2 investors observing $P_t$ of the form in (16) is equivalent to observing $\tilde{P}_t = \lambda_z Z_t$, where $\lambda_z = [\lambda_G, -\lambda_Y]$. Let $S_t = [\tilde{P}_t, D_t, Y_{2,t}]'$, $t \in T$, denote the signal they receive during the day, $S_t^* = [D_t, Y_{2,t}]'$, $t \in N$, the signal during the night. We have

$$t \in T : \quad dS_t = a_s Z_t dt + b_s dw_t$$

$$t \in N : \quad dS_t^* = a_s^* Z_t + b_s^* dw_t$$

where $a_s = \text{stack}(\lambda_z - \lambda_z a_z, [1, 0], [0, 0])$, $b_s = \text{stack}(\lambda_z b_z, b_D, b_Y)$, $a_s^* = \text{stack}([1, 0], [0, 0])$ and $b_s^* = \text{stack}(b_D, b_Y)$. Given the information of the uninformed investors, we can compute their conditional distribution of the unobserved state variables. Since both the state vector and the signal vector follow Gaussian processes, the conditional distribution is also Gaussian, which is fully characterized by its mean and variance [see, e.g., Liptser and Shiryayev (1974)]. We have the following result:

**Lemma 3** In a linear, periodic equilibrium in the form of (16), $\tilde{Z}_t$ and $\omega$ are governed by

$$t = t_k : \quad \left\{ \begin{array}{l} \tilde{Z}_{t_k} = \tilde{Z}_{t_k} + [\lambda_z(o(t_k^-) \omega_z(t_k))^{-1} o(t_k^-) \lambda_z'(t_k)] (P_{t_k} - E_{2,t_k^-} [P_{t_k}]) \\ \omega(t_{k+1}) = \omega(t_{k+1}) - o(t_{k+1}^-) \lambda_z'(t_{k+1}) \lambda_z(t_{k+1}) \omega(t_{k+1}) \end{array} \right.$$  

$$t \in [t_k, n_k] : \quad d\tilde{Z}_t = a_z \tilde{Z}_t dt + (\sigma_{zz} + o a_z) \sigma_z^{-1} (dS_t - E_{2,t}[dS_t])$$

$$\dot{o} = - (a_z o + o a_z) + \sigma_{zz} - (\sigma_{zz} + o a_z) (\sigma_z^{-1} a_z o)$$  

(17)

$$t = n_k : \quad \left\{ \begin{array}{l} \tilde{Z}_{n_k} = \tilde{Z}_{n_k}^+ \\ o(n_k) = o(n_k^+) \\ \right.$$

$$t \in (n_k, t_{k+1}) : \quad \left\{ \begin{array}{l} d\tilde{Z}_t = a_z \tilde{Z}_t dt + (\sigma_{zz} + o a_z) \sigma_z^{-1} (dS_t^* - E_{2,t}[dS_t^*]) \\ \dot{o} = - (a_z o + o a_z) + \sigma_{zz} - (\sigma_{zz} + o a_z) (\sigma_z^{-1} a_z o) \\ \right.$$  

and $\omega(t_{k+1}) = \omega(t_k)$. Furthermore, $\Psi_t$ follows a Gaussian Markov process.

Clearly, $\omega$ is deterministic (but time-dependent and periodic). Under the information of each class of investors, future payoffs of the stock and private technologies have Gaussian distributions. Their conditional means are fully specified by $\Psi_t$, which itself follows a Gaussian Markov process. Thus, $\Psi_t$ is a sufficient statistic for the investors’ information at $t$ on future asset payoffs.

Lemma 3 clearly shows the impact of stock market closures on the information flow in the economy and investors’ expectations. During the market open, the uninformed investors observe
the equilibrium stock price as given by (16). As mentioned earlier, \( P_t \) reveals \( \tilde{P}_t = \lambda_0 G_t - \lambda_1 Y_{1,t} \), which is a weighted sum of the two unobservables. Consequently, \( \tilde{P}_t = \lambda_0 \hat{G}_t - \lambda_1 \hat{Y}_{1,t} \) and \( \hat{G}_t \) and \( \hat{Y}_{1,t} \) are linearly dependent given \( P_t \). During the market close, the stock price as a source of information is no longer available. There is no simple linear dependence between \( \hat{G}_t \) and \( \hat{Y}_{1,t} \) as during market open. The uninformed investors update their expectations of \( G_t \) and \( Y_{1,t} \) separately based on realized dividends and shocks to their own technologies. When the market re-opens, the information revealed by the opening price re-enforces the relation between \( \hat{G}_t \) and \( \hat{Y}_{1,t} \), which causes them to change dis-continuously at the open. At the close, however, the investors’ expectations evolve continuously.

Given the stock price function and the investors’ expectations, we now consider their optimal policies. From (16) and (17), we can express the excess share return on the stock as follows

\[
dQ_t = a_Q X_{i,t} dt + \sigma_Q dw_t, \quad t \in T
\]

where \( a_Q \) and \( \sigma_Q \) are periodic, time-dependent matrices of proper rank. The expected excess share return for class-\( i \) investors (\( i = 1, 2 \)) is \( e_{i,t} = a_{i,Q} X_{i,t} \), which only depends on \( X_{i,t} \). (Here, \( a_{1,Q} = a_Q \) and \( a_{2,Q} = a_Q \) without the last element.) Since \( X_{i,t} \) follows a Gaussian Markov process under \( T_{i,t} \), \( i = 1, 2 \) (see appendix), it fully characterize investor \( i \)'s current and future investment opportunities. Consequently, investor \( i \)'s value function and optimal policies only depend on \( X_{i,t} \). Define \( a_{i,Q} = 1_{1,i+1}^{(1)} \), \( a_{i,W} = \text{stack}(a_{i,Q}, a_{i,Q}) \) and \( b_{i,W} = \text{stack}(b_{i,Q}, b_{i,Q}) \). We have Lemma 4.

**Lemma 4** In a linear, periodic equilibrium of the form in (16), investor \( i \)'s optimal policies and the value function are

\[
\begin{align*}
\begin{pmatrix}
\theta_{i,t} \\
y_{i,t}
\end{pmatrix} &= 
\begin{pmatrix}
h_{i,\theta} \\
h_{i,y}
\end{pmatrix} X_{i,t} \\
c_{i,t} &= -\frac{1}{\gamma} \ln \gamma + r W_{i,t} + \frac{1}{2\gamma} X_{i,t}^t v_i X_{i,t} \\
J_{i,t} &= -\exp \left\{-\rho t - r \gamma W_{i,t} - \frac{1}{2} X_{i,t}^t v_i X_{i,t} \right\}
\end{align*}
\]

\[
\begin{align*}
y_{i,t} &= h_{i,y}^* X_{i,t} + h_{i,\theta}^* \theta_{i,n_k} \\
c_{i,t} &= -\frac{1}{\gamma} \ln \gamma + r W_{i,t}^* + \frac{1}{2\gamma} \left(X_{i,t}^t v_i^* X_{i,t} - 2\theta_{i,n_k} u_i^* X_{i,t} - \theta_{i,n_k}^2 u_i^* u_i^* \right) \\
J_{i,t} &= -\exp \left\{-\rho t - r \gamma W_{i,t}^* - \frac{1}{2} X_{i,t}^t v_i^* X_{i,t} - \theta_{i,n_k} u_i^* X_{i,t} - \frac{1}{2} \theta_{i,n_k}^2 u_i^* \right\}
\end{align*}
\]

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where \( i = 1, 2, \) \( W_{i,t}^* = W_{i,t}^0 + \theta_{i,n_k} F_t, \) and

\[
\begin{align*}
t \in [t_k, n_k) : & \quad \begin{bmatrix} h_{i, \theta} \\ h_{i, y} \end{bmatrix} = \frac{1}{T^2} \sigma_{i,w}^{-1} (a_{i,w} - \sigma_{i,w} x_i) \\
& \quad \begin{bmatrix} \frac{1}{T} \sigma_{i,x}^{-1} (r \gamma \lambda_i - u_i^*) \\ \frac{1}{T} \sigma_{i,q}^{-1} (r \gamma \sigma_{i,q}^* - \sigma_{i,x}^* u_i^*) (r \gamma \lambda_i - u_i^*) \end{bmatrix} \\

\end{align*}
\]

\[
t = n_k : \quad \begin{bmatrix} h_{i, \theta} \\ h_{i, y} \end{bmatrix} = \begin{bmatrix} \frac{1}{T} \sigma_{i,q}^{-1} (a_{i,q} - \sigma_{i,x}^* v_i^*) \\ \frac{1}{T} \sigma_{i,q}^{-1} (r \gamma \sigma_{i,q}^* - \sigma_{i,x}^* u_i^*) (r \gamma \lambda_i - u_i^*) \end{bmatrix}
\]

\[
t \in (n_k, t_{k+1}) : \quad h_{i,y} = \frac{1}{T} \sigma_{i,q}^{-1} (a_{i,q} - \sigma_{i,x}^* v_i^*) , \quad h_i^* = \frac{1}{T} \sigma_{i,q}^{-1} (r \gamma \sigma_{i,q}^* - \sigma_{i,x}^* u_i^*)
\]

Here, \( v_i \) and \( v_i^* \) (both symmetric), \( u_i^* \), \( u_{i,0}^* \) are matrices of proper order define on \( T \) and \( N \), respectively, with periodicity \( T+N \).

As in the case of symmetric information, given \( \lambda \) (and \( o \)), \( v_i \), \( v_i^* \), \( u_i^* \) and \( u_{i,0}^* \) are determined by a set of first-order ODE’s, which is specified in Appendix C, with periodic boundary conditions.

Investor \( i \)'s optimal stock holding is a linear function of \( X_{i,t} \). An informed investor's stock position takes the following form:

\[
\theta_{1,t} = h_{1,\theta}^{(0)} + h_{1,\theta}^{(1)} v_{1,t} + h_{1,\theta}^{(2)} v_{2,t} + h_{1,\theta}^{(3)} (G_t - \tilde{G}_t).
\]

The first three terms correspond to, respectively, the unconditional, hedging and market-making components of his stock position, as in the case of symmetric information. The fourth term, \( h_{1,\theta}^{(3)} (G_t - \tilde{G}_t) \), arises from his private information about future stock payoffs. It gives the speculative component in his stock position. When \( G_t - \tilde{G}_t > 0 \), the uninformed investors are under estimating future dividend growth. The informed investor then goes long in the stock to capture expected returns from future appreciation of the stock. \( h_{1,\theta}^{(3)} \) characterizes the intensity of the informed investors' speculative trading. An uninformed investor's stock holding now takes the form

\[
\theta_{2,t} = h_{2,\theta}^{(0)} + h_{2,\theta}^{(1)} \tilde{v}_{1,t} + h_{2,\theta}^{(2)} \tilde{v}_{2,t}.
\]

Since the uninformed investor has no private information, his stock holding has only the unconditional component, the hedging component and the market-making component. Given that he does not observe the informed investors' trading motives, his market-making trade is based on his expectation about the informed investors' hedging needs. Part of his market-making trade actually corresponds to the informed investors' speculative trade, who take the other side.

The existence of an equilibrium in the form of (16) requires the existence of a (periodic) vector function of time \( \lambda \) such that the stock market clears (\( \forall t \in T \)). As in the case of symmetric information, the market clearing condition and the periodic condition can be reduced to a two-point boundary-value problem for \( \lambda \) (together with \( o, v_i, v_i^*, u_i^*, u_{i,0}^*, i = 1, 2 \)). Its solution leads to the equilibrium of the form in (16). We have the following existence result:

**Theorem 2** For \( \omega \) close to one, a linear, periodic equilibrium of the form in (16) exists, in
which the uninformed investors’ expectations are given by Lemma 3 and the optimal policies of both classes of investors are given by Lemma 4.

Here, the condition that \( \omega \) is close to one comes from the specific approach we use in the proof as opposed to economic reasons (see Appendix C). The proof relies on a continuity argument. It is first shown that at \( \omega = 1 \), a solution to the given system exists. Since the system is smooth with respect to \( \omega \), it is then shown that a solution also exists for \( \omega \) close to one. The proof itself, however, does not specify how close it has to be. In numerically solving the system, we always start with the solution at \( \omega = 1 \) and then obtain solutions for values of \( \omega \) close to one. Iteratively, we arrive at the solutions at desired values of \( \omega \).

In the remainder of this section, we examine the trading and return patterns under periodic market closures in the presence of asymmetric information. We first illustrate the nature of equilibrium using a specific set of parameter values, which is similar to the one used in Section 4. In particular, this set of parameter values generates return patterns consistent with the empirical ones. We then examine how different parameters change the return patterns. This exercise of comparative statics allows us to further understand the important factors in determining the return generating process. Finally, we discuss the discrete returns.

4.2. Trading and Return Patterns

Under asymmetric information, market closures directly affect the information flow in the economy, which determines the information asymmetry between the two classes of investors. There are two main effects: First, closures take away market prices as a source of information from the uninformed investors. Thus, the information asymmetry increases during market close as trading halts and tends to decrease during market open as trading resumes. Second, closures reduce the informed investors’ hedging trade during the open. The change in the informed investors’ hedging trade changes the information content in market prices. Note that in the current model, it is the informed investors’ hedging trade that prevents stock prices from fully revealing their speculative trade. By reducing the informed investors hedging trade, market closures make stock prices more informative about future payoffs, thereby reduce the information asymmetry between the two classes of investors.

A. Time Variation in Information Asymmetry

The degree of information asymmetry between the two classes of investors on the stock can be measured by the uninformed investors' conditional variance of the dividend growth, \( o_{11} \equiv E_{2,1} \left[ (G_t - \hat{G}_t)^2 \right] \). Figure 1-5 plots the time path of \( o_{11} \) through the course of a calendar day, where the time interval \((-0.5, 0)\) corresponds to the night and \([0, 0.5]\) corresponds to the following day. The parameters are set at the same values as in figures in the previous section except that the fraction of class-1 investors (the informed investors) is now \( \omega = 0.05 \) (instead of 0.5). Due to the periodicity of the equilibrium, \( o_{11} \) takes the same value at \( \tau = -0.5 \) and 0.5.
During the night, the uninformed investors lose market prices as a source of information. Given the continuous arrival of new information for the informed investors, the information asymmetry between the two classes of investors increases monotonically over the night. Trading at the open partially reveals the private information of the informed investors accumulated over the night and causes a discrete drop in $\sigma_{11}$. The continuous trading after the open further reveals the informed investors' private information and tend to reduce $\sigma_{11}$. At the close, two off-setting forces are at work. On the one hand, the informed investors cut back their hedging positions as discussed in the previous section. The reduction in the informed investors hedging positions also unveil their speculative positions and their private information. This tends to further decrease the information asymmetry at the close. On the other hand, the informed investors also cut back their speculative positions since the risk of carrying stock positions overnight is high. This makes the stock prices less informative about their information and increases the information asymmetry at the close. In the case shown in Figure 1-5, the former effect dominates and $\sigma_{11}$ decreases faster at the close.

**B. Stock Price Coefficients**

Given the time pattern of $\sigma_{11}$, we now examine the effect of time-varying information asymmetry on equilibrium prices. From (16), we can rewrite the equilibrium stock price as follows:

$$P_t = \hat{F}_t - \lambda_0 - \lambda_1 Y_{1,t} - \lambda_2 Y_{2,t} + \lambda_0 (G_t - \hat{G}_t), \quad t \in T$$

(19)

In the absence of informed investors, the stock price takes the form $P_t = \hat{F}_t - \lambda_0 - \lambda_2 Y_{2,t}$ where $\hat{F}_t$ is the fundamental value of the stock perceived by the uninformed investors and $\lambda_0 + \lambda_2 Y_{2,t}$ is

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20This behavior of the equilibrium near the close depends on the competitive assumption. Under the competitive assumption, the informed investors revise their hedging positions without taking into account the fact that revisions also reveal their speculative positions. In a non-competitive model [e.g. Kyle (1985)], the informed investors do take into account the information impact of their hedging activities, and optimally trade off the benefits and costs of cutting back hedging positions at the close.
the risk discount. When the informed investors are present, they make speculative trades based on their private information. For example, when the uninformed investors underestimate the dividend growth, $G_t - \hat{G}_t > 0$, the informed investors purchase the stock in expectation of future price increases, thereby drive up the price. This gives the term $\lambda_G(G_t - \hat{G}_t)$ in the stock price where $\lambda_G > 0$. The informed investors' hedging activities further gives the term $\lambda_1 Y_{1,t}$ in the risk discount. Figure 1-6 illustrates the time pattern of price coefficients $\lambda_0$, $\lambda_1$, $\lambda_2$ and $\lambda_G$ for the given set of parameters.

![Graphs](image)

**Figure 1-6:** Intraday price coefficients under partial revelation. The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.05$, $\gamma = 1000$, $r = 0.001$, $a_d = 0.08$, $a_G = 0.85$, $\sigma_G = 0.08$, $\sigma_Q = 0.5$, $a_y = 0.25$, $\sigma_1 = \sigma_2 = 7$, $\kappa_{DQ} = 0.5$, $\kappa_{12} = 0$.

The coefficient $\lambda_0$ gives the mean risk discount of the stock. After the open, more private information is revealed through trading, hence the uninformed investors' uncertainty about the stock's future payoffs decreases. Consequently, a lower risk discount is required in equilibrium, and $\lambda_0$ decreases after the market opens. Near the close, the risk discount decreases faster due to faster information revelation as the informed investors reduce their hedging positions. $\lambda_1$ decreases during the day as in the case of symmetric information, reflecting the anticipated reduction in the informed investors' hedging positions at the close. $\lambda_2$ now exhibits a U-shaped pattern, indicating active hedging activities by the uninformed investors around both the open and the close. $\lambda_G$ reflects the impact of the informed investors' informational trading on the equilibrium price. It increases monotonically during the day, indicating that the stock price is more informative about the dividend growth at the end of the day than at the beginning. This again is related to the decrease in the informed investors' allocational trading at the close.

**C. Investors' Trading Strategies**

In order to better understand the equilibrium behavior, let us now examine the investors' trading policies. Figure 1-7 illustrates the different components of the informed investors' stock holdings. Figure 1-7(a) gives an informed investor's mean stock position in excess of his market share. Two

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21 Another way to interpret the stock price is to rewrite it as $P_t = \lambda_G G_t + \left(\frac{1}{r+\lambda_G} - \lambda_G\right) \hat{G}_t - (\lambda_0 + \lambda_1 Y_{1,t} + \lambda_2 Y_{2,t})$. The first two terms then reflect how, respectively, the expectations of informed and uninformed investors on future stock payoffs affect the stock price. Thus, we expect $0 < \lambda_G < \frac{1}{r+\lambda_G}$. 

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factors affect the investors' mean stock position. The first factor is information asymmetry. Since the informed investors have better information about the dividend growth than the uninformed investors, they perceive lower risk for holding the stock, thereby hold more stock shares on average. Their excess stock positions tend to decrease after the market opens as information asymmetry decreases, and drop discretely at the close. The second factor is the allocation of risk across the two classes of investors. For Figure 1-7, \( \omega = 0.05 \) and the informed investors are only a small fraction of the market. With \( \sigma_1 = \sigma_2 \), each individual investor faces similar non-traded risk. But the non-traded risk of uninformed investors is less diversifiable because they are majority of the market and face the same risk. In other words, they bear most of the aggregate non-traded risk. Given the positive correlation between shocks to non-traded income and stock payoffs, the uninformed investors are less willing to hold the stock. Both of these factors tend to yield \( h_1^{(0)} - 1 > 0 \).

\[
\begin{align*}
\text{(a)} & \quad h_1^{(0)} - 1 \\
\text{(b)} & \quad h_1^{(1)} \\
\text{(c)} & \quad h_1^{(2)} \\
\text{(d)} & \quad h_1^{(3)}
\end{align*}
\]

**Figure 1-7:** Components of investor 1's stock holdings during the day under partial revelation. Their values at the close (\( \tau = T \)) are indicated by circles. The parameters are set at the following values: \( T = 0.5, N = 0.5, \omega = 0.05, \gamma = 1000, r = 0.001, \sigma_D = 0.08, a_\sigma = .85, \sigma_u = 0.08, \sigma_D = 0.5, a_\gamma = 0.25, \sigma_1 = \sigma_2 = 7, \kappa_{pq} = 0.5, \kappa_{12} = 0. \)

The informed investors' hedging activity, as characterized by \( h_1^{(1)} \), is shown in Figure 1-7(b). It decreases monotonically during the day and then abruptly drops at the close. Similar to the case of symmetric information, the informed investors actively trade in the stock in response to changes in their non-traded income, but reduce their positions abruptly at the close.

The coefficient \( h_1^{(2)} \) characterizes the market-making activity of informed investors, which in turn reflects the intensity of uninformed investors' hedging activities. Factors affecting the informed investors' hedging activities are also affecting the uninformed investors. There are, however, additional factors affecting the uninformed investors' hedging needs. Especially, the continuous revelation of information through the day reduces the risk of taking stock positions, thereby increases their hedging activities. Also, as the market closes, market prices cease to be a source of information about the stock's future payoffs, and the uninformed investors solely rely on other information such as realized dividends. Consequently, their expectations of the stock's future payoffs, now completely driven by dividend realizations, become more correlated with
shocks to their non-traded income. Thus, the stock is perceived to be a better hedging vehicle during the night than during the day. The uninformed investors may actually increase their hedging positions at close, opposite to what the informed investors do. Both factors mentioned above tend to increase the uninformed investors' hedging activity before the close, thereby they tend to increase $\lambda_2$ (see Figure 1-6).

Finally, we consider the speculative activities of the informed investors. The informed investors speculate actively during the day. At the close, the risk in carrying speculative positions overnight makes them abruptly reduce their positions. This pattern of the informed investors' speculative trading is illustrated by $h_1^{(3)}$ in Figure 1-7(d).

D. Time Pattern of Return Distributions

Given the investors' trading policies and the resulting equilibrium, we can easily calculate the intraday patterns in mean return and return volatility. Figure 1-8 illustrates these patterns for the given set of parameters. Both the mean excess share return and the return volatility exhibit U-shaped patterns, different from those under symmetric information. In particular, both the mean and the volatility of instantaneous stock returns are increasing over time right before the close, instead of decreasing as under symmetric information.

![Graphs](image_url)

**Figure 1-8:** Intraday patterns of instantaneous mean excess share return and return volatility under partial revelation. The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.05$, $\gamma = 1000$, $\tau = 0.001$, $\sigma_D = 0.08$, $a_D = .85$, $\sigma_G = 0.08$, $\sigma_Q = 0.5$, $a_V = 0.25$, $\sigma_1 = \sigma_2 = 7$, $\kappa_{DG} = 0.5$, $\kappa_{12} = 0$.

Two factors are driving the time variation in the return generating process, both associated with market closures. The first factor is the time variation in investors' hedging demand. As discussed in the case of symmetric information, investors reduce their hedging demand at the close. As a result, the stock price is lower and less responsive to investors' technological shocks at the close than at the open. Thus, the time variation in investors' hedging demand tends to make the mean and volatility of stock returns decrease during the day. The second factor is the time variation in information asymmetry. Under market closures, the information asymmetry on the stock (as measured by $\alpha_{11}$) increases during the night as trading stops and decreases during the day as trading continues (see Figure 1-5). Consequently, the stock price increases

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during the day as less premium is demanded on the stock by the uninformed investors, so does the mean return. Also, the price becomes more volatile as it reflects more information about the stock’s future payoffs. Thus, the time variation in information asymmetry tends to make the mean and volatility of stock returns increase during the day. For the parameters we have chosen, the effect of time-varying hedging demand tends to dominate early in the day and the effect of time-varying information asymmetry tends to dominate later in the day. Consequently, the U-shaped intraday patterns emerge for both the mean and volatility of returns.

4.3. Further Discussions

So far, we have used only one set of parameter values in illustrating the nature of equilibrium. In this subsection, we briefly analyze how the equilibrium changes as we vary some parameters. This analysis not only illustrates possible return patterns, but also helps to deepen our understanding of the model. We focus first on parameters affecting the level of information asymmetry and its dynamics and then on parameters affecting the interaction between market open and close.

A. Varying \( \sigma_{\sigma} \)

An important parameter in determining the level of information asymmetry is \( \sigma_{\sigma} \). When \( \sigma_{\sigma} = 0 \), there is no information asymmetry between the two classes of investors about future stock payoffs, and the situation is similar to the case of symmetric information.\(^{22}\) When \( \sigma_{\sigma} \) increases, the variability in \( G_t \) increases. The information asymmetry between the two classes of investors also increases, and so does its effect on the equilibrium and return distributions.

Figure 1-9(a) shows how \( \sigma_{11} \) changes with \( \sigma_{\sigma} \). When \( \sigma_{\sigma} \) is close to zero, \( \sigma_{11} \) is small throughout a calendar day (day and night). As \( \sigma_{\sigma} \) increases, the overall level of \( \sigma_{11} \) also increases. During the night, \( \sigma_{11} \) increases. For large values of \( \sigma_{\sigma} \), the dividend growth \( G_t \) can change a lot overnight and the informed investors accumulate a significant amount of private information during the night. Even though \( \sigma_{11} \) drops significantly at the open, it maintains a high level after the open and tend to decrease gradually over time as trading continues.

Figure 1-9(b) compares the level of \( \sigma_{11} \) under periodic market closures with its level under continuous trading (i.e., when \( N = 0 \)). For large values of \( \sigma_{\sigma} \), the informed investors accumulate a large amount of private information after the market closes. The value of \( \sigma_{11} \) can exceed its value under continuous trading during the night, but falls below during the day. For small values of \( \sigma_{\sigma} \), however, the value of \( \sigma_{11} \) can stay uniformly (day and night) below its value under continuous trading. In this case, by closing the market periodically, the uninformed investors can become more informed about the stock’s future payoffs. This result arises from the interaction

\(^{22}\) There is still information asymmetry between the two classes of investors about the informed investors’ technological shocks, which are not observable to the uninformed investors. During the day, this asymmetry vanishes since the stock price fully reveals \( Y_{1,t} \), and two classes have the same information in equilibrium. During the night, however, this asymmetry in information does appear. Given that the information asymmetry is only on private investment opportunities, it has little effect on the equilibrium.
between the investors’ hedging and speculative trade in our model. Since market closures increase the cost of hedging (especially the cost of maintaining hedging positions overnight), the informed investors become less active in hedging their non-traded risk. Their trades are more likely to be speculative, thereby are more informative about their private information on future payoffs. This result stands in contrast to the results from models of noisy rational expectations equilibrium in which the allocational trading is exogenously specified and independent from market closures [see, e.g., Slezak (1995)].

Figure 1-9(c) further illustrates the details of time variation in information asymmetry (normalized by and net of its level under continuous trading) for different values of $\sigma_G$. The dotted line is for $\sigma_G = 0.15$, the dashed line is for $\sigma_G = 0.08$, and the solid line is for $\sigma_G = 0.025$. For large value of $\sigma_G$ (0.15), the information asymmetry reaches high levels during the night, above its continuous-trading value. At the open, it drops significantly but remains above the continuous-trading level. As trading continues, it decreases below the continuous-trading level and further decreases as the closure approaches. For small value of $\sigma_G$ (0.025), the level of information asymmetry lies uniformly below the continuous-trading level. Although it drops at the open, the change is small. Furthermore, in the particular case we show here, the level of information asymmetry initially keeps increasing as trading resumes after the open. This may seem puzzling since the continuation of trading should reduce information asymmetry. Note that in our model, the true dividend growth evolves over time. At the previous close ($\tau = -0.5$), the informed investors’ cutting back of their hedging positions reveals a lot of information about the dividend growth then. As the night passes by, the dividend growth moves away from its value at previous close and $o_{11}$ increases. After the market opens, even though the prices start revealing information about the new values of the dividend growth, $o_{11}$ may still increase for a while as the value of old information (from prices of the previous day) diminishes. Eventually, the information asymmetry starts to decrease, especially as the next close approaches. For middle value of $\sigma_G$ (0.08), $o_{11}$ reaches above the continuous-trading level right before the market
reopens, drops below the continuous level at the open and continue to decreases during the day.

\[ e/e(N=0) \quad \text{and} \quad \sigma^2_\eta/\sigma^2_{\eta(N=0)} \]

Figure 1-10: Intraday patterns of mean access share return and return volatility for different values of \( \sigma_G \). (For easy comparison, the mean return and return volatility are normalized by their respective values under continuous trading.) Other parameters are set at the following values: \( T = 0.5, N = 0.5, \omega = 0.05, \gamma = 1000, r = 0.001, \sigma_D = 0.08, \alpha_G = 0.85, \sigma_\eta = 0.5, \alpha_\eta = 0.25, \sigma_1 = \sigma_2 = 7, \kappa_{pq} = 0.5, \kappa_{12} = 0. \)

By changing \( \sigma_G \), the relative effect of information asymmetry on equilibrium returns also change. For small values of \( \sigma_G \), the effect of information asymmetry is small. The return patterns are similar to those in the case of symmetric information: Both the mean return and return volatility decrease during the day, mainly reflecting the impact of market closures on investors’ hedging demand. For large values of \( \sigma_G \), the effect of time-varying information asymmetry dominates. The mean return and return volatility tend to increase as information asymmetry diminishes over the day. For the parameter values in Figure 1-10, the effect of time-varying hedging trade tends to be more important earlier in the day while the effect of time-varying information asymmetry tends to be more important later in the day. Thus, for the middle range of \( \sigma_G \), the mean return and return volatility both exhibit U-shaped intraday patterns.

B. Varying \( \alpha_G \) and \( \alpha_\eta \)

Two parameters are important in determining the evolution of information asymmetry over time, \( \alpha_G \) and \( \alpha_\eta \). The coefficient \( \alpha_G \) determines the persistence in dividend growth \( G_t \). For large values of \( \alpha_G \), the process of \( G_t \) is strongly mean reverting and any fluctuations in \( G_t \) from its unconditional mean die off quickly. When \( \alpha_G = 2 \), for example, the half-life of \( G_t \) fluctuations is 0.5 (day). In this case, the level of \( G_t \) at the close, for example, has little to do with its level at the following open. This fact implies that the informed investors' private information is short-lived. For small values of \( \alpha_G \), the process of \( G_t \) is closer to a random walk and fluctuations in \( G_t \) is more persistent. The level of \( G_t \) at the close is highly correlated with its level at the following open. In this case, the informed investors' private information is more long-lived. It is important to note that \( \alpha_G \) also affects the unconditional variance of \( G_t \). In particular, since
Var[G_t] = \sigma_G^2/(2a_G), the unconditional variance of G_t increases as a_G decreases.

Figure 1-11(a) illustrates the time pattern of \(o_{11}\) for different values of \(a_G\). During the night, \(o_{11}\) increases faster and reaches higher values before the open for smaller \(a_G\), reflecting the higher (unconditional) uncertainty in \(G_t\). But for smaller \(a_G\), \(o_{11}\) goes through a much larger drop at the open and maintains a lower level during the day. This may seem surprising since given the higher unconditional uncertainty in \(G_t\) for smaller \(a_G\), one would expect the level of information asymmetry to be higher even during the day. The reason why this is not true is as follows. As trading continues in the day, new prices are more informative about the true level of \(G_t\) if fluctuations in \(G_t\) are more persistent. In the extreme case when \(G_t\) follows a random walk, neighboring prices provide different signals (only partially correlated as we discuss later) about the same \(G_t\) for the uninformed investors. Consequently, a large fraction of the uncertainty about \(G_t\) is resolved by a given sequence of market prices. In particular, at the open, the previous closing price still provides information about the current \(G_t\). Combined with the opening price, they reveal a lot of information about the value of \(G_t\) at the open. In the other extreme case when \(a_G\) is very large, fluctuations in \(G_t\) have very short life time and past prices have little information about the current level of \(G_t\). A given sequence of market prices only resolve a small fraction of the uncertainty about \(G_t\) for the uninformed investors. Consequently, the drop in \(o_{11}\) at the open can be smaller and its level during the day higher.

![Figure 1-11: Intraday patterns of information asymmetry, mean return and return volatility for different values of \(a_G\). (For easy comparison, the mean return and return volatility are normalized by their respective values under continuous trading.) Other parameters are set at the following values: \(T = 0.5, N = 0.5, \omega = 0.05, \gamma = 1000, r = 0.001, \sigma_D = 0.08, \sigma_G = 0.08, \sigma_Q = 0.5, \sigma_V = 0.25, \sigma_1 = \sigma_2 = 7, \kappa_D = 0.5, \kappa_12 = 0\). Here, the mean return and return volatility are both normalized by their values under continuous trading.](image)

Figure 1-11(b) and (c) illustrate how the intraday patterns of return distributions change with \(a_G\). For most values of \(a_G\) within the range we consider, the U-shaped patterns appear for both the mean return and return volatility. However, as \(a_G\) becomes large, the effect due to information asymmetry becomes more important (as \(o_{11}\) increases with \(a_G\)), the increasing trend in the mean and volatility becomes more prominent.

The coefficient \(a_V\) determines the persistence in investors' technological shocks. Since these
shocks drive the investors’ hedging needs, $a_Y$ also determines the dynamics of investors’ hedging positions. As discussed earlier, it is the informed investors’ hedging trade that generates the “noise” in the stock price as a signal about true dividend rate $G_t$. The serial correlation in the informed investors’ hedging positions determines the serial correlation in the noise. At low levels of serial correlation in the noise, neighboring prices are close to independent signals (about $G_t$), which tend to reveal more of the informed investors’ private information on $G_t$. At high levels of serial correlation in the noise, however, neighboring prices are more correlated as signals, which tend to reveal less of the informed investors’ private information on $G_t$. Thus, as $a_Y$ increases, a series of market prices tend to be more informative about the true value of $G_t$. Figure 1-12(a) shows the time variation of $\sigma_{11}$ for different values of $a_Y$. Clearly, for larger values of $a_Y$, the drop in $\sigma_{11}$ at the open is larger since the prices at the open and previous close are less dependent as signals about $G_t$, and its level during the day is also lower. In addition, for larger values of $a_Y$, investors cut more of their hedging positions at the close since their hedging are more transitory while the positions have to be held overnight. More reduction in informed investors hedging positions can reveal more of their private information about stock payoffs. Thus, $\sigma_{11}$ decreases faster toward the end of the day for larger values of $a_Y$.

\[ \sigma_{11} \quad e/e(n=0) - 1 \quad \sigma_{22}/\sigma_{22}^{(n=0)} - 1 \]

(a) \quad (b) \quad (c)

**Figure 1-12:** Intraday patterns of information asymmetry, mean return and return volatility for different values of $a_Y$. (For easy comparison, the mean return and return volatility are normalized by their respective values under continuous trading.) Other parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.05$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $a_C = 0.85$, $\sigma_c = 0.08$, $\sigma_a = 0.5$, $\sigma_1 = \sigma_2 = 7$, $\kappa_D = 0.5$, $\kappa_2 = 0.5$.

Figure 1-12(b) and (c) show, respectively, the intraday patterns in mean stock return and return volatility for different values of $a_Y$. For most values of $a_Y$, the patterns are U-shaped. However, as $a_Y$ increases, the initial decrease in both the mean and volatility becomes less significant. As we discussed earlier, the U-shaped patterns appear when the hedging effect to be more important earlier in the day and the information effect to be more important later in the day. Moreover, the effect of hedging is mainly driven by the decrease in investors’ hedging demand at the close. When $a_Y$ becomes larger, the hedging demand at the close is less related to the hedging demand at the open. The hedging effect becomes less important earlier in the
day. Thus, the upward trend in the mean return and return volatility becomes more important as $a_Y$ increases.

C. Varying $T$ and $N$

We now examine how the length of day and night, $T$ and $N$, respectively, affects the interaction between market open and close. By changing $T$, we can examine how returns around the open are affected by the close later in the day. Similarly, by changing $N$, we can examine how returns around the close are affected by the open next day.

\[\sigma_{11}\]
\[\sigma\]

**Figure 1-13:** Intraday patterns of information asymmetry, mean access share return and return volatility for different values of $T$. The parameters are set at the following values: $N = 0.5$, $\omega = 0.05$, $\gamma = 1000$, $r = 0.001$, $\sigma_d = 0.08$, $a_G = 2$, $\sigma_G = 0.15$, $\sigma_Q = 0.5$, $a_Y = 1$, $\sigma_1 = \sigma_2 = 7$, $\kappa_{dQ} = 0.5$, $\kappa_{12} = 0$.

For small values of $T$, the day is short. In the limit $T = 0$, the economy approaches the situation of discrete trading. Given the limited trading opportunities and overnight risk of carrying stock positions, the informed investors reduce their hedging trade. As a result, the level of information asymmetry is lower for smaller $T$. Figure 1-13(a) plots the time variation of $\sigma_{11}$ for different values of $T$. In the figure, we normalize the time during the day by $2T$, twice the length of a day, hence the normalized length of a day is always 0.5. Clearly, during the day, $\sigma_{11}$ is smaller for smaller $T$. It also decreases less over the day since a shorter sequence of prices is observed.

Figure 1-13(b) and (c) plot the intraday pattern of stock returns. As discussed earlier, two effects are generating the intraday pattern: the time variation in investors' hedging demand tends to cause the mean and volatility to decrease over time while the time variation in information asymmetry tends to cause them to increase. Given the values of other parameters and for $T$ around 0.5, the hedging effect dominates early in the day while the information effect dominates late in the day, and we have U-shaped patterns for both the mean and volatility. For smaller $T$, both effects are squeezed into a smaller time interval. The resulting patterns become simpler, either decrease or increase monotonically, depending on which one of the two effects dominates. For the parameter values we use here, the information effect dominates and the patterns are all
down sloping. For large values of $T$, the hedging effect becomes less important in shaping the returns around the open. The information effect can dominate throughout the day and both the return and volatility patterns become more upward sloping.

Changing the value of $N$, i.e., the length of night, changes investors’ hedging cost. As $N$ increases, the cost of carrying a hedging position increases and investors reduce their positions more dramatically at the close. Given the persistence in their technological shocks, the overall level of their hedging activity is also lower. Thus, stock prices are more informative about future payoffs and the level of information $o_{11}$ is lower. Moreover, the decrease in $o_{11}$ over the day is larger. Figure 1-14(a) shows the time variation of $o_{11}$ for different values of $N$. For large values of $N$, the information asymmetry becomes the dominant factor in affecting the time variation in return distributions. Both the intraday patterns of mean and variance slope upwards. For smaller $N$, the hedging effect becomes relatively more important and the U-shaped patterns start to emerge in both mean return and return volatility. Of course, as $N \to 0$, we approach the continuous trading limit and the time variation in return distribution disappears.

Figure 1-14: Intraday patterns of information asymmetry, mean access share return and return volatility for different values of $N$. The parameters are set at the following values: $T = 0.5$, $\omega = 0.05$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $a_C = 1$, $\sigma_C = 0.08$, $\sigma_Q = 0.5$, $a_V = 0.95$, $\sigma_1 = \sigma_2 = 7$, $\kappa_{pq} = 0.5$, $\kappa_{12} = 0$.

D. Other Return Patterns

The time patterns of returns shown above (in both mean and volatility) are of three types: downward sloping, upward sloping and U-shaped. When the effect of time-varying hedging demand dominates, the mean and volatility tend to be downward sloping as in the case of symmetric information. When the effect of time-varying information asymmetry dominates, the mean and volatility tend to be upward sloping. When the effect of time-varying hedging demand dominates earlier in the day while the effect of time-varying information asymmetry dominates later in the day, the mean and volatility tend to decrease early in the day and to increase late in the day, exhibiting U-shaped patterns. Other patterns are also possible. For example, when the information effect dominates around the open while the hedging effect dominates around
the close, and the mean and volatility of stock returns exhibit inverted-U patterns, increasing first and then decreasing during the day. Figure 1-15 provides some examples.

![Figure 1-15: Intraday patterns of information asymmetry, mean access share return and return volatility for different values of $a_y$. (For easy comparison, the mean return and return volatility are normalized by their respective values under continuous trading.) The other parameters are set at the following values: $T = 0.5$, $N = 0.3$, $\omega = 0.2$, $\gamma = 1000$, $r = 0.001$, $\sigma_d = 0.15$, $\sigma_0 = 0.15$, $\gamma = 0.9$, $a_y = 0.85$, $\gamma_1 = 7$, $\gamma_2 = 0$, $\kappa_{dq} = 0.5$, $\kappa_{12} = 0$. The mean return and return volatility are normalized by their values under continuous trading.](image)

The parameter values used to generate Figure 1-15 are slightly different from those in Figure 1-10 and 1-12. Most importantly, $\sigma_2$ is set at zero here and the uninformed investors face no technological shocks. This eliminates the effect from the uninformed investors’ hedging demand on the stock price, which tend to cause mean return and return volatility to increase at the close [see Figure 1-6(c)]. The length of the night $N$ is set at 0.3, slightly smaller than the value of 0.5 used in previous figures. A smaller $N$ reduces the effect of negative effect of market closures on the informed investors’ hedging demand and information asymmetry. The value of $\sigma_d$ is also slightly increased to enhance the information asymmetry between the two classes of investors.

For smaller $a_y$, the informed investors’ hedging needs are more persistent. Expected decrease in hedging need at forthcoming close has a larger effect on the hedging need during the day. The effect of time-varying hedging demand tends to be more important around the open as well as around the close. Also, for smaller $a_y$, the rate at which information asymmetry decreases during the day is smaller since prices provide highly correlated signals. Hence, the effect of time-varying information is small. Consequently, the mean return and return volatility tend to decrease around both the open and the close, reflecting the effect of time-varying hedging demand. In the middle of the day, however, they tend to slightly increase, reflecting the effect of time-varying information asymmetry. As a result, the mean return and return volatility decrease after the market opens, then increase as trading continues, and decrease again as the closing approaches.

As $a_y$ increases, three things happen. First, the closure has less effect around the open since hedging needs are less correlated over time. Second, sequences of prices provide more independent signals about stock payoffs. The information asymmetry decreases at a faster
rate as trading continues. Hence, the information effect becomes more important. Third, less persistence in technological shocks increases the magnitude at which the informed investors cut their hedging positions (see Figure 1-4 and the related discussion). The first two factors make the effect of time-varying information asymmetry more important around the open and the third factor makes the effect of time-varying hedging demand more important around the close. Consequently, we see inverted-U patterns for intraday mean returns and return volatility for larger $a_Y$ in Figure 1-15. When $a_Y$ becomes too large, the information effect dominates and both the mean return and return volatility increase monotonically during the day.

4.4. Discrete Returns

We now consider the discrete returns in relating to some of the empirical results. We use the same parameter values as in Table 1.2 for the case of symmetric information. The first two moments are reported in Table 1.3 for the same set of parameters. The mean return over the trading periods is now positive while the mean return over the non-trading periods is negative. This is simply due to the fact that the risk discount on the stock $\lambda_0$ is higher at the open than at the close (see Figure 1-6); hence the (mean) stock price is lower at the open than at the close. From the periodicity of the equilibrium, it then follows that the daily return is positive and the nightly return is negative. As discussed earlier, two effects are contributing to this result. During market open, the uninformed investors face the adverse selection problem when they trade with the informed investors. This makes them less willing to hold the stock during the trading periods as shown in Figure 1-7(a). In addition, without observing the market prices over the non-trading periods, the uninformed investors perceive higher correlation between stock payoffs and non-traded income. This makes them more willing to hold the stock over the non-trading periods since they can adjust their private investments to better reduce the risk of their stock positions. In equilibrium, the return over the trading periods has to be higher that the return over the non-trading period.

<table>
<thead>
<tr>
<th>Returns</th>
<th>Mean ($10^{-4}$)</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{oo}$</td>
<td>0.00</td>
<td>0.0454</td>
</tr>
<tr>
<td>$R_{cc}$</td>
<td>0.00</td>
<td>0.0441</td>
</tr>
<tr>
<td>$R_{oc}$</td>
<td>4.80</td>
<td>0.0166</td>
</tr>
<tr>
<td>$R_{co}$</td>
<td>-4.80</td>
<td>0.0148</td>
</tr>
</tbody>
</table>

The parameters are set at the following values: $T = 0.5$, $N = 0.5$, $\omega = 0.05$, $\gamma = 1000$, $r = 0.001$, $\sigma_D = 0.08$, $a_G = .85$, $\sigma_G = 0.08$, $\sigma_Q = 0.5$, $a_Y = 0.25$, $\sigma_1 = \sigma_2 = 7$, $\kappa_D = 0.5$, $\kappa_{12} = 0$.

The return over the trading periods is more volatile than the return over the non-trading
periods. For the parameter values chosen here, the variance during market open is 0.0166 which is larger than the variance during market close which is 0.0148. The ratio between the variance of returns over the trading period and the non-trading period is 1.12. Note that we have chosen the length of trading periods and non-trading periods to be the same and exogenous information flow is constant over time. Under symmetric information, the volatility of return over the trading periods and the non-trading periods is the same (see Table 1.2). Under asymmetric information, trading among investors reveals investors' private information to the market. Consequently, returns over trading periods are more volatile than returns over non-trading periods since more information is impounded in the equilibrium price over the trading periods. Table 1.3 also shows that the return variance from open-to-open, 0.0454, is larger than the return variance from close-to-close, 0.0441. The ratio between the variances of the two returns is 1.03.

5. Conclusion

In this paper, we consider a continuous-time stock market with periodic market closures where investors trade for both hedging and speculative reasons. We show that periodic market closures can generate variations in trading and return distribution that are consistent with the empirical findings. In general, open-to-open returns are more volatile than close-to-close returns and intraday trading volume is U-shaped. When investors have asymmetric information about the stock’s future cash flows, the intraday returns and volatility can be U-shaped as well. Returns are positive over periods of market open but negative over periods of market close. Moreover, returns are more volatile over the trading periods than over the non-trading periods. Our results point to the importance of periodic closures in explaining documented intraday patterns.

The paper focuses solely on the impact of periodic market closures on trading and returns. Details of the market micro-structure concerning the actual trading mechanism are completely ignored. Also, the assumption of a competitive market rules out any strategic behavior on the part of investors in their speculative trading [see, e.g., Kyle (1985)]. Given that the patterns of interest here are on daily or weekly frequency, these factors can be important in fully understanding the empirical patterns. As mentioned in the introduction, some of these factors have been analyzed in the literature concerning their impact on intraday or intra-week returns [e.g., Admati and Pfleiderer (1988, 1989), Foster and Viswanathan (1990)].

Although the focus of this paper is on the time variation in trading and returns generated by market closures, the model developed here can also be used to analyze other issues and phenomenon associated with market closures such as welfare implications of market closures, over-night trading, the interaction across markets in different time-zones, etc. We leave these issues for future research.
6. Appendix

This appendix consists of five parts: Section 6.1. provides some technical results needed for later use. Section 6.2. gives the solution to an investor's control problem under the general assumption that asset returns (on the stock and private technologies) follow Gaussian Markov processes. The solution can be used in later proofs for the existence of a linear, periodic equilibrium in which asset returns do follow Gaussian Markov processes. Section 6.3. provides proofs of Lemma 1-2 and Theorem 1 concerning the equilibrium in the case of symmetric information. It also proves Propositions 1-4 in Section 4. Section 6.4. includes the proofs for Lemma 3-4 and Theorem 2 concerning the equilibrium in the case of asymmetric information. Section 6.5. discusses the numerical solutions to the equilibrium.

We begin by introducing some additional notation. Let \( \theta^{(m,n)} \) denote the zero matrix of order \((m \times n)\) and \( [m] \) the column matrix consists of the independent elements of matrix \( m \). A semi-positive (positive) definite matrix \( m \) is stated as \( m \geq (>0) \) and \( \|m\| = \max |m_{i,j}| \) defines the norm of \( m \). Let \( \Theta = \{ r > 0, \gamma > 0, a_\sigma > 0, a_\nu > 0, \sigma_D \geq 0, \sigma_G \geq 0, \sigma_1 \geq 0, \sigma_2 \geq 0, \sigma_q \geq 0, -1 \leq \kappa_{12} \leq 1, -1 \leq \kappa_{12} \leq 1, T \geq 0, N \geq 0 \} \) be the set of parameter values and \( \vartheta \in \Theta \) denote a generic element.

6.1. Mathematical Preliminaries

In deriving several results in the paper, the problem to be solved often reduces to a two-point boundary-value problem for a (vector) ordinary differential equation. Here, we give a formal and relatively general definition of the two-point boundary-value problem and state some known results concerning its solution.

**Definition 3** Let \( f : \mathbb{R}^+ \otimes \mathbb{R}^n \otimes \mathbb{R}^m \otimes \mathbb{R} \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \otimes \mathbb{R}^m \otimes \mathbb{R} \rightarrow \mathbb{R}^n \). A two-point boundary-value problem is defined as

\[
\begin{align*}
\dot{z} &= f(t, z; \vartheta, \omega), \quad t \in [0, T] \\
0 &= g[z(0), z(T); \vartheta, \omega]
\end{align*}
\]  

(20)

where \( T > 0, \vartheta \in \Theta \) and \( \omega \in [0, 1] \).

We also define the terminal value problem:

\[
\begin{align*}
\dot{z} &= f(t, z; \vartheta, \omega), \quad t \in [0, T] \\
z(T) &= z_T.
\end{align*}
\]

(21)

Under appropriate smoothness conditions on \( f(t, z; \vartheta, \omega) \), (21) has a unique solution \( z = z(t; \vartheta, \omega; z_T) \), which is differentiably in \( z_T \) [see Keller (1992), Theorem 1.1.1, p.2]. Solving the two-point
boundary-value (20) is to seek $z_T$ that solves

$$0 = g [z(0; \vartheta, \omega; z_T), z_T; \vartheta, \omega] \equiv g \circ z(z_T; \vartheta, \omega).$$

(22)

The existence of a root to (22) relies on properties of $g \circ z(z_T; \vartheta, \omega)$. Let $(g \circ z + 1)(z_T; \vartheta, \omega) \equiv g \circ z(z_T; \vartheta, \omega) + z_T$.

**Lemma 5** If $(g \circ z + 1)(\cdot; \vartheta, \omega) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a nonempty, closed, bounded, and convex subset of $\mathbb{R}^n$, $L$, such that $(g \circ z + 1)(\cdot; \vartheta, \omega)$ maps $L$ into itself, then (22) has a root and the two-point boundary value problem (20) has a solution.

**Proof.** Existence of a root to (22) follows from Brouwer's Fixed Point Theorem [see, e.g. Cronin (1994, p.352)]. $\Box$

The condition on $(g \circ z + 1)$ required by Lemma 5 is not always easy to verify, in which case the existence of a solution to (20) is not readily confirmed. However, if a solution exists for $\omega_0$, the existence of a solution for $\omega$ close to $\omega_0$ is easy to establish.

**Definition 4** $z = z(t; \vartheta; \omega_0)$ is an isolated solution of system (20) if the linearized system

$$\begin{align*}
\dot{y} &= \nabla_z f(t, z; \vartheta, \omega_0) y, \quad t \in [0, T] \\
0 &= \nabla_z g(z_0; \vartheta, \omega_0) y(0) + \nabla_z g(z_T; \vartheta, \omega_0) y(T)
\end{align*}$$

(23)

has $y = 0$ as the only solution, where $\nabla$ denotes the partial derivative operator.

**Lemma 6** Let (20) have an isolated solution $z = z(t; \vartheta, \omega_0)$ for $\omega = \omega_0$. Suppose that $f(t, z; \vartheta, \omega)$ and $g(z(0); \vartheta, \omega)$ are continuously differentiable in the neighborhood of $(t, z(t; \vartheta, \omega_0), \omega_0)$. Then, (20) has a solution for $\omega$ close to $\omega_0$.

**Proof.** See Keller (1992), p.199. $\Box$

We next state the general form of the boundary-value problems we encounter in the paper and show that they can be reduced to the two-point boundary-value problem (20). Let $j = a, b$ and $z = \text{stack}\{z_a, z_b\}$. Our boundary value problem is given by:

$$\dot{z} = f(z, \vartheta, \omega), \quad t \in [0, T]$$

(24a)

$$z_a(T) = g_a^c [z_a^*(T); \vartheta, \omega] \quad \text{and} \quad z_b(T) = z_b^*(T), \quad t = T$$

(24b)

$$\dot{z}_j^* = f_j^c (z_j^*, \vartheta, \omega), \quad t \in (T, T+N), \quad j = a, b$$

(24c)

$$z_a^*(T+N) = g_a^c [z_a(T+N); \vartheta, \omega] \quad \text{and} \quad z_b(T+N) = g_b^c [z_b^*(T+N); \vartheta, \omega], \quad t = T+N$$

(24d)

$$z(0) = z(T+N)$$

(24e)
where $f, g_j^c, f_j^c$, and $g_0^c$ are continuously differentiable. Given $z_a(0)$, the periodicity condition, (24e) and (24d) give $z_a^c(T+N) = g_0^c[z_a(0); \vartheta, \omega]$ as the terminal condition for (24c). Integrating (24c) yields $z_a^c(T; g_0^c[z_a(0); \vartheta, \omega])$. (24b) gives a two-point boundary condition for $z_a(0)$ and $z_a(T)$:

$$z_a(T) = g_0^c[z_a^c(T; g_0^c[z_a(0); \vartheta, \omega]); \vartheta, \omega] \equiv g_0^c \circ z_a^c \circ g_0^c(z_a(0); \vartheta, \omega). \quad (25)$$

Given $z_b(T)$, (24b)-(24c) lead to $z_b^c(T+N; z_b(T))$. (24d) gives $z_b(T+N) = g_0^c[z_b^c(T+N; z_b(T)); \vartheta, \omega]$. (24e) gives

$$z_b(0) = g_0^c[z_b^c(T+N; z_b(T)); \vartheta, \omega]; \vartheta, \omega] \equiv g_0^c \circ z_b^c(z_b(T); \vartheta, \omega). \quad (26)$$

Our boundary-value problem now reduces to the system given by (24a), (25) and (26), which is in the form of (20).

For future use, we also state two auxiliary lemmas.

**Definition 5** Let $a_j, j = 0, 2, r$ be symmetric and $a_0 \geq 0, a_2 > 0, r$ be symmetric, and $a_3(t, r)$ be a positive, linear operator mapping symmetric matrices into themselves. A matrix Ricatti differential equation is defined as

$$\dot{r} + a_0 + (a_1r + ra_1) - r^2a_2 + a_3(t, r) = 0 \text{ (a.e.), } t \in [0, T] \quad (27)$$

and $r(T) = r_T$.

**Lemma 7** For any given terminal value $r_T \geq 0$, the matrix Ricatti equation (27) has a unique, symmetric, positive, semi-definite solution for any given. Let $m$ be an arbitrary (bounded measurable) matrix defined on $[0, T]$ and $e$ be the solution of the following linear equation:

$$\dot{e} + a_0 + (a_1 - m)e + e(a_1 - m) + m'r_2m + a_3(t, e) = 0, \quad t \in [0, T] \quad (28)$$

where $e(T) = r_T$. If $r$ is the solution of (27), then $r \leq e$ for $t \in [0, T]$.

**Proof.** See Wonham (1968). □

**Lemma 8** Let $f : \mathcal{D} \to \mathbb{R}$ be a real analytic function, where $\mathcal{D} = D_1 \otimes \cdots \otimes D_n$ is an open subset of $\mathbb{R}^n$. Let $Z = \{x \in \mathcal{D} : f(x) = 0\}$ be its zero set. Then, either $Z = \mathcal{D}$ or $\mu_0(Z) = 0$ where $\mu_0$ is the $n$-dimensional Lebesgue measure.

**Proof.** We prove by induction. First note that $Z$ is closed and therefore measurable. For $n = 1$, $Z$ is either finite, or has an accumulation point. In the latter case, $f$ is identically zero on $\mathcal{D}$ [see, e.g., Ahlfors (1979)]. Noting that any finite set has zero Lebesgue measure concludes this part of the proof. Let us suppose that the conclusion of the lemma holds for certain $k \geq 1$ and prove it for $n = k + 1$. Denoting $f$ as a function of two variables $f(t, x)$ on $D_1 \otimes D_2$, where $D_2 = D_2 \otimes \cdots \otimes D_{k+1}$. Clearly, $f$ is a real analytic function in both $t$ and $x$ separately. Consider the set

50
\( S = \{ t \in D_1 : \forall x \in D_2, f(t, x) = 0 \} \). For \( t \notin S \), \( \int_{D_2} f(t, x)dx = 0 \) by the inductive assumption. If \( S \) is finite, it is of zero Lebesgue measure in \( D_1 \). Thus, \( \mu_n(Z) = \int_{D_1} \int_{D_2} f(t, x)dxdt = 0 \) by Fubini theorem [see, e.g., Doob (1991)]. If \( S \) is not finite, it has an accumulation point. From the result for \( n = 1, \forall x \in D_2, f(t, x) \) is identically zero in \( D_1 \). Thus, \( f(t, x) \) is identically zero on \( D = D_1 \otimes D_2 \). □

6.2. Solution to Investors’ Control Problem

In this appendix, we derive the solution to investors’ control problem under a specific class of stock return process. The actual control problems for both classes of investors to be solved in the paper are special cases of this more general problem.

Suppose that under the information of investor \( i, i = 1, 2 \), the stock return process is specified as follows.

1. The stock price is given by

\[
P_t = F_{i,t} - \lambda_i X_{i,t}, \quad t \in \mathcal{T}
\]

where \( F_{i,t} \) and \( X_{i,t} \) are the state variables of finite dimension.

2. The state variables follow Gaussian Markov processes:

\[
\begin{align*}
t = t_k : & \quad X_{i,t_k} = a_{i,X}^0 X_{i,t_k}^{-} + b_{i,X}^0 \epsilon_{i,t_k}, \quad \hat{F}_{i,t_k} = \hat{F}_{i,t_k}^{-} + b_{i,F}^0 \epsilon_{i,t_k} \\
t \in \mathcal{T}_k : & \quad dX_{i,t} = a_{i,X} X_{i,t}dt + b_{i,X} dw_{i,t}, \quad d\hat{F}_{i,t} = a_{i,F} \hat{F}_{i,t}dt + b_{i,F} dw_{i,t} \\
t = n_k : & \quad \tilde{X}_{i,n_k} = a_{i,X}^* X_{i,n_k}, \quad \tilde{F}_{i,n_k} = \tilde{F}_{i,n_k} \\
t \in \mathcal{N}_k : & \quad d\tilde{X}_{i,t} = a_{i,X}^* \tilde{X}_{i,t}dt + b_{i,X}^* dw_{i,t}, \quad d\hat{F}_{i,t} = a_{i,F} \hat{F}_{i,t}dt + b_{i,F}^* dw_{i,t}
\end{align*}
\]

where \( k = 0, 1, \cdots, w_{i,t} \) is a finite-dimensional Wiener process under \( \mathcal{I}_{i,t} \) and \( \epsilon_{i,t_k} \) is a (finite-dimensional) standard normal random variable conditional on \( \mathcal{I}_{i,t_k}^{-} \) with zero mean and identity covariance-matrix.

3. The coefficient matrices \( \lambda_i, a_{i,X}, b_{i,X}, a_{i,F}, b_{i,F} \) are periodic functions of time on \( \mathcal{T} \), \( a_{i,X}^*, b_{i,X}^*, a_{i,F}^*, b_{i,F}^* \) are periodic functions of time on \( \mathcal{N} \), and \( b_{i,X}^0, b_{i,F}^0 \) are constant matrices.

Clearly, \( F_{i,t} \) and \( X_{i,t} \) are continuous over time except at \( t = t_k \) (when the stock market reopens) since the information flow can be discrete then.

In the context of the paper, \( F_{i,t} = E_{i,t}[F_t] \) is investor \( i \)'s expectation of the fundamental and \( X_{i,t} \) the reduced state vector for investor \( i \). Different investors may have different information concerning the state of the economy. Thus, the reduced state vector \( X_{i,t} \) of investor \( i \) is in general different across investors. Since all investors observe the stock price, \( F_t = F_t - \lambda_1 X_{1,t} = \hat{F}_t - \lambda_2 X_{2,t} \). We later verify that in a linear equilibrium, investors’ expectations about the fundamental indeed takes the form of (30).
Given the above processes of the stock price, the state vector and investors’ expectations, the return processes of investor $i$ can be expressed as follows

$$dQ_t = \alpha_{i,Q} X_{i,t} + b_{i,Q} dw_{i,t}, \quad t \in \mathcal{T} \tag{31a}$$

$$dq_{i,t} = \alpha_{i,q} X_{i,t} + b_{i,q} dw_{i,t}, \quad t \in \mathcal{T} \cup \mathcal{N} \tag{31b}$$

$X_{i,t}$ governs the expected excess returns (on both the stock and private technology) investor $i$ faces. Given that $X_{i,t}$ follows a Gaussian Markov process, it fully determines investor $i$’s current and future investment opportunities. Also, $\alpha_{i,Q}$, $b_{i,Q}$ are periodic functions of time on $\mathcal{T}$, and $\alpha_{i,q}$, $b_{i,q}$, $b_{i,q}^0$ are constant matrices. Thus, the return process exhibits strict periodicity over time with periodicity $T+N$.

Given (29-31), investor $i$’s control problem as defined in Lemma 4 can be solved explicitly. We start by conjecturing that his value function has the following form:

$$t \in \mathcal{T}_k : J_{i,t} = -\exp \left\{ -\rho t - \gamma W_{i,t} - \frac{1}{2} X_{i,t}^t v_i X_{i,t} \right\} \tag{32a}$$

$$t \in \mathcal{N}_k : J_{i,t}^* = -\exp \left\{ -\rho t - \gamma W_{i,t}^* - \frac{1}{2} X_{i,t}^t v_i^* X_{i,t} + \theta_{i,n,k} u_i^* X_{i,t} + \frac{1}{2} \theta_{i,n,k}^2 u_i^* \right\} \tag{32b}$$

where $i = 1, 2, k = 0, 1, \cdots$, $W_{i,t}^* = W_{i,t}^0 + \theta_{i,n,k} F_{i,t}$, $W_{i,t}^0$ is investor $i$’s investment in the risk-free asset, $v_i$ (defined on $\mathcal{T}_k$), $v_i^*$, $u_i^*$ and $u_i^*_{i,0}$ (defined on $\mathcal{N}_k$) are all periodic functions of time of the proper order, where $v_i$ and $v_i^*$ are symmetric matrices. We show that it gives the solution to investor $i$’s control problem by verifying that it satisfies the Bellman equation:

$$t \in \mathcal{T}_k : \quad 0 = \sup_{c_{i,t}, y_i} \left\{ -e^{-\rho t - \gamma c_i} + E_{i,t} \left[ dJ_{i,t} \right] / dt \right\}$$

$$\text{s.t. } dW_{i,t} = (r W_{i,t} - c_{i,t}) dt + \theta_{i,t} dQ_t + y_{i,t} dq_{i,t}$$

$$t \in \mathcal{N}_k : \quad 0 = \sup_{c_{i,t}, y_i} \left\{ -e^{-\rho t - \gamma c_i} + E_{i,t} \left[ dJ_{i,t}^* \right] / dt \right\}$$

$$\text{s.t. } dW_{i,t}^0 = (r W_{i,t}^0 - c_{i,t}) dt + \theta_{i,n,k} dD_t + y_{i,t} dq_{i,t}.$$
\[ t \in [t_k, n_k]: \quad \begin{cases} \theta_{i,t} = h_{i,\theta} X_{i,t} \\ y_{i,t} = h_{i,y} \end{cases} \]
\[ c_{i,t} = \bar{c}_i + r W_{i,t} + \frac{1}{2\gamma} X_{i,t}^t v_i X_{i,t} \]
\[ (33) \]
\[ t \in (n_k, t_{k+1}]: \quad \begin{cases} y_{i,t} = h_{i,y}^* X_{i,t} + h_{i,\theta}^* \theta_{i,n_k} \\ c_{i,t} = \bar{c}_i^* + r W_{i,t}^* + \frac{1}{2\gamma} \left( X_{i,t}^t v_i^* X_{i,t}^* - 2\theta_{i,n_k}^* u_i^* X_{i,t}^* - \theta_{i,n_k}^* \theta_{i,0}^* \right) \end{cases} \]

where

\[ t \in [t_k, n_k):
\]
\[ \begin{bmatrix} h_{i,\theta} \\ h_{i,y} \end{bmatrix} = \frac{1}{\gamma} \sigma_{i,ww}^{-1} (a_{i,w} - \sigma_{i,w^x} v_i) \]
\[ t = n_k:
\]
\[ \begin{bmatrix} h_{i,\theta} \\ h_{i,y} \end{bmatrix} = \left[ \frac{1}{u_{i,0}^t} (r \gamma \lambda_i - u_i^* a_{i,x}^c) \right] \]
\[ \left[ \frac{(a_{i,q}^c - \sigma_{i,q^x} v_i^* a_{i,x}^c) - \frac{1}{u_{i,0}^t} (r \gamma \sigma_{i,qq}^* - \sigma_{i,q^x} u_i^*) (r \gamma \lambda_i - u_i^* a_{i,x}^c)}{(a_{i,q}^c - \sigma_{i,q^x} v_i^* a_{i,x}^c) - \frac{1}{u_{i,0}^t} (r \gamma \sigma_{i,qq}^* - \sigma_{i,q^x} u_i^*)} \right] \]
\[ t \in (n_k, t_{k+1}):
\]
\[ h_{i,y}^* = \frac{1}{\gamma} \sigma_{qq}^{-1} (a_{i,q}^c - \sigma_{i,q^x} v_i^*), \quad h_i^* = \frac{1}{\gamma} \sigma_{qq}^{-1} (r \gamma \sigma_{i,F}^* - \sigma_{i,q^x} u_i^*). \]

Let
\[
\delta_{i,2} = a_{i,x}^t v_i (t_{k+1}) b_{i,x}^t \left[ i + b_{i,x}^t v_i (t_{k+1}) b_{i,x}^t \right]^{-1} b_{i,x}^t v_i (t_{k+1}) a_{i,x}^t \]
\[
\delta_{i,1} = \left[ b_{i,F}^t - \lambda (t_{k+1}) b_{i,x}^t \right] \left[ i + b_{i,x}^t v_i (t_{k+1}) b_{i,x}^t \right]^{-1} b_{i,x}^t v_i (t_{k+1}) a_{i,x}^t \]
\[
\delta_{i,0} = b_{i,F}^t \left[ i + b_{i,x}^t v_i (t_{k+1}) b_{i,x}^t \right]^{-1} b_{i,F}'. \]

Substituting the optimal policies into the Bellman's equation, we obtain the equations for \( v_i, v_i^*, u_i^* \) and \( u_{i,0}^* \), we have

\[ t \in T_k: \quad 0 = \dot{v}_i - v_i \sigma_{i,x} v_i + (a_{i,w} - \sigma_{i,w^x} v_i)^t \sigma_{i,ww}^{-1} (a_{i,w} - \sigma_{i,w^x} v_i) - r v_i + (a_{i,x} v_i + v_i a_{i,x}^t) + [\tilde{v} + \text{tr}(\sigma_{i,xx} v_i)] 1_{11} \]
\[ (34a) \]
\[ t = n_k: \quad v_i = a_{i,x}^t v_i^* a_{i,x}^t + \frac{1}{u_{i,0}^t} (r \gamma \lambda - u^* a_{i,x}^c) (r \gamma \lambda - u^* a_{i,x}^c) \]
\[ (34b) \]
\[ t \in N_k: \quad \begin{cases} 0 = \dot{u}_i^* - v_i \sigma_{i,xx} v_i + (a_{i,q}^c - \sigma_{i,q^x} v_i^*) \sigma_{qq}^{-1} (a_{i,q}^c - \sigma_{i,q^x} v_i^*) - r u_i^* + (a_{i,x} v_i^* + v_i a_{i,x}^t) + [\tilde{v} + \text{tr}(\sigma_{i,xx} v_i)] 1_{11} \\
0 = \dot{u}_{i,0}^* + (r \gamma \sigma_{i,F}^* - \sigma_{i,q^x} u_i^*) \sigma_{qq}^{-1} (a_{i,q}^c - \sigma_{i,q^x} v_i^*) - u_{i,0}^* \sigma_{i,xx} v_i^* - r u_{i,0}^* + u_{i,0}^* a_{i,x}^t + r \gamma \sigma_{i,F}^* v_i^* \]
\[ (34c) \]

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where \( \bar{\nu} = 2(\rho + r \ln r - r) \). The solution to (34), including \( v_i, v_i^*, u_i^*, u_i^{*0} \), characterizes the optimal policies given by (33). In order to show that a solution exists, we need the following lemma.

**Lemma 9** Given terminal value, \( v_{i,n_k} \), (34a) has a solution \( v_i(t; v_{i,n_k}) \) which is symmetric, positive semi-definite and \( \|v_i(t; v_{i,n_k})\| \leq \alpha_i \|v_{i,n_k}\| + \beta_i \). Similarly, given terminal values, \( v_{i,t_{k+1}}^* \), \( u_{i,t_{k+1}}^*, u_{i0,t_{k+1}}^* \), solutions to (34c) exist. And \( v_i^*(t; v_{i,t_{k+1}}^*) \) is symmetric, positive semi-definite and \( \|v_i^*(t; v_{i,t_{k+1}}^*)\| \leq \alpha_i^* \|v_{i,t_{k+1}}^*\| + \beta_i^* \).

**Proof.** First, we verify this claim for (34a). Using the notation of Lemma 7, let \( r = v_i, a_0 = \sigma_{i,w}'i_{i,w}, a_1 = -\frac{1}{2}r_i + a_{i,x} - a_{i,w}'i_{i,w}, a_2 = \sigma_{i,xx} - \sigma_{i,w}i_{i,x}, a_3 = a_{i,x}, a_4 = a_{i,x}, \) and \( a_5 = a_1 \). It is easy to verify that \( a_2 \) is positive definite. Assume \( \bar{\nu} \geq 0 \) (this lemma can easily be shown to hold for \( \bar{\nu} < 0 \)). \( a_3(t,r) \) is a linear positive operator since trace is a linear operator. Hence, (34a) is a matrix Ricatti differential equation. By Lemma 7, given \( v_i(n_k) = v_{i,n_k} > 0, v_i(t; v_{i,n_k}) \) exists and is symmetric, positive semi-definite. Let \( \bar{a}^0 \) and \( \bar{a}^c \) be two arbitrary parameters. Let \( m = -(a_{i,w}'i_{i,w} + \bar{a}^c) \), then by Lemma 7, \( v_i(t; v_{i,n_k}) \leq e(t; v_{i,n_k}) \), where \( e \) is the solution to a linear system given in Lemma 7 with \( a_1 - m = -\frac{1}{2}(r + 2\bar{\nu})i + a_{i,x} < 0 \). By standard linear differential equation theory, \( \|e(t; v_{i,n_k})\| \leq \alpha_i \|v_{i,n_k}\| + \beta_i \), where \( \alpha_i = \exp\{-(r + 2a \bar{\nu} + 2\bar{\nu})\}(r - s)\} \) and \( \beta_i = \|f_0^T \exp\{-(1/2)(r + 2\bar{\nu})\}| + a_{i,x} = -(1/2)(r + 2\bar{\nu})| + a_{i,x} \}. \)

Next, consider (34c). Let \( r = v_i^*, a_0 = \sigma_{i,q}^{-1}i_{i,q} + a_{i,q}, a_1 = -\frac{1}{2}r_i + a_{i,x}, a_2 = \sigma_{i,xx}, a_3 = a_{i,x}, \) and \( a_5 = a_1 \). Apply the same reasoning as above and (34c) is a matrix Ricatti equation as well. Let \( v_i^*(t_{k+1}) = v_{i,t_{k+1}}^* > 0 \) and \( m = \bar{a}^c \), then it follows from above reasoning that \( \|v_i^*(t; v_{i,t_{k+1}}^*)\| \leq \alpha_i^* \|v_{i,t_{k+1}}^*\| + \beta_i^* \) where \( \alpha_i^* = \exp\{-(r + 2a \bar{\nu} + \bar{a}^c)(r - s)\} \) and \( \beta_i^* = \|f_0^T \exp\{-(1/2)(r + 2\bar{\nu})i + a_{i,x} = -(1/2)(r + 2\bar{\nu})i + a_{i,x} \} \}. \)

Finally, given \( v_i^*(t; v_{i,t_{k+1}}^*), u_{i,t_{k+1}}^* \) and \( u_{i0,t_{k+1}}^* \) \( u_i^* \) and \( u_{i0}^* \) exist since they satisfy linear differential systems.

**Lemma 10** The system (34) has a solution.

**Proof.** For \( i = 1, 2 \), let \( z_i = [v_i] \) for \( t \in \mathcal{T}, z_i^* = [v_i^*], [u_i^*], [u_i^{*0}] \), \( t \in \mathcal{N}, z_a = \text{stack}\{z_1, z_2\} \) and \( z_a^* = \text{stack}\{z_1^*, z_2^*\} \). First, the system conforms to the general boundary value problem (24). Next, since \( \sigma_{i,xx} = \text{diag}\{1, 0, 0, (\sigma_{i,xx}^2)/\sigma_{i,qq}\} > 0, \|u_i^*(t; v_{i,t_{k+1}}^*)\| \leq \beta \) where \( \beta = \exp\{-(1/2)(r + 2a \bar{\nu})(N - t)\} \|v_i^*(t_{k+1})\| + \|u_i^*(t_{k+1})\| + \|f_0^N \exp\{-(r + 2a \bar{\nu})(N - s)\} \|v_{i,x}^*(s_{i,x}, s_{i,q})^{-1}a_{i,q} + \sigma_{i,xx}\}ds\| \). Thus, \( u_i^*(n_k; v_{i,t_{k+1}}^*) \) is uniformly bounded in \( v_i(t; v_{i,n_k}) \). Let \( L_i \) be the space
of symmetric, positive semi-definite matrices such that their norms are less than \( \psi_i \) where \( \psi_i = \frac{\zeta_i}{1 - \eta_i} \), \( \eta_i < 1 \), \( \zeta_i = \alpha_i^{*} \beta_i (||a_{x,x}^{c}\|\|a_{x}^{c}\|\|a_{x,x}^{c}\||\|^2 + \alpha_i^{*} ||a_{x}^{c}\|\|a_{x}^{c}\||\|^2 + \beta) \) and \( \eta_i = \alpha_i^{*} \alpha_i (||a_{x}^{c}\|\|a_{x}^{c}\||\|^2) \). Here, \( \alpha_i, \alpha_i^{*}, \beta_i \) and \( \beta_i^{*} \) are given in Lemma 9 with \( \tilde{a}^{0} = ||a_{x}^{c}\||\|^2 \) and \( \tilde{a}^{c} = ||a_{x}^{c}\||\|^2 \). It is easy to verify that \( \eta_i < 1 \). Let \( M_i : L_i \to L_i \) be given by the right hand side of (34b). \( L_i \) is a nonempty, closed, bounded convex subset of a finite dimensional normed vector space. \( M_i \) is symmetric since the right hand side of (34b) is symmetric. \( M_i \geq 0 \) because \( u_{i,0}^{*}(n_k) > 0 \) is positive given its terminal value defined in (34d) and \( v_{i}^{*}(n_k) \geq 0 \) by Lemma 9. It is easy to show that \( ||M_i(u_{i,n_k})|| \leq \frac{\psi_i}{(1 - \eta_i)} \). By Lemma 5, the result follows. \( \square \)

6.3. Proofs for Section 3

In this section, we derive the equilibrium of the economy under symmetric information and provide proofs for Lemma 1, Lemma 2, Theorem 1, and Propositions 1-4.

**Proof of Lemma 1.** When \( I_{i,t} = I_t \), \( i = 1, 2 \), the state variables of the economy are \( D_t, G_t, Y_{1,t} \) and \( Y_{2,t} \), which are observable to all investors. \( X_{i,t} = X_t = [1 \ Y_{1,t} \ Y_{2,t}]' \). \( X_t \) then follows a continuous Gaussian process with \( a_{i,x}^{c} = a_{i,x}^{c} = \tilde{x}^{(0)}, b_{i,x}^{c} = 0, a_{i,x} = a_{i,x}^{c} = -\text{diag} \{a, \alpha_i \} \), \( b_{i,x} = b_{i,x}^{c} = b_x = \text{stack} \{0, b_1, b_2\} \), \( b_{i,f}^{c} = 0, a_{i,f} = a_{i,f}^{c} = -\frac{\tilde{a}^{0}}{r + \tilde{a}^{0}} \), \( b_{i,f} = b_{i,f}^{c} = b_f = \frac{1}{r + \tilde{a}^{0}} \). Applying (33), (34) and Lemma 10, the proof follows. \( \square \)

**Proof of Theorem 1.** For \( i = 1, 2 \), let \( z_{i} = [u_i] \) for \( t \in T \), \( z_{i}^{*} = \text{stack} \{[u_{i}^{*}], [u_{i}^{*}], u_{i,0}^{*}\} \), \( t \in T, i = 1, 2 \). Let \( z_a = \text{stack} \{\lambda', z_1, z_2\} \), \( z_a^{*} = \text{stack} \{z_{1}^{*}, z_{2}^{*}\} \), and \( z = z_a \). This conforms to boundary-value problem (24), where \( f, g_a^{c}, f_{a}^{c}, g_a^{0} \) are given by (11), (12), (13), and (14). Let \( \omega_0 = 1 \). Existence of \( z(t; \theta, \omega_0) \) is given by Proposition 1 and Lemma 10. It remains to verify that \( z(t; \theta, \omega_0) \) is an isolated solution. This is equivalent to showing that \( m(\theta, \omega_0) = \nabla z g(z_0) + \nabla z g(z_T) \exp \{\int_{t_k}^{T} \nabla z f \} \) is nonsingular [see Keller (1992), p. 191]. Clearly, \( m(\theta, \omega_0) \) is analytic. It is easy to show that \( \det(m(\theta_0, \omega_0)) \neq 0 \) for \( \theta_0 = [.001, 1000, 1/4, 1, 7, 7, .08, 1/2, 1, 2, 0] \). By Lemma 8, \( m(\theta_0, \omega_0) \) is generically nondegenerate. By Lemma 6, Theorem 1 holds. \( \square \)

**Proof of Proposition 1.** When \( \kappa_{12} = 1, Y_{1,t} = Y_{2,t} = Y_t \) and \( \theta_{i,t} = 1, i = 1, 2 \). Thus \( P_t = F_t - (\lambda_0 + \lambda Y_t) \) and \( \lambda_0 \) and \( \lambda \) are constants. The equilibrium becomes a stationary one and \( v, v^{*}, u^{*}, u_0^{*} \) are all constants over time. (Given that the two investors are identical, we can drop the index \( i \).) (34a) then reduces to the following algebraic equation:

\[
0 = rv - (a_{x}^{c}v + va_{x}^{c}) + v\sigma_{x,x}v - \frac{1}{\sigma_{x}^{2}}a_{1,q}^{c}a_{1,q} - \delta 1_{11} \\
v^{*} = v, \quad u^{*} = r \gamma \lambda, \quad u_0^{*} = r \gamma^2 (\tilde{a}^{02} + \lambda^2 \sigma_{\gamma}^{2})
\]

Given \( \lambda \), the equation for \( v \) has only two roots, one positive and one negative. The positive root corresponds to the optimal solution of the investors’ control problem since it gives higher expected utility. \( \lambda^* \) and \( \lambda^* \) can then be solved from \( \theta_{i,t} = 1 \). \( \square \)
Proof of Proposition 2. From Proposition 1, it is easy to show that $u^*_1 = r\gamma[0 \bar{\lambda} 0]$, $u^*_2 = r\gamma[0 0 \bar{\lambda}]$, and $u^*_0 = r\gamma^2[\bar{\sigma}_F^2 + (\bar{\lambda})^2 \sigma_F^2]$. Next $h(T) = -\bar{\lambda}/\bar{\lambda}_0$. It is straightforward from Proposition 2 that $\bar{\lambda}/\bar{\lambda}_0 < |\bar{\lambda}|$. Next, $\lambda_0$ is given by $r\lambda_0 - \bar{\lambda}_0 = r\gamma\bar{\sigma}_F^2$, with $\lambda_0(T) = \bar{\lambda}_0$, which has the solution $\lambda(t) = (\bar{\lambda}_0 - r\gamma\bar{\sigma}_F^2)e^{-r(t-T)} + \gamma\bar{\sigma}_F^2$. $\lambda(t)$ is strictly increasing over time. □

Proof of Proposition 3. Given the periodicity in equilibrium, $E[P_{tk}] = -\lambda_0(0)$ and $E[P_{nt}] = \lambda_0(T)$, independent of $k = 0, \cdots$. Thus $E[R^{co}] = E[R^{cc}] = 0$. Furthermore, $E[R^{co}] = \lambda_0(T) - \lambda_0(0) = -E[R^{cc}]$. For $\lambda_0(0) < \lambda_0(T)$, we have $E[R^{co}] > 0 > E[R^{cc}]$. □

Proof of Proposition 4. Since $P_t = F_t - \lambda_0(t) - \lambda(t)Y_t$ for $t \in T$, we have

$$R_{t,s} = \int_t^s dD_r d\tau + (F_s - F_t) - [\lambda(s)Y_s - \lambda(t)Y_t]$$

Note that $Y_s = e^{-\gamma(s-t)}Y_t + \int_t^s e^{-\gamma(s-r)}b_r dw_r$. It then follows that $\text{Var}[R^{co}] - \text{Var}[R^{cc}] = [\lambda^2(0) - \lambda^2(T)]\text{Var}[Y_T - Y_0] > 0$ for $\lambda(0) > \lambda(T)$ and $\text{Var}[R^{cc}] - \text{Var}[R^{co}] = 0$. □

6.4. Proofs for Section 4

We now consider the equilibrium under asymmetric information. Let $\tilde{G}_t = E_{2,t}[G_t]$, $\tilde{G}_t = E_{2,t}[G_t]$, and $\tilde{F}_t = E_{2,t}[F_t]$ denote investor 2’s expectations about unobserved state variables. The conjectured price function has the following form:

$$P_t = F_t - \lambda_1 X_{1,t} = \tilde{F}_t - \lambda_2 X_{2,t}, \quad t \in T \quad (35)$$

where $X_{1,t} = [1, Y_{1,t}, Y_{2,t}, G_t - \tilde{G}_t, \tilde{Y}_{1,t}, \tilde{Y}_{2,t}]$, $X_{2,t} = [1, \tilde{Y}_{1,t}, Y_{2,t}]$. Footnote 19 implies that we can also rewrite the price function as

$$P_t = F_t - \lambda[1 Y_{1,t} Y_{2,t} G_t - \tilde{G}_t]$$

where $\lambda = \lambda_1 = [\lambda_0, \lambda_1, \lambda_2, r + \rho_{\alpha} - \lambda_2]$ and $\lambda_2 = [\lambda_0, \lambda_1, \lambda_2]$. Observing $P_t$ is equivalent to observing $\tilde{F}_t = \lambda_0 G_t - \lambda_1 Y_{1,t}$. We need to verify that (35) leads to an equilibrium.

Proof of Lemma 2. Given the conjectured price function, let us first solve for investor 2’s conditional expectations of unobserved state variables, $Z_t = [G_t, Y_{1,t}]$, can be calculated from Theorem 10.2 of Lipster and Shiryaev (1977). □

For $k = 0, 1, \cdots$, define the following matrices:

$$t \in T_k: \quad k = \text{stack} \{k_1, k_2\} = (\sigma_{sz} + \alpha a_s')\sigma_{ss}^{-1}$$
$$t \in N_k: \quad k = \text{stack} \{k_1, k_2\} = (\sigma_{sz} + \alpha a_{s'})\sigma_{ss}^{-1}$$
$$t = t_{k+1} \quad k' = \text{stack} \{k_1', k_2'\} = [\lambda_z(t_k) o(t_k') \lambda_z'(t_k)]^{-1} o(t_k') \lambda_{z'}(t_k)$$

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Proof of Lemma 3. For the solution to the investors’ optimization problems, we can directly apply the results in Section 6.2. Let \( a_\Delta = a_\sigma + \lambda_\sigma k_{11} [a_\nu - a_\sigma (\dot{\lambda}_1 / \lambda_1 - \dot{\lambda}_\sigma / \lambda_\sigma)] + k_{12}, b_\Delta = b_\sigma - k_1 b_s, \)

\[ a_{1,x} = -\text{diag} \{0, a_\nu, a_\nu, a_\Delta\}, \quad a_{2,x} = -\text{diag} \{0, a_\nu, a_\nu\}, \]

\[ a^*_1 = \text{diag} \{a_{2,x}, a_z + k_1 (2,2)\}, \quad a^*_2 = a_{2,x}, \quad a_Q = \lambda_1 \left( r^* (5) - a_{1,x} \right) - \dot{\lambda}_1 \]

\[ a_{1,w} = \text{stack} \{a_Q, 1_{12}^{(1,4)}\}, \quad a_{2,w} = \text{stack} \{\lambda_2 \left( r^* (3) - a_{2,x} \right) - \dot{\lambda}_2, 1_{13}^{(1,3)}\} \]

\[ b_{1,x} = \text{stack} \{0, b_1, b_2, b_\Delta\}, \quad b_{2,x} = \text{stack} \{0, k_2 b_s, b_2\} \]

\[ b^*_1, x = \text{stack} \{0^{(1,5)}, b_1, b_2, b_G - k_1 b_s^*, b_1 - k_2 b_s^*\}, \quad b^*_2, x = \text{stack} \{0^{(1,5)}, k_2 b_s^*, b_2\} \]

\[ b^o_{1,F} = 0, \quad b^o_{2,F} = [\lambda_\sigma (t_k) k_1^o - \lambda_1 (t_k) k_2^o] \lambda_z (t_k) o(t_k^-)^{1/2} \]

\[ b^o_{1,x} = 0, \quad b^o_{2,x} = \text{stack} \{0, k_2^o \lambda_z (t_k), 0, 0\} \text{ stack} \{0^{(1,2)}, o(t_k^-)^{1/2}\} \]

\[ b^*_{1,F} = b_{1,F}, \quad b^*_{2,F} = b_F, \quad b^*_{2,F} = \frac{1}{r + a_\sigma} k_2 b_s, \quad b_Q = b_D + b_F - \lambda_1 b_{1,x}, \quad b_{i,w} = \text{stack} \{b_Q, b_Q\} \]

\[ a^c_2, x = a^o_{2,x} = i^{(3)} \]

\[ a^c_{1,x} = \text{diag} \{i^{(3)}, \lambda_\sigma (n_k) / \lambda_1 (n_k)\}, \quad a^c_{2,x} = \text{stack} \{[i^{(3)} 0^{(3,2)}], [0^{(1,3)} 1_{11}^{(1,2)} - k_1^o \lambda_z (t_k)]\} \]

and \( \epsilon_{t_k} \equiv o(t_k^-)^{-1/2} (Z_t^k - \hat{Z}_t^k). \) By Lemma 10, the results follow. □

Proof of Theorem 2. Given the investors’ optimal investment policies, the market clearing condition then becomes

\[ \omega \lambda_1 h_{1,\theta} + (1 - \omega) r' h_{2,\theta} = 1_{11}^{(4,1)}, \quad t \in T_k, \; k = 0, 1, \ldots \] (36)

where stack\{[\lambda_1, 0, 0, 0], [0, \lambda_1, 0, -\lambda_\sigma], [0, 0, \lambda_1, 0]\}. The periodic condition for the price function requires

\[ \lambda_i (t_{k+1}) = \lambda_i (t_k), \quad k = 0, 1, \ldots \] (37)

Let \( z_i = [v_i] \) for \( t \in T, \) \( z^*_i = \text{stack} \{[v^*_i], [u^*_i], u^*_i 0\}, \) \( t \in T, \) \( i = 1, 2. \) Let \( z_a = \text{stack} \{z^* \}, \) \( z_a^* = \text{stack} \{z^*_1, z^*_2\} \) and \( z_a^* = \text{stack} \{z^*_1, z^*_2\}. \) Let \( z_b = \text{stack} \{o^*\} \) and \( z_b^* = \text{stack} \{o^*\}. \) Then the problem conforms to boundary value problem (24). Let \( \omega_0 = 1. \) Existence of \( z(t, \theta, \omega_0) \) is given by Proposition 1 and Lemma 10. We first note that in the case of asymmetric information, \( \hat{z} \) and \( z \) are related implicitly: \( 0 = F(\hat{z}, z; \theta, \omega) \), \( t \in [t_k, n_k]. \) By the Implicit Function Theorem [see Protter and Morrey (1991)], \( F \) defines an implicit function: \( \hat{z} = f(z; \theta, \omega) \) if \( \nabla z F \) is nonsingular. For \( \theta_0 = \{0.01, 1000, 1/4, 1, 7, 7, .08, 1/2, 0, 1/2, 1/2, 0\} \) and \( \omega_0 = 1, \; \text{det}(\nabla z F) \neq 0. \) By Lemma 8, \( \hat{z} = f(z; \theta, \omega) \) exists generically. It remains to verify that \( z(t; \theta, \omega_0) \) is a isolated solution. This is equivalent to showing that \( m(\theta, \omega_0) = \nabla z g(z_0) + \nabla z g(z_T) \exp \{\int_{t_k}^{n_k} \nabla z f\} \) is nonsingular [see Keller (1992), p.191]. Clearly, \( m(\theta, \omega_0) \) is analytic. It is easy to show that \( \text{det}(m(\theta_0, \omega_0)) \neq 0. \) By Lemma 8, \( m(\theta, \omega_0) \) is generically nondegenerate. By Lemma 6, Theorem 2 holds. □
6.5. Numerical Procedure

We briefly discuss the numerical procedure used to solve for the periodic equilibrium. We use the Newton-Kantorovich method to solve this problem numerically [see, e.g., Kubicek (1983)]. This recursive method linearizes the system and the boundary conditions around a conjectured solution to the nonlinear problem at a discrete number of points in the interval $[t_k, n_k]$. Since the system is linearized, it is easy to calculate an updated solution that satisfies the linearized system and boundary conditions from the conjectured solution. The updated solution is then used as the conjectured solution to start the next recursion. It can be shown that the limit of this recursion converges to the solution of the nonlinear problem given that the initial conjectured solution is not too far away from the true solution.

This method requires a sufficiently accurate initial guess of the true solution. We obtain such a guess by starting the recursion at $\omega = 1$ for any given set of parameters since a solution exists at $\omega = 1$. In order to calculate a solution at $\omega_0 < 1$, we begin by using the solution at $\omega = 1$ as the initial guess to find a solution for an $\omega$ close to 1 and repeat the same procedure to move toward $\omega_0$.

Since we have no knowledge on the uniqueness of the solution, the above procedure also guarantees that we stay on the same branch of solutions. In particular, the solution gives the expected results when we take the limit $\omega \to 1$. We have also checked the solution by taking other limits in the parameter space such as $\sigma_2 \to 0$, $\kappa_{12} \to 1$ or $-1$ and obtained the expected results.
7. References


Chapter 2

A Dynamic Model Of Spot And Futures Markets

Motivated by regulatory concerns, the effects of trading in futures on the underlying cash market have drawn intense empirical interest. The central issue addressed by this empirical literature is whether introducing futures destabilizes spot prices. There is much evidence which indicates that futures trading tends to decrease spot return volatility in a number of storable commodities.¹ An interesting study by Netz (1995) of wheat and corn futures markets indicates that futures trading tends to simultaneously lower spot price variability and increase storage (i.e. price elasticities to demand and supply shocks fall while storage elasticities to these shocks rise). While much attention has been paid to such static effects of futures trading, much less attention has been devoted to the concomitant dynamic effects. Intuitively, since the correlatedness of spot and futures prices vary as a futures expires, the natural question is how spot return volatility and storage vary as a futures expires?²

This paper develops a model of a competitive futures market to analyze such dynamic effects of futures trading on the underlying cash market. In this economy, risk averse investors can simultaneously trade in both spot and futures. Holding spot positions (as opposed to futures positions) provides the owner with an exogenously specified, mean reverting convenience yield. Some investors are exposed to non-marketed risks, which are also mean reverting. They take positions in the spot to hedge their exposure to these risks. They take positions in futures to hedge their exposure to spot price risks due to movements in the stochastic convenience yield. Think of these investors as producers and storers with inherently different preferences for holding inventories (storage). It is assumed that the available set of spot and futures does not span these risks. Hence, futures are not redundant securities and cannot be priced by arbitrage. Instead, spot and futures prices are determined jointly.

The spot price fluctuates due to both convenience yield and non-marketed risk shocks. Spot return volatility is partly due to convenience yield shocks because the spot price discounts the future stream of convenience yield payoffs. Spot return volatility is also driven by non-marketed

¹See Rindi (1988) and Brorsen, et.al. (1989) for reviews of the evidence.
²See Bessembinder (1992) for some weak evidence that spot return volatility does not remain constant.
risk shocks because these shocks cause some investors to change their quantity demanded of the spot. Because investors are risk averse, they demand a risk premium to accommodate these changes in spot positions. The extent to which investors are willing to accommodate these changes are limited by spot price risks due to movements in the convenience yield. Hence, the more variable are convenience yield shocks, the more sensitivity the spot price becomes to non-marketed risk shocks as investors are less willing to accommodate these shocks.

Since the futures price is positively correlated with convenience yield movements, appropriate positions in futures reduces the price risk of taking spot positions. This tends to make investors more willing to take on spot positions (i.e. storage elasticities to shocks increase). As such, the spot price becomes less sensitive to non-marketed risk shocks. Hence, trading in futures tends to decrease the level of spot return volatility by increasing the willingness of investors to take on spot positions.

As a futures expires, its effectiveness in hedging spot price risks changes, leading to time variation in spot return volatility associated with its expiration. At its maturity, investors lose an important hedging vehicle. Thus, as the nearest to maturity futures expires, spot return volatility tends to rise in anticipation of the loss of this hedging vehicle. On the other hand, as the nearest to maturity futures expires, other futures tend to be introduced. In anticipation of these future trading opportunities, spot return volatility tends to fall as the nearest to maturity contract expires. Hence, as a contract rolls toward expiration, spot return volatility may initially rise and then fall.

In contrast, the return volatility of a futures monotonically rises as the contract expires. In the presence of mean-reverting shocks to spot prices (convenience yield and non-marketed risks), the price of a given futures is less sensitive to contemporaneous shocks than the spot price (e.g. Samuelson (1965)). These mean reverting shocks are less reflected in the price of the futures than the spot because they will die away by the time the contract expires. As a futures matures, its price elasticities to contemporaneous shocks increase and its return volatility rises. Hence, this model predicts that there need not be any relationship between the Samuelson effect in the futures market and the behavior of spot return volatility as a futures expires.

In addition to these return volatility patterns, there are also distinct patterns in spot positions (storage). When futures provide an imperfect hedge of spot commitments, the risk to taking spot positions may increase with the time-to-maturity of a given contract and hence spot positions may fall with the time-to-maturity of the contract. The rationale is that the futures price becomes relatively less sensitive to transitory convenience yield shocks the farther the contract is from maturity. Hence, the transitory component of the convenience yield becomes less hedgeable with the time-to-maturity of the contract. In addition to this time variation, this model also predicts that spot volume tends to rise at the expiration of a futures since investors cut back on their spot positions with the loss of a contract.

Since the level of open interest depends on the level of spot commitments, the fact that investors can simultaneously trade in both spot and futures leads to interesting patterns in the
open interest. Intuitively, the level of open interest in the futures is a function of the price elasticities of the futures to convenience yield shocks and the magnitude of investor a’s spot position. When the futures price is more sensitive to underlying shocks, a smaller position in the futures is required to hedge a given spot position. When the spot position increases, a larger position in the futures is required for a given price sensitivity of the futures to underlying shocks. Mean reversion in convenience yield shocks implies that the futures price elasticities are inversely related to time-to-maturity. Hence, holding fixed spot commitments, open interest increases with the contract’s time-to-maturity. However, the level of the spot position need not be fixed and can be inversely related to time-to-maturity. When the futures only provides an imperfect hedge of spot commitments, the risk to taking spot positions may increase with time-to-maturity and hence spot positions may fall with time-to-maturity. This tends to imply that lower levels of open interest are observed with time-to-maturity. The interaction of the time-to-maturity patterns in futures price elasticity and spot commitments can generate an inverted U-shaped time-to-maturity pattern in open interest.

The empirical literature has not focused on the dynamic effects of futures expiration on the cash market. Most existing theoretical models on this issue are static ones (see, e.g. Netz (1995) for a review). While there are dynamic models of futures trading (see, e.g. Breeden (1984), Duffie (1989), Duffie and Jackson (1990)), they generally assume that the spot price is exogenously specified. Of course, studying the dynamic effects of futures trading on the cash market requires that the spot price be endogenously determined with futures prices. It is the contention of this paper that modeling the interaction of these two markets in a dynamic setting offers a number of rich empirical predictions regarding the behavior of spot return volatility, spot positions (storage) and open interest. Our model is related Wang (1994).

Our model applies to storable commodities in which the presence of a convenience yield implies that the standard cost of carry formula does not apply. Hence, this model is more applicable for storable commodities such as crude oil, corn, wheat and less applicable for some metals and financials. Our model also leaves out another important function of futures markets—price discovery (see e.g. Grossman (1977, 1986, 1988)). When futures are also important sources of information, the effects of futures trading on the cash market become more complex (see e.g. Stein (1987) and Hong (1997)). We intentionally leave out these informational effects and focus on futures as risk management tools. The existing empirical evidence suggests that such allocational effects are quite prominent in a number of markets. The contribution of this paper is to develop additional testable predictions resulting from this particular function of futures.

This paper proceeds as follows. We present our model in Section 1 and discuss the definition and general solution of the equilibrium in Sections 2.1 and 2.2 respectively. Results are presented in Section 3. Section 4 contains a summary of the testable implications of this model.
1. The Model

We consider a single good economy defined on a discrete time, infinite horizon indexed by $t$ where $t \in \{0, 1, 2, \ldots, \infty\}$. The underlying uncertainty of the economy is generated by a vector of i.i.d. standard normal shocks denoted by $\epsilon_t$. There are two classes of investors in the economy, $i = a, b$, with population weights $\omega$ and $1 - \omega$, respectively, where $\omega \in [0, 1]$. Investors in class $i$ will also be referred to as investor $i$.

A. Competitively Traded Securities

There are three competitively traded securities. First, there is a riskless asset that pays a constant gross rate of return per period of $R = 1 + r$ ($r > 0$). The riskless interest rate $r$ is exogenously specified. Investors are initially endowed with $\bar{\theta}$ shares of the spot asset. This spot pays the owner $D_t$ each period:

$$D_{t+1} = \sum_{i=1}^{2} Z_{t,t+1} + b_d \epsilon_{t+1}$$

$$Z_{t+1} = a_z Z_t + b_z \epsilon_{t+1}$$

where $Z_t = [Z_{1,t}, Z_{2,t}]'$.\(^3\) It is competitively traded and has a price of $S_t$. Additionally, investors can also competitively trade in a futures written on the spot price. The futures is in zero net supply. The futures is also competitively traded and has a price of $H_t$. Since the riskless interest rate is constant in this economy, futures are equivalent to forwards.\(^4\)

The spot asset can be thought of as a storable commodity. The payoff stream to holding the commodity, $D_t$, is the "convenience yield" that accrues from holding inventories—the value of any benefits that inventories provide, including the ability to smooth production, avoid stockouts, and facilitate the scheduling of production and sales. Such an exogenous, stochastic convenience yield can be regarded as a reduced form of a more general model in which the convenience yield is determined endogenously by production and consumption decisions. It can be motivated by the empirically documented importance of convenience yields in driving spot and futures prices in numerous markets.\(^5\)

---

\(^3\) $b_d$, $a_z$, and $b_z$ are constant matrices. $D_t$ is composed of two components. The first is a stochastic growth rate (time-varying expected return) given by $\sum_{i=1}^{2} Z_{i,t}$. This time-varying expected return follows a two factor model, where the two factors follow a vector auto-regressive process. It is assumed that these two factors have differing degrees of persistence, denoted by $a_{z,i}$, where $0 < a_{z,i} \leq 1$ ($a_z = \text{diag}(a_{z,1}, a_{z,2})$). For a set of elements $e_1, e_2, \ldots, e_m$, $\text{diag}(e_1, e_2, \ldots, e_m)$ is a diagonal matrix with these elements as its diagonal elements. Hence, $D_t$ exhibits mean reversion. The matrices $b_d$ and $b_z$ determine the exposure of $D_t$ and $Z_{i,t}$ to the underlying shocks. $D_t$ has an exposure of $\sigma_d$ and $Z_{i,t}$ has an exposure of $\sigma_{z,i}$.

\(^4\) The importance of stochastic interest rates in futures pricing is discussed empirically in Fama and French (1987) and theoretically in Cox, Ingersoll, and Ross (1981, 1985a,b).

\(^5\) See Pindyck (1993) for a discussion on convenience yields. See Brennan (1991) for a motivation of an exogenous convenience yield in the context of commodity contingent claims pricing. See Fama and French (1987) for a study...
We adopt the specification given by (1) for the convenience yield for two reasons. First, this specification can be motivated by the empirically documented mean-reversion of prices in a number of futures markets.\textsuperscript{6} Second, the multi-factor model keeps markets incomplete when futures are traded.

\textbf{B. Non-marketed Risks And Storage}

In addition to these publicly traded securities, it is assumed that investor \( a \) receives a non-marketed income.\textsuperscript{7} Investor \( a \)'s non-marketed income has an excess return each period of \( q_t \) given by

\begin{align}
q_{t+1} &= Y_t' a_q Y_t + e Y_t b_q \varepsilon_{t+1} \\
Y_{t+1} &= a_Y Y_t + b_Y \varepsilon_{t+1},
\end{align}

(2a) \hspace{1cm} (2b)

where \( Y_t = [1, Y_{1,t}]' \).\textsuperscript{8} All shocks in the economy are uncorrelated except for innovations to the convenience yield and investor \( a \)'s non-marketed risks. These two shocks are assumed to be positively correlated, \( \text{Cov}(b_D \varepsilon_t, b_q \varepsilon_t) = \kappa_{pq} > 0 \).\textsuperscript{9} When \( Y_t > 0 \), we have that the local innovation to the convenience yield is correlated with that of investor \( a \)'s non-marketed risks.

Hence investor \( a \) shorts the spot to counteract her exposure to these non-marketed risks. Investor \( b \) makes the market for investor \( a \). By assuming some correlation between the convenience yield and investor \( a \)'s non-marketed risk shocks, we generate different preferences for the spot on the part of these two groups of investors. This leads to trade in the spot. The elasticity of investors' spot positions to the non-marketed risks shocks can be thought of as storage elasticities. Additionally, investors take positions in futures to hedge their resulting spot positions. Here, investor \( a \) plays the role of Keynesian hedgers who long futures to hedge their exposure to spot price risk, while investor \( b \) makes the market for investor \( a \). There is an obvious mapping from this model to more conventional models of risk averse storers as in Netz (1995).

\textsuperscript{6}See Bessembinder, et.al. (1995) for findings on mean reversion in spot and futures prices across a variety of futures markets.

\textsuperscript{7}We could also give investor \( b \) a non-marketed income as long as its payoffs are not perfectly correlated with investor \( a \)'s. But for expository simplicity, we do not. Our results are robust to this assumption.

\textsuperscript{8}\( a_q, b_q, a_Y, \) and \( b_Y \) are constant matrices. \( q_t \) has two components. The first component is a stochastic growth rate given by the quadratic form \( Y_t' a_q Y_t \). A special case of this specification is just the linear factor model which drives the time-varying expected return of the convenience yield. This form is the most general specification possible and yet still keep the problem tractable. Realizations from this specification can be positive or negative if \( a_q \) is an indefinite matrix. Here, the non-marketed income has a stochastic volatility given by a linear factor model: \( (1 + Y_{1,t}) b_q \varepsilon_{t+1} \) (\( e \) is \( (1,1,...,1) \) of proper order). The realizations of the drift and volatility is driven by \( Y_t \) which follows a vector auto-regressive process with differing degrees of persistence, \( 0 < \alpha_{Y_1} < 1 \) (\( a_Y = \text{diag}(1, \alpha_{Y_1}) \)). \( b_q \) and \( b_Y \) determine the exposure of the non-marketed risks to the underlying vector of uncertainty. \( q_t \) has an exposure given by \( a_q \) and \( Y_{1,t} \) has an exposure of \( \sigma_{Y_1} \).

\textsuperscript{9}It is unimportant whether this correlation is assumed to be positive or negative as long as it is not zero.
C. Timing of Futures Introduction and Expiration

Investors can trade in the riskless asset and the spot at all times. They can also trade in a futures with a maturity length of \( M \). Figure 1 illustrates the futures maturity cycle in this economy. The solid lines indicate when a particular contract is traded while the dotted lines indicate when it is not. At \( m_{k-1} \), the \((k-1)\)-th replication of the futures has matured. After a period of length \( N \) represented by the dotted lines, the \( k \)-th replication of the nearby begins trading at \( t_k \), and matures after a length of \( M \) at \( m_k \). This maturity and replication process cycles periodically from the \( k \)-th replication to the \((k+1)\)-th replication and so on.

This replication structure generates two trading regimes. Let \( \tau = o, p \) and \( \mathcal{M}_k(\tau) \) denote the trading regimes. \( \mathcal{M}_k(o) \equiv \{ t_k, \cdots, m_k - 1 \} \) denotes the set of dates in which the futures is traded during the \( k \)-th cycle. \( \mathcal{M}_k(p) \equiv \{ m_k, \cdots, t_{k+1} - 1 \} \) denotes the set of dates in which positions cannot be taken in the futures. Notice that by letting \( N \to 0 \), we obtain the special case of one futures always being traded. This periodic replication of futures is a simple way to get at the inherent effects of a futures maturing and being introduced into the economy. In this paper, we want to understand the allocational effects of a futures maturing (as the futures price converges to the spot price).

D. Information Endowments

The total set of information in the economy is composed of the spot price, futures price, realizations of the convenience yield and the stochastic growth rates of the convenience yield and investor \( a \)'s non-marketed risks:

\[
 t \in \{0, 1, 2, \cdots, \infty\} : \mathcal{I}_t \equiv \{ S_s, H_s, D_s, Z_s, Y_s, q_s : 0 \leq s \leq t \}. \tag{3}
\]

Let \( \mathcal{I}_{i,t} \) denote the information set of investor \( i \), \( i = a, b \). In general, \( \mathcal{I}_{i,t} \subseteq \mathcal{I}_t \), but we will adopt a complete, symmetric information structure, \( \mathcal{I}_{a,t} = \mathcal{I}_{b,t} = \mathcal{I}_t \).

E. Policies and Preferences

The investors choose consumption and investment policies to maximize their expected utility over life-time consumption. Let their consumption policies be denoted by \( \{ c_{i,t} : t \in \{0, 1, 2, \cdots, \infty\} \} \). Their investment policies in the spot and futures will depend on the different trading regimes, \( \mathcal{M}_k(\tau) \). Let \( \theta_{i,t}(\tau) \) be the holdings vector of investor \( i \). Recall that \( \tau = o, p \) denotes the particular

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trading regimes and hence the holdings vector will change dimensionality depending on whether the futures is traded. We assume that the investors' consumption and investment policies are adapted to their information sets \( I_{i,t} \).

The investors have CARA preferences and they maximize the expected utility of the form:

\[
i = a, b : \quad E_{i,t}\left[-\sum_{s=t}^{\infty} \rho^{(s-t)} e^{-\gamma c_{i,s}}\right],
\]

where \( E_{i,t} \) is the expectation operator conditional on investor \( i \)'s information set, \( I_{i,t} \). For simplicity, we assume that the time-discount factor, \( \rho \), and the relative risk-aversion coefficient, \( \gamma \), are identical across investors. Table 2.1 summarizes the parameters of the model.

### Table 2.1: Summary of Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population weight of type-1 investors</td>
<td>( \omega )</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>( r )</td>
</tr>
<tr>
<td>Volatility of convenience yield</td>
<td>( \sigma_p )</td>
</tr>
<tr>
<td>Mean reversion of convenience yield growth rate shocks for ( i = 1, 2 )</td>
<td>( a_{x_i} )</td>
</tr>
<tr>
<td>Volatility of convenience yield growth rate shocks for ( i = 1, 2 )</td>
<td>( \sigma_{x_i} )</td>
</tr>
<tr>
<td>Volatility of returns on investor ( a )'s non-marketed income</td>
<td>( \sigma_q )</td>
</tr>
<tr>
<td>Mean reversion of non-marketed income growth rate shock</td>
<td>( a_{y_i} )</td>
</tr>
<tr>
<td>Volatility of non-marketed income growth rate shock</td>
<td>( \sigma_{y_i} )</td>
</tr>
<tr>
<td>Correlation of convenience yield and non-marketed income shocks</td>
<td>( \rho_{Dq} )</td>
</tr>
<tr>
<td>Length of A Futures</td>
<td>( M )</td>
</tr>
<tr>
<td>Length of Nontrading In A Futures</td>
<td>( N )</td>
</tr>
</tbody>
</table>

### 2. Equilibrium

We first define what it means for the economy to be in equilibrium. In Lemma 1 we characterize the investors' value functions and optimal policies for given spot and futures price processes. We then prove in Theorem 1 that an equilibrium exists generically for \( \omega \) close to 1 (the fraction of type \( b \) investors in the economy is small).

#### 2.1. Definition of Equilibrium

An equilibrium of the economy specified above is defined by a spot price process \( \{S_t : t \in \{0, 1, 2, \ldots, \infty\}\} \) and a futures price process \( \{H_t : t \in \{t_k, \ldots, m_k\}\} \) for \( k = 0, 1, 2, \ldots \), such that investors follow their optimal policies, \( \theta_{i,t}(\tau) \), to maximize their expected utilities. In general,
the equilibrium spot and futures prices and investors' optimal policies can be expressed as a function of time and a relevant set of state variables, $\Psi_t$.

The dependence of the equilibrium prices and policies on time follows from the nature of a futures. The expiration of a futures implies that its price will depend not only on the evolving uncertainty in the economy but also on its time-to-maturity. Because some investors in the economy use futures to hedge their positions in the spot, this implies that the optimal policies of the investors will vary with the time-to-maturity of futures and hence so will the equilibrium spot price as well as the futures price.

Due to the nature of the periodic replication of futures after maturity, we will in this paper consider periodic equilibria in which the equilibrium price processes and investors' optimal policies exhibit periodicity in time. And due to our assumptions of a constant risk-free rate and constant absolute risk aversion in preferences, investors' demand of risky investments will be independent of their wealth. Hence, $\Psi_t$ will be independent of $W_{it}$. $\Psi_t$ will in general include the state variables $\{Z_t, Y_t\}$ which characterize the future payoffs of the investments. Since we assume that investors observe all of these variables, $\Psi_t = \{Z_t, Y_t\}$ is a set of sufficient statistics for the information sets of the investors, $\mathcal{I}_{it}$. Let $\bullet$ denote some subset of $\Psi_t$ that we will specify below. Finally, we restrict ourselves to a linear periodic equilibrium.

**Definition 1** In the economy defined above, a linear periodic equilibrium is defined by $S_t = S(\bullet; t), H_t = H(\bullet; t)$ where $S(\bullet; t)$ and $H(\bullet; t)$ are linear in $\bullet$ and periodic in time, i.e. for $k = 0, 1, 2, \ldots$

\[
\alpha \in \{0, 1, 2, \ldots, M + N\}: \quad S(\bullet; \alpha) = S(\bullet; t_k + \alpha) \quad (5a)
\]
\[
\alpha \in \{0, 1, 2, \ldots, M\}: \quad H(\bullet; \alpha) = H(\bullet; t_k + \alpha). \quad (5b)
\]

Additionally, investors' policy functions are such that (i) the policies maximize investors' expected utility, (ii) the spot and futures markets clear and (iii) the investors' policy functions are periodic in time.

In other words, $S_t$ is defined at all times. $H_t$ is only defined on $\{t_k, \ldots, m_k\}$. In addition, the futures price must converge to the spot price at maturity: for $k = 0, 1, \ldots$,

\[H_{mk} = S_{mk}. \quad (6)\]

This condition must be satisfied for there to be no arbitrage in the economy.\(^{10}\)

Observe that although the underlying uncertainty in the economy evolves homogeneously through time, the functional form of the equilibrium price processes have the same functional

\(^{10}\)It is possible in reality for the futures price not to converge to the spot price. This would require a different boundary condition.
form for every futures maturity cycle. The realized values of the spot and futures prices will be
different each cycle since the state variables can be different.

The return per share each period from investing in the spot, \( Q_{0,t} \), is composed of the stochastic
convenience yield, \( D_t \), and the discounted capital gains \( S_{t+1} - RS_t \): \( Q_{0,t} = S_{t-1} - RS_t + D_t \).
We assume that fluctuations in the futures price are settled each period and hence the return
each period from taking a position in a futures is \( Q_{1,t} = H_{t+1} - H_t \). The vector of return
processes is denoted by \( Q_t(\tau) \) which changes dimensionality depending on the trading regime.

We now state the investors’ control problem so as to elaborate on the nature of the periodic
equilibrium. Given the investors’ preferences in (4), the investors’ optimization problems are
given by: for \( i = a, b \)

\[
J_i\left(W_{i,t}, \cdot, t\right) \equiv \sup_{\{c_i, \theta_i(\tau)\}} \mathbb{E}_{i,t} \left[-\sum_{s=t}^{\infty} \rho^{(s-t)}e^{-\gamma c_{i,s}}\right]
\]

\[
\text{s.t. } W_{i,t+1} = (W_{i,t} - c_{i,t}) R + \theta_{i,t}(\tau)Q_t(\tau) + 1_i q_t,
\]

where \( 1_i = 1 \) if \( i = a \) and \( 1_i = 0 \) if \( i = b \), and \( J_i\left(W_{i,t}, \cdot, t\right) \) is investor \( i \)'s value function. \( \cdot \)
denotes the relevant state variables that characterize the return processes of the spot, futures
and their non-marketed risks given their information.

We assume that investors’ value functions will also exhibit periodicity: for \( k = 0, 1, 2, \cdots \)

\[
t \in \{0, 1, 2, \cdots, \infty\} : J_i (\cdot, ; t) = J_i (\cdot, ; t + M + N), \quad i = a, b.
\]

The periodicity condition for the value functions, (8), provides the necessary boundary conditions
for a periodic solution.

The spot market has to clear at all times and the futures are in zero net supply when traded:

\[
t \in \mathcal{M}_k(\tau) : \omega \theta_{a,t}(\tau) + (1-\omega) \theta_{b,t}(\tau) = \bar{\theta}(\tau)
\]

Here, \( \bar{\theta}(\tau) \) is a matrix denoting the supply of the securities in the economy and depends on the
trading regime. Thus, a periodic equilibrium is given by periodic price functions (5) such that
investors optimally solve (7)-(8), and the markets clear—(9) holds in equilibrium.

2.2. The General Solution

We assume that \( \mathcal{I}_{a,t} = \mathcal{I}_{b,t} = \mathcal{I}_t \), where \( \mathcal{I}_t \) is given by (3). We conjecture that the equilibrium
asset prices have the following linear form:

**Conjecture 1** Let \( P_t(\tau) \) be a subvector of \([S_t, H_t]^T\) which depends on the trading regime. A
linear periodic equilibrium is \( P_t \) such that: for \( \tau = o, p \) and \( k = 0, 1, 2, \cdots \)

\[
t \in \mathcal{M}_k(\tau) : P_t(\tau) = \lambda_{o,t}(\tau)X_t - \lambda_{p,t}(\tau) \epsilon_t.
\]

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Given the equilibrium prices, we now consider the optimal policies of the investors. We have the following results for investors’ optimal policies and value functions:

**Lemma 1** When \( \mathcal{I}_{a,t} = \mathcal{I}_{b,t} \equiv \mathcal{I}_t \) \( \forall \ t \geq 0 \), \( i = a, b \), and the spot and futures prices have the form in (10), investor \( i \)'s optimal policies and her value function have the form: for \( k = 0, 1, 2, \ldots \)

\[
\begin{align*}
\theta_{i,t}(\tau) &= \frac{R}{\tau} h_{i,t}(\tau) Y_t \\
\sigma_{i,t} &= -\frac{1}{\tau} \log \left[ \frac{1}{\tau} \frac{\partial J_{i,t}}{\partial W_t} \right] \\
J_{i,t} &= -\rho^t \exp \left\{ -\frac{\tau}{R} W_{i,t} - \frac{1}{2} \left( Y_t Y_{i,t} Y_t \right) \right\}
\end{align*}
\]

where \( h_{i,t}(\tau) \) are given in proof in appendix and \( v_{i,t} \) are symmetric positive semi-definite matrices which satisfy a system of ordinary difference equations given in the appendix.

Notice that the investors’ holdings are linear in the state variables \( Y_t \). The investors’ value functions are characterized by \( v_{i,t} \). Given the conjectured periodic equilibrium prices of (10), the above lemma expresses investor \( i \)'s policies as functions of \( v_{i,t} \). To show that a periodic equilibrium exists, we need to show that the conjectured prices of (10) clear the spot and futures markets. Solving for a linear, periodic equilibrium reduces to solving a system of nonlinear first order difference equations given by (22) and (26) subject to periodicity conditions given by (23) and (5). We prove the existence of such a linear periodic equilibrium in Theorem 1.

**Theorem 1** For \( \omega \) close to one, a linear periodic equilibrium of the form in (10) exists generically in which the optimal policies of both investors are given by Lemma 1.

In general, the model needs to be solved numerically. However, through studying a number of special cases of the model which yield closed form solutions and performing comparative statics, we are able to gain a relatively good understanding of the model. The numerical methods used to solve this system of nonlinear first order difference equations is standard and is discussed in the appendix.

### 3. Results

In this section, we study the properties of the solution described above. The discussion is organized around two topics: (1) spot and futures return volatility and (2) spot positions (storage) and open interest.

#### 3.1. Spot And Futures Return Volatility

In this subsection, we focus on the predictions of this model for the time-variation in spot and futures return volatility associated with futures expiration. We show that the behavior of spot
return volatility can take on complex patterns depending on a number of factors, while the return volatility of a futures monotonically rises as the contract expires.

A. Homogeneous Investors

We first consider a special case of our model in which the economy has only investors from class $a$ (or $\omega = 1$). In this setting, markets are effectively complete and the periodic replication of futures does not affect the spot price. Hence futures can be priced by arbitrage from the spot price.\textsuperscript{11} This setting is important as we can obtain closed form solutions for spot and futures prices. It also provides a benchmark to analyze the effects of heterogeneity in investors on the market equilibrium.

Suppose that the convenience yield and investor $a$'s non-marketed risks are deterministic.\textsuperscript{12} Here, the spot is analogous to a coupon bond. Its price is simply the present discounted value of the payoffs from the convenience yield, $D_t$:

$$ t \in \{0, 1, 2, \ldots, \infty\} : S_t = \sum_{\tau=1}^{\infty} R^{-\tau} e Z_{t+\tau} = \lambda_{0z} Z_t, \tag{11} $$

where $\lambda_{0z} = ea_x (Ri - a_x)^{-1}$. Hence, spot return volatility is constant through time and is unaffected by futures trading.

The price of the futures is given by the familiar cost of carry formula: for $k = 0, 1, 2, \ldots$,

$$ t \in \{t_k, \ldots, m_k\} : H_t = R^{(m_k-t)} \left( S_t - \sum_{s=t+1}^{m_k} R^{(t-s)} e Z_s \right) = \lambda_{1z,t} Z_t, \tag{12} $$

where $\lambda_{1z,t} = \lambda_{0z} a_x^{(m_k-t)}$. Observe that when $|a_x| < 1$, $\lambda_{1z,t} < \lambda_{0z}$ and the futures price converges to the spot price at maturity: $\lambda_{1z,t} \to \lambda_{0z}$ as $t \to m_k$. Hence, as the futures approaches maturity, its price elasticities to contemporaneous convenience yield payoffs, $Z_t$, increase. This is because a mean reverting convenience yield, $|a_x| < 1$, implies that future payoffs to holding the spot are decaying. So the current price of a futures reflects this depreciation and the price elasticities of a futures to these convenience yield payoffs decreases with time-to-maturity. The difference between the price elasticities of a futures and the spot to convenience yield payoffs is increasing in the time-to-maturity of the contract and decreasing in the magnitude of $a_x$. In the extreme, when $a_x = i$, there is no difference between them.

Hence, when there is mean reversion in convenience yield shocks, futures return volatility systematically rises with the passage of time, while spot return volatility remains constant. The

\textsuperscript{11}See Cox, Ingersoll and Ross (1985a,b) and Richard and Sundaresan (1981) for examples of the representative agent pricing method to futures.

\textsuperscript{12}This corresponds to setting the following parameters, $\sigma_D \equiv \text{Var}(\epsilon_{D,t})$, $\sigma_{z_1} \equiv \text{Var}(\epsilon_{z_1,t})$, $\sigma_{z_2} \equiv \text{Var}(\epsilon_{z_2,t})$, $\sigma_{y_1} \equiv \text{Var}(\epsilon_{y_1,t})$, to zero.
monotonic pattern in futures return volatility is known as the Samuelson effect. No such effect need hold for spot return volatility in this complete market setting.

Now suppose both the convenience yield and non-marketed risks are stochastic. The equilibrium spot and futures prices are stated in Proposition 1.

**Proposition 1** When \( \omega = 1 \), the economy has a unique linear periodic equilibrium. The equilibrium prices of the spot and the futures are: for \( k = 0, 1, 2, \cdots \),

\[
S_t = \lambda_{02} Z_t - \lambda_{00} - \lambda_{0Y_1} Y_{1,t}, \quad (13a)
\]

\[
H_t = \lambda_{1z,t} Z_t - \lambda_{10,t} - \lambda_{1Y_1,t} Y_{1,t}. \quad (13b)
\]

\( \lambda_{02} \) and \( \lambda_{1z,t} \) are given in (11) and (12). \( \lambda_{00}, \lambda_{0Y_1}, \lambda_{10,t}, \) and \( \lambda_{1Y_1,t} \) are given in the appendix. The spot price elasticity to the non-marketed risk shock, \( \lambda_{0Y_1,t} \), remains constant with the passage of time. The corresponding futures price elasticity, \( \lambda_{1Y_1,t} \), monotonically increases as the contract matures.

Since investors are risk averse, the spot and futures prices now contain an unconditional and a conditional risk premium in the presence of stochastic convenience yield and non-marketed risks. The unconditional premia, \( \lambda_{00} \) and \( \lambda_{10,t} \), reflect risk discounts for the volatility of the convenience yield and also the correlatedness of the convenience yield to payoffs of investor \( a \)'s non-marketed risks. When \( Y_{1,t} > 0 \) and \( \kappa_{Dq} > 0 \), investor \( a \) would like to short the spot to control the overall risk of her portfolio since shocks to the convenience yield are positively correlated to her non-marketed risks. Since there are homogeneous investors in the economy, she has to hold the risky spot in equilibrium; hence, its price falls in equilibrium by \( \lambda_{0Y_1} Y_{1,t} \) in the form of a risk discount ( \( \lambda_{0Y_1} > 0 \) if and only if \( \kappa_{Dq} > 0 \)). Hence, futures prices also fall when \( Y_{1,t} > 0 \) by \( \lambda_{1Y_1,t} Y_{1,t} \).

Spot return volatility once again remains constant with the passage of time. The futures price is less sensitive to \( Y_{1,t} \) the farther the contract is from maturity and the smaller the persistence of the non-marketed income shocks, \( \alpha_{Y_1} \). The reasoning is similar to the decreased sensitivity of the futures far from maturity to convenience yield shocks. The special case of \( \omega = 1 \) illustrates that the Samuelson effect can be generated not only from the fundamentals portion of futures prices, \( \lambda_{1z,t} Z_t \), but also from the risk premium portion of futures prices, \( \lambda_{1Y_1,t} Y_{1,t} \).

**B. Heterogeneous Investors**

When \( 0 < \omega < 1 \) (heterogeneity in investors), the convenience yield portion of the spot and futures prices remain identical to the case of \( \omega = 1 \). However, we no longer have closed form

---

13The monotonicity pattern is known as the Samuelson effect (or the Samuelson Hypothesis) because the first explanation is given by Samuelson (1965). See Bessembinder, et.al. (1996) for a recent review of evidence on the Samuelson effect across different markets. See Khoury and Youroung (1993) and Anderson (1985) for discussions on markets in which the Samuelson effect does not hold.
solutions for the risk premium portion as the spot and futures price elasticities to the non-marketed risks are now determined through allocational trades.

With heterogeneous investors, the spot price elasticity to the non-marketed risk shock, $Y_{1,t}$, is no longer invariant through time. It will vary as the futures expires. Hence, spot return volatility no longer remains invariant through time but also varies with the time-to-maturity of the futures. We illustrate the time patterns induced by futures expiration in two steps. First we consider what happens when there is only one trading opportunity in the futures (i.e. $N = \infty$). Then, we consider a case where $N$ is small.

Figure 2-2 illustrates the behavior of the price elasticities of the spot and futures to the underlying non-marketed income shock $Y_{1,t}$ ($\lambda_{0Y_{1,t}}, \lambda_{1Y_{1,t}}$). The figure is for $\omega = .1$ and $N = \infty$. Since the vector $\lambda_{0z}$ is constant across the life of the nearby, the time-to-maturity pattern of

![Spot Price Elasticity](image1) ![Futures Price Elasticity](image2)

Figure 2-2: Spot and futures price elasticities to non-marketed risk shock $Y_{1,t}$ ($\lambda_{0Y_{1,t}}, \lambda_{1Y_{1,t}}$). Time-to-maturity is number of trading periods until the futures expires. The parameters are set at the following values: $M = 60$, $N = \infty$, $\omega = .1$, $\gamma = 100$, $r = .0005$, $\kappa_{Dq} = .5$, $\sigma_{D} = .09$, $\sigma_{x} = .03$, $a_{x_1} = .8$, $\sigma_{x_1} = .1$, $a_{x_2} = .99$, $\sigma_{x_2} = .065$, $a_{y_1} = .91$, $\sigma_{y_1} = .03$.

spot return volatility is entirely driven by the time-to-maturity patterns in $\lambda_{0Y_{1,t}}$. Apparently, $\lambda_{0Y_{1,t}}$ is also monotonically increasing with futures expiration (from Figure 2-2(a)). $\lambda_{1Y_{1,t}}$ is also monotonically increasing as the contract expires, so that the Samuelson effect still holds (from Figure 2-2(b)).

Trading in the futures allows investor $a$ to more aggressively hedge her non-marketed risk shocks. The effect of this hedging possibility is to decrease the conditional premium demanded with respect to non-marketed income shocks: $|\lambda_{0Y_{1,t}}|$ decreases with futures trading. Hence, futures trading lowers spot return volatility when non-marketed income is not i.i.d..

As the futures rolls toward maturity, two effects are driving the time-to-maturity pattern of $\lambda_{0Y_{1,t}}$ and hence spot return volatility. As the contract expires, this price coefficient rises in anticipation of the loss of the futures. Hence, spot return volatility tends to rise as the futures rolls toward maturity. This effect is most dramatic for $N = \infty$, which is illustrated in Figure 2-2(a). The other effect is that a new futures will be introduced after the nearby expires. This
tends to imply that the spot price coefficient $\lambda_{0Y_1,t}$ will fall when the new contract is initiated. Since the spot price has to be continuous for there to be no arbitrage, these spot price coefficients fall in anticipation of the introduction of the new contract. Hence spot return falls as the nearby rolls near to maturity. This effect is most dramatic for $N$ small.

The interaction of these two effects implies that $\lambda_{0Y_1,t}$ (and hence spot return volatility) rises as the nearby initially matures (since the introduction of the next futures is far away) and then falls as it nears expiration (as the effect of the introduction of the next futures becomes more prominent). This time-to-maturity pattern for $\lambda_{0Y_1,t}$ is illustrated in Figure 2-3(a). The futures

![Spot Price Elasticity](image1)

![Futures Price Elasticity](image2)

**Figure 2-3:** Spot and futures price elasticities to non-marketed risk shock $Y_{1,t}$ ($\lambda_{0Y_1}$, $\lambda_{1Y_1}$). Time-to-maturity is number of trading periods before futures expires. The parameters are set at the following values: $M = 60$, $N = 10$, $\omega = .1$, $\gamma = 100$, $r = .0005$, $\kappa_{DQ} = 0.5$, $\sigma_D = .09$, $\sigma_q = .03$, $a_{z_1} = .8$, $\sigma_{z_1} = .1$, $a_{z_2} = .99$, $\sigma_{z_2} = .065$, $a_{\gamma_1} = .91$, $\sigma_{\gamma_1} = .03$.

price elasticity continues to increase as the contract expires. It is unaffected by the introduction of new contracts in the future.

Next, we consider what happens to these time-to-maturity patterns of spot and futures price elasticities when we vary $\omega$. Figure 2-4 illustrates these comparative statics. Spot and futures price elasticities decrease as $\omega$ decreases because with proportionately more investors from the other class to make the market for investors from class $a$, prices are less sensitive to the non-marketed shocks of investor $a$. Hence, the levels of spot and futures return volatility also fall with smaller $\omega$. Notice that for all values of $\omega$, the futures price elasticity is decreasing with time-to-maturity and so is futures return volatility. Hence, the Samuelson effect still applies in this setting ($\omega < 1$).

### 3.2. Spot Positions (Storage) And Open Interest

In this subsection, we focus on the predictions of this model for the effects of futures expiration on spot holdings (storage) and open interest. We show that spot holdings elasticities to non-marketed risk shocks tend to be higher when the futures expires. We also show that the open interest in a given contract can take on an inverted U-shaped time-to-maturity pattern
Spot Price Elasticity

Futures Price Elasticity

Figure 2-4: Variation in spot and futures price elasticities to non-marketed risk shock \( Y_{1,t} (\lambda_{0y_1}, \lambda_{1y_1}) \) with \( \omega \). Time-to-maturity is number of trading periods before futures expires.

The parameters are set at the following values: \( M = 60, N = 10, \gamma = 100, r = .0005, \kappa_{yq} = 0.5, \sigma_D = .09, \sigma_q = .03, a_{z_1} = .8, \sigma_{z_1} = .1, a_{z_2} = .99, \sigma_{z_2} = .065, a_{y_1} = .91, \sigma_{y_1} = .03. \)

independent of often cited liquidity reasons for the rolling of open interest from the nearby to the next maturing contract.\(^{14}\)

Without loss of generality, we define open interest in a contract to be investor \( a \)'s average position in the contract scaled by the population weight of investors in class \( a \). In our discussion, we will in general drop the scaling by the population weight of investor \( a \) in the economy as \( \omega \) will stay fixed for our comparative statics. To illustrate the intuition of the first mechanism, it is sufficient to consider a special case of our model. We assume in this subsection that the non-marketed risks are i.i.d. (i.e. \( \sigma_{y_1} = 0 \)). Assume without loss of generality that \( a_{z_1} > a_{z_2} \), so that \( Z_{1,t} \) is the more persistent component and \( Z_{2,t} \) the more transitory component of the convenience yield growth rate.

In this instance, investors in class \( a \) trade in the spot to hedge their exposure to non-marketed risks. The spot provides only an imperfect hedge of non-marketed risks, however, as spot prices are sensitive to fluctuations in the convenience yield growth rate. Since investors in class \( a \) incur a price risk in trading in the spot to hedge their non-marketed risks, they trade in the futures to hedge this price risk. Since there are two uncorrelated convenience yield shocks, \( Z_{1,t} \) and \( Z_{2,t} \), and only one futures traded, the nearby provides an imperfect hedge of this price risk (i.e. the net position of investor \( a \)'s spot and nearby holdings is subject to a basis risk).

A. One Factor Model of The Convenience Yield

We elaborate on the effect of futures trading by assuming that the convenience yield growth rate follows a one factor model, \( Z_{1,t} \). In this instance, appropriate positions in the spot and futures provide a perfect hedge of the fluctuations in the spot price due to movements in \( Z_{1,t} \). Proposition 2 describes the time-to-maturity patterns in spot volume.

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\(^{14}\)See Duffie (1990).
Proposition 2 Suppose that the convenience yield growth rate follows a one factor model and investor a’s non-marketed risk is i.i.d., then the magnitude of investor a’s spot hedging position is larger with futures trading and is constant with the time-to-maturity of the futures. This leads to higher volume at futures expiration. Open interest in the futures is increasing with its time-to-maturity.

Intuitively, trading in the futures allows investor a to take more aggressive hedging positions in the spot as the futures allows investor a to hedge the price risk due to fluctuations in $Z_{1,t}$. When the futures provides a perfect hedge of fluctuations in the expected return of the convenience yield, the risk to hedging in the spot is constant across the life of the futures. Hence investor a’s spot commitment is constant with the time-to-maturity of the futures. Hence, at futures expiration, investors cut back on their spot positions, leading to higher volume.

There is also time variation in open interest associated with futures expiration. Proposition 2 describes this time variation. When convenience yield shocks are mean reverting ($a_{Z_1} < 1$), the price elasticity of the futures to $Z_{1,t}$ is decreasing with its time-to-maturity. Hence, the magnitude of the optimal hedging position in the futures increases with its time-to-maturity. So, open interest is monotonically decreasing as the contract matures.

B. Two Factor Model of The Convenience Yield

We next assume that the convenience yield growth rate is driven by a two factor model. In this instance, appropriate positions in the spot and futures will not provide a perfect hedge of fluctuations in the spot price. Proposition 3 describes the time-to-maturity patterns in spot hedging commitments and open interest in this setting.

Proposition 3 Suppose that the convenience yield growth rate follows a two factor model and investor a’s non-marketed risk is i.i.d., then the magnitude of investor a’s spot hedging position is still higher with futures trading but now tends to decrease with the time-to-maturity of the futures. Open interest may take on an inverted U-shaped time-to-maturity pattern.

When the futures provides an imperfect hedge of spot commitments, the risk to taking positions in the spot is increasing with the contract’s time-to-maturity. The rationale is that the contract’s price becomes relatively less sensitive to the transitory component of the convenience yield, $Z_{2,t}$ and hence the transitory component becomes less hedgeable with the time-to-maturity of the contract.

Interestingly, in this case, endogenizing spot trades has an interesting effect on open interest. Proposition 3 describes this effect. As spot commitments fall with the time-to-maturity of the futures, the level of open interest in the futures also tends to fall with its time-to-maturity holding all else constant. This is because the optimal hedging position in the nearby falls with the level of spot hedging commitments holding fixed the price sensitivity of the nearby to underlying shocks. The interaction of the time-to-maturity patterns in the futures price elasticities and the
level of the spot hedging position can generate an inverted U-shaped time-to-maturity pattern in open interest.

It is intuitively clear that when \( \sigma_{z_2}^2 \) is close to zero (convenience yield growth rate follows a one factor model), open interest tends to be higher far from maturity since the price elasticity effect is more dominant. However, as \( \sigma_{z_2}^2 \) increases (convenience yield growth rate follows a two factor model), the effect of the time-to-maturity patterns in the spot commitments become more prominent. So we tend to see the effect of lower open interest far from maturity. Figure 2-5 demonstrates this fact. We have normalized the outstanding shares of the risky spot to

**Figure 2-5:** Variation in investor a's spot position and open interest with volatility of convenience yield shock \( Z_{2,t} \) (\( \sigma_{z_2} \)). Time-to-maturity is number of trading periods until the futures expires. The parameters are set at the following values: \( M = 60, N = 60, \omega = 0.5, \gamma = 100, r = .0005, \kappa_{pq} = 0.5, \sigma_d = .095, \sigma_q = .03, a_{z_1} = .995, \sigma_{z_1} = .065, a_{z_2} = .97. \)

\( \bar{\theta} = 1 \) and chosen an arbitrary set of parameters to demonstrate the effect of the interaction of differing persistence in the shocks. Observe in Figure 2-5(a) that as \( \sigma_{z_2} \) increases, the level of investor a's short position in the spot decreases as there is more basis risk in the economy. Also, as \( \sigma_{z_2} \) increases, investor a's short position changes from being relatively constant with time-to-maturity to decreasing with time-to-maturity. Next, notice as \( \sigma_{z_2} \) increases, open interest in the nearby moves from monotonically increasing with time-to-maturity to taking on an inverted U-shaped pattern (reflecting the decreased spot commitments far from maturity).

4. **Conclusion**

This paper develops a model of a competitive futures market in which the underlying spot and futures prices are determined simultaneously. This model emphasizes the interaction of spot and futures trades. This interaction generates interesting behavior for spot return volatility and holdings as well as open interest.

While this model is stylized and cannot capture many important features of futures and cash markets, it does generate some testable implications. The first testable implication relates to
the time-to-maturity pattern in the return volatility of the spot. It is argued that the expiration of the nearby produces an inverted U-shaped pattern in the spot return volatility. The second testable implication relates to the time-to-maturity patterns in spot holdings and volume. This model predicts that spot holdings tend to fall far from expiration and that there tends to be higher spot volume at futures expiration. With the available storage (inventory) data, we can test whether inventory varies as the nearby futures expires. The third testable implication relates to open interest. This paper predicts that open interest may take on an inverted U-shaped time-to-maturity pattern. The ability of investors to adjust their spot trades leads to this pattern. This distinction suggests that if inventory data is available (and assuming market participants use futures to hedge the price risk to their inventory), then analyzing whether inventory moves with the maturity of the nearby contract may provide a way to indentify this mechanism. Finally, all of these patterns are consistent with the Samuelson Effect holding in the futures market.
5. Appendix

This appendix consists of three parts. Section 5.1. provides some technical results needed for later use. Section 5.2. contains the proofs for all the results stated in Section 2.2. Section 5.3. contains proofs of results in Section 3.1 and 3.2. Section 5.4. discusses the numerical procedures.

5.1. Mathematical Preliminaries

For future use, we introduce some notation. A positive semi-definite (definite) matrix $m$ is $m \geq (>0)$ and $|m|$ denotes the spectral norm of $m$, defined as the absolute value of the numerically largest eigenvalue of $m$. Let $\Theta = \{ r > 0, \gamma > 0, 1 \geq a_{zi} > 0 (i = 1, 2), 1 > a_{\gamma_1} > 0, \sigma_D \geq 0, \sigma_q \geq 0, \sigma_{\gamma_1} \geq 0 (i = 1, 2), \sigma_{\gamma_1} \geq 0, \kappa_{Dq} > 0 \}$. Let $|m|$ be the column matrix consisting of the independent elements of matrix $m$. We now also state a definition and an auxiliary lemma.

**Definition 2** Let $r_t$ and $a_j$ for $j = 0, 1, 3$ be symmetric matrices and $a_0 \geq 0$, $a_3 > 0$. A discrete time matrix Riccati equation is defined as

$$ t \in \{1, T\} : \quad r_{t-1} = a_0 + a_1 r_t a_1' - a_1 r_t a_2' (a_3 + a_2 r_t a_2')^{-1} a_2 r_t a_1' $$

(14)

and $r_T = \bar{r}_T$.

**Lemma 2** For $r_T \geq 0$, (14) has a unique, symmetric, positive semi-definite solution, $r_t$. And $r_t \leq e_t$ where $e_t$ is the solution to the following matrix linear difference equation:

$$ t \in \{1, T\} : \quad e_{t-1} = (a_1 - ka_2) e_t (a_1 - ka_2)' + a_0 + ka_3 k', $$

(15)

where $e_T = r_T$ and $k$ is an arbitrary sequence of matrices.

**Proof.** See Caines and Mayne (1970). \(\square\)

In deriving several results in this paper, the problem to be solved often reduces to a two-point boundary-value problem for a (vector) ordinary difference equation. Here, we give a formal and relatively general definition of the two-point boundary-value problem and state some known results concerning its solution.

**Definition 3** Let $f : \mathbb{R}_+ \otimes \mathbb{R}^n \otimes \mathbb{R}^m \otimes \mathbb{R} \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^m \otimes \mathbb{R} \rightarrow \mathbb{R}^n$. A two-point boundary-value problem is defined as

$$
\begin{align*}
  u_{t-1} &= f(t, u_t; \vartheta, \omega), \quad t \in \{1, T\} \\
  0 &= g(u_0, u_T; \vartheta, \omega)
\end{align*}
$$

(16)

where $T > 0$, $\vartheta \in \Theta$ and $\omega \in [0, 1]$.
We also define the terminal value problem:

\[
\begin{align*}
  u_{t-1} &= f(t, u_t; \vartheta, \omega), \quad t \in \{1, T\} \\
  u_T &= \bar{u}_T.
\end{align*}
\]  

(17)

Under appropriate smoothness conditions on \( f(t, u_t; \vartheta, \omega) \), (17) has a unique solution denoted by \( u(t; \vartheta, \omega; \bar{u}_T) \), which is differentiable in \( \bar{u}_T \) [see, e.g. Agarwal (1992), p.224]. Solving the two-point boundary-value (16) is to seek \( \bar{u}_T \) that solves

\[
0 = g[u(0; \vartheta, \omega; \bar{u}_T), \bar{u}_T; \vartheta, \omega] \equiv g \circ u(\bar{u}_T; \vartheta, \omega).
\]  

(18)

The existence of a root to (18) relies on properties of \( g \circ u(\bar{u}_T; \vartheta, \omega) \). Let \( (g \circ u + 1)(\bar{u}_T; \vartheta, \omega) \equiv g \circ u(\bar{u}_T; \vartheta, \omega) + \bar{u}_T \).

**Lemma 3** If \( (g \circ u + 1)(\cdot; \vartheta, \omega) : \mathbb{R}^n \to \mathbb{R}^n \) is continuous and there exists a nonempty, closed, bounded, and convex subset of \( \mathbb{R}^n \), \( L \), such that \( (g \circ u + 1)(\cdot; \vartheta, \omega) \) maps \( L \) into itself, then (18) has a root and the two-point boundary value problem (16) has a solution.

**Proof.** Existence of a root to (18) follows from Brouwer's Fixed Point Theorem [see, e.g. Cronin (1994, p.352)]. □

The condition on \( (g \circ u + 1) \) required by Lemma 3 is not always easy to verify, in which case the existence of a solution to (16) is not readily confirmed. However, if a solution exists for \( \omega_0 \), the existence of a solution for \( \omega \) close to \( \omega_0 \) is easy to establish.

**Definition 4** \( u(t; \vartheta, \omega_0) \) is an isolated solution of system (16) if the linearized system

\[
\begin{align*}
  y_{t-1} &= \nabla u f(t, u_t; \vartheta, \omega_0) y_t, \quad t \in \{1, T\} \\
  0 &= \nabla u_0 g(u_0, u_T; \vartheta, \omega_0) y_0 + \nabla u_T g(u_0, u_T; \vartheta, \omega_0) y_T
\end{align*}
\]  

(19)

has \( y = 0 \) as the only solution, where \( \nabla \) denotes the partial derivative operator.

**Lemma 4** Let (16) have an isolated solution \( u(t; \vartheta, \omega_0) \) for \( \omega = \omega_0 \). Suppose that \( f(t, z; \vartheta, \omega) \) and \( g(z_0, z_T; \vartheta, \omega) \) are continuously differentiable in the neighborhood of \((t, u(t; \vartheta, \omega_0), \omega_0)\). Then, (16) has a solution for \( \omega \) close to \( \omega_0 \).

**Proof.** [See, Agarwal (1992), p.525]. □

**Lemma 5** Let \( f : D \to \mathbb{R} \) be a real analytic function, where \( D = D_1 \otimes \cdots \otimes D_n \) is an open subset of \( \mathbb{R}^n \). Let \( Z = \{ x \in D : f(x) = 0 \} \) be its zero set. Then, either \( Z = D \) or \( \mu_n(Z) = 0 \) where \( \mu_n \) is the \( n \)-dimensional Lebesgue measure.
Proof. [See, e.g. Hong and Wang (1997)]. □

We next state the general form of the boundary-value problem we encounter in this paper and show that it can be reduced to the two-point boundary-value problem (16). Our boundary value problem is given by:

\[ u^o_t = f^o(t + 1, u^o_{t+1}; \vartheta, \omega), \quad t \in \{t_k, m_k - 2\} \]
\[ u^p_t = g^{op}(u^p_{t+1}; \vartheta, \omega), \quad t = m_k - 1 \]
\[ u^p_t = f^p(t + 1, u^p_{t+1}; \vartheta, \omega), \quad t \in \{m_k, t_{k+1} - 2\} \]
\[ u^p_t = g^{po}(u^p_{t+1}; \vartheta, \omega), \quad t = t_{k+1} - 1 \]

and \( u^o_t = u^o_{t+1} \) (periodicity). And \( f^n \) for \( n = o, p, g^{op} \) and \( g^{po} \) are continuously differentiable.

Let \( u = \text{stack} \{u^o, u^p\} \), \( f = \text{stack} \{f^o, f^p\} \), and \( g = \text{stack} \{g^{op}, g^{po}\} \). Let

\[ \bar{u}_T = \text{stack} \{u^o_{m_k-1}, u^p_{t_{k+1}-1}\}. \]

This vectorized system conforms to our two-point boundary value problem.

5.2. Proofs of Results In Section 2.2

We introduce some notation for later use. Let \( \epsilon_i = [\epsilon_{d,1}, \epsilon_{q,1}, \epsilon_{x_1,1}, \epsilon_{x_2,1}, \epsilon_{y_1,1}, \epsilon_{y_2,1}]^T \). It is assumed that \( \hat{1}^{(i,j)}_{l,k} \) is an \( i \times j \) matrix with its \( l \times k \) element equal to one and all its other elements equal to zero. Then from the above discussion, define \( b_D = \sigma_{D,1,1}^{(1,5)} \), \( b_q = \sigma_{q,1,2}^{(1,5)} \), \( b_z = \text{stack}\{\sigma_{z,1}^{(1,5)}, \sigma_{z_2}^{(1,5)}\} \), \( b_Y = \text{stack}\{0^{(1,5)}, \sigma_{y,1}^{(1,5)}\} \).

Proof of Lemma 1. The excess share return of the spot and futures contracts can be derived from the evolution of the investors’ state variables and follows the following processes: for \( k = 0, 1, 2, \ldots \)

\[ t \in \mathcal{M}_k(\tau): \quad Q_t(\tau) = a_Q(\tau)Y_t + b_Q(\tau)\epsilon_{t+1}. \quad (21) \]

Let \( \bar{b}_{jz,t+1} = 1_j b_D + \left(1_j e^{(2)} + \lambda_{jz,t+1}\right) b_z \) and for \( j = 0, 1 \), where \( 1_j = 1 \) if \( j = 0 \) and \( 1_j = 0 \) if \( j = 1 \). Define

\[ a_{Q_0} = R\lambda_{0Y,t} - \lambda_{0Y,t+1}a_Y, \]
\[ a_{Q_1} = \lambda_{1Y,t} - \lambda_{1Y,t+1}a_Y, \]
\[ b_{Q_j} = \bar{b}_{jz,t+1} - \lambda_{jY,t+1}b_Y. \]

Hence \( a_{Q}(o) = \text{stack}\{a_{Q_0}, a_{Q_1}\} \) and \( b_{Q}(o) = \text{stack}\{b_{Q_0}, b_{Q_1}\} \). And \( a_{Q}(p) \) and \( b_{Q}(p) = b_{Q_0} \).

Given the state variables processes and the return processes, we now derive the system of nonlinear difference equations which govern \( u_{t,t} \). For simplicity, we will derive the recursive
relation generally: dropping superscripts denoting trading regimes. The specific system cited in Lemma 1 can be derived from this recursive relation by making the appropriate substitutions. Suppose that at \( t \), investor \( i \)'s state variable process is \( Y_{t+1} = a_Y Y_t + b_Y \epsilon_{t+1} \) and value function given by \( J_{i,t}(W_{i,t}; Y_i; t) \). Then we can rewrite \( q_{t+1} = Y_t \bar{a}_q Y_t + \bar{b}_q Y_t b_q \epsilon_{t+1} \). Let \( \alpha = \frac{\gamma}{\kappa} \). Now define the following matrices:

\[
\Xi_{i,t+1} = \left[ (\sigma_{\epsilon})^{-1} + b_Y \nu_{i,t+1} b_Y \right]^{-1} \Omega_{i,t+1}(\tau) = (b_Q(\tau) \Xi_{i,t+1} b_Q(\tau))^{-1} \nu_{iab,t+1} = a_Y \nu_{i,t+1} b_Y.
\]

And also define:

\[
t \in \{ t_k, m_k - 1 \} : \quad h_{i,t}(o) = \Xi_{i,t+1}(o) [a_Q(o) - b_Q(o) \Xi_{i,t+1} + \alpha \bar{b}_q]' \n
t \in \{ m_k, t_k - 1 \} : \quad h_{i,t}(p) = \Xi_{i,t+1}(p) [a_Q(p) - b_Q(p) \Xi_{i,t+1} + \alpha \bar{b}_q]'.
\]

Using the above notation but dropping the superscripts for the trading regime, both the informed and the uninformed investors' optimization problems can be expressed in the form of the Bellman's equation:

\[
0 = \sup_{\{c_i, \theta_i\}} \left\{ -\rho^t e^{-\gamma c_i,t} + E_{i,t} [J(W_{i,t+1}; Y_{t+1}; t+1)] - J(W_{i,t}; Y_i; t) \right\}
\]

s.t. \( W_{i,t+1} = (W_{i,t} - c_{i,t}) R + \theta_{i,t}' (a_Q Y_t + b_Q \epsilon_{t+1}) + 1_i (Y_t \bar{a}_q Y_t + \bar{b}_q Y_t b_q \epsilon_{t+1}) \).

Consider the following trial solution for the value function:

\[
J_{i,t}(W_{i,t}; Y_i; t) = -\rho^t \exp \left\{ -\alpha W_{i,t} - \frac{1}{2} (Y_t' \nu_{i,t} Y_t) \right\},
\]

where \( \nu_{i,t} \) is a symmetric matrix. Let \( d_i = |\Xi_{i,t+1}^{-1} \sigma_{\epsilon}|^{-\frac{1}{2}} \) and \( v_{iaa,t+1} = a_Y \nu_{i,t+1} a_Y \). It follows from normality of \( \epsilon_{t+1} \) that

\[
E_{i,t}[J_{i,t+1}] = -d \rho^{t+1} \exp \left\{ -\alpha R (W_{i,t} - c_{i,t}) - \alpha Y_t' h_{i,t} \theta_{i,t} + \frac{1}{2} \alpha^2 \theta_{i,t}' \Omega_{i,t+1}^{-1} \theta_{i,t} \right. \\
- \frac{1}{2} Y_t' \left[ v_{iaa,t+1} - (v_{iab,t+1} + \alpha 1_i \bar{b}_q) \Xi_{i,t+1} (v_{iab,t+1} + \alpha 1_i \bar{b}_q)' + 1_i \bar{b}_q \right] Y_t \right\}.
\]

The first order conditions for the optimal investment-consumption policies are \( \theta_{i,t} = h_{i,t} Y_t \) and

\[
c_{i,t} = \tilde{c}_t + \frac{\alpha R}{\gamma + \alpha R} W_{i,t} + \frac{1}{2(\gamma + \alpha R)} Y_t' g_i, \quad t+1 Y_t,
\]

where \( \tilde{c}_t = \frac{1}{\gamma + \alpha R} \log \left( \frac{\gamma}{\alpha R d_i} \right) \) and

\[
g_{i,t+1} = v_{iaa,t+1} - (v_{iab,t+1} + \alpha 1_i \bar{b}_q) \Xi_{i,t+1} (v_{iab,t+1} + \alpha 1_i \bar{b}_q)' + 1_i \bar{a}_q + h_{i,t}' \Omega_{i,t+1} h_{i,t}.
\]
Lemma 6 $v_{i,t}$ satisfies the following nonlinear two-point boundary value problem:

$$
t = t_k : \quad R v_{i,t-1} = a_Y v_{i,t} a_Y + h_{i,t-1} (p) \Omega_{i,t}(p) h_{i,t-1}(p)' + \bar{v}_i (v_{iab,t} + \alpha \bar{b}_q)' \Xi_{i,t} (v_{iab,t} + \alpha \bar{b}_q)'
$$

$$
t \in \{t_k + 1, m_k - 1\} \quad R v_{i,t-1} = a_Y v_{i,t} a_Y + h_{i,t-1}(o) \Omega_{i,t}(o) h_{i,t-1}(o)' + \bar{v}_i (v_{iab,t} + \alpha \bar{b}_q)' \Xi_{i,t} (v_{iab,t} + \alpha \bar{b}_q)'
$$

$$
t = m_k : \quad R v_{i,t-1} = a_Y v_{i,t} a_Y + h_{i,t-1}(o) \Omega_{i,t}(o) h_{i,t-1}(o)' + \bar{v}_i (v_{iab,t} + \alpha \bar{b}_q)' \Xi_{i,t} (v_{iab,t} + \alpha \bar{b}_q)'
$$

$$
t \in \{m_k + 1, t_{k+1} - 1\} \quad R v_{i,t-1} = a_Y v_{i,t} a_Y + h_{i,t-1}(p) \Omega_{i,t}(p) h_{i,t-1}(p)' + \bar{v}_i (v_{iab,t} + \alpha \bar{b}_q)' \Xi_{i,t} (v_{iab,t} + \alpha \bar{b}_q)'
$$

and for $k = 0, 1, 2, \ldots$

$$
v_i(t_k) = v_i(t_{k+1}). \quad \tag{23}
$$

Proof of Lemma 6. This gives the following recursion for $v_{i,t}$ given $v_{i,t+1}$: $\frac{1}{\bar{R}} \bar{g}_{i,t+1} - v_{i,t} + \left[ \gamma \bar{c}_i + \log \left( \frac{r}{\bar{R}} \right) \right] 1_{i,1}^{(2,2)} = 0$. Let $\bar{v}_i = R \left[ \gamma \bar{c}_i + \log \left( \frac{r}{\bar{R}} \right) \right] 1_{i,1}^{(2,2)} + 1_i \bar{a}_q$, then it follows that

$$
R v_{i,t-1} = v_{iab,t} - (v_{iab,t} + \alpha 1_i \bar{b}_q)' \Xi_{i,t} (v_{iab,t} + \alpha 1_i \bar{b}_q)' + h_{i,t-1}' \Omega_{i,t} h_{i,t-1} + \bar{v}_i. \quad \tag{24}
$$

System (22) follows from this. □

Lemma 7 Given terminal value $v_{i,t_k+1} \geq 0$, (22) has a solution $v_i(t; v_{i,t_k+1})$ which is symmetric, positive semi-definite and $|v_i(t; v_{i,t_k+1})| \leq \beta_0 |v_{i,t_k+1}| + \beta_1$ where $0 < \beta_0 < 1$ and $\beta_1 > 0$.

Proof of Lemma 7. We prove existence of the terminal value problem for (22) by bounding (lower and upper) its solution by the solutions to two particular matrix Riccati difference equations. First we specify our assumption about $a_q$. It is assumed that

$$
a_q = \text{stack}\{[0, (\alpha \sigma_q)^2], [(\alpha \sigma_q)^2, 2(\alpha \sigma_q)^2]\}.
$$

This assumption is sufficient to satisfy assumption in Lemma 2 for positive semi-definite solutions to the Riccati equation. First the lower bound. Consider (24) on $t \in \{0, T\}$. Suppose $v_{i,T} \geq v_{i,T}$,
then it follows that $v_{i,t} \geq v_{i,t}^*$ where $v_{i,t}^*$ satisfies

$$Rv_{i,t-1}^* = v_{iab,t}^* - \left( v_{iab,t} + \alpha L_i \right) \Xi_{i,t} \left( v_{iab,t} + \alpha L_i \right)' + \bar{v}_i^*.$$ 

This follows from fact that $h_{i,t-1}^* \Omega_{i,t} h_{i,t-1} \geq 0$. Next, an upper bound. Suppose $v_{i,t} \leq v_{i,t}^*$. Suppose $v_{i,t}^*$ satisfies the following:

$$Rv_{i,t-1}^* = a_{i,y} v_{i,t}^* + \left( a_{i,y} v_{i,t}^* b_{i,t} + \bar{b}_q \right) \Xi_{i,t} \left( a_{i,y} v_{i,t}^* b_{i,t} + \bar{b}_q \right)' + \bar{v}_i^*,$$ 

where $a_{i,y} = a_{i,y} + b_{i,y}' k^*$ and $k^* = a_{i,y} \Xi_{i,t} b_{i,y} \Xi_{i,t}^*$ if $v_{i,t} > v_{i,t+1}$, and $k^* = 0$ otherwise. Substituting $h_{i,t-1}$ into (24) and multiplying out the matrices, it can be shown that $v_{i,t} \leq v_{i,t}^*$. Note that $\Xi_{i,t} = (b_{i,y} \Xi_{i,t}^*)' \Omega_{i,t} (b_{i,y} \Xi_{i,t}) > 0$, so that (24) is negative in the quadratic terms involving $v_{i,t}$. So, this keeps the solution bounded and hence is bounded by a solution to a particular matrix Riccati equation. Now, apply Lemma 2 to (25) by setting $k = k^*$ and it follows that $v_{i,t}$ is bounded by the solution to a matrix linear equation whose linear term is $a_{i,y} v_{i,t}^* b_{i,t}$ and the constant term is independent of $v_{i,t}$, $\forall t$. Hence, since $|a_{i,y}| < 1$, it follows that we can find $\beta_0$ and $\beta_1$ such that the result holds. \(\square\)

**Lemma 8** (22) and (23) has a symmetric, positive semi-definite periodic solution.

**Proof of Lemma 8.** Given the bound established in Lemma 7, let $L$ be the space of symmetric, positive semi-definite matrices such that their norms are less than $\psi$, where $\psi = \beta_1/(1 - \beta_0)$. Then define the following mapping: $M : L \rightarrow L$ where $M$ is given by Equation (22) as a mapping of $v_{i,t+1}$ to $v_{i,t}$. $L$ is a nonempty, closed, bounded, convex subset of a finite dimensional normed vector space. $M$ is clearly symmetric, positive semi-definite by Lemma 2. It is easy to calculate that $|M(v_{i,t})| \leq \beta_1/(1 - \beta_0)$. By Lemma 3, the result follows. \(\square\)

**Proof of Theorem 1.** In the case of $\omega = 1$, we can find closed form solutions for the equilibrium price functions and the value function of the informed investor (see Proposition 1).

The existence of a periodic solution to the value function of the uninformed investors follows from Lemma 8. We next show that there exists a solution close for $\omega$ close to $\omega_0 = 1$. The market clearing conditions which determine the price coefficients $\lambda_j(\tau)$ for $j = 0, 1$ and $\tau = i, p$ are such that the following nonlinear equations hold: for $k = 0, 1, 2, \cdots$

$$t \in \mathcal{M}_k(\tau) : \omega h_{a,t}(\tau) + (1 - \omega) h_{b,t}(\tau) = \bar{\theta}(\tau).$$

It is not hard to show that $\lambda_{j,t}(\tau)$ and $\lambda_{j,t+1}(\tau)$ are related by $\lambda_{j,t}(\tau) = f(\lambda_{j,t+1}(\tau); \theta, \omega)$. Let

$$\bar{\theta}_0 = [.05, 100, .99, .95, .9, .9, .05, .025, .025, .05, .025, .05].$$

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Now define the following
\[ u^a = \text{stack} \{ \lambda_0, \lambda_1, [v_a], [v_b] \} \]
\[ u^p = \text{stack} \{ \lambda_0, [v_a], [v_b] \} \]

It is now straightforward to relate system (20) to the equilibrium system given by (22) and (26) subject to periodicity conditions given by (23) and (5). Let \( \omega_0 = 1 \). Since we have existence of \( u(t; \theta, \omega_0) \), it remains to verify that \( u(t; \theta, \omega_0) \) is an isolated solution. This is equivalent to showing [see, e.g. Agarwal (1992)] that \( m(\theta, \omega_0) = \nabla_{\omega_0} g(\omega_0) + \nabla_{u_T} g(\omega_T) \quad (\nabla_{u} f)^T \) is nonsingular where \( T \) is the number of periods over which the system is defined. Clearly, \( m(\theta, \omega_0) \) is analytic. It is easy to show that \( \det(m(\theta, \omega_0)) \neq 0 \). By Lemma 5, \( m(\theta, \omega_0) \) is generically nondegenerate. By Lemma 4, Theorem 1 follows. \( \square \)

5.3. Proofs of Results In Section 3.1, 3.2

Proof of Proposition 1. When \( \omega = 1 \), it is easy to verify that when
\[ v = \left[ v_0 + \sqrt{v_0^2 + 4R\sigma_1^2 \alpha \left( 2a_q - \alpha \sigma_q^2 \right)} \right] / (2R\sigma_1^2), \]

where \( v_0 = \sigma_1^2 + R + \alpha \sigma_1^2 \left( 2a_q - \alpha \sigma_q^2 \right) \) and \( a_q = 2(\alpha \sigma_q)^2 \) that the price function satisfies needed conditions. It is easy to show that: \( \lambda_0, \gamma = R^{-1} \left( \zeta \left( \sigma_2^2 + \lambda_0 \sigma_1^2 \right) + \sigma_{pq} \right) \), \( \lambda_0, \gamma = \gamma R^{-1} \left( e^{(3)} \bar{a}_z \lambda_0 \sigma_1 \right) \), and \( \lambda_1, \gamma = \gamma R^{-1} \left( e^{(3)} \bar{a}_z \lambda_0 \sigma_1 \right) \). Here, \( \bar{a}_z = a(z_1 - \sigma_1^2 v) \) and \( 0 \leq \bar{a}_z \leq 1 \). \( \sigma_1^2 \) and \( v \) are given in the appendix. \( \square \)

Proof of Proposition 2. Apply Lemma 1. Let \( T \) be length of futures contract. Let \( \eta_0 = \frac{\lambda_2^2 \sigma_2^2}{\lambda_1 \sigma_1^2}, \eta_1 = \frac{1+\lambda_2}{1+\lambda_2 \lambda_1} \lambda_2 \sigma_2^2, \) and \( \eta_2 = \frac{(1+\lambda_2) \lambda_2 \sigma_2^2}{(1+\lambda_2 \lambda_1) \lambda_1 \sigma_1^2} \). Investor a's spot holding is for \( t \in \{ t_k, m_k - 1 \} \):
\[ \theta_{a,t} = \zeta - \frac{\left\{ 1 + \eta_0 \left( a_{z_2}/a_{z_1} \right)^2 (T-t) \right\} \sigma_{pq}(1-\omega)}{\sigma_2^2 \left\{ 1 + \eta_0 \left( a_{z_2}/a_{z_1} \right)^2 (T-t) \right\} + \left\{ 1 - \eta_1 \left( a_{z_2}/a_{z_1} \right)^2 (T-t) \right\}^2 (1 + \lambda_z^2) \sigma_2^2}. \quad (27) \]

Here holdings of the spot for \( t \in \{ m_k, t_{k+1} - 1 \} \):
\[ \theta_{a,t} = \zeta - \frac{\sigma_{pq}(1-\omega)}{\sigma_2^2}. \quad (28) \]
Her holdings in futures contract \( #1 \) for \( t \in \{t_k, m_k - 1\} \):

\[
\varphi_{a,t} = \frac{(1 + \lambda_{z_1}) \left\{ 1 + \eta_2 \left( z_{z_2}/a_{z_1} \right)^{(T-t)} \right\} \sigma_D (1 - \omega)}{\lambda_{z_1} a_{z_1}^{(T-t)} \left\{ \sigma_D^2 + 1 + \eta_0 \left( z_{z_2}/a_{z_1} \right)^{(T-t)} \right\}^2 (1 + \lambda_{z_2})^2 \sigma_{z_2}^2}.
\]

When \( \sigma_{z_2} = 0 \), it is easy to see from (27) that during futures market open, investor a’s spot holdings are constant: \( \vartheta_{a,t} = \zeta - \frac{\sigma_{D_2}(1-\omega)}{\sigma_D} \). Since \( \sigma_{z_2}^2 > \sigma_D^2 \), it follows from (28) that investor a reduces her short position in the spot upon maturity of the futures. Investor a’s positions in the futures is \( \varphi_{a,t} = \frac{(1+\lambda_{z_1})\sigma_{D_2}(1-\omega)}{\lambda_{z_1} a_{z_1}^{(T-t)} \sigma_D^2} \) during futures market open. Notice that \( \varphi_{a,t} \) is proportional to the investor’s spot commitments: \( \frac{\sigma_{D_2}(1-\omega)}{\sigma_D^2} \). \( \square \)

**Proof of Proposition 3.** It is easy to show that \( \frac{\partial \varphi_{a,t}}{\partial (T-t)} > 0 \). When \( \sigma_{z_2}^2 > 0 \) and \( a_{z_1} = 1 \), it is easy to see from Proposition 2 that as \( T - t \to \infty \), \( \varphi_{a,t} \to \frac{(1+\lambda_{z_1})\sigma_{D_2}(1-\omega)}{\lambda_{z_1} (\sigma_D^2 + (1+\lambda_{z_2})^2 \sigma_{z_2}^2)} \). Notice that her futures position far from maturity is proportional to the level of her spot hedging positions far from maturity, \( \frac{\sigma_{D_2}(1-\omega)}{\sigma_D^2 + (1+\lambda_{z_2})^2 \sigma_{z_2}^2} \). Notice also that investor a’s futures position far from maturity is less than near maturity. Hence, her futures position must initially rise as the contract matures, leading to the inverted U-shaped pattern. \( \square \)

### 5.4. Numerical Procedure

We briefly discuss the numerical procedure used to solve for the periodic equilibrium. We use the Newton-Kantorovich method to solve this problem numerically [see, e.g., Agarwal (1992)]. This recursive method linearizes the system and the boundary conditions around a conjectured solution to the nonlinear problem at a discrete number of points in the interval \([t_k, t_{k+1}]\). Since the system is linearized, it is easy to calculate an updated solution that satisfies the linearized system and boundary conditions from the conjectured solution. The updated solution is then used as the conjectured solution to start the next recursion. It can be shown that the limit of this recursion converges to the solution of the nonlinear problem given that the initial conjectured solution is not too far away from the true solution.

This method requires a sufficiently accurate initial guess of the true solution. We obtain such a guess by starting the recursion at \( \omega = 1 \) for any given set of parameters since a solution exists at \( \omega = 1 \). In order to calculate a solution at \( \omega_0 < 1 \), we begin by using the solution at \( \omega = 1 \) as the initial guess to find a solution for an \( \omega \) close to 1 and repeat the same procedure to move toward \( \omega_0 \).

Since we have no knowledge on the uniqueness of the solution, the above procedure also guarantees that we stay on the same branch of solutions. In particular, the solution gives the expected results when we take the limit \( \omega \to 1 \). We have also checked the solution by taking other limits in the parameter space such as \( \sigma_{z_1} \to 0 \) and obtained the expected results.
6. References


Chapter 3

A Model Of Returns And Trading In Futures Markets Under Asymmetric Information

A variety of return volatility and open interest patterns are observed across futures markets. In some markets, the return volatility of a futures increases as the contract expires; whereas in others, it may exhibit non-monotonic patterns.1 And in many markets, the open interest of a contract initially rises with time and then falls as it approaches expiration.2 Along with these time-to-maturity patterns of a given contract, there are also differences in the maturity length of futures traded across markets. For example, there is little trade in S&P contracts of longer than six months; whereas in crude oil, contracts of five years or longer are traded.3

Motivated by these observations, this paper develops an equilibrium model of a competitive futures market to study the patterns in return volatility and open interest associated with futures. In this economy, investors can simultaneously trade in both spot and futures. Holding spot positions (as opposed to futures positions) provides the owner with an exogenously specified convenience yield, which is stochastic and mean reverting. Some investors are exposed to non-marketed risks, which also have varying degrees of persistence. Investors trade in spot and futures to hedge their exposure to these risks. Some investors have private information about future payoffs of the convenience yield. They trade in spot and futures to speculate on their private information. It is assumed that the available set of spot and futures does not span the risks in the economy. Hence, futures are not redundant securities and cannot be priced by arbitrage. Instead, spot and futures prices are determined jointly by the hedging and speculative trades of investors.

1The monotonicity pattern is known as the Samuelson effect (or the Samuelson Hypothesis) because the first explanation is given by Samuelson (1965). See Bessembinder, et.al. (1996) for a recent review of evidence on the Samuelson effect across different markets. See Khoury and Yourougou (1993) and Anderson (1985) for discussions on markets in which the Samuelson effect does not hold.

2Milonas (1986) reviews the evidence on open interest and volume patterns in a variety of futures markets.

3Trading in S&P futures is limited to the nearest two contracts which comprise 97% of total open interest across contracts of all maturity lengths. In crude oil, open interest in the nearest two contracts accounts for only 44.2% of total open interest. These figures are calculated from Bloomberg's on September 1, 1996 (S&P is from CME and crude oil is from NYMEX).
When investors trade for only hedging reasons, the return volatility of a futures rises as it matures. In the presence of mean-reverting shocks to spot prices (convenience yield and non-marketed risks), the price of a given futures is less sensitive to contemporaneous shocks than the spot price (e.g. Samuelson (1965)). These mean reverting shocks are less reflected in the price of the futures than the spot because they will die away by the time the contract expires. As a futures matures, its price elasticities to contemporaneous shocks increase and its return volatility rises.

When investors also trade to speculate on their private information, the return volatility of a futures also depends on the level of adverse selection in the economy (i.e. the degree of information asymmetry among investors regarding future convenience yield payoffs). The presence of privately informed investors generates an adverse selection cost to trading for the uninformed. When uninformed investors become less informed about future convenience yield payoffs (i.e. adverse selection in the economy rises all else equal), futures prices may move less because uninformed investors' expectations about these payoffs are less variable.

The level of adverse selection in the economy in turn depends on how informative spot and futures prices are about future convenience yield payoffs. Indeed, informed investors' speculative trades transmit their information regarding these payoffs into spot and futures prices. Since these prices may also fluctuate because of shocks to investors' non-marketed risks that cause them to rebalance their holdings, spot and futures prices provide noisy signals for uninformed investors. The informativeness of these noisy price signals depends on the time-to-maturity of futures. Intuitively, since the price of a given contract must converge to the spot price when it expires, the information that it conveys is redundant (given the spot price) at its maturity. Hence, information asymmetry should be higher near the maturity than far from the maturity of a futures since investors lose an important signal at expiration.

How the level of adverse selection in the economy changes across the life of a contract is more complicated than what this intuition might suggest. The time-to-maturity pattern in the level of adverse selection in the economy also depends on how the futures price approaches the spot price. Suppose that convenience yield shocks are close to random walks while shocks to non-marketed risks are mean-reverting. In this instance, the price of the futures is equally sensitive to convenience yield shocks across the life of the contract. However, its sensitivity to non-marketed risk shocks is monotonically decreasing with the time-to-maturity of the contract. Far from maturity, the futures price is very informative about future convenience yield payoffs since it is relatively inelastic to non-marketed risk shocks (the noise in the futures price signal). As the contract matures, its sensitivity to this noise becomes relatively more important and hence adverse selection rises. Conversely, when convenience yield shocks are mean reverting and shocks to non-marketed risks are close to random walks, the reverse is true and the level of

\[\text{The importance of informational trades in futures markets is apparently supported by Roll (1984) for frozen orange juice.}\]
adverse selection in the economy falls as the contract rolls toward maturity.

This information effect of a futures expiring may counteract or reinforce the Samuelson effect. Suppose convenience yield shocks are relatively more persistent than non-marketed risk shocks. In this instance, there is relatively little adverse selection far from the maturity of the futures. The Samuelson effect dominates and the return volatility of the futures rises as it initially matures. As the contract approaches expiration, the level of adverse selection increases. This information effect may counteract the Samuelson effect, leading to a fall in the return volatility of the contract before it expires. Hence, the return volatility of the futures may initially rise with time and then fall before it expires. Conversely, when convenience yield shocks are relatively less persistent than shocks to non-marketed risks, adverse selection falls as a contract expires. In this instance, the information effect of a contract expiring reinforces the Samuelson effect.

The interaction of hedging and speculative trades also affects the open interest of a futures. It can cause the open interest of a futures to initially rise with time and then fall before it expires (an inverted U-shaped time-to-maturity pattern). Suppose that investors can trade in only one futures to hedge their positions in the spot. Under symmetric information, the optimal position in the futures to hedge a given spot position is roughly proportional to the regression coefficient of changes in the spot price on changes in the futures price multiplied by the magnitude of the spot position. In the presence of mean reverting convenience yield shocks to the spot price, this regression coefficient increases with the time-to-maturity of the contract (since the futures price is less sensitive to shocks far from maturity.) Hence open interest increases with the contract’s time-to-maturity (or falls as the contract matures). Under asymmetric information, the level of open interest observed will also vary with the adverse selection cost to trading: the higher the cost, the lower the level of open interest observed. When the adverse selection cost to trading in a particular contract is monotonically increasing with time-to-maturity, holding all else equal, open interest tends to fall with time-to-maturity. The interaction of mean reverting convenience yield shocks and the informational effect of a contract maturing can generate an inverted U-shaped time-to-maturity pattern in open interest.

In addition to these time-to-maturity patterns in the open interest of a particular contract, this model can also speak to the distribution of open interest across contracts of differing maturities at a given point in time. Under symmetric information, when the nearby contract provides an imperfect hedge of spot positions, investors trade in a distant contract to hedge the basis risk of trading in the nearby. When convenience yield shocks are relatively more permanent, rolling over in the nearby is a good substitute for trading in distant contracts. Hence, all else equal, markets with more mean reverting convenience yield shocks tend to have more trading in distant contracts. Under asymmetric information, since nearby and distant contracts have different time-to-maturity, the adverse selection cost to trading in each need not be the same. It is shown that adverse selection may lead to less open interest in distant contracts. Whether information asymmetry leads to a distribution of open interest that is skewed toward the nearby depends on a number of factors. One factor is the relative importance of hedging trades in nearby
versus distant contracts under symmetric information. When there is relatively little hedging trades in distant contracts, then under asymmetric information, positions in distant contracts are more likely to be speculative or information driven than those in the nearby. This results in a disproportionate decrease of open interest in the distant contracts to total open interest as information asymmetry rises. Hence, asymmetric information reinforces the pattern observed under symmetric information that markets with more mean reverting convenience yield shocks tend to observe more trading in distant contracts.

Finally, it is important to emphasize that these seemingly separate predictions regarding violations of the Samuelson Effect and the distribution of open interest across contracts of differing maturity lengths are really related. Since markets where convenience yield shocks are relatively more persistent than non-marketed income shocks are more likely to observe violations of the Samuelson Effect and less trading in distant contracts, it follows that the Samuelson effect is more likely to be violated in markets with less trading in distant contracts.

This paper proceeds as follows. In the next section, we relate our work to the theoretical literature on futures returns and trading. We present our model in Section 2 and discuss the definition and general solution of the equilibrium in Sections 3.1 and 3.2 respectively. Results are presented in Section 4. Section 5 contains a summary of the testable implications of this model.

1. Related Literature

There is a vast theoretical literature on futures returns and trading. Under symmetric information, Samuelson (1965, 1976) assumes a stationary spot price process and derives the behavior of futures prices under arbitrage. That the return volatility of a futures falls with its time-to-maturity is one pattern rationalized in his models. The other is that futures returns far from maturity are less volatile than those close to maturity. Rutledge (1976) provides an example of a non-stationary spot price process in which the return volatility of a futures may rise with its time-to-maturity. Anderson and Danthine (1983) argue that futures return volatility and open interest are sensitive to exogenous time-variation in the underlying uncertainty. Duffie and Jackson (1990) derive closed form solutions for optimal futures hedging positions assuming various exogenous spot price processes. Hirshleifer (1990, 1991) studies optimal futures hedging in the presence of effectively complete markets and differences in the resolution of uncertainty among market participants. Kamara (1993) studies futures returns and trading in the presence of production inflexibilities. Under asymmetric information, the notion of futures playing the dual role of conveying information as well as allocating risk dates back to Grossman (1977, 1986, 1988). Henrotte (1992) considers the effect of information asymmetry between a risk averse manager and traders on the real production decisions of a good in which a forward contract is traded.

While our model is related to these papers in a number of ways, there are also some distinct differences. One important difference is that our discussion emphasizes the interaction of alloca-
tional and informational trades in determining equilibrium returns and trading. Several benefits flow from this modeling effort. First, we can simultaneously speak to both return volatility and open interest. There has been little analysis of these two objects in tandem. Studying these two objects in tandem provides additional testable implications. Another benefit is that by allowing speculative trades in both spot and futures, we can study the natural variation in the informativeness of futures prices across the life of a contract. While the role of futures prices conveying information is well understood in a static setting, its role in a dynamic setting is less understood. A contribution of this paper is to model how this adverse selection cost varies across the life of a given contract and across contracts with different time-to-maturity at a given time. Finally, our model is related to Hong (1997). Hong (1997) focuses on futures as risk management tools and considers less studied issues such as the time-to-maturity patterns in spot return volatility and volume when a futures matures. Our model is also related to Wang (1994) and Huang and Wang (1996) under asymmetric information.

2. The Model

We consider a single good economy defined on a discrete time horizon indexed by \(t\) where \(t \in \{0, 1, 2, ..., \infty\}\). The underlying uncertainty of the economy is generated by a vector of i.i.d. standard normal shocks denoted by \(\epsilon_t\). There are two classes of investors in the economy, \(i = a, b\), with population weights \(\omega\) and \(1 - \omega\), respectively, where \(\omega \in [0, 1]\). Investors in class \(i\) will also be referred to as investor \(i\).

A. Publicly Traded Securities

There are four publicly traded securities: a riskless asset, a spot asset and two futures written on the spot price. Investors are initially endowed with \(\tilde{\theta}\) shares of the spot asset, while the two futures are in zero net supply. The payoffs to these investments are as follows:

- The riskless asset pays a constant gross rate of return per period of \(R = 1 + r\) \((r > 0)\).\(^5\)
- Holding the spot is different from investing in futures as the owner of the spot obtains a payoff of \(D_t\) each period:

\[
D_{t+1} = e Z_{t+1} + b_D \epsilon_{t+1}
\]

\[
Z_{t+1} = a_Z Z_t + b_Z \epsilon_{t+1}
\]

where \(Z_t = [Z_{1,t}, Z_{2,t}]'\). \(e\) is the unit row vector of appropriate order. \(b_D, a_Z,\) and \(b_Z\) are constant matrices.

\(^5\)The importance of stochastic interest rates in futures pricing is discussed empirically in Fama and French (1987) and theoretically in Cox, Ingersoll, and Ross (1981, 1985a,b).
$D_t$ is composed of two components. The first is a stochastic growth rate (time-varying expected return) given by $eZ_t \equiv \sum_{i=1}^{2} Z_{i,t}$. This time-varying expected return follows a two factor model, where the two factors follow a vector auto-regressive process. It is assumed that these two factors have differing degrees of persistence, denoted by $a_{z_i}$, where $0 < a_{z_1} \leq 1 \ (a_z = \text{diag}(a_{z_1}, a_{z_2}))$.\(^6\) Hence, $D_t$ exhibits mean reversion. The matrices $b_D$ and $b_z$ determine the exposure of $D_t$ and $Z_{i,t}$ to the underlying shocks. $D_t$ has an exposure of $\sigma_D$ and $Z_{i,t}$ has an exposure of $\sigma_{Z_i}$.

This spot asset can be thought of as a storable commodity. The payoff stream to holding the commodity, $D_t$, is the “convenience yield” that accrues from holding inventories—the value of any benefits that inventories provide, including the ability to smooth production, avoid stockouts, and facilitate the scheduling of production and sales. Such an exogenous, stochastic convenience yield can be regarded as a reduced form of a more general model in which the convenience yield is determined endogenously by production, consumption and storage decisions. It can be motivated by the empirically documented importance of convenience yields in driving spot and futures prices in numerous markets.\(^7\)

We adopt the specification given by (1) for the convenience yield for two reasons. First, this specification can be motivated by empirically documented mean-reversion of prices in a number of futures markets.\(^8\) Second, the multi-factor model keeps markets incomplete when futures are traded and hence the equilibrium prices from being fully revealing under asymmetric information.

**B. Non-marketed Risks**

In addition to these publicly traded securities, it is assumed that investor $a$ receives a non-marketed income with the following payoff:\(^9\)

- Investor $a$’s non-marketed income has an excess return each period of $q_t$ given by

  
  \[
  q_{t+1} = Y_t' a_q Y_t + e Y_t b_q e_{t+1} \\
  Y_{t+1} = a_y Y_t + b_y e_{t+1},
  \]

  \[
  \text{(2a)} \quad \text{and} \quad \text{(2b)}
  \]

  where $Y_t = [1, Y_{1,t}, Y_{2,t}]'$. $a_q$, $b_q$, $a_y$, and $b_y$ are constant matrices.

\(^6\)For a set of elements $e_1, e_2, \cdots, e_m$, $\text{diag}(e_1, e_2, \cdots, e_m)$ is a diagonal matrix with these elements as its diagonal elements.

\(^7\)See Pindyck (1993) for a discussion on convenience yields. See Brennan (1991) for a motivation of an exogenous convenience yield in the context of commodity contingent claims pricing. See Fama and French (1987) for a study of the relative importance of convenience yields across a variety of futures markets.

\(^8\)See Bessembinder, et.al. (1995) for findings on mean reversion in spot and futures prices across a variety of futures markets. See Gibson and Schwartz (1990) for evidence of a mean-reverting convenience yield in crude oil futures.

\(^9\)We could also give investors $b$ a non-marketed income as long as its payoffs are not perfectly correlated with investor $a$’s. But for expositional simplicity, we do not. Our results are robust to this assumption.
$q_t$ has two components. The first component is a stochastic growth rate given by the quadratic form $Y_t' a_q Y_t$. A special case of this specification is just the linear factor model which drives the time-varying expected return of the convenience yield. This form is the most general specification possible and yet still keep the problem tractable. Realizations from this specification can be positive or negative if $a_q$ is an indefinite matrix. Here, the non-marketed income has a stochastic volatility given by a linear factor model: $(1 + Y_{1,t} + Y_{2,t}) b_q \epsilon_{t+1}$. The realizations of the drift and volatility is driven by $Y_t$ which follows a vector auto-regressive process with differing degrees of persistence, $0 < a_{Y_i} < 1$ for $i = 1, 2$ ($a_Y = \text{diag}(1, a_{Y_1}, a_{Y_2})$). $b_q$ and $b_Y$ determine the exposure of the non-marketed risks to the underlying vector of uncertainty. $q_t$ has an exposure given by $\sigma_q$ and $Y_{t,t}$ has an exposure of $\sigma_{Y_t}$.

All shocks in the economy are uncorrelated except for innovations to the convenience yield and investor $a$'s non-marketed risks. These two shocks are assumed to be positively correlated, Cov$(b_q \epsilon_t, b_Y \epsilon_t) = \kappa_{dq} > 0$.\footnote{It is unimportant whether this correlation is assumed to be positive or negative as long as it is not zero.} When $Y_t > 0$, we have that the local innovation to the convenience yield is correlated with that of investor $a$'s non-marketed risks. Hence investor $a$ short the spot to counteract her exposure to these non-marketed risks. Investor $b$ makes the market for investor $a$. Investors trade in futures to hedge their spot hedging positions.

\textit{C. Timing of Futures Introduction and Expiration}

Investors can trade in the riskless asset and the spot at all times. They can also trade in two futures of identical maturity lengths of $M$. Figure 1 illustrates the futures maturity cycle in this economy. The solid lines indicate when a particular contract is traded while the dotted lines indicate when it is not. The top line is the maturity cycle for the “nearby”, while the bottom line represents the “distant”. At $m_{1,k-1}$, the $(k-1)$-th replication of the nearby has matured. Since the $(k-1)$-th replication of the distant is initiated after the nearby, it does not mature until $m_{2,k-1}$. After a period of length $N$ represented by the dotted lines, the $k$-th replication of the nearby begins trading at $t_{1,k}$, and matures after a length of $M$ at $m_{1,k}$. The distant starts trading at $t_{2,k}$, a period of length $L$ after the nearby started trading, and matures at $m_{2,k}$. This maturity and replication process cycles periodically from the $k$-th replication to the $(k+1)$-th replication and so on. We call the first contract the “nearby” and the second contract

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{futures_maturity_cycle.png}
\caption{Futures Maturity Cycle}
\end{figure}
the “distant” since the first contract has a shorter time-to-maturity when both are traded.

This replication structure generates four trading regimes which depend on how many futures are traded within the k-th maturity cycle. Let \( \tau = o, p, q, r \) and \( \mathcal{M}_k(\tau) \) denote the trading regimes. \( \mathcal{M}_k(o) \equiv \{t_{1,k}, \ldots, t_{2,k} - 1\} \) denotes the set of dates in which only the nearby is traded during the k-th cycle. \( \mathcal{M}_k(p) \equiv \{t_{2,k}, \ldots, m_{1,k} - 1\} \) denotes the set of dates in which positions in both contracts can be opened during the k-th cycle. \( \mathcal{M}_k(q) \equiv \{m_{1,k}, \ldots, m_{2,k} - 1\} \) is those dates in which only the distant can be traded during the k-th cycle. And \( \mathcal{M}_k(r) \equiv \{m_{2,k}, \ldots, t_{1,k+1} - 1\} \) is the set of dates in which no contracts can be traded during the k-th cycle. Notice that by letting \( N \to 0 \), we obtain the special case of one futures always being traded (i.e. \( \mathcal{M}_k(r) \) is the empty set).

Let \( S_t \) denote the spot price, \( H_{1,t} \) the price of the nearby and \( H_{2,t} \) the price of the distant. \( S_t \) is defined at all times. \( H_{j,t} \) for \( j = 1, 2 \) are only defined on \( \{t_{j,k}, \ldots, m_{j,k}\} \) and must converge to the spot price at maturity: for \( k = 0, 1, \ldots \),

\[
    j = 1, 2 : H_{j,m_{j,k}} = S_{m_{j,k}}. \tag{3}
\]

This condition must be satisfied for there to be no arbitrage in the economy.

This periodic replication of futures is a simple way to get at the inherent effects of a futures maturing and being introduced into an economy. In this paper, we want to understand the informational effects of a futures maturing (as futures prices converge to spot prices). We also want to understand how information asymmetry affects the distribution of open interest across contracts of differing maturities. The first of these two goals can actually be analyzed with just one futures trading since we are interested in the inherent effects on the level of information asymmetry as the price of the nearby approaches the spot price. Having more futures traded would not change the intuition.\(^{11}\) By analyzing the behavior of prices and trading strategies in regime \( \mathcal{M}_k(p) \) (when both the nearby and distant are traded), we can understand the inherent differences in the risks that these two contracts hedge and in the information that they convey. Correctly, one could argue that many more contracts are traded in reality. While we can allow for trading in more contracts, the essential intuition behind the distribution of open interest across contracts of differing maturities can be illustrated with just two contracts.

D. Information Endowments

The publicly available set of information in the economy, \( \mathcal{N}_t \), consists of the spot prices, futures prices, and convenience yield:

\[
    t \in \{0, 1, 2, \ldots, \infty\} : \mathcal{N}_t \equiv \{S_s, H_{1,s}, H_{2,s}, D_s : 1 \leq s \leq t\}. \tag{4}
\]

\(^{11}\)The informational effect that we want to highlight is distinct from the effect of introducing another futures into the economy. Of course, the introduction of a futures has distinct informational effects since it provides a new source of information. This effect can also be analyzed within our model.
The total set of information in the economy is composed of the public information and possibly private information regarding the stochastic growth rates of the convenience yield and investor $a$'s non-marketed risks:

$$t \in \{0, 1, 2, \cdots, \infty]\ : \ I_t \equiv \mathcal{N}_t \otimes \{Z_s, Y_s, q_s : 1 \leq s \leq t\}. \quad (5)$$

Let $I_{i,t}$ denote the information set of investor $i$, $i = a, b$. In general, $I_{i,t} \subseteq I_t$, but we will adopt a simple nested information structure in this paper where $I_{a,t} = I_t$ and $I_{b,t} \subseteq I_{a,t}$. Under symmetric information, $I_{b,t} = I_{a,t}$, while under asymmetric information, $I_{b,t} = I_{b,0} \otimes \mathcal{N}_t$, where $I_{b,0}$ denotes investor $b$'s prior information on $Z_0$ and $Y_0$.

### E. Policies and Preferences

The investors choose consumption and investment policies to maximize their expected utility over life-time consumption. Let their consumption policies be denoted by $\{c_{i,t} : t \in \{0, 1, 2, \cdots, \infty\}\}$. Their investment policies in the spot and futures will depend on the different trading regimes, $\mathcal{M}_k(\tau)$. Let $\theta_{i,t}(\tau)$ be the holdings vector of investor $i$. Recall that $\tau = o, p, q, r$ denotes the particular trading regimes and hence the holdings vector will change dimensionality depending on how many futures are traded. We assume that the investors' consumption and investment policies are adapted to their information sets $I_{i,t}$.

The investors have CARA preferences and they maximize the expected utility of the form:

$$i = a, b : \quad \mathbb{E}_{i,t}\left[ -\sum_{s=t}^{\infty} \rho^{(s-t)} e^{-\gamma c_{i,s}} \right], \quad (6)$$

where $\mathbb{E}_{i,t}$ is the expectation operator conditional on investor $i$'s information set, $I_{i,t}$. For simplicity, we assume that the time-discount factor, $\rho$, and the relative risk-aversion coefficient, $\gamma$, are identical across investors.

Table 3.1 summarizes the parameters of the model.

### 3. Equilibrium

We first define what it means for the economy to be in equilibrium. Then, we specify in Lemma 1 the evolution of the state variables that characterize the uncertainty in the economy when there is asymmetric information among the investors. In Lemma 2 we characterize the investors' value functions and optimal policies for given spot and futures price processes. We then prove in Theorem 1 that an equilibrium exists generically for $\omega$ close to 1 (the fraction of uninformed in the economy is small).
Table 3.1: Summary of Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population weight of type-1 investors</td>
<td>$\omega$</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>$\tau$</td>
</tr>
<tr>
<td>Volatility of convenience yield</td>
<td>$\sigma_D$</td>
</tr>
<tr>
<td>Mean reversion of convenience yield growth rate shocks for $i=1,2,3$</td>
<td>$aZ_i$</td>
</tr>
<tr>
<td>Volatility of convenience yield growth rate shocks for $i=1,2,3$</td>
<td>$\sigma_{Z_i}$</td>
</tr>
<tr>
<td>Volatility of returns on investor $a$'s non-marketed income</td>
<td>$\sigma_q$</td>
</tr>
<tr>
<td>Mean reversion of non-marketed income growth rate shocks for $i=1,2$</td>
<td>$aY_i$</td>
</tr>
<tr>
<td>Volatility of non-marketed income growth rate shocks for $i=1,2$</td>
<td>$\sigma_{Y_i}$</td>
</tr>
<tr>
<td>Correlation of convenience yield and non-marketed income shocks</td>
<td>$\kappa_{Dq}$</td>
</tr>
<tr>
<td>Length of A Futures</td>
<td>$M$</td>
</tr>
<tr>
<td>Length of Nontrading In A Futures</td>
<td>$N$</td>
</tr>
<tr>
<td>Lag In Initiation Of Nearby and Distant</td>
<td>$L$</td>
</tr>
</tbody>
</table>

3.1. Definition of Equilibrium

An equilibrium of the economy specified above is defined by a spot price process \( \{S_t : t \in \{0,1,2,\ldots,\infty\}\} \) and futures price processes \( \{H_{1,t} : t \in \{t_{1_k},\ldots,m_{1_k}\}\} \) and \( \{H_{2,t} : t \in \{t_{2_k},\ldots,m_{2_k}\}\} \) for \( k = 0,1,2,\ldots \), such that investors follow their optimal policies, \( \theta_{i,t}(\tau) \), to maximize their expected utilities. In general, the equilibrium spot and futures prices and investors' optimal policies can be expressed as a function of time and a relevant set of state variables, \( \Psi_t \).

The dependence of the equilibrium prices and policies on time follows from the nature of a futures. The expiration of a futures implies that its prices will depend not only on the evolving uncertainty in the economy but also on its time-to-maturity. Because some investors in the economy use futures to hedge their positions in the spot, this implies that the optimal policies of the investors will vary with the time-to-maturity of futures and hence so will the equilibrium spot price as well as futures prices.

Due to the nature of the periodic replication of futures after maturity, we will in this paper consider periodic equilibria in which the equilibrium price processes and investors' optimal policies exhibit periodicity in time. And due to our assumptions of a constant risk-free rate and constant absolute risk aversion in preferences, investors' demand of risky investments will be independent of their wealth. Hence, \( \Psi_t \) will be independent of \( W_{i,t} \). \( \Psi_t \) will in general include the state variables \( \{Z_t,Y_t\} \) which characterize the future payoffs of the investments. Since we assume that investor \( b \) need not observe all of these variables, investor \( b \)'s expectations about these variables, \( \hat{Z}_t \equiv E_{b,t}[Z_t] \), and \( \hat{Y}_t \equiv E_{b,t}[Y_t] \), will also determine the distribution of equilibrium prices and optimal policies. Thus, \( \Psi_t = \{Z_t,Y_t,\hat{Z}_t,\hat{Y}_t,\cdots\} \) is a set of sufficient statistics
for the information sets of the investors, $\mathcal{I}_{i,t}$. Let $\bullet$ denote some subset of $\mathcal{U}_t$ that we will specify below in the cases of symmetric and asymmetric information. Finally, we restrict ourselves to a linear periodic equilibrium.

**Definition 1** In the economy defined above, a linear periodic equilibrium is defined by $S_t = S(\bullet; t)$, $H_{1,t} = H_1(\bullet; t)$ and $H_{2,t} = H_2(\bullet; t)$ where $S(\bullet; t)$ and $H_j(\bullet; t)$ for $j = 1, 2$ are linear in $\bullet$ and periodic in time, i.e. for $k = 0, 1, 2, \cdots$

\begin{align}
\alpha \in \{0, 1, 2, \cdots, M + N\} :& \quad S(\bullet; \alpha) = S(\bullet; t_{1,k} + \alpha) \quad \text{(7a)} \\
\alpha \in \{0, 1, 2, \cdots, M\} :& \quad H_j(\bullet; \alpha) = H_j(\bullet; t_{j,k} + \alpha). \quad \text{(7b)}
\end{align}

Additionally, investors' policy functions are such that (i) the policies maximize investors' expected utility, (ii) the spot and futures markets clear and (iii) the investors' policy functions are periodic in time.

Observe that although the underlying uncertainty in the economy evolves homogeneously through time, the functional form of the equilibrium price processes have the same functional form for every futures maturity cycle. The realized values of the spot and futures prices will be different each cycle since the state variables can be different.

The return per share each period from investing in the spot, $Q_{0,t}$, is composed of the stochastic convenience yield, $D_t$, and the discounted capital gains $S_{t+1} - RS_t$. $Q_{0,t} = S_{t+1} - RS_t + D_t$. We assume that fluctuations in the futures prices are settled each period and hence the return each period from taking a position in a futures is $Q_{j,t} = H_{j,t+1} - H_{j,t}$ for $j = 1, 2$. The vector of return processes is denoted by $Q_t(\tau)$ which changes dimensionality depending on the trading regime.

We now state the investors' control problem so as to elaborate on the nature of the periodic equilibrium. Given the investors' preferences in (6), the investors' optimization problems are given by: for $i = a, b$

\begin{align}
J_i(W_{i,t}, \bullet, t) \equiv \sup_{\{c_t, \theta_t(\tau)\}} E_{i,t} \left[ -\sum_{s=t}^{\infty} \rho^{(s-t)} e^{-\gamma c_{i,s}} \right] \\
\text{s.t.} \quad W_{i,t+1} = (W_{i,t} - c_{i,t}) R + \theta_{i,t}(\tau) Q_t(\tau) + 1_i q_{t,i}. \quad \text{(8a)}
\end{align}

where $1_i = 1$ if $i = a$ and $1_i = 0$ if $i = b$, and $J_i(W_{i,t}, \bullet, t)$ is investor $i$'s value function. $\bullet$ denotes the relevant state variables that characterize the return processes of the spot, futures and their non-marketed risks given their information.

We assume that investors' value functions will also exhibit periodicity: for $k = 0, 1, 2, \cdots$

\begin{align}
t \in \{0, 1, 2, \cdots, \infty\} : J_i(\cdot, \cdot; t) = J_i(\cdot, \cdot; t + M + N), \quad i = a, b. \quad \text{(9)}
\end{align}
The periodicity condition for the value functions, (9), provides the necessary boundary conditions for a periodic solution.

The spot market has to clear at all times and the futures are in zero net supply when traded:

\[ t \in M_k(\tau) : \omega \theta_{a,t}(\tau) + (1 - \omega) \theta_{b,t}(\tau) = \bar{\theta}(\tau) \] (10)

Here, \( \bar{\theta}(\tau) \) is a matrix denoting the supply of the securities in the economy and depends on the trading regime. Thus, a periodic equilibrium is given by periodic price functions (7) such that investors optimally solve (8)-(9), and the markets clear—(10) holds in equilibrium.

### 3.2. The General Solution

We assume that investor \( a \) is endowed with information set \( I_{a,t} = I_t \), where \( I_t \) is given by (5) while investor \( b \) is endowed with \( I_{b,t} = I_{b,0} \otimes N_t \), where \( N_t \) is given by (4). Since investor \( b \) is uninformed about the stochastic growth rates which drive the payoffs of the various investments, she will form expectations about them conditional on her information set. Her expectations will influence the prices of the assets. Let \( X_t = [Y_t', (Z_t - \tilde{Z}_t)', (Y_t - \tilde{Y}_t)', X_t(\tau)]' \) and \( X_t(\tau) \) denote some sub-vector of \( X_t \) which depends on the trading regime. We conjecture that the equilibrium asset prices have the following linear form:

**Conjecture 1** Let \( P_t(\tau) \) be a subvector of \([S_t, H_{1,t}, H_{2,t}]'\) which depends on the trading regime. A linear periodic equilibrium is \( P_t(\tau) \) such that: for \( \tau = o, p, q, r \)

\[ t \in M_k(\tau) : P_t(\tau) = \lambda_{z,t}(\tau)Z_t - \lambda_{x,t}(\tau)X_t(\tau). \] (11)

Let \( F_t = [Z_{1,t}, Z_{2,t}, Y_{1,t}, Y_{2,t}]' \) be the vector of processes which the uninformed investor forms conditional expectations on given market prices and realizations of convenience yields. Let \( \tilde{F}_t = E_{b,t}[F_t] \) be the conditional expectations formed by the uninformed investor. We can calculate the evolution of the conditional expectations and conditional variances formed by investor \( b \): \( \tilde{F}_t \) and \( \sigma_t = E_{b,t}[\Delta_t \Delta_t'] \). Notice that the state variables \( F_t \) follow a Gaussian Markov process,

\[ t \in \{0, \cdots, \infty\} : F_t = a_F F_{t-1} + b_F \epsilon_t, \] (12)

where \( a_F \) and \( b_F \) are given in proof of Lemma 1 and the signal vector also follows a Gaussian Markov process, for \( k = 0, 1, 2, \cdots \)

\[ t \in M_k(\tau) : N_t(\tau) = a_{N,t}(\tau)F_t + b_N(\tau)\epsilon_t, \] (13)

where \( a_{N,t}(\tau) \) and \( b_N(\tau) \) can be derived from (11) and are given in proof of Lemma 1. Thus, we have a Kalman filtering problem whose solution is given in Lemma 1.
Lemma 1 In a linear periodic equilibrium of the form of (11), \( \hat{F}_t \) and \( o_t \) are governed by

\[
\begin{align*}
 t \in M_k(\tau) : \quad \begin{cases} 
 \hat{F}_t = a_F \hat{F}_{t-1} + k_t(\tau) (N_t(\tau) - E_{b,t}[N_t(\tau)]) \\
 o_t = (t - k_t(\tau) a_{N,t}(\tau)) (a_F o_{t-1} a_F' + \sigma_{FF})
\end{cases}
\end{align*}
\]

for \( k = 0, 1, 2, \ldots \), and the periodicity condition is given by

\[
o(t_{1,k}) = o(t_{1,k+1}).
\]

Furthermore, \( \Psi_t \) follows a Gaussian Markov process. \( k_t(\tau) \) is given in proof in appendix. \( i \) is the identity matrix of appropriate order.

Given the equilibrium prices, we now consider the optimal policies of the investors. Let \( X_{a,t}(\tau) = X_t(\tau) \) and \( X_{b,t}(\tau) = \text{stack} \{1, \hat{Y}_{1,t}, \hat{Y}_{2,t}\} \) for \( \tau = o, p, q, r \). From (14), we can derive the evolution of the investors' state variables, \( X_{i,t}(\tau) \). This is done in the proof of Lemma 2. Given (11) and (25), we can also derive the excess share returns on the spot and futures. We have the following results for investors' optimal policies and value functions:

Lemma 2 When \( I_{a,t} = I_t \) and \( I_{b,t} = I_{b,0} \otimes N_t \), \( \forall t \geq 0, i = a, b \), and the spot and futures prices have the form in (11), investor \( i \)'s optimal policies and her value function have the form: for \( k = 0, 1, 2, \ldots \)

\[
\begin{align*}
 t \in M_k(\tau) : \quad \begin{cases} 
 \theta_{i,t}(\tau) = \frac{R}{\tau} h_{i,t}(\tau) X_{i,t}(\tau) \\
 c_{i,t} = -\frac{1}{\tau} \log \left[ \frac{\partial J_{i,t}}{\partial W_{i,t}} \right] \\
 J_{i,t} = -\rho' \exp \left\{ -\alpha W_{i,t} - \frac{1}{2} \left( X_{i,t}(\tau)' v_{i,t}(\tau) X_{i,t}(\tau) \right) \right\}
\end{cases}
\end{align*}
\]

where \( h_{i,t}(\tau) \) are given in proof in appendix and \( v_{i,t}(\tau) \) are symmetric positive semi-definite matrices which satisfy a system of ordinary difference equations given in Lemma 7.

Notice that the investors' holdings are linear in the state variables \( X_{i,t}(\tau) \). The investors' value functions are characterized by \( v_{i,t}(\tau) \) for \( \tau = o, p, q, r \). Given the conjectured periodic equilibrium prices of (11), the above lemma expresses investor \( i \)'s policies as functions of \( v_{i}(\tau) \) for \( \tau = o, p, q, r \). To show that a periodic equilibrium exists, we need to show that the conjectured prices of (11) clear the spot and futures markets. Solving for a linear, periodic equilibrium reduces to solving a system of nonlinear first order difference equations given by (14), (27) and (31) subject to periodicity conditions given by (15), (28) and (7). We prove the existence of such a linear periodic equilibrium in Theorem 1.

Theorem 1 For \( \omega \) close to one, a linear periodic equilibrium of the form in (11) exists generically in which the uninformed investors' expectations are given by Lemma 1 and the optimal policies of both investors are given by Lemma 2.
In general, the model needs to be solved numerically. However, through studying a number of special cases of the model which yield closed form solutions and performing comparative statics, we are able to gain a relatively good understanding of the model. The numerical methods used to solve this system of nonlinear first order difference equations is standard and is discussed in the appendix.

Before proceeding with our discussion, we describe why we choose certain parameter values for our comparative statics. Empirical findings place constraints on some of the parameters in our model. While we try to be sensitive to these constraints, we do not try to calibrate the model. We want each period of trading to correspond roughly to a day. The most important parameters are the risk-free rate and parameters of mean reversion governing the convenience yield and non-marketed risk shocks. Throughout, we set the constant risk-free rate per period, \( r \), to be .05%. The mean reversion parameters for the convenience yield and the non-marketed risk shocks are harder to choose since there is a huge dispersion in the mean reversion of spot and futures prices across futures markets. For instance, there is substantial mean reversion in crude oil and little in S&P.\(^{12}\) Hence, we try to choose the mean reversion coefficients to be as large as possible (i.e. as little mean reversion as possible) and still illustrate the intuition that we want to develop with this model. In general, the mean reversion coefficients for the convenience yield, \( a_{Z_t} \), will be no smaller than .99. The only exceptions are when we vary one of these parameters to illustrate intuition. The mean reversion coefficients for the non-marketed risk shocks, \( a_{\nu_t} \), are set to be no smaller than .95. The parameter of absolute risk aversion, \( \gamma \), is set to be 100. The maturity length for a given contract will vary from sixty to one-hundred twenty periods depending on the particular situation analyzed. The remaining parameters are chosen to illustrate the main insights of the model.

4. Results

In this section, we study the properties of the solution described above. The discussion is organized around four topics: (1) the time-to-maturity pattern in futures return volatility, (2) the time-to-maturity pattern in open interest, and (3) the distribution of open interest across contracts of different maturities. For each topic, we develop the intuition for the symmetric information benchmark and then consider the effect of asymmetric information on this benchmark.

4.1. Time-To-Maturity Pattern In Futures Return Volatility

In this subsection, we focus on the predictions of this model for the time-to-maturity pattern in the return volatility of a given futures. Since the intuition we want to develop is independent

\(^{12}\)See Bessembinder, et.al. (1995).
of how many contracts are traded, we limit our discussion to the case where only the nearby is traded. When investors trade for only hedging reasons, we show that the return volatility of the contract monotonically rises as the contract expires. When investors also trade for speculative reasons, the return volatility of the contract may initially rise as it matures and then fall before it expires (i.e. it exhibits an inverted U-shaped time-to-maturity pattern).\textsuperscript{13}

\subsection{A. Symmetric Information}

Under symmetric information, the time-to-maturity pattern of futures return volatility is particularly simple. Recall that the futures price is given by: for $k = 0, 1, 2, \ldots$

\[ H_{1,t} = \lambda_{12,t} Z_{1,t} - \lambda_{10,t} - \lambda_{11,t} Y_{1,t} - \lambda_{12,t} Y_{2,t}. \]

Since all the shocks are homogeneous through time, the return volatility of the futures is determined by the time variation in the price elasticities: $\lambda_{12,t}$, $\lambda_{11,t}$ and $\lambda_{12,t}$. It is not hard to show that $\lambda_{12,t}$ monotonically increases as the futures expires (i.e. the futures price becomes more sensitive to convenience yield shocks as the contract expires).\textsuperscript{14} This is just the Samuelson Effect: intuitively, think of the futures price as approximately the spot price at some future date. If shocks are mean-reverting, shocks which occur from the expiration date die out by the time the contract expires. Hence, they get less impounded into the futures price far from maturity.

This intuition also extends to the non-marketed risks shocks as well. We assume for simplicity that the expected return of the convenience yield is driven only by $Z_{1,t}$ whereas the expected return and volatility on investor $\alpha$'s non-marketed risks are driven by two shocks, $Y_{1,t}$ and $Y_{2,t}$.\textsuperscript{15} We will use this case as a benchmark for our discussion when investors are asymmetrically informed.

Figure 3-2 illustrates the behavior of the price elasticity of the nearby to the underlying non-marketed income shock $Y_{1,t}$ ($\lambda_{11,t}$) and the time-to-maturity pattern of the return volatility of this contract. $\lambda_{11,y_{2,t}}$ is also monotonically increasing as the contract expires, although at a different rate. The level of the price elasticity decrease as $\omega$ decreases because with proportionately more investors from the other class to make the market for investors from class $\alpha$, prices are less sensitive to the non-marketed shocks of investor $\alpha$. Hence, futures return volatility also falls with smaller $\omega$. Notice that for all values of $\omega$, the price elasticity is decreasing with time-to-maturity and hence so is futures return volatility. Hence, the Samuelson effect still applies in this setting.

\begin{itemize}
  \item Many studies have tested the Samuelson effect (i.e. considered linear specifications for the return volatility of a contract as it expires). For studies, see Grammatikos and Saunders (1986) in currency futures, Bessembinder and Seguin (1992) in the S&P futures, Barnhill, et.al. (1987) in treasury-bond futures, and Gibson and Schwartz (1990) in crude oil futures. But few have consider nonlinear specifications. This paper provides a particular non-linear specification for testing.
  \item See Proposition 1 of Hong (1997).
  \item This is equivalent to assuming that $\sigma_{2} = 0$ while $\sigma_{1} > 0$ and $\sigma_{2} > 0$. Assuming that the expected return is driven only by one shock is without lost of generality since it is sufficient to generate the Samuelson effect.
\end{itemize}
Figure 3-2: Variation in futures price elasticity to non-marketed risk shock $Y_{1,t}$ ($\lambda_{1Y_{1}}$) and futures return volatility with population weight of investors in class $a$ ($\omega$). Time-to-maturity is number of trading periods until the futures expires. The parameters are set at the following values: $M = 60$, $N = 60$, $\gamma = 100$, $r = .0005$, $\kappa_{Dq} = 0.5$, $\sigma_{D} = 0.1$, $\sigma_{q} = 0.025$, $a_{x_{1}} = 0.995$, $\sigma_{x_{1}} = 0.025$, $a_{y_{1}} = 0.9925$, $\sigma_{y_{1}} = 0.075$, $a_{v_{2}} = 0.95$, $\sigma_{v_{2}} = 0.05$.

($\omega < 1$).

B. Asymmetric Information

Uninformed investors in this economy use convenience yield payoffs, spot prices and futures prices to learn about the true growth rate of the convenience yield. To the extent that futures prices must converge to spot prices at maturity, the informational content of futures prices vary with the time-to-maturity of the futures. This naturally induces time-to-maturity patterns in the level of information asymmetry (or the adverse selection cost to trading for the uninformed).

This variation in the level of information asymmetry in turn induces important effects on the volatility of futures returns. Since the growth rate of the convenience yield is driven solely by $Z_{1,t}$ whereas the hedging trades of the informed investors are driven by $Y_{1,t}$ and $Y_{2,t}$, a natural measure of information asymmetry among agents regarding the future payoffs of the convenience yield is $\sigma_{11} = E_{b,t} \left[ (Z_{1,t} - \hat{Z}_{1,t})^2 \right]$, the conditional variance of the uninformed investor’s estimation error regarding the convenience yield growth rate. The time variation of this measure affects returns return volatility in the following way: when $\sigma_{11}$ increases, investors are less certain about the growth rate of the convenience yield and hence prices may move less because the uninformed investors’ expectations about future payoffs of the convenience yield are less variable when they become more uninformed. A fall in $\sigma_{11}$ yields the opposite effect.

Since the information contained in the futures price is perfectly correlated with the spot price at maturity, it follows that investors in the economy lose an important source of information with the maturity of the nearby. One natural effect is for investors to be less informed near the maturity of the contract than far from maturity all else equal (i.e. $\sigma_{11}$ is higher near maturity than far from maturity).
Interestingly, even though futures prices become redundant signals at maturity, information asymmetry, \( o_{11} \), need not be higher closer to maturity than far from maturity. The reason is that time variation in information asymmetry also depends on the time-to-maturity pattern of the following linear signal: 
\[
\tilde{H}_{1,t} = \lambda_{121,t}^* Z_{1,t} - \lambda_{1Y_{1,t}} Y_{1,t} - \lambda_{1Y_{2,t}} Y_{2,t}.
\]
Whether \( o_{11} \) increases or decreases with time-to-maturity depends on the relative rates with which the price coefficients increase as the contract expires. Holding all else constant, when \( \lambda_{1Z_{1,t}} \) increases, the futures price signal is more sensitive to and hence reveals more about the value of \( Z_{1,t} \); on the other hand, when \( \lambda_{1Y_{1,t}} \) and \( \lambda_{1Y_{2,t}} \) increase, the “noise” in the futures price signal becomes more prominent and hence the futures price signal reveals less about the true growth rate of the convenience yield.

When the shocks driving the informed investors’ hedging trades are more mean reverting, say \( a_{z_2} \) is small relative to \( a_{z_1} \), our intuition developed from the symmetric case suggests that \( \lambda_{1Z_{1,t}} \) tends to be greater than \( \lambda_{1Y_{2,t}} \) far from maturity and \( \lambda_{1Y_{2,t}} \) increases at a faster rate than \( \lambda_{1Z_{1,t}} \).\(^{16}\) Thus, futures prices are more revealing about the true convenience yield growth rate far from maturity than near (\( o_{11} \) is smaller far from maturity); since \( \lambda_{1Y_{2,t}} \) increases at a faster rate than \( \lambda_{1Z_{1,t}} \), the futures price signal is less informative about the growth of the convenience yield near maturity since the futures price signal becomes much more sensitive to the shocks driving the hedging trades with maturity. Conversely, if \( a_{z_1} \) is smaller than \( a_{z_2} \), one would expect to see information asymmetry higher far from maturity and declining as the contract expires since the futures price signal becomes more sensitive to \( Z_{1,t} \) at a much faster rate than \( Y_{2,t} \).

Whether futures return volatility takes on an inverted U-shaped time-to-maturity pattern largely depends on two factors: the level of information asymmetry, the magnitude of \( o_{11} \), and the time-to-maturity pattern of \( o_{11} \). First, the level of information asymmetry must be sufficiently large for the effect of informational trades to matter for futures return volatility. And given that \( o_{11} \) is sufficiently large, \( o_{11} \) needs to be increasing at a sufficiently fast rate as the contract expires to produce non-monotonic patterns. The rationale being that if \( o_{11} \) were decreasing as the contract expired, the informational and allocational effects both lead to increasing futures return volatility as the contract expired. The rate of information asymmetry must rise sufficiently fast to offset the natural tendency for futures return volatility to increase (the Samuelson effect).

We now detail the mechanism behind this non-monotonic time-to-maturity pattern of futures return volatility by considering a number of comparative statics to illustrate the intuition developed above. First, we highlight how the level of information asymmetry is crucial to producing non-monotonic patterns by changing \( \sigma_D \), the volatility of payoffs to holding the spot contract. Since uninformed investors also learn from realizations of the convenience yield, higher levels of \( \sigma_D \) imply that the convenience yield provides a noisier signal about its true growth rate, \( Z_{1,t} \). Figure 3-3(a) illustrates the time-to-maturity pattern of information asymmetry while Figure 3-

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\(^{16}\)Here, \( \lambda_{1z_{1,t}} \) behaves similarly to \( \lambda_{1z_{1,t}} \). When there is no adverse selection, \( \lambda_{1z_{1,t}} = \lambda_{1z_{1,t}} \).
3(b) shows the corresponding futures return volatility pattern. Notice that as \( \sigma_D \) rises, the level

![Adverse Selection (\( o_{11} \))](a)

![Futures Return Volatility](b)

**Figure 3-3: Variation in level of adverse selection (\( o_{11} \)) and futures return volatility with volatility of innovations to convenience yield (\( \sigma_D \)).** Time-to-maturity is number of trading periods until the futures expires. The parameters are set at the following values: \( M = 60, N = 60, \omega = .05, \gamma = 100, r = .0005, \kappa_{Dq} = 0.5, \sigma_\gamma = .025, a_{z_1} = .995, \sigma_{z_1} = .025, a_{\gamma_1} = .9925, \sigma_{\gamma_1} = .075, a_{\gamma_2} = .95, \sigma_{\gamma_2} = .05. \)

of information asymmetry, \( o_{11} \), also rises since payoffs to the convenience yield provide noisier signals about \( Z_{1,t} \) with larger values of \( \sigma_D \). The level of futures return volatility falls with \( \sigma_D \) for the following reason. The effect of a more variable convenience yield on futures return volatility is through the uninformed investors' forecast errors, \( Z_{1,t} - \hat{Z}_{1,t} \). Since a larger \( o_{11} \) implies that uninformed investors are less certain about future payoffs of the convenience yield, futures prices move less. Next, we turn to the time-to-maturity patterns of information asymmetry and futures return volatility. Information asymmetry, \( o_{11} \), rises as a contract expires and does so at a faster rate when \( \sigma_D \) is large. And as \( \sigma_D \) rises, futures return volatility moves from having a monotonically increasing maturity pattern to an inverted U-shaped pattern, rising initially and then falling before expiring. When \( \sigma_D \) is small, information asymmetry is less important and the Samuelson effect dominates; hence, futures return volatility rises as the contract expires. When \( \sigma_D \) is large, information asymmetry increases dramatically near maturity and counteracts the Samuelson effect, leading to futures return volatility that actually falls near maturity.

Next, we demonstrate the importance of the persistence of shocks to investor \( a \)'s hedging trades, \( a_{\gamma_1} \) and \( a_{\gamma_2} \), on the time-to-maturity pattern of information asymmetry and futures return volatility. We continue to assume that \( Y_{1,t} \) is the more persistent component of the shocks to the investor \( a \)'s hedging trades, \( a_{\gamma_1} > a_{\gamma_2} \). Holding fixed \( a_{\gamma_1} \), decreasing \( a_{\gamma_2} \) implies that changes in \( Y_{2,t} \) become less serially correlated and hence the noise in the price signals becomes less serially correlated. Hence, the uninformed investors learn more from a sequence of prices at lower levels of \( a_{\gamma_2} \). In the extreme, when \( a_{\gamma_2} \) is close to zero, neighboring prices provide close to independent signals (about \( Z_{1,t} \)) which tend to reveal more of the informed investors' private information about \( Z_{1,t} \). Figure 3-4(a) illustrates the time-to-maturity pattern of \( o_{11} \) for
varying levels of persistence in the shocks to the hedging trades, $a_{Y_2}$. Figure 3.4(b) shows the corresponding time-to-maturity patterns for futures return volatility. Notice that as $a_{Y_2}$ falls,

![Adverse Selection (a11)](image1)

![Futures Return Volatility](image2)

Figure 3.4: Variation in level of adverse selection ($a_{11}$) and futures return volatility with persistence of non-marketed risk shock $Y_{2,t}$ ($a_{Y_2}$). Time-to-maturity is number of trading periods until the futures expires. The parameters are set at the following values: $M = 60$, $N = 60$, $\omega = .05$, $\gamma = 100$, $r = .0005$, $\kappa_{Dq} = .5$, $\sigma_D = .1$, $\sigma_g = .025$, $a_{z_1} = .995$, $\sigma_{z_1} = .025$, $a_{Y_1} = .9925$, $\sigma_{Y_1} = .075$, $\sigma_{Y_2} = .05$.

the level of information asymmetry falls as investors learn more from futures and spot prices. The level of futures return volatility changes only slightly because changing $a_{Y_2}$ has complex effects on the level of futures return volatility. While increasing $a_{Y_2}$ tends to lower information asymmetry (which increases futures return volatility), it also tends to make prices less sensitive to $Y_{2,t}$ as uninformed investors demand a lower conditional premium to make the market for the informed investors’ allocational trades (which tends to lower futures return volatility). What is most prominent about the change in $a_{Y_2}$ is the rate at which information asymmetry increases as the contract expires. Notice that as $a_{Y_2}$ falls, information asymmetry increases at a much faster rate near maturity. This is consistent with the intuition developed above. For $a_{Y_2}$ small, $\lambda_{1Y_2}$ is less than $\lambda_{1Z_1}$ far from maturity and hence far from maturity, futures prices are more revealing for smaller values of $a_{Y_2}$; and $a_{11}$ increases at a faster rate close to maturity as $\lambda_{1Y_2}$ increases at a faster rate than $\lambda_{1Z_1}$ with maturity. We attain an inverted U-shaped pattern for futures return volatility for $a_{Y_2}$ smaller as the rate at which information asymmetry increases near maturity dominates the Samuelson effect. In this sense, we are more likely to see an inverted pattern in futures return volatility in markets whose futures prices are more mean reverting, controlling for the mean reversion in the convenience yield.

Next, we consider the impact of changing the mean reversion in the growth rate of the convenience yield, $a_{z_1}$, on the time-to-maturity pattern of information asymmetry and futures return volatility. When $a_{z_1}$ rises and $Z_{1,t}$ is close to having a unit root, fluctuations in $Z_{1,t}$ are more persistent and neighboring prices provide different signals about essentially the same $Z_{1,t}$; thus, more uncertainty about $Z_{1,t}$ is resolved by a given sequence of prices. Figure 3-
5(a) illustrates the maturity pattern of the information asymmetry while Figure 3-5(b) shows the corresponding futures return volatility pattern. As $a_{z_1}$ decreases, the level of information asymmetry rises since a sequence of prices reveals a smaller fraction of uncertainty about $Z_{1,t}$. Futures return volatility falls because spot prices are less sensitive to shocks in the growth rate of the convenience yield due to discounting as described above and because at higher levels of information asymmetry, prices move less. Notice that as $a_{z_1}$ falls, the rate at which information asymmetry rises as the contract expires decreases since $\lambda_{1z_1}$ now increases at a faster rate and hence reveals more information about $Z_{1,t}$. More importantly, futures return volatility moves from having an inverted U-shape for large values of $a_{z_1}$ to monotonically increasing at small values of $a_{z_1}$. So, even though we have higher levels of information asymmetry (which is necessary to induce non-monotonic variations in futures return volatility) as $a_{z_1}$ falls, futures return volatility is monotonically increasing for small $a_{z_1}$ as the rate at which $o_{11}$ increases is decreasing with a less mean-reverting growth rate for the convenience yield. This confirms that while a sufficiently high level of information asymmetry is necessary to achieve non-monotonic variations in futures return volatility, it is far from sufficient. More crucially, it depends on the interaction of the persistence of shocks to the growth rates of the convenience yield and the informed investors’ hedging trades. Markets that have more mean-reverting convenience yields are less likely to have futures return volatility which take on non-monotonic patterns.

This intuition is made particularly clear by considering a setting in which convenience yield shocks are more temporary than shocks to the the non-marketed risks in the economy. When $a_{z_1} < a_{y_2}$, prices are less informative far from maturity than near. The reason follows straightforwardly from the above discussion. When $a_{z_1} < a_{y_1}$, $\lambda_{1z_1,t}$ increases at a faster rate with maturity. Hence, far from maturity, prices are not very informative about $Z_{1,t}$ and $o_{11}$ is large.
As the contract matures, however, \( \lambda_{1Z_{1,t}} \) increases faster than \( \lambda_{1Y_{2,t}} \) and hence \( o_{11} \) falls. This

![Adverse Selection \((o_{11})\) and Futures Return Volatility](image)

**Figure 3-6:** Variation in level of adverse selection \((o_{11})\) and futures return volatility with persistence of convenience yield shock \( Z_{1,t} \) \((a_{z_1})\). Time-to-maturity is number of trading periods until the futures expires. The parameters are set at the following values: \( M = 60, N = 60, \omega = .05, \gamma = 100, r = .0005, \kappa_{Dq} = 0.5, \sigma_D = .1, \sigma_q = .025, \sigma_{z_1} = .025, a_{y_1} = .9925, \sigma_{y_1} = .075, a_{y_2} = .95, \sigma_{y_2} = .05.\)

intuition is illustrated in Figure 3-6. Notice that when \( a_{z_1} \) is large, \( o_{11} \) is U-shaped. However, as \( a_{z_1} \) decreases, this U-shaped becomes just strictly downward sloping, as the above intuition would suggest. Here, futures return volatility is everywhere where increasing since the effect of information asymmetry is to accentuate the Samuelson effect of higher futures return volatility with maturity.

**4.2. Time-To-Maturity Pattern in Open Interest**

In this subsection, we focus on the predictions of this model for the time-to-maturity pattern in the open interest of a given futures. We show that the open interest in a given contract can take on an inverted U-shaped time-to-maturity pattern independent of often cited liquidity reasons for the rolling of open interest from the nearby to the next maturing contract. Hence, throughout this subsection, we restrict our discussion to the case in which only the nearby is traded.

Without loss of generality, we define open interest in a contract to be investor \( a \)'s average position in the contract scaled by the population weight of investors in class \( a \). In our discussion, we will in general drop the scaling by the population weight of investor \( a \) in the economy as \( \omega \) will stay fixed for our comparative statics.

**A. Symmetric Information**

Intuitively, the level of open interest in the nearby is a function of the price elasticities of the nearby to convenience yield shocks and the magnitude of investor \( a \)'s spot hedging position.
When the nearby’s price is more sensitive to underlying shocks, a smaller position in the nearby is required to hedge a given spot hedging position. When the spot hedging position increases, a larger position in the nearby is required for a given price sensitivity of the nearby to underlying shocks. Mean reversion in convenience yield shocks implies that the nearby’s price elasticities are inversely related to time-to-maturity. Hence, holding fixed spot hedging commitments, open interest in the nearby increases with its time-to-maturity. This is no longer true under asymmetric information.

B. Asymmetric Information

Under asymmetric information, open interest can take on an inverted U-shaped time-to-maturity pattern for an informational reason. Since the adverse selection cost to trading in this contract varies across the life of the contract, the open interest in this contract will also vary. The reason follows from the fact that the higher the level of information asymmetry (or adverse selection cost), the fewer the number of contracts opened. Since this cost is different depending on the time-to-maturity of the contract, the number of contracts opened will vary with time-to-maturity. In particular, when convenience yield shocks are less persistent than non-marketed income shocks, the adverse selection cost of trading is higher far from maturity and decreases as the contract matures. Hence, holding fixed the price elasticity of futures contracts, this information effect of a futures contract maturing suggests that we see fewer number of contracts opened far from maturity. On the other hand, the futures price elasticities with respect to convenience yield shocks are decreasing with time-to-maturity. This tends to suggest that more contracts are opened far from maturity. The interaction of these two effects can generate an inverted U-shaped time-to-maturity pattern in open interest.

We elaborate on this informational mechanism. Here we assume that \( \sigma_{\nu_1} \) and \( \sigma_{\nu_2} \) are positive so that the equilibrium is not fully revealing. Figure 3-7(a) shows the time-to-maturity pattern in the adverse selection cost to trading. As \( \sigma_D \) increases, the level of information asymmetry rises since realizations of the convenience yield are noisier signals of the true growth rate. Notice that we have chosen parameters \( a_{\nu_1} = .985 \), \( a_{\nu_2} = .995 \) and \( a_{\nu_2} = .9925 \). Hence convenience yield shocks are much less persistent than non-marketed income shocks. So information asymmetry is higher far from maturity and falls as the contract matures. This time-to-maturity pattern in the adverse selection cost to trading has important effects on the time-to-maturity pattern in open interest. Notice that since \( a_{\nu_1} \) is getting larger with higher levels of \( \sigma_D \), open interest levels fall with \( \sigma_D \). Now, since the adverse selection cost to trading is increasing with time-to-maturity, notice that the time-to-maturity pattern in open interest changes from monotonically increasing with time-to-maturity to an inverted U-shaped pattern. This reflects the interaction of the time-to-maturity patterns in futures price elasticity and information asymmetry. At low levels of information asymmetry (or \( \sigma_D \) small), the first effect dominates and open interest

\(^{17}\text{See Hong (1997).}\)
Adverse Selection ($o_{11}$)  

Open Interest

Figure 3-7: Variation in level of adverse selection ($o_{11}$) and open interest with volatility of innovations to convenience yield ($\sigma_D$). Time-to-maturity is the number of trading periods until the futures expires. The parameters are set at the following values: $M = 60$, $N = 60$, $\omega = .05$, $\gamma = 100$, $r = .0005$, $\kappa_{Dq} = 0.5$, $\sigma_q = .03$, $a_{z_1} = .985$, $\sigma_{z_1} = .065$, $a_{\gamma_1} = .995$, $\sigma_{\gamma_1} = .075$, $a_{\gamma_2} = .9925$, $\sigma_{\gamma_2} = .08$.

is monotonically increasing with time-to-maturity. When information asymmetry is large ($\sigma_D$ large), the information effect dominates and open interest declines far from maturity since $o_{11}$ is higher far from maturity than near.

Hence, the time-to-maturity pattern in the adverse selection cost to trading results in another mechanism which can generate an inverted U-shaped time-to-maturity pattern in open interest independent of liquidity reasons. The difference between the mechanism under symmetric and asymmetric information is that the former relies on the level of spot hedging positions falling with time-to-maturity whereas the latter does not.

4.3. Distribution Of Open Interest Across Contracts

When both contracts are traded, this model can speak to not only the time-to-maturity patterns in the open interest of each contract but also the ratio of their open interest. In other words, we can examine the distribution of open interest across different maturities and relate this distribution to adverse selection in the economy. We show that adverse selection can lead to more trading in the either the nearby or distant contracts depending on the level of hedging trades in these respective contracts under symmetric information.

A. Symmetric Information

We first analyze the determinants of the relative levels of open interest in the nearby and distant contracts when investors trade in these contracts only to hedge their risks. When the convenience yield growth rate follows a two factor model, positions in the nearby is subject to a basis risk. Because there is a basis risk in rolling over positions in the nearby contract, the distant contract
provides a vehicle to hedge this basis risk. The relative persistence of shocks \( Z_{1,t} \) and \( Z_{2,t} \) determines the distribution in open interest across these two contracts. We assume without loss of generality that \( a_{Z_2} < a_{Z_1} \). And by fixing \( a_{Z_1} \), we can speak of the convenience yield being more mean-reverting as \( a_{Z_2} \) decreases toward zero. Proposition 1 describes the holdings of investor \( a \) when both futures are traded.

**Proposition 1** When the convenience yield growth rate follows a two-factor model and shocks to non-marketed risks are i.i.d., the open interest in the nearby is larger than in the distant.

With two shocks and two futures, she can perfectly hedge away both shocks to the convenience yield growth rate. She goes long in the nearby contract (a sign opposite from her short position in the spot) and takes a short position in the distant contract equal (approximately) to the difference between her positions in the spot and nearby. Her position in the nearby is always larger than her position in the distant.

The ratio of open interest in these two contracts is increasing in the mean reversion of the convenience yield (\( a_{Z_2} \) approaches zero).

**Proposition 2** Given assumptions of Proposition 1, the proportion of open interest in the distant contract is increasing in the mean reversion of the expected return of the convenience yield.

This reflects the fact that when \( Z_{2,t} \) is more persistent (or \( a_{Z_2} \rightarrow a_{Z_1} \)), futures and spot prices are more correlated and hence the nearby contract is more effective in hedging both convenience yield shocks, \( Z_{1,t} \) and \( Z_{2,t} \). As \( a_{Z_2} \) decreases, the nearby contract is a less effective hedge and there is more need to use the distant contract to hedge the basis risk. In this setting the ratio of open interest in the distant to nearby contracts varies with time-to-maturity. In general, this ratio is larger farther from maturity and decreases with maturity because there is more basis risk with a mean reverting convenience yield far from maturity.\(^{18}\)

Hence, when investors’ non-marketed risks are an i.i.d. or equivalently when non-marketed risk shocks are very temporary (i.e. \( a_V \) is small), open interest in the nearby is greater than in the distant contract. This need not be the case however when \( a_V \) becomes larger. When non-marketed risks become more persistent, the distant contract is no used merely to hedge the basis risk in the nearby. It, along with the nearby, may also be used to dynamically hedge the stochastic investment opportunity set (e.g. Breeden (1979)). In this instance, there may be more open interest in the nearby than in the distant under symmetric information.

**B. Asymmetric Information**

\(^{18}\)This type of hedging is known in the futures trading literature as a spread trade since each futures contract attracts an offsetting position in other futures contracts. This model predicts that futures markets with more mean reverting convenience yields tend to see more trading in distant contracts because of these spread trades which hedge the basis risk. There is anecdotal evidence that energy futures markets which have more mean reverting convenience yields also tend to see more spread trades. See Brown and Errera (1987).
In this subsection, we ask how the distribution of open interest across differing maturities vary with changes in the level of information asymmetry in the economy. In the presence of information asymmetry, uninformed investors face an adverse selection cost to trading since they might be on the other side of informationally driven trades. These adverse selection costs vary across contracts of different maturities. The reason being that the level of allocational trades vary across contracts of differing maturities as the analysis under the case of symmetric information suggests. Suppose that the open interest in the nearby contract under symmetric information is substantially larger than the distant contract (from the above analysis, this would correspond to a case when \( a_T \) is small or when convenience yield shocks are persistent). Since investors in this economy behave competitively, the uninformed investors can infer that the probability of an allocational trade being placed in the distant contract is much lower than in the nearby contract. Hence, the adverse selection cost of trading in the distant contract is higher than trading in the nearby contract since the probability of an informationally motivated trade is higher for the distant contract. So, when information asymmetry among investors rises, the open interest in both contracts tend to decline since there is now a higher adverse selection cost to trading in both contracts; however, the decrease in open interest of the distant contract will be proportionately larger than in the nearby contract, resulting in an increase in the ratio of open interest in the nearby to distant contract.

We consider an example in which the adverse selection cost to trading is higher for the distant contract than for the nearby contract. We vary the level of information asymmetry by increasing the volatility of the payoffs to the spot contract, \( \sigma_D \). Increases in \( \sigma_D \) makes realizations of the convenience yield noisier signals about the true growth rate of payoffs to the spot contract. To analyze the effect of information asymmetry on the dispersion of open interest across futures contracts, we need first to understand the behavior of this ratio under symmetric information for a given set of parameters. Figure 3-8 provides the benchmark ratio of open interest in the nearby contract to that of the distant contract. Notice that far from maturity, the ratio of open interest in nearby to distant is near one as there is a larger basis risk to hedging with the nearby contract far from maturity and hence larger positions in the distant contract is used to hedge this basis risk. As the nearby contract matures, the ratio of open interest in the nearby to distant contracts increases since this basis risk gradually diminishes with maturity. As \( \sigma_D \) increases, this ratio rises, reflecting the increased covariability of futures and spot prices as \( \sigma_{Dq} \) increases and hence the increased effectiveness of the front contract as a hedging vehicle for positions in the risky security.

Given this benchmark for the ratio of open interest in the nearby to distant, we now consider how this ratio changes under asymmetric information. Figure 3-9(a) shows that information asymmetry is rising with \( \sigma_D \) as realizations of the convenience yield provide noisier signals for the future payoffs to holding the spot contract. Figure 3-9(b) shows the corresponding change in the ratio. Here, for a fixed \( \sigma_D \), the difference in the ratio of open interest in the nearby to distant under asymmetric information is greater than under symmetric information. This implies that
Figure 3-8: Variation in ratio of open interest in the nearby to distant contract under symmetric information with volatility of innovations to convenience yield ($\sigma_D$). The parameters are set at the following values: $T = 60$, $M = 120$, $N = 60$, $\omega = .05$, $\gamma = 100$, $r = .005$, $\kappa_{Dq} = 0.5$, $\sigma_q = .03$, $a_{z_1} = .995$, $\sigma_{z_1} = .065$, $a_{z_2} = .99$, $\sigma_{z_2} = .065$, $a_{\gamma_1} = .99$, $\sigma_{\gamma_1} = .09$, $a_{\gamma_2} = .95$, $\sigma_{\gamma_2} = .04$.

Figure 3-9: Variation in level of adverse selection ($\sigma_{11}$) and the difference in open interest in the nearby to distant contract under asymmetric and symmetric information with the volatility of innovations to convenience yield with $\sigma_D$. Time-to-maturity is the number of trading periods until the futures expires. The parameters are set at the following values: $T = 60$, $M = 120$, $N = 60$, $\omega = .05$, $\gamma = 100$, $r = .005$, $\kappa_{Dq} = 0.5$, $\sigma_q = .03$, $a_{z_1} = .995$, $\sigma_{z_1} = .065$, $a_{z_2} = .99$, $\sigma_{z_2} = .065$, $a_{\gamma_1} = .99$, $\sigma_{\gamma_1} = .09$, $a_{\gamma_2} = .95$, $\sigma_{\gamma_2} = .04$. 

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information asymmetry leads to concentration of open interest in the nearby contract. Notice that this difference increases with as the contracts expire: this is consistent with our intuition that the adverse selection cost to trading in the distant contract is higher closer to maturity than far from maturity as the probability of seeing allocational trades in the distant contract close to maturity is much lower than far from maturity. Next, notice that as $\sigma_D$ increases, this difference also increases. This is consistent with both an increase in asymmetric information and that probability of seeing allocational trades in the distant contract decreases with $\sigma_D$.

It is important to note that asymmetric information can also lead to a concentration of trading in the distant. This would occur were we to choose parameters in the model to lead to more trading in the distant under symmetric information. This would correspond to more a more persistent non-marketed income process ($\alpha_Y$ larger).

5. Conclusion

This paper develops a model of a competitive futures market in which investors are asymmetrically informed. This model emphasizes the interaction of allocational and informational trades. This interaction generates interesting behavior for futures return volatility and open interest. While this model is stylized and cannot capture many important features of futures returns and trading, it does generate some testable implications.

The first testable implication relates to the time-to-maturity pattern in the return volatility of the nearby. It is argued that the expiration of the nearby produces an information effect that interacts with Samuelson’s effect to produce an inverted U-shaped time-to-maturity pattern in the return volatility of the contract. Existing empirical tests generally assume a linear relation between the time-to-maturity and return volatility of a futures. This paper suggests that a quadratic specification should also be used.

The second testable implication relates to the time-to-maturity pattern in the open interest of a futures. This paper predicts that open interest may take on an inverted U-shaped time-to-maturity pattern due to information asymmetry. This can be tested in conjunction with the predictions about futures return volatility.

Along with these predictions, this model also generates cross-sectional patterns in the distribution of open interest across contracts of different maturities. This model predicts that markets with more mean-reverting convenience yield shocks tend to have a more dispersed open interest distribution for both allocational and informational reasons.
6. Appendix

This appendix consists of three parts. Section 6.1. provides some technical results needed for later use. Section 6.2. contains the proofs for all the results stated in Section 3. Section 6.3. contains proofs of results in Section 4. Section 6.4. discusses the numerical procedures.

6.1. Mathematical Preliminaries

For future use, we introduce some notation. A positive semi-definite (definite) matrix \( m \) is \( m \geq (>) 0 \) and \( |m| \) denotes the spectral norm of \( m \), defined as the absolute value of the numerically largest eigenvalue of \( m \). Let \( \Theta = \{ r > 0, \gamma > 0, 1 \geq a_{xi} > 0 (i = 1, 2), 1 > a_{yi} > 0 (i = 1, 2), \sigma_{D} \geq 0, \sigma_{C} \geq 0, \sigma_{Zi} \geq 0 (i = 1, 2), \sigma_{Yi} \geq 0 (i = 1, 2), \kappa_{pq} > 0 \} \). Let \( [m] \) be the column matrix consisting of the independent elements of matrix \( m \). We now also state a definition and an auxiliary lemma.

Definition 2 Let \( r_t \) and \( a_j \) for \( j = 0, 1, 3 \) be symmetric matrices and \( a_0 \geq 0, a_3 > 0 \). A discrete time matrix Riccati equation is defined as

\[
t \in \{1, T\} : \quad r_{t-1} = a_0 + a_1 r_t a_1' - a_1 r_t a_2' (a_3 + a_2 r_t a_2')^{-1} a_2 r_t a_1'
\]

and \( r_T = \tilde{r}_T \).

Lemma 3 For \( r_T \geq 0 \), (16) has a unique, symmetric, positive semi-definite solution, \( r_t \). And \( r_t \leq e_t \) where \( e_t \) is the solution to the following matrix linear difference equation:

\[
t \in \{1, T\} : \quad e_{t-1} = (a_1 - ka_2) e_t (a_1 - ka_2)' + a_0 + ka_3 k',
\]

where \( e_T = r_T \) and \( k \) is an arbitrary sequence of matrices.


In deriving several results in this paper, the problem to be solved often reduces to a two-point boundary-value problem for a (vector) ordinary difference equation. Here, we give a formal and relatively general definition of the two-point boundary-value problem and state some known results concerning its solution.

Definition 3 Let \( f : \mathbb{R}^+ \otimes \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R} \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R} \rightarrow \mathbb{R}^n \). A two-point boundary-value problem is defined as

\[
\begin{align*}
    u_{t-1} &= f(t, u_t; \vartheta, \omega), \quad t \in \{1, T\} \\
    0 &= g(u_0, u_T; \vartheta, \omega)
\end{align*}
\]

where \( T > 0, \vartheta \in \Theta \) and \( \omega \in [0, 1] \).
We also define the terminal value problem:

\[
\begin{aligned}
    u_{t-1} &= f(t, u_t; \vartheta, \omega), \quad t \in \{1, T\} \\
    u_T &= \bar{u}_T.
\end{aligned}
\] (19)

Under appropriate smoothness conditions on \( f(t, u_t; \vartheta, \omega) \), (19) has a unique solution denoted by \( u(t; \vartheta, \omega; \bar{u}_T) \), which is differentiable in \( \bar{u}_T \) [see, e.g. Agarwal (1992), p.224]. Solving the two-point boundary-value (18) is to seek \( \bar{u}_T \) that solves

\[
0 = g[u(0; \vartheta, \omega; \bar{u}_T), \bar{u}_T; \vartheta, \omega] \equiv g \circ u(\bar{u}_T; \vartheta, \omega).
\] (20)

The existence of a root to (20) relies on properties of \( g \circ u(\bar{u}_T; \vartheta, \omega) \). Let \( (g \circ u + 1)(\bar{u}_T; \vartheta, \omega) \equiv g \circ u(\bar{u}_T; \vartheta, \omega) + \bar{u}_T \).

**Lemma 4** If \( (g \circ u + 1)(\cdot; \vartheta, \omega) : \mathbb{R}^n \to \mathbb{R}^n \) is continuous and there exists a nonempty, closed, bounded, and convex subset of \( \mathbb{R}^n \), \( L \), such that \( (g \circ u + 1)(\cdot; \vartheta, \omega) \) maps \( L \) into itself, then (20) has a root and the two-point boundary value problem (18) has a solution.

**Proof.** Existence of a root to (20) follows from Brouwer's Fixed Point Theorem [see, e.g. Cronin (1994, p.352)]. □

The condition on \( (g \circ u + 1) \) required by Lemma 4 is not always easy to verify, in which case the existence of a solution to (18) is not readily confirmed. However, if a solution exists for \( \omega_0 \), the existence of a solution for \( \omega \) close to \( \omega_0 \) is easy to establish.

**Definition 4** \( u(t; \vartheta, \omega_0) \) is an isolated solution of system (18) if the linearized system

\[
\begin{aligned}
    y_{t-1} &= \nabla_u f(t, u_t; \vartheta, \omega_0) y_t, \quad t \in \{1, T\} \\
    0 &= \nabla_{u_0} g(u_0, u_T; \vartheta, \omega_0) y_0 + \nabla_{u_T} g(u_0, u_T; \vartheta, \omega_0) y_T
\end{aligned}
\] (21)

has \( y = 0 \) as the only solution, where \( \nabla \) denotes the partial derivative operator.

**Lemma 5** Let (18) have an isolated solution \( u(t; \vartheta, \omega_0) \) for \( \omega = \omega_0 \). Suppose that \( f(t, z; \vartheta, \omega) \) and \( g(z_0, z_T; \vartheta, \omega) \) are continuously differentiable in the neighborhood of \( (t, u(t; \vartheta, \omega_0), \omega_0) \). Then, (13) has a solution for \( \omega \) close to \( \omega_0 \).

**Proof.** [See, Agarwal (1992), p.525]. □

**Lemma 6** Let \( f : D \to \mathbb{R} \) be a real analytic function, where \( D = D_1 \otimes \cdots \otimes D_n \) is an open subset of \( \mathbb{R}^n \). Let \( Z = \{ x \in D : f(x) = 0 \} \) be its zero set. Then, either \( Z = D \) or \( \mu_n(Z) = 0 \) where \( \mu_n \) is the \( n \)-dimensional Lebesgue measure.
\textbf{Proof.} [See, e.g. Huang and Wang (1996)]. □

We next state the general form of the boundary-value problem we encounter in this paper and show that it can be reduced to the two-point boundary-value problem (18). Our boundary value problem is given by:

\begin{align*}
  u^o_t &= f^o(t + 1, u^o_{t+1}; \theta, \omega), \quad t \in \{t_{1,k}, t_{2,k} - 2\} \\
  u^p_t &= g^{op}(u^p_{t+1}; \theta, \omega), \quad t = t_{2,k} - 1 \\
  u^p_t &= f^p(t + 1, u^p_{t+1}; \theta, \omega), \quad t \in \{t_{2,k}, m_{1,k} - 2\} \\
  u^q_t &= g^{pq}(u^q_{t+1}; \theta, \omega), \quad t = m_{1,k} - 1 \\
  u^q_t &= f^q(t + 1, u^q_{t+1}; \theta, \omega), \quad t \in \{m_{1,k}, m_{2,k} - 2\} \\
  u^r_t &= g^{qr}(u^r_{t+1}; \theta, \omega), \quad t = m_{2,k} - 1 \\
  u^r_t &= f^r(t + 1, u^r_{t+1}; \theta, \omega), \quad t \in \{m_{2,k}, t_{1,k+1} - 1\} \\
  u^o_t &= g^{ro}(u^o_{t+1}; \theta, \omega), \quad t = t_{1,k+1} - 1
\end{align*}

(22)

and $u^o_{t_{1,k}} = u^o_{t_{1,k+1}}$ (periodicity). And $f^n$ for $n = o, p, q, r$, $g^{op}$, $g^{pq}$, $g^{qr}$ and $g^{ro}$ are continuously differentiable. Let $u = \text{stack} \{u^o, u^p, u^q, u^r\}$, $f = \text{stack} \{f^o, f^p, f^q, f^r\}$, and $g = \text{stack} \{g^{op}, g^{pq}, g^{qr}, g^{ro}\}$. Let

$$
\tilde{u}_t = \text{stack} \{\tilde{u}^o_{2,k-1}, \tilde{u}^p_{m_{1,k}-1}, \tilde{u}^q_{m_{2,k}-1}, \tilde{u}^r_{t_{1,k+1}-1}\}.
$$

This vectorized system conforms to our two-point boundary value problem.

6.2. Proofs of Results in Section 3

We introduce some notation for later use. Let $\epsilon_t = [\epsilon_D, t, \epsilon_q, t, \epsilon_z, t, \epsilon_{1,t}, \epsilon_{2,t}]$. It is assumed that $\mathbb{I}_{t, l}^{1(t,j)}$ is an $i \times j$ matrix with its $l \times k$ element equal to one and all its other elements equal to zero. Then from the above discussion, define $b_D = \sigma_D 1^{1(6)}_l$, $b_q = \sigma_q 1^{1(2)}_l$, $b_z = \text{stack} \{\sigma_{z_1} 1^{1(3)}_l, \sigma_{z_2} 1^{1(4)}_l\}$, $b_Y = \text{stack} \{0^{1(6)}_l, \sigma_{1} 1^{1(5)}_l, \sigma_{2} 1^{1(6)}_l\}$. $a_F = \text{diag} \{a_1, a_2, \cdots, a_{12}\}$. $a_F = \text{stack} \{\sigma_{z_1} 1^{1(3)}_l, \sigma_{z_2} 1^{1(4)}_l, \sigma_{1} 1^{1(5)}_l, \sigma_{2} 1^{1(6)}_l\}$. Also let $a_N(r) \equiv a^T_N$ for $r = o, p, q, r$. And $b_N(r) \equiv b^T_N$ for $r = o, p, q, r$. And $m_4(r) \equiv m^T_4$ for $r = o, p, q, r$.

We first define the notation introduced in Conjecture 1. Let $\lambda_{jz} = [\lambda_{jz_1}, \lambda_{jz_2}]$ for $j = 0, 1, 2$. Let

$$
X_t(o) \equiv X^o_t = [1, Y_{1,t}, Y_{2,t}, Z_{1,t} - \tilde{Z}_{1,t}, Z_{2,t} - \tilde{Z}_{2,t}]',
$$

$$
X_t(p) \equiv X^p_t = [1, Y_{1,t}, Y_{2,t}, Z_{1,t} - \tilde{Z}_{1,t}]'.
$$

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\[ X_t(q) = X_t^q = [1, Y_{1,t}, Y_{2,t}, Z_{1,t} - \tilde{Z}_{1,t}, Z_{2,t} - \tilde{Z}_{2,t}]', \]
\[ X_t(r) = X_t^r = [1, Y_{1,t}, Y_{2,t}, Z_{1,t} - \tilde{Z}_{1,t}, Z_{2,t} - \tilde{Z}_{2,t}, Y_{1,t} - \tilde{Y}_{1,t}]'. \]

Define the price coefficients as follows: for \( j = 0, 1, 2 \)
\[
\lambda_{jx}^q = \left[ \lambda_{j0}, \lambda_{jy_1}, \lambda_{jy_2}, (\lambda_{jz_1} - \lambda_{jz_2}), (\lambda_{jz_2} - \lambda_{jz_3}) \right]
\]
\[
\lambda_{jx}^p = \left[ \lambda_{j0}, \lambda_{jy_1}, \lambda_{jy_2}, (\lambda_{jz_1} - \lambda_{jz_2}) \right]
\]
\[
\lambda_{jx}^q = \left[ \lambda_{j0}, \lambda_{jy_1}, \lambda_{jy_2}, (\lambda_{jz_1} - \lambda_{jz_2}), (\lambda_{jz_2} - \lambda_{jz_3}) \right]
\]
\[
\lambda_{jx}^r = \left[ \lambda_{j0}, \lambda_{jy_1}, \lambda_{jy_2}, (\lambda_{jz_1} - \lambda_{jz_2}), (\lambda_{jz_2} - \lambda_{jz_3}), (\lambda_{jy_1} - \lambda_{jy_2}) \right].
\]

Then the vector price function in (11) can be rewritten as
\[
t \in M_k(o) : \quad P_{j,t} = \lambda_{jz} Z_t - \lambda_{jx}^o X_t^o, \quad j = 0, 1 \tag{23a}
\]
\[
t \in M_k(p) : \quad P_{j,t} = \lambda_{jz} Z_t - \lambda_{jx}^p X_t^p, \quad j = 0, 1, 2 \tag{23b}
\]
\[
t \in M_k(q) : \quad P_{j,t} = \lambda_{jz} Z_t - \lambda_{jx}^q X_t^q, \quad j = 0, 1 \tag{23c}
\]
\[
t \in M_k(r) : \quad P_{j,t} = \lambda_{jz} Z_t - \lambda_{jx}^r X_t^r, \quad j = 0. \tag{23d}
\]

Given the conjectured price functions, we can calculate the conditional expectations of investor \( b \) regarding the state variables. Observing spot prices and prices of contract #1 is equivalent to observing for \( j = 0, 1 \) \( \tilde{P}_{j,t} = \lambda_{jF} F_t \), where \( \lambda_{jF} = \left[ \lambda_{jz_1}, \lambda_{jz_2}, -\lambda_{jy_1}, -\lambda_{jy_2} \right] \). Hence, \( N_t(o) = \left[ \tilde{P}_{1,t}, \tilde{P}_{0,t}, D_t \right]' \) for \( t \in M_k(o) \) and for \( k = 0, 1, 2, \cdots \)
\[
t \in M_k(o) : \quad \Delta_t = m_t(o) \left[ (Z_{1,t} - \tilde{Z}_{1,t}), (Z_{2,t} - \tilde{Z}_{2,t}) \right]' \tag{24}
\]

The existence of \( m_t(o) \) follows from the linearity of the price signals. When futures contract #2 is introduced into the economy, notice that the state variable \( X_t^p \) is only affected by \( (Z_{1,t} - \tilde{Z}_{1,t}) \) instead of \( \left[ (Z_{1,t} - \tilde{Z}_{1,t}), (Z_{2,t} - \tilde{Z}_{2,t}) \right]' \). This follows from the fact that contract #2 provides an additional linear signal. A similar analysis holds for each trading regime and we can find \( m_t(p), m_t(q), \) and \( m_t(r) \) for the remaining trading regimes.

**Proof of Lemma 1.** It follows from above discussion that the coefficients of the signal process
are:

\[a_N(o) = \text{stack}\{\lambda_{1F}, \lambda_{0F}, [1, 1, 0, 0]\}, \quad b_N(o) = \text{stack}\{0^{(1,6)}, 0^{(1,6)}, 1^{(1,6)}\},\]

\[a_N(p) = \text{stack}\{\lambda_{2F}, \lambda_{1F}, \lambda_{0F}, [1, 1, 0, 0]\}, \quad b_N(p) = \text{stack}\{0^{(1,6)}, 0^{(1,6)}, 0^{(1,6)}, 1^{(1,6)}\},\]

\[a_N(q) = \text{stack}\{\lambda_{2F}, \lambda_{0F}, [1, 1, 0, 0]\}, \quad b_N(q) = \text{stack}\{0^{(1,6)}, 0^{(1,6)}, 1^{(1,6)}\},\]

\[a_N(r) = \text{stack}\{\lambda_{0F}, [1, 1, 0, 0]\}, \quad b_N(r) = \text{stack}\{0^{(1,6)}, 1^{(1,6)}\}.\]

Suppose that the state variables, \(F_t\), follow the Gaussian Markov process specified in (12) and the signal vectors follow the Gaussian Markov process specified in (13), then calculating \(\hat{F}_t\) and \(o_t\) is a Kalman filtering problem and their dynamics are given in (14). Then the Kalman gain is:

\[k_t(\tau) = (\alpha F o_{t-1} o_F + \sigma F F) a_{N,t}(\tau)' [a_{N,t}(\tau) (\alpha_F o_{t-1} o_F + \sigma F F) a_{N,t}(\tau)' + \sigma_{N,t}]^{-1}.\]

**Proof of Lemma 2.** Given the evolution of the uninformed investors' conditional expectations, it is easy to derive that the investors' state variables follow:

\[t = t_{1,k} : \quad X_{i,t}^o = a_{oX}^o X_{i,t-1} + b_{oX}^o e_{i,t}^o \quad (25a)\]

\[t \in \{t_{1,k} + 1, t_{2,k} - 1\} : \quad X_{i,t}^o = a_{oX}^o X_{i,t-1}^o + b_{oX}^o e_{i,t}^o \quad (25b)\]

\[t = t_{2,k} : \quad X_{i,t}^p = a_{oX}^p X_{i,t-1}^p + b_{oX}^p e_{i,t}^p \quad (25c)\]

\[t \in \{t_{2,k} + 1, m_{1,k} - 1\} : \quad X_{i,t}^p = a_{oX}^p X_{i,t-1}^p + b_{oX}^p e_{i,t}^p \quad (25d)\]

\[t = m_{1,k} : \quad X_{i,t}^q = a_{oX}^o X_{i,t-1}^q + b_{oX}^q e_{i,t}^q \quad (25e)\]

\[t \in \{m_{1,k} + 1, m_{2,k} - 1\} : \quad X_{i,t}^r = a_{oX}^r X_{i,t-1}^r + b_{oX}^r e_{i,t}^r \quad (25f)\]

\[t = m_{2,k} : \quad X_{i,t}^r = a_{oX}^r X_{i,t-1} + b_{oX}^r e_{i,t}^r \quad (25g)\]

\[t \in \{m_{2,k} + 1, t_{1,k+1} - 1\} : \quad X_{i,t} = a_{oX}^r X_{i,t-1} + b_{oX}^r e_{i,t}^r \quad (25h)\]

where \(e_{i,t}^n = e_t\) for \(n = o, p, q, r\) and \(e_{i,t}^b = \begin{bmatrix} Z_{1,t} - \hat{Z}_{1,t}, Z_{2,t} - \hat{Z}_{2,t}, e_t' \end{bmatrix}
\]

\(e_{b,t} = \begin{bmatrix} Z_{1,t} - \hat{Z}_{1,t}, Z_{2,t} - \hat{Z}_{2,t}, Y_{t-1} - \hat{Y}_{t-1}, e_t' \end{bmatrix}
\]

For \(i = a, b, a_{oX}^n, b_{oX}^n\) for \(n = o, p, q, r\) and \(a_{oX}^{op}, a_{oX}^{pq}, a_{oX}^{qr}, b_{oX}^{op}, b_{oX}^{pq}, b_{oX}^{qr}\) and \(b_{oX}^{qr}\) are deterministic, time-dependent matrices of appropriate periodicities.

We first derive (25d). The remaining sub-equations of (25) follow straightforwardly. Let \(\Delta_t \equiv F_t - \hat{F}_t\). It is easy to find \(m_t(p)\) such that for \(t \in M_k(p)\), \(\Delta_t = m_t(p) \left(Z_{1,t} - \hat{Z}_{1,t}\right)\). It follows from (14) that

\[t \in \{t_{2,k} + 1, m_{1,k} - 1\} : \quad \hat{F}_t = a_F \hat{F}_{t-1} + b_F e_{b,t}^b\]

\[\text{[See, e.g. Theorem 7.2 in Jazwinski (1970)].}\]
where \( b_{\Delta,t} = \left( k_t^P a_N^P a_F m_{t-1}^P \right) \{ k_t^P (a_N^P b_F + b_N^P) \} \) and \( \epsilon_{t,t} = \left[ (Z_{1,t} - \hat{Z}_{1,t}), \epsilon_t \right]^T. \) (14) also implies that
\[
t \in \{ t_{2,k} + 1, m_{1,k} - 1 \} : \quad \Delta_t = a_{\Delta,t}^P \Delta_{t-1} + b_{\Delta,t}^P \epsilon_t,
\]
where \( a_{\Delta,t}^P = \left( i^{(4)} - k_t^P a_N^P \right) a_F m_{t-1}^P \) and \( b_{\Delta,t}^P = k_t^P (a_N^P b_F + b_N^P). \) Hence, (25d) follows by letting \( a_{aX} = \text{diag} \{ 1, a_{Y1}, a_{Y2}, a_{aX(t+1)} \}, \) \( a_{bX} = a_Y. \) And additionally, let \( b_{aX}^P = \text{stack} \{ b_r, b_{aX(t+1)}^P \}, \) and \( b_{bX}^P = \text{stack} \{ 0^{(1,8)}, b_{bX(t+1)}^P \}. \)

The excess share return of the spot and futures contracts can be derived from the evolution of the investors' state variables and follows the following processes:

\[
t \in \mathcal{M}_k(o) : \quad \Delta Q_{j,t} = a_{iQ,j}^P X_{i,t}^P + b_{iQ,j}^P \epsilon_{i,t+1}, \quad j = 0, 1 \tag{26a}
\]
\[
t \in \mathcal{M}_k(p) : \quad \Delta Q_{j,t} = a_{iQ,j}^P X_{i,t}^j + b_{iQ,j}^P \epsilon_{i,t+1}, \quad j = 0, 1, 2 \tag{26b}
\]
\[
t \in \mathcal{M}_k(q) : \quad \Delta Q_{j,t} = a_{iQ,j}^q X_{i,t}^q + b_{iQ,j}^q \epsilon_{i,t+1}, \quad j = 0, 2 \tag{26c}
\]
\[
t \in \mathcal{M}_k(r) : \quad \Delta Q_{j,t} = a_{iQ,j}^r X_{i,t}^r + b_{iQ,j}^r \epsilon_{i,t+1}, \quad j = 0 \tag{26d}
\]

Given the state variables process, we derive the returns processes, (26b). Let \( \tilde{b}_{jz,t+1} = 1_j b_D + \left( 1_j e^{(2)} + 1_j \lambda_{jx,t+1} \right) b_Z. \) We first derive return process for investor \( a: \) for \( j = 1, 2 \)
\[
t \in \{ t_{2,k}, m_{1,k} - 2 \} : \quad a_{aQ,o} = R a_{0x,t} - \lambda_{0x,t+1} a_{ax} \quad a_{aQ,j} = \lambda_{0x,t} - \lambda_{0x,t+1} a_{ax}
\]
\[
t = m_{1,k} - 1 : \quad a_{aQ,o} = R a_{0x,t} - \lambda_{0x,t+1} a_{aX} \quad a_{aQ,j} = \lambda_{0x,t} - \lambda_{0x,t+1} a_{aX}.
\]

And for \( t \in \{ t_{2,k}, m_{1,k} - 2 \}, \) \( b_{aQ,j} = \tilde{b}_{jz,t+1} - \lambda_{jx,t+1} b_{aX} \) for \( j = 0, 1, 2. \) And for \( t = m_{2,k} - 1, \)
\( b_{aQ,j} = \tilde{b}_{jz,t+1} - \lambda_{jx,t+1} b_{aX}^{(1,2)} \) for \( j = 0, 1, 2. \) Next, investor \( b \)’s returns process is:
\[
t \in \{ t_{1,k}, m_{2,k} - 2 \} : \quad a_{bQ,j} = a_{aQ,j}^{(3,1,3)} \quad \text{stack} \{ 3^{(3)}, 0^{(1,3)} \}
\]
\[
t = m_{1,k} - 1 : \quad a_{bQ,j} = a_{aQ,j}^{(3,1,3)} \quad \text{stack} \{ 3^{(3)}, 0^{(1,3)} \}.
\]

And for \( t \in \{ t_{2,k}, m_{1,k} - 2 \}, \)
\( b_{bQ,j} = \left[ \{ a_{aQ,j} \text{ stack} \{ 0, m_{t_{2,1}, m_{t_{3,1}}, 1} \} \} \right] b_{aQ,j} \) for \( j = 0, 1, 2. \) And for \( t = m_{1,k} - 1, \)
\( b_{bQ,j} = \left[ \{ a_{aQ,j} \text{ stack} \{ 0, m_{t_{2,1}, m_{t_{3,1}}, 1} \} \} \right] b_{aQ,j} \) for \( j = 0, 1, 2. \)

Given the return processes, we can derive the wealth processes of the investors. Let \( a_{iw} = \text{stack} \{ a_{iQ,o}, a_{iQ,q} \} \) and \( b_{iw} = \text{stack} \{ b_{iQ,o}, b_{iQ,q} \}. \) Let \( a_{iw} = \text{stack} \{ a_{iQ,o}, a_{iQ,q}, a_{iQ,2} \} \) and \( b_{iw} = \text{stack} \{ b_{iQ,o}, b_{iQ,q}, b_{iQ,2} \}. \) Also define as follows: \( a_{iw} = \text{stack} \{ a_{iQ,o}, a_{iQ,q}, a_{iQ,2} \}, b_{iw} = \text{stack} \{ b_{iQ,o}, b_{iQ,q}, b_{iQ,2} \}, \)
\( a_{iw} = a_{iQ,o}, b_{iw} = b_{iQ,o}. \)

Given the state variables processes, the return processes and the wealth processes, we now derive the system of nonlinear difference equations which govern \( v_{i,t}, \) for \( n = o, p, q, r. \)
For simplicity, we will derive the recursive relation generally: dropping superscripts denoting trading regimes. The specific system cited in Lemma 2 can be derived from this recursive relation by making the appropriate substitutions. Suppose that at \( t \), investor \( i \)'s state variable process is \( X_{i,t+1} = a_{ix}X_{i,t} + b_{ix}\varepsilon_{i,t+1} \) and value function given by \( J_{i,t}(W_{i,t}; X_{i,t}; t) \). Suppose \( X_{i,t} \) is \( n_i \times 1 \). Then we can rewrite \( q_{t+1} = X_{a,t}'\tilde{a}_qX_{a,t} + \tilde{b}_q'X_{a,t}d_{q,t+1} \), where \( \tilde{a}_q = \text{stack} \left\{ a_q, 0^{(3,n_a-3)}, 0^{(n_a-3,n_a)} \right\}, b_{qq} = c^{(3)}, 0^{(1,n_a-3)} \right\}' \), and \( \tilde{b}_q = b_{qq}b_q \).

Now define the following matrices:

\[
\Xi_{n_i,t+1}^{n} = \left( \frac{\Xi_{n_i,t+1}^{n}}{\Sigma_{ii,t+1}^{n}} \right)^{-1} + \left( \frac{b_{ix}v_{i,t+1}^{n}b_{ix}'}{\Sigma_{ii,t+1}^{n}} \right)
\]

\[
\Xi_{i,t+1}^{op} = \left( \frac{\Xi_{i,t+1}^{op}}{\Sigma_{ii,t+1}^{op}} \right)^{-1} + \left( \frac{b_{ix}v_{i,t+1}^{op}b_{ix}'}{\Sigma_{ii,t+1}^{op}} \right)
\]

\[
\Xi_{i,t+1}^{pq} = \left( \frac{\Xi_{i,t+1}^{pq}}{\Sigma_{ii,t+1}^{pq}} \right)^{-1} + \left( \frac{b_{ix}v_{i,t+1}^{pq}b_{ix}'}{\Sigma_{ii,t+1}^{pq}} \right)
\]

\[
\Xi_{i,t+1}^{qr} = \left( \frac{\Xi_{i,t+1}^{qr}}{\Sigma_{ii,t+1}^{qr}} \right)^{-1} + \left( \frac{b_{ix}v_{i,t+1}^{qr}b_{ix}'}{\Sigma_{ii,t+1}^{qr}} \right)
\]

\[
\Xi_{i,t+1}^{ro} = \left( \frac{\Xi_{i,t+1}^{ro}}{\Sigma_{ii,t+1}^{ro}} \right)^{-1} + \left( \frac{b_{ix}v_{i,t+1}^{ro}b_{ix}'}{\Sigma_{ii,t+1}^{ro}} \right)
\]

Then let

\[
h_{i,t}(\tau) \equiv h_{i,t}^{\tau}
\]

\[
t \in \{t_{1,k}, t_{2,k} - 2\} : h_{i,t}^{\tau} = \Xi_{i,t+1}^{\tau} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{\tau} \left( v_{iab,t+1}^{\tau} + \alpha b_{q}^{\tau} \right) \right]
\]

\[
t = t_{2,k} - 1 : h_{i,t}^{\tau} = \Xi_{i,t+1}^{op} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{op} \left( v_{iab,t+1}^{op} + \alpha b_{q}^{op} \right) \right]
\]

\[
t \in \{t_{2,k}, m_{1,k} - 2\} : h_{i,t}^{p} = \Xi_{i,t+1}^{p} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{p} \left( v_{iab,t+1}^{p} + \alpha b_{q}^{p} \right) \right]
\]

\[
t = m_{1,k} - 1 : h_{i,t}^{p} = \Xi_{i,t+1}^{pq} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{pq} \left( v_{iab,t+1}^{pq} + \alpha b_{q}^{pq} \right) \right]
\]

\[
t \in \{m_{1,k}, m_{2,k} - 2\} : h_{i,t}^{q} = \Xi_{i,t+1}^{q} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{q} \left( v_{iab,t+1}^{q} + \alpha b_{q}^{q} \right) \right]
\]

\[
t = m_{2,k} - 1 : h_{i,t}^{q} = \Xi_{i,t+1}^{ro} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{ro} \left( v_{iab,t+1}^{ro} + \alpha b_{q}^{ro} \right) \right]
\]

\[
t \in \{m_{2,k}, t_{1,k+1} - 2\} : h_{i,t}^{r} = \Xi_{i,t+1}^{r} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{r} \left( v_{iab,t+1}^{r} + \alpha b_{q}^{r} \right) \right]
\]

\[
t = t_{1,k+1} - 1 : h_{i,t}^{r} = \Xi_{i,t+1}^{ro} \left[ a_{iw} - b_{iw} \Xi_{i,t+1}^{ro} \left( v_{iab,t+1}^{ro} + \alpha b_{q}^{ro} \right) \right]
\]

Using the above notation but dropping the superscripts for the trading regime, both the informed and the uninformed investors’ optimization problems can be expressed in the form of
the Bellman’s equation:

\[
0 = \sup_{\{c_i, \theta_i\}} \left\{ -\rho e^{-\gamma c_i} + E_{i,t} [J(W_i,t+1; X_{i,t+1}; t + 1)] - J(W_i,t; X_{i,t}; t) \right\}
\]

s.t. \( W_{i,t+1} = (W_{i,t} - c_{i,t}) R + \theta_{i,t}' (a_{iw} X_{i,t} + b_{iw} \epsilon_{i,t+1}) + 1_i (X_{i,t}' \bar{a}_q X_{i,t} + \bar{b}_q' X_{i,t} b_q \epsilon_t) \).

Consider the following trial solution for the value function:

\[
J_{i,t}(W_{i,t}; X_{i,t}; t) = -\rho \exp \left\{ -\alpha W_{i,t} - \frac{1}{2} (X_{i,t}' v_{i,t} X_{i,t}) \right\},
\]

where \( v_{i,t} \) is a symmetric matrix. Let \( d_i = |\Xi_{i,t+1} \sigma_{ii}|^{-\frac{1}{2}} \) and \( v_{iaa,t+1} = a_{ix}' v_{i,t+1} a_{ix} \). It follows from normality of \( \epsilon_{i,t+1} \) that

\[
E_{i,t} [J_{i,t+1}] = -\rho \rho^{t+1} \exp \left\{ -\alpha R (W_{i,t} - c_{i,t}) - \alpha X_{i,t}' h_{i,t}' \theta_{i,t} + \frac{1}{2} \alpha^2 \theta_{i,t}' \Omega_{i,t+1}^{-1} \theta_{i,t} \right. \]

\[
-\frac{1}{2} X_{i,t}' [v_{iaa,t+1} - (v_{iab,t+1} + \alpha 1_i \bar{b}_q) \Xi_{i,t+1} (v_{iab,t+1} + \alpha 1_i \bar{b}_q)' + 1_i \bar{a}_q] X_{i,t} \left\}.
\]

The first order conditions for the optimal investment-consumption policies are \( \theta_{i,t} = h_{i,t} X_{i,t} \) and

\[
c_{i,t} = \bar{c}_i + \frac{\alpha R}{\gamma + \alpha R} W_{i,t} + \frac{1}{2(\gamma + \alpha R)} X_{i,t}' g_{i,t+1} X_{i,t},
\]

where \( \bar{c}_i = \frac{1}{\gamma + \alpha R} \log \left( \frac{\gamma R d_i}{\alpha R d_i} \right) \) and

\[
g_{i,t+1} = v_{iaa,t+1} - (v_{iab,t+1} + \alpha 1_i \bar{b}_q) \Xi_{i,t+1} (v_{iab,t+1} + \alpha 1_i \bar{b}_q)' + 1_i \bar{a}_q + h_{i,t}' \Omega_{i,t+1} h_{i,t}.
\]

\[\square\]

**Lemma 7** \( v_{i,t}(\tau) \equiv v_{i,t}^\tau \) for \( \tau = o, p, q, r \) in Lemma 2 satisfy the following nonlinear two-point
boundary value problem:

\begin{align*}
t = t_{1,k} : & \quad Rv^{r}_{i,t-1} = a^{r}_{ix} v^{r}_{i,t} a^{r}_{ix} + h^{r}_{i,t-1} \Omega^{r}_{i,t} h^{r}_{i,t-1} + \bar{v}^{r}_{t} \\
& \quad \left( v^{r}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{r}_{i,t} \left( v^{r}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t \in \{ t_{1,k} + 1, t_{2,k} - 1 \} & \quad Rv^{o}_{i,t-1} = \alpha^{op}_{i} v^{p}_{i,t} a^{p}_{ix} + h^{o}_{i,t-1} \Omega^{p}_{i,t} h^{o}_{i,t-1} + \bar{v}^{p}_{t} \\
& \quad \left( v^{p}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{o}_{i,t} \left( v^{p}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t = t_{2,k} : & \quad Rv^{p}_{i,t-1} = \alpha^{op}_{i} v^{p}_{i,t} a^{p}_{ix} + h^{o}_{i,t-1} \Omega^{p}_{i,t} h^{o}_{i,t-1} + \bar{v}^{p}_{t} \\
& \quad \left( v^{p}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{p}_{i,t} \left( v^{p}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t \in \{ t_{2,k} + 1, m_{1,k} - 1 \} & \quad Rv^{p}_{i,t-1} = \alpha^{pq}_{i} v^{q}_{i,t} a^{p}_{ix} + h^{q}_{i,t-1} \Omega^{p}_{i,t} h^{q}_{i,t-1} + \bar{v}^{q}_{t} \\
& \quad \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{p}_{i,t} \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t = m_{1,k} : & \quad Rv^{p}_{i,t-1} = \alpha^{pq}_{i} v^{q}_{i,t} a^{p}_{ix} + h^{p}_{i,t-1} \Omega^{pq}_{i,t} h^{p}_{i,t-1} + \bar{v}^{p}_{t} \\
& \quad \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{p}_{i,t} \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t \in \{ m_{1,k} + 1, m_{2,k} - 1 \} & \quad Rv^{q}_{i,t-1} = \alpha^{q}_{i} v^{q}_{i,t} a^{q}_{ix} + h^{q}_{i,t-1} \Omega^{q}_{i,t} h^{q}_{i,t-1} + \bar{v}^{q}_{t} \\
& \quad \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{p}_{i,t} \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t = m_{2,k} - 1 : & \quad Rv^{q}_{i,t-1} = \alpha^{qr}_{i} v^{q}_{i,t} a^{qr}_{ix} + h^{q}_{i,t-1} \Omega^{qr}_{i,t} h^{q}_{i,t-1} + \bar{v}^{q}_{t} \\
& \quad \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{q}_{i,t} \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

\begin{align*}
t \in \{ m_{2,k} + 1, t_{1,k+1} - 1 \} & \quad Rv^{q}_{i,t-1} = \alpha^{qr}_{i} v^{q}_{i,t} a^{qr}_{ix} + h^{q}_{i,t-1} \Omega^{qr}_{i,t} h^{q}_{i,t-1} + \bar{v}^{q}_{t} \\
& \quad \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \Xi^{q}_{i,t} \left( v^{q}_{lab,t} + \alpha b^{q}_{t} \right) \end{align*}

and for \( k = 0, 1, 2, \ldots \)

\[ v^{q}_{t}(t_{1,k}) = v^{q}_{t}(t_{1,k+1}) \]  \( (28) \)

Here, \( h^{r}_{i,t} \) for \( r = o, p, q, r, \Xi^{n}_{i,t} \) and \( \Omega^{n}_{i,t} \) for \( n = qo, op, p, pq, q, qr, r \) are given in proof in of Lemma 2. And \( \bar{b}^{q}_{t} \) for \( n = o, p, q, r \) and \( \bar{v}^{q}_{t} \) for \( n = o, p, q, op, pq, qr, r, ro \) are also given in proof of Lemma 2.

**Proof of Lemma 7.** This gives the following recursion for \( v_{i,t} \) given \( v_{i,t+1} : \frac{1}{2} g_{i,t+1} - v_{i,t} + [\gamma c_{i} + \log \left( \frac{R}{\Lambda} \right) ] 1^{(n_{i},n_{i})}_{i} = 0 \). Let \( \bar{v}_{i} = R [\gamma c_{i} + \log \left( \frac{R}{\Lambda} \right) ] 1^{(n_{i},n_{i})}_{i} + 1 \bar{b}_{q} \), then it follows that

\begin{align*}
Rv_{i,t-1} = v_{iab,t} - (v_{lab,t} + \alpha 1_{1} b_{q}) \Xi_{i,t} (v_{lab,t} + \alpha 1_{1} b_{q})' + h_{i,t-1}' \Omega_{i,t} h_{i,t-1} + \bar{v}_{i}. \quad (29)
\end{align*}
System (27) follows from this. \(\square\)

**Lemma 8** Given terminal value \(v_{i,t_1,k+1} > 0\), (27) has a solution \(v_i(t; v_{i,t_1,k+1})\) which is symmetric, positive semi-definite and \(|v_i(t; v_{i,t_1,k+1})| \leq \beta_0|v_{i,t_1,k+1}| + \beta_1\) where \(0 < \beta_0 < 1\) and \(\beta_1 > 0\).

**Proof of Lemma 8.** We prove existence of the terminal value problem for (27) by bounding (lower and upper) its solution by the solutions to two particular matrix Riccati difference equations. First we specify our assumption about \(a_q\). It is assumed that

\[
a_q = \text{stack}\left\{ \left[0, (\alpha \sigma_q)^2, (\alpha \sigma_q)^2\right], \left[(\alpha \sigma_q)^2, 2(\alpha \sigma_q)^2, 0\right], \left[(\alpha \sigma_q)^2, 0, 2(\alpha \sigma_q)^2\right] \right\}.
\]

This assumption is sufficient to satisfy assumption in Lemma 3 for positive semi-definite solutions to the Riccati equation. First the lower bound. Consider (29) on \(t \in \{0, T\}\). Suppose \(v_{i,t} \geq v_{i,t}^*\), then it follows that \(v_{i,t} \geq v_{i,t}^*\) where \(v_{i,t}^*\) satisfies

\[
Rv_{i,t-1}^* = v_{i\alpha,t}^* - \left(v_{iab,t}^* + \alpha \tilde{b}_q\right) \Xi_{i,t} \left(v_{iab,t}^* + \alpha \tilde{b}_q\right)' + \bar{v}_i^*.
\]

This follows from fact that \(h_{i,t-1}^* \Omega_i h_{i,t-1}^* \geq 0\). Next, an upper bound. Suppose \(v_{i,t} \leq v_{i,t}^*\). Suppose \(v_{i,t}^*\) satisfies the following:

\[
Rv_{i,t-1}^* = a_{i,x}^* v_{i,t}^* a_{i,x}^* - \left(a_{i,x}^* v_{i,t}^* b_{i,x} + \tilde{b}_q\right) \Xi_{i,t} \left(a_{i,x}^* v_{i,t}^* b_{i,x} + \tilde{b}_q\right)' + \bar{v}_i^*,
\]

(30)

where \(a_{i,x}^* = a_{i,x} + b_{i,x}'k^*\) and \(k^* = a_{i,w}^* \Xi_i \Xi_i^*\) if \(v_{i,t} > v_{i,t+1}\), and \(k^* = 0\) otherwise. Substituting \(h_{i,t-1}^*\) into (29) and multiplying out the matrices, it can be shown that \(v_{i,t} \leq v_{i,t}^*\). Note that \(\Xi_{i,t} - (b_{i,w} \Xi_{i,t})'(\Xi_{i,t}) (b_{i,w} \Xi_{i,t}) > 0\), so that that (29) is negative in the quadratic terms involving \(v_{i,t}\). So, this keeps the solution bounded and hence is bounded by a solution to a particular matrix Riccati equation. Now, apply Lemma 3 to (30) by setting \(k = k^*\) and it follows that \(v_{i,t}\) is bounded by the solution to a matrix linear equation whose linear term is \(a_{i,x} \Xi_{i,t}^* a_{i,x}'\) and the constant term is independent of \(v_{i,t}\). Hence, since \(|a_{i,x}| < 1\), it follows that we can find \(\beta_0\) and \(\beta_1\) such that the result holds. \(\square\)

**Lemma 9** (27) and (28) has a symmetric, positive semi-definite periodic solution.

**Proof of Lemma 9.** Given the bound established in Lemma 8, let \(L\) be the space of symmetric, positive semi-definite matrices such that their norms are less than \(\psi\), where \(\psi = \beta_1/(1 - \beta_0)\). Then define the following mapping: \(M : L \rightarrow L\) where \(M\) is given by Equation (27) as a mapping of \(v_{i,t_1,k+1}\) to \(v_{i,t_1,k}\). \(L\) is a nonempty, closed, bounded, convex subset of a finite dimensional normed vector space. \(M\) is clearly symmetric, positive semi-definite by Lemma 3. It is easy to calculate that \(|M(o_{i_1,k+1})| \leq \beta_1/(1 - \beta_0)\). By Lemma 4, the result follows. \(\square\)
Proof of Theorem 1. In the case of \( \omega = 1 \), we can find closed form solutions for the equilibrium price functions and the value function of the informed investor (see Proposition 1).

Given this equilibrium price function, we now show that there exists a symmetric, covariance matrix, \( \sigma_{t+1} \), such that \( \sigma_t \) follows dynamics specified in (14) and \( \sigma_{t+1} = \sigma_t \) (i.e. we seek a periodic solution to the filtering problem). Our time-varying dynamic system satisfies a “uniform detectability” condition (see Anderson and Moore (1981)) at \( \omega = 1 \). Given that \( a_r \) and \( b_r \) are uniformly bounded in time, this condition essentially says that the price function communicates information about each of the state variables of interest (i.e. \( a_{N,t} \) be nonsingular and uninformly bounded in time (see Nahi (p. 103-105))— it is easy to verify this condition for a given equilibrium price functional which we can get in closed form for the case of \( \omega = 1 \)). In fact, our price function in equilibrium for \( \omega = 1 \) will satisfy “uniform observability” and “uniform controllability” conditions (see Jazwinski (1970)). From Lemma 5.1 of Anderson and Moore (1981) and Jazwinski (p. 231-234), we conclude that \( |\sigma_t| \leq \beta_0 |\sigma_{t+1}| + \beta_1 \), where \( 0 < \beta_0 < 1 \) and \( \beta_1 > 0 \). The result follows for the filtering problem.

The existence of a periodic solution to the value function of the uniformed investors follows from Lemma 9. We next show that there exists a solution close for \( \omega \) close to \( \omega = 1 \). The market clearing conditions which determine the price coefficients \( \lambda_{jX,t}^n \) for \( j = 0,1,2 \) and \( n = o,p,q,r \) are such that the following nonlinear equations hold: for \( k = 0,1,2, \cdots \)

\[
t \in \mathcal{M}_k(\tau) : \quad \omega h_{a,t}(\tau) + (1 - \omega) h_{b,t}(\tau) \delta_t(\tau) = \tilde{\theta}(\tau). \tag{31}\]

It is easy to find \( \delta_t(\tau) \) given \( m_t(\tau) \) for \( n = o,p,q,r \). We first note that \( \lambda_{jX,t}^n \) and \( \lambda_{jX,t+1}^n \) are related implicitly for \( n = o,p,q,r \) and \( j = 0,1,2 \). For example, taken \( n = p \) and let \( \lambda_{jX,t}^p = \text{stack}\{\lambda_{0X,t}^p, \lambda_{1X,t}^p, \lambda_{2X,t}^p\} \). (31) defines the following implicit system: \( 0 = F_p(\lambda_{jX,t}^p, \lambda_{jX,t+1}^p; \vartheta, \omega) \).

By the implicit function theorem [see e.g. Protter and Morrey (1991)], \( F^o \) defines an implicit function: \( \lambda_{X,t}^p = f(\lambda_{X,t+1}^p; \vartheta, \omega) \) if \( \nabla \lambda_{X,t}^p \) is nonsingular. Let

\[
\vartheta_0 = \{0.05, 0.10, 0.99, 0.95, 0.9, 0.875, 0.8, 0.75, 0.6, 0.5, 0.5, 0.25, \cdots \}.
\]

For \( \vartheta = \vartheta_0 \) and \( \omega = 1 \), \( \det(\nabla \lambda_{X,t}^p) \neq 0 \). So, by Lemma 6, \( \lambda_{X,t}^p = f(\lambda_{X,t+1}^p; \vartheta, \omega) \) exists generically. Now define the following

\[
u^o = \text{stack}\{\lambda_{0X,t}^o, \lambda_{1X,t}^o, [o], [v_{a,o}], [v_{b,o}]\}
\]

\[
u^p = \text{stack}\{\lambda_{0X,t}^p, \lambda_{1X,t}^p, \lambda_{2X,t}^p, [o], [v_{a,p}], [v_{b,p}]\}
\]

\[
u^q = \text{stack}\{\lambda_{0X,t}^q, \lambda_{2X,t}^q, [o], [v_{a,q}], [v_{b,q}]\}
\]

\[
u^r = \text{stack}\{\lambda_{0X,t}^r, [o], [v_{a,r}], [v_{b,r}]\}.
\]

It is now straightforward to relate system (22) to the equilibrium system given by (14), (27)
and (31) subject to periodicity conditions given by (7), (15), and (28). Let $\omega_0 = 1$. Since we have existence of $u(t; \vartheta, \omega_0)$, it remains to verify that $u(t; \vartheta, \omega_0)$ is an isolated solution. This is equivalent to showing [see, e.g., Agarwal (1992)] that $m(\vartheta, \omega_0) = \nabla u_0 g(u_0) + \nabla u_2 g(u_T) (\nabla u_T)^T$ is nonsingular where $T$ is the number of periods over which the system is defined. Clearly, $m(\vartheta, \omega_0)$ is analytic. It is easy to show that $\det(m(\vartheta_0, \omega_0)) \neq 0$. By Lemma 6, $m(\vartheta, \omega_0)$ is generically nondegenerate. By Lemma 5, Theorem 1 follows. □

6.3. Proofs of Results In Section 4

Proof of Proposition 1. Let $\eta_0 = \frac{\lambda z_2(1+\lambda z_1)}{\lambda z_1(1+\lambda z_2)}$ and proof follows from Proposition 2. The holdings of investor a when both futures contracts are traded are for $t \in M_k^T$:

$$\vartheta_{a,t} = \zeta - \frac{\sigma_{Dq}(1-\omega)}{\sigma_D^2}$$ (32)

$$\psi_{a,t} = (1+\lambda z_2) \left( \frac{a_{z_1}^{(M_2-t)} - \eta_0 a_{z_2}^{(M_2-t)}}{\lambda z_2 \left( \frac{a_{z_1}^{(M_2-t)} a_{z_1}^{(M_1-t)}}{a_{z_2}^{(M_2-t)} a_{z_2}^{(M_1-t)}} \right)} \right) \sigma_{Dq}(1-\omega)$$ (33)

$$\varphi_{a,t} = \frac{-(1+\lambda z_2) \left( a_{z_1}^{(M_1-t)} - \eta_0 a_{z_2}^{(M_1-t)} \right)}{\lambda z_2 \left( \frac{a_{z_1}^{(M_2-t)} a_{z_1}^{(M_1-t)}}{a_{z_2}^{(M_2-t)} a_{z_2}^{(M_1-t)}} \right)} \sigma_{Dq}(1-\omega)$$ (34)

$\psi_{a,t} < 0$ (her position in th distant), $\varphi_{a,t} > 0$ (her position in the nearby). It is not hard to show that $|\varphi_{a,t}| > |\psi_{a,t}|$. □

Proof of Proposition 2. Given results in Proposition 1, take the derivative with respect to $a_{z_2}$ of the ratio of the distant to total open interest. □

6.4. Numerical Procedure

We briefly discuss the numerical procedure used to solve for the periodic equilibrium. We use the Newton-Kantorovich method to solve this problem numerically [see, e.g., Agarwal (1992)]. This recursive method linearizes the system and the boundary conditions around a conjectured solution to the nonlinear problem at a discrete number of points in the interval $[t_{1,k}, t_{1,k+1}]$. Since the system is linearized, it is easy to calculate an updated solution that satisfies the linearized system and boundary conditions from the conjectured solution. The updated solution is then used as the conjectured solution to start the next recursion. It can be shown that the limit of this recursion converges to the solution of the nonlinear problem given that the initial conjectured solution is not too far away from the true solution.

This method requires a sufficiently accurate initial guess of the true solution. We obtain such a guess by starting the recursion at $\omega = 1$ for any given set of parameters since a solution
exists at $\omega = 1$. In order to calculate a solution at $\omega_0 < 1$, we begin by using the solution at $\omega = 1$ as the initial guess to find a solution for an $\omega$ close to 1 and repeat the same procedure to move toward $\omega_0$.

Since we have no knowledge on the uniqueness of the solution, the above procedure also guarantees that we stay on the same branch of solutions. In particular, the solution gives the expected results when we take the limit $\omega \to 1$. We have also checked the solution by taking other limits in the parameter space such as $\sigma_{z_1} \to 0$ and obtained the expected results.
7. References


