ALGEBRAIC CHARACTERIZATIONS OF ALMOST INVARIANCE

by

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ABSTRACT

This paper stresses the algebraic (as opposed to analytic) nature of the concept of almost invariance, which has been introduced by J.C. Willems as a means of studying asymptotic aspects of linear systems. Based on a discrete-time interpretation, it is shown that the basic properties and a number of characterizations of the four fundamental classes of subspaces can be derived in a relatively simple and purely algebraic fashion. Among other things we extend the results of Hautus on the frequency-domain interpretation of ordinary (A,B)-invariant subspaces, we re-derive the pencil characterization of Jaffe and Karcanias, and we obtain a new "hybrid" type of characterization. The last result also leads to a rank test for almost invariance.

1. Introduction

The concepts of almost (A,B)-invariant subspaces and almost controllability subspaces were introduced and studied by J.C. Willems in a series of papers (Willems 1980, 1981, 1982). The concepts are intended to serve as theoretical tools for the study of the many problems in linear system theory where asymptotic properties are involved (high gain feedback, singular optimal control, 'almost' solvability of design problems, etc.). This goal is reflected in the original definition given by Willems (1980), which refers to a standard continuous-time finite-dimensional linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t) \in X, \ u(t) \in U. \tag{1.1}$$

Willems defines a linear subspace V_a of the state space X to be <u>almost (A,B)-invariant</u> if for every $x_0 \in V_a$ and for every $\varepsilon > 0$ there exists a trajectory $x_\varepsilon : \mathbb{R} \to X$ which satisfies (1.1) for some $u(\cdot)$, and for which we have

$$x_\varepsilon(0) = x_0 \tag{1.2}$$

$$\sup_{t \in \mathbb{R}} \ \inf_{x \in V_a} \ ||x_\varepsilon(t)-x|| \leq \varepsilon \ . \tag{1.3}$$

Although this definition has an analytic flavor, due to the appearance of the small number ε, one can see from the fact that the condition (1.3) must hold for all positive ε that the concept itself is algebraic (i.e., depends only on the given mappings A and B and on the subspace V_a). The idea of this definition is to catch the impulsive trajectories by approximating them with smooth trajectories. Another approach, explored in Willems (1981), consists of using a distributional set-up in order to describe

impulsive trajectories directly. Of course, this still brings in a fair
amount of analysis.

In the present paper, we propose a completely algebraic treatment
of the concept of almost invariance. The main motivation for taking this
approach is its simplicity. Of course, an algebraic treatment of almost
invariance could be based purely on the algorithms given by Willems (1980),
but in this way one would lose the intuitive feel that is associated with
thinking in terms of trajectories. Our solution to the problem will be to
use the discrete-time context. To avoid any misunderstanding, let us
emphasize at this point that the discrete-time, algebraic treatment of
almost invariance proposed here prejudices in no way the use of this con-
cept in a continuous-time, analytic context. The point is that even though
it is likely that almost invariant subspaces will be employed in situations
where limits are being taken etcetera, the concept of almost invariance
itself is algebraic, and we are free to use any framework we prefer (for
simplicity, intuitive guidance, or other reasons) to investigate the
algebraic properties associated with it.

We shall present the discrete-time set-up and the basic definitions
in the next section. In section 3, it will be shown that the fundamental
properties of and relations among the various types of subspaces can be
derived easily from this set-up. Here, it will also be proved that our
definitions coincide with the ones given by Willems.

Next, we shall show that the frequency-domain characterization of
(A,B)-invariant subspaces given by Hautus (1980) can be extended in a
natural way to cover also almost (A,B)-invariant subspaces and almost
controllability subspaces. Then, in Section 5, we shall give equivalent

characterizations in terms of a certain pair of rational matrix equations.
Section 6 is devoted to the restriction pencil (Karcanias 1979) associated
with the state-input pair (A,B) and a given subspace K. It will be shown
that the characterizations in terms of the invariants of this pencil, as
given by Jaffe and Karcanias (1980), can be derived from the present
framework in a straightforward manner. A new type of characterization
is presented in section 7. This characterization is stated in terms of
subspaces but it also involves a complex parameter, and therefore we
have termed it a "hybrid" characterization. As a corollary, we obtain
a rank test for almost invariance. Conclusions follow in Section 8.

This paper will not deal with applications of almost invariance.
For this, we refer to Willems (1980, 1981, 1982), Jaffe and Karcanias
(1981), Schumacher (1982), and further references given in these papers.
The standard reference on the geometric approach to linear systems is
Wonham (1979), where one can find definitions, properties and applications
of (A,B)-invariant and controllability subspaces.

The following conventions will be used. Vector spaces will be
denoted by script capitals, and italic capitals are used for linear trans-
formations. All spaces and transformations will be real, but where needed
we shall use the obvious complexifications, without change of notation.
Also, the same symbol will be used for a linear transformation and for
its matrix with respect to a specified basis. If a basis is specified,
then a basic matrix K for a subspace K is a matrix whose column vectors
span K. The image and the kernel of a linear mapping T will be denoted
by im T and ker T, respectively. $\mathbb{R}$ is the real number field, $\mathbb{C}$ is the
complex number field, and $\mathbb{Z}$ is the ring of integers.

2. Definitions

Throughout the paper, we shall consider the following linear, finite-dimensional, time-invariant, discrete-time system (with $x_k \in X$, $u_k \in U$, $A: X \to X$, $B: U \to X$):

$$x_{k+1} = A x_k + B u_k + \delta_{0k} x_{in} \qquad (k \in \mathbb{Z}) \qquad (2.1)$$

Here, the symbol δ_{0k} is defined by

$$\delta_{0k} = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k \neq 0 . \end{cases} \qquad (2.2)$$

The vector $x_{in} \in X$ is called the initial value; note, however, that the time axis is $\mathbb{Z}$. As control sequences we shall admit any mapping $\underline{u}: \mathbb{Z} \to U$, written, in obvious notation, as

$$\underline{u} = (\ldots, u_{-1}, u_0, u_1, \ldots), \qquad (2.3)$$

for which there exists an integer r such that

$$u_k = 0 \qquad (k \leq r). \qquad (2.4)$$

Given a control sequence $\underline{u}$ and an initial value x_{in}, there exists a unique sequence $\underline{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ for which there exists an integer r such that

$$x_k = 0 \qquad (k \leq r) \qquad (2.5)$$

and which satisfies (2.1). A trajectory will be any sequence that arises in this way.

Of course, instead of control sequences and trajectories we might just as well speak of the formal power series defined by

$$\underline{x}(z) = \sum_{k \in \mathbb{Z}} x_k z^{-k} \quad , \qquad \underline{u}(z) = \sum_{k \in \mathbb{Z}} u_k z^{-k} \quad . \tag{2.6}$$

With the usual conventions, the solution of (2.1) can then be written down as

$$\underline{x}(z) = (zI-A)^{-1} (B\underline{u}(z) + x_{in}) \quad . \tag{2.7}$$

Although the difference is only terminological, we shall still prefer to use the language of trajectories since, in the author's opinion, this leads to a better intuitive grasp.

A discrete-time set-up has also been discussed briefly in Willems (1982), who uses the idea of "reverse time". This is different from what we do here, be it that certain relations can be established.

Now, let K be a given subspace. A trajectory $\underline{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ will be called a __trajectory in K__ if $x_k \in K$ for all k. We make the following definition.

__Definition 2.1.__ The set of all $x \in K$ that can serve as initial value of a trajectory in X is denoted by $V_a^*(K)$.

Note that the set of all trajectories is linear, and so is the set of all trajectories in K. From this, it is immediate that $V_a^*(K)$ is a linear set. We also define:

__Definition 2.2:__ A subspace K is __almost (A,B)-invariant__ if $V_a^*(K) = K$. Below, we shall show that this definition is equivalent to the one given by Willems (1980). Closely related to Def. 2.1 is the following:

__Definition 2.3.__ The set of all $x \in X$ that can serve as initial value of a trajectory in K is denoted by $V_b^*(K)$.

A trajectory $\underline{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ will be called a <u>polynomial</u> <u>trajectory</u> if $x_k = 0$ for all $k \geq 1$. Also, we shall say that $\underline{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)$ is a <u>regular control sequence</u> if $u_k = 0$ for all $k < 0$. We now define:

<u>Definition 2.4.</u> The set of all $x \in K$ that can serve as initial value of a polynomial trajectory in K is denoted by $R_a^*(K)$.

<u>Definition 2.5.</u> A subspace K is an <u>almost controllability subspace</u> if $R_a^*(K) = K$.

<u>Definition 2.6.</u> The set of all $x \in X$ that can serve as initial value of a polynomial trajectory in K is denoted by $R_b^*(K)$.

<u>Definition 2.7.</u> The set of all $x \in K$ for which there exists a regular control sequence, such that the trajectory resulting from this sequence and the initial value x is in K, is denoted by $V^*(K)$.

<u>Definition 2.8.</u> A subspace K is <u>(A,B)-invariant</u> if $V^*(K) = K$.

<u>Definition 2.9.</u> $R^*(K) = V^*(K) \cap R_a^*(K)$.

<u>Definition 2.10.</u> A subspace K is a <u>controllability subspace</u> if $R^*(K) = K$.

Again, it is immediately verified that all sets introduced here are linear sets. Note that it would not make a difference if we would replace "$x \in K$" by "$x \in X$" in Def. 2.7, because $x_{in} = x_1 \in K$ in this case. We shall now proceed to derive the elementary properties of the types of subspaces introduced above, and to justify our notation and terminology by showing that our definitions are equivalent to those of Wonham (1979) and Willems (1980).

3. Properties

Let K be a given subspace. The following observation is quite useful.

Lemma 3.1. Let $\underline{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)$ be a control sequence, and let $\underline{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ be the trajectory corresponding to $\underline{u}$ and the initial value x_{in}. Suppose that $\underline{x}$ is a trajectory in K. Then $x_k \in \mathcal{R}_a^*(K)$ for all $k \leq 0$, and $x_k \in \mathcal{V}^*(K)$ for all $k \geq 1$. If we also suppose that $x_{in} \in K$, then we have, moreover, $Ax_0 + Bu_0 \in \mathcal{R}_a^*(K)$.

Proof. Let k be a fixed nonpositive integer. We have, of course,

$$0 = Ax_{k-1} + Bu_{k-1} - x_k. \tag{3.1}$$

Define a new control sequence $\underline{\tilde{u}}$ by $\tilde{u}_j = \tilde{u}_{j+k-1}$ $(j \leq 0)$ and $\tilde{u}_j = 0$ $(j > 0)$. Also, define $\tilde{x}_{in} = -x_k$. Then it is clear that the trajectory corresponding to $\underline{\tilde{u}}$ and $\tilde{x}_{in}$ is a polynomial trajectory in K. It follows that $x_{in} \in \mathcal{R}_a^*(K)$, and hence also $x_k \in \mathcal{R}_a^*(K)$.

Next, take $k \geq 1$. Define a new control sequence $\underline{\hat{u}}$ by $\hat{u}_j = u_{j+k-1}$ $(j \geq 1)$ and $\hat{u}_j = 0$ $(j \leq 0)$. Also, set $\hat{x}_{in} = x_k$. The trajectory corresponding to $\underline{\hat{u}}$ and $\hat{x}_{in}$ is $\underline{\hat{x}}$, with $\hat{x}_j = 0$ $(j \leq 0)$ and $\hat{x}_j = x_{j+k-1}$ $(j \geq 1)$. Clearly, this is a trajectory in K. Because $\underline{u}$ is regular, we get $x_k = \hat{x}_{in} \in \mathcal{V}^*(K)$.

Finally, suppose now that also $x_{in} \in K$. Then we have $Ax_0 + Bu_0 \in K$ too, because $Ax_0 + Bu_0 = x_1 - x_{in}$. Define a new control sequence $\underline{u}'$ by $u'_j = u_j$ $(j \leq 0)$ and $u'_j = 0$ $(j \geq 1)$, and set $x'_{in} = -(Ax_0 + Bu_0)$. The resulting trajectory is a polynomial trajectory in K, so it follows that $Ax_0 + Bu_0 \in \mathcal{R}_a^*(K)$.

As an immediate consequence of the lemma, we have:

Corollary 3.2: $\mathcal{V}_a^*(K) = \mathcal{R}_a^*(K) + \mathcal{V}^*(K)$.

Proof. It is obvious from the definitions that $R_a^*(K) \subset V_a^*(K)$ and

$V^*(K) \subset V_a^*(K)$; hence, also $R_a^*(K) + V^*(K)$ must be contained in $V_a^*(K)$. On

the other hand, take $x \in V_a^*(K)$. Then $x \in K$, and there exists a control

sequence $\underline{u}$ such that the trajectory corresponding to $\underline{u}$ and the initial

value x is in K. In particular,

$$x_1 = Ax_0 + Bu_0 + x \tag{3.2}$$

where, according to the lemma, $x_1 \in V^*(K)$ and $Ax_0 + Bu_0 \in R_a^*(K)$. Clearly,

$x \in V^*(K) + R_a^*(K)$, which completes the proof.

It is also clear from the lemma that every trajectory in K is in

fact a trajectory in $V_a^*(K)$. This leads at once to the conclusion that

every initial value for a trajectory in K is also an initial value for

a trajectory in $V_a^*(K)$, or:

Corollary 3.3. $V_a^*(V_a^*(K)) = V_a^*(K)$.

In other words, $V_a^*(K)$ is almost (A,B)-invariant. In fact:

Corollary 3.4. $V_a^*(K)$ is the largest almost (A,B)-invariant subspace

contained in K.

Proof. Let L be an almost (A,B)-invariant subspace, and suppose that

$K \supset L$. Then $V_a^*(K) \supset V_a^*(L) = L$.

In the same way, one proves that $R_a^*(K)$ $(V^*(K))$ is the largest

almost controllability ((A,B)-invariant) subspace in K. It then follows

easily that $R^*(K)$ is the largest controllability subspace in K.

Each of the four classes of subspaces is closed under taking sums. This is immediate from the definitions, using the linearity of the set of trajectories, or one can use the following type of reasoning. (Exactly the same argument holds for the other three classes.)

Proposition 3.5. If K_1 and K_2 are both almost (A,B)-invariant, then so is $K_1 + K_2$.

Proof. We have both $V_a^*(K_1 + K_2) \supset V_a^*(K_1) = K_1$ and $V_a^*(K_1 + K_2) \supset V_a^*(K_2) = K_2$. Hence, $V_a^*(K_1 + K_2) = K_1 + K_2$.

The relation between $R_a^*(K)$ and $R_b^*(K)$ is as follows:

Corollary 3.6. $R_b^*(K) = A R_a^*(K) + \text{im } B$.

Proof. Take $x \in R_b^*(K)$; then x acts as initial value for some polynomial trajectory in K, say $\underline{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$. We have

$$0 = x_1 = Ax_0 + Bu_0 + x \tag{3.3}$$

where $x_0 \in R_a^*(K)$, according to Lemma 3.1. We see that $x \in A R_a^*(K) + \text{im } B$.

On the other hand, let $x = A\tilde{x} + B\tilde{u}$ where $\tilde{x} \in R_a^*(K)$. Say that $\tilde{x}$ is the initial value for a polynomial trajectory $\underline{x}$ in K produced by a control sequence $\underline{u}$. Define a new control sequence $\underline{\tilde{u}}$ by $\tilde{u}_j = u_{j+1} (j<0)$, $\tilde{u}_0 = -\tilde{u}$, and $\tilde{u}_j = 0 (j>0)$. Set $\tilde{x}_{in} = x$. Then the trajectory $\underline{\tilde{x}}$ produced by $\underline{\tilde{u}}$ and $\tilde{x}_{in}$ is a polynomial trajectory in K (in particular, we have $\tilde{x}_0 = -\tilde{x} \in K$), which proves that $x = x_{in} \in R_b^*(K)$.

Proposition 3.7. $R_a^*(K) = R_b^*(K) \cap K$.

Proof. This is obvious from the definitions.

Also, the relation $V_a^*(K) = V_b^*(K) \cap K$ is obvious. We can get $V_b^*(K)$ from $V_a^*(K)$ by the following rule:

Corollary 3.8. $V_b^*(K) = V^*(K) + R_b^*(K)$.

Proof. Take $x \in V_b^*(K)$; then x is initial value for a trajectory in K, say $\underline{x}$. In particular, we have

$$x_1 = Ax_0 + Bu_0 + x \tag{3.4}$$

where $x_1 \in V^*(K)$ and $x_0 \in R_a^*(K)$, so that $Ax_0 + Bu_0 \in R_b^*(K)$. It follows that $V_b^*(K) \subset V^*(K) + R_b^*(K)$. The reverse inclusion is immediate.

Next, let us discuss how to compute the six subspaces associated with a given subspace K. From Def. 2.9, Cor. 3.2, Prop. 3.7 and Cor. 3.8 it is clear that we can compute all these subspaces if we can compute $V^*(K)$ and $R_b^*(K)$. To do this, we introduce the following subspaces, for each k>1.

Definition 3.1. $V^k(K)$ = the set of all $x \in K$ for which there exists a regular control sequence such that the trajectory $\underline{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ resulting from this control sequence and the initial value x satisfies $x_j \in K$ for all $j \le k$.

Definition 3.2. $R_b^k(K)$ = the set of all $x \in X$ that can serve as initial value of a polynomial trajectory $\underline{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ in K that satisfies $x_j = 0$ for all $j \le -k+1$.

It is easily verified that $V^k(K)$ and $R_b^k(K)$ can be computed recursively from the following equations:

$$V^1(K) = K; \quad V^{k+1}(K) = K \cap A^{-1}(V^k(K) + \text{im } B) \tag{3.5}$$

$$R_b^1(K) = \text{im } B; \quad R_b^{k+1}(K) = A(R_b^k(K) \cap K) + \text{im } B. \tag{3.6}$$

A way to describe the results of such iterations is given by the following lemma.

Lemma 3.9. Let X be a finite-dimensional linear space, and let Φ be an order-preserving mapping from the set of subspaces of X into itself (i.e., $\Phi(L_1) \subset \Phi(L_2)$ if $L_1 \subset L_2$). Consider the iterations

$$L_0 = \{0\} ; \quad L_{k+1} = \Phi(L_k) \quad (k \geq 0) \tag{3.7}$$

$$L^0 = X; \quad L^{k+1} = \Phi(L^k) \quad (k \geq 0) \tag{3.8}$$

The sequence $\{L_k\}$ is non-decreasing and converges after a finite number of steps; the limit subspace L_∞ can be characterized as the unique smallest element of the set of subspaces $\{L | \Phi(L) \subset L\}$. The sequence $\{L^k\}$ is non-increasing and converges after a finite number of steps; the limit subspace L^∞ can be characterized as the unique largest element of the set of subspaces $\{L | \Phi(L) \supset L\}$.

Proof. Obviously, we have $L_0 \subset L_1$. The fact that $L_k \subset L_{k+1}$ for all $k \geq 0$ then follows by iterating Φ on both sides of this inclusion. Because X is finite-dimensional, convergence must take place after a finite number of steps, and the limit subspace L_∞ satisfies $\Phi(L_\infty) = L_\infty$. Let L be a subspace such that $L \supset \Phi(L)$. By induction, we shall show that $L \supset L_k$

for all k. The inclusion $L \supset L_0$ is immediate. Suppose that $L \supset L_k$; then also $L \supset \Phi(L) \supset \Phi(L_k) = L_{k+1}$. It follows that $L \supset L_\infty$. The proof for the other sequence is entirely analogous (in fact, the statements are dual).

If we set $V^0(K) = X$ and $R_b^0(K) = \{0\}$, then the lemma applies to the sequences defined by (3.5) and (3.6). (Note that the mappings Φ_1 and Φ_2, defined by

$$\Phi_1(L) = K \cap A^{-1}(L + \text{im } B) \tag{3.9}$$

$$\Phi_2(L) = A(L \cap K) + \text{im } B \tag{3.10}$$

are both order-preserving.) Denote the limits of these sequences by $V^\infty(K)$ and $R_b^\infty(K)$, respectively. Then the lemma shows:

Corollary 3.10: $V^\infty(K)$ is the largest element of set of subspaces L that satisfy

$$L \subset K \text{ and } AL \subset L + \text{im } B. \tag{3.11}$$

Moreover, $R_b^\infty(K)$ is the smallest element of the set of subspaces L that satisfy

$$L \supset \text{im } B \text{ and } A(L \cap K) \subset L. \tag{3.12}$$

Finally, we make the following identifications.

Proposition 3.11. $V^\infty(K) = V*(K)$ and $R_b^\infty(K) = R_b^*(K)$.

Proof. Since $V*(K) \subset V^k(K)$ for all k, it is clear that $V*(K) \subset V^\infty(K)$. To prove the reverse inclusion, take $x \in V^\infty(K)$. According to Cor. 3.10, there exists u_1 such that $x_2: = Ax_1 + Bu_1 \in V^\infty(K)$. Again applying Cor. 3.10, we find that there exists u_2 such that $x_3: = Ax_2 + Bu_2 \in V^\infty(K)$.

Going on in this way, we construct a regular control sequence such that

the corresponding trajectory (with initial value x) is in $V^\infty(K)$ and hence

certainly in K. It follows that $x \in V*(K)$.

From the definitions, it is clear that $R_b^\infty(K) \subset R_b^*(K)$. Take $x \in R_b^*(K)$.

Then x is initial value for a polynomial trajectory in K. By our definition

of trajectories, there must be a k such that $x \in R_b^k(K)$. It follows that

$x \in R_b^\infty(K)$.

The proposition not only shows how to compute $V*(K)$ and $R_b^*(K)$, but

it also establishes the fact that the definitions of the various classes

of subspaces as presented here coincide with the ones given by Wonham and

Willems, since the algorithms in Wonham (1979; p.91) and Willems (1981) are

the same as those given here. In closing, we note the following immediate

consequence of Cor. 3.10 and Prop. 3.11.

<u>Corollary 3.12.</u> The subspace K is (A,B)-invariant if and only if

$$A K \subset K + \text{im } B. \tag{3.13}$$

This is the well-known 'geometric' characterization of (A,B)-invariance

(Wonham 1979; p.88). Such simple formulas cannot be given for the other

three classes of subspaces, however. We shall study other types of

characterizations below, which will apply to all four classes in a likewise

manner.

4. A Frequency Domain Description

The following theorem is an extension of a result of Hautus (1980).
We shall use the following notational conventions: If Z is a vector
space, then $Z[s]$ is the set of polynomials with values in Z, $Z(s)$ is the
set of rational functions with values in Z, and $Z_+(s)$ is the set of
strictly proper rational functions with values in Z.

Theorem 4.1. The following relations hold.

$$V^*_a(K) = \{x \in K \mid \exists \, \xi \in K(s), \, \omega \in U(s): \quad x = (sI-A)\xi(s) + B\omega(s) \} \qquad (4.1)$$

$$V^*_b(K) = \{x \in X \mid \exists \, \xi \in K(s), \, \omega \in U(s): \quad x = (sI-A)\xi(s) + B\omega(s) \} \qquad (4.2)$$

$$R^*_a(K) = \{x \in K \mid \exists \, \xi \in K[s], \, \omega \in U[s]: \quad x = (sI-A)\xi(s) + B\omega(s) \} \qquad (4.3)$$

$$R^*_b(K) = \{x \in X \mid \exists \, \xi \in K[s], \, \omega \in U[s]: \quad x = (sI-A)\xi(s) + B\omega(s) \} \qquad (4.4)$$

$$V^*(K) = \{x \in K \mid \exists \, \xi \in K_+(s), \, \omega \in U_+(s): \quad x = (sI-A)\xi(s) + B\omega(s) \} \qquad (4.5)$$

$$R^*(K) = \{x \in K \mid \exists \, \xi_1 \in K_+(s), \, \xi_2 \in K[s], \, \omega_1 \in U_+(s), \, \omega_2 \in U[s]:$$

$$x = (sI-A)\xi_1(s) + B\omega_1(s) = (sI-A)\xi_2(s) + B\omega_2(s) \} \qquad (4.6)$$

Proof. The equality (4.5) has been proved by Hautus (1980) who used
the same kind of framework as is employed here. To show that (4.4) holds,
suppose first that $x = (sI-A)\xi(s) + B\omega(s)$, where

$$\xi(s) = x_{-k}s^k + x_{-k+1}s^{k-1} + \ldots + x_0 \qquad x_i \in K \qquad (4.7)$$

$$\omega(s) = u_{-k-1}s^{k+1} + u_{-k}s^k + \ldots + u_0 \qquad u_i \in U \qquad (4.8)$$

Then we can write down the following relations:

$$x_{-k} = -Bu_{-k-1}$$

$$x_{-k+1} = Ax_{-k} - Bu_{-k}$$
$$\vdots$$
$$x_0 = Ax_{-1} - Bu_{-1}$$

$$0 = Ax_0 - Bu_0 + x \qquad (4.9)$$

From this, it is clear that x acts as initial value for a polynomial

trajectory in K. It follows that $x \in R_b^*(K)$.

Conversely, if we take $x \in R_b^*(K)$, then we can set up a series of

relations as in (4.9). It is then clear that $x = (sI-A)\xi(s) + B\omega(s)$

where ξ and ω are defined by (4.7) and (4.8).

The inclusion "$\supset$" in (4.2) can be shown by an argument similar

to the one used above. The reverse inclusion is obvious from (4.4),

(4.5) and Cor. 3.8. Finally, the equalities (4.1), (4.3) and (4.6)

are now immediate from the definitions.

The treatment here has been kept coordinate-free, but we shall turn

to the matrix terminology in the next section.

5. The Associated Rational Matrix Equations

Let us now assume that we have chosen fixed bases for X and for U. The subspace K can then be represented by a basis matrix K. If L is another subspace represented by a basis matrix L, it is clear from Thm. 4.1 that, for instance, $L \subset R_b^*(K)$ if and only if there exist polynomial matrices Z(s) and U(s) such that

$$L = (sI-A)KZ(s) + BU(s) . \qquad (5.1)$$

It is convenient to eliminate U(s) from such expressions, and this can be done by introducing an annihilator N for B (i.e., if B is an n×m-matrix, then N is an (n-m)×n-matrix such that NB = 0). Operating with N on both sides of (5.1) gives

$$NL = (sNK - NAK)Z(s) . \qquad (5.2)$$

We therefore introduce the matrices

$$E: = NK, \quad H := NAK \qquad (5.3)$$

and we consider the rational matrix equations

$$E = (sE-H)Z_1(x) \qquad (5.4)$$

$$H = (E-s^{-1}H)Z_2(s) . \qquad (5.5)$$

Now we can formulate the following result.

Theorem 5.1. The subspace K is:

(i)   an almost (A,B)-invariant subspace if and only if (5.4) has a

        rational solution $Z_1(s)$, or, equivalently, if and only if (5.5)

        has a rational solution $Z_2(s)$.

(ii)   an almost controllability subspace if and only if (5.4) has a

        polynomial solution $Z_1(s)$.

(iii)   an (A,B)-invariant subspace if and only if (5.5) has a proper

        rational solution $Z_2(s)$.

(iv)   a controllability subspace if and only if (5.4) has a polynomial

        solution $Z_1(s)$, and (5.5) has a proper rational solution $Z_2(s)$.

Proof Claim (i). According to Thm. 4.1, K is almost (A,B)-invariant if

and only if there exist rational matrices Z(s) and U(s) such that

$$K = (sI-A) KZ(s) + BU(s). \tag{5.6}$$

Note that the content of this statement does not change if we only postulate

that Z(s) must be rational, since the rationality of U(s) is then automatic

from (5.6). So we conclude that K is almost (A,B)-invariant if and only

if there exists a rational matrix Z(s) such that

$$NK = N(sI-A) KZ(s) \tag{5.7}$$

i.e., if and only if (5.4) has a rational solution.

        As for the second part of the claim, it is easy to verify that if

$Z_1(s)$ is a solution of (5.4), then

$$Z_2(s) = s^2 Z_1(s) - sI \tag{5.8}$$

solves (5.5). Conversely, if $Z_2(s)$ is a solution of (5.5), then

$$Z_1(s) = s^{-1}I + s^{-2}Z_2(s) \tag{5.9}$$

is a solution of (5.4).

Claim (ii). The argument is the same as in the first paragraph of the proof, with "rational" replaced by "polynomial".

Claim (iii). If (5.5) has a proper rational solution $Z_2(s)$, then (5.9) shows that (5.4) has a strictly proper rational solution $Z_1(s)$ which has the property that the coefficient of s^{-1} in its development around infinity is equal to the identity. It then follows that the matrix $U(s)$ that satisfies

$$K = (sI-A)KZ_1(s) + BU(s) \tag{5.10}$$

is in fact a strictly proper rational matrix. By Thm. 4.1, this proves that K is (A,B)-invariant. The argument can be reversed to complete the proof.

Claim (iv). This is immediate from the above and the definitions.

The conditions of the theorem all ask for the existence of a solution of a certain type for some matrix equation, where the type of the solution can be specified by the absence of zeros in a given region of the extended complex plane. There is a general theory available which gives necessary and sufficient conditions for such solutions to exist: see, for instance, Verghese and Kailath (1981). An application of this theory would lead to statements in terms of the finite and infinite

zeros of the rational matrices $sE-H$ and $E-s^{-1}H$. But, of course, the rational matrix equations (5.4) and (5.5) are of a very special and simple type, and one can get fully detailed information by invoking the Kronecker normal form for matrix pencils, as we shall do in the next section. Here, we just mention one particularly simple conclusion that can be drawn immediately from the theorem.

Corollary 5.2. The subspace K is almost (A,B)-invariant if and only if

$$\text{rank}[E;\ sE-H] = \text{rank}[sE-H] \tag{5.11}$$

where the rank is taken over the field of rational functions.

Remark. One can also eliminate $Z(s)$ from (5.1), rather than $U(s)$. In fact, this is the procedure that is naturally suggested by the formula (2.7). If C is a mapping such that $K = \ker C$, then it is clear from (2.7) that the input sequence $\underline{u}$ and the initial value x_{in} give rise to a trajectory in K if and only if

$$0 = C(zI-A)^{-1}B\underline{u}(z) + C(zI-A)^{-1}x_{in}. \tag{5.12}$$

However, the existence of a polynomial solution $\underline{u}$ to this equation, for a given x_{in}, only guarantees that the corresponding trajectory is in the unobservable subspace of the pair (C,A) for positive values of the time parameter k, not that this trajectory is polynomial. Hence, the existence of a polynomial solution $U(s)$ to the rational matrix equation

$$C(sI-A)^{-1}L = C(sI-A)^{-1}BU(s) \tag{5.13}$$

is equivalent to

$$L \subset R_b^*(K) + N \qquad\qquad (5.14)$$

where N denotes the unobservable subspace of the pair (C,A).

This result was first proven by Bengtsson (1975a) in an unpublished report, and later re-derived (under an implicit observability assumption) by Willems (1982). Both authors also give similar expressions for the other types of subspaces (see also Bengtsson (1975b)). For the strictly proper/(A,B)-invariant case, the result was re-derived by Emre and Hautus (1980), and, using a simpler method, by Hautus (1980).

6. Pencil Characterization

The introduction of the form sE-H related to a given pair (A,B) and a given subspace K, as done in the previous section, is due to Karcanias (1979), who termed this form the "restriction pencil". A classification of the various classes of subspaces related to the pair (A,B) on the basis of the invariants of this pencil (Gantmacher 1959) has been undertaken by Jaffe and Karcanias (1981). We will now show that this classification can be readily derived from the framework presented here. The derivation will also be helpful to arrive at the "hybrid" characterization which will be discussed in the next section.

Kronecker's basic result is that the pencil sE-H can be brought into a block quasi-diagonal form by suitable basis transformations, "quasi" meaning that the blocks appearing on the diagonal are not necessarily square. The blocks each have a special form, related to the "invariants" of the pencil. To the diagonalization of sE-H there corresponds a diagonalization of E, of course, and as a result we can break down the equation (5.4) into a series of simpler equations. A complete treatment of this procedure is given in Gantmacher (1959, pp. 35-40; cf. also pp. 45-49).

The resulting equations divide into five classes, corresponding to the pencil invariants. Corresponding to the zero row and column minimal indices, there is one equation of the form

$$
\begin{pmatrix} 0\ldots.0 \\ \vdots \quad \vdots \\ 0\ldots.0 \end{pmatrix} = \begin{pmatrix} 0\ldots.0 \\ \vdots \quad \vdots \\ 0\ldots.0 \end{pmatrix} z_j(s) \tag{6.1}
$$

This equation is obviously satisfied for any $Z_j(s)$ of compatible size.

Corresponding to each nonzero column minimal index, there is an equation of the form

$$\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} s & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & s & 1 \end{pmatrix} \; Z_j(s) \tag{6.2}$$

One solution of this equation is given by

$$Z_j(s) = \left(\begin{array}{c|c} \begin{matrix} 0 & \cdots & 0 \\ & & \\ & S^{-1} & \end{matrix} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right) \;, \qquad S = \begin{pmatrix} 1 & \cdots & & 0 \\ s & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & s & 1 \end{pmatrix} . \tag{6.3}$$

Note that S is of the form I-sM where M is nilpotent, so $S^{-1} = (I-sM)^{-1} = I + sM + \ldots + s^r M^r$ is polynomial. Hence, the solution given by (6.3) is polynomial. Another solution of (6.2) is given by

$$Z_j(s) = \left(\begin{array}{c|c} T^{-1} & t(s) \\ \hline 0 \cdots \cdots 0 & s^{-1} \end{array} \right) \;, \qquad T = \begin{pmatrix} s & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & \cdots & & s \end{pmatrix} \tag{6.4}$$

$$t(s) = -T^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s^{-1} \end{pmatrix} = \begin{pmatrix} (-1)^{r+1} s^{-r} \\ \vdots \\ \vdots \\ -s^{-2} \end{pmatrix} .$$

Here, T is of the form sI-M where M is nilpotent. So we see that $T^{-1} = (sI-M)^{-1} = s^{-1}I + \ldots + s^{-(r+1)}M^r$ is strictly proper. Hence, the solution given by (6.4) is strictly proper, and, moreover, the leading coefficient of its development around infinity equals the identity.

Corresponding to each nonzero row minimal index, there is an equation of the form

$$
\begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \\ 0 & \ldots & 0 \end{pmatrix}
\begin{pmatrix} s & \ldots & 0 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & s \\ 0 & \ldots & 1 \end{pmatrix}
z_j(s) \ . \tag{6.5}
$$

This equation has no rational solution, because

$$
\text{rank}
\begin{pmatrix} 1 & s & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & s \\ 0 & & \ldots & 1 \end{pmatrix}
> \text{rank}
\begin{pmatrix} s & \ldots & 0 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & s \\ 0 & \ldots & 1 \end{pmatrix}
\tag{6.6}
$$

(the rank being taken over the field of rational functions).

Corresponding to a finite elementary divisor at $\alpha \in \mathbb{C}$, we get an equation of the form

$$
\begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ 0 & \ldots & 1 \end{pmatrix}
=
\begin{pmatrix} s-\alpha & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \ldots & \ldots & s-\alpha \end{pmatrix}
z_j(s) \ . \tag{6.7}
$$

Clearly, this equation has a unique solution, which is strictly proper rational and whose leading coefficient in the development around infinity equals the identity.

Finally, corresponding to infinite elementary divisors we have equations of the form

$$
\begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & & \ldots & 0 \end{pmatrix}
=
\begin{pmatrix} 1 & s & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & s \\ 0 & & \ldots & 1 \end{pmatrix}
z_j(s) \ . \tag{6.8}
$$

This equation has a unique solution, which is given by

$$
Z_j(s) = \begin{pmatrix} \begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & s^{-1} \\ \hline 0 & 0 \ \ldots \ 0 \end{array} \end{pmatrix}
\tag{6.9}
$$

where S is of the same form as in (6.3). Hence, this solution is polynomial.

We can now read off immediately the following result (Jaffe and

Karcanias 1981):

<u>Theorem 6.1.</u> Let K be a subspace and let sE-H be the associated restriction

pencil, defined through (5.3). Then the following holds:

(i) K is almost (A,B)-invariant if and only if the associated pencil

has no nonzero row minimal indices.

(ii) K is an almost controllability subspace if and only if the

associated pencil has no nonzero row minimal indices and no

finite elementary divisors.

(iii) K is an (A,B)-invariant subspace if and only if the associated

pencil has no nonzero row minimal indices and no infinite elementary

divisors.

(iv) K is a controllability subspace if and only if the associated

pencil has no nonzero row minimal indices and no elementary

divisors (finite or infinite).

<u>Proof.</u> The result follows from Thm. 5.1 and the above analysis, under

the observation that (5.4) has a strictly proper solution with identity

leading coefficient in the development around infinity if and only if

(5.5) has a proper solution (see (5.8) and (5.9)).

In the next section, we shall use the analysis via the Kronecker normal form to find out what happens if we look upon (5.4) as an equation over the field of scalars, for each separate $s \in \mathbb{C}$.

7. Hybrid Characterization

Carrying the analysis one step further, we obtain the following result.

Theorem 7.1. Let K be a given subspace. Then:

(i) K is almost (A,B)-invariant if and only if

$$(sI-A)K + \text{im } B \supset K + \text{im } B \qquad \text{for some } s \in \mathbb{C} \; . \qquad (7.1)$$

(ii) K is an almost controllability subspace if and only if

$$(sI-A)K + \text{im } B \supset K + \text{im } B \qquad \text{for all } s \in \mathbb{C} \; . \qquad (7.2)$$

(iii) K is (A,B)-invariant if and only if

$$(sI-A)K + \text{im } B = K + \text{im } B \qquad \text{for some } s \in \mathbb{C} \qquad (7.3)$$

(iv) K is a controllability subspace if and only if

$$(sI-A)K + \text{im } B = K + \text{im } B \qquad \text{for all } s \in \mathbb{C} \; . \qquad (7.4)$$

Moreover, both of the following conditions are equivalent to (7.3):

$$(sI-A)K + \text{im } B \subset K + \text{im } B \qquad \text{for some } s \in \mathbb{C} \qquad (7.5)$$

$$(sI-A)K + \text{im } B \subset K + \text{im } B \qquad \text{for all } s \in \mathbb{C} \; . \qquad (7.6)$$

Proof Claim (i). If (7.1) holds, then the matrix equation (5.4), viewed
as an equation over the field of scalars, is solvable for some $s \in \mathbb{C}$. Since
the rank inequality (6.6) holds for each s separately, the matrix equation
(6.5) is not solvable at any point in the complex plane. In view of this
fact, (7.1) implies that the associated pencil has no nonzero row minimal
indices, which means, by Thm. 6.1, that K is almost (A,B)-invariant.

Reversing the argument shows that the implication also holds the other way around.

Claim (ii). If (7.4) holds, then the matrix equation (5.4) is solvable at all s ∈ ℂ. It follows that the associated pencil does not have any nonzero row minimal indices, and also that it has no finite elementary divisors, for the equation (6.7) is not solvable at α ∈ ℂ. By Thm. 6.1, this implies that K is an almost controllability subspace. Again, the reverse argument proves the reverse implication.

Claim (iii). Suppose that K is (A,B)-invariant. Then, by Cor. 3.12, the condition (7.5) holds. It is easy to verify that (7.5) implies (7.6). Since an (A,B)-invariant subspace is certainly also almost (A,B)-invariant, (7.1) holds too. It follows that (7.3) is true. Conversely, (7.3) implies (7.5) which implies (7.6). In particular, the inclusion of (7.6) holds at s=0, and involving Cor. 3.12 again, we find that K is (A,B)-invariant. In passing, we have also proved the final part of the theorem.

Claim (iv). By the definition, and by what has been proved above, the condition for K to be a controllability subspace is obtained by combining (7.2) and (7.6). This immediately leads to (7.4).

Even for the cases (iii) and (iv), the characterizations given above are new. It follows from the proof that, if (7.1) is true, the inclusion does in fact hold for all except a finite number of points in the complex plane, corresponding to the finite elementary divisors of the restriction pencil. A similar remark holds with respect to (7.3). One can show that in both cases the exceptional values of s coincide with the transmission zeros of the system (C,A,B) (where C is such that K = ker C), as defined

in Wonham (1979; p. 113).

The theorem indicates that the four classes of subspaces that we
have been considering in this paper are indeed basic classes, as they are
characterized by the existence of inclusion relations between K and $(sI-A)K$,
modulo im B, at either some or all points of the complex plane. The next step
would be to consider subspaces that are characterized by a certain region of the
complex plane where inclusion or equality must hold. The right half plane
is a natural candidate, and from this choice one obtains the classes of
stabilizability subspaces (Hautus 1980) and almost stabilizability subspaces
(Schumacher 1982).

The characterizations of this section have been termed "hybrid",
because they are stated in terms of subspaces but also involve the
complex parameter s. A somewhat more computational form can be obtained
through the following corollary.

Cor. 7.2. Let K be a given subspace. Then K is an almost controllability
subspace if and only if the equality

$$\dim[(sI-A)K + \text{im B})] = \dim(K + AK + \text{im B}) \tag{7.7}$$

holds for all $s \in \mathbb{C}$, and K is almost (A,B)-invariant if and only if (7.7)
holds for some $s \in \mathbb{C}$.

Proof. We obviously have, for all $s \in \mathbb{C}$,

$$(sI-A)K + \text{im B} \subset K + AK + \text{im B}, \tag{7.8}$$

so (7.7) is equivalent to

$$K + AK + \text{im } B \subseteq (sI-A)K + \text{im } B. \tag{7.9}$$

Because $AK \subseteq K + (sI-A)K$, (7.9) is equivalent to

$$K + \text{im } B \subseteq (sI-A)K + \text{im } B. \tag{7.10}$$

The result is now immediate from Thm. 7.1.

We now want to rephrase this corollary in matrix terms. Let us write $\dim K = k$, $\dim(K + AK + \text{im } B) = r$, $\dim \text{im } B = m$. We can select a basis for X such that the first k basis vectors span K and the first r basis vectors span $K + AK + \text{im } B$. With respect to this basis, we can write down a basis matrix K for K and matrices for A and B which have the following form:

$$K = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{23} & A_{33} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}. \tag{7.11}$$

Consider the following $r \times (k+m)$ polynomial matrix:

$$M(s) = \begin{pmatrix} sI-A_{11} & B_1 \\ -A_{21} & B_2 \end{pmatrix}. \tag{7.12}$$

The matrix version of Cor. 7.2 is now the following.

Corollary 7.3. K is an almost controllability subspace if and only if the rank of the matrix M(s) defined above is equal to r for all $s \in \mathbb{C}$. K is almost (A,B)-invariant if and only if the normal rank of M(s) (i.e., the rank that M(s) has everywhere in $\mathbb{C}$ except possibly at a finite number of points where drop-off takes place) is equal to r.

So here we have a rank test to determine whether a given subspace
is almost (A,B)-invariant or is an almost controllability subspace. To
find out about plain (A,B)-invariance, one can use the simple test of
Cor. 3.12. This test can be combined with the rank test in order to investi-
gate whether a given subspace is a controllability subspace. One then
obtains a test which is new and convenient, because it does not require
the computation of an F such that $(A+BF) K \subset K$ (cf. Wonham (1979; p. 104)).

The first condition of the corollary can be checked by computing
the Smith form of M(s) (MacDuffee 1950; p.41). Indeed, a polynomial matrix
is of full row rank for all $s \in \mathbb{C}$ if and only if its Smith form is (I 0).
To find the normal rank it is sufficient to compute the Hermite form
(MacDuffee 1950; p. 32). It is obviously less work to compute these
normal forms than to compute the full Kronecker normal form for the
restriction pencil.

The result of Cor. 7.3 is reminiscent of the well-known Hautus
test (Hautus 1969) and, in fact, reduces to it in the special case $K = X$.
As one easily verifies, the statement "(A,B) is a controllable pair" is
equivalent to the statement "the state space X is an almost controllability
subspace". Note that the alternative way of checking controllability,
via the matrix $(B \quad AB...A^{n-1}B)$, is obtained as a special case from the
algorithm (3.6). So the test of Cor. 7.3 relates to the algorithms of
Section 3 precisely as the Hautus test relates to the test via the
controllability matrix.

8. Conclusions

We have presented a purely algebraic treatment of almost invariance. By using a discrete-time interpretation, we were still able to retain the intuitive feel associated with a dynamic system. In this way, key results could be obtained with relatively little effort. We re-derived the basic properties studied first by Willems; we extended the results of Hautus on the frequency-domain characterization of (A,B)-invariant subspaces; we placed the results of Jaffe and Karcanias within this framework; and finally, we derived a "hybrid" characterization which could be transformed into a rank test for almost invariance.

We have not discussed the numerical feasibility of any of the characterizations. Much work in this area remains to be done. Possibly, it will turn out that a meaningful evaluation of this aspect can only be made in the context of specific applications.

Some of the characterizations we obtained are easily dualized, some are not. From Thm. 7.1 it is quite immediate that, for example, a subspace T is a "complementary almost observability subspace" as defined in Willems (1982) if and only if

$$T \cap \ker C \supset (sI-A)^{-1} T \cap \ker C \qquad \text{for all } s \in \mathbb{C} . \qquad (8.1)$$

While a formal derivation of such results via transposition is perfectly well feasible, an _ab initio_ treatment of the four classes of subspaces related to a given state-output pair, as done in the present paper for a given state-input pair, has yet to be undertaken.

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