Low Complexity Quantized Controllers for LTI Systems: Peak-to-peak Performance Guarantees

by

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Submitted to the Department of Electrical Engineering and Computer Science
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Abstract

In this thesis, we propose a novel feedback control scheme for unstable LTI systems
performing noise attenuation via a finite-rate digital channel.

In the first part of the thesis, we introduce the structure of the control system as
well as the encoder and decoder used to transmit the required control signals along
the digital channel. The performances of the proposed algorithm are then evaluated
by providing explicit bounds on the peak-to-peak noise attenuation, in regards to the
induced $l_\infty$ gain of the closed loop. This result is obtained by constructing a new
class of storage functions that can be employed to verify the dissipativity of the closed
loop system with respect to a suitable supply rate function.

In the second part of the thesis, we examine the trade-off between the closed loop
performances and the required rate of the channel. While the digital channel imposes
some limitations on the achievable induced $l_\infty$ gain, we show how the performances
of the proposed scheme can still approximate those achievable without communica-
tion constraints provided that the rate of the channel is large enough. A numerical
optimization problem is then devised to design the parameters of the control scheme
in order to minimize the strain on the channel while matching some prescribed con-
straints on the closed loop induced $l_\infty$ gain.

Thesis Supervisor: Professor Munther Dahleh
Title: Professor of Electrical Engineering
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Preface

Motivation

Feedback control systems permeate our modern technological life. They are at work in our homes, our cars, our factories, our transportation systems, our defense systems – everywhere we look. Qualitatively speaking, their main purpose is to gather data using sensors, process it and send commands to actuators capable of interacting with the real world system they are trying to regulate.

Given the required interplay of different physical components, control systems have always inherently relied on some type of communication systems, from the wiring in an embedded system, and data buses in large factories, to wireless sensors networks and the internet for autonomous vehicles and telerobotics.

Despite the interconnections between control and communication, both sciences have historically developed independently. Communication theory is mainly concerned with the transmission of information from one point to another, without much regard to its final purpose. Control theory, on the other end, aims at using the information in a feedback loop to achieve some required performances without taking into consideration the effects introduced by communication. One of the main differences between these two branches of engineering lies in the importance of delays. The most famous results in communication theory are asymptotical in nature, and practical implementations rely on block coding using large blocks in order to achieve higher performances at the expense of large delays. On the other hand, control theory shows that time delays in the feedback loop can have a detrimental effect on the performances of the controlled plant.
In systems with a large communication bandwidth, neglecting the interplay between control and communication is a safe assumption and allows for a simplified analysis and design of both controllers and communication systems.

The validity of this modular approach has been questioned by some recent emerging applications. As the complexity and size of modern systems grows, so does the number of sensors and actuator required to control them. Since the cost of additional communication infrastructure is often prohibitive, the communication medium is shared among different components, thus decreasing the total bandwidth associated with each communication link. This is particularly true in large factories where controllers for different processes may share the same communication bus, or in wireless sensor network applications where the total bandwidth is allotted a priori regardless of the number of nodes.

These emerging scenarios require an understanding of how communication impacts the effectiveness of controllers designed to rely on it. While large networked systems with multiple sensors and actuators are the driving motivation for this research, in this thesis, we will focus our attention only on the simplest possible network topology, consisting of one controller and one dynamical system connected by a noiseless digital feedback loop with a given data rate in bits per unit time. Our first objective is to provide a new characterization of the achievable performances and illustrate their connection to the capabilities of the communication link.

Motivated by the widespread use of digital controllers, our second goal is to provide a control algorithm readily implementable on a digital platform. We strived for simplicity and a reasonably low requirement in terms of computational power and memory usage. In line with the choice of a digital controller, we model our communication system as a digital link. Even though many communication medium are intrinsically analog in nature, this assumption is generally true for a digital platform whose interface with the communication link is ultimately digital in nature.

In reality, real life communication systems may present a variety of other challenges like randomness, packet loss and delays. We hope that by providing these results and theoretical tools suitable in a simpler scenario, insights can be gained about more
In the last twenty years, there has been a significant research devoted to the study of control under communication constraints associated to the network topology presented in Figure 0-1.

In this scenario, the sensors and actuators are not co-located and, therefore, a communication channel is required in the feedback loop. The encoder and decoder pair can be designed arbitrarily and will also incorporate the required estimation and control tasks. Even though some attention has been recently devoted to nonlinear models for the plant, the results presented on this brief overview are restricted to linear plants of the type:

\[
\begin{align*}
x^+ &= Ax + Bu + w \\
e &= C_x x + Du u \\
y &= C_y x + v
\end{align*}
\]

to be consistent with the assumptions in the remainder of the thesis.

The first question that can be asked in this setting is: what are the minimal performances of the communication channel ensuring the existence of an Encoder/Decoder...
pair capable of stabilizing the plant when it is unperturbed \((w,v \equiv 0)\)? The answer to this question depends heavily on the adopted model for the communication channel and on the particular notion of stability. The initial works in [61, 63, 67] have all assumed a deterministic finite rate channel and investigated the problem of obtaining asymptotic stability for the plant state, i.e. \(\|x(k)\| \to 0\) as \(k \to \infty\). The authors have shown that this is indeed possible if and only if\footnote{To be precise the inequality \(\leq\) is necessary while the inequality \(<\) is sufficient. Many results from the literature present this negligible gap in the characterization.} the following condition is matched:

\[
H = \sum_{\lambda_i \in \Lambda(A):|\lambda_i|>1} \log_2 |\lambda_i| < C,
\]

where \(C\) is the Shannon capacity of the channel and \(H\) refers to the intrinsic entropy of the plant and measures its level of instability. The same result has been obtained when trying to achieve almost sure asymptotic stability, first when using a discrete memoryless channel [44], and then through a wide class of noisy channels including delays, erasures and AWGN [62].

The problem becomes more complex when a perturbation \(w,v \neq 0\) is injected into the plant. For this case, the works in [43, 50] point out that requiring almost sure boundedness of \(\|x\|\) becomes too restrictive. If the model of the channel is still deterministic with a finite rate, the work in [46] shows that a second order moment stability can be achieved if and only if condition (1) is matched. For a general channel, the two works in [50, 51] provide a complete characterization in order to achieve bounded moments of order alpha:

\[
\exists \text{ Encoder/Decoder: } \sup_k \mathbb{E}[\|x(k)\|^\alpha] < \infty \iff H < C_{\text{Any}} (\alpha \max |\lambda_i|).
\]
decreases exponentially as the time passes.

The second fundamental question related to control under communication constraints in the given setting, centers around performances. In the case when \( w, v \equiv 0 \), of interest is how to drive the state to zero more so than simply driving it to zero. The works in [48, 52] assume a deterministic noiseless finite rate channel as the communication link and weigh the performances with a quadratic cost:

\[
J = \sum_{k=0}^{\infty} [x(k)^T S x(k) + u(k)^T R u(k)].
\]

In [52] the authors prove that the resulting LQR problem can be solved, with the optimal solution arbitrarily close to that obtained without communication constraint, if and only if condition (1) is satisfied, provided that \( C \) is interpreted as an average capacity over time. In [48] the authors consider only scalar plants and assume the rate to be constrained at every instant of time. In this case, an additional term in \( J_{opt} \) appears due to the communication constraint.

In the case when \( w, v \neq 0 \), much research has been devoted to investigate how the restricted information flow impinges on the effects that a stochastic input has on the controlled signal \( e \). A simple finite horizon version of this problem has been studied in [42] while significant results in the infinite-horizon setting first appeared in [64] where the authors considered the minimization of the cost

\[
J = \limsup_{T} \frac{1}{T} E \sum_{k=0}^{T-1} [x(k)^T S x(k) + u(k)^T R u(k)].
\]

When using a deterministic finite rate channel, they proved that a lower bound on the rate of the channel to achieve a cost of \( J_{opt} + D \), is given by:

\[
\frac{1}{2} \log \left| A A^T + \frac{n}{D} \Pi \right|
\]

where \( n \) is the size of the plant and \( \Pi \) is the covariance matrix of the process noise and \( J_{opt} \) is the optimal cost without communication constraints. Unfortunately, this result is tight only for \( D \to 0 \). With the same setting, in [23] a simple En-
coder/Decoder scheme based on a uniform quantizer is capable of achieving the cost
\[ J_{\text{opt}} + cN^{-2} \log N \] when the rate \( R = \log_2 N \) of the channel satisfies:

\[ 2^R = N \gg \left( \lambda_{\text{max}}(A^T A) \right)^{\frac{3}{2}}. \]

Notice that, even in this case, the result holds for high-rate configurations.

The work in [64] has also shed light on the LQG control for AWGN channels. Unfortunately, positive results were proven only when the output of the channel and the control signal are available at the encoder, usually a very demanding condition. If the information pattern is more restrictive, the authors pointed out that non-linear encoding scheme are required to obtain a tight characterization of the achievable performances.

To overcome these difficulties, great effort has recently been devoted to the topic of control with signal-to-noise ratio (SNR) constraint. In this framework, where the channel is AWGN and the input has again a stochastic description, the main assumption is that the encoder and decoder are LTI systems themselves. This assumption simplifies the LQG problem as the SNR constraint at the channel can be expressed as the norm of a transfer function, thus leading to tractable solutions.

In this framework, once again, the results first covered stabilizability, while the results for the optimal LQG problem have become available more recently. In regards to stabilizability, to be interpreted here as the uniform boundness of the second moment of \( x \), the most comprehensive results are contained in [56]. The condition:

\[ \sigma^2 > \prod_{\lambda_i \in \Lambda(A) : |\lambda_i| > 1} |\lambda_i|^2 - 1, \quad (3) \]

is necessary and sufficient for stabilization when the channel has a perfect feedback whereas the condition:

\[ \sigma^2 > \prod_{\lambda_i \in \Lambda(A) : |\lambda_i| > 1} |\lambda_i|^2 - 1 + \eta + \delta, \]
is the equivalent one when such feedback is not available. Notice that condition (3) is equivalent to the one in (1), while the two additional terms in (3) take into account the detrimental contribution of non minimal phase zeros and the relative degree of the plant.

The optimal solution to the LQG problem with SNR constraint is presented in [31, 32, 55]. The optimal solution can be achieved by solving a convex optimization problem whose solution defines both the encoder and decoder. Note that the optimal solution is, in general, infinite-dimensional and further approximations are required for its implementation.

We conclude this section by emphasizing that, with the exception of the SNR framework, many control schemes cited so far do not assume a bound on the available amount of memory or computational power at the Encoder and Decoder. Since this feature may lead to solutions that are difficult to implement in practice, a large amount research has been devoted to the design of Encoder/Decoder schemes where the encoding scheme, with respect to a noiseless digital channel, involves simple quantizers and a limited amount of memory. Along this line, the works in [11, 22, 24, 27, 35] present control schemes based on log quantizers with an infinite number of levels whereas the works in [6, 25, 26, 29, 30, 33, 37] are based on finite level quantizers and make use of a dynamical variable to scale the control signal before coding it. Within this second group of contributions, the ones in [33, 37] are those most intimately related to this thesis. The aforementioned papers present a simple control scheme implementable via a finite rate channel and capable of achieving a finite induced gain for the closed loop, without any further assumption or prior knowledge of \( w, v \). This type of characterization is extremely powerful because it allows for the modelling of input signals that are not disturbances. Unfortunately, the results in [33, 37] are rather limited since the transient time can be arbitrarily large and, most importantly, do not provide an expression for the induced gain. The negative result presented in [40], which shows that such gain behaves poorly for very small and very large inputs, partially justifies the lack of recent developments along this line of research. The goal of this thesis, therefore, is two-fold: one, to provide schemes for which such
gain is computable, and two, to illustrate how to optimize it while satisfying the rate constraint due to the channel.

Outline and Contributions

This section contains an outline of the thesis and a summary of the contributions.

Chapter 1 presents known results about standard optimal $l_1$ control for linear systems. In particular, most of the chapter is devoted to the design of sub-optimal controllers that will constitute the cornerstone of the results presented in this dissertation. While the majority of the material is covered in [1, 7, 8, 49], further constraints, in the form of additional LMIs, are added to the original formulation. The purpose of such conditions is to bound properties of the linear regulators that will be useful in the design of a finite rate controller.

Chapter 2 formulates the control problem and provides a constructive solution built upon a linear controller. While based on the concept of dynamic quantization already found in the literature [6, 25, 26, 33, 37], our solution differs from the ones previously proposed in that the control signal is not always scaled before transmission, but is sent as is when the state magnitude is within certain prescribed thresholds. Moreover, when the scaling factor becomes too large, instead of decreasing it through a geometric progression, its value is quickly reset to avoid long transients. These features are critical in showing that the proposed solution exhibits performances comparable to those of the underlying linear controller.

In the second part of Chapter 2, the obtained performances are assessed by estimating the effects that an external perturbation $w$ would have on a regulated output $e$. This estimation is done by proving that the closed loop system is dissipative with respect to a suitable supply rate function and, consequently, by constructing a function $\Gamma$ such that, when the closed loop system operates out of a zero initial condition, we have

$$\|e\|_\infty \leq \Gamma(\|w\|_\infty).$$

Previous related works have either proven the existence of such type of bounds without
providing an explicit expression [33, 37] or have managed to provide partial bounds by assuming a fixed statistical description of the input [23, 64]. Within the framework of control via a finite rate channel, this is the first time that an explicit expression for $\Gamma$ is provided without any assumption on $w$ other than being bounded. We would like to point out that our results can be used to derive the same type of bounds already presented in the aforementioned works and, in addition, can be used in situations where little is known about the input $w$ or it is unreasonable to use a stochastic model to describe it.

Chapter 3 presents an optimization problem that can be solved to design the parameters of the proposed control scheme so that the strain on the channel is minimized while some performance requirements for $\Gamma$ are satisfied. It is shown that, provided the bitrate of the communication link is large enough, the gain function $\Gamma$ can approximate arbitrarily well the performances of the underlying linear controller over any range of noise amplitudes. The obtained optimization problem is non-convex, but involves only five variables, and can be solved in reasonable time as some examples at the end of the chapter indicate.

The material in this thesis is therefore presented in logical order and, while a large portion of the manuscript is devoted to the introduction of new technical tools and the proof of the main statements, we hope that the first part of Chapter 2 together with the optimization problem in Chapter 3 will provide practical guidance for the design of finite rate controllers.

Finally, while most of the results presented in this dissertation are still unpublished at the time of writing, earlier ideas and results are presented in [3].
Chapter 1

Peak-to-peak Control of LTI Systems

1.1 Optimal $l_1$ Control

Consider a finite dimensional LTI system described by:

$$
\Sigma_O: \begin{cases} 
  x^+ &= Ax + Bu + Buw \\
  e &= C_x x + D_{eu} u + D_{eu} w \\
  y &= x 
\end{cases}
$$

(1.1)

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the available input for control, $w \in \mathbb{R}^q$ is the external input acting on the system and $e \in \mathbb{R}^p$ describes an error signal that, desirably, is maintained be small when the system is operating properly. Notice that, since in this dissertation we will only cover full state feedback controllers, the measurement available to the controller is the full unperturbed state of the plant.

With reference to Figure 1-1, the problem of optimal $l_1$ control lies in designing a controller $K$, which maps the available state into a control action $u$, such that the induced $l_\infty$ norm of the closed loop system is minimized. Formally, if $T_K$ is the operator mapping $w$ into $e$ obtained when using the controller $K$, this problem can be stated as:

$$
K = \arg \min_K \sup_{w: \|w\|_{l_\infty} < \infty} \frac{\|T_K(w)\|_{l_\infty}}{\|w\|_{l_\infty}},
$$

(1.2)
Figure 1-1: Feedback scheme for the optimal $l_1$ control problem

where

$$\|w\|_{l_\infty} = \sup_{k \geq 0} \|w(k)\|_{\infty}$$

is the standard supremum norm for discrete time signals and the argument of the minimization in (1.2) is called the induced $l_\infty$ norm of the operator $\mathcal{T}_K$. Given that the induced $l_\infty$ norm links the maximum amplitude of an input signal with that of the corresponding output, it is usually referred as the peak-to-peak gain of the involved operator.

It is easy to show [14] that if $\mathcal{T}_K$ is a linear operator, its induced $l_\infty$ norm coincides with the $l_1$ norm of its impulse response $T_K(k) \ k \geq 0$, thus motivating the name $l_1$ optimal control when the induced $l_\infty$ norm is the objective to be minimized.

The problem of minimizing the induced $l_\infty$ norm of an operator is of practical importance when the input $w$ does not have a finite energy and is best modelled as a persistent signal. Such situation arises, for example, when $w$ is a perturbation noise and the corresponding maximum error is of interest. Moreover, the $l_\infty$ induced norm can provide a bound on the tracking performances of a system with respect to the whole trajectories, thus maintaining its validity during transient phases where other approaches may fail to provide adequate bounds.

The optimal $l_1$ problem has attracted the attention of researchers since the mid-eighties [66], with one of the first solutions appearing in [12] and leveraging duality theory to construct a controller via linear programming. Unlike the minimization of the induced $l_2$ norm, where optimal controllers are always linear and of the same order
as the plant, the optimal $l_1$ control poses some challenges. It has been shown that in its full generality, when only partial observations from the plant are available, this problem may be solved by a nonlinear controller even when the plant is linear [60]. Moreover, when the full state is available at the controller, as is the case throughout this dissertation, there is always an optimal solution that is linear [13], but its order may be arbitrarily large [18]. Finally, in this last scenario, it may be computationally convenient to implement a nonlinear optimal solution as shown in [53, 54].

Delving into the details of how to solve the optimal $l_1$ control problem is beyond the scope of this brief introduction. However, since a linear controller achieving a prescribed induced $l_\infty$ norm will act as the cornerstone for the algorithm proposed in this thesis, the design of simpler and, possibly, sub-optimal controllers will be revisited in the next section.

### 1.2 $*$-norm Minimization

One of the aspects that make the optimal $l_1$ control problem intrinsically challenging is the computational cost of estimating the $l_1$ norm of the impulse response of an LTI systems. Given the consequent complex solutions to the optimal $l_1$ control problem, much research has been devoted to the development of sub-optimal design techniques via tractable approximations. In this section, we will review one of these approaches with the intent of explaining how sub-optimal controllers can be designed efficiently using SDP optimization techniques. While most of the material contained in this section is already well known, we decided to report some of the related proofs for completeness and underline the connections with the ideas presented later in the thesis.

One approach to develop sub-optimal solutions to the optimal $l_1$ control problem is based on the so-called star-norm of an LTI system, which has been initially developed in [1] and [8], respectively, in continuous and discrete time settings. Similar to the minimum invariant set approach used in solving the optimal $l_1$ control problem in the general case, the star-norm of an LTI system is computed by restricting the sets of
feasible invariant sets to ellipsoids and finding the one that yields the best bound on the $l_1$ norm.

$$\Sigma : \begin{cases} x^+ = Ax + Bw, \\ e = C_x + D_{ew}w \end{cases} \quad (1.3)$$

The idea behind this approach is summarized in the following lemma where conditions based on two matrix inequalities are exploited to provide a description of an invariant ellipsoid and the corresponding bound for the induced $l_\infty$ norm of a stable system. We would like to stress out that this result already appeared in [8] and is reported here with a full proof for completeness and to avoid repeating it with minor differences in Lemma 2.1.1.

**Lemma 1.2.1.** Consider the stable LTI system described by (1.3), where $w \in l_\infty$.

Assume there exists a matrix $Q = Q^T > 0$, and constants $\eta < 1$, $\gamma, \mu \geq 0$ such that

$$\begin{bmatrix} Q & AQ & Bw \\ QA^T & \eta^2 Q & 0 \\ B_w^T & 0 & \mu^2 I \end{bmatrix} \geq 0 \quad (1.4)$$

$$\begin{bmatrix} (1 - \eta^2)Q & 0 \\ 0 & (\gamma^2 - \mu^2)I \\ C_xQ & D_{ew} \end{bmatrix} \geq 0 \quad (1.5)$$

then, if we introduce $P = Q^{-1}$, we have:

1. $$(x^+)^T P x^+ \leq \eta^2 x^T P x + \mu^2 w^T w \quad (1.6)$$

2. $$e^T e \leq (1 - \eta^2)x^T P x + (\gamma^2 - \mu^2)w^T w \quad (1.7)$$

3. $$\|e(k)\|^2 \leq (1 - \eta^2)(\eta^2)^k x(0)^T P x(0) + \gamma^2 \|w\|_{\infty}^2 \quad (1.8)$$
Proof. Let us begin by pointing out that condition (1.4) is equivalent to

\[
\begin{bmatrix}
P & PA & PB_w \\
A^TP & \eta^2P & 0 \\
B_w^TP & 0 & \mu^2I \\
\end{bmatrix} \succeq 0
\]  

(1.9)

as it can be obtained by multiplying the matrix in (1.4) by

\[
\begin{bmatrix}
P & 0 & 0 \\
0 & P & 0 \\
0 & 0 & I \\
\end{bmatrix}
\]
on the left and its transpose on the right.

Employing the same reasoning, condition (1.5) is equivalent to

\[
\begin{bmatrix}
(1 - \eta^2)P & 0 & C_e^T \\
0 & (\gamma^2 - \mu^2)I & D_{ew}^T \\
C_e & D_{ew} & I \\
\end{bmatrix} \succeq 0
\]  

(1.10)

with the equivalence matrix given by

\[
\begin{bmatrix}
P & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I \\
\end{bmatrix}
\]

To obtain (1.6) simply apply Schur complement to equation (1.9) to obtain:

\[
\begin{bmatrix}
\eta^2P & 0 \\
0 & \mu^2I \\
\end{bmatrix} - \begin{bmatrix} A^TP \\ B_w^TP \end{bmatrix} P^{-1} \begin{bmatrix} PA & PB_w \end{bmatrix} = \begin{bmatrix}
\eta^2P - A^TPA & -A^TPB_w \\
-B_w^TPA & \mu^2I - B_w^TPB_w \\
\end{bmatrix} \succeq 0.
\]

By pre and post multiplying the last matrix inequality by \([x^Tw^T]\) and \([x^Tw^T]^T\) respectively, we obtain:

\[
x^T(\eta^2P - A^TPA)x - x^T A^TPB_w w - w^TB_w^TPA x + w^T(\mu^2I - B_w^TPB_w)w \geq 0
\]
\[ (Ax + B_w w)^T P (Ax + B_w w) \leq \eta^2 x^T P x + \mu^2 w^T w, \]

which is equivalent to (1.6) given the linear dynamic in (1.3).

To obtain (1.7) we apply again Schur complement to equation (1.10) and obtain:

\[
\begin{bmatrix}
(1-\eta^2)P & 0 \\
0 & (\gamma^2 - \mu^2)I
\end{bmatrix}
- \begin{bmatrix}
C_e^T \\
D_{ew}^T
\end{bmatrix}
\begin{bmatrix}
C_e & D_{ew}
\end{bmatrix}
\geq \begin{bmatrix}
(1-\eta^2)P - C_e^T C_e & -C_e^T D_{ew} \\
-D_{ew}^T C_e & (\gamma^2 - \mu^2)I - D_{ew}^T D_{ew}
\end{bmatrix} \geq 0.
\]

By pre and post multiplying the last matrix inequality by \([x^T \, w^T]^T\) and \([x^T \, w^T]^T\) respectively, we obtain:

\[
x^T((1-\eta^2)P - C_e^T C_e)x - x^T C_e^T D_{ew} w - w^T D_{ew}^T C_e x + w^T((\gamma^2 - \mu^2)I - D_{ew}^T D_{ew})w \geq 0
\]

which is equivalent to (1.7) given the linear dynamic in (1.3)

Finally, the result in (1.8) is a consequence of (1.6) and (1.7). If fact, equation (1.6) implies:

\[
(x^+)^T P x^+ \leq \eta^2 x^T P x + \mu^2 \|w\|_{l_\infty}^2,
\]

which, applied repeatedly from time 0 to \(k - 1\), yields:

\[
x(k)^T P x(k) \leq (\eta^2)^k x(0)^T P x(0) + \mu^2 \|w\|_{l_\infty}^2 \sum_{i=0}^{k-1} \eta^{2i} \leq (\eta^2)^k x(0)^T P x(0) + \frac{\mu^2}{1 - \eta^2} \|w\|_{l_\infty}^2.
\]

Substituting the last inequality in (1.7) finally gives (1.8).

\[\Box\]

The result in Lemma 1.2.1 guarantees that, for any solution of the inequalities (1.4) and (1.5), \(\gamma\) is an upper bound to the induced \(l_\infty\) norm of the given system and
that the ellipsoid:
\[
E = \left\{ x \in \mathbb{R}^n : x^TPx \leq \frac{\mu^2}{1 - \eta^2} \|w\|_{\infty}^2 \right\}
\]
is an attractive invariant set under the given dynamics.

The so-called star-norm of a linear system can then be defined as the minimum \( \gamma \) obtained while enforcing the constraints (1.4) and (1.5):

\[
\min \gamma^2 \quad \text{s.t.} \quad \eta^2 < 1, \quad Q > 0, \quad (1 - \eta^2)Q \geq 0
\]

\[
\|\Sigma\|_\star^2 = \left\{ \begin{array}{c}
\min_{\mu^2, \gamma^2, \eta^2, Q} \gamma^2 \\
\text{s.t.}
\eta^2 < 1 \\
Q > 0 \\
Q A^T \eta^2 Q \geq 0 \\
B_w^T 0 \mu^2 I \\
(1 - \eta^2)Q 0 Q C_e^T \\
0 (\gamma^2 - \mu^2) I D_{ew}^T \\
C_e Q D_{ew} I
\end{array} \right\} \geq 0
\]

While the optimization problem appearing in the definition of the star-norm is not convex due to the product between \( Q \) and \( \eta^2 \), it can still be solved efficiently by solving an LMI with a fixed value of \( \eta \) followed by a line search in \( \eta \). It is also possible to prove, as shown in [8] and [7], that the line search over \( \eta \) is itself a tractable problem being convex if \( D_{ew} = 0 \) or quasi-convex if \( D_{ew} \neq 0 \).

One of the advantages of the tractable definition of the star norm introduced in (1.11) is that it allows for an efficient computation of controllers capable of minimizing it, thus providing easy sub-optimal solutions to the optimal \( l_1 \) control problem. With reference to Figure 1-1, where the plant is described by equations (1.1), the problem of optimal star-norm control can be formulated as finding the controller \( K \) that internally stabilized the closed loop system \( \Sigma_C \) and minimizes the star-norm \( \|\Sigma_C\|_\star \). In other words:

\[
K = \arg \min_{K: \Sigma_C \text{ stable}} \|\Sigma_C\|_\star
\]
It is worth pointing out that, although generalizations are possible, our definition of star-norm in (1.11) can be applied only to finite dimensional linear systems and, therefore, the search in (1.12) will be restricted to finite-dimensional linear controllers. In the case of full feedback control, it can be shown that the optimal linear star-norm controller is static and can be found using an approach similar to that used in computing the star norm of a given system in (1.11). The following lemma proves this result and provides a characterization of all linear controller achieving a closed loop star-norm less than a prescribed value. Once again, this result can be found in [8] and is reported here with a full proof for completeness and to avoid repeating it, with only minor modifications, in the proof of Lemma 2.1.1.

**Lemma 1.2.2.** Consider the open loop system described by (1.1), where the pair \((A, B_u)\) is stabilizable. Then the following statements are equivalent:

1. There exists a finite-dimensional LTI controller \(K\) that stabilizes the plant and such that the star-norm of the closed loop system \(\Sigma_C\) satisfies \(\|\Sigma_C\|_* \leq \gamma\)

2. There exists constants \(\eta^2 < 1\) and \(\mu^2\) as well as matrices \(Q = Q^T > 0\) and \(L\) such that the following inequalities hold:

\[
\begin{bmatrix}
Q & AQ + B_uL & B_w \\
QA^T + L^TB_u^T & \eta^2Q & 0 \\
B_w^T & 0 & \mu^2I
\end{bmatrix} \geq 0 \quad (1.13)
\]

\[
\begin{bmatrix}
(1 - \eta^2)Q & 0 & QC_e^T + L^TD_{eu}^T \\
0 & (\gamma^2 - \mu^2)I & D_{eu}^T \\
C_eQ + D_{eu}L & D_{eu} & I
\end{bmatrix} \geq 0 \quad (1.14)
\]

**Proof.** \((1 \Rightarrow 2)\) Let:

\[
\begin{cases}
x_c^+ = A_cx_c + B_cx \\
u = C_cx_c + D_cx
\end{cases}
\]

be a minimal realization of the controller satisfying the first statement, then the
The corresponding closed loop system is described by:

\[
\Sigma_C : \begin{cases}
\begin{bmatrix}
  x \\
  x_c
\end{bmatrix}^+ = \begin{bmatrix}
  A + B_u D_c & B_u C_c \\
  B_c & A_c
\end{bmatrix} \begin{bmatrix}
  x \\
  x_c
\end{bmatrix} + \begin{bmatrix}
  B_w \\
  0
\end{bmatrix} w \\
  e = \begin{bmatrix}
  C_c + D_{eu} D_c & D_{eu} C_c
\end{bmatrix} \begin{bmatrix}
  x \\
  x_c
\end{bmatrix} + D_{ew} w
\end{cases}
\]

Since \( \| \Sigma_C \|_\star \leq \gamma \) by assumption, the definition of star-norm in (1.11) guarantees that, for such a system, the hypothesis of Lemma 1.2.1 holds true. Writing down those equations after partitioning \( Q \) as

\[
Q = \begin{bmatrix}
  Q_{11} & Q_{12} \\
  Q_{12}^T & Q_{22}
\end{bmatrix},
\]

yields the matrix inequalities:

\[
\begin{bmatrix}
  Q_{11} & Q_{12} & (A + B_u D_c)Q_{11} + B_u C_c Q_{12}^T & (A + B_u D_c)Q_{12} + B_u C_c Q_{22} & B_w \\
  Q_{12}^T & Q_{22} & B_c Q_{11} + A_c Q_{12}^T & B_c Q_{12} + A_c Q_{22} & 0 \\
  * & * & \eta^2 Q_{11} & \eta^2 Q_{12} & 0 \\
  * & * & \eta^2 Q_{12}^T & \eta^2 Q_{22} & 0 \\
  * & * & * & * & \mu^2 I
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
  (1 - \eta^2)Q_{11} & (1 - \eta^2)Q_{12} & * & * \\
  (1 - \eta^2)Q_{12}^T & (1 - \eta^2)Q_{22} & * & * \\
  0 & 0 & (\gamma^2 - \mu^2)I & * \\
  (C_c + D_{eu} D_c)Q_{11} + D_{eu} C_c Q_{12}^T & (C_c + D_{eu} D_c)Q_{12} + D_{eu} C_c Q_{22} & D_{ew} & I
\end{bmatrix} \geq 0.
\]

The first inequality reduces to (1.13) after removing the second and fourth row/column and setting \( Q = Q_{11} \) and \( L = D_c Q_{11} + C_c Q_{12}^T \). The second inequality, with the same choice of \( Q \) and \( L \), reduces to (1.14) after removing the second row/column.

(2 \( \Rightarrow \) 1) The controller that would satisfy the first statement is the static feedback controller given by:

\[
u = Kx = LQ^{-1}x.
\]
This can be easily seen by considering that the closed-loop dynamic when using such controller is described by

\[
\Sigma_C : \begin{cases} 
    x^+ = (A + LQ^{-1})x + B_w w \\
    e = (C_c + D_{eu}LQ^{-1})x + D_{ew} w 
\end{cases},
\]

and, for this system, the hypothesis of Lemma 1.2.1 becomes exactly conditions (1.13) and (1.14), which are true by assumption. Hence, by applying the aforementioned lemma together with the definition of star-norm in (1.11), we conclude that \( \|\Sigma_C\|_* \leq \gamma \).

The result in Lemma 1.2.2 shows that a static controller would always perform as well as any dynamic one and provides the expression \( K = LQ^{-1} \) to recover the static gain from the solutions of the matrix inequalities. The optimal star-norm controller is then found by minimizing \( \gamma \) while enforcing the constraints (1.13) and (1.14):

\[
\begin{aligned}
\min_{K: \Sigma_C \text{stable}} \|\Sigma_C\|_*^2 = \min_{\mu^2, \gamma^2, \eta^2, Q, L} & \gamma^2 \\
\text{s.t.} & 
\begin{cases} 
    \eta^2 < 1 \\
    Q > 0 \\
    \begin{bmatrix} 
        Q & AQ + B_u L & B_w \\
        QA^T + LB_u^T & \eta^2 Q & 0 \\
        B_w^T & 0 & \mu^2 I \\
        (1 - \eta^2)Q & 0 & QC_c^T + L^T D_{ew}^T \\
        0 & (\gamma^2 - \mu^2)I & D_{ew}^T \\
        C_c Q + D_{cu} L & D_{ew} & I 
    \end{bmatrix} \geq 0 
\end{cases}
\end{aligned}
\]

Similarly to the computation of the star norm, the optimization problem appearing in (1.15) is not convex due to the product between \( Q \) and \( \eta^2 \) and must solved by first solving an LMI with a fixed value of \( \eta \) followed by a line search in \( \eta \). The works in [8] and [7], showed that even for this problem, the line search over \( \eta \) is itself a tractable problem being convex if \( D_{ew} = 0 \) or quasi-convex if \( D_{ew} \neq 0 \).
We conclude this section by illustrating how it is possible to introduce further constraints on the set of feasible controllers considered in the optimization problem (1.15). The main reason for adding extra constraints, which will become evident in the next chapter, is related to bounding the number of bits required to implement our proposed control algorithm over a finite-rate channel. In particular, condition (1.16) in the next lemma shows how it is possible to bound the rate of expansion of the invariant ellipsoid when the system runs in open loop, while conditions (1.17) and (1.18) show how it is possible to bound the two scalar quantities $B_u^T P B_u$ and $K P^{-1} K^T$ which will useful in the following chapters.

Lemma 1.2.3. Consider a quintuple $(\mu^2, \gamma^2, \eta^2, Q, L)$ belonging to the feasible set of the optimization problem (1.15). If $K = LQ^{-1}$ is the corresponding controller and $P = Q^{-1}$ is the matrix describing the corresponding invariant set, then the following statements holds:

1. 
\[ A^T P A \leq \sigma^2 P \iff \begin{bmatrix} Q & AQ \\ QA^T & \sigma^2 Q \end{bmatrix} \succeq 0 \]  
\[ (1.16) \]

2. 
\[ B_u^T P B_u \leq \lambda_u^2 \iff \begin{bmatrix} Q & B_u \\ B_u^T & \lambda_u^2 \end{bmatrix} \succeq 0 \]  
\[ (1.17) \]

3. 
\[ K P^{-1} K^T \leq \lambda_k^2 \iff \begin{bmatrix} Q & L^T \\ L & \lambda_k^2 \end{bmatrix} \succeq 0 \]  
\[ (1.18) \]

Proof. To prove the first statement, let us start by noticing that
\[
\begin{bmatrix} Q & AQ \\ QA^T & \sigma^2 Q \end{bmatrix} \succeq 0 \iff \begin{bmatrix} P & PA \\ A^T P & \sigma^2 P \end{bmatrix} \succeq 0,
\]

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where the equivalence is proved by multiplying the first matrix on the left by
\[
\begin{bmatrix}
P & 0 \\
0 & P
\end{bmatrix}
\]
and by its transpose on the right. Finally, a simple application of the Schur complement yields:
\[
\begin{bmatrix}
P & PA \\
A^TP & \sigma^2 P
\end{bmatrix} \geq 0 \iff \sigma^2 P - A^T P P^{-1} P A \geq 0 \iff A^T P A \leq \sigma^2 P
\]

The second statement is a direct application of the Schur complement:
\[
\begin{bmatrix}
Q & B_u \\
B_u^T & \lambda_u^2
\end{bmatrix} \geq 0 \iff \lambda_u^2 - B_u^T Q^{-1} B_u \geq 0 \iff B_u^T P B_u \leq \lambda_u^2
\]

The third statement can be proven by first using the expression for $K$ to obtain
\[
KP^{-1}K^T = LQ^{-1}QQ^{-1}L^T = LQ^{-1}L^T,
\]
and then applying the Schur complement again to the matrix in (1.18) to get:
\[
\begin{bmatrix}
Q & L^T \\
L & \lambda_k^2
\end{bmatrix} \geq 0 \iff \lambda_k^2 - LQ^{-1}L^T \geq 0 \iff LQ^{-1}L^T = KP^{-1}K^T \leq \lambda_k^2
\]
Chapter 2

A Novel Finite-rate Control Scheme

2.1 Introduction

In this chapter, we will introduce and analyse a control scheme capable of stabilizing a discrete time LTI plant through a finite rate channel while achieving some prescribed performances with respect to the induced $l_{\infty}$ norm of the closed loop system. The given plant dynamics to be regulated is assumed to be in the form:

\[
\begin{align*}
\Sigma: \quad x^+ &= Ax + B_u u + B_w w \\
e &= C_e x + D_{eu} u + D_{ew} w \\
y &= x
\end{align*}
\]  \tag{2.1}

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $e \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and the involved matrices have suitable dimensions. In addition to having only a scalar control signal $u$ and the full unperturbed state being available as a measurement, we will further assume that $B_w$ is full column rank, $A$ is unstable and the pair $(A, B_u)$ is stabilizable.

The proposed control scheme will be based upon linear controllers guaranteeing a prescribed induced $l_{\infty}$ norm of the closed loop system. In addition to this property, such controllers will be designed to be robust against gain uncertainties in the feedback loop to accommodate the unavoidable quantization error. The gain uncertainty is
Figure 2-1: Feedback scheme for robust stabilization of $\Sigma$ with additive uncertainty on the control input characterized by a scalar, time varying parameter $\Delta$ with bounded amplitude:

$$|\Delta| \leq \delta$$

and its impact on the closed loop system can be represented as shown in Figure 2-1.

The design of linear controllers satisfying these two properties is based on a set of LMIs similar to those appearing in Lemma 1.2.2. As shown in the following lemma, a minor modification of such conditions allows for the design of robust controllers achieving a specific bound on the induced $l_\infty$ norm of the closed loop system while coping with the bounded gain uncertainty.

**Lemma 2.1.1.** Consider the open loop system described by (2.1), where the pair $(A, B_u)$ is stabilizable. If there exists constants $\eta^2 < 1$, $\mu^2$, $\delta < 1$ as well as matrices $Q = Q^T > 0$ and $L$ such that the following inequalities hold:

\[
\begin{bmatrix}
Q & AQ + (1 - \delta)B_uL & B_w \\
AQ^T + (1 - \delta)L^TB_u^T & \eta^2Q & 0 \\
B_w^T & 0 & \mu^2 I
\end{bmatrix} \geq 0, \quad (2.2)
\]

\[
\begin{bmatrix}
Q & AQ + (1 + \delta)B_uL & B_w \\
AQ^T + (1 + \delta)L^TB_u^T & \eta^2Q & 0 \\
B_w^T & 0 & \mu^2 I
\end{bmatrix} \geq 0, \quad (2.3)
\]
then the static controller \( u = Kx \) robustly stabilizes the system in Figure 2-1 with \( |\Delta| \leq \delta \) and is such that the induced \( l_\infty \) norm of the closed loop system is less than \( \gamma \).

**Proof.** The proof mimics that of Lemma 1.2.2 where in conditions (1.13) and (1.14) the matrices \( B_u \) and \( D_{eu} \) now appear with a scalar factor \((1 + \Delta)\) and need to be satisfied \( \forall |\Delta| \leq \delta \). Those two conditions then become a polytopic LMI that is satisfied if and only it is satisfied at the two vertices. Since conditions (2.2)-(2.5) are exactly the LMIs obtained in those vertices, the proof is complete.

With the same technique used in Lemma 1.2.1, it is possible to show that the constraints (2.2)-(2.5) are equivalent to the following two conditions:

\[
W^+ \leq \eta^2 W + \mu^2 \|w\|^2 \quad \forall |\Delta| \leq \delta, \tag{2.6}
\]

\[
\|e\|^2 \leq (1 - \eta^2) W + (\gamma^2 - \mu^2) \|w\|^2 \quad \forall |\Delta| \leq \delta, \tag{2.7}
\]

where \( W = x^T P x = x^T Q^{-1} x \).

Assuming \( w \in l_\infty \), equation (2.6) implies that

\[
W(x(k)) \leq (\eta^2)^k W(x(0)) + \frac{\mu^2}{1 - \eta^2} \|w\|^2_\infty,
\]

which, combined with (2.6), yields a bound on the induced \( l_\infty \) gain of the closed loop system:

\[
\|e(k)\|^2 \leq (1 - \eta^2)(\eta^2)^k W(x(0)) + \gamma^2 \|w\|^2_\infty
\]

Our goal is to show that, building upon linear controllers satisfying conditions
(2.2) through (2.5) or, equivalently, conditions (2.6) and (2.7), we can design a control scheme such that:

- It is easily implementable over a finite rate channel.
- Has an explicit bound on the required minimum bitrate.
- Guarantees explicit $l_\infty$ performances for the closed loop system in the form

\[ \|e(k)\|^2 \leq (1 - \tilde{\eta}^2)(\tilde{\eta}^2)^k \phi(\|x(0)\|) + \Gamma(\|w\|_\infty) \]  

(2.8)

with $\tilde{\eta} < 1$, $\phi \geq 0$ a suitable function such that $\phi(s) \to 0$ as $s \to 0$ and $\Gamma : \mathbb{R} \mapsto \mathbb{R}$ an increasing function representing the impact of the input on the output.

- As shown in [40], due to the communication constraint, the nonlinear gain $\Gamma$ will have unbounded gain at 0 and $\infty$ but, compatibly with this requirement, can still approximate the optimal gain $\gamma^2$ achievable without communication constraints to an arbitrary precision and in an arbitrarily large window of noise amplitudes as the bitrate increases. A more precise definition of what is the set of achievable gain functions $\Gamma$ will be introduced in Chapter 3.

- The involved parameters can be designed to minimize the required bitrate while achieving specific requirements on $\Gamma$ by solving a "not too hard" nonconvex problem.

### 2.2 Algorithm Description

The proposed control scheme is represented in Figure 2-2, where the channel is assumed to be a finite rate (noiseless) channel capable of transmitting $N$ symbols per transmission with a bitrate of $\log_2 N$ bits per transmission. The encoder and decoder are the subject of our design and will make use of a linear controller satisfying the robust conditions (2.2)-(2.5) as well as some additional parameters to be designed.
2.2.1 Parameter definitions and constraints

Before looking at the structure of the Encoder and Decoder, we will define the additional parameters upon which they depend. Since the construction is based upon a given robust linear controller, assume that $\delta, L, Q > 0$, $\eta, \mu$ and $\gamma$ have already been selected to satisfy conditions (2.2)-(2.5) and let $P = Q^{-1}$ and $K = LQ^{-1}$ denote the invariant ellipsoid matrix and the corresponding controller.

Let us start by defining the following quantities:

\[
\sigma^2 = \inf \{ \tilde{\sigma}^2 \in \mathbb{R} : A^T P A \leq \tilde{\sigma}^2 P \} \quad (2.9)
\]

\[
\lambda_w^2 = \inf \{ \tilde{\lambda}_w^2 \in \mathbb{R} : B_w^T P B_w \leq \tilde{\lambda}_w^2 I \} \quad (2.10)
\]

\[
\lambda_u^2 = B_u^T P B_u \quad (2.11)
\]

\[
\lambda_k^2 = KP^{-1}K^T \quad (2.12)
\]

and observe that, due to the instability of $A$, we have $\sigma > 1$, while condition (2.6) implies $\lambda_w \leq \mu$.

The proposed algorithm has eight further scalar parameters to be designed plus two given parameters, $\rho_1 \in (0, 1)$ and $\rho_2 > 0$ that are considered given and will determine the asymptotic behavior of the nonlinear gain $\Gamma$ in (2.8). In particular, we
will obtain

\[ \Gamma(s) \approx s^{1-\rho_1} \quad \text{for } s \to 0 \]

\[ \Gamma(s) \approx s^{1+\rho_2} \quad \text{for } s \to \infty \]

The first two design parameters, \( \theta \in (0, 1) \) and \( \tau > 0 \), are selected to satisfy the condition:

\[ (1 + \tau)\eta^2 + (1 + \tau^{-1})\theta^2 < 1, \quad (2.13) \]

which can always be satisfied since \( \eta < 1 \) by assumption.

Before introducing the other six parameters let us define, for convenience, the following quantities:

\[ \tilde{\sigma}^2 = (1 + \tau^{-1})\sigma^2 \]

\[ \tilde{\lambda}_w^2 = (1 + \tau)\lambda_w^2 \]

\[ \tilde{\mu}^2 = (1 + \tau)\mu^2 \]

\[ \tilde{\eta}^2 = (1 + \tau)\eta^2 + (1 + \tau^{-1})\theta^2 \]

The remaining six parameters \( \alpha, \beta, \gamma, \tilde{r}, \tilde{r}_1, \tilde{r}_2 \in \mathbb{R}^+ \) are selected to satisfy the following conditions:

\[
\begin{cases}
  \beta \geq \frac{\alpha}{\eta} \\
  \tilde{r}_1 \geq \tilde{r} \left( \frac{\tilde{\sigma}}{\eta} \right) \frac{1}{\rho_1} \\
  \tilde{r}_2 > \tilde{r}_1 \\
  \tilde{r} \geq \tilde{r}_2 \left( \frac{\tilde{\sigma}}{\eta} \right)^{\frac{1}{\rho_2}}
\end{cases} \quad (2.14)
\]

Notice that conditions (2.14) imply

\[ \tilde{r} > \left( \frac{\tilde{\sigma}}{\eta} \right)^{\frac{1}{\rho_1} + \frac{1}{\rho_2}} \geq \left( \frac{\tilde{\sigma}}{\eta} \right)^{1 + \frac{1}{\rho_2}} \]

Finally let us define, for convenience, the following quantities:

\[ r_1 = \tilde{r} \left( \frac{\tilde{\sigma}}{\eta} \right)^{\frac{1}{\rho_1}} \quad r_2 = \tilde{r} \left( \frac{\tilde{\sigma}}{\eta} \right)^{\frac{1}{\rho_2}}. \quad (2.15) \]
2.2.2 Dynamics Description

The works in [6, 25, 33, 37], from which this dissertation stems, all present algorithms performing plant regulation via finite rate channels that are based upon a common simple concept. The main idea is to introduce a scalar dynamical variable $\xi$ that, during the evolution of the system, estimates the magnitude of the plant state and is used to scale appropriately the control signal at the encoder and then rescale it at the decoder.

The algorithm we are proposing uses the same technique and, since the scaling variable is required on both sides of the channel, encoder and decoder possess each a local variable $\xi_e, \xi_d$ and a synchronization mechanism is provided to enforce $\xi_e(k) = \xi_d(k) \quad \forall k \geq 0$. Such mechanism relies on setting the same initial condition at the encoder and decoder side $\xi_e(0) = \xi_d(0) = \xi(0)$ and then providing an update law that relies only on the transmitted symbols. Since the transmitted and received symbols are the same, both encoder and decoder will keep updating their own private copy of $\xi$ in the same way, thus maintaining synchronization. Given that the two private scaling variables are maintained equal, for the sake of exposition, in the remaining of the dissertation we will always refer to a single scalar variable $\xi$ with the understanding that a private copy is available on both sides of the channel.

Analog signals that require discretization will be quantized through a finite level quantizer obtained by truncating a logarithmic quantizer and described by:

\[
Q_{\delta_1, \delta_2, e,M}(u) = \begin{cases} 
-\frac{Q_1}{2} (\frac{Q_1}{2}) & \text{if } u < 0 \\
0 & \text{if } 0 \leq u < \epsilon \\
\epsilon \delta_2 \left( \frac{\delta_2}{\delta_1} \right) \left| \log \frac{u}{\epsilon} \right| & \text{if } \epsilon \leq u < M \\
0 & \text{if } u \geq M
\end{cases}
\]

(2.16)

where $\epsilon$ defines the width of the dead-zone, $M$ defines the saturation level, while $\delta_1$ and $\delta_2$ specify the cone containing the levels. A graphical depiction of this type of quantizer is represented in Figure 2-3.

The encoding scheme will make use of the scaling variable $\xi$ and the weighted
norm of the plant state $\|x\|_p$ to partition the entire state space into nine regions. To fully describe the dynamics in the $(x, \xi)$ space, we will now introduce said regions and illustrate the idea behind the corresponding dynamics.

Region 1

$$R_1 = \begin{cases} 
\|x\|_p \geq r\xi \\
\|x\|_p \leq \bar{r}\xi 
\end{cases}$$

$$\xi^+ = \xi Q_{\frac{\beta}{\bar{r}\xi + \beta}, \frac{\bar{r}\xi}{\xi}, \bar{r}, \bar{r}, \bar{r}, \bar{r}}\left(\frac{\|x\|_p}{\xi}\right)$$

$$u = \xi Q_{1-\delta,1+\delta,\epsilon_1, M_1}(K_r)$$

where $\epsilon_1$ and $M_1$ are defined as:

$$\epsilon_1 = \frac{\theta r}{\lambda_u}, \quad M_1 = \bar{r} \lambda_k.$$ 

In this region, the size of the scaling variable is comparable to the norm of the state plant; hence, we can use it to scale the control input before coding it and rescale it back at the decoder side. In this region, the dynamics behaves similarly
to that obtained using the robust regulator. To make this statement more rigorous, the following lemma shows that, in region \( R_1 \), the norm of the state is subject to a bound similar to that achieved by the robust controller in (2.6). This result will be used later in Section 2.4 to show that the obtained closed loop system is dissipative.

**Lemma 2.2.1.** When the system state is in region \( R_1 \), the dynamics is such that:

\[
\|x^+\|_P^2 \leq \eta^2 \|x\|_P^2 + \mu^2 \|w\|^2
\]

**Proof.** Let us begin with the expression for \( x^+ \) when the dynamics is in \( R_1 \):

\[
x^+ = Ax + B_u \xi \mathcal{Q}_{1-\delta,1+\delta_1,M_1} \left( \frac{Kx}{\xi} \right) + B_w w.
\]

From the definition of region \( R_1 \) we obtain:

\[
\frac{x^T P x}{\xi^2} \leq \bar{r}^2 \Rightarrow \left( \frac{Kx}{\xi} \right)^2 = \frac{x^T K^T K x}{\xi^2} \leq \frac{x^T P x}{\xi^2} K P^{-1} K^T \leq \bar{r}^2 \lambda_k^2 = M_1^2,
\]

which guarantees that, whenever the system is in region \( R_1 \), the quantizer is either in the dead zone or in the pseudo linear one.

If the quantizer is in the pseudo linear zone, then

\[
\mathcal{Q}_{1-\delta,1+\delta_1,M_1} \left( \frac{Kx}{\xi} \right) = \frac{Kx}{\xi} (1 + \Delta)
\]

for some \( \Delta : |\Delta| \leq \delta \). It follows that the update law for the plant state is given by:

\[
x^+ = Ax + B_u Kx (1 + \Delta) + B_w w
\]

and matches exactly that obtained when using the robust linear controller \( K \). Hence, since such controller satisfies condition (2.6) by assumption, we have

\[
\|x^+\|_P^2 \leq \eta^2 \|x\|_P^2 + \mu^2 \|w\|^2 \leq \bar{r}^2 \|x\|_P^2 + \bar{r}^2 \|w\|^2.
\]

If the quantizer is in the dead zone, we have:

\[
\left| \frac{Kx}{\xi} \right| \leq \epsilon_1, \quad \mathcal{Q}_{1-\delta,1+\delta_1,M_1} \left( \frac{Kx}{\xi} \right) = 0,
\]
from which we can derive:

\[
\|x^+\|^2_P = \|Ax + B_w w\|^2_P - \|Ax + B_w w + B_u K x - B_u K x\|^2_P \\
\leq (1 + \tau)\|Ax + B_u K x + B_w w\|^2_P + (1 + \tau^{-1})\|B_u K x\|^2_P \\
= (1 + \tau)\|Ax + B_u K x + B_w w\|^2_P + (1 + \tau^{-1})\xi^2 \left|\frac{K x}{\xi}\right|^2 \|B_u\|^2_P \\
\leq (1 + \tau) \left( \eta^2 \|x\|^2_P + \mu^2 \|w\|^2 \right) + (1 + \tau^{-1})\frac{\|x\|^2_P}{L^2} \xi^2 \lambda_u^2 \\
= \left[ (1 + \tau) \eta^2 + (1 + \tau^{-1}) \frac{\xi^2 \lambda_u^2}{L^2} \right] \|x\|^2_P + (1 + \tau)\mu^2 \|w\|^2 \\
= \eta^2 \|x\|^2_P + \mu^2 \|w\|^2.
\]

\[
\square
\]

Remark 1. We would like to point out that, unlike previous works in the literature where $\xi$ is updated by multiplying it by a fixed constant less than one when it is smaller than the state norm and a constant greater than one in the opposite situation, in region $R_1$ the norm of the state is coded and transmitted along the control input to improve upon the update law for $\xi$. Intuitively this is done to track more accurately the norm of the state space and, crucially, it will prove pivotal in obtaining a finite $l_{\infty}$ gain.

Region 2

\[
R_2 = \left\{ \|x\|_P \geq \bar{\alpha} \xi \cup \left\{ \alpha \leq \|x\|_P \leq \beta \right\} \right\} \\
\left\{ \xi^+ = Q_{\frac{\xi}{\eta^2}, \frac{\xi}{\mu^2}, \alpha, \beta} (\|x\|_P) \right\} \\
u = Q_{1-\delta, 1, \eta \xi, \xi} (K x)
\]

(2.18)
where $e_2$ and $M_2$ are defined as:

$$e_2 = \frac{\theta \alpha}{\lambda_u}, \quad M_2 = \beta \lambda_k.$$ 

In this region, the scaling variable $\xi$ does not provide an indication of the amplitude of the state norm. Despite this, when such norm is within given limits, the dynamics still emulates that obtained in the communication constraint free case. As in region $R_1$, to make this statement more rigorous, the following lemma shows that the norm of the state is subject to a bound similar to that achieved by the robust controller in (2.6). This result will be used later in Section 2.4 to show that the obtained closed loop system is dissipative.

**Lemma 2.2.2.** When the system state is in region $R_2$, the dynamics is such that:

$$\|x^+\|^2_p \leq \tilde{\eta}^2 \|x\|^2_p + \tilde{\mu}^2 \|w\|^2$$

**Proof.** Let us begin with the expression for $x^+$ when the dynamics is in $R_2$:

$$x^+ = Ax + BuQ_{1-\delta,1+\delta,0,M_2}(Kx) + Bw.$$ 

From the definition of region $R_2$ we obtain:

$$x^TPx \leq \beta^2 \Rightarrow (Kx)^2 = x^TK^TKx \leq (x^TPx)(KP^{-1}K^T) \leq \beta^2 \lambda_k^2 = M_2^2,$$

which guarantees that, whenever the system is in region $R_2$, the quantizer is either in the dead zone or in the pseudo linear one.

If the quantizer is in the pseudo linear zone, then

$$Q_{1-\delta,1+\delta,e_2,M_2}(Kx) = Kx(1 + \Delta)$$

for some $\Delta : |\Delta| \leq \delta$. It follows that the update law for the plant state is given by:

$$x^+ = Ax + BuKx(1 + \Delta) + Bw$$
and matches exactly that obtained when using the robust linear controller \( K \). Hence, since such controller satisfies condition (2.6) by assumption, we have

\[
\|x^+\|_P^2 \leq \eta^2 \|x\|_P^2 + \mu^2 \|w\|^2 \\
\leq \tilde{\eta}^2 \|x\|_P^2 + \tilde{\mu}^2 \|w\|^2.
\]

If the quantizer is in the dead zone we have:

\[
|Kx| \leq \epsilon_2, \quad Q_{1-\delta_1,\epsilon_2,M_2}(Kx) = 0,
\]

from which we can derive:

\[
\|x^+\|_P^2 = \|Ax + Bw_w\|_P^2 \\
= \|Ax + Bw_w + BwKx - BwKx\|_P^2 \\
\leq (1 + \tau)\|Ax + BwKx + Bw_w\|_P^2 + (1 + \tau^{-1})\|BwKx\|_P^2 \\
\leq (1 + \tau)\|Ax + BwKx + Bw_w\|_P^2 + (1 + \tau^{-1})\|\frac{\|x\|_P^2}{\alpha^2} |Kx|^2 \|Bw\|_P^2 \\
\leq (1 + \tau)\left(\eta^2 \|x\|_P^2 + \mu^2 \|w\|^2\right) + (1 + \tau^{-1})\|\frac{\|x\|_P^2}{\alpha^2} \epsilon_2^2 \lambda_a^2 \\
= \left[(1 + \tau)\eta^2 + (1 + \tau^{-1})\frac{\epsilon_2^2 \lambda_a^2}{\alpha^2}\right] \|x\|_P^2 + (1 + \tau)\|w\|^2 \\
= \tilde{\eta}^2 \|x\|_P^2 + \tilde{\mu}^2 \|w\|^2.
\]

\[\square\]

Remark 2. The introduction of region \( R_2 \) contrasts with the simpler dynamics proposed in [6], [25], [37], and [33] where, in similar conditions, the state dynamics was left in an open loop by setting \( u = 0 \). In particular region \( R_2 \), together with region \( R_1 \), aim at emulating the linear controller, when the state magnitude is within given limits. This will prove to be pivotal in achieving an \( l_{\infty} \) gain \( \Gamma \) arbitrarily close to that achievable without communication constraints.
Region 3

In this region, the scaling variable $\xi$ is too small compared to the norm of the state and it is also too small to increase it simply by multiplying by a constant greater than 1. In this case, the plant dynamics is run in open-loop ($u = 0$) while $\xi$ is reset to a larger value in an effort to quickly catch up with $\|x\|_p$. This choice of dynamics also appears in [37] and [33] and is required to obtain finite $l_\infty$ gains.

Region 4

The reasoning behind the dynamics in region $R_4$ is the same as the one for region $R_3$. The only difference is that the reset value for $\xi$ is larger given that the state norm is also larger in $R_4$. 

\begin{align*}
R_3 &= \begin{cases} 
\|x\|_p > \bar{r}\xi \\
\|x\|_p \leq \alpha 
\end{cases} \\
&= \begin{cases} 
\bar{\xi} = \frac{\bar{\alpha}}{\bar{r}^2} \\
u = 0
\end{cases} 
\end{align*}

\begin{align*}
R_4 &= \begin{cases} 
\|x\|_p > \bar{r}\xi \\
\|x\|_p \geq \beta 
\end{cases} \\
&= \begin{cases} 
\bar{\xi} = \frac{\bar{\beta}}{\bar{r}} \\
u = 0
\end{cases} 
\end{align*}
Region 5

In region $R_5$ the scaling variable $\xi$ is too small compared to the norm of the state, however it is sufficiently far away from zero that it could be increased via multiplication by a constant greater than 1. In particular, the plant dynamics is run in open-loop ($u = 0$) while $\xi$ is multiplied by a constant greater than the natural expansion rate of $\|x\|_P$ in open loop, $\sigma$, in order to catch up to the state norm. This choice of dynamics is fairly standard and is key to even achieve stability.

Region 6

In region $R_6$ the scaling variable $\xi$ is itself relatively small but still too large compared to the norm of the state to provide accurate scaling of the control input. As a result, the plant dynamics is run in open-loop ($u = 0$) while $\xi$ is multiplied by a constant less than one in order to catch up to the state norm. This choice of dynamics is also fairly standard and is key to even achieve stability.
Region 7

\[ R_7 = \begin{cases} 
\|x\|_P < \alpha \\
\xi > \frac{\alpha}{\xi} \\
\xi^+ = \frac{\delta\alpha}{r_2} \\
u = 0
\end{cases} \tag{2.23} \]

In region \( R_7 \) the scaling variable \( \xi \) is too large compared to the norm of the state to provide accurate scaling of the control input. As a result, the plant dynamics is run in open-loop (\( u = 0 \)) while \( \xi \) is reset to a small positive constant in order to catch up to the state norm.

Remark 3. This type of behavior was not introduced in [37] and [33] and is yet crucial to obtain an \( l_{\infty} \) bound of the type (2.8) where the transient terms cannot be arbitrarily large when \( x(0) \) is small. In contrast, the work in [37] proves only the existence of an asymptotic \( l_{\infty} \) bound where the transient term can be extremely large even if \( x(0) = 0 \).

Region 8

\[ R_8 = \begin{cases} 
\|x\|_P < \xi \xi \\
\|x\|_P > \beta \\
\|x\|_P \geq \Phi \xi \frac{\xi}{r_{1+p}^{\xi}} \\
\xi^+ = \frac{\delta\xi}{r_1} \\
u = 0
\end{cases} \tag{2.24} \]

with \( \Phi > 0 \) any constant satisfying

\[ \Phi \leq \beta r_1^{\rho_2} \min \left\{ \left( \frac{\bar{\eta}r_1^\rho_1}{\bar{\sigma}} \right) \left( \frac{r_1^{\rho_1+r_2}}{\rho_1+r_2} \right) \left( \frac{r_1^{\rho_1+r_2}}{\rho_1+r_2} \right), \left( \frac{1}{\rho_2^{\rho_2(1-r_2)}} \left( \frac{\bar{\eta}}{\bar{\sigma}} \right) \right) \right\} \tag{2.25} \]
In region $R_8$ the scaling variable $\xi$ is too large compared to the norm of the state to provide accurate scaling of the control input but is not large enough to simply reset it to a constant value and is, therefore, multiplied by a constant less than one. In order to achieve the desired asymptotic behavior for $\Gamma$, the threshold between reset and scaling is described by an equation of the type $\|x\|_P \approx \xi^{\frac{p_1}{p_1+p_2}}$ where the multiplicative constant $\Phi$ is selected to be as large as possible while still ensuring that a dissipation inequality can be proven. Notice that, since the dynamics in region $R_8$ and $R_6$ is the same, those two regions are coded with the same symbol by the encoder.

Region 9

![Diagram](image)

$$R_9 = \begin{cases} 
\|x\|_P < \xi^\alpha & \\
\|x\|_P \geq \beta & \\
\|x\|_P \leq \Phi\xi^{\frac{p_1}{p_1+p_2}} 
\end{cases}$$

(2.26)

where $\Phi$ was defined in region 8.

In region $R_9$ the scaling variable $\xi$ is too large compared to the norm of the state to provide accurate scaling of the control input and can be safely reset to a constant value. The same situation was true in region $R_7$ but the reset value, in this case, is different.
2.3 Performances Analysis

In this section we will analyse the performances of the control scheme depicted in Figure 2-2 and the algorithm introduced in the previous section.

When dealing with the design of a control scheme via a communication channel, the two main performance figures that need to be assessed are the strain on the communication link and the performances of the regulator with respect to the exogenous input $w$.

The performances with respect to the exogenous input $w$ are contingent on showing that an $\ell_\infty$ bound of the type in (2.8) exists. Namely, if there exists a function $\phi : \mathbb{R} \mapsto \mathbb{R}^+$ such that $\lim_{s \to 0} \phi(s) = 0$ and a nonlinear function $\Gamma : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that:

$$\|e\|^2(k) \leq (1 - \eta^2)(\tilde{\eta}^2)^k \phi(\|x(0)\|) + \Gamma(\|w\|_\infty^2),$$  \hspace{1cm} (2.27)

then the performances with respect to the input $w$ will be quantified via the nonlinear gain $\Gamma$. Notice that the function $\Gamma$ express the peak-to-peak gain from $w$ to $e$ of the closed loop system.

With regard to the exploitation of the channel, the other performance figure, it can be easily quantified, for a finite rate channel, in terms of the minimum bitrate required for the algorithm to be implemented.

These two figure of merit are trivially competing; the larger the required bitrate the better performances we may expect in terms of the nonlinear function $\Gamma$. In this section, we will directly quantify this correlation for the proposed algorithm while the optimization of such trade-off will be investigated in the next chapter.

2.3.1 Required Bit Rate

Given the structure of our proposed algorithm, the required minimum bitrate can be easily evaluated in terms of the parameters and constants used in its description. The result is formalized in the following lemma where the simplicity of an encoding based on a log quantizer is exploited to count how many symbols can be transmitted per unit of time.
Theorem 2.3.1. The number of symbols required to implement the proposed algorithm is given by:

\[ N = 4 + \left( 1 + 2 \left[ \frac{\log \frac{\frac{r}{\epsilon}}{\log \frac{1+\delta}{1-\delta}}}{\log \frac{r_2}{r_1}} \right] \right) \left[ \frac{\log \frac{\frac{r}{\epsilon}}{\log \frac{1+\delta}{1-\delta}}}{\log \frac{r_2}{r_1}} \right] + \left( 1 + 2 \left[ \frac{\log \frac{\frac{r}{\epsilon}}{\log \frac{1+\delta}{1-\delta}}}{\log \frac{r_2}{r_1}} \right] \right) \left[ \frac{\log \frac{\frac{r}{\epsilon}}{\log \frac{1+\delta}{1-\delta}}}{\log \frac{r_2}{r_1}} \right] \] (2.28)

Proof. Let us begin by considering the generic quantizer \( Q_{\delta_1, \delta_2, \epsilon, M} (u) \).

From the definition in (2.16), it is possible to compute the number of quantization levels required to fill the (positive) pseudo-linear zone of such quantizer by computing the minimum integer \( N \) such that:

\[ \epsilon \left( \frac{\delta_2}{\delta_1} \right)^N \geq M, \]

thus obtaining:

\[ \tilde{N} = \left[ \frac{\log M}{\frac{\delta_2}{\delta_1}} \right]. \]

Let us now consider the coding of region \( R_1 \). From the definition of the dynamics in (2.17), it transpires that two independent codings are necessary: one for the ratio \( \frac{\frac{x}{\xi}}{\delta} \) and one for the scaled control input \( \frac{K_x}{\xi} \). By definition of region \( R_1 \), the ratio \( \frac{\frac{x}{\xi}}{\delta} \) always falls in the positive pseudo-linear zone of \( Q_{\delta_2, \delta_2, \epsilon, \epsilon, \epsilon} (\cdot) \) and can be coded using:

\[ \left[ \frac{\log \frac{\frac{r}{\epsilon}}{\log \frac{1+\delta}{1-\delta}}}{\log \frac{r_2}{r_1}} \right] \]

symbols. The scaled control input, as shown during the proof of Lemma (2.2.1), can instead be either in the positive or negative pseudo-linear zone as well as in the dead-zone of \( Q_{1-\delta, 1+\delta, \epsilon, M_1} (\cdot) \) and can therefore be coded using

\[ 1 + 2 \left[ \frac{\log \frac{\frac{r}{\epsilon}}{\log \frac{1+\delta}{1-\delta}}}{\log \frac{r_2}{r_1}} \right] \]
symbols. Combining these results, the coding of region $R_1$ requires:

$$N_1 = \left( 1 + 2 \left[ \frac{\log M_1}{\log 1 + \delta} \right] \right) \left[ \frac{\log \frac{r}{r_1}}{\log 1 + \delta} \right] = \left( 1 + 2 \left[ \frac{\log \frac{\lambda \lambda_0}{\theta}}{\log 1 + \delta} \right] \right) \left[ \frac{\log \frac{r}{r_1}}{\log 1 + \delta} \right]$$

symbols.

Using the same reasoning, it is possible to show that the coding of region $R_2$ requires

$$N_2 = \left( 1 + 2 \left[ \frac{\log M_2}{\log 1 + \delta} \right] \right) \left[ \frac{\log \frac{\beta}{\alpha}}{\log 1 + \delta} \right] = \left( 1 + 2 \left[ \frac{\log \frac{\lambda \lambda_0}{\theta}}{\log 1 + \delta} \right] \right) \left[ \frac{\log \frac{\beta}{\alpha}}{\log 1 + \delta} \right]$$

symbols, where the product is, again, a consequence of the independent quantization of $\|x\|_p$ and the control input $Kx$.

The expression in (2.28) then follows since coding the remaining seven regions requires one symbol for each different dynamics, hence regions $R_6$ and $R_8$, region $R_3$ and $R_7$ as well as regions $R_4$ and $R_9$ can actually be coded using one symbol per pair.

It is worth pointing out that the necessity to code independently both the state magnitude and the control input lead to the main multiplications in (2.28). Consequently, when only stability is required, our algorithm is easily outperformed in terms of bitrate by other simpler implementations already available ([25]). The possibility of controlling the response to external inputs, though, compensates for the increased minimal bandwidth.

### 2.3.2 The Candidate Storage Function

The condition in (2.27) is an instance of an input-to-output stability (IOS) property that we wish to prove for the considered closed loop system. The study of this particular type of property in the particular case when $e = x$ (ISS) has originated in the late eighties with the work in [57] and has then been further developed to include outputs, discrete time systems and discontinuous dynamics [28, 36, 58, 59].
The main contributions of these works is proving that such a property is equivalent to the existence of a non-negative function of the whole state \( V \) and a pair of \( \mathcal{K} \)-functions \( s_1, s_2 \) such that:

\[
V^+ - V \leq s_1(\|w\|) - s_2(\|e\|). \tag{2.29}
\]

Condition (2.29) is called a dissipation inequality as the integral over time of the right-hand side expression is stored up in the given storage function \( V \). The right-hand side of condition (2.29) is called supply rate and, by changing its expression, different types of dissipativity can be proven [20, 34]. In this dissertation we will focus on dissipation inequalities like the one in (2.29) as this type of supply function leads to the required bounds on the \( l_\infty \) induced gain.

The key to obtaining tight bounds for the peak-to-peak measure \( \Gamma \) in (2.27) is to use the right storage function \( V \) in proving the dissipation inequality (2.29):

Apart from some standard categories like polynomial or piecewise polynomial, constructing suitable storage functions can be very challenging and is even considered an art.

Inspired by the fact that the proposed algorithm attempts to mimic the dynamics of a robust controller in some regions of the state space, our approach is to mimic the behavior and meaning of the corresponding storage function \( W = x^TPx \) used to prove the robust \( l_\infty \) gain in the communication constraint free case.

Let us begin by recalling the two conditions used to derive a robust linear \( l_\infty \) gain already presented at the beginning of this chapter:

\[
W^+ \leq \eta^2 W + \mu^2\|w\|^2 \quad \forall |\Delta| \leq \delta, \tag{2.30}
\]

\[
\|e\|^2 \leq (1 - \eta^2)W + (\gamma^2 - \mu^2)\|w\|^2 \quad \forall |\Delta| \leq \delta.
\]

While the second of these equations is simply an algebraic relationship between \( W \) and the regulated output \( e \), the first condition is a dissipation inequality that describes how \( W \) evolves in time and can be used to understand which features to look for in a good candidate storage function.

In the context of the proposed algorithm, a dissipation inequality similar to the
first one of (2.30) has been proven valid within regions $R_1$ and $R_2$ where, by virtue of Lemmas 2.2.1 and 2.2.2 we obtained that:

$$W^+ \leq \tilde{\eta}^2 W + \tilde{\mu}^2 \|w\|^2.$$  

From such relationship, we can deduce two important properties of $W$:

- If $w \equiv 0$, then $W(x(k))$ is the maximum value of $x^TPx$ along the future part of the trajectory, i.e. $W(x(k)) = \sup\{x(k)^TPx(k), \ k \geq \hat{k}\}$. This fact is not so surprising, given that $W$ can be used to provide peak-to-peak gain performances.

- If $w \equiv 0$, then, in the worst case, $W$ decreases by a factor $\tilde{\eta}^2$.

While it is natural to choose

$$V_1 = V_2 = x^TPx$$

in regions $R_1$ and $R_2$ where the linear dynamics is mimicked, we will construct a storage function that shows the same two properties in the other seven regions.

The most interesting regions are those where the plant is left in open loop while the scaling variable $\xi$ is multiplied by a constant to catch up with the plant state. It turns out that, given the simplicity of the dynamics, it is possible to compute an upper bound of the supremum over the trajectories of $x^TPx$ as a function of $(x, \xi)$ and derive from it the value of a good candidate function. Let us first examine region $R_5$, where the dynamics (2.21) is such that:

$$\left\{ \begin{align*}
\|x\|^+_P &\leq \hat{\sigma} \|x\|_P \\
\xi^+ &= \frac{\hat{\sigma} x}{\tilde{r}^2} \end{align*} \right. \Rightarrow \left( \frac{\|x\|_P}{\xi} \right)^+ \leq \frac{\tilde{r}^2}{\hat{\sigma}} \left( \frac{\|x\|_P}{\xi} \right),$$

and a worst-case trajectory beginning from such region is depicted in Figure 2-4. If $(x, \xi) \in R_5$ and knowing that $R_5$ is characterized by $\frac{\|x\|_P}{\xi} > \hat{r}$, by using the relationship just found we can upper bound the number of steps before the state exits $R_5$ as the
smallest integer $N$ such that:

$$\left(\frac{r_2}{r}\right)^N \frac{\|x\|_P}{\xi} \leq \bar{r},$$

thus obtaining:

$$\tilde{N} = \left\lceil \frac{\log \frac{r_2}{r_2^2}}{\log \frac{r_2}{r_2}} \right\rceil \leq 1 + \frac{\log \frac{\|x\|_P}{r_2^2}}{\log \frac{r_2}{r_2}} = \frac{\log \frac{\|x\|_P}{r_2^2}}{\log \frac{r_2}{r_2}} = \frac{\log \left(\frac{\|x\|_P}{r_2^2}\right)^{r_2}}{\log \left(\frac{\tilde{\eta}}{\eta}\right)^{r_2}},$$

where we used the definition of $r_2$ in (2.15). Since, after exiting region $R_5$, the state will land in region $R_1$, the value of the storage function at that time will be bounded by $\tilde{\eta}^{2N}\|x\|_p^2$. Finally, taking into account that the storage function must shrink by a factor $\tilde{\eta}^2$ at every time step, a good candidate storage function for $(x, \xi) \in R_5$ is given by:

$$V_5 = \left(\frac{\tilde{\eta}}{\eta}\right)^{2N} \|x\|_P^2 = \left(\frac{\tilde{\eta}}{\eta}\right)^{2N} \|x\|_P^2 = \left(\frac{\|x\|_P}{r_2^2}\right)^{2N} \|x\|_P^2 = \left(\frac{x^T P x}{r_2^2}\right)^{1+p_2}.$$  

(2.32)

In regions $R_6$ and $R_8$ a similar reasoning can be exploited to construct a candidate storage function. In those regions the dynamics is such that:

$$\begin{cases}
\|x\|_P^2 \leq \tilde{\eta} \|x\|_P \\
\xi^+ = \frac{r_1}{r_2} \xi
\end{cases} \Rightarrow \left(\frac{\|x\|_P}{\xi}\right)^+ \leq \frac{r_1}{r_2} \left(\frac{\|x\|_P}{\xi}\right),$$

and an example of a worst-case trajectory starting in $R_8$ is depicted in Figure 2-4. If $(x, \xi) \in R_6$ ($R_8$) and knowing that such region is characterized by $\|x\|_P < r_1$, by using the relationship just found we can upper bound the number of steps before the state exits $R_6$ ($R_8$) as the smallest integer $\tilde{N}$ such that:

$$\left(\frac{r_1}{r_2}\right)^{\tilde{N}} \|x\|_P \leq \bar{r}.$$
Figure 2-4: Worst case trajectories when $w \equiv 0$ starting from region $R_5$, $R_6$ and $R_8$. 

thus obtaining:

$$N = \left[ \frac{\log \frac{r_6}{\|x\|_P}}{\log \frac{r_4}{r_1}} \right] \leq 1 + \frac{\log \frac{r_6}{\|x\|_P}}{\log \frac{r_4}{r_1}} = \frac{\log \frac{r_6}{\|x\|_P}}{\log \frac{r_4}{r_1}} = \frac{\log \frac{r_6}{\|x\|_P}}{\log \left( \frac{\tilde{\sigma}}{\eta} \right)} = \log \left( \frac{\tilde{\sigma}}{\eta} \right),$$

where we used the definition of $r_2$ in (2.15). As with region $R_5$, after exiting region $R_6$ ($R_8$), the state will land in region $R_1$ and the value of the storage function at that time will be bounded by $\tilde{\sigma}^{2N} \|x\|_{P}^{2}$. Taking into account that the storage function must shrink by a factor $\tilde{\eta}^{2}$ at every time step, good candidate storage functions inside $R_6$ and $R_8$ are given, respectively, by:

$$V_6 = \left( \frac{\tilde{\sigma}}{\tilde{\eta}} \right)^{2N} \|x\|_{P}^{2} = \left( \frac{\tilde{\sigma}}{\tilde{\eta}} \right) \left( \frac{r_1 \xi}{\|x\|_P} \right)^{2\rho_1} \|x\|_{P}^{2} = \left( \frac{r_1 \xi}{\|x\|_P} \right)^{2\rho_1} \|x\|_{P}^{2} = (r_1 \xi)^{2\rho_1} (x^T P x)^{1-\rho_1}$$

$$V_8 = V_6 = (r_1 \xi)^{2\rho_1} (x^T P x)^{1-\rho_1} \quad (2.34)$$

From regions $R_3$ and $R_7$, the dynamics move to a different one in one step. While knowing exactly in which region the state will land is not possible, we construct our
candidate storage function assuming\(^1\) that the state lands in \(R_6\). In this case the storage function at the next instant of time will be bounded by:

\[
V_6\left(\tilde{\sigma}X, \frac{\tilde{\sigma}X}{r_2}\right) = \left(\frac{r_1}{r_2}\right)^{2p_1} \tilde{\sigma}^2 \alpha^{2p_1}(x^TPx)^{1-p_1} \leq \tilde{\sigma}^2 \alpha^{2p_1}(x^TPx)^{1-p_1},
\]

thus a good guess for the storage function in regions \(R_3\) and \(R_7\) is given by:

\[
V_3 = \frac{\tilde{\sigma}^2 \alpha^{2p_1}}{\tilde{\eta}^2} (x^TPx)^{1-p_1},
\]

and

\[
V_7 = \frac{\tilde{\sigma}^2 \alpha^{2p_1}}{\tilde{\eta}^2} (x^TPx)^{1-p_1}.
\]

A similar reasoning can be done for regions \(R_4\) and \(R_9\), from which the state moves to another region in one step. In this case we will construct the storage function assuming that the state lands in \(R_5\) so that an upper bound to the value at the next instant of time is given by:

\[
V_5\left(\tilde{\sigma}X, \frac{\tilde{\sigma}X}{r_2}\right) = \left(\frac{r\tilde{\sigma}}{r_2}\right)^{2\rho_2} \tilde{\sigma}^2 (x^TPx)^{1+\rho_2} \leq \frac{\tilde{\sigma}^2 (x^TPx)^{1+\rho_2}}{\beta^{2\rho_2}},
\]

where the last inequality holds since \(r_2 \geq r\tilde{\sigma}\) is implied by (2.14). Taking into account the required contraction factor of \(\tilde{\eta}^2\), a good guess for the storage function in regions \(R_4\) and \(R_9\) is given by:

\[
V_4 = \frac{\tilde{\sigma}^2 (x^TPx)^{1+\rho_2}}{\tilde{\eta}^2 \beta^{2\rho_2}}
\]

and

\[
V_9 = \frac{\tilde{\sigma}^2 (x^TPx)^{1+\rho_2}}{\tilde{\eta}^2 \beta^{2\rho_2}}
\]

In conclusion, our candidate storage function can be concisely described by:

\[
V(x, \xi) = V_i(x, \xi) \quad \text{if } (x, \xi) \in R_i,
\]

where the regions \(R_i\) have been introduced in Section 2.2 and the functions \(V_i\) are

\(^1\)The accurate analysis of all possible transitions is carried out in the next section.
defined in (2.31)-(2.38).

**Remark 4.** The candidate storage function defined in (2.39) does not fall into any of the canonical families described in the literature as it is piecewise transcendental and of the form:

\[ V_i = c_i(x^{2a_i})(x^T P x)^{1-b_i} \quad \text{if} \quad (x, \xi) \in R_i, \]

with \( a_i, b_i \in \{0, \rho_1, -\rho_2\} \). The coefficients appearing in its definition, as well as those appearing in the definition of the dynamics, have been already selected in terms of the given parameters. Although this particular choice may be sub-optimal and a numerical optimization could provide tighter results, it is sufficient to prove that the performances of our algorithm are close to those attainable without communication constraints.

We will conclude this section by proving an important property of the proposed storage function that will be useful in the following sections. The next lemma shows that it is possible to obtain an upper and a lower bound on \( V \) as a function of \( x^T P x \) only. This result is key to obtaining an \( l_\infty \) performance bound of the type (2.27) where the contribution given by the initial condition is uniform in \( \xi \) and depends only on the initial condition of the plant.

**Lemma 2.3.2.** For the candidate storage function defined in (2.39), the following bounds hold:

\[ x^T P x \leq V(x, \xi) \leq \phi(x^T P x), \]

where the function \( \phi : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) is given by:

\[ \phi(s) = \max \left\{ s, \frac{\tilde{\sigma}^2 \alpha^{2\rho_1}}{\tilde{\eta}^2} s^{1-\rho_1}, \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^{2\rho_2}} s^{1+\rho_2} \right\} \]

**Proof.** Since the bounds clearly hold within regions \( R_1 \) and \( R_2 \), we will show their validity inside the remaining regions.

Let us begin with the lower bound \( x^T P x \leq V(x, \xi) \). Using the bounds imposed on \( \xi \) and \( \|x\|_p \) by the respective regions and the definitions in (2.15), we obtain the
following inequalities:

\[ V_3 = \frac{\hat{\sigma}^2 \alpha^{2\rho_1}}{\hat{\eta}^2} (x^T P x)^{1-\rho_1} \geq \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{valid since } \|x\|_P \leq \alpha \text{ in } R_3 \]

\[ V_4 = \frac{\hat{\sigma}^2}{\hat{\eta}^2} \frac{(x^T P x)^{1+\rho_2}}{\beta^{2\rho_2}} \geq \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{valid since } \|x\|_P \geq \beta \text{ in } R_4 \]

\[ V_5 = \frac{(x^T P x)^{1+\rho_2}}{(r_2 \xi)^{2\rho_2}} \geq \frac{(x^T P x)^{1+\rho_2}}{(r_2 \|x\|_P)^{2\rho_2}} = \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{valid since } \xi \leq \frac{\|x\|_P}{r} \text{ in } R_5 \]

\[ V_6 \geq \left( r_1 \frac{\|x\|_P}{r} \right)^{2\rho_1} (x^T P x)^{1-\rho_1} = \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{valid since } \xi \geq \frac{\|x\|_P}{r} \text{ in } R_6 \]

\[ V_7 = \frac{\hat{\sigma}^2 \alpha^{2\rho_1}}{\hat{\eta}^2} (x^T P x)^{1-\rho_1} \geq \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{same reasoning as for } V_3 \]

\[ V_8 \geq \left( r_1 \frac{\|x\|_P}{r} \right)^{2\rho_1} (x^T P x)^{1-\rho_1} = \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{same reasoning as for } V_6 \]

\[ V_9 = \frac{\hat{\sigma}^2}{\hat{\eta}^2} \frac{(x^T P x)^{1+\rho_2}}{\beta^{2\rho_2}} \geq \frac{\hat{\sigma}^2}{\hat{\eta}^2} x^T P x \quad \text{same reasoning as for } V_4 \]

which complete the proof of the lower bound since \( \frac{\xi}{\eta} \geq 1 \).

To prove the upper bound in (2.40), observe that the functions \( V_3 \) and \( V_7 \) are defined as one of the functions appearing in the definition of \( \phi \) in (2.41), while for \( V_4 \) and \( V_9 \) we have the following simple bound:

\[ V_4 = V_9 = \frac{\hat{\sigma}^2}{\hat{\eta}^2} \frac{(x^T P x)^{1+\rho_2}}{\beta^{2\rho_2}} \leq \frac{\hat{\sigma}^2}{\hat{\eta}^2} (x^T P x)^{1+\rho_2}. \]

Exploiting the bounds imposed on \( \xi \) and \( \|x\|_P \) by regions \( R_5 \) and \( R_6 \), we obtain:

\[ V_5 = \frac{(x^T P x)^{1+\rho_2}}{(r_2 \xi)^{2\rho_2}} \leq \frac{(x^T P x)^{1+\rho_2}}{(r_2 \beta)^{2\rho_2}} \leq \frac{\hat{\sigma}^2}{\hat{\eta}^2} \frac{(x^T P x)^{1+\rho_2}}{(r_2 \beta)^{2\rho_2}} \quad \text{valid since } \xi \geq \frac{\beta}{r} \text{ in } R_5 \]
\[ V_6 \leq \left( \frac{r_1 \alpha}{\tau} \right)^{2\rho_1} (x^TPx)^{1-\rho_1} = \frac{\tilde{\sigma}^2 \alpha^{2\rho_1}}{\tilde{\eta}^2} (x^TPx)^{1-\rho_1} \quad \text{valid since } \xi \leq \frac{\alpha}{\tau} \text{ in } R_6 \]

while the bound in region \( R_8 \) yields:

\[
V_8 = (r_1 \xi)^{2\rho_1} (x^TPx)^{1-\rho_1} \leq \left( r_1 \left( \frac{\|x\|_P}{\Phi} \right)^{\rho_1+\rho_2} \right)^{2\rho_1} (x^TPx)^{1-\rho_1} = \\
= \frac{r_1^{2\rho_1}}{\Phi^{2(\rho_1+\rho_2)}} (x^TPx)^{1+\rho_2} \leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^4} \left( \frac{r_1 \tilde{\sigma}}{r_1} \right)^{2\rho_1+2\rho_2} (x^TPx)^{1+\rho_2} \leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^4} (x^TPx)^{1+\rho_2},
\]

where the first inequality is due to the bound \( \xi \leq \left( \frac{\|x\|_P}{\Phi} \right)^{\rho_1+\rho_2} \) valid in region \( R_8 \), the second is obtained using the first argument of the minimum appearing in (2.25) while the last holds since \( r_1 \geq \tilde{\sigma} \) is implied by (2.15).

\[
2.3.3 \quad \text{Obtained } l_\infty \text{ Bound}
\]

In Section 2.1, it has been shown that the two conditions (2.6) and (2.7) guarantee that any robust linear controller satisfying them yields a corresponding induced \( l_\infty \) gain for the closed loop system. While condition (2.6) is a type of dissipation inequality describing how the input signal affects the storage function over time, the condition in (2.7) algebraically bound the output signal by a linear combination of the storage function and the input signal.

We will show that two similar conditions hold for the dynamical system obtained when applying the algorithm proposed in Section 2.2. Using these conditions, we will derive an \( l_\infty \) bound similar to that in (2.27) that can be used to characterize the peak-to-peak gain of the closed loop.

The condition that more easily extends to the considered nonlinear system is an algebraic relationship between input signal, output signal and candidate storage function similar to condition (2.7). The following proposition shows how, by adding a simple additional constraint to the linear controller, the obtained relationship is the
same as the one obtained in the linear setting.

**Proposition 2.3.3.** Consider the dynamical system presented in Section 2.2 where the robust linear controller satisfies conditions (2.2)-(2.5) as well as

\[
\begin{bmatrix}
(1 - \eta^2)Q & 0 & QC_e^T \\
0 & (\gamma^2 - \mu^2)I & D_{ew}^T \\
C_eQ & D_{ew} & I
\end{bmatrix} \geq 0.
\] (2.42)

If \( V \) is the storage function defined in (2.39), then the input signal \( w \), the output signal \( e \) and \( V \) satisfy the following constraint at any instant of time:

\[
\|e\|^2 \leq (1 - \eta^2)V + (\gamma^2 - \mu^2)\|w\|^2.
\]

**Proof.** The proof is a direct consequence of the fact that the proposed controller is either applying a control signal similar to that of the linear controller \((u \approx Kx)\) or is leaving the plant to evolve in open loop \((u = 0)\).

As shown in Lemmas 2.2.1 and 2.2.2, in regions \( R_1 \) and \( R_2 \) the quantizer used to code the control input is either in the dead zone or in the pseudo linear one. When such quantizer is in pseudo linear zone we have \( u = (1 + \Delta)Kx \) for some \( \Delta \in [-\delta, \delta] \) and therefore we can use the result of Lemma 2.1.1 to conclude that

\[
\|e\|^2 \leq (1 - \eta^2)x^TPx + (\gamma^2 - \mu^2)\|w\|^2.
\]

When the quantizer in regions \( R_1 \) and \( R_2 \) is in the dead zone, or when the state belongs to \( R_3 \) through \( R_9 \), the plant is left in open loop via \( u = 0 \). In this case we can leverage the additional condition on the controller imposed in (2.42). Such condition is obtained from (2.4) by setting \( D_{cu} = 0 \) and, hence, guarantees the same bound:

\[
\|e\|^2 \leq (1 - \eta^2)x^TPx + (\gamma^2 - \mu^2)\|w\|^2,
\]

when \( e \) does not depend on \( u \) thus making it valid for \( u = 0 \).
The proof is then completed by using the bound $x^T P x \leq V$ derived in Lemma 2.3.2.

The extra condition required in Proposition 2.3.3 is completely redundant whenever $D_{eu} = 0$ as, in this case, it reduces to conditions (2.4) and (2.5). Moreover, when $D_{eu} \neq 0$, it is still not strictly necessary to bound the output signal as a function of $w$ and $V$. A more complex non-linear bound could be derived by analysing the behavior for $\|x\|_P \leq \alpha$ and $\|x\|_P \geq \beta$ separately while also proving that the effect of the quantizer dead zone in regions $R_1$ and $R_2$ can be made arbitrarily small with $\theta$. While such analysis would allow us to show that the proposed algorithm has a gain arbitrarily close to the achievable star norm of the given system even when $D_{eu} \neq 0$, it causes the optimization of the involved parameters to be less tractable and is not pursued in this dissertation.

We now introduce a dissipation inequality for the proposed algorithm that plays a role analogous to that of condition (2.6) in the linear setting. Such inequality describes how the storage function evolves from one instant of time to the next one as a function of the original state and the perturbing noise and can be used to draw some conclusion on the overall dependence of $V$ on the input signal. Since the space state is partitioned into nine regions and the dissipation inequality must always hold, it is necessary to verify its validity for any possible transition between two regions. Given the large number of regions, we will state the obtained result in the following theorem and postpone its proof to Section 2.4 for readability reasons.

**Theorem 2.3.4.** Consider the dynamical system presented in Section 2.2 and the storage function defined in (2.39). Then

$$\Delta V = V^+ - \tilde{\eta}^2 V \leq \tilde{\Gamma}(\|w\|^2),$$

where

$$\tilde{\Gamma}(s) = \max \{ \tilde{\Gamma}_1(s), \tilde{\Gamma}_2(s), \tilde{\Gamma}_3(s), \tilde{\Gamma}_4(s), \tilde{\Gamma}_5(s) \},$$

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\[ \hat{G}_1(s) = \hat{\mu}^2 s \]

\[ \hat{G}_2(s) = \begin{cases} 
\frac{\sigma^2 \alpha^{2p_1}}{\eta^2 (1 - \left(\frac{\sigma}{\sigma_2}\right)^2)^{1-p_1}} \left(\hat{\mu}^2 s\right)^{1-p_1} - \frac{(\hat{\mu}^2 s)^2}{1 - \left(\frac{\sigma}{\sigma_2}\right)^2} \hat{\mu}^2 s & \text{if } s \leq \left(1 - \frac{\sigma^2}{\mu^2}\right) \frac{\sigma^2 (1-p_1)}{\eta^2} \frac{1}{p_1} \\
\alpha^2 p_1 \left(\frac{\hat{\mu}^2 s}{\eta^2}\right)^{1-p_1} + \hat{\mu}^2 s & \text{if } s \geq \frac{\sigma^2}{\mu^2} \left(\frac{\sigma^2 (1-p_1)}{\eta^2}\right) \frac{1}{p_1} \\
\alpha^2 p_1 \left(\frac{\hat{\mu}^2 s}{\eta^2}\right)^{1-p_1} & \text{otherwise}
\end{cases} \]

\[ \hat{G}_3(s) = \begin{cases} 
0 & \text{if } s \leq \frac{\sigma^2}{\mu^2} \left(1 - \frac{\sigma^2}{\mu^2}\right) \\
\frac{\sigma^2}{\eta^2 \beta^{2p_2}} \left(\beta^2 \hat{\mu}^2 s + \hat{\mu}^2 s\right)^{1+p_2} - \beta^2 \hat{\mu}^2 s & \text{if } s \geq \frac{\sigma^2}{\mu^2} \left(1 - \frac{\sigma^2}{\mu^2}\right)
\end{cases} \]

\[ \hat{G}_4(s) = \begin{cases} 
0 & \text{if } \sqrt{s} \leq \frac{\sigma}{\mu w} \left(1 + \tau^{-1}\right) \left(\frac{1}{2} - 1\right) \\
\frac{\sigma^2}{\beta^{2p_2}} \left(\frac{\beta^2 \hat{\mu}^2 s}{(1 + \tau^{-1})^{1+p_2}}\right)^{1+p_2} - \beta^2 (1+p_2) & \text{if } \sqrt{s} \geq \frac{\sigma}{\mu w} \left(1 + \tau^{-1}\right) \left(\frac{1}{2} - 1\right)
\end{cases} \]

\[ \hat{G}_5(s) = \begin{cases} 
0 & \text{if } \sqrt{s} \leq \frac{\sigma}{\mu w} \left(1 + \tau^{-1}\right) \left(\frac{1}{2} - 1\right) \\
\frac{\sigma^2 \alpha^{2p_1}}{\eta} \left(\frac{\hat{\mu}^2 s}{\sqrt{1 + \tau^{-1}}}\right)^{2(p_1+2)} \left[\frac{\beta + \frac{\alpha^2}{\mu^2} \sqrt{s}}{\sqrt{1 + \tau^{-1}}}\right]^{2(1-p_1)} - \beta^2 (1+p_1) & \text{if } p_1 \geq \frac{1}{2}, \sqrt{s} \geq \frac{\sigma}{\mu w} \left(1 + \tau^{-1}\right) \left(\frac{1}{2} - 1\right)
\end{cases} \]

The results from Proposition 2.3.3 and Theorem 2.3.4 can easily be combined to obtain the desired relationship between input and output signals in an \( l_\infty \) type inequality. The following theorem highlight this result and provides an explicit de-
dependence of $e$ on the initial condition of the system and the applied disturbance $w$.

**Theorem 2.3.5.** Consider the dynamical system presented in Section 2.2 and the storage function $V$ defined in (2.39).

Then the following inequality holds:

$$\|e(k)\|^2 \leq (1 - \eta^2) \tilde{\eta}^{2k} V(x(0), \xi(0)) + \Gamma (\|w\|_\infty^2),$$

where

$$\Gamma (s) = \frac{1 - \eta^2}{1 - \tilde{\eta}^2} \tilde{\Gamma} (s) + (\gamma^2 - \mu^2) s$$

(2.44)

**Proof.** From Proposition 2.3.3 we obtain:

$$\|e(k)\|^2 \leq (1 - \eta^2) V(k) + (\gamma^2 - \mu^2) \|w(k)\|^2$$

$$\leq (1 - \eta^2) V(k) + (\gamma^2 - \mu^2) \|w\|_\infty^2,$$

(2.45)

where $V(k) = V(x(k), \xi(k))$ for brevity.

Applying Theorem 2.3.4 we also have:

$$V^+ \leq \tilde{\eta}^2 V + \tilde{\Gamma} (\|w\|_\infty^2)$$

$$\leq \tilde{\eta}^2 V + \tilde{\Gamma} (\|w\|_\infty^2),$$

which, iterated from time 0 to time $k - 1$, yields:

$$V(k) \leq \tilde{\eta}^{2k} V(0) + \tilde{\Gamma} (\|w\|_\infty^2) \sum_{i=0}^{k-1} \tilde{\eta}^{2i}$$

$$\leq \tilde{\eta}^{2k} V(0) + \frac{1}{1 - \tilde{\eta}^2} \tilde{\Gamma} (\|w\|_\infty^2).$$

(2.46)

The claim is then obtained by combining the bounds in (2.45) and (2.46). 

The result of Theorem 2.3.5 shows that the magnitude of the output signal at time $k$ depends both on the initial condition, as measured by $V(x(0), \xi(0))$, and on the amplitude of the applied input signal as measured by $\Gamma (\|w\|_\infty^2)$. By combining
such result with that presented in Lemma 2.3.2 we obtain:

\[ \|e(k)\|^2 \leq (1 - \eta^2)\bar{\eta}^{2k} \phi(\|x(0)\|_p^2) + \Gamma(\|w\|_\infty^2), \]

thus showing that the initial condition on the scaling factor \( \xi(0) \) does not affect the qualitative behavior of the output. Since \( \phi(0) = 0 \), the term \( \Gamma(\|w\|_\infty^2) \) measures the amplitude of the output as a function of the input when the initial condition on the plant is zero as, in this case, we can easily derive:

\[ \|e\|_\infty^2 \leq \Gamma(\|w\|_\infty^2). \]

The function \( \Gamma \) precisely describes how much the input affects the output under zero initial condition and, hence, represents the induced \( l_\infty \) gain of the obtained closed loop. Based on its definition in (2.44), such gain can be described by:

\[ \Gamma(s) = \max \{ \Gamma_1(s), \Gamma_2(s), \Gamma_3(s), \Gamma_4(s), \Gamma_5(s) \}, \]

where

\[ \Gamma_i(s) = \frac{1 - \eta^2}{1 - \bar{\eta}^2} \Gamma_i(s) + (\gamma^2 - \mu^2)s \quad i = 1, \ldots, 5. \]  

(2.47)

By studying the behavior of the different functions \( \Gamma_i \), we can draw some conclusions on the overall gain function \( \Gamma \):

- The function \( \Gamma_1 \) is linear and, in particular, we obtain:

\[ \Gamma_1(s) = \left( \frac{1 - \eta^2}{1 - \bar{\eta}^2} \bar{\mu}^2 + \gamma^2 - \mu^2 \right) s. \]  

(2.48)

Since \( \bar{\eta}^2 = (1 + \tau)\eta^2 + (1 + \tau^{-1})\theta > \eta^2 \) and \( \bar{\mu}^2 = (1 + \tau)\mu^2 > \mu^2 \), the slope of \( \Gamma_1 \) is always greater than \( \gamma^2 \), the slope obtained in the linear case. Moreover, if \( \theta \) and \( \tau \) are carefully selected to be small, such slope can be arbitrarily close to the limit value \( \gamma^2 \).

Notice that \( \Gamma \) will exhibit the same linear slope in the region where \( \Gamma_1 \) is domi-
nant over the remaining $\Gamma_i$.

- As anticipated at the beginning of this chapter, the function $\Gamma$ has an unbounded gain for very small and very large values of its argument $s$. In particular, since $\Gamma_2$ is dominant for small values of $s$, we have:

$$\Gamma(s) \approx s^{1-\rho_1} \quad \text{for } s \to 0.$$  

Similarly, for large values of $s$, one among $\Gamma_3$, $\Gamma_4$ and $\Gamma_5$ becomes dominant and, since all of them have the same asymptotic behavior, we have:

$$\Gamma(s) \approx s^{1+\rho_2} \quad \text{for } s \to \infty.$$  

- It is straightforward to verify that:

$$s \leq \frac{\alpha^2}{\mu^2} \left( \frac{\bar{\sigma}}{\bar{\eta}} \right)^{\frac{2}{\rho_1}} \Leftrightarrow \Gamma_2(s) \geq \Gamma_1(s),$$

$$s \geq \frac{\beta^2}{\bar{\mu}^2} \left( 1 - \frac{\bar{r}_2^2}{\bar{r}^2} \right) \Leftrightarrow \Gamma_3(s) \geq \Gamma_1(s),$$

$$s \leq \frac{\beta^2 \sigma^2}{\lambda_w^2} \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2 \Rightarrow \begin{cases} \Gamma_4(s) \leq \Gamma_1(s) \\ \Gamma_5(s) \leq \Gamma_1(s) \end{cases}.$$  

These conditions prove that, by properly selecting $\alpha$ and $\beta$, it is possible to guarantee that $\Gamma_1$ is dominating over an arbitrarily large range of disturbance amplitudes, thus making $\Gamma$ linear in the same region.

A depiction of the obtained gain $\Gamma$ for a typical choice of the parameters is presented in Figure 2-5. In the next chapter we will show how the involved parameters can be selected to minimize the required minimal rate of the channel while guaranteeing that $\Gamma$ has a prescribed linear slope in a given interval.
2.4 A Proof of Dissipativity

In this section, we will provide a complete proof of Theorem 2.3.4. Together with the result in Proposition 2.3.3, this Theorem implies that the closed loop system obtained when applying the proposed finite rate control scheme is dissipative and satisfies an inequality as the one introduced in (2.29).

This result can be shown by upper bounding the increment over one step of the storage function $V$ defined in (2.39):

$$V^+ - V \leq -(1 - \eta^2) V + \hat{\Gamma} (\|w\|^2)$$

$$\leq -\frac{1 - \eta^2}{1 - \eta^2} (\|e\|^2 - (\gamma^2 - \mu^2)\|w\|^2) + \hat{\Gamma} (\|w\|^2)$$

$$= \frac{1 - \eta^2}{1 - \eta^2} \left( -\|e\|^2 + (\gamma^2 - \mu^2)\|w\|^2 + \frac{1 - \eta^2}{1 - \eta^2} \hat{\Gamma} (\|w\|^2) \right)$$

$$= \frac{1 - \eta^2}{1 - \eta^2} (\hat{\Gamma} (\|w\|^2) - \|e\|^2),$$
which shows that the scaled storage function $\tilde{V} = \frac{1-\eta}{1-\eta^2} V$ satisfies:

$$\tilde{V}^+ - \tilde{V} \leq \Gamma (\|w\|^2) - \|e\|^2,$$

which is exactly the condition given in (2.29) with supply function $s = \Gamma (\|w\|^2) - \|e\|^2$.

Given that the dynamics describing the finite-rate controller can be divided into nine regions, the remaining of this section will prove that the inequality (2.43) holds for each of the eighty one possible transition between regions. Before going into the details of every transition, the following lemma allows us to reduce the number of possible combinations to be considered.

**Lemma 2.4.1.** Consider the dynamical system presented in Section 2.2 and the storage function $V$ defined in (2.39).

Whenever the state moves from any region among $R_4$, $R_5$, $R_8$ and $R_9$ to any region among $R_3$, $R_6$ and $R_7$ in one step, the dissipation inequality in (2.43) yields:

$$V^+ - \tilde{\eta}^2 V \leq 0$$

**Proof.** It is easy to verify that:

$$\min_{(x,\xi)\in R_4\cup R_5\cup R_8\cup R_9} V = \frac{\beta^2 \tilde{\sigma}^2}{\tilde{\eta}^2},$$

$$\max_{(x,\xi)\in R_3\cup R_6\cup R_7} V = \frac{\alpha^2 \tilde{\sigma}^2}{\tilde{\eta}^2}.$$

Combining these observation with the first of conditions (2.14), the statement follows immediately. \qed

### 2.4.1 Transitions Starting from Region $R_1$

The proof of Lemma 2.2.1 shows that the effects of the dead zone in the quantizer are neglectable and the dynamics is similar to that of the robust linear controller. Supported by this observation and with a slight abuse of notation for simplicity
reasons, we will indicate the update law for $x$ in region $R_1$ with

$$x^+ = Ax + B_u \xi \mathcal{Q}_{1-\delta,1+\delta,M_1} \left( \frac{K x}{\xi} \right) + B_u w \triangleq A_1(\Delta)x + B_u w,$$

and we will make use of such notation when proving dissipation inequalities involving region $R_1$.

We will also make extensive use of the following two inequalities:

$$\tilde{r}_1 \xi^+ \leq \tilde{\eta} \|x\|_P \leq \tilde{r}_2 \xi^+, \quad (2.51)$$

which are a consequence of the update law for $\xi$ in (2.17) and the fact that the argument of the quantizer is always in the pseudo-linear zone due to how $R_1$ is defined.

**Transition** $R_1 \rightarrow R_1$

$$V^+ - \tilde{\eta}^2 V = (A_1(\Delta)x + B_u w)^T P (A_1(\Delta)x + B_u w) - \tilde{\eta}^2 x^T P x$$

$$\leq \tilde{\eta}^2 x^T P x + \tilde{\mu}^2 \|w\|^2 - \tilde{\eta}^2 x^T P x$$

$$= \tilde{\Gamma}_1(\|w\|^2)$$

where the upper bound is a direct application of Lemma 2.2.1.

**Transition** $R_1 \rightarrow R_2$

This transition yields a decrease in the storage function identical to that obtained in the transition $R_1 \rightarrow R_1$ since the storage function in $R_1$ and $R_2$ is defined in the same way.

Hence:

$$V^+ - \tilde{\eta}^2 V \leq \tilde{\mu}^2 \|w\|^2 = \tilde{\Gamma}_1(\|w\|^2)$$
Transition $R_1 \rightarrow R_3$

\[
V^+ - \hat{\eta}^2 V = \frac{\delta^2 \alpha_2 \rho_1}{\eta^2} - ((A_1(\Delta)x + B_w w)^TP(A_1(\Delta)x + B_w w))^{1-\rho_1} - \hat{\eta}^2 x^TPx \\
\leq \frac{\delta^2 \alpha_2 \rho_1}{\eta^2} (\hat{\eta}^2 x^TPx + \hat{\mu}^2 \|w\|^2)^{1-\rho_1} - \hat{\eta}^2 x^TPx
\]

(2.52)

Since, by assumption, the system lands in region $R_3$, we can leverage the inequality:

\[
\hat{\eta}^2 x^TPx + \hat{\mu}^2 \|w\|^2 \geq \|A_1(\Delta)x + B_w w\|^2 \geq \bar{r}^2 (\xi^+)^2,
\]

which, together with the upper bound in (2.51), yields an upper bound on the maximum value of $\hat{\eta}^2 x^TPx$:

\[
\hat{\eta}^2 x^TPx \in \left[0, \frac{(\frac{\bar{r}}{\eta})^2}{1 - (\frac{\bar{r}}{\eta})^2} \hat{\mu}^2 \|w\|^2 \right] = E_{13}.
\]

The derivative of (2.52) with respect to $\hat{\eta}^2 x^TPx$ yields:

\[
\frac{d \Delta V}{d (\hat{\eta}^2 x^TPx)} = \frac{\delta^2 \alpha_2 \rho_1}{\eta^2} (1 - \rho_1)(\hat{\eta}^2 x^TPx + \hat{\mu}^2 \|w\|^2)^{-\rho_1} - 1,
\]

from which we can infer that:

- if $\hat{\mu}^2 \|w\|^2 \geq \left(\frac{\delta^2 \alpha_2 \rho_1}{\eta^2}(1-\rho_1)\right)^{\frac{1}{\rho_1}}$ then $\Delta V$ is always decreasing in $E_{13}$ hence its maximum is attained for $\hat{\eta}^2 x^TPx = 0$ and is equal to: $\frac{\delta^2 \alpha_2 \rho_1}{\eta^2} (\hat{\mu}^2 \|w\|^2)^{1-\rho_1}$.

- if $\hat{\mu}^2 \|w\|^2 \leq \left(1 - \left(\frac{\bar{r}}{\eta}\right)^2\right)\left(\frac{\delta^2 \alpha_2 \rho_1}{\eta^2}(1-\rho_1)\right)^{\frac{1}{\rho_1}}$ then $\Delta V$ is always increasing in $E_{13}$ hence its maximum is attained for $\hat{\eta}^2 x^TPx = \left(\frac{\bar{r}}{\eta}\right)^2 \hat{\mu}^2 \|w\|^2$ and is equal to: $\frac{\delta^2 \alpha_2 \rho_1}{\eta^2(1 - \left(\frac{\bar{r}}{\eta}\right)^2)^{1-\rho_1}} (\hat{\mu}^2 \|w\|^2)^{1-\rho_1} - \left(\frac{\bar{r}}{\eta}\right)^2 \hat{\mu}^2 \|w\|^2$.

- if $\left(1 - \left(\frac{\bar{r}}{\eta}\right)^2\right)\left(\frac{\delta^2 \alpha_2 \rho_1}{\eta^2}(1-\rho_1)\right)^{\frac{1}{\rho_1}} \leq \hat{\mu}^2 \|w\|^2 \leq \left(\frac{\delta^2 \alpha_2 \rho_1}{\eta^2}(1-\rho_1)\right)^{\frac{1}{\rho_1}}$ then $\Delta V$ attains its maximum for $\hat{\eta}^2 x^TPx = \left(\frac{\delta^2 \alpha_2 \rho_1}{\eta^2}(1-\rho_1)\right)^{\frac{1}{\rho_1}} - \hat{\mu}^2 \|w\|^2 \in E_{13}$ and it is equal to: $\frac{\rho_1}{1-\rho_1} \left(\frac{\delta^2 \alpha_2 \rho_1}{\eta^2}(1-\rho_1)\right)^{\frac{1}{\rho_1}} + \hat{\mu}^2 \|w\|^2$. 

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These observations allow us to conclude that, in this case, we have:

\[ V^* - \eta^2 V \leq \hat{\Gamma}_2(\|w\|^2) \]

**Transition** \( R_1 \rightarrow R_4 \)

\[
V^* - \eta^2 V = \frac{\sigma^2}{\eta^2 \beta^2 p_2} ((A_1(\Delta)x + B_ww)^T P(A_1(\Delta)x + B_ww))^{1+p_2} - \eta^2 x^T P x \\
\leq \frac{\sigma^2}{\eta^2 \beta^2 p_2} (\eta^2 x^T P x + \mu^2 \|w\|^2)^{1+p_2} - \eta^2 x^T P x
\]

Since, by assumption, the system lands in \( R_4 \), we can enforce the constraints

\[
\begin{cases}
\xi^+ \leq \frac{\beta}{\tau} \\
\|A_1(\Delta)x + B_ww\|_p^2 \geq \beta^2 
\end{cases} \Rightarrow \begin{cases}
\xi^+ \leq \frac{\beta}{\tau} \\
\eta^2 \|x\|_p^2 \geq \beta^2 - \tilde{\mu}^2 \|w\|^2
\end{cases}
\]

which, together with the constraints in (2.51), yield the feasible region represented in Figure 2-6.

As a consequence of the existing constraints, the feasible region is empty, and
hence the transition $R_1 \rightarrow R_4$ is not admissible, if

$$\frac{\beta^2 - \tilde{\mu}^2\|w\|^2}{\tilde{\eta}^2} > \frac{\beta^2 \tilde{\tau}^2}{\tilde{\eta}^2 \tilde{\tau}^2} \Leftrightarrow \|w\|^2 < \frac{\beta^2}{\tilde{\mu}^2} \left(1 - \frac{\tilde{\tau}^2}{\tilde{\tau}^2}ight).$$

In addition we can infer that $\|x\|_P$ is bounded from above and below by

$$\max\left\{0, \frac{\beta^2 - \tilde{\mu}^2\|w\|^2}{\tilde{\eta}^2}\right\} \leq x^TPx \leq \frac{\beta^2 \tilde{\tau}^2}{\tilde{\eta}^2 \tilde{\tau}^2}$$

and, since the obtained bound on the dissipation inequality can be proven to be increasing in that interval, we have:

$$\Delta V \leq -\frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^2} \left(\frac{\beta^2 \tilde{\tau}^2}{\tilde{\tau}^2} + \tilde{\mu}^2\|w\|^2\right)^{1+\rho_2} - \frac{\beta^2 \tilde{\tau}^2}{\tilde{\tau}^2} \text{ if } \|w\|^2 \geq \frac{\beta^2}{\tilde{\mu}^2} \left(1 - \frac{\tilde{\tau}^2}{\tilde{\tau}^2}\right),$$

thus proving that

$$\Delta V \leq \tilde{\Gamma}_3(\|w\|^2).$$

**Transition** $R_1 \rightarrow R_5$

$$V^+ - \tilde{\eta}^2 V = \frac{(A_1(\Delta)x + B_u w)^TP(A_1(\Delta)x + B_u w))^{1+\rho_2}}{(r_2 \xi^+)^{2\rho_2}} - \tilde{\eta}^2 x^TPx \leq \frac{(\tilde{\eta}^2 x^TPx + \tilde{\mu}^2\|w\|^2)^{1+\rho_2}}{(r_2 \xi^+)^{2\rho_2}} - \tilde{\eta}^2 x^TPx$$

By taking into account that the system lands in $R_5$, we can enforce the following two conditions:

$$\begin{cases} 
\xi^+ \geq \frac{\tilde{\beta}}{\tilde{\tau}} \\
\|A_1(\Delta)x + B_u w\|^2 \geq \tilde{\tau}^2(\xi^+)^2 
\end{cases} \Rightarrow \begin{cases} 
\xi^+ \geq \frac{\tilde{\beta}}{\tilde{\tau}} \\
\tilde{\eta}^2 \|x\|^2 \geq \tilde{\tau}^2(\xi^+)^2 - \tilde{\mu}^2\|w\|^2 
\end{cases}$$

which, together with the constraints in (2.51), yield the feasible region represented in Figure 2-7. We would like to point out that such feasible set can be empty if the contribution of the noise $w$ is too small. In particular it is empty, and hence the
transition $R_1 \to R_5$ is not admissible, if
\[
\frac{\beta^2 - \bar{\mu}^2 \|w\|^2}{\eta^2} > \frac{\beta^2 r_2}{\bar{\eta}^2 r_2^2} \iff \|w\|^2 < \frac{\beta^2}{\bar{\mu}^2} \left( 1 - \frac{r_2^2}{r^2} \right).
\]

It is easy to realize that the obtained bound on the increment of the storage function is decreasing in $\xi^+$ and, hence, it is maximized either for $\xi^+ = \frac{\tilde{x}}{\tilde{r}}$ or for $\xi^+ = \frac{\tilde{r}}{\tilde{r}}$, which correspond to the two thicker lines in Figure 2-7.

In the case $\xi^+ = \frac{\tilde{x}}{\tilde{r}}$ we obtain:
\[
V^+ - \tilde{\eta}^2 V \leq \left( \frac{r_2}{r} \right)^{2\rho_2} \left( \tilde{x}^T P x + \tilde{\mu}^2 \|w\|^2 \right)^{1+\rho_2} - \tilde{\eta}^2 x^T P x \\
\leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^2 \rho_2} \left( \beta^2 \frac{r_2^2}{r^2} + \bar{\mu}^2 \|w\|^2 \right)^{1+\rho_2} - \beta^2 \frac{r_2^2}{r^2}
\]

where we recognized that the maximum is obtained for $\|x\|_P = \frac{\beta r_2}{\bar{\eta} r}$ by virtue of Lemma A.0.2.

In the case $\xi^+ = \frac{\tilde{r}}{\tilde{r}}$ we get:
\[
V^+ - \tilde{\eta}^2 V \leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^2 \rho_2} \left( \tilde{x}^T P x + \bar{\mu}^2 \|w\|^2 \right)^{1+\rho_2} - \tilde{x}^T P x,
\]

Figure 2-7: The feasible set in the transition $R_1 \to R_5$
where $\|x\|_P$ is bounded from above and below by:

$$\max \left\{ \frac{\beta^2 r_1}{\eta^2 r^2}, \frac{\beta^2 - \bar{\mu}^2 \|w\|^2}{\bar{\eta}^2} \right\} \leq x^T P x \leq \frac{\beta^2 r_2}{\bar{\eta}^2 r^2}$$

and, since the obtained bound can be proven to be increasing in that interval, it is maximized for $\|x\|_P = \frac{\beta^2 r_2}{\bar{\eta}^2 r^2}$ and yields the same bound as in the previous case.

In conclusion, for the transition $R_1 \rightarrow R_5$, we obtain the following bound:

$$\Delta V \leq \frac{\bar{\sigma}^2}{\bar{\eta}^2 \beta^2 r_2} \left( \frac{\beta^2 r_2}{r^2} + \bar{\mu}^2 \|w\|^2 \right)^{1+\rho_2} - \beta^2 \frac{r_2}{r^2} \left( 1 - \frac{r_2}{r^2} \right),$$

thus proving that

$$\Delta V \leq \tilde{\Gamma}_3(\|w\|^2).$$

**Transition** $R_1 \rightarrow R_6$

\[ V^+ - \tilde{\eta}^2 V = (r_1 \xi^+)^{2\rho_1} ((A_1(\Delta)x + B_w w)^T P (A_1(\Delta)x + B_w w))^{1-\rho_1} - \bar{\eta}^2 x^T P x \]

\[ \leq (r_1 \xi^+)^{2\rho_1} \left( \bar{\eta}^2 x^T P x + \bar{\mu}^2 \|w\|^2 \right)^{1-\rho_1} - \bar{\eta}^2 x^T P x \]

\[ \leq \left( \frac{r_1}{\bar{r}_1} \right)^{2\rho_1} \left( \bar{\eta}^2 x^T P x + \bar{\mu}^2 \|w\|^2 \right)^{1-\rho_1} - \bar{\eta}^2 x^T P x \]

where the last inequality holds in virtue of the lower bound in (2.51).

Since $r_1 \leq \bar{r}_1$, by applying Lemma A.0.3, we obtain:

$$V^+ - \bar{\eta}^2 V \leq \left( \frac{r_1}{\bar{r}_1} \right)^{2\rho_1} (1 - \rho_1) \bar{\mu}^2 \|w\|^2$$

\[ \leq \bar{\mu}^2 \|w\|^2 = \tilde{\Gamma}_1(\|w\|^2) \]

**Transition** $R_1 \rightarrow R_7$

\[ V^+ - \bar{\eta}^2 V = \frac{\bar{\eta}^2 \alpha^{2\rho_1}}{\bar{\eta}^2} ((A_1(\Delta)x + B_w w)^T P (A_1(\Delta)x + B_w w))^{1-\rho_1} - \bar{\eta}^2 x^T P x \]

\[ \leq \frac{\bar{\sigma}^2 \alpha^{2\rho_1}}{\bar{\eta}^2} \left( \bar{\eta}^2 x^T P x + \bar{\mu}^2 \|w\|^2 \right)^{1-\rho_1} - \bar{\eta}^2 x^T P x \]
Since, by assumption, we land in $R_7$, we can use the inequality $\xi^+ \geq \frac{q}{\tau}$ to obtain:

$$\frac{\tilde{r}^2 \alpha^{2p_1}}{\tilde{\eta}^2} = \alpha^{2p_1} \left(\frac{\tau_1}{\tau}\right)^{2p_1} = \left(\frac{\tau_1}{\tau}\right)^{2p_1} \leq (r_1 \xi^+)^{2p_1}.$$  

The dissipation inequality then becomes:

$$V^+ - \tilde{\eta}^2 V \leq \left(\frac{r_1}{\tilde{r}_1}\right)^{2p_1} \left(\tilde{\eta}^2 x^T P x + \tilde{\mu}^2 \|w\|^2\right)^{1-p_1} - \tilde{\eta}^2 x^T P x$$

$$\leq \left(\frac{r_1}{\tilde{r}_1}\right)^{2p_1} \left(\tilde{\eta}^2 x^T P x + \tilde{\mu}^2 \|w\|^2\right)^{1-p_1} - \tilde{\eta}^2 x^T P x$$

where the last inequality holds in virtue of the lower bound in (2.51).

Since $r_1 \leq \tilde{r}_1$, by applying Lemma A.0.3, we finally obtain:

$$V^+ - \tilde{\eta}^2 V \leq \tilde{\mu}^2 \|w\|^2 = \tilde{\Gamma}_1(\|w\|^2)$$

**Transition $R_1 \rightarrow R_8$**

This transition yields a decrease in the Lyapunov function identical to that obtained in the transition $R_1 \rightarrow R_6$ and it can be proved with the same reasoning. Hence,

$$V^+ - \tilde{\eta}^2 V \leq \tilde{\mu}^2 \|w\|^2 = \tilde{\Gamma}_1(\|w\|^2)$$

**Transition $R_1 \rightarrow R_9$**

$$V^+ - \tilde{\eta}^2 V = \tilde{\sigma}^2 \left\{ \left( A_1(\Delta)x + B_w w \right)^T P \left( A_1(\Delta)x + B_w w \right) \right\}^{1+p_2} - \tilde{\eta}^2 x^T P x$$

$$\leq \tilde{\sigma}^2 \left\{ \left( A_1(\Delta)x + B_w w \right)^T P \left( A_1(\Delta)x + B_w w \right) \right\}^{1+p_2} - \tilde{r}_1^2 (\xi^+)^2$$

where the last inequality holds in virtue of the lower bound in (2.51).

Since, by assumption, the system lands in $R_9$, we can leverage the following bound:

$$\| A_1(\Delta)x + B_w w \|_p \leq \Phi(\xi^+)^{\frac{q}{\tilde{r}_1^2 + p_2}}.$$  

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and derive the following inequality:

\[
V^+ - \eta^2 V \leq \frac{\hat{\sigma}^2}{\eta^2 \beta^2 p_2} \phi^{2(1+p_2)}(\xi^+) \phi^{\frac{2p_1(1+p_1)}{p_1+p_2}} - \tilde{r}_1^2(\xi^+)^2
\]

\[
\leq (\xi^+) \phi^{\frac{2p_1(1+p_1)}{p_1+p_2}} \left[ \frac{\hat{\sigma}^2}{\eta^2 \beta^2 p_2} \phi^{2(1+p_2)} - \tilde{r}_1^2 \right] (\xi^+) \phi^{\frac{2p_1(1+p_1)}{p_1+p_2}}
\]

\[
\leq (\xi^+) \phi^{\frac{2p_1(1+p_1)}{p_1+p_2}} \left[ \frac{\hat{\sigma}^2}{\eta^2 \beta^2 p_2} \phi^{2(1+p_2)} - \tilde{r}_1^2 \left( \frac{\beta}{\tau} \right) \right]
\]

where we used the inequality \( \xi^+ \geq \frac{\beta}{\tau} \) because, by assumption, the system lands in \( R_3 \).

We can now conclude

\[
V^+ - \eta^2 V \leq 0,
\]

since the inequality

\[
\frac{\hat{\sigma}^2}{\eta^2 \beta^2 p_2} \phi^{2(1+p_2)} - \tilde{r}_1^2 \left( \frac{\beta}{\tau} \right) \leq 0 \iff \Phi \leq \frac{\beta^{\frac{p_2}{p_1+p_2}}}{\eta^2 \beta^{\frac{2p_1(1+p_1)}{p_1+p_2}} (\frac{\eta}{\tilde{r}_1})^{\frac{1}{p_1+p_2}}}
\]

is implied by our choice of \( \Phi \) according to (2.25).

### 2.4.2 Transitions Starting from Region \( R_2 \)

The bounds obtained when the system is in region \( R_2 \) are the same as those obtained in region \( R_1 \). This is not only due to the definition of the storage function, identical in regions \( R_1 \) and \( R_2 \), but also to the properties of the proposed dynamics. In particular, not only does the bound:

\[
\|x^+\|_p^2 \leq \eta^2 \|x\|_p^2 + \tilde{\mu}^2 \|w\|^2
\]

hold also in region \( R_2 \) due to Lemma 2.2.2, but the update law for \( \xi \) in (2.18) is also such that:

\[
\tilde{r}_1 \xi^+ \leq \tilde{\eta} \|x\|_\tau \leq \tilde{r}_2 \xi^+
\]

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due to the fact that the argument of the quantizer is in the pseudo-linear zone because of how $R_2$ is defined.

Since all the proofs starting from region $R_1$ make use of these constraints together with the definitions of the regions the state lands in, it is easy to check that for region $R_2$ we obtain the same bounds when landing in $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, $R_6$ and $R_7$.

The only substantial difference with region $R_1$ is that the transitions toward regions $R_8$ and $R_9$ are illegal. In fact, from the update law for $\xi$, we obtain:

$$\xi^+ \leq \frac{\beta \tilde{\eta}}{\tilde{r}_1} < \frac{\beta}{r},$$

thus violating the constraint for $\xi$ in the definition of $R_8$ and $R_9$.

### 2.4.3 Transitions Starting from Region $R_3$

In region $R_3$, the update law for $\xi$ is given by:

$$\xi^+ = \frac{\alpha \tilde{\sigma}}{\tilde{r}_2},$$

and, since it is easy to verify that:

$$\frac{\alpha}{r} \leq \frac{\alpha \tilde{\sigma}}{\tilde{r}_2} \leq \frac{\alpha}{\bar{r}},$$

the state at the next time can only belong to regions $R_1$, $R_2$, $R_4$, $R_5$ and $R_6$, where the transition to $R_4$ can happen only if $\frac{\alpha}{r} > \frac{\alpha}{\tilde{r}_2}$ while the transition to $R_5$ can occur only if $\frac{\alpha}{r} \leq \frac{\alpha \tilde{\sigma}}{\tilde{r}_2}$

**Transition $R_3 \rightarrow R_1$**

$$V^+ - \tilde{\eta}^2 V = (Ax + B_w w)^T P (Ax + B_w w) - \alpha^{2 + \rho_1} \tilde{\sigma}^2 (x^T P x)^{1 - \rho_1}$$

$$\leq \tilde{\sigma}^2 x^T P x - \alpha^{2 + \rho_1} \tilde{\sigma}^2 (x^T P x)^{1 - \rho_1} + \tilde{\lambda}_w^2 \|w\|^2$$

Since $\|x\|_p \leq \alpha$, the first two terms in the obtained bound always add up to a
non-positive number, thus we obtain:

\[ V^+ - \tilde{\eta}^2 V \leq \tilde{\lambda}_w^2 \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2) \]

**Transition** \( R_3 \to R_2 \)

This transition yields a decrease in the storage function identical to that obtained in the transition \( R_3 \to R_1 \) since the storage function in \( R_1 \) and \( R_2 \) is defined in the same way.

Hence:

\[ V^+ - \tilde{\eta}^2 V \leq \tilde{\lambda}_w^2 \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2) \]

**Transition** \( R_3 \to R_4 \)

\[
V^+ - \tilde{\eta}^2 V = \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^2 \rho_2} ((Ax + B_w w)^T P(Ax + B_w w))^{1+\rho_2} - (\tilde{\sigma} \alpha)^{2\rho_1}(\tilde{\sigma} x^T P x)^{1-\rho_1} \\
\leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^2 \rho_2} (\tilde{\sigma}^2 x^T P x + \tilde{\lambda}_w^2 \|w\|^2)^{1+\rho_2} - \tilde{\sigma}^2 x^T P x,
\]

where we used the inequality \((\tilde{\sigma} \alpha)^{2\rho_1}(\tilde{\sigma}^2 x^T P x)^{1-\rho_1} \geq \tilde{\sigma}^2 x^T P x\) which holds due to condition \(\|x\|_P \leq \alpha\) which is inherited from the definition of \( R_3 \).

Since the system state lands in \( R_4 \), we can enforce the following constraint:

\[ \|Ax + B_w w\|_P^2 \geq \beta^2 \Rightarrow \tilde{\sigma}^2 x^T P x \geq \beta^2 - \tilde{\lambda}_w^2 \|w\|^2 \]

which, together with the condition \(\|x\|_P \leq \alpha\), yields the feasible region:

\[ \max\{0, \beta^2 - \tilde{\lambda}_w^2 \|w\|^2\} \leq \tilde{\sigma}^2 x^T P x \leq \tilde{\sigma}^2 \alpha^2. \]

Let us point out that the feasible region is empty and, hence, the transition
$R_3 \rightarrow R_4$ does not occur if
\[
\beta^2 - \lambda^2_w \|w\|^2 > \sigma^2 \alpha^2 \iff \|w\|^2 < \frac{\beta^2}{\lambda^2_w} \left(1 - \frac{\alpha^2 \sigma^2}{\beta^2}\right).
\]

Moreover, a simple analysis shows that the obtained bound for the increment in the storage function is always increasing $\forall \sigma^2 x^T P x \geq \max\{0, \beta^2 - \lambda^2_w \|w\|^2\}$.

By virtue of these two observations and that $\alpha^2 \sigma^2 < \frac{\beta^2 \sigma^2}{\tau^2}$ holds whenever a transition $R_3 \rightarrow R_4$ is possible, we obtain the bound:

$$
\Delta V \leq \begin{cases} 
\frac{\sigma^2}{\eta^2 \beta^2 \rho^2} \left(\beta^2 \frac{\sigma^2}{\tau^2} + \lambda^2_w \|w\|^2\right)^{1+\rho^2} - \beta^2 \frac{\sigma^2}{\tau^2} & \text{if } \|w\|^2 \geq \frac{\sigma^2}{\lambda^2_w} \left(1 - \frac{\alpha^2 \sigma^2}{\beta^2}\right) \\
0 & \text{otherwise}
\end{cases}
$$

$$
\leq \begin{cases} 
\frac{\sigma^2}{\eta^2 \beta^2 \rho^2} \left(\beta^2 \frac{\sigma^2}{\tau^2} + \lambda^2_w \|w\|^2\right)^{1+\rho^2} - \beta^2 \frac{\sigma^2}{\tau^2} & \text{if } \|w\|^2 \geq \frac{\sigma^2}{\beta^2} \left(1 - \frac{\tau^2}{\beta^2}\right) \\
0 & \text{otherwise}
\end{cases}
$$

$$
= \tilde{\Gamma}_3(\|w\|^2).
$$

**Transition $R_3 \rightarrow R_5$**

\[
V^+ - \eta^2 V = \frac{(Ax + B_w w)^T P (Ax + B_w w))^{1+\rho^2}}{(r_2 \tilde{\sigma} \alpha)^{2\rho^2}} - (\tilde{\sigma} \alpha)^{2\rho^1} (\tilde{\sigma}^2 x^T P x)^{1-\rho^1}
\leq \left(\frac{\tilde{\tau}_2}{r_2 \tilde{\sigma} \alpha}\right)^{2\rho^2} (\tilde{\sigma}^2 x^T P x + \lambda^2_w \|w\|^2)^{1+\rho^2} - \tilde{\sigma}^2 x^T P x,
\]

where we used the inequality $(\tilde{\sigma} \alpha)^{2\rho^1} (\tilde{\sigma}^2 x^T P x)^{1-\rho^1} \geq \tilde{\sigma}^2 x^T P x$ which holds due to condition $\|x\|_P \leq \alpha$ which is inherited from the definition of $R_3$.

Since the system state lands in $R_5$, we can enforce the following constraint:

$$
\|Ax + B_w w\|^2 \geq \tilde{r}^2 (\tilde{\tau}^2)^2 = \left(\frac{\tilde{r} \tilde{\sigma} \alpha}{\tilde{r}_2}\right)^2 \Rightarrow \tilde{\sigma}^2 x^T P x \geq \left(\frac{\tilde{r} \tilde{\sigma} \alpha}{\tilde{r}_2}\right)^2 - \lambda^2_w \|w\|^2
$$

which, together with the condition $\|x\|_P \leq \alpha$, yields the feasible region:

$$
\max \left\{0, \left(\frac{\tilde{r} \tilde{\sigma} \alpha}{\tilde{r}_2}\right)^2 - \lambda^2_w \|w\|^2\right\} \leq \tilde{\sigma}^2 x^T P x \leq \tilde{\sigma}^2 \alpha^2.
$$

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Let us point out that the feasible region is empty and, hence, the transition $R_3 \rightarrow R_5$ does not occur if

$$\left(\frac{\tilde{r} \tilde{\alpha}}{\tilde{r}_2}\right)^2 - \tilde{\chi}_w^2 \|w\|^2 > \tilde{\sigma}^2 \alpha^2 \Leftrightarrow \|w\|^2 < \frac{\tilde{\sigma}^2 \alpha^2}{\tilde{\chi}_w^2} \left(\frac{\tilde{r}_2^2}{\tilde{r}_2^2} - 1\right).$$

Moreover a simple derivative based analysis shows that the obtained bound for the increment in the storage function is always increasing in the feasible region.

By virtue of these two observations, we obtain the bound:

$$\Delta V \leq \begin{cases} \left(\frac{\tilde{r}_2}{\tilde{r}_2}\right)^{2\rho_2} \left(\tilde{\sigma}^2 \alpha^2 + \tilde{\chi}_w^2 \|w\|^2\right)^{1+\rho_2} - \beta^2 \tilde{r}_2^2 \rho_2 & \text{if } \|w\|^2 \geq \frac{\tilde{\sigma}^2 \alpha^2}{\tilde{\chi}_w^2} \left(\frac{\tilde{r}_2^2}{\tilde{r}_2^2} - 1\right), \\
0 & \text{otherwise} \end{cases}$$

which, by using Lemma A.0.2 and the fact that $\alpha^2 \tilde{\sigma}^2 \geq \beta^2 \tilde{r}_2^2 \rho_2$ holds whenever a transition $R_3 \rightarrow R_5$ is possible, yields:

$$\Delta V \leq \begin{cases} \frac{\tilde{\sigma}^2}{\tilde{r}_2^2 \beta \rho_2} \left(\tilde{\sigma}^2 \tilde{r}_2^2 + \tilde{\chi}_w^2 \|w\|^2\right)^{1+\rho_2} - \beta^2 \tilde{r}_2^2 \rho_2 & \text{if } \|w\|^2 \geq \frac{\tilde{\sigma}^2 \alpha^2}{\tilde{\chi}_w^2} \left(\frac{\tilde{r}_2^2}{\tilde{r}_2^2} - 1\right), \\
0 & \text{otherwise} \end{cases}$$

$$\leq \begin{cases} \frac{\tilde{\sigma}^2}{\tilde{r}_2^2 \beta \rho_2} \left(\tilde{\sigma}^2 \tilde{r}_2^2 + \tilde{\mu}^2 \|w\|^2\right)^{1+\rho_2} - \beta^2 \tilde{r}_2^2 \rho_2 & \text{if } \|w\|^2 \geq \frac{\tilde{\sigma}^2}{\rho_2} \left(1 - \frac{\tilde{r}_2^2}{\tilde{r}_2^2}\right) \\
0 & \text{otherwise} \end{cases}$$

$$= \tilde{\Gamma}_3(\|w\|^2).$$

**Transition** $R_3 \rightarrow R_6$

$$V^+ - \tilde{\eta}^2 V = \left(\frac{\tilde{\sigma} \alpha}{r_1 \tilde{r}_2}\right)^{2\rho_1} ((Ax + B_w w)^T P(Ax + B_w w))^{1-\rho_1} - \alpha^2 \tilde{\sigma}^2 (x^T P x)^{1-\rho_1}$$

$$\leq (\tilde{\sigma} \alpha)^{2\rho_1} \left[\tilde{\sigma}^2 x^T P x\right]^{1-\rho_1} - (\alpha \tilde{\sigma})^{2\rho_1} (\tilde{\sigma}^2 x^T P x)^{1-\rho_1}$$

$$= (\tilde{\sigma} \alpha)^{2\rho_1} \tilde{\chi}_w^2 \|w\|^2 \leq \frac{\tilde{\sigma}^2 \alpha^{2\rho_1}}{\tilde{\eta}^2} \|w\|^2 \leq \tilde{\Gamma}_2(\|w\|^2)$$

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2.4.4 Transitions Starting from Region $R_4$

Since the dynamics in region $R_4$ resets the value of $\xi$ to $\frac{\hat{\xi}}{\epsilon}$, the only regions in which the state can transition are $R_1$, $R_2$, $R_5$ and $R_7$.

**Transition $R_4 \rightarrow R_1$**

\[
V^+ - \eta^2 V = (Ax + B_w w)^T P (Ax + B_w w) - \frac{\delta^2 (x^T P x)^{1+\rho_2}}{\beta^{2\rho_2}} \\
\leq \delta^2 x^T P x - \frac{\delta^2}{\beta^{2\rho_2}} (x^T P x)^{1+\rho_2} + \lambda^2_w \|w\|^2
\]

Since $\|x\|_P \geq \beta$ given that the state belongs to $R_4$, the first two terms in the obtained bound always add up to a non-positive number and thus we obtain:

\[
V^+ - \eta^2 V \leq \lambda^2_w \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2)
\]

**Transition $R_4 \rightarrow R_2$**

This transition yields a decrease in the Lyapunov function identical to that obtained in the transition $R_4 \rightarrow R_1$ as the proof is the same. Hence

\[
V^+ - \eta^2 V \leq \lambda^2_w \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2)
\]

**Transition $R_4 \rightarrow R_5$**

\[
V^+ - \eta^2 V = \frac{((Ax + B_w w)^T P (Ax + B_w w))^{1+\rho_2}}{\beta^{2\rho_2}} - \frac{\delta^2 (x^T P x)^{1+\rho_2}}{\beta^{2\rho_2}} \\
\leq \frac{\delta^2}{\beta^{2\rho_2}} \left[ (\frac{(Ax + B_w w)^T P (Ax + B_w w))^{1+\rho_2}}{\beta^{2(1+\rho_2)}} - (x^T P x)^{1+\rho_2} \right] \\
\leq \frac{\delta^2}{\beta^{2\rho_2}} \left[ (\frac{\|x\|_P + \lambda_w \|w\|)^{2(1+\rho_2)}}{\beta^{2(1+\rho_2)}} - \|x\|_P^{2(1+\rho_2)} \right] \\
= \frac{\delta^2}{\beta^{2\rho_2}} \left[ (\frac{\|x\|_P + \lambda_w \|w\|)^{2(1+\rho_2)}}{(1 + \tau^{-1})^{2(1+\rho_2)}} - \|x\|_P^{2(1+\rho_2)} \right],
\]
where the first inequality holds due to the condition \( r_2 \geq \hat{\sigma}_T \) which is implied by (2.14).

Since we know that \( \|x\|_P \geq \beta \) because the state belongs to \( R_4 \), we can now apply the result of Lemma A.0.1 to the expression in the square brackets, thus obtaining

\[
\Delta V \leq \begin{cases} 
\frac{\bar{\alpha}^2}{\bar{\beta}^2} \frac{(\frac{1}{2}+\|u\|^2)^{1+\rho_2}}{(1+r^{-1})^{1+\rho_2} - 1} & \text{if } \|w\| \geq \frac{\beta \sigma}{\lambda_w} \left( (1 + r^{-1})^{1+\rho_2} - 1 \right) \\
\frac{\bar{\alpha}^2}{\bar{\beta}^2} \frac{(\beta + \frac{\beta \sigma}{\lambda_w} \|u\|)^{2(1+\rho_2)}}{(1+r^{-1})^{1+\rho_2} - \beta^2(1+\rho_2)} & \text{otherwise}
\end{cases}
\]

\[
\leq \tilde{\Gamma}_4(\|w\|^2)
\]

**Transition \( R_4 \rightarrow R_7 \)**

This transition is among those considered in Lemma 2.4.1 and, as such, yields:

\[
V^+ - \eta^2 V \leq 0
\]

**2.4.5 Transitions Starting from Region \( R_5 \)**

From region \( R_5 \) it is possible to reach any other region other than \( R_3 \) and \( R_4 \). This is due to the fact that the update law for \( \xi \) implies \( \xi^+ \geq \xi \) and \( \xi \) is greater than \( \frac{\beta}{\xi} \) in \( R_5 \) while it is less than such quantity in \( R_3 \) and \( R_4 \).

**Transition \( R_5 \rightarrow R_1 \)**

\[
V^+ - \eta^2 V = (Ax + B_uw)^T P(Ax + B_uw) - \eta^2 (x^T P x)^{1+\rho_2} \\
\leq \tilde{\sigma}^2 x^T P x + \tilde{\beta}^2 \|w\|^2 - \tilde{\eta}^2 x^T P x \left( \frac{r_2}{r_2} \right)^{2\rho_2} \\
\leq \tilde{\sigma}^2 x^T P x + \tilde{\beta}^2 \|w\|^2 - \tilde{\eta}^2 x^T P x \left( \frac{r_2}{r_2} \right)^{2\rho_2} \\
= \tilde{\sigma}^2 x^T P x + \tilde{\beta}^2 \|w\|^2 - \tilde{\sigma}^2 x^T P x \leq \tilde{\Gamma}_1(\|w\|^2)
\]

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**Transition** $R_5 \rightarrow R_2$

This transition yields a decrease in the Lyapunov function identical to that obtained in the transition $R_5 \rightarrow R_1$ as the storage function in $R_1$ and $R_2$ is the same. Hence

$$V^+ - \tilde{\eta}^2 V \leq \tilde{\lambda}_w^2 \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2)$$

**Transition** $R_5 \rightarrow R_5$

$$V^+ - \tilde{\eta}^2 V = \frac{((Ax + B_w w)^T P (Ax + B_w w))^{1+\rho_2}}{(r_2 \tilde{\xi})^{2\rho_2}} - \frac{\tilde{\eta}^2 (x^T P x)^{1+\rho_2}}{(r_2 \tilde{\xi})^{2\rho_2}}$$

$$= \frac{\tilde{\eta}^2}{(r_2 \tilde{\xi})^{2\rho_2}} \left[ \frac{((Ax + B_w w)^T P (Ax + B_w w))^{1+\rho_2}}{(\tilde{\sigma}^2)^{1+\rho_2}} - (x^T P x)^{1+\rho_2} \right]$$

$$\leq \frac{\tilde{\eta}^2}{(r_2 \tilde{\xi})^{2\rho_2}} \left[ \frac{((Ax + B_w w)^T P (Ax + B_w w))^{2(1+\rho_2)}}{(1 + \tau^{-1})^{1+\rho_2}} - \|x\|^2_{\tilde{\rho}_2} \right]$$

$$\leq \frac{\tilde{\eta}^2}{(r_2 \tilde{\xi})^{2\rho_2}} \left[ \frac{((Ax + B_w w)^T P (Ax + B_w w))^{2(1+\rho_2)}}{(1 + \tau^{-1})^{1+\rho_2}} - \|x\|^2_{\tilde{\rho}_2} \right]$$

$$= \frac{\tilde{\sigma}^2}{\beta^{2\rho_2}} \left[ \frac{((Ax + B_w w)^T P (Ax + B_w w))^{2(1+\rho_2)}}{(1 + \tau^{-1})^{1+\rho_2}} - \|x\|^2_{\tilde{\rho}_2} \right],$$

where the last upper bound is to be considered valid only when the quantity in the square brackets is positive. Naturally when such quantity is negative we have a zero bound.

Since we know that $\|x\|_{\tilde{\rho}} \geq \beta$ because the state belongs to $R_5$, we can now apply the result of Lemma A.0.1 to the expression in the square brackets, thus obtaining

$$\Delta V \leq \begin{cases} 
0 & \text{if } \|w\| \leq \frac{\beta^2}{\lambda_w} (1 + \tau^{-1})^{\frac{1}{2}} - 1 \\
\frac{\tilde{\sigma}^2}{\beta^{2\rho_2}} \left[ \frac{(\beta + \lambda_w \|w\|)^{2(1+\rho_2)}}{(1 + \tau^{-1})^{1+\rho_2} - \beta^{2(1+\rho_2)}} \right] & \text{if } \|w\| \geq \frac{\beta^2}{\lambda_w} (1 + \tau^{-1})^{\frac{1}{2}} - 1 \\
\frac{\tilde{\sigma}^2}{\beta^{2\rho_2}} \left[ \frac{(\beta + \lambda_w \|w\|)^{2(1+\rho_2)}}{(1 + \tau^{-1})^{1+\rho_2} - \beta^{2(1+\rho_2)}} \right] & \text{otherwise} \\
\tilde{\Gamma}_4(\|w\|^2) & \text{otherwise}
\end{cases}$$

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Transitions \( R_5 \to R_6 \) and \( R_5 \to R_7 \)

These two transitions are among those considered in Lemma 2.4.1 and, as such, they yield:

\[
V^+ - \eta^2 V \leq 0
\]

Transition \( R_5 \to R_8 \)

\[
V^+ - \eta^2 V = \left( r_1 \frac{\tilde{\sigma}^2}{r_2} \right)^{2\rho_1} \left( \frac{\tilde{\sigma}^2}{r_2} \right)^{2(1-\rho_1)} \left( Ax + B_w w \right)^T P (Ax + B_w w) \left( 1 - \rho_1 \right) - \eta^2 \frac{(x^T P x)^{1+\rho_2}}{(r_2 \xi)^{2\rho_2}}
\]

Since, by assumption, the system lands in \( R_8 \), we can leverage the following bound

\[
\| Ax + B_w w \|_p \leq r_2 \xi = \frac{\tilde{\sigma}^2}{r_2} \xi,
\]

to obtain:

\[
V^+ - \eta^2 V \leq \left( r_1 \frac{\tilde{\sigma}^2}{r_2} \right)^{2\rho_1} \left( \frac{\tilde{\sigma}^2}{r_2} \right)^{2(1-\rho_1)} \left( Ax + B_w w \right)^T P (Ax + B_w w) \left( 1 - \rho_1 \right) - \eta^2 \frac{(x^T P x)^{1+\rho_2}}{(r_2 \xi)^{2\rho_2}}
\]

\[
= \tilde{\sigma}^2 r_1^{2\rho_1} \left( \frac{\tilde{\sigma}^2}{r_2} \xi \right)^{2(1-\rho_1)} - \eta^2 \frac{(x^T P x)^{1+\rho_2}}{(r_2 \xi)^{2\rho_2}}
\]

\[
\leq \tilde{\sigma}^2 \left( \frac{\tilde{\sigma}^2}{\eta} \right)^2 \frac{2\rho_1}{r_2} \xi^2 - \frac{2\rho_2}{\xi^2} \tilde{\sigma}^2 \xi^2
\]

\[
= \tilde{\sigma}^2 \xi^2 \left[ \left( \frac{\tilde{\sigma}^2}{\eta} \right)^{2+\rho_2} \frac{2\rho_1}{r_2} \xi^2 - \frac{2\rho_2}{\xi^2} \tilde{\sigma}^2 \xi^2 \right]
\]

\[
\leq 0
\]

where the last inequality holds because of (2.14).
Transition $R_5 \to R_9$

\[
V^+ - \eta^2 V = \tilde{\sigma}^2 \frac{(Ax + B_w)P(Ax + B_w))^{1+\rho_2}}{\eta^2 \beta^{3\rho_2}} - \eta^2 \frac{(x^T P x)^{1+\rho_2}}{(r_2 \xi)^{3\rho_2}}
\]

\[
= \tilde{\sigma}^2 \frac{(Ax + B_w)TP(Ax + B_w))^{1+\rho_2}}{\eta^2 \beta^{3\rho_2}} - \tilde{\sigma}^2 x^T P x \frac{(x^T P x)^{\rho_2}}{(r_2 \xi)^{3\rho_2}}
\]

\[
\leq \tilde{\sigma}^2 \frac{(Ax + B_w)TP(Ax + B_w))^{1+\rho_2}}{\eta^2 \beta^{3\rho_2}} - \tilde{\sigma}^2 \xi^2
\]

Since, by assumption, the system lands in $R_9$, we can leverage the following bound:

\[
\|Ax + B_w\|_p \leq \Phi(\xi^+)^{\frac{\rho_1}{\rho_1 + \rho_2}}
\]

and derive the following inequality:

\[
V^+ - \eta^2 V \leq \frac{\tilde{\sigma}^2}{\eta^2 \beta^{3\rho_2}} \Phi^{2(1+\rho_2)}(\xi^+)^{2\phi(1+\rho_2)}_{\rho_1 + \rho_2} - \eta^2 \xi^2
\]

\[
\leq (\xi^+)^{2\phi(1+\rho_2)}_{\rho_1 + \rho_2} \left[ \frac{\tilde{\sigma}^2}{\eta^2 \beta^{3\rho_2}} \Phi^{2(1+\rho_2)} - \frac{\tilde{\sigma}^2}{\eta^2 \beta^{3\rho_2}} \Phi^{2(1+\rho_2)}_{\rho_1 + \rho_2} \right]
\]

\[
\leq (\xi^+)^{2\phi(1+\rho_2)}_{\rho_1 + \rho_2} \left[ \frac{\tilde{\sigma}^2}{\eta^2 \beta^{3\rho_2}} \Phi^{2(1+\rho_2)} - \frac{\beta}{r} \right]
\]

where we used the inequalities $r_2 \geq \tilde{r}_1$ from (2.14) and $\xi^+ \geq \frac{\beta}{r}$ since, by assumption, the system lands in $R_9$.

We can now conclude

\[
V^+ - \eta^2 V \leq 0,
\]

since the inequality

\[
\frac{\tilde{\sigma}^2}{\eta^2 \beta^{3\rho_2}} \Phi^{2(1+\rho_2)} - \frac{\beta}{r} \left( \frac{\Phi}{\Phi} \right)^{\frac{\rho_1}{\rho_1 + \rho_2}} \leq 0 \Leftrightarrow \Phi \leq \frac{\beta}{r} \left( \frac{\Phi}{\Phi} \right)^{\frac{\rho_1}{\rho_1 + \rho_2}}
\]

is implied by our choice of $\Phi$ according to (2.25).
2.4.6 Transitions Starting from Region $R_6$

In region $R_6$ the dynamics for $\xi$ is such that $\dot{\xi}^+ = \frac{C}{r_1} \xi$. Since $\frac{C}{r_1} \leq 1$ due to (2.14), $\xi$ decreases thus precluding transitions toward regions $R_7$, $R_8$ and $R_9$. Furthermore, the transition toward region $R_5$ is possible only if the parameters are selected such that $\frac{\dot{r}_1}{\dot{r}_5} \leq \alpha$.

**Transition $R_6 \rightarrow R_1$**

\[
V^+ - \tilde{\eta}^2 V = (Ax + B_w)^T P (Ax + B_w) - \tilde{\eta}^2 (r_1 \xi) 2^{\rho_1} (x^T P x) 1^{-\rho_1}
\leq \tilde{\sigma}^2 x^T P x + \tilde{\lambda}_w^2 \|w\|^2 - \tilde{\sigma}^2 (r_1 \xi) 2^{\rho_1} (x^T P x) 1^{-\rho_1}
= \tilde{\sigma}^2 x^T P x \left[ 1 - \left( \frac{x^T P x}{x^T P x} \right)^{\rho_1} \right] + \tilde{\lambda}_w^2 \|w\|^2
\]

Using the first two constraints in the definition of $R_6$ (2.24), we obtain:

\[
V^+ - \tilde{\eta}^2 V \leq \tilde{\lambda}_w^2 \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2)
\]

**Transition $R_6 \rightarrow R_2$**

This transition yields a decrease in the Lyapunov function identical to that obtained in the transition $R_6 \rightarrow R_1$ as the storage function is defined in the same way in $R_1$ and $R_2$. Hence

\[
V^+ - \tilde{\eta}^2 V \leq \tilde{\lambda}_w^2 \|w\|^2 \leq \tilde{\Gamma}_1(\|w\|^2)
\]

**Transition $R_6 \rightarrow R_3$**

\[
V^+ - \tilde{\eta}^2 V = \frac{\tilde{\sigma}^2 \alpha_{2p_1}}{\tilde{\eta}^2} ((Ax + B_w w)^T P (Ax + B_w w)) 1^{-\rho_1} - \tilde{\eta}^2 (r_1 \xi)^2 2^{\rho_1} (x^T P x) 1^{-\rho_1}
\leq \frac{\tilde{\sigma}^2 \alpha_{2p_1}}{\tilde{\eta}^2} (\tilde{\sigma}^2 x^T P x + \tilde{\lambda}_w^2 \|w\|^2) 1^{-\rho_1} - \tilde{\eta}^2 (r_1 \xi)^2 2^{\rho_1} (x^T P x) 1^{-\rho_1}
\leq \frac{\tilde{\sigma}^2 \alpha_{2p_1}}{\tilde{\eta}^2} (\tilde{\sigma}^2 x^T P x + \tilde{\mu}^2 \|w\|^2) 1^{-\rho_1} - \tilde{\sigma}^2 x^T P x,
\]

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where the last inequality is justified by using the bound $\xi \geq \frac{\xi_r}{r}$, valid within region $R_6$, and the definition of $r_1$ in (2.15).

Since, by hypothesis, the system lands in region $R_3$, we can leverage the inequality:

$$\sigma^2 x^T P x + \tilde{\mu}^2 \|w\|^2 \geq \|Ax + B_w w\|^2 \geq \tilde{\sigma}^2 (\xi^+)^2 = \frac{\tilde{\sigma}^2 \xi^2}{r^2} \xi^2,$$

which, together with the constraint imposed by region $R_6$:

$$\sigma^2 x^T P x \leq \sigma^2 \xi^2 \xi^2,$$

yields an upper bound on the maximum value of $\sigma^2 x^T P x$:

$$\sigma^2 x^T P x \leq \left(\frac{\tilde{\sigma}^2}{1 - \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2}}\right) \tilde{\mu}^2 \|w\|^2.$$

Using the inequality $r_1 \leq \tilde{r}_2$, inherited from (2.14), we can relax this bound to finally obtain:

$$\sigma^2 x^T P x \in \left[0, \left(\frac{\tilde{\sigma}^2}{1 - \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2}}\right) \tilde{\mu}^2 \|w\|^2 \right] = E_{63}.$$

From this point we can repeat the same reasoning utilized at the end of the proof relative to the transition from region $R_1$ to $R_3$ to obtain:

$$V^+ - \eta^2 V \leq \tilde{\Gamma}_2 (\|w\|^2).$$

**Transition $R_6 \rightarrow R_4$**

$$V^+ - \eta^2 V = \frac{\sigma^2 ((Ax + B_w w)^T (Ax + B_w w))^{1+\rho_2}}{\tilde{\sigma}^2 (r_1 \xi)^{2\rho_1}} - \eta^2 (\tilde{\sigma}^2 (r_1 \xi)^{2\rho_1}) x^T P x^{1-\rho_1}$$

$$\leq \frac{\sigma^2}{\tilde{\sigma}^2 \tilde{\mu}^2} (\sigma^2 x^T P x + \tilde{\mu}^2 \|w\|^2)^{1+\rho_2} - \sigma^2 (\tilde{\sigma}^2 (r_1 \xi)^{2\rho_1}) x^T P x^{1-\rho_1}$$

$$\leq \frac{\sigma^2}{\tilde{\sigma}^2 \tilde{\mu}^2} (\sigma^2 x^T P x + \tilde{\mu}^2 \|w\|^2)^{1+\rho_2} - \sigma^2 (\tilde{\sigma}^2 (r_1 \xi)^{2\rho_1}) x^T P x^{1-\rho_1}$$

By taking into account that the system lands in $R_4$, we can enforce the following
two conditions on the state variables:

\[
\begin{align*}
\xi^+ & \leq \frac{\beta}{\gamma} \\
\|Ax + Bu\|^2 & \geq \beta^2 \\
\sigma^2 \|x\|^2 & \geq \beta^2 - \tilde{\mu}^2 \|w\|^2
\end{align*}
\]

which, together with the definition of region \( R_6 \), restrict the feasible set to that represented in Figure 2-8. We would like to point out that, depending on the magnitude of the noise, the feasible set can be empty. In particular, it is empty and hence the transition \( R_6 \to R_4 \) is not admissible, if

\[
\frac{\beta^2 - \tilde{\mu}^2 \|w\|^2}{\sigma^2} \geq \frac{\beta^2 \eta_1^2}{r \sigma^2} \iff \|w\| \leq \frac{\beta}{\tilde{\mu}} \sqrt{1 - \left(\frac{\eta_1}{r}\right)^2}.
\]

It is easy to observe that the obtained bound on the increment of the storage function is decreasing in \( \xi \) and, hence, it is maximized for \( \xi = \frac{\|x\|^2}{r} \), which corresponds to the thicker line in Figure 2-8. Under such restriction, we obtain:

\[
V^+ - \eta^2 V \leq \frac{\tilde{\sigma}^2}{\eta^2 \beta_2 \eta_2} \left( \sigma^2 x^T P x + \tilde{\mu}^2 \|w\|^2 \right)^{1+p_1} \sigma^2 x^T P x,
\]

which, by means of a simple derivative in \( \sigma^2 \|x\|^2 \), can be shown to be always increasing for \( \sigma^2 \|x\|^2 \geq \beta^2 - \tilde{\mu}^2 \|w\|^2 \) and can therefore be bounded using the constraint \( \|x\|^2 \leq \frac{\sigma_1}{\sigma_2} \leq \frac{\sigma_2}{\gamma} \) valid in the considered feasible region:
\[ \Delta V \leq \begin{cases} 0 & \text{if } \|w\| < \frac{\beta}{\mu} \sqrt{1 - \left(\frac{\xi}{\nu}\right)^2} \\ \frac{\beta^2 r_2^2}{\nu^2 \beta^2 r_2^2} \left( \beta^2 r_2^2 + \tilde{r}^2 \|w\|^2 \right)^{1+\rho_2} - \beta^2 r_2^2 & \text{if } \|w\| \geq \frac{\beta}{\mu} \sqrt{1 - \left(\frac{\xi}{\nu}\right)^2} \end{cases} \]

Transition \( R_6 \rightarrow R_5 \)

\[ V^+ - \tilde{\eta}^2 V = \frac{((Ax + B_w w)^T P (Ax + B_w w))^{1+\rho_2}}{(r_2 \tilde{r} \xi)^{2\rho_2}} \leq \frac{r_1^2 (r_1 \xi)^{2\rho_1} (x^TPx)^{1-\rho_1}}{(\tilde{\nu} \xi)^{2\rho_2}} \]

By taking into account that the system lands in \( R_5 \), we can enforce the following two conditions on the state variables:

\[ \begin{cases} \xi^+ \geq \frac{\beta}{r} \\ \|Ax + B_w w\|_P^2 \geq \tilde{r}^2 (\xi^+)^2 \end{cases} \Rightarrow \begin{cases} \xi \geq \frac{\tilde{p}_1}{\tilde{r}^2} \\ \tilde{\sigma}^2 \|x\|^2 P \geq \left( \frac{\tilde{p}_1}{\tilde{r}_1} \right)^2 \xi^2 - \tilde{r}^2 \|w\|^2 \end{cases} \]

which, together with the definition of region \( R_6 \), restrict the feasible set to that
represented in Figure 2-9. We would like to point out that the inequality \( \frac{\beta_1}{r^{3/2}} \leq \frac{a}{r} \) must be satisfied for this transition to be allowed and that the feasible set can be empty if the contribution of the noise \( w \) is too small. In particular, it is empty and hence the transition \( R_6 \to R_5 \) is not admissible, if

\[
\frac{\beta^2 - \mu^2 \|w\|^2}{\hat{\sigma}^2} > \frac{\beta^2 r_1^2}{\hat{\sigma}^2} \Leftrightarrow \|w\| \leq \frac{\beta}{\hat{\mu}} \sqrt{1 - \left( \frac{r_1}{r} \right)^2}.
\]

It is easy to observe that the obtained bound on the increment of the storage function is decreasing in \( \xi \) and, hence, it is maximized either for \( \xi = \frac{\|x\|}{x} \) or for \( \xi = \frac{\beta_1}{r^{3/2}} \) corresponding to the two thicker lines in Figure 2-9.

In the case \( \xi = \frac{\|x\|}{x} \) the increment in the storage function becomes:

\[
V^+ - \tilde{\eta}_k V \leq \left( \frac{r_1}{\hat{\sigma} r_2} \right)^{2\rho_2} \left( \tilde{\sigma}^2 P x + \tilde{\mu}^2 \|w\|^2 \right)^{1 + \rho_2} - \tilde{\sigma}^2 (\|x\|)_{\hat{\sigma}^2} (x^T P x)^{1 - \rho_1} = \left( \frac{r_1}{r_2} \right)^{2\rho_2} \left( \tilde{\sigma}^2 P x + \tilde{\mu}^2 \|w\|^2 \right)^{1 + \rho_2} - \tilde{\sigma}^2 P x
\]

which, by virtue of Lemma A.0.2, is maximized when \( \tilde{\sigma}^2 P x = \beta^2 \frac{r_1^2}{r^2} \) and yields:

\[
V^+ - \tilde{\eta}_k V \leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^{2\rho_2}} \left( \beta^2 \frac{r_1^2}{r^2} + \tilde{\mu}^2 \|w\|^2 \right)^{1 + \rho_2} - \beta^2 \frac{r_1^2}{r^2} \tilde{\eta}_k^2 V \leq \left( \frac{\beta_1}{\hat{\sigma} r^{3/2}} \right)^{2\rho_1} (\tilde{\sigma}^2 P x)^{1 - \rho_1} = \left( \frac{r_1}{\hat{\sigma} r^{3/2}} \right)^{2\rho_1} (\tilde{\sigma}^2 P x)^{1 - \rho_1} (2.53)
\]

In the case \( \xi = \frac{\beta_1}{r^{3/2}} \) the increment in the storage function becomes:

\[
V^+ - \tilde{\eta}_k V \leq \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^{2\rho_2}} \left( \tilde{\sigma}^2 P x + \tilde{\mu}^2 \|w\|^2 \right)^{1 + \rho_2} - \tilde{\sigma}^2 P x = \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^{2\rho_2}} \left( \tilde{\sigma}^2 P x + \tilde{\mu}^2 \|w\|^2 \right)^{1 + \rho_2} - \tilde{\sigma}^2 P x,
\]

where the last inequality is justified since \( \|x\|_{\hat{\sigma}^2} \leq \frac{\beta_1}{r^{3/2}} \). As in transition \( R_6 \to R_4 \), such bound is maximized for \( \|x\|_{\hat{\sigma}^2} = \frac{\beta_1}{r^{3/2}} \) and yields the same result as in (2.53).

In conclusion, we obtain the following bound:
\[ \Delta V \leq \begin{cases} 
0 & \text{if } \|w\| < \frac{\tilde{\sigma}}{\mu} \sqrt{1 - \left( \frac{r_1}{r_2} \right)^2} \\
\frac{\tilde{\sigma}^2}{\eta^2 \beta^2 \rho^2} \left( \beta^2 \frac{r_1^2}{r_2} + \tilde{\mu}^2 \|w\|^2 \right)^{1+\rho_2} - \beta^2 \frac{r_1^2}{r_2} & \text{if } \|w\| \geq \frac{\tilde{\sigma}}{\mu} \sqrt{1 - \left( \frac{r_1}{r_2} \right)^2} 
\end{cases} \]

where the second inequality is justified since the expression

\[ \frac{\tilde{\sigma}^2}{\eta^2 \beta^2 \rho^2} \left( y + \tilde{\mu}^2 \|w\|^2 \right)^{1+\rho_2} - y \]

is monotone increasing in \( y \geq \beta^2 \frac{r_1^2}{r_2} \) whenever \( \|w\|^2 \geq \frac{\tilde{\sigma}^2}{\mu^2} \left( 1 - \frac{r_1^2}{r_2^2} \right) \).

**Transition** \( R_6 \rightarrow R_6 \)

\[ V^+ - \tilde{\eta}^2 V = \left( r_1 \frac{\tilde{\sigma} \xi}{r_2} \right)^{2\rho_1} \left( (Ax + B_ww)^T P (Ax + B_ww) \right)^{1-\rho_1} - \tilde{\eta}^2 \left( r_1 \xi \right)^{2\rho_1} \left( x^T P x \right)^{1-\rho_1} \]

\[ \leq \frac{\tilde{\eta}^2}{\tilde{\sigma}^2 (1-\rho_1)} \left( r_1 \xi \right)^{2\rho_1} \left( \tilde{\sigma}^2 x^T P x + \tilde{\lambda}_w^2 \|w\|^2 \right)^{1-\rho_1} - \tilde{\eta}^2 \left( r_1 \xi \right)^{2\rho_1} \left( x^T P x \right)^{1-\rho_1} \]

\[ \leq \frac{\tilde{\eta}^2}{\tilde{\sigma}^2 (1-\rho_1)} \left( r_1 \xi \right)^{2\rho_1} \left( \tilde{\lambda}_w^2 \|w\|^2 \right)^{1-\rho_1} \]

\[ \leq \frac{\tilde{\eta}^2}{\tilde{\sigma}^2 (1-\rho_1)} \left( r_1 \xi \right)^{2\rho_1} \left( \tilde{\lambda}_w^2 \|w\|^2 \right)^{1-\rho_1} \]

\[ \leq \left( \tilde{\sigma} \alpha \right)^{2\rho_1} \left( \tilde{\lambda}_w^2 \|w\|^2 \right)^{1-\rho_1} \]

\[ \leq \frac{\tilde{\sigma}^2 \alpha^{2\rho_1}}{\tilde{\eta}^2} \left( \tilde{\mu}^2 \|w\|^2 \right)^{1-\rho_1} \leq \tilde{\Gamma}_2(\|w\|^2) \]
2.4.7 Transitions Starting from Region $R_7$

The bounds obtained when the systems is in region $R_7$ are the same as those obtained in region $R_3$ and can be proven in the same way. This is not surprising since the two regions have the same dynamics (hence they are really coded with the same symbol), the same storage function and the bounds derived for $R_3$ only took advantage of the bound $\|x\|_P \leq \alpha$ which is true for $R_7$ as well.

2.4.8 Transitions Starting from Region $R_8$

In region $R_8$ the dynamics for $\xi$ is such that $\xi^+ = \frac{\beta}{r_1} \xi$. It is easy to see that the transitions toward $R_3$ and $R_4$ are illegal since those regions are characterized by $\xi \leq \frac{\beta}{r}$ while using the dynamics for $\xi$ and the condition $\xi \geq \frac{\beta}{r}$, valid within $R_8$, we obtain:

$$\frac{r \beta}{r_1} \frac{\beta}{r} \geq \frac{\beta}{r} \Leftrightarrow \frac{\beta}{r_1} \geq 1$$

which is clearly true.

**Transition $R_8 \rightarrow R_1$**

$$V^+ - \tilde{\eta}^2 V = (Ax + B_w w)^T P (Ax + B_w w) - \tilde{\eta}^2 (r_1 \xi)^{2\rho_1} (x^T P x)^{1-\rho_1}$$

$$\leq \sigma^2 x^T P x + \overline{\lambda}_w \|w\|^2 - \sigma^2 (r_1 \xi)^{2\rho_1} (x^T P x)^{1-\rho_1}$$

$$= \sigma^2 x^T P x \left[ 1 - \left( \frac{L \xi^2}{x^T P x} \right)^{\rho_1} \right] + \overline{\lambda}_w \|w\|^2$$

Using the first two constraints in the definition of $R_8$ (2.24), we obtain:

$$V^+ - \tilde{\eta}^2 V \leq \overline{\lambda}_w \|w\|^2 \leq \tilde{\Gamma}_1 (\|w\|^2)$$

**Transition $R_8 \rightarrow R_2$**

This transition yields a decrease in the Lyapunov function identical to that obtained in the transition $R_8 \rightarrow R_1$ since the storage function is defined in the same way in $R_1$ and $R_2$. Hence
Transition $R_8 \rightarrow R_5$

\[ V^+ - \tilde{\eta}^2 V \leq \frac{\lambda^2_v \|w\|^2}{\tilde{\Gamma}_1 (\|w\|^2)} \]

The obtained bound is decreasing in $\xi$ and, since the state is in region $R_8$, it can be maximized for $\xi = \frac{\|x\|_P}{x}$ to obtain:

\[ V^+ - \tilde{\eta}^2 V \leq \left( \frac{\tilde{\sigma}^2 x^T P x + \tilde{\mu}^2 \|w\|^2}{\tilde{\sigma}^2 x^T P x} \right)^{1+p_2} - \tilde{\eta}^2 (r_1 \xi)^{2p_1} (x^T P x)^{1-p_1} \]

Since $r_1 \leq r_2$, Lemma A.0.2 guarantees that the obtained bound is non-increasing in $\tilde{\sigma}^2 x^T P x$. Therefore, by using the inequality $\tilde{\sigma}^2 x^T P x \geq \beta^2 \frac{r_2^2}{r_1^2}$, valid within region $R_8$, we obtain the bound:

\[ V^+ - \tilde{\eta}^2 V \leq \left( \frac{r_1}{r_2} \right)^{2p_2} \left( \frac{\beta^2 \frac{r_2^2}{r_1^2} + \tilde{\mu}^2 \|w\|^2}{\beta^2 \frac{r_2^2}{r_1^2}} \right)^{1+p_2} - \beta^2 \frac{r_2^2}{r_1^2} \tilde{\sigma}^2 \frac{r_2^2}{r_1^2} \tilde{\mu}^2 \|w\|^2 \]

Finally, we will prove that the considered transition is not valid if $\|w\|$ is too small. By taking into account that the system lands in $R_5$, we can enforce the following two
conditions on the state variables:

\[
\begin{cases}
\xi^+ \geq \frac{\beta}{r} \\
\|Ax + B_w w\|^2 \geq \tilde{\mu}^2 (\xi^+)^2 
\end{cases}
\Rightarrow \begin{cases}
\tilde{\sigma}^2 \|x\|^2 \geq \left(\frac{\sigma}{r_1}\right)^2 \xi^2 - \tilde{\mu}^2 \|w\|^2 
\end{cases},
\]

which, together with the definition of region \(R_s\), restrict the feasible set to that represented in Figure 2-10. Such set is empty, and hence the transition \(R_s \rightarrow R_5\) is not admissible, if

\[
\frac{\beta^2 - \tilde{\mu}^2 \|w\|^2}{\tilde{\sigma}^2} > \frac{\beta^2 r_1^2}{\tilde{\sigma}^2} \Leftrightarrow \|w\| \leq \frac{\beta}{\tilde{\mu}} \sqrt{1 - \left(\frac{r_1}{r}\right)^2}.
\]

Thus we conclude that:

\[
\Delta V \leq \begin{cases}
0 & \text{if } \|w\| < \frac{\beta}{\tilde{\mu}} \sqrt{1 - \left(\frac{r_1}{r}\right)^2} \\
\frac{\tilde{\sigma}^2}{r^2} \left(\beta^2 r_1^2 + \tilde{\mu}^2 \|w\|^2\right)^{1 + \rho_2} - \beta^2 r_1^2 & \text{if } \|w\| \geq \frac{\beta}{\tilde{\mu}} \sqrt{1 - \left(\frac{r_1}{r}\right)^2}
\end{cases}
\]

\[
\leq \begin{cases}
0 & \text{if } \|w\| < \frac{\beta}{\tilde{\mu}} \sqrt{1 - \left(\frac{r_1}{r}\right)^2} \\
\frac{\tilde{\sigma}^2}{r^2} \left(\beta^2 r_1^2 + \tilde{\mu}^2 \|w\|^2\right)^{1 + \rho_2} - \beta^2 r_1^2 & \text{if } \|w\| \geq \frac{\beta}{\tilde{\mu}} \sqrt{1 - \left(\frac{r_1}{r}\right)^2}
\end{cases}
\]

\[= \tilde{\Gamma}_3(\|w\|^2)\]
Transitions $R_6 \rightarrow R_6$ and $R_8 \rightarrow R_7$

These two transitions are among those considered in Remark 2.4.1 and, as such, they yield:

$$V^+ - \hat{\eta}^2 V \leq 0$$

**Transition $R_8 \rightarrow R_8$**

$$\Delta V = \left( \frac{\sigma T_1}{r_1 \xi} \right)^{2\rho_1} \left( (Ax + B_ww)^T P (Ax + B_ww) \right)^{1-\rho_1} - \hat{\eta}^2 (r_1 \xi)^{2\rho_1} (x^T Px)^{1-\rho_1}$$

$$\leq \hat{\eta}^2 (r_1 \xi)^{2\rho_1} \left[ \left( \frac{\sigma \|x\|_P + \lambda_w \|w\|}{\sqrt{1+\tau^{-1}}} \right)^{2(1-\rho_1)} - \|x\|_P^{2(1-\rho_1)} \right].$$

Notice that the quantity in the square brackets is positive if and only if

$$\|x\|_P < \frac{\lambda_w \|w\|}{\sqrt{1+\tau^{-1}} - 1},$$

and, given that $\|x\|_P \geq \beta$ in region $R_8$, we have $\Delta V \leq 0$ whenever

$$\frac{\lambda_w \|w\|}{\sqrt{1+\tau^{-1}} - 1} \leq \beta \Leftrightarrow \|w\| \leq \frac{\beta \sigma}{\lambda_w} \left( \sqrt{1+\tau^{-1}} - 1 \right).$$

For the remaining values of $\|w\|$ we can bound the increment of the storage function as follows:

$$\Delta V \leq \frac{\hat{\eta}^2 r_1^{2\rho_1} \|x\|_P^{2(\rho_1+\rho_2)}}{\Phi^{2(\rho_1+\rho_2)}} \left[ \left( \frac{\|x\|_P + \lambda_w \|w\|}{\sqrt{1+\tau^{-1}}} \right)^{2(1-\rho_1)} - \|x\|_P^{2(1-\rho_1)} \right]$$

$$\leq \frac{\hat{\eta}^2 r_1^{2\rho_1} \lambda_w \|w\|}{(\Phi(\sqrt{1+\tau^{-1}} - 1))^{2(\rho_1+\rho_2)}} \left[ \left( \frac{\|x\|_P + \lambda_w \|w\|}{\sqrt{1+\tau^{-1}}} \right)^{2(1-\rho_1)} - \|x\|_P^{2(1-\rho_1)} \right],$$

where the first inequality is obtained by maximizing $\xi$ according to the constraint
\[ \| x \|_p \geq \Phi^{\frac{-\rho_1}{\rho_1 + \rho_2}} \] defining region \( R_8 \), while the second inequality is obtained by upper bounding \( \| x \|_p \) with the expression in (2.54) since the maximum is reached before such point given that the expression in the square brackets is negative afterwards.

In order to maximize the obtained bound on \( \Delta V \) for \( \| x \|_p \geq \beta \), let us consider the derivative of the expression in the square bracket with respect to \( \| x \|_p \). Since we obtain:

\[
2(1 - \rho_1) \left[ \frac{1}{(1 + \tau^{-1})^{\rho_1 + \rho_2}} \left( \| x \|_p + \frac{\lambda_w}{\sigma} \| w \| \right)^{1 - \rho_1} - \| x \|_p^{1 - \rho_1} \right],
\]

it is clear that we must distinguish between two scenarios. If \( \rho_1 \geq \frac{1}{2} \) such derivative is always negative and \( \Delta V \) is maximized for \( \| x \|_p = \beta \) thus yielding:

\[
\Delta V \leq -\eta^2 \tau_1^{2\rho_1} \left( \frac{\lambda_w \| w \|}{\sigma} \right)^{2(\rho_1 + \rho_2)} \left[ \left( \frac{\beta + \lambda_w \| w \|}{\sqrt{1 + \tau^{-1}}} \right)^{2(1 - \rho_1)} - \beta^{2(1 - \rho_1)} \right].
\]

If \( \rho_1 < \frac{1}{2} \) such derivative is positive up to

\[
\| x \|_p = \frac{\lambda_w \| w \|}{(1 + \tau^{-1})^{1 - \rho_1} - 1}
\]

and negative afterwards. Since the position of the maximum is greater than \( \beta \) when

\[
\frac{\lambda_w \| w \|}{(1 + \tau^{-1})^{1 - \rho_1} - 1} \geq \beta \iff \| w \| \geq \frac{\beta \sigma}{\lambda_w} (1 + \tau^{-1})^{\frac{1}{1 - \rho_1} - 1},
\]

for those values of \( \| w \| \), the bound on \( \Delta V \) is maximized for \( \| x \|_p = \| x \|_p^\ast \) and yields:

\[
\Delta V \leq -\eta^2 \tau_1^{2\rho_1} \left( \frac{\lambda_w \| w \|}{\sigma} \right)^{1 + \rho_2} \left[ \frac{\beta \sigma}{\lambda_w} (1 + \tau^{-1})^{\frac{1}{1 - \rho_1} - 1} \right]^{1 - \rho_1},
\]

while for the remaining values of \( \| w \| \) it is maximized for \( \| x \|_p = \beta \) and yields, once

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From these considerations, we conclude that:

$$\Delta V \leq \tilde{\Gamma}_5(\|w\|^2)$$

**Transition** $R_8 \to R_9$

$$V^+ - \tilde{\eta}^3 V = \tilde{\sigma}^2 \left( (Ax + Bw)^T P (Ax + Bw) \right)^{1+\rho_2} \frac{\left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{\rho_1+\rho_2}}{\tilde{\eta}^2 \beta^{2(p_2)}} - \tilde{\eta}^2 (r_1 \xi)^{2\rho_1} (x^T P x)^{1-\rho_1}.$$

Since, by assumption, $x \in R_8$ and $Ax + Bw \in R_9$, we can use the third constraint defining $R_8$ and $R_9$ to bound from above and below, respectively, the first and second terms in (2.55), thus achieving:

$$V^+ - \tilde{\eta}^3 V \leq \tilde{\sigma}^2 \left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{\rho_1+\rho_2} - \tilde{\eta}^2 (r_1 \xi)^{2\rho_1} \left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{1-\rho_1}$$

$$= \frac{\tilde{\sigma}^2}{\tilde{\eta}^2 \beta^{2(p_2)}} \Phi^{2(1+\rho_2)} \left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{\rho_1+\rho_2} - \tilde{\eta}^2 (r_1 \xi)^{2\rho_1} \Phi^{2(1-\rho_1)} \left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{1-\rho_1}$$

$$= \Phi^{2(1-\rho_1)} \xi^{2(p_1+\rho_2)} \left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{\rho_1+\rho_2}$$

$$\leq 0$$

where the last bound is justified since the inequality

$$\frac{\tilde{\sigma}^2 \Phi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} - \tilde{\eta}^2 (r_1 \xi)^{2\rho_1} \leq 0 \Leftrightarrow \Phi \leq \frac{\beta^{\rho_2}}{\tilde{\eta}^2 (r_1 \xi)^{2\rho_1}} \left( \frac{\tilde{\sigma}^2 \Phi \xi^{2(p_1+\rho_2)} r_1}{r_1 + \rho_2} \right)^{\rho_1+\rho_2}$$

is implied by our choice of $\Phi$ according to (2.25).
2.4.9 Transitions Starting from Region $R_9$

The bounds obtained when the system is in region $R_9$ are the same as those obtained in region $R_4$ and can be proven in the same way. This is not surprising since the two regions have the same dynamics (hence they are really coded with the same symbol), the same storage function and the bounds derived for $R_4$ only took advantage of the bound $\|x\|_p \geq \beta$ which is true for $R_9$ as well.
Chapter 3

Parameter Optimization and Examples

Standard optimal control techniques for linear systems characterize the performances of a given controller via the induced $l_p$ norm of the corresponding closed loop system. Since the norm is a scalar quantity, an optimal controller is then computed in order to minimize it. As mentioned in the previous chapters, when a finite rate channel limits the information flow in the closed loop, a characterization based on the induced norm is not feasible. In this situation it is still possible to obtain a finite $l_p$ gain for the closed loop system, but such gain cannot be bounded via any linear function and, therefore, does not lend itself to any natural optimization.

To overcome this intrinsic limitation, this chapter will introduce a set of performance specifications that are meaningful when dealing with control via finite rate channels. Moreover, we will show that the algorithm proposed in Chapter 2 can be designed to satisfy such conditions and its parameters can be further optimized to minimize the requirement on the communication channel.

The obtained optimization problem, albeit non-convex, is still tractable due its low dimensionality regardless of the size of the plant. The effectiveness of the algorithms designed by solving this optimization problem is then shown for some typical plants taken from the literature.
3.1 Performances Specifications

Let us consider the problem of controlling a given plant

\[
\begin{align*}
\Sigma : & \quad x^+ = Ax + B_u u + B_w w \\
& \quad e = C_e x + D_{eu} u + D_{ew} w \\
& \quad y = x
\end{align*}
\]

via a finite rate channel in a feedback configuration like the one represented in Figure 2-2.

Assume that an Encoder/Decoder pair has been designed so that the closed loop is stable and, when starting from a zero initial condition, the regulated output \( e \) is bounded by the input signal \( w \) according to

\[
\|e\|_\infty^2 \leq \Gamma(\|w\|_\infty^2).
\]

In this section we will specify what conditions can be enforced on the nonlinear gain \( \Gamma \) so that they can be achieved by using the Encoder/Decoder pair described in Chapter 2.

Before defining what performances we can expect in terms \( \Gamma \), let us point out that no controller subject to a communication constraint can outperform the optimal controller designed without such constraint. By virtue of this simple observation, the performances of the optimal \( l_1 \) controller as defined in Section 1.1 constitutes a trivial lower bound to any controller designed while accommodating a finite rate channel in the feedback loop. Moreover, since the algorithm presented in Chapter 2 relies on a class of controllers achieving suboptimal performances \( \Gamma(\|w\|_\infty^2) = \gamma^2 \|w\|_\infty^2 \) even in the linear setting, it is expected that the same gain cannot be outperformed by our proposed scheme.

The suboptimal gain \( \gamma \) can be computed by solving the following optimization
Notice that the problem in (3.1) is a slight variation of the star norm design problem defined in (1.15), with the only difference the addition of the last matrix inequality. Such inequality has been incorporated to comply with the requirements of proposition (2.3.3) which, in turn, was used to obtain the induced \( l_\infty \) gain of the proposed control scheme.

*Remark 5.* We would like to point out that, when \( D_{eu} = 0 \), the problems in (3.1) and (1.15) are equivalent and yield the same value for the star norm. If \( D_{eu} \neq 0 \) then (3.1) yields a solution not better than the one in (1.15) and there is, in general, a further gap between the optimal \( l_1 \) controller and the one obtained by solving (3.1).

Keeping in mind that the performances of the algorithm proposed in Chapter 2 cannot achieve the linear bound \( \gamma_* \), and that according to [40] the finite rate channel causes the attainable \( \Gamma \) to have an unbounded rate at 0 and \( \infty \), we propose the
following conditions for the closed loop gain $\Gamma$:

\[
\begin{align*}
\Gamma(s) &\leq \gamma_r^2 s \quad \forall s \in [a^2, b^2] \\
\lim_{s \to 0} \frac{\log \Gamma(s)}{\log s} &\to 1 - \rho_1 \\
\lim_{s \to \infty} \frac{\log \Gamma(s)}{\log s} &\to 1 + \rho_2
\end{align*}
\] (3.2)

where $a < b$, $\gamma_r > \gamma_*$, $\rho_1 \in (0, 1)$ and $\rho_2 > 0$ are given parameters.

The first condition in (3.2) specifies a range of input amplitudes for which the closed loop system should behave linearly; the interval $[a^2, b^2]$ and the corresponding gain $\gamma_r$ will be chosen according to the particular control application and should take into consideration any previous knowledge of the input $w$. The second and third conditions in (3.2) allow the designer to specify the asymptotic rates for $\Gamma$. This information is useful in situations where there is little knowledge about $w$ and a worst case scenario needs to be evaluated or when the designed system is further interconnected.
with another nonlinear operator in a feedback configuration and stability is to be proven using the small gain theorem\(^1\). A graphical depiction of how the constraints in (3.2) affect the closed loop gain \(\Gamma\) is represented in Figure 3-1. Notice that, since the imposed conditions only specify the asymptotic rate of \(\Gamma\) at 0 and \(\infty\), the exact asymptotic behavior is only known up to a constant factor and, outside the region \([a^2, b^2]\), \(\Gamma\) is only constrained to be non-decreasing.

3.2 Parameter Optimization

In this section we will show how to design the Encoder/Decoder pair introduced in Chapter 2 so that the obtained closed loop gain \(\Gamma\) satisfies the performance requirements introduced in the previous section and summarized in (3.2).

Since different solutions are possible, our design will focus on reducing the strain on the communication channel by minimizing the required minimum bitrate. The formalization of this statement will lead to the definition of an optimization problem whose solution provides the parameters required to implement the desired Encoder/Decoder pair.

Assume that a plant \(\Sigma = (A, B_u, B_w, C_e, D_{eu}, D_{ew})\) whose dynamic is described by:

\[
\begin{align*}
\Sigma : \quad & x^+ = Ax + B_u u + B_w w \\
& e = C_e x + D_{eu} u + D_{ew} w \\
& y = x
\end{align*}
\]

is assigned and the corresponding \(\gamma_*\) has been computed by solving the problem in (3.1). Assume also that a set of parameters \(a < b, \gamma_r > \gamma_*\), \(\rho_1 \in (0, 1)\) and \(\rho_2 > 0\) are also specified with the requirement that the closed loop gain satisfies the following

\(^1\)In this case we could derive necessary conditions on the asymptotic gains for the added feedback block.
conditions:

\[
\begin{align*}
\Gamma(s) & \leq \gamma^2_s \quad \forall s \in [a^2, b^2], \\
\lim_{s \to 0} \frac{\log \Gamma(s)}{\log s} & = 1 - \rho_1, \\
\lim_{s \to \infty} \frac{\log \Gamma(s)}{\log s} & = 1 + \rho_2
\end{align*}
\] (3.3)

If we implement the Encoder/Decoder pair described in Chapter 2, the obtained closed loop gain function \( \Gamma \) is defined by:

\[
\Gamma(s) = \max \{ \Gamma_1(s), \Gamma_2(s), \Gamma_3(s), \Gamma_4(s), \Gamma_5(s) \},
\]

with the \( \Gamma_i \) defined in (2.47), while the required minimum number of levels is given by:

\[
N = 4 + \left( 1 + 2 \left[ \frac{\log \frac{\lambda \lambda_0}{\theta}}{\log \frac{1+\delta}{1-\delta}} \right] \right) \left[ \frac{\log \frac{\xi}{r}}{\log \frac{\tilde{r}}{r_1}} \right] + \left( 1 + 2 \left[ \frac{\log \frac{\lambda_0 \lambda}{\theta}}{\log \frac{1+\delta}{1-\delta}} \right] \right) \left[ \frac{\log \frac{\beta}{\alpha}}{\log \frac{\tilde{r}_2}{r_2}} \right].
\]

Both the expression for \( \Gamma \) and \( N \) depend on the particular choice of the design parameters:

\( \eta, \delta, \theta, \tau, \alpha, \beta, \tilde{r}, \tilde{r}_1, \tilde{r}_2 \)

as well as the matrices \( P \) and \( K \) chosen among the solutions of the LMI defined by conditions (2.2)-(2.5) and (2.42). Before optimizing the expression for \( N \), we will impose some extra constraints on these parameters so that the conditions in (3.3) are satisfied.

Regarding the second and third conditions in (3.3), no additional constraints are required since the construction of the Encoder/Decoder pair described in Chapter 2 already relies on the given exponents \( \rho_1 \) and \( \rho_2 \) and, as pointed out after Theorem 2.3.5, yields a gain function \( \Gamma \) that automatically satisfies the aforementioned constraints.

Keeping in mind that \( \Gamma \) is the maximum among \( \Gamma_i \) for \( i = 1, \ldots, 5 \) and that \( \Gamma_1 \)
is the only linear function among them, the first condition in (3.3) is satisfied by guaranteeing that the slope of $\Gamma_1$ matches the desired one and that it is dominant among the others functions $\forall s \in [a^2, b^2]$.

The condition on the slope is easily determined by using the expression for $\Gamma_1$ in (2.48) and yields the following condition:

$$\frac{1 - \eta^2}{1 - \eta^2 \mu^2} + \gamma^2 - \mu^2 \leq \gamma_r^2.$$  

(3.4)

Ensuring that $\Gamma_1$ is dominant on a prescribed interval is less straightforward; this will be achieved by introducing a sufficient condition on $\tau$. Examining the functions $\Gamma_i$ and using the first two conditions in (2.49), we infer that imposing the constraints:

$$\left\{ \begin{array}{l}
\frac{a^2}{\hat{\sigma}^2} \left( \frac{\hat{\sigma}}{\eta} \right)^{\frac{2}{\gamma_1}} \leq a^2 \\
\frac{\beta^2}{\hat{\sigma}^2} \left( 1 - \frac{\hat{\sigma}_2}{\tau^2} \right) \geq b^2
\end{array} \right. \quad \text{or} \quad (3.5)$$

is equivalent to claiming that $\Gamma_1$ is dominant over $\Gamma_2$ and $\Gamma_3 \forall s \in [a^2, b^2]$. By using the third condition in (2.49), dominance over $\Gamma_4$ and $\Gamma_5$ is assured if

$$\frac{\beta^2 \sigma^2}{\lambda_{\tau}^2} \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2 \geq b^2,$$

which, by virtue of the following lemma, is automatically implied by the second condition in (3.5) provided that $\tau \leq \bar{\tau} \approx 0.42$.

---

\(^2\text{This is a sufficient condition}\)
Lemma 3.2.1. If

$$\tau \leq \hat{\tau} = \frac{1}{9} \left[ 1 + \frac{3\sqrt{13 + 3\sqrt{33}}}{2^{2/3}} - \frac{2^{5/3}}{\sqrt[3]{13 + 3\sqrt{33}}} \right]^2 \approx 0.4196 \tag{3.6}$$

then

$$\frac{\beta^2 \sigma^2}{\lambda^2_w} \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2 \geq \frac{\beta^2}{\mu^2} \left( 1 - \frac{\tau^2}{\bar{\tau}} \right)$$

Proof. Observe that

$$\frac{\beta^2 \sigma^2}{\lambda^2_w} \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2 \geq \frac{\beta^2}{\mu^2} \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2 = \frac{\beta^2}{\mu^2} (1 + \tau) \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2.$$ 

The thesis is then valid if

$$(1 + \tau) \left( \sqrt{1 + \tau^{-1}} - 1 \right)^2 \geq 1,$$

or, equivalently, if

$$-2\sqrt{\tau^3} + 2\sqrt{\tau^2} - 2\sqrt{\tau} + 1 \geq 0.$$ 

After finding the roots of the cubic equation and solving for $\tau$, we obtain the bound in (3.6)

$$\square$$

The optimization of the number of levels $N$ is done by minimizing a differentiable upper bound for its expression. This solution concedes some conservativeness on the obtained results to achieve a faster and more reliable solution to the optimization problem. By using the bound $[x] \leq 1 + x$ to the expression of $N$ in (2.28), we obtain the following upper bound:

$$\tilde{N} = 4 + \left( 3 + 2 \frac{\log \frac{\beta \lambda \alpha}{\alpha}}{\log \frac{\beta}{\log \frac{\beta}{\log \beta}}} \right) \left( 1 + \frac{\log \frac{\beta}{\log \frac{\beta}{\log \beta}}}{\log \frac{\beta}{\log \frac{\beta}{\log \beta}}} \right) + \left( 3 + 2 \frac{\log \frac{\beta \lambda \alpha}{\alpha}}{\log \frac{\beta}{\log \frac{\beta}{\log \beta}}} \right) \left( 1 + \frac{\log \frac{\beta}{\log \frac{\beta}{\log \beta}}}{\log \frac{\beta}{\log \frac{\beta}{\log \beta}}} \right), \tag{3.7}$$

which can be minimized by solving a suitable optimization problem as shown in the following Theorem.
Theorem 3.2.2. The minimization of $\bar{N}$ in (3.7) subject to the constraints in (2.13), (2.14), (2.2)-(2.5), (2.42), (3.4), (3.5) and (3.6), is obtained by solving the following optimization problem:

$$
\begin{align*}
\min \varphi_2(\delta, \eta, \sigma, \lambda_k, \nu) \\
\text{s.t.} \\
0 < \delta \leq \left( \prod_{i=1}^{\lambda_k(A)} |\lambda_i(A)| \right)^{-1} \\
0 < \eta < 1 \\
\sigma > 1 \\
\lambda_k > 0 \\
\nu > 0
\end{align*}
$$

where the cost function $\varphi_2$ is obtained by solving the following optimization problem:

$$
\begin{align*}
\min \frac{2 \log \left( c_1 c_2^b \sqrt{\frac{1}{1-\zeta^2}} \right) + 6 \log \frac{c_3}{\zeta} + 4 \log c_2 + 4c_3}{\log \frac{1+\delta}{1-\delta}} \\
\text{s.t.} \\
\tau \leq \min \left\{ \frac{1-\eta^2}{\nu-1+\eta^2}, -1 + \sqrt{\frac{1+\nu}{1-\eta^2}}, \tilde{\tau} \right\} \\
\zeta \leq \left[ \frac{\sqrt{\nu-\tau+(1+\tau)\eta^2}}{(1+\tau-1)\eta^2(1+\nu)} \right]^{\frac{1}{1+\nu}}
\end{align*}
$$

with

$$
c_1 = \frac{(1+\tau^{-1})\sigma^2(1+\nu)}{\nu-\tau+(1+\tau)\eta^2} \tag{3.10}
$$

$$
c_2 = \lambda_k \sqrt{\varphi_1(\delta, \eta, \sigma, \lambda_k, \nu)} \sqrt{\frac{(1+\tau^{-1})(1+\nu)}{\nu-\tau-\nu(1+\tau)\eta^2}}
$$

$$
c_3 = \sqrt{3 \log \frac{1+\delta}{1-\delta} + 2 \log \left( \frac{c_1 c_2^b}{\sqrt{1-\zeta^2}} \right) \log \left( \frac{c_1^{b+1}}{\sqrt{1-\zeta^2}} \right) + \left[ 3 \log \frac{1+\delta}{1-\delta} + 2 \log \frac{c_1 c_2}{\zeta} \right] \log \frac{c_1}{\zeta}}
$$

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and, finally, the function $\varphi_1$ is evaluated by solving the following SDP problem:

$$
\varphi_1(\delta, \eta, \sigma, \lambda_k, \nu) = \begin{cases}
\min \lambda_u^2 \\
\text{s.t.} \\
\gamma^2 - \gamma^2 \geq \nu \mu^2 \\
Q > 0 \\
\begin{bmatrix} 
Q & AQ + (1 - \delta)B_uL & B_u \\
QA^T + (1 - \delta)L^TB_u^T & \eta^2Q & 0 \\
B_u^T & 0 & \mu^2I \\
Q & AQ + (1 + \delta)B_uL & B_u \\
QA^T + (1 + \delta)L^TB_u^T & \eta^2Q & 0 \\
B_u^T & 0 & \mu^2I \\
(1 - \eta^2)Q & 0 & QC_e^T + (1 - \delta)L^TD_{eu}^T \\
0 & (\gamma^2 - \mu^2)I & D_{eu}^T \\
C_eQ + (1 - \delta)D_{eu}L & D_{eu} & I \\
(1 - \eta^2)Q & 0 & QC_e^T + (1 + \delta)L^TD_{eu}^T \\
0 & (\gamma^2 - \mu^2)I & D_{eu}^T \\
C_eQ + (1 + \delta)D_{eu}L & D_{eu} & I \\
(1 - \eta^2)Q & 0 & QC_e^T \\
0 & (\gamma^2 - \mu^2)I & D_{eu}^T \\
C_eQ & D_{eu} & I \\
Q & AQ \\
QA^T & \sigma^2Q \\
Q & B_u \\
B_u^T & \lambda_u^2 \\
Q & LT \\
L & \lambda_k^2 \\
\end{bmatrix} \geq 0 
\end{cases}
$$

(3.11)

Proof. Let us begin by noticing that all the given constraints and the cost function in (3.7) depend only on the relative amplitude of the parameters $\tau$, $\tilde{r}$, $\tilde{r}_1$ and $\tilde{r}_2$, thus
allowing them to be arbitrarily rescaled. Without loss of generality, we set:

$$\tilde{r}_1 = 1,$$  \hspace{1cm} (3.12)

which leads to the following expression for $\tilde{N}$

$$\tilde{N} = 4 + \left( 3 + 2 \frac{\log \frac{r \Lambda \theta}{\theta}}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \frac{r}{\eta}}{\log \tilde{r}_2} \right) + \left( 3 + 2 \frac{\log \frac{r \Lambda \theta}{\theta}}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \frac{\beta}{\alpha}}{\log \tilde{r}_2} \right).$$  \hspace{1cm} (3.13)

Observe now that the expression in (3.13) is decreasing in $r$ and can be minimized by selecting the largest $r$ satisfying the second condition in (2.14). With our choice for $\tilde{r}_1$, such minimizer is given by

$$r = \left( \frac{\tilde{\sigma}}{\tilde{\eta}} \right)^{-\frac{1}{r_1}},$$  \hspace{1cm} (3.14)

and yields:

$$\tilde{N} = 4 + \left( 3 + 2 \frac{\log \left( \frac{1}{\tilde{\sigma}} \right)^{\frac{1}{r_1}} \frac{\Lambda \theta}{\theta} \tilde{r} }{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \left( \frac{1}{\tilde{\sigma}} \right)^{\frac{1}{r_1}} \tilde{r} }{\log \tilde{r}_2} \right) + \left( 3 + 2 \frac{\log \frac{\beta \Lambda \theta}{\theta}}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \frac{\beta}{\alpha}}{\log \tilde{r}_2} \right).$$  \hspace{1cm} (3.15)

It is straightforward to realize that the expression for $\tilde{N}$ in (3.15) is monotone increasing in $\beta$ and can be minimized by choosing the smallest value of $\beta$ and the largest value of alpha satisfying, respectively, the first and second condition in (3.5). The obtained conditions:

$$\alpha = a \tilde{\mu} \left( \frac{\tilde{\sigma}}{\tilde{\eta}} \right)^{-\frac{1}{r_1}}, \quad \beta = \frac{b \tilde{\mu}}{\sqrt{1 - \frac{r_1}{\tilde{\beta}^2}}},$$  \hspace{1cm} (3.16)

satisfy the first constraint in (2.14) since

$$\frac{\beta}{\alpha} = \frac{b}{a} \left( \frac{\tilde{\sigma}}{\tilde{\eta}} \right)^{-\frac{1}{r_1}} \geq \left( \frac{\tilde{\sigma}}{\tilde{\eta}} \right)^{-\frac{1}{r_1}} \geq \frac{\tilde{\sigma}}{\tilde{\eta}} \geq \frac{1}{\tilde{\eta}},$$

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and yield the following expression for the corresponding number of levels:

\[
\tilde{N} = 4 + \left( 3 + 2 \frac{\log \left( \frac{x}{\eta} \right) \lambda_k \lambda_u \sqrt{\frac{(1+\tau)(1+\nu)}{(1-\tau)(1+\nu)}} + \log \tilde{r}_2}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \left( \frac{x}{\eta} \right) \lambda_k \lambda_u \sqrt{\frac{(1+\tau)(1+\nu)}{(1-\tau)(1+\nu)}}}{\log \frac{1+\delta}{1-\delta}} \right) + \left( 3 + 2 \frac{\log \frac{h}{\theta} \sqrt{\frac{1}{1-\zeta^2}} \lambda_k \lambda_u \sqrt{\frac{(1+\tau)(1+\nu)}{(1-\tau)(1+\nu)}}}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \frac{h}{\theta} \sqrt{\frac{1}{1-\zeta^2}}}{\log \frac{1+\delta}{1-\delta}} \right).
\]

(3.17)

Given that the obtained expression in (3.17) is decreasing in \( \theta \), it can be minimized by selecting the largest value of \( \theta \) satisfying conditions (2.13) and (3.4). From condition (3.4) we obtain

\[
\frac{1 - \eta^2}{1 - \eta^2} \mu^2 + \gamma^2 - \mu^2 \leq \gamma^2 \iff \frac{1 - \eta^2}{1 - \eta^2} (1 + \tau) - 1 \leq \frac{\eta^2}{\mu^2} \leq \frac{1 + \nu}{1 + \tau} \iff \\
\tilde{\eta} \leq 1 - (1 - \eta^2) \frac{1 + \tau}{1 + \nu} \iff (1 + \tau)(1 - \eta^2) \leq (1 + \tau - 1)(1 + \nu) \iff \\
\theta^2 \leq \frac{1 - (1 + \tau)(1 - \eta^2) \left( 1 + \frac{\tau}{1 + \nu} \right)}{1 + \tau - 1} = \frac{\nu(1 - (1 + \tau)(1 - \eta^2))}{(1 + \nu)(1 + \tau - 1)},
\]

where \( \nu = \frac{\gamma^2 - \eta^2}{\mu^2} \) has been introduced for convenience. Since from the obtained chain of inequalities it is clear that condition (3.4) implies (2.13), we can safely set

\[
\theta = \sqrt{\frac{1 - (1 + \tau)(1 - \eta^2) \left( 1 + \frac{\tau}{1 + \nu} \right)}{1 + \tau - 1}} = \sqrt{\frac{\nu(1 - (1 + \tau)(1 - \eta^2))}{(1 + \nu)(1 + \tau - 1)}},
\]

(3.18)

which yields:

\[
\tilde{N} = 4 + \left( 3 + 2 \frac{\log \left( \frac{x}{\eta} \right) \lambda_k \lambda_u \sqrt{\frac{(1+\tau)(1+\nu)}{(1-\tau)(1+\nu)}} + \log \tilde{r}_2}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \left( \frac{x}{\eta} \right) \lambda_k \lambda_u \sqrt{\frac{(1+\tau)(1+\nu)}{(1-\tau)(1+\nu)}}}{\log \frac{1+\delta}{1-\delta}} \right) + \left( 3 + 2 \frac{\log \frac{h}{\theta} \sqrt{\frac{1}{1-\zeta^2}} \lambda_k \lambda_u \sqrt{\frac{(1+\tau)(1+\nu)}{(1-\tau)(1+\nu)}}}{\log \frac{1+\delta}{1-\delta}} \right) \left( 1 + \frac{\log \frac{h}{\theta} \sqrt{\frac{1}{1-\zeta^2}}}{\log \frac{1+\delta}{1-\delta}} \right),
\]

(3.19)

\(^3\text{Notice that } \theta \text{ appears in (3.17) also in the definition of } \tilde{\eta}^2 = (1 + \tau)(1 - \eta^2) + (1 + \tau - 1)\theta^2. \text{ When } \theta \text{ increases, so does } \tilde{\eta}, \text{ contributing even more to the decrease of the cost function } \tilde{N}.\)
where \( c_1 \) has been defined in (3.10) and

\[
\zeta = \frac{\bar{\tau}_2}{\bar{\tau}} \quad (3.20)
\]

needs to satisfy the fourth constraint in (2.14) with our choice of \( \theta \):

\[
\zeta \leq \left( \frac{\nu - \tau + (1 + \tau)\eta^2}{(1 + \tau^{-1})\sigma^2(1 + \nu)} \right)^{\frac{1}{2\nu^2}} \quad (3.21)
\]

Before proceeding with the optimization of the remaining parameters, we can provide a tighter bound on \( \tau \) by exploiting some simple observations. First, notice that the expression for \( \theta \) in (3.18) is meaningful if and only if

\[
\frac{\nu(1 - (1 + \tau)\eta^2) - \tau}{(1 + \nu)(1 + \tau^{-1})} \geq 0 \Leftrightarrow \tau \leq \frac{1 - \eta^2}{\nu^{-1} + \eta^2},
\]

then observe that:

\[
\frac{d}{d\tau} \frac{(1 + \tau^{-1})}{\nu - \tau + (1 + \tau)\eta^2} = \frac{\tau^2(1 - \eta^2) + 2\tau(1 - \eta^2) - (\nu + \eta^2)}{(\tau(\nu + \eta^2) - \tau^2(1 - \eta^2))^2} \geq 0 \Leftrightarrow \tau \geq -1 + \sqrt{\frac{1 + \nu}{1 - \eta^2}},
\]

\[
\frac{d}{d\tau} \frac{(1 + \tau^{-1})}{\nu - \tau - \nu(1 + \tau)\eta^2} = \frac{\tau^2(1 + \nu\eta^2) + 2\tau(1 + \nu\eta^2) - \nu(1 - \eta^2)}{(\tau(\nu - \tau) - \nu(\tau + \tau^2)\eta^2)^2} \geq 0 \Leftrightarrow \tau \geq -1 + \sqrt{\frac{1 + \nu}{1 + \nu\eta^2}},
\]

which proves that the expression for \( \tilde{N} \) in (3.19) is monotone increasing for

\[
\tau \geq -1 + \sqrt{\frac{1 + \nu}{1 - \eta^2}},
\]

hence we can exclude those values of \( \tau \) for minimization purposes. In conclusion, by using these arguments together with the bound in (3.6), we obtain:

\[
\tau \leq \min \left\{ \frac{1 - \eta^2}{\nu^{-1} + \eta^2}, -1 + \sqrt{\frac{1 + \nu}{1 - \eta^2}}, \bar{\tau} \right\}. \quad (3.22)
\]

We can now optimize the expression in (3.19) by minimizing \( \lambda_u \) while enforcing the additional constraints imposed by \( \eta, \delta, \sigma, \nu \) and \( \lambda_k \) on the robust linear controller.
satisfying the LMIs in (2.2)-(2.5) and (2.42). It is straightforward\(^4\) to verify that the minimum of (3.19), subject to the conditions in (3.21) and (3.22), yields a quantity that is monotone increasing in \(\lambda_u, \lambda_k, \sigma\) and decreasing in \(\nu\) and, hence, can be optimized by minimizing \(\lambda_u\) while enforcing the following inequality constraints:

\[
\|B_u\|_P \leq \lambda_u, \quad \|K\|_{P^{-1}} \leq \lambda_k, \quad A^T P A \leq \sigma^2 P, \quad \frac{\gamma^2 - \gamma^2}{\mu^2} \geq \nu.
\]

Using Lemma (1.2.3), this problem can be posed as the SDP in (3.11) and yields the following expression for the required number of levels:

\[
\tilde{N} = 4 + \left(3 + 2 \frac{\log \frac{c_1 c_2}{\zeta} + \log \tilde{r}_2}{\log \frac{1+\delta}{1-\delta}}\right) \left(1 + \frac{\log \frac{c_1}{\zeta} + \log \tilde{r}_2}{\log \tilde{r}_2}\right) + \left(3 + 2 \frac{\log \frac{c_1 c_2}{\sqrt{1-\zeta^2}}}{\log \frac{1+\delta}{1-\delta}}\right) \left(1 + \frac{\log \frac{c_1}{\sqrt{1-\zeta^2}}}{\log \tilde{r}_2}\right).
\]

Furthermore, the expression for \(\tilde{N}\) in (3.23) is a rational function in \(\log \tilde{r}_2\) that can be minimized analytically by choosing:

\[
\log \tilde{r}_2 = \frac{1}{2} \sqrt{\left[3 \log \frac{1+\delta}{1-\delta} + 2 \log \left(\frac{c_1 c_2 \frac{b}{a}}{\sqrt{1-\zeta^2}}\right)\right] \log \left(\frac{c_1 \frac{b}{a}}{\sqrt{1-\zeta^2}}\right) + \left[3 \log \frac{1+\delta}{1-\delta} + 2 \log \frac{c_1 c_2}{\zeta}\right] \log \frac{c_1}{\zeta}}
\]

and yields exactly the expression appearing as the cost function in (3.9).

Finally notice that the upper bound on \(\delta\) in (3.8) is a product of the work in [22] and allows to reduce the feasible region.

\[
\square
\]

\textit{Remark 6.} We would like to point out that, from the solution of the problem in (3.8), the optimal parameters of the control scheme can be readily recovered. In particular, \(\tilde{r}_2\) can be recovered from (3.24), then \(\tilde{r}\) can be retrieved using (3.20), \(\theta\) is obtained from (3.18), \(\tilde{r}_1\) and \(\tilde{r}_1\) are recovered from (3.14) and (3.12) and, finally, \(\alpha\) and \(\beta\) are

\footnotesize
\(^4\)The cost function is increasing in \(\lambda_u, \lambda_k, \sigma\) and decreasing in \(\nu\). Moreover the feasible set shrinks as \(\sigma\) increases and and \(\nu\) decreases.
Remark 7. While the computation of the cost functions $\varphi_1$ and $\varphi_2$ can be done efficiently due to the convexity\textsuperscript{5} of the corresponding problems, the optimization problem in (3.8) is not convex. Given the limited number of decision variables, a solution can be obtained by generating a coarse grid within the feasible region and then conducting a local search from the most promising points. Notice that if the optimization problem defining $\varphi_1(\delta, \eta, \sigma, \lambda_k, \nu)$ is not feasible, then the quintuple $(\delta, \eta, \sigma, \lambda_k, \nu)$ is considered not feasible for the original problem.

3.3 Numerical Examples

In this section we present some examples of how to apply the results of this dissertation. The first two examples are relatively simple and aim at demonstrating the features and limitations of the proposed control scheme while the third shows how our control scheme can be designed and used to control a realistic mechanical system.

3.3.1 Discontinuity on the obtained $l_\infty$ gain

We will now examine a numerical example to gain insight on where our analysis is more conservative and what type of trajectories can be obtained when using our control scheme.

Consider the scalar plant $\Sigma$ described by the matrices:

$$ A = 2 \quad B_u = B_w = C_e = 1 \quad D_{eu} = D_{ew} = 0. $$

In the communication constraints free setting, the performances of standard controllers can be easily assessed. Since the plant is scalar and any full state regulator can therefore cancel the effects of all the past inputs, the optimal $l_1$ controller is the deadbeat one and yields a closed loop $l_\infty$ norm of 1. By solving the problem

\textsuperscript{5}The convexity of the problem in (3.9) has been checked numerically by minimizing the smallest eigenvalue of the Hessian matrix over all the possible choices of the involved parameters. We are currently working on a more rigorous proof.
in (3.1), we obtained the same deadbeat controller with a tight upper bound on the performances $\gamma_\ast = 1$.

Given that $\Sigma$ is scalar, its intrinsic entropy is given by $H = \log_2 |A| = 1$ bit. From the result of Theorem 1, we can infer that any Encoder/Decoder pair stabilizing the plant, would require the alphabet of the channel to contain at least $N = 2$ symbols. A natural question that arises in this context, is how our proposed scheme fares against this fundamental limit.

Although our design and analysis is geared toward achieving particular closed loop performances, it still makes sense to compute the minimum rate of the channel required by our proposed algorithm when the performances on the closed loop are extremely loose so that only stability is required. To obtain this result, we solved the optimization problem in (3.8) with the requirements on the closed loop $l_\infty$ gain given by:

$$\rho_1 = 1 \quad \rho_2 = 1000 \quad b = a = 1 \quad \gamma_r = 2000\gamma_\ast$$

and obtained:

$$\log_2 N \approx 5.78 \text{ bits.}$$

Unsurprisingly, the required bitrate is quite conservative when compared to the fundamental limit and even to the requirements of other schemes guaranteeing only stability.$^6$

We then solved the problem in (3.8) with some more realistic requirements on the closed loop performances. In particular, after setting:

$$\rho_1 = \frac{1}{2} \quad \rho_2 = 1 \quad a = 1 \quad b = 2 \quad \gamma_r = \frac{3}{2}\gamma_\ast,$$

we obtained:

$$\log_2 N \approx 8.50 \text{ bits}$$

$^6$As a comparison, the required bitrate when implementing the scheme proposed in [25] is 2 bits.
and the parameters of the corresponding minimizing controller are given by:

\[ K = -2, \quad P = 1.402, \quad \eta = 0.536, \quad \delta = 0.137, \]

\[ \tau = 0.101, \quad \theta = 0.044, \quad \alpha = 0.011, \quad \beta = 2.905, \]

\[ \bar{c} = 0.007, \quad \bar{r}_1 = 1, \quad \bar{r}_2 = 4145, \quad \bar{r} = 4.718 \times 10^4, \]

while the corresponding closed loop gain function \( \Gamma \) and the linear star norm \( \gamma_* \) are depicted in Figure 3-2.

By carefully exciting the closed loop system obtained with this last controller, we would like to assess the conservativeness of the obtained gain \( \Gamma \). In particular, we will focus our attention on the asymptotic behavior around 0 and \( \infty \) as well as the value \( \|u\|_\infty = b \), where \( \Gamma \) exhibits a discontinuity.

Let us begin with the discontinuity in \( b \) where the proposed analysis and definition of \( \Gamma \) in (2.44) predict an unavoidable discontinuity. Informally speaking, this happens as the noise level is large enough to drive the system state into regions \( R_4 \) or \( R_5 \) where the plant will operate in open loop for at least one step, thus amplifying the given input. By examining the main proof in the previous chapter, it is evident that such behavior is exacerbated when \( \xi^+ = \frac{\beta}{\bar{r}} \), thus we constructed some adversarial input designed to strike the system exactly in this configuration. In Figure 3-2, along with the obtained gain, we reported the obtained infinity norm of the constructed input and of the corresponding obtained state trajectory. While the location of the discontinuity is not far from the desired value of 2, the jump in amplitude is quite conservative. We would also like to point out that, for some parameters configurations, such discontinuity may not even be present, in particular the amplitude after the \( b \) is related to the instability level of the plant while the value before \( b \) is related to the required robust performances in the linear zone. Since these two quantities are not directly related, a tighter analysis may provide bounds that are not necessarily discontinuous.

Regarding the behavior for noise level arbitrarily small, a close inspection of the
Figure 3-2: Comparison between the obtained closed loop gain $\Gamma$ and the ideal one $\gamma^*$ for the given plant and the design parameters $p_1 = \frac{1}{2}$, $p_2 = 1$, $a = 1$, $b = 2$, $\frac{2c}{\gamma^*} = \frac{3}{2}$.

Proof of Theorem 2.3.4 reveals that transitions between regions $R_1$ and $R_3$ yield the largest amplification for small noise. To excite this behavior, we used the following adversarial\(^7\) input:

$$w(k) = \begin{cases} (0.9)^k & \text{if} \quad |2x(k) + u(k) + (0.9)^k| \geq \frac{r_\xi(k+1)}{\sqrt{P}} , \\ 0 & \text{otherwise} \end{cases} , \quad (3.25)$$

and we plotted the obtained trajectories in Figure 3-3. Notice that, since $\eta^2 \approx 0.33$, the transients due to non-zero initial condition are much faster than the decay in the amplitude of noise, thus making $\Gamma(|w(k)|)$ our best estimate of the effects of $w(k)$ on the state amplitude after just few steps. The obtained results show not only that the given analysis is conservative in terms of multiplicative constants but also that the involved exponents have been overestimated. This is due to conservativeness in

\(^7\)Note that, in this context, adversarial refers to the fact that the input is a function of the system state. It would be possible, but more cumbersome, to design a suitable input without knowledge of the state.
estimating how many steps are required to exit the regions where the plant is left in open loop.

Similarly, for noise level arbitrarily large, a close inspection of the proof of Theorem 2.3.4 reveals that transitions between regions $R_1$ and $R_4$ yield the largest amplification for large noise. To excite this behavior, we used the following adversarial input:

$w(k) = \begin{cases} 
(1.2)^k & \text{if } \xi(k) \leq \frac{\beta}{\gamma}, \\
0 & \text{otherwise}
\end{cases}$

and we plotted the obtained trajectories in Figure 3-4. Once again the gap between our provided upper bound $\Gamma(|w(k)|)$ and the actual maximum of $|x(k)|$ grows as $w$ increases, indicating that the exponent $1 + \rho_2 = 2$ is conservative.
3.3.2 Conservativeness on the required rate

We will now apply the proposed controller design to a plant first introduced and analysed in [18]. We will use this example to gain some qualitative insight on how the required bitrate changes depending on the given closed loop requirements.

Consider a plant $\Sigma$ described by the matrices:

$$A = \begin{bmatrix} 2.7 & -23.5 & 4.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B_u = B_w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $\kappa \in \mathbb{R}$ is a parameter. Notice that, since $\Lambda(A) = \{0.2, 1.25 - 4.63i, 1.25 + 4.63i\}$, the intrinsic entropy of $\Sigma$ is given by $H = 2 \log_2 4.79 = 4.52$ bits.

In [18] the authors showed that, if $\kappa \in (1.5, 3)$, the optimal $l_1$ controller is not
static. Moreover, as $\kappa \to 1.5$, the order of such controller diverges and the optimal cost tends to 3. As the linear controller found solving problem (3.1) is constrained to be static, it is instructive to assess how its performances index $\gamma_*$ compares to the gain of the optimal $l_1$ controller as $\kappa$ changes. This comparison is presented in Table 3.1 where, along with $\gamma_*$ and the order $nK$ and performances $\gamma_{opt}$ of the optimal $l_1$ controller, we present the minimum required bitrate computed by solving the problem in (3.8) with the following requirements on the closed loop gain:

$$\rho_1 = \frac{1}{2}, \rho_2 = 1, a = 1, b = 2, \gamma_\gamma = 2\gamma_*.$$ 

<table>
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<th>$\kappa$</th>
<th>$nK$</th>
<th>$\gamma_{opt}$</th>
<th>$\gamma_*$</th>
<th>$\log_2 N$</th>
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</tr>
<tr>
<td>1.500</td>
<td>$\infty$</td>
<td>3.00</td>
<td>3.90</td>
<td>12.97</td>
</tr>
</tbody>
</table>

Table 3.1: Comparison between optimal $l_1$ controller, *-norm controller and bitrate required to obtain a closed loop gain with $\rho_1 = \frac{1}{2}, \rho_2 = 1, a = 1, b = 2, \frac{\gamma_\gamma}{\gamma_*} = 2$. 

Notice that the *-norm based controllers do not appear to be affected by the singularity behavior of the underlying optimal $l_1$ problem and provide bounds on the performances that are within 30% of the optimal value. As the proposed control scheme is based on such controllers, the required bitrate also appears unaffected as $\kappa$ approaches the critical value 1.5.

For the remaining part of this study, assume $\kappa = 2$. In this case it is interesting to compute the minimum rate of the channel required by our proposed algorithm to achieve mere stability. This rate can be computed by solving the optimization problem in (3.8) when the performances on the closed loop are extremely loose. With the choice $\rho_1 = 1, \rho_2 = 100, b = a = 1, \gamma_\gamma = 10000\gamma_*$, we obtained:

$$\log_2 N \approx 10.70 \text{ bits.}$$
For a comprehensive examination of how the requirements on the closed loop performances affect the required bitrate, we fixed three among the requirements $\rho_1, \rho_2, \frac{b}{a}, \frac{\tau_c}{\tau}$ while changing the remaining one. The results are reported in Figures 3-5 through 3-8.

Figure 3-5: Required minimum bitrate as a function of $\frac{b}{a}$ when the remaining requirements on the closed loop gain are set at $\rho_1 = \frac{1}{2}, \rho_2 = 1, \frac{\tau_c}{\tau} = 2$. 
Figure 3-6: Required minimum bitrate as a function of $\rho_1$ when the remaining requirements on the closed loop gain are set at $\rho_2 = 1, \frac{b}{a} = 2, \frac{\gamma}{\gamma_*} = 2$.

Figure 3-7: Required minimum bitrate as a function of $\rho_2$ when the remaining requirements on the closed loop gain are set at $\rho_1 = \frac{1}{2}, \frac{b}{a} = 2, \frac{\gamma}{\gamma_*} = 2$. 

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Figure 3-8: Required minimum bitrate as a function of $\frac{x}{\gamma}$ when the remaining requirements on the closed loop gain are set at $\rho_1 = \frac{1}{2}$, $\rho_2 = 1$, $\frac{b}{a} = 2$. 
3.3.3 Inverted Pendulum on a Cart

We now examine a more realistic dynamical system to illustrate how to practically design rate constrained control schemes using the algorithm presented in Chapter 2. The concept of an augmented plant will be adopted from standard $\mathcal{H}_\infty / \mathcal{H}_2$ control theory and utilized to help specify the desired performances of the closed loop system.

Consider the classical inverted pendulum on a cart represented in Figure 3-9, where a control input $u$ and a disturbance force $w_c$ act on the cart while a disturbing torque $w_\varphi$ acts on the pendulum. The numerical values of the involved constants are given in the following table:

<table>
<thead>
<tr>
<th>Constant</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>2</td>
<td>Kg</td>
</tr>
<tr>
<td>$m$</td>
<td>0.5</td>
<td>Kg</td>
</tr>
<tr>
<td>$l$</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81</td>
<td>Kg-m/s²</td>
</tr>
</tbody>
</table>

Table 3.2: Numerical values for the considered inverted pendulum.

If $x_c$ represents the position of the cart, the dynamic of such system is described by

$$
\begin{align*}
\dot{x}_c &= \frac{ml\varphi^2 \sin \varphi - mg \cos \varphi \sin \varphi + u + w_x - \frac{w_\varphi}{l} \cos \varphi}{M + m(1 - \cos^2 \varphi)} \\
\dot{\varphi} &= \frac{-ml\varphi^2 \sin \varphi \cos \varphi - \cos \varphi (u + w_x) + (M + m) g \sin \varphi + \frac{M+m}{ml} w_\varphi}{Ml + ml(1 - \cos^2 \varphi)}
\end{align*}
$$

where $g$ is the gravitational acceleration.

If we introduce the vector $x = [x_c, \dot{x}_c, \varphi, \dot{\varphi}]^T$, after linearizing the equations in (3.27) around the vertical point ($\varphi = 0$), we obtain:

$$
\dot{x} = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -\frac{mg}{M} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \frac{(m+M)g}{Ml} & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x} \\
\varphi \\
\dot{\varphi} \\
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
\frac{1}{M} \\
0 \\
-\frac{1}{Ml} \\
\end{bmatrix}
\begin{bmatrix}
u \\
\frac{1}{M} \\
0 \\
-\frac{1}{Ml} \\
\end{bmatrix}
\begin{bmatrix}
w_c \\
w_\varphi \\
\end{bmatrix}.
$$
We will now develop a controller capable of maintaining the pendulum in a vertical configuration while rejecting the disturbances $w_\varphi$ and $w_c$. In addition, we would like for the position of the cart $x$ to track a given reference signal $r$.

Before proceeding with the design, let us first derive the DT version of the equations in (3.28). By using an exact continuous to discrete time conversion and a sampling time of $T_s = 0.05s$ we obtained the following DT time model $P$:

$$
x^+ = \begin{bmatrix} 1 & 0.05 & -0.0031 & -0.0001 \\ 0 & 1 & -0.1239 & -0.0031 \\ 0 & 0 & 1.0308 & 0.0505 \\ 0 & 0 & 1.2388 & 1.0308 \end{bmatrix} x + \begin{bmatrix} 0.0006 & 0.0006 & -0.0013 \\ 0.0251 & 0.0251 & -0.0505 \\ -0.0013 & -0.0013 & 0.0126 \\ -0.0505 & -0.0505 & 0.5051 \end{bmatrix} \begin{bmatrix} u \\ w_c \\ w_\varphi \end{bmatrix}
$$

Notice that the intrinsic entropy of $P$ is given by $H = \log_2 1.2809 \approx 0.36$ bits.

To obtain the desired control objectives, we construct a linear system $\Sigma$ that incorporates the plant $P$ and the given requirements. This new system is traditionally called augmented plant and contains all the original inputs of $P$ as well as any other input required to perform the control objectives. The regulated outputs of $\Sigma$ are constructed so that the given control objectives are achieved whenever they are small.
The complete structure of the adopted augmented plant for this example is given in Figure 3-10. The reasoning behind this particular choice can be summarized in a few points:

- Along with the disturbances acting on the plant $P$, another input required to perform tracking is the given trajectory. Hence $r$ is among the inputs of $\Sigma$.

- Given that we are required to track a given trajectory, the tracking error $e_1$ is a natural choice for one the output signals. Additionally, a scaled version of the control input $u$ constitutes a second output $e_2$. By changing the corresponding scaling factor, it is possible to prevents the control input to grow too large during the evolution of the system.

- The given reference signal $r$ is processed through a low pass filter $G$ before being compared with the current position. This component is more critical for $l_\infty$ controllers as the inertia of the plant would make impossible to track a fast changing trajectory during the transients and the controller would likely just stabilize the plant without tracking the given reference.

- The multiplicative constants $c_u$, $c_\varphi$ and $c_c$ are usually substituted by dynamical filters to isolate the high frequency components. Given that the proposed con-
trollers rely on the knowledge of the full state vector and the state of such filters would not be available, we are limited to use multiplicative factors.

Designing the values for the additional filters and constants defining the augmented plant in Figure 3-10 usually involves some knowledge about the input signals. Since finding the most appropriate choice for such parameters is beyond the scope of this small example, let us assume that the following values have been chosen:

\[ G(z) = \frac{-0.1462z + 0.1791}{z^2 - 1.637z + 0.6703} \quad c_u = 10^{-3} \quad c_v = c_r = 10^{-2}. \]

After solving the problem in (3.1), we obtained the following linear controller:

\[ K = \begin{bmatrix} 1164.6 & 473.9 & 1202.3 & 270.5 & -500.4 & -1040.1 \end{bmatrix}, \]

which exhibits a guaranteed linear \( l_\infty \) gain of slope \( \sqrt{3} \gamma_* = 0.85 \). Keeping in mind that the actual \( l_1 \) norm of the closed loop obtained when using the *-norm controller yields a gain of 0.73, this bound is more than acceptable.

We can now proceed with the design of a finite rate controller according to the scheme introduced in Chapter 2. For this purpose, we decided to set the closed loop requirements at:

\[ \rho_1 = \frac{1}{2}, \quad \rho_2 = 1, \quad a = 1, \quad b = 2, \quad \gamma_c = 1.2\gamma_*, \]

thus assuming that relevant range of amplitudes for the input signals lie in the interval \([1, 2]\).

After solving problem (3.8) with the given specifics, we obtained

\[ \log_2 N = 10.79 \text{ bits}, \]

and the parameters of the corresponding minimizing controller are given by:

\[ K = \begin{bmatrix} 778.8 & 353.9 & 919.7 & 206.6 & -448.3 & -686.1 \end{bmatrix} \]
\[
P = \begin{bmatrix}
67.2 & 18.5 & 42.1 & 9.5 & -3.4 & -63.3 \\
18.5 & 5.5 & 12.5 & 2.8 & -2.4 & -17.4 \\
42.1 & 12.5 & 28.7 & 6.5 & -5.6 & -39.6 \\
9.5 & 2.8 & 6.5 & 1.5 & -1.3 & -9.0 \\
-3.4 & -2.4 & -5.6 & -1.3 & 7.1 & 4.1 \\
-63.3 & -17.4 & -39.6 & -9.0 & 4.1 & 60.4 \\
\end{bmatrix}
\]

\(\eta = 0.8566 \quad \delta = 0.0446\)

\(\tau = 0.0261 \quad \theta = 0.0106 \quad \alpha = 4.2992 \times 10^{-4} \quad \beta = 1.1144\)

\(\tau = 7.7191 \times 10^{-4} \quad \tilde{r}_1 = 1 \quad \tilde{r}_2 = 3.6066 \times 10^5 \quad \tilde{r} = 1.2987 \times 10^7\)

In Figures 3-11 and 3-12, we present a comparison between the closed loop responses to a unitary step when using the \(\ast\)-norm controller and the proposed finite rate algorithm. The disturbances \(w_\varphi\) and \(w_c\) are assumed to be Gaussian white noise of variance 0.01. Notice that similar results are obtained when the full nonlinear equations (3.27) are used in place of the linearized plant, thus showing that the proposed controller is somewhat robust to model uncertainties.
Figure 3-11: Unitary step (at $t = 1$) response when using different controllers and models for the dynamics of the cart.

Figure 3-12: Tracking error (with respect to $\hat{r}$) when using different controllers and models for the dynamics of the cart.
Chapter 4

Conclusions and Future Directions

In this thesis we presented a new feedback control scheme to regulate unstable discrete time LTI systems via a finite rate channel. The scheme is applicable when the full state is available and the control input is scalar. The digital implementation of the control algorithm requires a limited amount of resources and its performances are assessed via the induced $l_\infty$ gain of the closed loop system.

In Chapter 2, we described the proposed control scheme in detail and showed that the obtained closed loop is dissipative. As a consequence, the effects of the input noise $w$ on the regulated output $e$ are described by a nonlinear function $\Gamma$ such that

$$\|e\|_\infty^2 \leq \Gamma(\|w\|_\infty^2)$$

whenever the evolution starts from a zero initial condition.

In Chapter 3, we showed how to design the parameters of the proposed control scheme so that the strain on the communication channel is minimized while satisfying some given constraints on the nonlinear function $\Gamma$. We also showed that, given an arbitrarily large capacity of the digital link, the performances of standard controllers without communication constraints can be retrieved.

Many improvements can be made to the current design and, correspondingly, different types of performances could be addressed. Some aspects we are currently investigating include:
• **Noisy channels:** One of the drawbacks of the proposed scheme is that the Encoder and Decoder need to maintain synchronization between their internal representation of the same scaling factor $\xi$. This feature makes the control scheme fragile to communication errors in the channel. Since these errors are an intrinsic component of communication systems, we are currently investigating improved schemes capable of coping with communication mistakes. One possible solution is to allow for the scaling factors at the encoder and decoder, $\xi_e$ and $\xi_d$, to be different and provide a dynamics for the Encoder that is capable of estimating the scaling factor at the Decoder. Since the full state is available, the following quantity:

$$\frac{B^T_u}{B^T_u B_u}(x(k) - Ax(k - 1)) = f_{k-1}\xi_d(k - 1) + \frac{B^T_u B_w}{B^T_u B_u}w(k - 1),$$

where $f_{k-1}$ is a value depending on the received symbol, could be computed at the encoder side and would provide some (possibly corrupted) information about the scaling factor at the Decoder side.

We are hoping to prove that the dissipation inequality in Theorem 2.3.4 can be replaced by:

$$\mathbb{E}[V^+|x, \xi_e, \xi_d] \leq \lambda V + \tilde{\Gamma}(\|w\|^2) \quad (\lambda < 1)$$

in the stochastic settings, thus paving the way to prove bounds on the second moments of the regulated output $e$ when a deterministic uniform bound on the input $w$ is known.

• **Output feedback regulators:** Another limitation that we are currently trying to remove is the need for the full state to be available for control. When the channel is noiseless, a basic implementation of a Luenberger observer would suffice and, instead of the norm of the state, an estimation could be used in the coding function. If the channel makes mistakes, the actual input to the plant is not known on the encoder side and a Luenberger observer would not work. In this scenario, an observer with unknown inputs [65] could be used to invert
the dynamics and retrieve a corrupted version of the scaling factor used by the decoder. This procedure would also add a delay comparable to the relative degree of the plant and would require a modification of the original design to compensate for it. Moreover, if the plant is not minimal-phase, the inversion of the dynamics cannot be done using any type of observer but, perhaps, resorting to nonlinear (piecewise) observers would still provide the required estimation at the encoder side.

- **Induced $l_2$ gain:** One of the first approaches pursued while designing finite rate controllers was to obtain a finite $l_2$ gain. The advantages of this choice is that linear controllers without communication constraints that are optimal in the $l_2$ sense are simple to design via SDP programming. In addition, the minimization of the $l_2$ gain usually leads to better tracking performances and simpler specifications in terms of the augmented plant. Unfortunately, when the $l_2$ gain is non-linear, the involved dissipation inequality becomes more convoluted. If we assume the input has a finite energy and we iterate equation (2.50) from time 0 to $\infty$ we obtain the bound:

$$\|e\|_{l_2}^2 \leq \bar{V}(x(0), \xi(0)) + \sum_{k=0}^{\infty} \Gamma (\|w(k)\|_{l_2}^2) \neq \bar{V}(x(0), \xi(0)) + \Gamma (\|w(k)\|_{l_2}^2),$$

where the last equality is false because $\Gamma$ is non-linear. To obtain a finite $l_2$ gain, following the concepts introduced in [19], it is possible to augment the dynamics of the systems to account for the total energy supplied up to time $k$ by introducing an extra variable $E$ whose dynamics is described by:

$$E^+ = E + w^T w.$$ 

If the augmented system admits a storage function $V$ and a function $\Gamma$ such that the following dissipation inequality is satisfied:

$$V^+ - V \leq \Gamma(E^+) - \Gamma(E) - \|e\|^2,$$
then the trajectories are such that

$$\|e\|_{l_2}^2 \leq V(x(0), \xi(0)) + \Gamma(E(\infty)) - \Gamma(E(0)) = \tilde{V}(x(0), \xi(0)) + \Gamma(\|w\|_{l_2}^2)$$

and \(\Gamma\) is indeed the induced \(l_2\) of the system. These ideas were not implemented in this dissertation because of the increased complexity due to the extra variable \(E\).
Lemma A.0.1. Given $c \geq 0$, $w \geq 0$ and $\rho_2 > 0$, consider the scalar real function:

$$f(x) = c(x + w)^{2(1+\rho_2)} - x^{2(1+\rho_2)} \quad x \in \mathbb{R}^+$$

If $c < 1$, then

$$\max_{x \geq \beta} f(x) = \begin{cases} 
\frac{cw^{2(1+\rho_2)}}{(1-c)^{1+2\rho_2}} & \text{if } \beta \leq \frac{c^{1+2\rho_2}}{1-c^{1+2\rho_2}} w \\
(c(\beta + w)^{2(1+\rho_2)} - \beta^{2(1+\rho_2)} & \text{otherwise}
\end{cases}$$

Proof. Let us compute $f'(x)$. We obtain:

$$f'(x) = 2(1 + \rho_2) \left[ c(x + w)^{1+2\rho_2} - x^{1+2\rho_2} \right],$$

from which we infer that $f$ increases up to

$$x^* = \frac{c^{1+2\rho_2}}{1 - c^{1+2\rho_2}} w,$$

and then decreases. Since

$$f(x^*) = \frac{cw^{2(1+\rho_2)}}{(1 - c^{1+2\rho_2})^{1+2\rho_2}},$$
the statement follows immediately. \qed

**Lemma A.0.2.** Given $c \geq 0$, $w \geq 0$ and $\rho_2 > 0$, consider the scalar real function:

$$f(x) = c \frac{(x + w)^{1+\rho_2}}{x^{\rho_2}} - x \quad x \in \mathbb{R}^+$$

If $c \leq 1$, then $f$ is monotone non-increasing.

**Proof.** Let us start by computing $f'(x)$. We obtain:

$$f'(x) = c \frac{(1 + \rho_2)x^\rho_2(x + w)^{\rho_2} - \rho_2(x + w)^{1+\rho_2}x^{\rho_2-1}}{x^{2\rho_2}} - 1$$

$$= c \left( \frac{1 + \rho_2}{x^{\rho_2}} + \frac{\rho_2 - \rho_2(x + w)x^{-1}}{x^{\rho_2}} \right) - 1$$

Then the statement is correct if the function $g(y) = (1 + y)^{\rho_2}(1 - \rho_2y)$ is always not larger than 1 for $y \geq 0$. This is indeed the case since $g(0) = 1$ and

$$g'(y) = -\rho_2(1 + \rho_2)y(1 + y)^{\rho_2-1}$$

is always non-positive. \qed

**Lemma A.0.3.** Given $c \geq 0$, $w \geq 0$ and $0 < \rho_1 < 1$, consider the scalar real function:

$$f(x) = cx^{\rho_1}(x + w)^{1-\rho_1} - x \quad x \in \mathbb{R}^+$$

If $c \leq 1$, then

$$f(x) \leq c(1 - \rho)w \quad \forall x \in \mathbb{R}^+$$

**Proof.** we aim at proving that

$$g(w) = cx^{\rho_1}(x + w)^{1-\rho_1} - x - c(1 - \rho)w \leq 0 \quad \forall w, x \in \mathbb{R}^+,$$

where we emphasized that it is convenient to think of the left-hand side of such inequality as a function of $w$ parametrized by $x$. 136
Computing \( g'(w) \) yields:

\[
g'(w) = c(1 - \rho_1) x^{\rho_1} (x + w)^{-\rho_1} - c(1 - \rho_1) \\
= c(1 - \rho_1) \left[ \left( \frac{x}{x + w} \right)^{\rho_1} - 1 \right] \leq 0,
\]

and, given that

\[ g(0) = cx - x \leq 0, \]

then \( g \) is always non-positive, thus concluding the proof.
Bibliography


