A Novel Torsional Spring Design for Knee Prostheses and Exoskeletons

by

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Abstract

In this thesis, a novel torsion spring design for use in knee prostheses and exoskeletons is presented and analyzed. The planar spring design features an outer hub and an inner hub, which are connected by slender beams and store torsion energy in beam bending. The beams are fixed to the outer hub on one end and attached to the inner hub by a pin and slot on the other. The modeled spring design is capable of deflecting $\pm \frac{\pi}{6}$ radians, higher than any existing planar torsion spring designs, and is capable of providing 100 N·m of torque. The maraging steel spring is predicted to have a total diameter of 0.112 meters, width of 0.005 meters, and mass of 98 grams. With this form factor, the planar spring design provides a more compact alternative to elastic elements currently used in series elastic actuators. From the presented models, the design dimensions, material, and slot geometry can be parametrized to design springs that meet specific requirements for different applications. In addition to quantifying performance, the models presented provide the foundation for further weight, efficiency, and performance optimization.

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Chapter 1

Introduction

As the fields of rehabilitation robotics, legged robots, prostheses, and exoskeletons, continue to grow, series elastic actuators (SEA) are increasingly utilized. Because applications where the compliance provided by an SEA is desired are so diverse, much research in the past decade has been dedicated to developing custom SEAs to meet the specific requirements of different applications.

However, due to the mechanical complexity of a passive, elastic element, existing SEAs are typically heavy, bulky, and not well-suited for applications such as exoskeletons and prostheses where there exists strict weight and form-factor constraints. The goal of this thesis was to develop a novel torsion spring design for use in a knee-joint exoskeleton or prostheses. This required a torsion spring design that is capable of higher angular deflections, able to withstand high torques, and has a much more compact form factor than existing solutions.

This thesis provides a basic framework and understanding of the design from which further optimization for specific applications can be made. We will first present the various approaches that were taken in attempting to find the optimal torsion spring design. After presenting the design, the mathematical models that were used to understand the effects of design parameters and analyze spring performance are discussed.
1.1 Series Elastic Actuators

A series elastic actuator (SEA) consists of a stiff actuator with a spring in series between the actuator and the load as shown in Figure 1-1. While a stiff actuator operating independently is capable of moving to and maintaining desired positions or following predefined trajectories, an SEA will allow deviation from an equilibrium position [5]. Stiff actuators are well-suited for position controlled applications where accurate point and trajectory tracking is required, but are less-suited for applications where spring-like behavior similar to those found in biological systems are desired [9].

![Figure 1-1: A schematic of a series elastic actuator showing a spring placed in series between the motor and the load. [1]](image)

Compared to stiff actuators, the compliance afforded by SEAs allow for exoskeleton and rehabilitation robotic systems to absorb large positional errors that occur due to human-system interfaces, preventing damage to the system and injury to the user [1]. The elastic element allows for energy to be stored and released mechanically, which is more efficient than using electric actuators as generators [9]. Furthermore, in legged robotics and rehabilitation applications, SEAs reduce shock loading on the transmission that may occur during operation.

The smoothness of force transmission of the actuator becomes much less significant since the series elasticity acts as a transducer between the actuator output position and load force. As a result, the actuator's required force fidelity is decreased while force control stability is improved [10]. In force control applications, the deflection of the elastic element can be measured and used as a feedback mechanism in force controllers. [5].
1.2 Existing Torsion Elastic Elements

The existing elastic elements for SEAs can be categorized into three main groups: planar springs, mechanisms that utilize an arrangement of compression springs, and more complex stiffness-controlled systems. While there is a relatively large diversity of planar torsion spring designs, they all typically are monolithic springs that store energy in beam bending as the outer hub rotates with respect to the inner hub. Shown in Figure 1-2 (a) [3], planar torsion springs can be configured in parallel or series to meet the differing requirements for specific applications. Another approach to providing rotary compliance is to use a configuration of linear springs, an example of which is shown in 1-2 (b) [11]. The last category refers to the large number of custom controllable stiffness actuators that have been designed for various robotic applications. These include equilibrium-controlled stiffness, antagonistic-controlled stiffness, and structure-controlled stiffness actuators. A specific variable stiffness actuator design can be seen in Figure 1-2 (c) in which three pulleys and two servo motors are used to control equilibrium position and actuator stiffness [5].

![Figure 1-2](a) (b) (c)

Figure 1-2: Examples of three different types of series elastic elements are shown. (a) An example of a monolithic torsion spring that provides compliance as the outer hub rotates with respect to the inner hub [3]. (b) An example of arranging linear springs that provide rotary compliance is shown [11]. (c) A variable stiffness actuator is shown that utilizes tensioning systems and pulleys to vary the stiffness of the SEA [5].
1.3 Design Requirements

Each of the discussed torsion spring designs have their advantages and disadvantages with regards to size, versatility, adaptability, dynamics, and torque response. In pursuing a torsion spring design for this thesis, we will be exploring the optimization and geometry of a planar torsion spring design due to the limitations of the other types of compliant systems for light-weight, wearable applications.

The purpose of this design project was to develop a compact, torsionally compliant element for an SEA that will be used in the knee joint of an exoskeleton design. This particular application results in the three main functional requirements that the presented design seeks to satisfy:

- torque response of 100 N·m [7]
- deflection of $\pm \frac{\pi}{6}$ radians [2]
- minimize resulting design's size and mass

In addition to the biomechanical functional requirements, wearable robotics require a compact form factor that is comfortable to wear and does not disturb the natural movement of users. Essentially, the ideal torsion spring can provide biologically appropriate deflections and torques while minimizing width, diameter, and mass.

1.3.1 NASA Planar Torsion Spring

While a single elastic element may not satisfy both the torque and deflection requirements, in evaluating the spring design, the existing spring shown in Figure 1-3 is used as a baseline from which performance metrics are compared. The torsion spring has a generally planar, disc shape and was developed by NASA (US Patent 8176809 B2) for use with a robotic arm. It is consisted of an outer mounting hub that is concentric to an inner mounting hub from which two splines extend radially. The splines vary in width with the length, having a decreased average width towards the
middle of the segment. The inner hub would be actively rotated by an actuator or
drive components, rotating it to move relative to the outer segment, which is attached
to the robotic arm. Aspects of this design, such as the spring width, spline widths,
spline shape, and material can be changed to obtain the stiffness desired for different
applications [6].

Figure 1-3: The NASA planar torsion spring design features concentric inner and
outer hubs that are connected by splines [6].

In order to properly evaluate the developed spring design, several additional physical
constraints were applied in the design of the new spring in order to make comparative
analysis more analogous. The current configuration of the NASA planar torsion
spring is capable of deflecting up to $\pm \frac{\pi}{36}$ radians.

- material is maraging steel
- maximum diameter of 0.085 meters
- maximum width (planar thickness) of 0.005 meters
Chapter 2

Mechanical Energy Storage

There are two types of mechanical energy storage in materials: hydrostatic energy and shear energy. However, in designing a compact spring, it is extremely difficult to apply hydrostatic forces and appropriately constrain the material. The first design approach was to minimize the amount of material in a spring while maximizing energy storage. However, due to the differences in types of loading, both of which result in material shear energy storage, the final design approach focused on minimizing stiffness in loading to maximize deflection rather than maximizing energy storage.

2.1 Energy Storage Density of Different Materials

The Von Mises Yield Criterion helps provide an understanding of how materials store energy and how materials yield. In the derivation of the Von Mises Yield Criterion, a material yields due to maximum shear energy. Since the Von Mises stress is calculated from distortion energy, or the amount of shear energy before failure, hydrostatic energy is disregarded. Therefore, it is extremely mechanically difficult, but theoretically possible to store incredibly large amounts of energy in a material through hydrostatic forces. Any stress states with the same distortion energy will have the same Von Mises stress, and the material fails when the Von Mises stress exceeds the yield strength of the material.

In exploring the max energy storage, the amount of energy stored before failure in
different materials was explored. The approximate modulus of resilience, which is the maximum energy that can be absorbed per unit volume without creating permanent distortions, was calculated by

\[ U_r = \frac{\sigma_y^2}{2E} \]  

where \( \sigma_y \) is the yield stress and \( E \) is the Young’s Modulus. It is important to note that in using Equation 2.1, the Young’s Modulus is assumed to be linear, and therefore is only accurate as an approximation for materials such as rubber, which have a non-linear Young’s modulus.

Figure 2-1: The modulus of resistance for various materials are presented and compared.

In calculating the modulus or resistance of materials, we can better compare the amount of energy that a material can store before it fails. From Figure 2-1, it can
be seen that traditional materials such as spring-tempered steel or even a titanium alloy can only store a tenth of the amount of energy per unit volume that materials such as aramid or rubber can. However, it is important to note that the modulus of resilience calculates the tensile energy stored before failing and is a poor estimation of maximum shear energy for non-isotropic materials such as aramid, which fail at much lower stresses in other loading conditions.

### 2.2 Beam Bending vs Axial Loading

Because hydrostatic loading on a material is extremely difficult to implement, springs store shear energy. To this end, there are two main types of loads to store energy: axial loading and beam bending. In most existing planar torsion springs, beams, which are fixed to an inner hub at one end and an outer hub on the other, provide energy storage through bending. In designing a spring for this particular application, high deflections are desirable, and therefore, stiffness needs to be minimized. For equivalent axial loading and bending loads on identical beams, the beam undergoing bending sees higher deflections. The analysis and comparison of these two types of loading on a simple beam is as follows:

For axial loading:

\[
F = \frac{EA}{L} \delta
\]

where \( F \) is the load force, \( A \) is the beam cross sectional area, \( L \) is the beam length, and \( \delta \) is the beam deflection at the end. From this, the stiffness is

\[
k_{\text{axial}} = \frac{EA}{L}
\]

For beam bending:

\[
F = \frac{3EI}{L^3} \delta
\]

where \( I \) is the second moment of area of a rectangular beam

\[
I = \frac{bh^3}{12}
\]
in which \( b \) is the width and \( h \) is the height of the beam. The stiffness is defined by

\[
k_{\text{bend}} = \frac{3EI}{L^3}
\]  

(2.6)

In the case where the beams have an \( L = 0.035 \) meters, \( b = 0.005 \) meters, and \( h = 0.001 \) meters, the bending stiffness is approximately 4000 times less than that of the axial stiffness. Because deflection is directly proportional to force in both beam bending and axial loading, the lower bending stiffness will result in much higher deflections at equivalent loads. Since high deflections are desired, the design approached storing torsion energy through beam bending. The modeling of such a spring design’s performance was based on derivations using the Euler-Bernoulli beam theory [4].
Chapter 3

Beam Modeling & Analysis

3.1 Beam Bending Boundary Conditions

In pursuing a planar torsion spring design in which the beams store energy in beam bending, mathematical modeling of beam bending is utilized to best determine beam boundary conditions that would maximize deflection before yielding. The three beam bending boundary conditions explored are shown in Figure 3-1. For each of the beam boundary conditions, deflection profiles, max stresses, energy stored, and stiffnesses are modeled.

(a) Fixed, Fixed-Roller

(b) Fixed, Pin-Roller

(c) Fixed, Free

In order to provide analogous comparison between different beam conditions, all beams have the listed properties. The dimensions of the beam used in the models are the same as those of the final presented spring design.

- Dimensions:
  - Length: 0.035 meters
  - Width: 0.005 meters
Height: 0.001 meters

- Material: Maraging Steel
  - Young's Modulus: $210 \times 10^9$ Pascals
  - Yield Stress: $2.0 \times 10^9$ Pascals
  - Ultimate Yield Stress: $3.5 \times 10^9$ Pascals

![Figure 3-1: (a) Fixed, Fixed-Roller; (b) Fixed, Pin-Roller; (c) Fixed, Free](image)

3.1.1 Fixed, Fixed-Roller Beam

The case in which the beam is fixed on one end and fixed-roller on the other is shown in Figure 3-1 (a). For the existing planar springs that use beam spokes, the fixed, fixed-roller boundary condition approximates the loading and stress characteristics. We will first derive the equations that describe the beam deflection, beam slope, and
beam bending moment. The amount of energy stored in bending and in tension are then be calculated, from which, stiffness can be found and compared.

In order to model the system, the deflection profile of the beam is first derived using the generalized equation for neutral axis deflection with respect to $x$

$$w(x) = Ax^3 + Bx^2 + Cx + D \quad (3.1)$$

where $x$ is the position along the length of the beam and $A$, $B$, $C$, and $D$ are constants that are dependent on the end conditions of the beam [4]. The derivative of the beam deflection equation

$$\ddot{w}(x) = 3Ax^2 + 2Bx + C \quad (3.2)$$

gives the slope of the beam as a function of position along the length. The second derivative of beam deflection is proportional to the bending moment along the length of the beam.

$$\dddot{w}(x) = 6Ax + 2B \quad (3.3)$$

From these three generalized equations, the following boundary conditions can be applied for a beam undergoing a bending deflection of $\delta$.

$$w(0) = \dot{w}(0) = 0 \quad (3.4)$$

due to the fixed condition at $x = 0$ and

$$w(L) = \delta; \quad \dot{w}(L) = 0 \quad (3.5)$$

due to the fixed, roller condition at $L = 0$. From the boundary conditions, the generalized constants can be solved and substituted for equations (3.1), (3.2), and (3.3).

$$w(x) = \frac{-2\delta}{L^3}x^3 + \frac{3\delta}{L^2}x^2 \quad (3.6)$$
\[
\dot{w}(x) = -\frac{6\delta}{L^3} x^2 + \frac{6\delta}{L^2} x 
\]

(3.7)

\[
\ddot{w}(x) = -\frac{12\delta}{L^3} x + \frac{6\delta}{L^2}
\]

(3.8)

With the generalized constants solved in terms of \( \delta \), Equation (3.6) can be plotted with the beam undergoing 0.01 meters of deflection.

Due to the fixed condition at each end of the beam, there is an inflection point at \( x = \frac{L}{2} \), where the change in slope of the beam is zero. The fixed condition and fixed distance between the ends of the beam make it such that as the beam deflects, the elongation of the beam due to bending increases the axial loading of the beam at high deflections. The equation for the elongated beam length is

\[
S = \int_0^L \sqrt{1 + \dot{w}(x)^2} \, dx
\]

(3.9)

where \( \dot{w}(x) \) is the slope of the beam as a function of distance along the length.
solved in Equation 3.7 [4]. The resulting elongation of the beam will be used to calculate and compare the stiffnesses and stresses of the different beams.

### 3.1.2 Fixed, Pinned-Roller Beam

In modeling the fixed, pinned-roller beam shown in Figure 3-1 (b), a similar approach was taken. In this case, while the boundary conditions due to the fixed end at $x = 0$ is the same as the fixed, fixed-roller beam,

\[
w(0) = \dot{w}(0) = 0 \tag{3.10}
\]

the boundary conditions at $x = L$ are

\[
w(L) = \delta; \quad \dot{w}(L) = 0 \tag{3.11}
\]

due to the pin. These boundary conditions, when used to solve for the generalized constants result in the following equations where

\[
w(x) = -\frac{\delta}{2L^3} x^3 \Bigg. + \frac{3\delta}{2L^2} x^2 \tag{3.12}
\]

describes the deflection as a function of position along the length of the beam,

\[
\dot{w}(x) = -\frac{3\delta}{2L^3} x^2 + \frac{3\delta}{L^2} x \tag{3.13}
\]

describes the slope of the beam, and

\[
\ddot{w}(x) = -\frac{3\delta}{L^3} x + \frac{3\delta}{L^2} \tag{3.14}
\]

describes the bending moment in the beam for a specific deflection, $\delta$.

Undergoing a deflection of 0.01 meters, the deflection profile of the beam can be seen in Figure 3-3

It is important to note that due to the boundary conditions at the pinned end, $w(L) = \delta; \quad \dot{w}(L) = 0$, the deflection profile of the fixed, pinned-roller beam is identical.
Figure 3-3: A fixed, pinned-roller beam undergoing 0.01 meters of deflection is plotted. to that of the fixed, free cantilever beam. However, unlike the fixed-free cantilever beam, the fixed distance between the fixed end and the pinned end result in an increase in axial stresses in the beam at high deflections. Similar to the fixed, fixed-roller beam, the equation for beam elongation is given by

\[ S = \int_0^L \sqrt{1 + \dot{w}(x)^2} \, dx \]  

(3.15)

where the different boundary conditions of the fixed, pinned-roller beam result in a different \( \dot{w}(x) \), solved in Equation 3.13.

### 3.1.3 Fixed, Free Beam

In the fixed, free beam, which is more commonly referred to as a cantilever beam, the boundary conditions are the same as that of the fixed, pin-roller beam.

\[ w(0) = \dot{w}(0) = 0 \]  

(3.16)
This results in the following equations and an identical beam deflection profile, shown in Figure 3-4.

\[ w(x) = \frac{-\delta}{2L^3} x^3 + \frac{3\delta}{2L^2} x^2 \]  
\[ \dot{w}(x) = \frac{-3\delta}{2L^3} x^2 + \frac{3\delta}{L^2} \]  
\[ \ddot{w}(x) = \frac{-3\delta}{L^3} x + \frac{3\delta}{L^2} \]

Figure 3-4: The deflection profile of a fixed, free cantilever beam undergoing 0.01 meters of deflection.

As modeled, the fixed, free beam is identical to the fixed, pinned-roller beam in deflection profile, beam slope, and beam bending moments. However, it is important to note that in the case of the cantilever beam, the axial elongation is zero and does not affect the stresses in the beam.
3.2 Beam Stresses and Stiffness Comparison

From the equations $w(x)$, $\dot{w}(x)$, and $\ddot{w}(x)$ for each beam, analysis on the amount of stress, bending energy, and tensile energy in each beam undergoing 0.01 meters of deflection can be performed.

3.2.1 Max Beam Stresses

Using superposition of axial and bending stresses in the beam, shown in Figure 3-5, the resulting maximum stress in each beam condition can be calculated. In the case of the fixed, fixed-roller beam and the fixed, pinned-roller beam, the maximum stress is equal to the sum of the bending stress and tensile stress. The tensile stress results from the elongation of the beam as it undergoes bending.

\begin{equation}
\sigma_{\text{total}} = \sigma_{\text{bend}} + \sigma_{\text{axial}}
\end{equation}

where

\begin{equation}
\sigma_{\text{bend}}(x, y) = \frac{M(x)y}{I}; \quad \sigma_{\text{axial}} = E\epsilon
\end{equation}

in which $x$ is the distance along the beam, $y$ is the distance from the neutral axis, and $\epsilon$ is the axial strain [4]. The bending stresses in each of the three beams is defined as

\begin{equation}
M(x) = -EI\ddot{w}(x)
\end{equation}

Figure 3-5: For the cases in which the beam is undergoing both bending and tensile loading, superposition of the stresses can be applied to calculate max stresses. As shown, the maximum stresses will occur on the top surface of the loaded beam.
in which $E$ is the Young’s Modulus of maraging steel, $I$ is the area moment of inertia of a rectangular cross section, and $\ddot{w}(x)$ were solved for each beam in Equations 3.8, 3.14, and 3.20 [4].

The max bending stress occurs at

$$x = 0; \quad y = \frac{h}{2}$$

(3.24)

for all beam cases, resulting in

$$\sigma_{bend} = \frac{M(0)\frac{h}{2}}{I}$$

(3.25)

In addition to bending stresses, the fixed, fixed-roller and fixed, pinned-roller beams also undergo axial stresses at higher deflections due to the elongation of the beam. The resulting axial stress is defined as

$$\sigma_{axial} = \frac{E(S - L)}{L}$$

(3.26)

where $S$ was solved for in Equations 3.9 and 3.15 for the fixed, fixed-roller and fix, pinned-roller beams, respectively. Using the superposition of stresses, the total max stress of each beam undergoing 0.01 meters of deflection is plotted in Figure 3-6.

From this comparison, it can be seen that at very small deflections, all beams increase in stress very similarly. However, as the deflection increases, the tensile stresses begin to dominate, and the fixed, fixed-roller beam and fixed, pinned-roller beam begin to see much higher maximum stresses. The rate of max stress increase is higher for the beams with more constraints at $x = L$.

The fixed, pinned-roller beam, while having the same deflection profile, begins seeing higher max stresses at high deflections. As expected, the max stresses of the fixed, pinned-roller beam is equal to that of the fixed, free beam for higher deflections than the fixed, fixed-roller beam.
3.2.2 Beam Stiffnesses

While the max stresses provide us valuable insight on the beams as they undergo deflection, it is important to understand the stiffnesses of each beam and how it changes with deflection. The stiffnesses of each beam was calculated by taking the numerical derivative of the energy stored in each beam. First, the total amount of energy stored in a beam as a function of deflection was calculated.

\[ U_{total}(\delta) = U_{bend}(\delta) + U_{axial}(\delta) \]  \hspace{1cm} (3.27)

\[ U_{bend}(\delta) = \frac{EI}{2} \int_{0}^{L} \ddot{w}(x)^2 dx \]  \hspace{1cm} (3.28)

where \( \ddot{w}(x) \) is defined by Equations 3.8, 3.14, and 3.20 for the fixed, fixed-roller
beam; fixed, pinned-roller beam; and fixed, free beam, respectively. Additionally, in the case of the fixed, fixed-roller beam and the fixed, pinned-roller beam, tensile energy is defined by

\[ U_{axial}(\delta) = \frac{AE}{2} \int_0^L \epsilon^2 dx \]  

(3.29)

After calculating the total amount of energy stored in each beam for \( 0 < \delta < 0.01 \) meters, the numerical derivatives were taken.

\[ \frac{dU_{total}}{d\delta} = F(\delta) \]  

(3.30)

\[ \frac{d^2U_{total}}{d\delta^2} = k(\delta) \]  

(3.31)

where \( k(\delta) \) is the stiffness of the beam as a function of deflection. From Figure 3-7, it can be seen that all three beams provide the same force when undergoing small deflections. However, as the fixed-roller and pinned-roller beams begin to undergo axial strain at higher deflections, the forces begin to differ drastically from that of the fixed-free beam which is undergoing pure bending.

The effect of axial loading does not become significant until approximately 0.002 meters of deflection. In Figure 3-6, this is also the deflection at which the max stresses of the three beams begin to diverge. However, from Figure 3-8, the axial loading’s effect on the stiffnesses of the beams is apparent at much lower deflections than 0.002 meters.

Figure 3-9 shows that the stiffness of the fixed, free cantilever beam is constant, as expected. Additionally, the stiffnesses of the fixed, free beam is up to 3 orders of magnitude less than that of the other two beams.
Figure 3-7: Plotting the beam force as a function of deflection, it can be seen that the fixed, fixed-roller beam and fixed, pinned-roller beam forces begin to increase drastically at higher deflections.
Figure 3-8: The stiffness of each of the three beams as a function of deflection.
Figure 3-9: The stiffness of a cantilever beam is independent of deflection.
Chapter 4

Spring Modeling

In designing a planar torsion spring that is capable of large angular deflections, it is desirable that the beams bending to store the torsional energy be as close to the fixed, free beam condition as possible. From analyzing the various beam bending conditions, such a beam configuration is desired to decrease stiffness, especially at high deflections. In pursuing such a design, the fixed, pinned-slotted beam design was explored, the first of which had the end of the beam following a straight, radial slot as the beam deflects (Figure 4-1). After exploring the efficiency and torque performance of this pinned, straight-slotted beam design, a more complex curved slot design was modeled and analyzed.

4.1 Pinned, Straight-Slot Constrained Beam

In the first fixed, pinned-slotted beam design, the slot was straight and allowed the pinned beam end to move radially as the inner hub of the spring turned.

4.1.1 Beam End Trajectory

In order to model the beam bending and forces on the pin, the trajectory of the beam end of an unconstrained cantilever beam was first calculated. In calculating this trajectory, it is assumed that the force required for deflection is applied at the
tip of the beam and the force is always perpendicular to the changing neutral axis of the beam.

Figure 4-1: A basic schematic of the beam and inner hub radius shows the pinned, straight-slot beam design.

Figure 4-2: The trajectory of the beam tip as it bends is calculated as a function of beam tip angle, deflection, and beam elongation due to bending.

The trajectory of the beam tip is calculated and plotted with the origin at the hub. In this calculation, the x and y-component of the end trajectory is calculated to be

\[ x = -R_{\text{inner}} - ((S - L)\cos(\theta_{b})) \]  \hspace{1cm} (4.1)

\[ y = \delta \]  \hspace{1cm} (4.2)
where $R_{inner} = 0.015$ meters and $S$ is the projected elongated length of a constrained beam undergoing bending. It is important to note here that the cantilever beam is not undergoing elongation because the beam is unconstrained at $x = L$.

$$S = \int_0^L \sqrt{1 + \dot{w}(x)^2} \, dx$$

(4.3)

where $\theta_b$ is the calculated beam angle with respect to the neutral axis at $x = L$.

$$\theta_b = \arctan(\dot{w}(x))$$

(4.4)

Figure 4-3: The trajectory of the beam tip undergoing 0.01 meters of deflection is shown. In calculating the trajectory, the origin is set at the center of the inner hub.

### 4.1.2 Beam and Slot Forces

In order to understand the torque response of this pinned, slotted beam design, the forces acting on the pin must be calculated. It is important to note that as the
inner radius turns and deflects the beam, the effective radius on which the forces act changes.

Figure 4-4: As the inner hub undergoes angular deflection, the resulting beam deflection changes the radius vector on which the torque is acting.

The deflected beam trajectories in Equations 4.1 and 4.2 were calculated with respect to the hub center as origin, and therefore are the x and y components of $\vec{R}_{vector}$.

$$R_x = -R_{inner} - ((S - L)\cos(\theta_b)); R_y = -\delta$$

From this, the $\theta_{turn}$ can be calculated.

$$\theta_{turn} = \arctan\left(\frac{R_y}{R_x}\right)$$

Figure 4-5 shows the two forces that act on the pin, and Figure 4-6 and Figure 4-7 shows the decomposition of these forces as they act on the pin and slot, respectively. $\vec{F}_{beam}$ and $\vec{F}_{slot}$ are both vectors that are dependent on $\theta_{turn}$.

$\vec{F}_{beam}$ is a result of the beam bending force and axial force. $\vec{F}_{slot}$ is a result of the friction force that acts on the pin, which acts along the slot, and the force that acts normal to the slot. The pin was modeled as having a zero diameter.

$$\vec{F}_{beam} + \vec{F}_{slot} = 0$$
Figure 4-5: The forces that act on the pin at the tip of the beam.

Figure 4-6: The forces due to each body acting on the pin is decomposed into their respective parts.

Of these forces, both the direction and magnitude of $\vec{F}_{bend}$ is known. For $\vec{F}_{axial}$, direction is known, but magnitude is unknown. Similarly, only the directions are known for both $\vec{F}_{friction}$ and $\vec{F}_{normal}$. In order to characterize the torque response of the beam, $\vec{F}_{slot}$ as a function of $\delta$ is required. From Equation 4.7 and what is known about the direction of the forces, the following equation is derived.

$$
|\vec{F}_{axial}| \begin{bmatrix} -\hat{F}_{bend-y} \\ \hat{F}_{bend-x} \end{bmatrix} + |\vec{F}_{slot}| \begin{bmatrix} \hat{F}_{slot-x} \\ \hat{F}_{slot-y} \end{bmatrix} + |\vec{F}_{bend}| \begin{bmatrix} \hat{F}_{bend-x} \\ \hat{F}_{bend-y} \end{bmatrix} = 0 \tag{4.8}
$$

where the unit vectors of $\vec{F}_{slot}$ are

$$
\hat{F}_{slot-x} = |\vec{F}_{normal}| \hat{R}_y + \mu |\vec{F}_{normal}| \hat{R}_x \tag{4.9}
$$
Figure 4-7: The forces acting on the slot are shown and are decomposed into their respective parts.

\[ \vec{F}_{\text{slot}-y} = -|\vec{F}_{\text{normal}}| \hat{R}_x + \mu |\vec{F}_{\text{normal}}| \hat{R}_y \]  

(4.10)

and

\[ |\vec{F}_{\text{bend}}| = 3EI \frac{\delta}{L^3} \]  

(4.11)

Substituting this into and rearranging Equation 4.8

\[
\begin{bmatrix}
-\vec{F}_{\text{bend}-y} & |\vec{F}_{\text{normal}}| \hat{R}_y + \mu |\vec{F}_{\text{normal}}| \hat{R}_x \\
\vec{F}_{\text{bend}-x} & -|\vec{F}_{\text{normal}}| \hat{R}_x + \mu |\vec{F}_{\text{normal}}| \hat{R}_y
\end{bmatrix}
\begin{bmatrix}
|\vec{F}_{\text{axial}}| \\
|\vec{F}_{\text{slot}}|
\end{bmatrix}
= -|\vec{F}_{\text{bend}}|
\begin{bmatrix}
\vec{F}_{\text{bend}-x} \\
\vec{F}_{\text{bend}-y}
\end{bmatrix}
\]  

(4.12)

\[
\begin{bmatrix}
|\vec{F}_{\text{axial}}| \\
|\vec{F}_{\text{slot}}|
\end{bmatrix}
= \begin{bmatrix}
-\vec{F}_{\text{bend}-y} & |\vec{F}_{\text{normal}}| \hat{R}_y + \mu |\vec{F}_{\text{normal}}| \hat{R}_x \\
\vec{F}_{\text{bend}-x} & -|\vec{F}_{\text{normal}}| \hat{R}_x + \mu |\vec{F}_{\text{normal}}| \hat{R}_y
\end{bmatrix}^{-1}
\begin{bmatrix}
-|\vec{F}_{\text{bend}}| \\
-|\vec{F}_{\text{bend}}|
\end{bmatrix}
\]  

(4.13)

From Equation 4.13, the magnitudes of \( \vec{F}_{\text{axial}} \) and \( \vec{F}_{\text{slot}} \) are calculated where \( \mu \) is the coefficient of friction between the pin and the slot. Using this, the entirety of \( \vec{F}_{\text{slot}} \) vector can be calculated for all \( \delta \).
\[
\vec{F}_{\text{slot}} = -\vec{F}_{\text{bend}} - \vec{F}_{\text{axial}}
\] (4.14)

### 4.1.3 Torque and Efficiency

From the $\vec{F}_{\text{slot}}$ calculated in Equation 4.14, the torque resulting from a single pinned, slotted beam is

\[
\vec{\tau} = \vec{R}_{\text{vector}} \times \vec{F}_{\text{slot}}
\] (4.15)

In the following case, $\mu = 0.2$ which is the coefficient of friction for lubricated steel-on-steel contact [8]. In order to simulate angular deflection in the opposite direction, $\mu = -0.2$ is used. Assuming that the torsion spring design has 10 beams, all acting in parallel, the torque response of one planar torsion spring is shown in Figure 4-8.

![Figure 4-8: With 10 beams acting in parallel, the torque response for turning the spring and then returning it to equilibrium is shown.](image)

In plotting the torque response, the effect of hardening can be observed. The
stiffness of the beams increase as the beams begin to see tensile stresses at higher deflections. Also, as expected, the torque response for $\mu = 0.2$ is higher than that of $\mu = -0.2$. When deflecting the beams in one direction, the effect of friction on the torque is additive, while in reversing the deflection, the effect is subtractive.

From the data presented in Figure 4-8, the efficiency of the spring as a function of deflection can also be calculated and plotted. Figure 4-9 shows the efficiency as a function of angular rotation of the inner hub for various $\mu$. This efficiency is calculated by taking the ratio of torque resulting from negative $\mu$ to torque resulting from positive $\mu$ at each deflection.

Figure 4-9: The efficiency of the spring as a function of angular rotation on the inner hub for various coefficients of friction is shown and compared.

It is demonstrated that efficiency is highly dependent on $\mu$ with lower efficiencies seen at higher $\mu$. If the spring is being designed for applications in which high efficiency is desired, lubrication and pin material are extremely important. However, as seen in Figure 4-10, higher $\mu$ allow for higher torque responses, at the cost of efficiency, especially at high deflections. Depending on the application of the torsion
spring, these parameters can be optimized to obtain the desired spring characteristics, whether it be high torque response or high efficiency.

![Torque Response of a Beam Spring](image)

Figure 4-10: The torque response of a 10 beam spring is shown for various coefficients of friction.

### 4.1.4 Max Stress

In order to estimate the max stress in the beam, Equation 3.25 was used. At a maximum angular deflection of $\pm \frac{\pi}{6}$ radians, the max stress in the cantilever beam is 2.4 GPa. For maraging steel, $\sigma_{ult} = 3.5$ GPa.

It is important to note that while the pinned, slotted beam used in this spring design mimics the behavior of a cantilever beam, there are axial stresses in the beam that are not estimated by this simple estimation. Therefore, it should be expected that max stresses be higher in the actual spring spokes. In order to decrease the max stress in a beam, the equation for moment about the neutral axis, which was solved in Equation 3.23, can be explored. It can be seen that, $M(x)$ and in turn, the max stress can be decreased as $L$ is increased. This has a quadratic effect on the max stress in the bending cantilever beam. Furthermore, a variable cross-sectional area
beam can be explored to further decrease stiffness and mass.

4.2 Curved Slotted Design

In designing the spring for exoskeleton applications, efficiency is an important factor that should be optimized, especially at higher deflections. In the straight-slot design, higher deflections resulted in drastically lower efficiencies. In attempting to optimize the slot design, the use of a curved slot was explored. Shown in Figure 4-11, the curved slot is configured such that at any given angular deflection of the inner hub, the slot at that point is angled $\theta_{slot}$ with respect to the radius vector to the beam's end. This results in a curved slot design similar to that shown in Figure 4-12.

![Figure 4-11: The curvature of the slot is such that at any given $\theta_{turn}$, the angle between the slot at that point and the radius vector, $\theta_{slot}$, is constant.](image)

4.2.1 Beam and Slot Forces

In analyzing the forces that acts on the pin, the approach was very similar to that of the straight slot done in Equations 4.1 - 4.14, except where before, the slot was along
the same vector as $\vec{R}_{vector}$, the slot vector is now angled with respect to the $\vec{R}_{vector}$. In this curved slot case, $\vec{F}_{friction}$ and $\vec{F}_{normal}$ now act on the angled slot vector, as shown in Figure 4-13.

\begin{align*}
\vec{C}_x &= -\cos(\theta_{\text{turn}} + \theta_{\text{slot}}) \\
\vec{C}_y &= -\sin(\theta_{\text{turn}} + \theta_{\text{slot}})
\end{align*}

Similar to the calculations done for the straight slot, the magnitude and direction is known for $\vec{F}_{bend}$ as $\delta$ increases, but for the $\vec{F}_{axial}$, $\vec{F}_{friction}$, and $\vec{F}_{normal}$ vectors, only
direction is known. In order to characterize the torque response of the beam, $\bar{F}_{slot}$ as a function of angular deflection of the spring must be calculated.

Similar to Equation 4.7, force balance on the slot gives us the following.

\[
|\bar{F}_{axial}| \begin{bmatrix} -\hat{F}_{bend-y} \\ \hat{F}_{bend-x} \end{bmatrix} + |\bar{F}_{slot}| \begin{bmatrix} \hat{F}_{slot-x} \\ \hat{F}_{slot-y} \end{bmatrix} + |\bar{F}_{bend}| \begin{bmatrix} \hat{F}_{bend-x} \\ \hat{F}_{bend-y} \end{bmatrix} = 0 \tag{4.18}
\]

However, in curved slot case, the components of $\bar{F}_{slot}$ are defined as

\[
\hat{F}_{slot-x} = |\bar{F}_{normal}| \hat{C}_y + \mu |\bar{F}_{normal}| \hat{C}_x \tag{4.19}
\]

\[
\hat{F}_{slot-y} = -|\bar{F}_{normal}| \hat{C}_x + \mu |\bar{F}_{normal}| \hat{C}_y \tag{4.20}
\]

Substituting this into and rearranging Equation 4.18

\[
\begin{bmatrix} -\hat{F}_{bend-y} \\ \hat{F}_{bend-x} \end{bmatrix} \begin{bmatrix} |\bar{F}_{normal}| \hat{C}_y + \mu |\bar{F}_{normal}| \hat{C}_x \\ \hat{F}_{normal}| \hat{C}_x + \mu |\bar{F}_{normal}| \hat{C}_y \end{bmatrix} \begin{bmatrix} |\bar{F}_{axial}| \\ |\bar{F}_{slot}| \end{bmatrix} = -|\bar{F}_{bend}| \begin{bmatrix} |\bar{F}_{bend-x}| \\ |\bar{F}_{bend-y}| \end{bmatrix} \tag{4.21}
\]

\[
\begin{bmatrix} |\bar{F}_{axial}| \\ |\bar{F}_{slot}| \end{bmatrix} = \left[ \begin{bmatrix} -\hat{F}_{bend-y} \\ \hat{F}_{bend-x} \end{bmatrix} - \begin{bmatrix} |\bar{F}_{normal}| \hat{C}_y + \mu |\bar{F}_{normal}| \hat{C}_x \\ |\bar{F}_{normal}| \hat{C}_x + \mu |\bar{F}_{normal}| \hat{C}_y \end{bmatrix} \right]^{-1} \begin{bmatrix} -|\bar{F}_{bend}| \\ \bar{F}_{slot} \end{bmatrix} \tag{4.22}
\]

From Equation 4.22, the magnitudes of $\bar{F}_{axial}$ and $\bar{F}_{slot}$ are calculated where $\mu$ is the coefficient of friction between the pin and the slot. Using this, the entirety of $\bar{F}_{slot}$ vector can be calculated for all deflections.

\[
\bar{F}_{slot} = -\bar{F}_{bend} - \bar{F}_{axial} \tag{4.23}
\]
4.2.2 Efficiency

In order to calculate the efficiency contour shown in Figure 4-14, the efficiency is calculated for turning the spring in one direction and then back to zero for $\mu = 0.2$, which is the coefficient of friction for lubricated steel-on-steel contact [8]. Efficiency was calculated by taking the ratio of torque resulting from $-\mu$ to torque resulting from $\mu$ at each $\theta_{\text{turn}}$ for $-0.5$ radians $< \theta_{\text{slot}} < 0.5$ radians. The results shown in Figure 4-14 can be used to design a slot geometry function that optimizes efficiency for a particular range of motion depending on the application.

Figure 4-14: The efficiency contour plot shows the effect of $\theta_{\text{turn}}$ and $\theta_{\text{slot}}$ on the efficiency of the spring. A $\theta_{\text{slot}} = 0$ shows the efficiency of the straight slot spring design discussed in Section 4.1.
Figure 5-1: The 10-beamed, straight slotted torsion spring design is shown. In the resulting design $D_{outer} = 0.112$ meters, $D_{inner} = 0.05$ meters, and $L = 0.035$ meters.

In the resulting spring design, the beams that undergo bending to provide the angular deflection are fixed onto the outer hub on one end but are constrained using a pin and slot to the inner hub on the other. Torsional compliance is provided as the outer hub rotates with respect to the inner hub, bending the slender beams. The
resulting maraging steel torsion spring has a mass of 98 grams, outer diameter of 0.112 meters, and width of 0.005 meters. The spring uses slender beams which have a length of 0.035 meters, height of 0.001 meters, and width of 0.005 meters. The resulting spring design is predicted to be capable of rotating $\pm \frac{\pi}{6}$. This max angular rotation is 6 times that of the NASA planar torsion spring, which has a slightly smaller diameter of 0.085 meters.

While the mathematical models for the torsion spring design presented provide a good foundation from which to move forward, the next step is to create a physical prototype of the straight-slotted spring design and perform testing. Through testing, the actual torque responses and efficiencies can be explored, especially at higher angular rotations, and the model revised as necessary. For applications in prosthesis and exoskeletons, efficiency is also of importance and features such as using a bearing at the pin-slot interface or using another method of providing rolling contact to reduce frictional losses can also be explored.

A fully parametrized model can be developed such that the effects of material, beam width, beam thickness, and slot design on efficiency, torque response and deflection are understood. This will create a baseline understanding from which this particular torsion spring design can be customized for specific applications. Such a model could also help optimize the spring size, weight, max stresses, and stiffness. In addition to improving the model, design features can be explored to further minimize the mass and size of the spring. One possible route would be to design the shape of the beams to make more efficient use of the mass by equalizing the stress along the surface of the beam, where the max stress occurs for beam bending.
Chapter 6

Conclusion

The objective of this thesis was to develop a torsion spring for a knee joint exoskeleton application. The resulting functional requirements that resulted from this specific application were:

- provide torque response of 100 N·m
- capable of max deflections of ± 6 π radians
- minimize resulting design’s size and weight

The resulting spring design has a mass of 98 grams, which includes the mass of the inner hub, outer hub, and slender beams. It has an outer diameter of 0.112 meters and width of 0.005 meters. With a similar form factor and material, the resulting pinned, slotted beam spring design provided higher deflections than the existing NASA planar torsion spring.

- material is maraging steel
- maximum diameter of 0.085 meters
- maximum width (planar thickness) of 0.005 meters

In the compact design of a planar torsion spring, it is important to note that the spring thickness can be adjusted to obtain the desired stiffness and maximum torque.
While having a pin, straight-slot constrain on the inner hub has the disadvantage of friction forces and efficiency losses, it allows the spring to undergo much higher angular deflections than existing planar torsion springs. Furthermore, this novel design opens up an entire design space with potential optimization and performance trade-offs that fixed, fixed beam torsion springs lack. The main advantage of the presented spring design is the ability to undergo comparatively higher angular deflections, and this thesis provides the fundamentals required to further parametrize and optimize the torsion spring design for specific applications.
Bibliography


