Problem Set 3

Due: February 26

Reading: Chapter 6, *Induction*; Chapter 7, *Partial Orders*, §§1–3.

Problem 1.
For any sets, A, and B, let $[A \to B]$ be the set of total functions from A to B. Prove that if A is not empty and B has more than one element, then $\text{NOT}(A \text{ surj } [A \to B])$.

Hint: Suppose there is a function, σ, that maps each element $a \in A$ to a function $\sigma_a : A \to B$. Pick any two elements of B; call them 0 and 1. Then define

$$\text{diag}(a) ::= \begin{cases} 0 \text{ if } \sigma_a(a) = 1, \\ 1 \text{ otherwise.} \end{cases}$$

Problem 2.
Fibonacci numbers are defined as follows:

$$\begin{aligned} F(0) &::= 0, \\ F(1) &::= 1, \\ F(n) &::= F(n-1) + F(n-2) \qquad (\text{for } n \geq 2). \end{aligned} \tag{1}$$

Thus, the first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, and 21. Prove by induction that for all $n \geq 1$,

$$F(n-1) \cdot F(n+1) - F(n)^2 = (-1)^n. \tag{2}$$

Problem 3.
For any binary string, α, let $\text{num}(\alpha)$ be the nonnegative integer it represents in binary notation. For example, $\text{num}(10) = 2$, and $\text{num}(0101) = 5$.

An $n+1$-*bit adder* adds two $n+1$-bit binary numbers. More precisely, an $n+1$-bit adder takes two length $n+1$ binary strings

$$\begin{aligned} \alpha_n &::= a_n \dots a_1 a_0, \\ \beta_n &::= b_n \dots b_1 b_0, \end{aligned}$$

and a binary digit, c_0, as inputs, and produces a length $n+1$ binary string

$$\sigma_n ::= s_n \dots s_1 s_0,$$

and a binary digit, c_{n+1}, as outputs, and satisfies the specification:

$$\text{num}(\alpha_n) + \text{num}(\beta_n) + c_0 = 2^{n+1}c_{n+1} + \text{num}(\sigma_n). \tag{3}$$

There is a straighforward way to implement an $n+1$-bit adder as a digital circuit: an $n+1$-bit *ripple-carry circuit* has $1 + 2(n+1)$ binary inputs

$$a_n, \ldots, a_1, a_0, b_n, \ldots, b_1, b_0, c_0,$$

and $n+2$ binary outputs,

$$c_{n+1}, s_n, \ldots, s_1, s_0.$$

As in Problem 3.5, the ripple-carry circuit is specified by the following formulas:

$$s_i ::= a_i \text{ XOR } b_i \text{ XOR } c_i \tag{4}$$

$$c_{i+1} ::= (a_i \text{ AND } b_i) \text{ OR } (a_i \text{ AND } c_i) \text{ OR } (b_i \text{ AND } c_i),. \tag{5}$$

for $0 \le i \le n$.

(a) Verify that definitions (4) and (5) imply that

$$a_n + b_n + c_n = 2c_{n+1} + s_n. \tag{6}$$

for all $n \in \mathbb{N}$.

(b) Prove by induction on n that an $n+1$-bit ripple-carry circuit really is an $n+1$-bit adder, that is, its outputs satisfy (3).

Hint: You may assume that, by definition of binary representation of integers,

$$\text{num}(\alpha_{n+1}) = a_{n+1}2^{n+1} + \text{num}(\alpha_n). \tag{7}$$

Problem 4.

Let R and S be transitive relations on a set, A. For each of the relations below, either prove that it is transitive, or give a counter-example showing that it may *not* be transitive.

- R^{-1}
- $R \cap S$
- $R \cup S$
- $R \circ S$

Student's Solutions to Problem Set 3

Your name:	
Due date:	February 26
Submission date:	
Circle your TA/LA:	Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '10 course materials.

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