Problem Set 11

Due: April 30

Reading: Redefine the \reading command to get this week's reading assignment here!

Problem 1.
Let $x_0 ::= 0, x_1 ::= 1$ and for $n \geq 2$, let x_n be defined by the linear recurrence:

$$x_n = 3x_{n-1} - 2x_{n-2} + n.$$

Find a closed form expression for x_n.

Problem 2.
[The Four-Door Deal]

Let's see what happens when *Let's Make a Deal* is played with **four** doors. A prize is hidden behind one of the four doors. Then the contestant picks a door. Next, the host opens an unpicked door that has no prize behind it. The contestant is allowed to stick with their original door or to switch to one of the two unopened, unpicked doors. The contestant wins if their final choice is the door hiding the prize.

Use The Four Step Method of Section 18.1 to find the following probabilities. The tree diagram may become awkwardly large, in which case just draw enough of it to make its structure clear.

(a) Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that Stu wins the prize?

(b) Contestant Zelda, an alien abduction researcher from Helena, Montana, switches to one of the remaining two doors with equal probability. What is the probability that Zelda wins the prize?

Problem 3.
Suppose $\Pr\{\} : \mathcal{S} \rightarrow [0, 1]$ is a probability function on a sample space, $\mathcal{S}$, and let B be an event such that $\Pr\{B\} > 0$. Define a function $\Pr_B\{\cdot\}$ on events outcomes $w \in \mathcal{S}$ by the rule:

$$\Pr_B\{w\} ::= \begin{cases} \Pr\{w\} / \Pr\{B\} & \text{if } w \in B, \\ 0 & \text{if } w \notin B. \end{cases} \tag{1}$$

(a) Prove that $\Pr_B\{\cdot\}$ is also a probability function on $\mathcal{S}$ according to Definition 18.2.2.

(b) Prove that

$$\Pr_B\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

for all $A \subseteq \mathcal{S}$.

Problem 4.

Taking derivatives of generating functions is another useful operation. This is done termwise, that is, if

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots,$$

then

$$F'(x) ::= f_1 + 2f_2 x + 3f_3 x^2 + \cdots.$$

For example,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{(1-x)}\right)' = 1 + 2x + 3x^2 + \cdots$$

so

$$H(x) ::= \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots$$

is the generating function for the sequence of nonnegative integers. Therefore

$$\frac{1+x}{(1-x)^3} = H'(x) = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots,$$

so

$$\frac{x^2+x}{(1-x)^3} = xH'(x) = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \cdots + n^2 x^n + \cdots$$

is the generating function for the nonnegative integer squares.

(a) Prove that for all $k \in \mathbb{N}$, the generating function for the nonnegative integer kth powers is a quotient of polynomials in x. That is, for all $k \in \mathbb{N}$ there are polynomials $R_k(x)$ and $S_k(x)$ such that

$$[x^n]\left(\frac{R_k(x)}{S_k(x)}\right) = n^k. \tag{2}$$

Hint: Observe that the derivative of a quotient of polynomials is also a quotient of polynomials. It is not necessary work out explicit formulas for R_k and S_k to prove this part.

(b) Conclude that if $f(n)$ is a function on the nonnegative integers defined recursively in the form

$$f(n) = af(n-1) + bf(n-2) + cf(n-3) + p(n)\alpha^n$$

where the $a, b, c, \alpha \in \mathbb{C}$ and p is a polynomial with complex coefficients, then the generating function for the sequence $f(0), f(1), f(2), \ldots$ will be a quotient of polynomials in x, and hence there is a closed form expression for $f(n)$.

Hint: Consider

$$\frac{R_k(\alpha x)}{S_k(\alpha x)}$$

Student's Solutions to Problem Set 11

Your name:

Due date: April 30

Submission date:

Circle your TA/LA: Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '10 course materials.