ABSTRACT

This paper examines robustness issues in Model Reference Adaptive Control systems in the presence of unmodeled dynamics and output disturbances. We present an approximate technique, trend analysis, by which we can study the evolution of the parameter error trajectory under periodic excitation. This analysis provides new insights upon the size and spectral content of the excitation sufficient to guarantee local stability.

I. INTRODUCTION

Several similar Model Reference Adaptive Controllers (MRAC's) have been shown to be globally stable under certain restrictive assumptions, including the assumption that the order and relative degree of the plant are exactly known, and that no disturbances are present ([1]-[3]). Under certain "sufficiently rich" excitation conditions, the origin is globally asymptotically stable for the adaptive controller parameter error ([5],[6]).

When the restrictive assumptions are violated, as they always are in practice, no proof of stability exists [4],[9],[10]. Furthermore, instability can occur under excitations which are "sufficiently rich" in the sense mentioned above. A stronger definition of sufficient richness is required to guarantee stability of the adaptive controller in the presence of unmodeled dynamics and disturbances.

The key factor to the stability of the adaptive controller is the time-evolution of the parameter error vector. This time-evolution is described by a complicated set of time-varying nonlinear differential equations, putting a closed-form analytic solution out of reach. The relevant theorems are stated here and proven in [8]. Thus the analysis technique can be termed a trend analysis for slowly adapting controllers.

The trend analysis provides a vector field, defined as a function of the command input, which approximates (in a long term sense) the time-derivative of the parameter error vector. From this vector field one can determine potential limit sets of the parameter error vector, and associated regions of attraction and approximate rates of convergence. New excitation conditions are defined which are sufficient for stability of the adaptive controller in the presence of unmodeled dynamics. Direction of future research are indicated which are required to enable analytic evaluation of some of the sufficient condition for a controller with a given adaptive gain.

At the present state of our studies, the new richness conditions provide substantial insight into the type and size of excitation required for stability. With reasonable designer-known information, one can determine a desirable frequency range for the command input energy, and the most desirable subset of this range. One can also estimate the amplitude of command input required to prevent a dangerous slow drift of the parameter error due to an output disturbance of any frequency.

We present the problem and establish notation in Section II. The approximate analysis technique is presented in Section III, and theorems which justify its use are stated. Sufficient excitation conditions for stability are given and discussed. The primary insights and results derived from the analysis are briefly stated and discussed.

II. THE ADAPTIVE CONTROL SYSTEM

The designer assumes that the low frequency portion of the single-input-single-output plant can be described by

\[ y(s) = \hat{G}(s)u(s) \]  

where \( \hat{G}(s) \) is of order \( n \). The poles and zeroes of \( \hat{G}(s) \) are unknown but are assumed to lie within some known bounds.

The actual plant input/output relationship is

\[ y(s) = G(s)u(s) + d(s) \]  

where

\[ G_p(s) = \hat{G}(s)(1+E(s)) \]  

The quantity \( d(s) \) is an output disturbance, and \( E(s) \) is a modeling error due to neglected high-frequency plant dynamics. An upper bound \( E_p(\omega) \) on the magnitude of \( E_p(\omega) \) is generally known in practice [7].

A model output is constructed:

\[ y_M(s) = G_M(s)r(s) \]  

where \( r(s) \) is the command input. The plant control input for adaptive control is

\[ u(t) = W(t)k(t) \]  

where \( k(t) \) is a 2n-vector of adjustable parameters, and

\[ W(t) = H(t)* \begin{bmatrix} x(t) \n u(t) \n y(t) \end{bmatrix} \]  

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where * denotes convolution and H(t) is the impulse response of a known causal linear time-invariant system. The vector \( \mathbf{w}(t) \) is constructed by the adaptive controller.

If \( k(t) \) is assigned a constant value \( k_0 \), a time-invariant closed-loop plant results.

Classical robustness techniques can be used to determine which values of \( k_0 \) result in guaranteed stability of the closed-loop plant. Define \( K_0 \) to be the set of all such stabilizing \( k_0 \).

The adjustable parameter vector \( k(t) \) can be considered a sum of a constant desired value and an error:

\[
k(t) = k^* + \Delta k(t)
\]

\( k^* \) is such that \( \Delta k(t) = 0 \) and \( E_0(s) = 0 \) results in \( y(s)/r(s) = y_0(s)/r(s) \). With an appropriately chosen model, such model matching can result in a desirable performance improvement over a nonadaptive alternative design [8].

An error signal is defined:

\[
e(t) = y(t) - y_0(t) = T_1(t)w(t)T_2(t)k(t) + T_3(t)d(t) (8)
\]

where \( T_1(t), T_2(t), \) and \( T_3(t) \) are impulse responses of causal linear systems. The Laplace transform \( T_1(s) \) of \( T_1(t) \) is of the form

\[
T_1(s) = T_0(s)(1 + E_0(s))
\]

where \( T_0(s) \) is a strictly positive real known linear system, \( E_0(s) \) is unknown but shares the characteristic shape of \( E_0(s) \). By choosing a grid on the space of possible plant parameters, one can numerically map the known upper bound on \( |E_0(j\omega)|, |T_2(j\omega)|, \) and \( |T_3(j\omega)| \). When \( E_0(j\omega) = 0 \), \( E_0(j\omega) = T_2(j\omega) = 0 \). These upper bounds are important in the application of the trend analysis.

The parameter update law is

\[
\dot{k}(t) = \gamma e(t) (10)
\]

where the adaptive gain \( \gamma \) is any strictly positive scalar constant.

Equations (8) and (10) characterize the parameter error behavior as a function of \( w(t) \). When \( E_0(s) = 0 \) and \( d(s) = 0 \), the adaptive control system is stable in the sense that \( e(t) \) is bounded and all signals are uniformly bounded. With nonzero unmodeled dynamics, no stability proof has been given, and potential instability has been demonstrated [4, 9, 10].

B. Presentation and Validation of the Trend Analysis

The trajectory of the adjustable parameter vector \( k(t) \) is critical to the stability of the system. If \( k(t) \) approaches a limit inside \( K_0 \), the system is stable. If \( k(t) \) strays outside of \( K_0 \), a rapid onset of instability may occur and has been observed.

The differential equations describing the time evolution of \( k(t) \) are highly nonlinear and time-varying, and cannot be solved analytically, hence we employ approximate analysis techniques. A vector \( k^*(t) \) is defined which has a much simpler trajectory and which represents the long term trends of the vector \( k(t) \). The accuracy of the trend analysis is shown to improve as the rate of adaptation is reduced.

The trend vector trajectory is a function of the system excitation. The command input studied is periodic and of the form

\[
r(t) = r(0) + \frac{1}{2} \sum_{i=1}^{N} \sin(\omega_i t + \theta_i), \omega \in \omega_0
\]

To show the relationship between the trajectory of the trend vector \( k^*(t) \) and the actual trajectory of \( k(t) \), and to aid the definition of the trend, we first present the limiting case in which the rate of adaptation is zero:

\[
k(t) = k_0 \in K_0 \forall t
\]

The usual parameter update law (10) may be modified to function as an observer:

\[
\dot{k} = -y_0 T(t) e(t)
\]

With \( r(t) \) as in (11), equations (8), (13) and the identity

\[
sin(x+y) = \sin(x) \cos(y) + \cos(x)\sin(y)
\]

yield

\[
\dot{k} = \gamma (\Delta d_c \Delta a_c(t)) k_0 + \beta d_c + \beta a_c(t)
\]

where \( \Delta d_c, \Delta a_c(t) \) are 2nx2n and \( b_{d_c}, b_{a_c}(t) \) are 2nx1. The elements of \( \Delta d_c \) and \( \Delta a_c \) are constants, and the elements of \( b_{d_c}(t) \) and \( b_{a_c}(t) \) are sums of sinusoids of nonzero frequency. \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \) are readily-computed functions of accessible signals \( y(t) \) which is a function of \( r(t) \) and \( y(t) \) and as such can be considered user-known. Aiding computation and analysis is the property that each of the frequency components present in \( r(t) \) contributes independently to \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \).

In the absence of an output measurement, \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \) are a function of \( r(t) \) and \( k_0 \), and are defined for all \( r(t) \) of the form of equation (11) and all \( k_0 \in K_0 \).

Exact computation of \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \) is no longer so easy. Nominal values of \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \) can be computed a priori for each possible \( k_0 \in K_0 \). The actual values of \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \) will be a perturbed version of these nominal values.

A bound on the perturbation can be calculated. The sign of the real part of the eigenvalues of the computed \( \Lambda_{d_c} \) can never be changed by inclusion of any stable unmodeled dynamics, but the presence of unmodeled dynamics in \( T_1(s) \) can cause such a change.

To avoid confusion between the various sources of error, we will subsequently ignore the issue of a priori calculation of \( \Lambda_{d_c} \) and \( \Lambda_{a_c} \) without an output measurement, and shall concentrate only on the errors relevant...
The bounds $z_1$ and $z_2$ are known to exist for sufficiently small $\gamma$, and in fact approach zero as $\gamma$ approaches zero.
Condition 2:

If condition 2 is satisfied, then

\[ \| \tilde{z}_2 - z \|_2 \leq z_1 - z_1 \]  

(32)

Therefore

\[ \tilde{x}_m \in \beta_1, \quad m = 0, 1, 2, \ldots \]  

(33)

and \( \beta_1 \) is a locally stable region for \( \tilde{x}(t) \). Completion of the proof would involve a demonstration that equation (34) and \( \beta_1 \) implies boundedness of other system signals.

In effect, condition 2 requires that \( F \) be oriented inward around the boundary of a region, and that this inward orientation exceed the error bound using a particular standard of measurement. Condition 3 to follow has a similar interpretation but employs a different standard of measurement.

For any positive \( \rho_3 > 0 \), define

\[ \tilde{z}_3 = \frac{\tilde{z}_3}{\| \tilde{z}_3 \|_2} \]  

(34)

\[ \tilde{z}_4 = \frac{\tilde{z}_4}{\| \tilde{z}_4 \|_2} \]  

(35)

\[ z_3 = \max_{\tilde{x}_m \in \beta_1} \arccos \left( \frac{\tilde{x}_1^T \tilde{x}_3}{\| \tilde{x}_1 \| \| \tilde{x}_3 \|} \right) \]  

(36)

That is, \( z_3 \) is an upper bound on the angle between \( \tilde{x}_1 \) and \( \tilde{z}_3 \) for all \( \tilde{x}_m \) in \( \beta_1 \). For a sufficiently small \( \gamma \), \( z_3 \) exists; as \( \gamma \) approaches zero, \( z_3 \) approaches zero.

Condition 3:

If \( \tilde{x}(t) \) is sufficient for local stability of the adaptive controller if there exists \( \rho_3 > 0 \) and \( \tilde{x} \) such that

\[ \rho_3 \tilde{x}_m \in \beta_3 \]  

(37)

Partial proof: The use of \( \beta_3 \) rather than \( \beta_1 \) in the condition is a subtlety that shall not be discussed here.

Denote

\[ \tilde{k}(t_0) = \tilde{k}_0 \]  

(29)

\[ \tilde{k}(k + r) = \tilde{k}_1 = \tilde{k}_1 + \Delta \]  

(30)

Thus

\[ \| \tilde{k}_1 - \tilde{k} \|_2 \leq \| \tilde{k}_1 - \tilde{k} \|_2 + \| \tilde{k}_1 - \tilde{k} \|_2 < z_1 + \rho_1 - z_1 \]  

(31)

\[ \| \tilde{k}_1 - \tilde{k} \|_2 \leq \| \tilde{k}_1 - \tilde{k} \|_2 \]  

(32)

Therefore

\[ \tilde{k}_m \in \beta_1, \quad m = 0, 1, 2, \ldots \]  

(33)

and \( \beta_1 \) is a locally stable region for \( \tilde{x}(t) \). Completion of the proof would involve a demonstration that equation (33) and \( \beta_1 \) implies boundedness of other system signals.

The proof is simple and is left to the reader. A few words must now be said about the practical application of these conditions, lest the reader be misled as to the present state of our research.

Testing of condition 1 for stability in the limit is currently feasible. Equilibrium points \( \tilde{x}_E \) are defined by

\[ F_L (\tilde{x}_E, z) = 0 \]  

(37)

Since \( F_L (\tilde{x}_E, z) \) is continuous, one can check for local stability of an equilibrium point through linearization.

Testing of conditions 2 and 3 is not feasible without further research. Computation of the bounds \( z_1 \), \( z_2 \), and \( z_3 \) is not currently possible. The theorems contained herein guarantee that any desired value for any of these bounds can be achieved through the use of some sufficiently small \( \gamma \), but the value of \( \gamma \) required is not clear due to the nature of the proofs.

The stabilization technique employed in [8] involves selection of a reference input which is sufficient for stability in the limit as \( \gamma \) approaches zero, followed by an auxiliary selection of a "small" value for \( \gamma \). Simulations indicate that 1) this process is often very overconservative, and 2) the meaning of "small" changes substantially with different plants, models, and excitations. Tight bounds on \( z_1 \), \( z_2 \), \( z_3 \) as a function of these different elements shows promise of allowing the use of much a larger \( \gamma \) (hence faster adaptation) with confidence.

D. Trend Analysis Results

The trend analysis technique constitutes an analytic quantification of the relationship between parameter convergence and the spectral components of the excitation. As such it has provided new information and insights into the stability of model reference adaptive control systems. Some of these results shall be stated here without derivation.

Though the stability problems of adaptive controllers has been well publicized recently, it is worth emphasizing that the trend analysis provides analytic verification of the potential instability of MRAC's. It also provides verification of a potential stability of the controllers and a means of obtaining stability without a modification of the basic algorithm. One can determine a periodic excitation which is capable of stabilizing the system, and add this excitation to the command input.

The excitation should be chosen to provide stability in the limit (condition 1), and allow wide margins for the errors with the unknown bounds \( z_1 \), \( z_2 \), and \( z_3 \). Determination of such an excitation is simplified by an empirically observed property of the matrix \( A_{dc} (\tilde{x}_E) \). Given \( z \), \( A_{dc} \) and \( B_{dc} \) vary relatively slowly with \( \tilde{x} \), such that the properties of \( F_L (\tilde{x}_E, z) \) in some neighborhood of an equilibrium point \( \tilde{x}_E \) are revealed by considering the \( F_L \) resulting from the re-defined trend

\[ \dot{\tilde{x}}' = A_{dc} (\tilde{x}_E, z) \tilde{x}' + B_{dc} \]  

(38)

Hence we have, as a minimal condition for asymptotic stability in the limit of an equilibrium point \( \tilde{x}_E \), the requirement that all the eigenvalues of \( A_{dc} (\tilde{x}_E, z) \) lie in the open left half plane.

We can place further requirements on \( A_{dc} \) (and hence indirectly on \( r(t) \)) to allow a margin for errors of the type appearing in conditions 2 and 3. For example, a large margin for \( z_3 \) of equation (36) would
be provided by an \(A_{dc} \) which gives a small value (large in the negative direction) to \(z_4 \), where

\[
z_4 = \max \left(\frac{T_{\text{dc}}}{K} \right)^x \left(\frac{T}{K} \right)^x \tag{39}
\]

For the case of two adjustable parameters, the smallest values of \(z_4 \) are obtained by choice of a command with frequency content concentrated near the model bandwidth. Spectral components of the command above a threshold frequency \(\omega_T \) serve to increase \(z_4 \). Acting alone these high frequency components yield an \(A_{dc} \) with positive eigenvalues.

The threshold frequency \(\omega_T \) is the frequency at which \(T_1(j\omega) \) has a phase of 90° (see equation (8)). This has an interesting correspondence to the strictly positive citation required. Further research is required to develop such conditions through the approximate analysis technique sketched in this paper. Valuable insights have resulted as to the type and size of excitation required. Further research is required to enhance the usefulness of the new richness conditions.

IV. CONCLUSIONS

For several promising adaptive algorithms, in the absence of unmodeled dynamics and output disturbances, weak richness conditions on the excitation guarantee convergence of the adaptive controller parameter error to zero. Under more realistic assumptions, convergence to zero parameter error must be given up. To achieve the weaker goal of stability and bounded parameter error without a modification of the algorithms, one must impose stronger conditions on the excitation. One can develop such conditions through the approximate analysis technique sketched in this paper. Valuable insights have resulted as to the type and size of excitation required. Further research is required to enhance the usefulness of the new richness conditions.

REFERENCES


