Equivariant Quantum Cohomology and the Geometric Satake Equivalence

by

Michael Viscardi

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

Recent work on equivariant aspects of mirror symmetry has discovered relations between the equivariant quantum cohomology of symplectic resolutions and Casimir-type connections (among many other objects). We provide a new example of this theory in the setting of the affine Grassmannian, a fundamental space in the geometric Langlands program. More precisely, we identify the equivariant quantum connection of certain symplectic resolutions of slices in the affine Grassmannian of a semisimple group $G$ with a trigonometric Knizhnik-Zamolodchikov (KZ)-type connection of the Langlands dual group of $G$. These symplectic resolutions are expected to be symplectic duals of Nakajima quiver varieties, and thus our result is an analogue of (part of) the work of Maulik and Okounkov in the symplectic dual setting.

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Chapter 1

Introduction

The problem considered in this thesis lies at the intersection of three major dualities of modern mathematics and physics. In roughly chronological order, the three dualities are:

- **Geometric Langlands duality.** The Langlands program, a vast program connecting number theory and representation theory, was initiated by Langlands in the 1960s [40]. Its analogue in the geometry, the geometric Langlands program, was initiated by Laumon, Beilinson, Drinfeld, Ginzburg, and others [41, 4, 26], and is intensely studied today both in mathematics and in physics [39, 24, 37]. The basic objects of study are a reductive algebraic group $G$ (such as $GL_n$ or $SO_n$) and its Langlands dual group $G^\vee$ (obtained by interchanging the systems of roots and coroots).

- **(Equivariant) mirror symmetry.** Originally a duality in string theory, mirror symmetry was set on mathematical footing in the 1990s, and is a fundamental duality in modern mathematical physics [31]. An equivariant version is currently being developed [56, 44, 7]. In its original form, mirror symmetry predicts an equality between the quantum connection (or A-model) of a Calabi-Yau 3-fold $M$, and the Gauss-Manin connection (or B-model) of its mirror $M^\vee$. 
• **Symplectic duality.** Also originally a duality in physics [32], a large part of this duality has been recently set on mathematical footing in [10] and [9], and is continuing to be developed in current work of Braverman, Finkelberg, and Nakajima [49]. It replaces the theory of semisimple Lie algebras $\mathfrak{g}$ with a more general and geometric theory of symplectic resolutions $X$ and their symplectic duals $X^\vee$.

We now describe the problem more precisely. Let $G$ be a connected semisimple algebraic group over $\mathbb{C}$. A fundamental object in the geometric Langlands program is the affine Grassmannian $\text{Gr}_G$, a certain ind-scheme that geometrically “encodes” the Langlands dual group $G^\vee$. In [35], the authors identify certain symplectic slices $X_0$ inside of $\text{Gr}_G$, and construct $T \times \mathbb{C}^*$-equivariant symplectic resolutions $X \to X_0$ of these slices. In types ADE, these symplectic resolutions are conjectured in [10] to be symplectic dual to certain Nakajima quiver varieties (see Remark 3.12).

The main result of this thesis, Theorem 4.6, computes the $T \times \mathbb{C}^*$-equivariant small quantum connection of these symplectic resolutions (modulo certain parameters) in the case that they are Picard rank 1. (Following the strategy of [13], we expect to show that the general case can be derived from this one; see section 4.7.) In particular, we observe that this connection is closely related to the trigonometric Knizhnik-Zamolodchikov (KZ) connection of $G^\vee$, a fundamental system of differential equations with regular singularities arising in conformal field theory [18]. The proof strategy is classical: we show that our spaces have finitely many torus-invariant curves, and identify each of their contributions to the quantum corrections using virtual localization.

Our result provides a new example in the program of Bezrukovnikov, Braverman, Etingof, Maulik, Okounkov, Toledano-Laredo, and others [2, 6] describing the relation between equivariant quantum cohomology of symplectic resolutions and Casimir-type connections (among many other objects). Previously, this relation has been studied in the settings of the Springer resolution [13], Nakajima quiver varieties [43], and
hypsotropic varieties [44].

Due to the conjectural symplectic duality between quiver varieties and slices in the affine Grassmannian, our work can also be thought of as a symplectic dual analogue of (part of) the work of [43] on quiver varieties. We expect that there is a precise relation in the context of equivariant quantum K-theory [53]. Equivariant quantum K-theory associates two difference equations to a conical symplectic resolution, one in the Kähler parameters and the other in the equivariant parameters. In the recent paper [1], it is conjectured, and proven in the case of quiver varieties, that symplectic duality interchanges these two difference equations. We expect that the full picture can be summarized as follows:

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Thus, we conjecture that the quantum K-theoretic difference equations for our symplectic resolutions in the Kähler and equivariant parameters are, respectively, the trigonometric qKZ equations and the trigonometric dynamical difference equations. These are the same difference equations appearing in the quiver variety picture, but with the Kähler and equivariant parameters interchanged. The first part of this conjecture is consistent with Theorem 4.6 since the trigonometric qKZ equations naturally degenerate to the trigonometric KZ equations. Thus, the result in this thesis can be viewed as a step toward completing the understanding of quantum K-theory.
in the context of symplectic duality for quiver varieties and affine Grassmannians.

The paper is organized as follows. In Chapter 2, we recall some basic results on quantum cohomology we will need. In Chapter 3, we recall from [35] how to construct symplectic resolutions from the affine Grassmannian, and identify those which are Picard rank $1$. Finally, in Chapter 4, we compute the quantum connections of these Picard 1 spaces (modulo certain parameters), and relate them to a trigonometric KZ-type connection.
Chapter 2

Recollections on quantum cohomology

2.1 The equivariant small quantum connection

All varieties and cohomology are over $\mathbb{C}$. Let $G$ be a reductive algebraic group and $X$ a smooth quasi-projective $G$-variety such that the fixed locus $X^G$ is compact. By definition, the operator of quantum multiplication by a class $\gamma \in H^*_G(X)$ has matrix elements

$$(\gamma \cdot \gamma_1, \gamma_2) = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \langle \gamma, \gamma_1, \gamma_2 \rangle_\beta,$$

where $(\cdot, \cdot)$ denotes the standard inner product on $H^*_G(X)$ and the angle brackets denote 3-point, genus 0, degree $\beta$, $G$-equivariant Gromov-Witten invariants on $X$.

Consider the trivial bundle with fiber $H^*_G(X)$ and base $H^2(X)$. Define the $G$-equivariant quantum connection of $X$ to be the connection $\nabla_{quantum}$ on this bundle defined, for any $D \in H^2(X)$, by

$$\nabla^quantum_D = d_D - D^\bullet,$$
where \( d_D \) denotes the derivative in the direction of \( D \).

Assume that \( H^*_G(X) \) is a free module over \( H^*_G(\text{pt}) \). Then we have a natural \( H^*_G(\text{pt}) \)-module isomorphism \( H^*_G(X) \cong H^*(X) \otimes H^*_G(\text{pt}) \). Thus, we can view \( \nabla \) as a family of connections, parametrized by equivariant parameters in \( H^*_G(\text{pt}) \), on the trivial bundle with fiber \( H^*(X) \) and base \( H^2(X) \). It is well-known that each connection in this family is flat [31].

### 2.2 Symplectic resolutions

We recall some standard facts on symplectic singularities and symplectic resolutions; see [33, 9] for further details.

Recall that a smooth variety \( X \) is said to be a **symplectic resolution** if \( X \) is equipped with an algebraic symplectic form \( \omega \) and the map to its affinization \( X_0 = \text{Spec}(\mathcal{O}_X) \) is proper and birational. A symplectic resolution \( X \rightarrow X_0 \) is said to be **conical** if there is a \( \mathbb{C}^* \) which acts compatibly on the resolution and base, contracting the base to a point, and which acts with positive weight on the symplectic form \( \omega \). Conical symplectic resolutions include several examples of spaces that commonly arise in geometric representation theory [9]:

**Example 2.1.** For a reductive algebraic group \( G \) with a Borel subgroup \( B \) and nilpotent cone \( \mathcal{N} \subset \mathfrak{g} \), the Springer resolution \( T^*(G/B) \rightarrow \mathcal{N} \) is a conical symplectic resolution, where \( \mathbb{C}^* \) acts fiberwise on \( T^*(G/B) \).

**Example 2.2.** The Hilbert scheme \( \text{Hilb}_n(\mathbb{C}^2) \) of \( n \) points in \( \mathbb{C}^2 \) with its Hilbert-Chow morphism to \( \text{Sym}^n(\mathbb{C}^2) \) forms a conical symplectic resolution, with \( \mathbb{C}^* \)-action on \( \text{Hilb}_n(\mathbb{C}^2) \) induced from the standard one on \( \mathbb{C}^2 \). More generally, the Hilbert scheme \( \mathcal{H}(k,n) \) of \( n \) points on a crepant resolution of \( \mathbb{C}^2/(\mathbb{Z}/k\mathbb{Z}) \), where \( \mathbb{Z}/k\mathbb{Z} \) acts symplectically on \( \mathbb{C}^2 \), is a conical symplectic resolution.

**Example 2.3.** Let \( \mathcal{M}(k,n) \) denote the moduli space of torsion-free sheaves \( \mathcal{E} \) on \( \mathbb{P}^2 \) with rank \( \mathcal{E} = k \) and \( c_2(\mathcal{E}) = n \) together with a framing \( \mathcal{E}|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus k} \). Then \( \mathcal{M}(k,n) \)
mapping to its affinization, together with a suitable $\mathbb{C}^*$ action, is a conical symplectic resolution.

**Example 2.4.** Nakajima quiver varieties mapping to their affinizations are known to be conical symplectic resolutions. When the quiver is affine type $A$, these generalize the previous two examples.

**Example 2.5.** The spaces considered in this paper, resolutions of slices in the affine Grassmannian, admit the structure of a conical symplectic resolution (see Section 3.5).

In [10, 9], the notion of *symplectic duality* between two conical symplectic resolutions is introduced. One associates a Koszul "category $\mathcal{O}$" to any symplectic resolution (e.g. the standard category $\mathcal{O}$ for the Springer resolution), and roughly, two symplectic resolutions are said to be symplectic dual if their corresponding categories $\mathcal{O}$ are Koszul dual.

**Example 2.6.** By [5], the Springer resolution is symplectic self-dual (or more canonically, $T^*(G/B)$ is symplectic dual to $T^*(G'/B')$, where $G'$ is the Langlands dual group of $G$).

**Example 2.7.** By [9, Corollary 10.11], the Hilbert schemes $\mathcal{H}(k, n)$ and framed instanton moduli spaces $\mathcal{M}(k, n)$ in Examples 2.2 and 2.3 are symplectic dual. For $k = 1$ they are isomorphic.

**Example 2.8.** The spaces considered in this paper (Example 2.5) are expected to be symplectic dual to certain Nakajima quiver varieties; see Remark 3.12.

### 2.3 Quantum cohomology of symplectic resolutions

Any conical symplectic resolution $X$ may be deformed to an affine variety (namely, the generic fiber of its universal Poisson deformation), so that the ordinary Gromov-
Witten theory of $X$ is trivial. However, such a deformation is not in general $\mathbb{C}^*$-equivariant, so the equivariant Gromov-Witten theory of $X$ can be highly non-trivial.

In the notation of Section 2.1, we will take $X$ to be a conical symplectic resolution and $G = T \times \mathbb{C}^*$ for a Hamiltonian torus $T$ (i.e. $T$ stabilizes the symplectic form $\omega$). By [10, Prop. 2.5], a symplectic resolution $X$ has no odd-dimensional cohomology, so the assumption that $H^i_G(X)$ is a free module over $H^4_G(pt)$ is satisfied [29].

In [13], it is shown that the equivariant quantum connection of a symplectic resolution $X \to X_0$ can be expressed via Lagrangian Steinberg correspondences, i.e. Lagrangian components of the Steinberg variety $Z := X \times_{X_0} X$. It is also shown how the general computation can be reduced to one for symplectic resolutions whose Picard group has rank 1. We simply summarize here the fundamental ideas: the divisor equation reduces us to the study of 2-point Gromov-Witten invariants. Any curve in $X$ must lie in a fiber of $X \to X_0$ since $X_0$ is affine, so the evaluation map from the space of stable 2-pointed maps to $X \times X$ factors through the Steinberg variety. Finally, a dimension count yields that the pushforward of the reduced virtual fundamental class (see the next section) to $Z$ can be expressed as a non-equivariant rational linear combination of Lagrangian Steinberg components. One may then apply deformation invariance of ordinary Gromov-Witten invariants to reduce the computation of these non-equivariant constants to the Picard rank 1 case.

For the Springer resolution, the equivariant quantum connection is determined in [13] to be equivalent to the trigonometric Dunkl/affine KZ connection [17]. For quiver varieties, the equivariant quantum connection is determined in [43] to be equivalent to a trigonometric Casimir-type connection [58]. These two results encompass Examples 2.1 - 2.4. The main result of this thesis identifies the answer in Example 2.5 with a trigonometric KZ-type connection.
2.4 Reduced virtual fundamental class

Suppose $X \to X_0$ is a conical symplectic resolution. Let $\beta$ be an effective curve class on $X$, and let $\overline{\mathcal{M}}_{0,2}(X, \beta)$ be the moduli space of 2-pointed stable maps to $X$ of class $\beta$. Then it is well-known that there is a so-called reduced virtual fundamental class $[\overline{\mathcal{M}}_{0,2}(X, \beta)]^{\text{red}}$ of dimension $\dim X$ which satisfies

$$[\overline{\mathcal{M}}_{0,2}(X, \beta)]^{\text{red}} = -h[\overline{\mathcal{M}}_{0,2}(X, \beta)]^{\text{vir}},$$

where $[\overline{\mathcal{M}}_{0,2}(X, \beta)]^{\text{vir}}$ is the usual virtual fundamental class, and $h$ is the weight of the symplectic form under the $\mathbb{C}^*$-action.

2.5 Unbroken maps

For convenience, we recall the notion of unbroken maps here from [43, Section 7.3]. Let $X$ be a conical symplectic resolution with an action of a torus $T$ that preserves the symplectic form. Let $f : C \to X$ be a $T$-fixed point of $\overline{\mathcal{M}}_{0,2}(X, \beta)$ such that the domain $C$ is a chain of rational curves

$$C = C_1 \cup \ldots \cup C_k$$

with the two marked points lying on $C_1$ and $C_k$, respectively.

Note that all nodes are fixed by $T$. We say that $f$ is an unbroken chain if at every node of $C$, the tangent weights of the two branches are opposite and non-zero.

More generally, we say that $f$ is an unbroken map if it satisfies one of the following three conditions:

1. $f$ arises from a map $C \to X^T$,

2. $f$ is an unbroken chain, or
3. The domain $C$ is a chain of rational curves

$$C = C_0 \cup C_1 \cup \ldots \cup C_k$$

such that $C_0$ is contracted by $f$, the two marked points lie on $C_0$, and the remaining curves form an unbroken chain.

A map is said to be broken if it does not satisfy any of the above conditions. The following is proven in [54, Section 3.8.3]:

**Lemma 2.9.** Every map in a given connected component of $\overline{M}_{0,2}(X, \beta)^T$ is either unbroken or broken. Only unbroken components contribute to the $T$-equivariant localization of the reduced virtual fundamental class.
Chapter 3

Symplectic resolutions of slices in the affine Grassmannian

Let $G$ be a connected semisimple algebraic group over $\mathbb{C}$. We choose a maximal torus $T$ and a Borel subgroup $B$ containing it. Let $r$ be the rank of $G$, and let $\alpha_1, ..., \alpha_r$ denote the simple roots. Let $\Lambda^\vee$ denote the coweight lattice of $G$, and let $\Lambda^\vee_+$ denote the dominant coweights. Let $Q^\vee$ denote the coroot lattice of $G$. Finally, let $\Delta$ denote the set of roots of $G$, $\Delta_+$ the set of positive roots, and $\Sigma$ the set of simple roots.

3.1 Geometric Satake equivalence

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Recall that the affine Grassmannian $\text{Gr}_G$ is the ind-scheme defined by $G(\mathcal{K})/G(\mathcal{O})$.

The Langlands dual group $G^\vee$, defined by interchanging the root and coroot systems of $G$, is of fundamental importance in the Langlands program and geometric representation theory. By the Tannakian formalism, this group can be reconstructed from its tensor category of finite-dimensional representations $(\text{Rep}(G^\vee), \otimes)$. The geometric Satake equivalence [42, 26, 45] describes this category in terms of the geometry
of the affine Grassmannian of $G$: namely, it gives an isomorphism of tensor categories

$$\text{Perv}_{G(O)}(\text{Gr}_G) \simeq \text{Rep}(G^\vee),$$

where the left-hand side denotes the category of left $G(O)$-equivariant perverse sheaves on $\text{Gr}_G$, with tensor structure given by the convolution product (see section 3.4).

It is well-known that the $G(O)$-orbits on $\text{Gr}_G$ are indexed by the dominant coweights $\Lambda_+^\vee$ of $G$, which can be identified with the dominant weights of $G^\vee$. For $\lambda \in \Lambda_+^\vee$, we denote the corresponding $G(O)$-orbit by $\text{Gr}^\lambda$. If we let $t^\lambda$ denote the image of $t$ under the map

$$\mathbb{C}^*(t) = G_m(K) \to G(K) \to \text{Gr}_G,$$

then $\text{Gr}^\lambda = G(O) \cdot t^\lambda$. Under the geometric Satake equivalence, the intersection cohomology sheaf $IC_\lambda := IC(\text{Gr}^\lambda)$ is sent to the irreducible highest-weight representation $V_\lambda$ of $G^\vee$.

Let $s \in \mathbb{C}^*$ act on $\text{Gr}$ by "loop rotation" $t \mapsto st$. This $\mathbb{C}^*$ action contracts each $G(O)$-orbit $\text{Gr}^\lambda$ to the $G$-orbit $G/P_\lambda$, where $P_\lambda = \text{Stab}(t^\lambda)$ is the parabolic subgroup of $G$ spanned by the root subgroups $U_\alpha$ for $\alpha$ such that $(\alpha, \lambda) \leq 0$.

### 3.2 Minuscule cells

Of particular importance to us will be the orbits $\text{Gr}^\lambda$ with $\lambda$ minuscule, that is, minimal in $\Lambda_+^\vee$ with respect to the standard partial ordering. It is well-known that this is equivalent to saying that the pairing of $\lambda$ with any root of $G$ is -1, 0, or 1. Since $\overline{\text{Gr}}^\lambda = \bigcup_{\mu \leq \lambda} \text{Gr}^\mu$, it follows that $\overline{\text{Gr}}^\lambda = \text{Gr}^\lambda$ is smooth for $\lambda$ minuscule; by the previous section, we have $\text{Gr}^\lambda \simeq G/P_\lambda$.

It follows that every non-zero minuscule coweight is fundamental. By definition, minuscule coweights are indexed by $\Lambda_+^\vee/Q^\vee$. In particular, there are no non-zero.

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minuscule coweights in types $E_8, F_4, \text{ or } G_2$. In the remaining types, the non-zero minuscule coweights, and the corresponding representations of $G^\vee$, are as follows, where the $\omega_i$ are fundamental coweights [8, Section 7.3]:

Type $A_n$: $\omega_1, \ldots, \omega_n$ (exterior powers of the vector representation)
Type $B_n$: $\omega_1$ (the vector representation)
Type $C_n$: $\omega_n$ (the spin representation)
Type $D_n$: $\omega_1, \omega_{n-1}, \omega_n$ (the vector and half-spin representations)
Type $E_6$: $\omega_1, \omega_6$ (the two 27-dimensional representations)
Type $E_7$: $\omega_7$ (the 56-dimensional representation)

We now give some examples of the corresponding minuscule Schubert cells:

**Example 3.1.** The minuscule coweights for $G = PGL_n$ are $0, \omega_1, \omega_2, \ldots, \omega_{n-1}$, where $\omega_i$ is the $i$th fundamental weight of $SL_n = (PGL_n)^\vee$. The corresponding Schubert cells are $Gr^0 = pt$ and $Gr^{\omega_i} = Gr(i, n), 1 \leq i \leq n - 1$.

**Example 3.2.** For $G$ of type $B_n$ and of adjoint type, the minuscule Schubert cell $G/P_{\omega_1}$ is isomorphic to $Gr_{\omega}(n, 2n)$, the Grassmannian of maximal isotropic subspaces in a $2n$-dimensional symplectic vector space.

**Example 3.3.** For $G$ of type $E_6$ and of adjoint type, the minuscule Schubert cell $G/P_{\omega_1}$ is isomorphic to $OP^2$, the 16-dimensional projective plane over the complexified octonions.

For a complete description of minuscule Schubert cells in all types, see [16].

### 3.3 Gradings

For any $\lambda \in \Lambda_+^\vee$, the geometric Satake equivalence gives an isomorphism of vector spaces $H^*(IC_\lambda) = V_\lambda$. This vector space has two natural gradings: one by coho-
logical degree in $H^*(IC_{\lambda})$, and one by weight spaces of $V_{\lambda}$. We describe how each of these gradings is transported to the other side of the equivalence.

Fix a regular nilpotent element $e \in \mathfrak{g}^\vee$, and complete it to an $\mathfrak{sl}_2$-triple $e, f, h$ using the Jacobson-Morozov theorem [19]. For example, one can take $e = \sum_{\alpha \in \Delta} e_{\alpha}$; then $h = \sum_{\alpha \in \Delta} \alpha$. Then:

**Lemma 3.4.** [26] Under the geometric Satake equivalence,

$$H^i(IC_{\lambda}) \simeq \{v \in V_{\lambda} : h \cdot v = iv\}, \ i \in \mathbb{Z}.$$  

**Remark 3.5.** While the right-hand side of the lemma depends on the choice of $h$ and the left-hand side does not, the isomorphism in the lemma may also be conjugated by $h$ via its action on $\text{Rep}(G^\vee)$.

**Example 3.6.** For $\mathfrak{g}^\vee = \mathfrak{sl}_n(\mathbb{C})$, we may take $e$ to be the matrix with 1's above the diagonal and 0's elsewhere, and $h$ the diagonal matrix with diagonal entries $n - 1, n - 3, \ldots, -n + 1$. Let $\lambda = \omega_1$, so that $\text{Gr}^\lambda \simeq \mathbb{P}^{n-1}$ and $V_{\lambda} \simeq \mathbb{C}^n$. Then $IC_{\lambda}$ is the constant perverse sheaf $\mathbb{C}_{\mathbb{P}^{n-1}[n-1]}$, which has non-zero cohomology in degrees $n - 1, n - 3, \ldots, -n + 1$.

Choose $\mu \leq \lambda$. Observe that, since each $\text{Gr}^\mu$ is simply connected, $IC_{\lambda}|_{\text{Gr}^\mu}$ is a constant complex, i.e. a graded vector space. Define the *Brylinski filtration* on $V_{\lambda}$ associated to $e$ by

$$F^i(V_{\lambda}) = \{x \in V_{\lambda} : e^i \cdot x = 0\}.$$  

This filtration is independent of $e$ since all such $e$ are conjugate under $T$. Then we have:

**Lemma 3.7.** [26] There is a canonical isomorphism

$$(IC_{\lambda}|_{\text{Gr}^\mu})_{-2i-(2p,\mu)} \simeq \text{gr}_i^F(V_{\lambda}[\mu]),$$
where $V_{\lambda}[\mu]$ denotes the $\mu$ weight space in $V_{\lambda}$. In particular, $I\!C_{\lambda}|_{G^{T^{\omega}}}$ and $V_{\lambda}[\mu]$ are isomorphic as ordinary vector spaces.

### 3.4 Convolution

By the isomorphism

$$\frac{G/H \times G/H}{G} \simeq H\backslash G/H$$

in the setting $G = G(K)$ and $H = G(O)$, the $G(K)$-orbits in $\text{Gr} \times \text{Gr}$ are also indexed by $\Lambda'_{\gamma}$. Given $(L_1, L_2) \in \text{Gr} \times \text{Gr}$ and $\lambda \in \Lambda'_{\gamma}$, we say that $L_1$ and $L_2$ are in relative position $\lambda$ and write $L_1 \xrightarrow{\lambda} L_2$ if $(L_1, L_2)$ lies in the $G(K)$-orbit corresponding to $\lambda$. Let $L_0 = \text{Gr}^0$. Given $\lambda_1, \ldots, \lambda_n \in \Lambda'_{\gamma}$, we define the (closed) convolution diagram

$$\widetilde{\text{Gr}}^{\lambda_1} \times \cdots \times \widetilde{\text{Gr}}^{\lambda_n} := \{(L_1, \ldots, L_n) \in \text{Gr}^n : L_0 \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} L_2 \xrightarrow{\lambda_3} \cdots \xrightarrow{\lambda_n} L_n\}.$$  

We will also denote this space by $\text{Gr}^{\lambda_1, \ldots, \lambda_n}$. Define the convolution morphism

$$m : \widetilde{\text{Gr}}^{\lambda_1, \ldots, \lambda_n} \to \text{Gr}^{\lambda_1 + \cdots + \lambda_n}$$

by $m(L_1, \ldots, L_n) = L_n$. The map $m$ is known to be semismall [45], and when the $\lambda_i$ are minuscule, $\text{Gr}^{\lambda_1, \ldots, \lambda_n}$ is smooth (since each $\text{Gr}^{\lambda_i}$ is), and hence $m$ is a resolution of singularities.

**Example 3.8.** If $G = PGL_2$, then $\text{Gr}^{\omega} \times \text{Gr}^{\omega}$ is the Hirzebruch surface $F_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}) = T^*\mathbb{P}^1$, and the map $m : \text{Gr}^{\omega} \times \text{Gr}^{\omega} \to \text{Gr}^{2\omega}$ is the contraction of the zero section.

Given $I\!C_{\lambda_1}, \ldots, I\!C_{\lambda_n} \in \text{Perv}_{G(O)}(\text{Gr}_G)$, define the convolution product

$$I\!C_{\lambda_1} \ast \cdots \ast I\!C_{\lambda_n} := m_!(I\!C(\widetilde{\text{Gr}}^{\lambda_1, \ldots, \lambda_n})).$$  

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Since $m$ is semismall, we have

$$IC_{\lambda_1} \ast \ldots \ast IC_{\lambda_n} \in \text{Perv}_{G(\sigma)}(\text{Gr}_G),$$

which under geometric Satake corresponds to the tensor product of representations $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$.

### 3.5 Transversal slices

Consider the group $G[t^{-1}] \subset G((t))$. Let $G_1[t^{-1}]$ denote the kernel of the natural ("evaluation at $\infty$") homomorphism $G[t^{-1}] \to G$. For $\mu \in \Lambda^\vee_+$, set $\text{Gr}_\mu = G_1[[t^{-1}]] \cdot t^\mu$.

Finally, for $\lambda, \mu \in \Lambda^\vee_+$ with $\lambda \geq \mu$, set

$$\overline{\text{Gr}}^\lambda_\mu = \overline{\text{Gr}}^\lambda \cap \text{Gr}_\mu.$$

By Lemma 2.9 in [11], $\overline{\text{Gr}}^\lambda_\mu$ is a transversal slice to $\text{Gr}^\mu$ inside of $\overline{\text{Gr}}^\lambda$, and by Theorem 2.7 in [35], $\overline{\text{Gr}}^\lambda_\mu$ has symplectic singularities. We now describe an instance in which $\overline{\text{Gr}}^\lambda_\mu$ has a symplectic resolution.

Suppose $\lambda_1, \ldots, \lambda_n \in \Lambda^\vee_+$ are minuscule coweights, and consider the convolution morphism

$$m : \overline{\text{Gr}}^{\lambda_1, \ldots, \lambda_n} \to \overline{\text{Gr}}^{\lambda_1+\ldots+\lambda_n}.$$

For $\mu \in \Lambda^\vee_+$ with $\mu \leq \lambda_1 + \ldots + \lambda_n$, consider the transversal slice $\overline{\text{Gr}}^{\lambda_1, \ldots, \lambda_n}_\mu$, and set

$$\overline{\text{Gr}}^{\lambda_1, \ldots, \lambda_n}_\mu = m^{-1}(\overline{\text{Gr}}^{\lambda_1+\ldots+\lambda_n}_\mu).$$

Then by [35, Theorem 2.9],

$$m : \overline{\text{Gr}}^{\lambda_1, \ldots, \lambda_n}_\mu \to \overline{\text{Gr}}^{\lambda_1+\ldots+\lambda_n}_\mu.$$
is a symplectic resolution.

**Example 3.9.** If $G = PGL_2$, then $\widetilde{Gr}_0^{\omega,\omega} \simeq T^*\mathbb{P}^1$, and the map $\widetilde{Gr}_0^{\omega,\omega} \to \overline{Gr}_0^{2\omega}$ is the contraction of the zero section (or alternatively, the Springer resolution for $sl_2$).

**Example 3.10.** [35] More generally, if $G = PGL_2$, $\lambda_1 = \ldots = \lambda_n = \omega$, and $\mu = (n-2)\omega$, then $\widetilde{Gr}_0^{\lambda_1,\ldots,\lambda_n} \to \overline{Gr}_0^{\lambda_1+\ldots+\lambda_n}$ is a resolution of the $A_{n-1}$ singularity $\mathbb{C}^2/(\mathbb{Z}/n)$.

Note that the groups $G$ and $\mathbb{C}^*$ act compatibly on both the resolution and the base. It is clear that the $\mathbb{C}^*$-action contracts the base to the point $t^\mu$, hence contracts the resolution to the fiber $m^{-1}(t^\mu)$. This $\mathbb{C}^*$-action makes $m$ into a conical symplectic resolution (see Section 2.2).

**Remark 3.11.** If the coweights $\lambda_i$ are not all minuscule, the variety $\widetilde{Gr}_\mu^{\lambda_1,\ldots,\lambda_n}$ will be singular; however, it is known to have symplectic singularities [35, Theorem 2.9].

**Remark 3.12.** Suppose $G$ is simply-laced. Then to $\lambda$ and $\mu$ one can associate a quiver variety $Q(\lambda, \mu)$, whose graph is the Dynkin diagram of $G$ with some choice of orientation, and with dimension vector $v$ and framing vector $w$ defined by $\lambda = \sum w_i \omega_i$ and $\mu = \lambda - \sum v_i \alpha_i$. Then it is conjectured [9, Example 10.27] that $\widetilde{Gr}_\mu^{\lambda_1,\ldots,\lambda_n}$ and $Q(\lambda, \mu)$ are symplectic duals.

**Remark 3.13.** In type $A$, the space $\widetilde{Gr}_\mu^{\lambda_1,\ldots,\lambda_n}$ is known to be isomorphic to a quiver variety, as well as a resolution of a Slodowy slice intersected with a nilpotent orbit; for details, see [46].

In the special case $n = 2$, we have the following:

**Lemma 3.14.** The central fiber $m^{-1}(t^\mu)$ of $\widetilde{Gr}_\mu^{\lambda_1,\lambda_2}$ is irreducible.

**Proof.** By [61, Cor. 5.1.5], the irreducible components of $m^{-1}(t^\mu)$ of dimension $\langle 2\rho, \lambda_1 + \lambda_2 - \mu \rangle$ are in bijection with a basis for the $\mu$ multiplicity space

$$\text{Hom}(V_\mu, V_{\lambda_1} \otimes V_{\lambda_2}).$$
But since $\lambda_1$ and $\lambda_2$ are minuscule, [30] gives that $m^{-1}(\mu)$ is equidimensional, and [52, Lemma 10.2] gives that the multiplicity space is 1-dimensional. This proves the lemma. 

\[ \square \]

### 3.6 Picard group

Recall that the affine Grassmannian of $G$ is naturally equipped with an ample $G(\mathcal{O})$-equivariant line bundle $\mathcal{L} := \mathcal{O}(1)$, which generates the Picard group if $G$ is simply connected [61, Theorem 2.4.2].

There are $n$ natural line bundles defined on $\widetilde{\text{Gr}}_{\mu}^{\lambda_1, \ldots, \lambda_n}$ as follows. Consider the $n$-fold iterated convolution morphism

\[
m : \widetilde{\text{Gr}} = G(K) \times^{G(\mathcal{O})} \text{Gr} \times^{G(\mathcal{O})} \ldots \times^{G(\mathcal{O})} \text{Gr} \to \text{Gr}.
\]

Let $\pi_1$ be the projection to the first factor of $\text{Gr}$, and define $\mathcal{L}_i = \pi_1^* \mathcal{L}$. For $2 \leq i \leq n$, the $G(\mathcal{O})$-equivariant line bundle $\mathcal{L}$ on the $i$th convolution factor induces a line bundle $\mathcal{L}_i$ on $\text{Gr}$. It is then known that $\mathcal{L}$ satisfies naturality with respect to convolution as follows (see, e.g. [59]):

\[
m^* \mathcal{L} = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n.
\]

We now restrict everything to the resolution of a slice $m : \widetilde{\text{Gr}}_{\mu}^{\lambda_1, \ldots, \lambda_n} \to \text{Gr}_{\mu}^{\lambda_1 + \ldots + \lambda_n}$. Then $\mathcal{L}$ restricted to the affine variety $\text{Gr}_{\mu}^{\lambda_1 + \ldots + \lambda_n}$ is trivial, so we conclude that, on $\widetilde{\text{Gr}}_{\mu}^{\lambda_1, \ldots, \lambda_n}$,

\[
\mathcal{O} = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n.
\]

Let $D_i = c_1(\mathcal{L}_i)$ be the corresponding divisor classes, so that we have

\[
D_1 + \ldots + D_n = 0.
\]

Note that for $G$ not simply connected, $\mathcal{L}$ may not be equivariant on each connected
component of $\text{Gr}_G$, in particular on $\text{Gr}^{\lambda_i}$. However, a suitable multiple $L^\otimes N$ will be $G(O)$-equivariant, e.g. for $N = |A^V/Q^V|$. So the above construction may be applied to $L^\otimes N$ to yield the same result.

The rank of the Picard group of $\tilde{\text{Gr}}^{\lambda_1,\ldots,\lambda_n}$ is not known in general. However, in the case $n = 2$ we have the following:

**Lemma 3.15.** For $\mu < \lambda_1 + \lambda_2$, the space $\tilde{\text{Gr}}^{\lambda_1,\lambda_2}$ has Picard rank 1.

**Proof.** Recall from [10, Prop. 2.5] that a conical symplectic resolution $X$ has no odd cohomology groups (this is also clear in our setting since $\text{Gr}$ is a union of even-dimensional cells), and hence that $\text{Pic}(X) \otimes \mathbb{C} \simeq H^2(X, \mathbb{C})$. So we have

$$\text{Pic}(\tilde{\text{Gr}}^{\lambda_1,\lambda_2}) \otimes \mathbb{C} \simeq H^2(\tilde{\text{Gr}}^{\lambda_1,\lambda_2}, \mathbb{C}) \simeq H^2(m^{-1}(t^\mu), \mathbb{C}).$$

Note that $m^{-1}(t^\mu)$ is a $B$-invariant irreducible (by Lemma 3.14) subvariety of $\text{Gr}^{\lambda_1} \simeq G/P_{\lambda_1}$. This maximal parabolic flag variety has an affine paving by $B$-orbits with a single codimension 1 stratum; thus, we obtain an affine paving of $m^{-1}(t^\mu)$ with a single codimension 1 stratum, so $H^2(m^{-1}(t^\mu), \mathbb{C})$ is 1-dimensional. 

In the case $n = 2$, we let $D := D_1 - D_2 = 2D_1 = -2D_2$.

### 3.7 Equivariant cohomology

We first compute the ordinary cohomology of our symplectic resolutions:

**Lemma 3.16.** We have

$$H^*(\tilde{\text{Gr}}^{\lambda_1,\ldots,\lambda_n}) = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}[\mu].$$

**Proof.** As a vector space, the stalk of $IC_{\lambda_1} \ast \ldots \ast IC_{\lambda_n} = m_* IC(\tilde{\text{Gr}}^{\lambda_1,\ldots,\lambda_n})$ at $t^\mu$ is known [26] to be isomorphic to the weight space $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}[\mu]$; see Lemma 3.7.
for the case \( n = 1 \). Since \( \tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n} \) is smooth, we conclude that

\[
H^*(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n}) = H^*(m^{-1}(t^\mu)) = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}[\mu].
\]

By Lemma 4.1, the space \( \tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n} \) is \( \tilde{T} \)-equivariantly formal, so

\[
H^*_\tilde{T}(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n}) \simeq H^*(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n}) \otimes_{\mathbb{C}[\alpha_1,...,\alpha_r,h]} \mathbb{C}[c_1,...,c_r]
\]

and localized equivariant cohomology is given by

\[
H^*_\tilde{T}(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n})_{\text{loc}} \simeq V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}[\mu] \otimes \mathbb{C}[\alpha_1,...,\alpha_r,h].
\]

The weight spaces in each \( V_{\lambda_i} \) are obviously 1-dimensional, so the fixed-point basis in \( H^*_\tilde{T}(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n})_{\text{loc}} \) may be indexed as \( v_{(\nu_1,...,\nu_n)} \), where \( \nu_i \in W \cdot \lambda_i \) and \( \sum \nu_i = \mu \).

For non-localized equivariant cohomology, we have the following result (referring to [28] for more details):

**Lemma 3.17.**

\[
H^*_\tilde{T}(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n}) \gamma = \text{Hom}_{U_h(g)}(M_\gamma, V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \otimes M_{\gamma-\mu}),
\]

where \( U_h(g) \) is the asymptotic universal enveloping algebra of \( g \), \( \gamma \) is an equivariant parameter, and \( M_\gamma \) and \( M_{\gamma-\mu} \) are universal Verma modules for \( U_h(g) \).

**Proof.** We have that \( H^*_\tilde{T}(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n}) \) is Poincaré dual to \( H^*_{\text{BM}}(\tilde{\text{Gr}}_{\mu}^{\lambda_1,...,\lambda_n}) = H^*_{\text{BM}}(m^{-1}(t^\mu)) \).

The lemma now follows from [28, Prop. 8.1.5]. \( \square \)
For $n = 2$, we will be interested in the truncated Casimir operator

$$ \Omega^{\text{trunc}} := \sum_{\alpha \in \Delta} e_\alpha \otimes e_{-\alpha} $$

which acts on $H^*_T(\Gr_{\lambda_1, \lambda_2})$ by acting factor-wise on $V_{\lambda_1} \otimes V_{\lambda_2}$. 
Chapter 4

Calculation of the equivariant quantum connection

From now on, we restrict to the case $n = 2$, i.e. the resolution of a slice $m : \widetilde{\text{Gr}}^{\lambda_1,\lambda_2}_{\mu} \rightarrow \text{Gr}^{\lambda_1+\lambda_2}_{\mu}$.

4.1 Torus-invariant curves

We first determine the torus-fixed points and torus-invariant curves of $\widetilde{\text{Gr}}^{\lambda_1,\lambda_2}_{\mu}$. For brevity, we set $\tilde{T} = T \times \mathbb{C}^*$.

**Lemma 4.1.** 1. The $\tilde{T}$-fixed points in $\widetilde{\text{Gr}}^{\lambda_1,\lambda_2}_{\mu}$ are indexed by pairs of weights $(\nu_1,\nu_2)$ of $G^\vee$ such that $\nu_1 \in W \cdot \lambda_1$, $\nu_2 \in W \cdot \lambda_2$, and $\nu_1 + \nu_2 = \mu$. In particular, there are finitely many such points.

2. Let $(\nu_1,\nu_2)$ be a $\tilde{T}$-fixed point of $\widetilde{\text{Gr}}^{\lambda_1,\lambda_2}_{\mu}$, and let $\alpha^\vee$ be a root of $G^\vee$ such that $\nu_1 + \alpha^\vee \in W \cdot \lambda_1$ and $\nu_2 - \alpha^\vee \in W \cdot \lambda_2$. Then there is a unique $\tilde{T}$-invariant curve $C_{\nu_1,\nu_2,\alpha}$ connecting the points $(\nu_1,\nu_2)$ and $(\nu_1 + \alpha^\vee,\nu_2 - \alpha^\vee)$, and all $\tilde{T}$-invariant curves are of this form. In particular, there are finitely many such curves.

**Proof.** Since each $\lambda_i$ is minuscule, the set of weights of the representation $V_{\lambda_i}$ of $G^\vee$
is simply $W \cdot \lambda_i$. It is well-known that the $T$-fixed points of $\text{Gr}^\lambda_i$ are then given by $t^\nu$ with $\nu \in W \cdot \lambda_i$, and the $T$-fixed points of $\tilde{\text{Gr}}^{\lambda_1,\lambda_2}$ are given by $(t^{\nu_1}, t^{\nu_1+\nu_2})$ with $\nu_1 \in W \cdot \lambda_1$ and $\nu_2 \in W \cdot \lambda_2$. Hence, the $T$-fixed points of the fiber of the convolution morphism $m^{-1}(t^\mu)$ are indexed by such pairs $(\nu_1, \nu_2)$ with $\nu_1 + \nu_2 = \mu$. Finally, since the $\mathbb{C}^*$-action contracts $\tilde{\text{Gr}}^{\lambda_1,\lambda_2}_\mu$ to its central fiber $m^{-1}(t^\mu)$, the $\tilde{T}$-fixed points of $\tilde{\text{Gr}}^{\lambda_1,\lambda_2}_\mu$ are the same as the $T$-fixed points of $m^{-1}(t^\mu)$. This proves (1).

Likewise, the $\tilde{T}$-invariant curves in $\tilde{\text{Gr}}^{\lambda_1,\lambda_2}_\mu$ are the same as the $T$-invariant curves in $m^{-1}(t^\mu)$. Note that, by definition, $m^{-1}(t^\mu)$ is a subvariety of $\text{Gr}^\lambda_i \simeq G/P_\lambda_i$. It is well-known that this parabolic flag variety has finitely many $T$-invariant curves, given by $(SL_2)_\alpha$-orbits, where $\alpha$ is any root of $G$ belonging to $G/P_\lambda_i$. Such a curve connects the fixed points corresponding to $\nu_1$ and $s_\alpha(\nu_1)$, and since $m^{-1}(t^\mu)$ is a $T$-invariant (in fact $G(O)$-invariant) subvariety of $\text{Gr}^\lambda_i$, this curve must lie entirely inside of $m^{-1}(t^\mu)$. Now, we have $s_\alpha(\nu_1) = \nu_1 - (\nu_1, \alpha) \alpha'$, and since $\nu_1$ is minuscule and the curve connects distinct points, $s_\alpha(\nu_1) = \nu_1 \pm \alpha'$. Thus, for roots $\alpha'$ of $G'$ such that $\nu_1 + \alpha' \in W \cdot \lambda_i$, there is a unique $T$-invariant curve connecting $t^\nu$ and $t^{\nu_1+\alpha'}$ in $\text{Gr}^\lambda_i$. Since $\nu_1 + \nu_2 = \mu$, this curve connects the $T$-fixed points $(\nu_1, \nu_2)$ and $(\nu_1 + \alpha', \nu_2 - \alpha')$ in $m^{-1}(t^\mu)$. This proves (2).

\[ \square \]

### 4.2 Unbroken maps

Recall from section 2.5 the definition of unbroken maps.

**Lemma 4.2.** The unbroken maps $f : C \to \tilde{\text{Gr}}^{\lambda_1,\lambda_2}_\mu$ are of the following two types:

1. $C \simeq \mathbb{P}^1$ and $f$ is a multiple cover of a $\tilde{T}$-invariant curve $C_{\nu_1,\nu_2,\alpha}$ branched over its two endpoints.

2. $C \simeq C_0 \cup C_1$, where $C_0$ is a rational curve contracted to a $\tilde{T}$-fixed point, the two marked points lie on $C_0$, and $C_1 \simeq \mathbb{P}^1$ is a multiple cover of a $\tilde{T}$-invariant curve $C_{\nu_1,\nu_2,\alpha}$ branched over its two endpoints.
Proof. The curve $C_{\nu_1,\nu_2,\alpha}$ has $T$-weight $\alpha$ at $(\nu_1, \nu_2)$, since it is an $(SL_2)\alpha$-orbit and since the loop rotation $\mathbb{C}^*$ acts trivially on the central fiber $m^{-1}(t^\mu)$ in which $C_{\nu_1,\nu_2,\alpha}$ is contained. Since these are linearly independent, the only unbroken chains are $\mathbb{P}^1$'s, which by definition yields the lemma. \qed

As noted in [43, §7.3.1], the contribution of the second type of unbroken map is diagonal in the fixed-point basis, so we will focus on the first type of map.

4.3 The tangent bundle on fixed points

We begin with the following elementary result:

**Lemma 4.3.** For any coweights $\mu \leq \lambda$ of $G$, the tangent space to $Gr^\lambda$ at $t^\mu$ is isomorphic to

$$\bigoplus_{\beta \in \Delta} \bigoplus_{n=0}^{\max(0,(\beta,\mu))} g_\beta t^n$$

**Proof.** We have

$$T_{t^\mu} Gr^\lambda \simeq g(\mathcal{O})/(g(\mathcal{O}) \cap \text{Stab}(\mu)) \simeq g(\mathcal{O})/(g(\mathcal{O}) \cap g(\mathcal{O})^\mu),$$

where $g(\mathcal{O})^\mu := \text{ad}_{t^{-\mu}} g(\mathcal{O})$. For $X$ in a root subspace $g_\beta$, we have $\text{ad}_{t^{-\mu}} X = t^{-\langle \beta, \mu \rangle} X$, so we can decompose

$$g(\mathcal{O}) = \bigoplus_\beta g_\beta(\mathcal{O})$$

$$g(\mathcal{O})^\mu = \bigoplus_\beta t^{-\langle \beta, \mu \rangle} g_\beta(\mathcal{O}).$$

Substituting these into the above yields the lemma. \qed

We can now determine the $T$-weights of tangent bundle at a $\widetilde{T}$-fixed point:
Lemma 4.4. Let \((\nu_1, \nu_2) \in \text{Gr}^\lambda_{\mu,1,2}\) be a \(\tilde{T}\)-fixed point. Then the \(T\)-weights of the tangent space \(T_{(\nu_1, \nu_2)}\text{Gr}^\lambda_{\mu,1,2}\) are pairs of roots \(\beta, -\beta\) such that \(\langle \beta, \nu_1 \rangle = 1\) and \(\langle \beta, \nu_2 \rangle = -1\).

Proof. Since \(\text{Gr}^\lambda_{\mu} = G/P_{\lambda_1}\), the \(T\)-weights of \(T_{\nu_1} \text{Gr}^\lambda_{\mu}\) are all roots \(\beta\) such that \(\langle \beta, \nu_1 \rangle > 0\). Since each \(\nu_i\) is minuscule, this implies \(\langle \beta, \nu_i \rangle = 1\). Hence, the \(T\)-weights of \(T_{(\nu_1, \nu_2)}(\text{Gr}^\lambda_{\mu,1,2})\) are \(\beta\) such that either \(\langle \beta, \nu_1 \rangle = 1\) or \(\langle \beta, \nu_2 \rangle = 1\). There is a split exact sequence

\[
0 \to T_{(\nu_1, \nu_2)}(\text{Gr}^\lambda_{\mu,1,2}) \to T_{(\nu_1, \nu_2)}(\text{Gr}^\lambda_{\mu,1,2}) \to T_{\mu}(\text{Gr}^\mu) \to 0
\]

since \(\nu_1 + \nu_2 = \mu\). By Lemma 4.3, the \(T\)-weights of \(T_{\mu}(\text{Gr}^\mu)\) are roots \(\beta\) with \(\langle \beta, \mu \rangle > 0\). So the above sequence gives that the \(T\)-weights of \(T_{(\nu_1, \nu_2)}(\text{Gr}^\lambda_{\mu,1,2})\) are \(\beta\) such that either \(\langle \beta, \nu_1 \rangle = 1\) or \(\langle \beta, \nu_2 \rangle = 1\), and \(\langle \beta, \nu_1 \rangle + \langle \beta, \nu_2 \rangle = \langle \beta, \mu \rangle \leq 0\). But since \(\langle \beta, \nu_i \rangle \in \{-1, 0, 1\}\), these conditions in fact imply that \(\langle \beta, \mu \rangle = 0\). This proves the lemma.

\[\square\]

4.4 The tangent bundle on fixed curves

We now decompose the tangent bundle \(T(\text{Gr}^\lambda_{\mu,1,2})\) into \(T\)-equivariant line bundles on each \(\tilde{T}\)-invariant curve \(C_{\nu_1, \nu_2, \alpha}\).

Define \(L_{\beta}\) as the \(T\)-equivariant line bundle on \(C_{\nu_1, \nu_2, \alpha}\) whose fibers at \((\nu_1, \nu_2)\) and \((\nu_1 + \alpha^\vee, \nu_2 - \alpha^\vee)\) have \(T\)-weights \(\beta\) and \(s_\alpha(\beta)\), respectively. Since \(s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha\), the degree of \(L_{\beta}\) is \(-\langle \beta, \alpha^\vee \rangle\).

Lemma 4.5. The restriction of \(T(\text{Gr}^\lambda_{\mu,1,2})\) to \(C_{\nu_1, \nu_2, \alpha}\) decomposes as a direct sum of \(T\)-equivariant line bundles

\[
\bigoplus_{\beta} L_{\beta}
\]

where \(\beta\) ranges over the \(T\)-weights of the tangent space \(T_{(\nu_1, \nu_2)}(\text{Gr}^\lambda_{\mu,1,2})\), i.e. all roots satisfying \(\langle \beta, \nu_1 \rangle = \pm 1\) and \(\langle \beta, \nu_2 \rangle = \mp 1\). For \(\beta\) appearing in the "horizontal" direc-
tion $T_{\nu_1} \Gr^\lambda$, the degree of $L_\beta$ is non-negative, and for $\beta$ appearing in the "vertical" direction $T_{\nu_2} \Gr^\lambda$, the degree of $L_\beta$ is non-positive.

Proof. The tangent bundle of $\Gr_{\mu}^{\lambda_1, \lambda_2}$ is equivariant with respect to the reductive part of the stabilizer of $t^\mu$, in particular with respect to $SL(2)$. So the Borel of $sl(2)$ acts on the tangent space at $(\nu_1, \nu_2)$. The element $e_\alpha$ in the radical of that Borel must act trivially, since it cannot be that $\langle \beta, \nu_1 \rangle = \pm 1$ and $\langle \beta + \alpha, \nu_1 \rangle = \pm 1$. Thus, the tangent bundle decomposes as an $SL(2)$-equivariant bundle into the direct sum of $L_\beta$, where $\beta$ ranges over the $T$-weights of the tangent space at $(\nu_1, \nu_2)$ obtained in Lemma 4.4, as desired.

By the proof of Lemma 4.1, $\alpha$ satisfies $\langle \alpha, \nu_1 \rangle = -1$ and $\langle \alpha, \nu_2 \rangle = 1$. The horizontal roots $\beta$ satisfy $\langle \beta, \nu_1 \rangle = 1$, so $\langle \beta - \alpha, \nu_1 \rangle = 2$, and thus $\beta - \alpha$ cannot be a root since $\nu_1$ is minuscule. Considering root strings, we conclude that $\langle \beta, \alpha^\vee \rangle \leq 0$, so $\deg L_\beta = -\langle \beta, \alpha^\vee \rangle \geq 0$. The vertical roots are simply the negatives of the horizontal ones by Lemma 4.4, so the claim for vertical roots follows.

4.5 Localization

We can now calculate the contribution of $C_{\nu_1, \nu_2, \alpha}$ to the quantum product by the divisor class $D$ in the fixed-point basis $\nu_{(\nu_1, \nu_2)}$. Let $f : C \to \Gr_{\mu}^{\lambda_1, \lambda_2}$ be a degree $d$ cover of $C_{\nu_1, \nu_2, \alpha}$, i.e. the first type of unbroken map on $\Gr_{\mu}^{\lambda_1, \lambda_2}$ identified in Lemma 4.2. There is a single quantum parameter $q = q^{[C_{\nu_1, \nu_2, \alpha}]}$. We want to calculate

$$(D \bullet \nu_{(\nu_1, \nu_2)}, \nu_{(\nu_1 + \alpha^\vee, \nu_2 - \alpha^\vee)})$$

$$=-h \sum_{d>0} (D, d[C_{\nu_1, \nu_2, \alpha}])(\alpha_{\nu_1, \nu_2}) \nu_{(\nu_1 + \alpha^\vee, \nu_2 - \alpha^\vee)}q^d.$$
Note that a factor of $d$ can be pulled out from $(D, d[C_{\nu_1, \nu_2, \alpha}])$. By virtual localization, modulo $h$ we have:

$$d(\text{ev}_*[\mathcal{M}_{0,2}(\overline{\text{Gr}}_{\mu}^{\lambda_1, \lambda_2}, d[C_{\nu_1, \nu_2, \alpha}])_{\text{red}}, v_{(\nu_1, \nu_2)} \otimes v_{(\nu_1 + \alpha', \nu_2 - \alpha')})$$

$$= \frac{d}{d} e'(T_{(\nu_1, \nu_2)} \overline{\text{Gr}}_{\mu}^{\lambda_1, \lambda_2}) e'(T_{(\nu_1 + \alpha', \nu_2 - \alpha')} \overline{\text{Gr}}_{\mu}^{\lambda_1, \lambda_2}) \frac{H^1(C, f^*\overline{T\text{Gr}}_{\mu}^{\lambda_1, \lambda_2})}{H^0(C, f^*\overline{T\text{Gr}}_{\mu}^{\lambda_1, \lambda_2})},$$

where the factor of $1/d$ comes from $\text{Aut}(f) = \mathbb{Z}/d\mathbb{Z}$, and $e'$ denotes the product of non-zero $T$-weights.

Let $S = \{ \beta \in \Delta : \langle \beta, \nu_1 \rangle = 1 \text{ and } \langle \beta, \nu_2 \rangle = -1 \}$, and set

$$\mathcal{T} = \bigoplus_{\beta \in S} f^*L_{\beta},$$

so that, by Lemma 4.5, we have $f^*\overline{T\text{Gr}}_{\mu}^{\lambda_1, \lambda_2} = \mathcal{T} \oplus \mathcal{T}^*$. Thus, by Lemma 11.1.3 in [43], we obtain that the above quantity is equal to $\pm 1$; more precisely, it is given by

$$(-1)^{\text{rank } \mathcal{T} + \text{deg } \mathcal{T} + \# z},$$

where $\# z$ denotes the number of 0 weights in $\mathcal{T} \oplus \mathcal{T}^*$. Each of the quantities in the exponent is easily calculated; we obtain

$$\text{rank } \mathcal{T} = |S| = \frac{1}{2} \dim \overline{\text{Gr}}_{\mu}^{\lambda_1, \lambda_2} = \langle \rho, \lambda_1 + \lambda_2 - \mu \rangle$$

$$\text{deg } \mathcal{T} = -d \left( \sum_{s} \beta_s \alpha^s \right)$$

$$\# z = 1.$$

It is also easy to check that $\text{deg } \mathcal{T}$ is even.

Finally, note that the effective curve class $[C_{\nu_1, \nu_2, \alpha}]$ generates $H_2(G/P_{\lambda_1})$, and
\( D = 2D_1 \) restricts to \( O(2) \) on \( G/P_{\lambda_1} \), so

\[
(D, [C_{\nu_1, \nu_2, a}]) = 2.
\]

We conclude that

\[
(D \bullet v(\nu_1, \nu_2), v(\nu_1 + a\nu, \nu_2 - \alpha \nu_2)) = -(-1)^{\rho \lambda_1 + \lambda_2 - \mu} 2\hbar \sum_{d > 0} q^d = -(-1)^{\rho \lambda_1 + \lambda_2 - \mu} 2\hbar \frac{q}{1 - q}.
\]

Summing over all torus-invariant curves, we finally obtain:

**Theorem 4.6.**

\[
D* = D \cup -2\hbar \frac{q}{1 - q} \tilde{\Omega} + \ldots
\]

where the dots denote a scalar operator, and \( \tilde{\Omega} \) acts on the fixed point basis \( v(\nu_1, \nu_2) \in H^*(Gr_{\mu})_{\text{loc}} \) by

\[
\tilde{\Omega}(v(\nu_1, \nu_2)) = (-1)^{\rho \lambda_1 + \lambda_2 - \mu} \sum_{\alpha} v(\nu_1 + \alpha \nu, \nu_2 - \alpha \nu) \mod \hbar,
\]

where the sum is taken over all \( \alpha \in \Delta \) such that \( \nu_1 + \alpha \nu \in W \cdot \lambda_1 \) and \( \nu_2 - \alpha \nu \in W \cdot \lambda_2 \).

On the other hand, the truncated Casimir operator \( \Omega^{\text{trunc}} \) identified in Section 3.7 acts on the fixed-point basis with the same non-zero matrix elements as \( \tilde{\Omega} \). So up to rescaling coefficients and modulo \( \hbar \), we have \( \tilde{\Omega} = \Omega^{\text{trunc}} \). We conjecture that the two operators are, in fact, equal.

By Theorem 4.6, the \( T \)-equivariant quantum connection of \( Gr_{\mu}^{\lambda_1, \lambda_2} \) is given by

\[
\nabla_D^{\text{quantum}} = d_D - D \cup +2\hbar \frac{q}{1 - q} \tilde{\Omega} + \ldots
\]

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4.6 Trigonometric KZ connection

Let $G^\vee$ be a semisimple algebraic group of rank $r$ with maximal torus $T^\vee$, and let $\mathfrak{g}^\vee$ and $t^\vee$ denote their Lie algebras. Choose simple roots $\alpha_i$ and root vectors $e_{\alpha_i} \in \mathfrak{g}$ for $i = 1, \ldots, r$. Let $\Delta_+$ denote the set of all positive roots. Let $x_1, \ldots, x_r$ be an orthonormal basis in $t^\vee$. Let

$$\Omega = \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Delta_+} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha)$$

be the corresponding Casimir element in $U(\mathfrak{g}^\vee)^{\otimes 2}$, and decompose $\Omega$ into positive and negative parts:

$$\Omega^+ = \frac{1}{2} \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Delta_+} e_\alpha \otimes e_{-\alpha}, \quad \Omega^- = \frac{1}{2} \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Delta_+} e_{-\alpha} \otimes e_\alpha.$$

Define the trigonometric $r$-matrix

$$r(u) = \frac{\Omega^+ e^u + \Omega^-}{e^u - 1}.$$

Let $V_1, \ldots, V_n$ be representations of $G^\vee$. Consider the trivial vector bundle with fiber $V_1 \otimes \ldots \otimes V_n$ and base $\mathbb{C}^n$ with coordinates $(u_1, \ldots, u_n)$. For $a \in t$ and $h \in \mathbb{C}$, define the trigonometric KZ connection $\nabla^{KZ}(a, h)$ on this vector bundle by (see, e.g. [57])

$$\nabla^{KZ}(a, h) = \partial - 2h \left( \sum_{i<j} r^{ij}(u_i - u_j) d(u_i - u_j) + \sum_i a^{(i)} du_i \right),$$

where $r^{ij}$ denotes $r$ acting on the $i$th and $j$th tensor factors, and likewise for $a^{(i)}$. Note that this is invariant under the action of the diagonal, so we may translate all the $u_i$ so that $u_1 + \ldots + u_n = 0$.

Note that $\Omega^+$ and $\Omega^-$ preserve the weight decomposition of $V_1 \otimes \ldots \otimes V_n$. Hence, for any $\mu$ appearing as a weight of $V_1 \otimes \ldots \otimes V_n$, the trigonometric KZ connection
restricts to a connection on the subbundle with fiber $V_1 \otimes \ldots \otimes V_n[\mu]$.

Specializing to $n = 2$ and setting $u = u_1 - u_2$, we can write differentiation in the
direction $u$ as

$$\nabla^K_Z = d_u - 2\hbar(a^{(1)} + a^{(2)}) + 2\hbar \left( \frac{\Omega^+ e^u + \Omega^-}{1 - e^u} \right)$$

$$= d_u - 2\hbar(a^{(1)} + a^{(2)} - \Omega^-) + 2\hbar \frac{e^u}{1 - e^u} \Omega.$$  

Note that the quantum connection and trigonometric KZ connection cannot match
exactly, since the latter is trivial for $\hbar = 0$ while the former is not (due to classical
multiplication). However, identifying $q = e^u$, we see that, assuming our conjecture
from the previous section, the purely quantum part matches the corresponding part
of the trigonometric KZ connection, but with $\Omega$ replaced by $\Omega^{trunc}$. We thus view
the quantum connection as a "truncated" trigonometric KZ-type connection.

Finally, it is natural to conjecture that $D\cup = 2(a^{(1)} + a^{(2)} - \hbar \Omega^-)$, and that the
full connections are equal under the identification of equivariant parameters $\lambda = \hbar a$.

### 4.7 Reduction to Picard rank 1

We briefly describe how we expect the strategy of [13] allows us to reduce to the case
$n = 2$.

The semi-universal Poisson deformation of $\tilde{\text{Gr}}_{\lambda_1, \ldots, \lambda_n}$ is the corresponding Beilinson-
Drinfeld Grassmannian $\text{Gr}^{\lambda_1, \ldots, \lambda_n}_{A^n}$ over $A^n = \mathbb{C}(x_1, \ldots, x_n)$ (see [34]). For the $x_i$ pairwise distinct, the fiber is the ordinary product $\text{Gr}^{\lambda_1} \times \ldots \times \text{Gr}^{\lambda_n}$. Over a generic point of the hyperplane $\{x_i = x_j\}, i \neq j$, the fiber is the convolution product $\text{Gr}^{\lambda_i} \times \text{Gr}^{\lambda_j}$ times the ordinary product of the remaining factors.

As described in [35], this restricts to a Poisson deformation of $\tilde{\text{Gr}}_{\mu}^{\lambda_1, \ldots, \lambda_n}$. For
the $x_i$ pairwise distinct, the fiber is affine. Over a generic point of the hyperplane
$\{x_i = x_j\}$, we expect that the fiber to be related to a $\tilde{\text{Gr}}_{\mu'}^{\lambda_i, \lambda_j}$-bundle over an affine
variety for some $\mu'$. Applying the argument described in [13] for general symplectic resolutions, we would obtain that the quantum corrections are given by a sum over $1 \leq i < j \leq n$, of the quantum corrections for $\widetilde{\text{Gr}}_{\mu'}^{\lambda_i, \lambda_j}$, yielding each of the summands in the trigonometric KZ connection.
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