Quantum intertwiners and integrable systems

by

Yi Sun

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Signature redacted

Department of Mathematics
April 25, 2016

Signature redacted

Certified by........................................

Pavel Etingof
Professor
Thesis Supervisor

Signature redacted

Accepted by .......................................

Alexei Borodin
Co-Chair, Department Committee on Graduate Students
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Abstract

We present a collection of results on the relationship between intertwining operators for quantum groups and eigenfunctions for quantum integrable systems.

First, we study the Etingof-Kirillov Jr. expression of Macdonald polynomials as traces of intertwiners of quantum groups in the Gelfand-Tsetlin basis. In the quasi-classical limit, we obtain a new Harish-Chandra type integral formula for Heckman-Opdam hypergeometric functions. This formula is related to an integral formula appearing in recent work of Borodin-Gorin by integration over Liouville tori of the Gelfand-Tsetlin integrable system. At the quantum level, we obtain a new proof of the branching rule for Macdonald polynomials which transparently relates branching of Macdonald polynomials to branching of quantum group representations.

Second, we study traces of intertwiners for quantum affine algebras. In the \( \mathfrak{sl}_2 \) case, we show that, when valued in the three-dimensional evaluation representation, such traces converge in a certain region of parameters and provide a representation-theoretic construction of Felder-Varchenko's hypergeometric solutions to the \( q \)-KZB heat equation. This gives the first proof that such a trace function converges and resolves the first case of a conjecture of Etingof-Varchenko. As an application, we prove Felder-Varchenko's conjecture that their elliptic Macdonald polynomials are related to Etingof-Kirillov Jr.'s affine Macdonald polynomials. In the general case, we modify the setting of the work of Etingof-Schiffmann-Varchenko to show that traces of such intertwiners satisfy four commuting systems of \( q \)-difference equations – the Macdonald-Ruijsenaars, dual Macdonald-Ruijsenaars, \( q \)-KZB, and dual \( q \)-KZB equations.

Thesis Supervisor: Pavel Etingof
Title: Professor
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Chapter 1

Introduction

This thesis presents a collection of results on the relationship between intertwining operators for quantum groups and eigenfunctions for quantum integrable systems. Viewed from mathematical physics, such intertwiners are the basic building blocks of conformal field theory and give rise to solutions to KZ-type systems. Viewed from special functions, the property of being an intertwiner implies symmetry relations on matrix elements and traces which manifest as differential and $q$-difference equations comprising integrable systems.

We apply techniques from representation theory to establish and exploit this connection in a variety of situations. At the trigonometric level, we apply this philosophy to finite-type quantum groups to give a representation-theoretic explanation of the structure of Macdonald polynomials and Heckman-Opdam hypergeometric functions. At the elliptic level, we connect elliptic versions of KZ and Macdonald-Ruijsenaars systems to traces of intertwiners of quantum affine algebras and provide the first integral formula for such a trace.

Chapters 2 and 3 are devoted to a study of branching rules and integral formulas for Macdonald polynomials and Heckman-Opdam hypergeometric functions. These functions are eigenfunctions of the Macdonald-Ruijsenaars and trigonometric Calogero-Moser systems, quantum integrable systems describing interacting particles in a potential well, and they generalize many important families of special functions appearing in mathematical physics and probability theory such as Schur functions, Jack polynomials, and Hall-Littlewood polynomials. Our approach hinges on the interaction between the Etingof-Kirillov Jr. construction of Macdonald polynomials as traces of intertwiners of $U_q(g_l^n)$, its quasiclassical limit to an integral with respect to Liouville measure over a dressing orbit, and the corresponding Gelfand-Tsetlin basis and Gelfand-Tsetlin integrable system.

In Chapter 2, we provide Harish-Chandra type formulas for the multivariate Bessel functions and Heckman-Opdam hypergeometric functions as representation-valued integrals over dressing orbits. Our expression is the quasi-classical limit of the realization of Macdonald polynomials as traces of intertwiners of quantum groups given by Etingof-Kirillov Jr. in [26]. Integration over the Liouville tori of the Gelfand-Tsetlin integrable system and adjunction for higher Calogero-Moser Hamiltonians recovers and gives a new proof of the integral realization over Gelfand-Tsetlin polytopes which
appeared in the recent work [9] of Borodin-Gorin on the \( \beta \)-Jacobi corners ensemble. Chapter 2 is based on the paper


In Chapter 3, we give a new representation-theoretic proof of the branching rule for Macdonald polynomials using the Etingof-Kirillov Jr. expression for Macdonald polynomials as traces of intertwiners of \( U_q(\mathfrak{gl}_n) \) given in [26]. In the Gelfand-Tsetlin basis, we show that diagonal matrix elements of such intertwiners are given by application of Macdonald’s operators to a simple kernel. An essential ingredient in the proof is a map between spherical parts of double affine Hecke algebras of different ranks based upon the Dunkl-Kasatani conjecture of [18, 20, 34, 66]. Chapter 3 is based on the paper


Chapters 4 and 5 are concerned with the study of intertwining operators for quantum affine algebras and their connection to integrable systems of Knizhnik-Zamolodchikov type. Such equations form an important family of integrable systems of differential and \( q \)-difference equations appearing first in the study of Wess-Zumino-Novikov-Witten (WZNW) conformal field theory in string theory. As surveyed in [24, 50], KZ-type systems come in three types, rational, trigonometric, and elliptic, corresponding to the classification of solutions to the dynamical Yang-Baxter equation with spectral parameter. The \( q \)-KZB (Knizhnik-Zamolodchikov-Bernard) equations introduced in [42] are the \( q \)-difference elliptic version of these systems. These chapters are devoted to the relation between traces of intertwiners for quantum affine algebras and solutions to the \( q \)-KZB equations and similar integrable systems.

In Chapter 4, we show that the traces of \( U_q(\mathfrak{sl}_2) \)-intertwiners of [33] valued in the three-dimensional evaluation representation converge in a certain region of parameters and give a representation-theoretic construction of Felder-Varchenko’s hypergeometric solutions to the \( q \)-KZB heat equation given in [48]. This gives the first proof that such a trace function converges and resolves the first case of the Etingof-Varchenko conjecture of [37].

As applications, we prove a symmetry property for traces of intertwiners and prove Felder-Varchenko’s conjecture in [49] that their elliptic Macdonald polynomials are related to the affine Macdonald polynomials defined as traces over irreducible integrable \( U_q(\mathfrak{sl}_2) \)-modules in [28]. In the trigonometric and classical limits, we recover results of [29, 37]. Our method relies on an interplay between the method of coherent states applied to the free field realization of the \( q \)-Wakimoto module of [77], convergence properties given by the theta hypergeometric integrals of [48], and rationality properties originating from the representation-theoretic definition of the trace function. Chapter 4 is based on the paper

In Chapter 5, we modify and give complete proofs for the results of Etingof-Schiffmann-Varchenko in [33] on traces of intertwiners of untwisted quantum affine algebras in the opposite coproduct and with the standard grading instead of the principal grading. More precisely, we show that certain renormalized generalized traces $F_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k)$ for $U_q(\mathfrak{g})$ solve four commuting systems of $q$-difference equations: the Macdonald-Ruijsenaars, dual Macdonald-Ruijsenaars, $q$-KZB, and dual $q$-KZB equations. In addition, we show a symmetry property for these renormalized trace functions. Our modifications are motivated by their appearance in the recent work [96]. Chapter 5 is based on the paper

Chapter 2

Integral formula for Heckman-Opdam hypergeometric functions

2.1 Introduction

The Heckman-Opdam hypergeometric functions are a family of real-analytic symmetric functions introduced by Heckman-Opdam in \[60, 59, 84, 85\] as joint eigenfunctions of the trigonometric Calogero-Moser integrable system. The latter is a quasi-classical limit of the Macdonald-Ruijsenaars integrable system, and in \[9\], Borodin-Gorin realized the Heckman-Opdam hypergeometric function as a limit of the Macdonald polynomials under the quasi-classical scaling. By applying their limit transition to Macdonald’s branching rule, they obtained a new formula for the Heckman-Opdam hypergeometric functions as an integral over Gelfand-Tsetlin polytopes.

The purpose of the present chapter is to provide new Harish-Chandra type integral formulas for the Heckman-Opdam hypergeometric functions as representation-valued integrals over dressing orbits of \(U_N\). Our formulas are the quasi-classical limits of the expression given by Etingof-Kirillov Jr. in \[26\] for Macdonald polynomials as representation-valued traces of \(U_q(\mathfrak{g}_N)\)-intertwiners. In this limit, traces over irreducible representations become integrals with respect to Liouville measure on the corresponding dressing orbit.

Integrating our formulas over Liouville tori of the Gelfand-Tsetlin integrable system yields an expression for Heckman-Opdam hypergeometric functions as an integral of \(U_N\)-matrix elements over the Gelfand-Tsetlin polytope. We identify these matrix elements as an application of higher Calogero-Moser Hamiltonians to an explicit kernel. Taking adjoints of these Hamiltonians recovers and gives a new proof of the formula of \[9\]. Our techniques involve a relation between spherical parts of rational Cherednik algebras of different rank which is of independent interest.

In the remainder of the introduction, we summarize our motivations, give precise statements of our results, and explain how they relate to other recent work. This chapter is based on the paper \[93\].
2.1.1 Heckman-Opdam hypergeometric functions

Fix a complex number $k$ and a positive integer $N$. The rational and trigonometric Calogero-Moser integrable systems in the variables $\{\lambda_i\}_{1 \leq i \leq N}$ are the quantum integrable systems with quadratic Hamiltonians

$$
L_{p_2}(k) = \sum_i \partial_i^2 + 2k(1-k) \sum_{i<j} \frac{1}{(\lambda_i - \lambda_j)^2}
$$

and

$$
L_{p_2}^{\text{trig}}(k) = \sum_i \partial_i^2 + k(1-k) \sum_{i<j} \frac{1}{\sinh^2 \left( \frac{\lambda_i - \lambda_j}{2} \right)}.
$$

They are completely integrable systems, meaning that $L_{p_2}(k)$ and $L_{p_2}^{\text{trig}}(k)$ fit into families $L_{p}(k)$ and $L_{p}^{\text{trig}}(k)$ of commuting Hamiltonians defined for each symmetric polynomial $p$. Define conjugated versions of these Hamiltonians by

$$
\overline{L}_p(k) = \Delta(\lambda)^{-k} \circ L_p(k) \circ \Delta(\lambda)^k
$$

and

$$
\overline{L}_p^{\text{trig}}(k) = e^{\frac{(N-1)k}{2}} \sum_i \Delta(e^{\lambda})^{-k} \circ \overline{L}_p^{\text{trig}}(k) \circ e^{-\frac{(N-1)k}{2}} \sum_i \lambda_i \Delta(e^{\lambda})^k,
$$

where for a set of variables $x$, we denote by $\Delta(x)$ the Vandermonde determinant $\Delta(x) = \prod_{i<j} (x_i - x_j)$. For each $s = (s_1, \ldots, s_N)$, the hypergeometric system corresponding to $s$ was introduced in [60, 59, 84, 85] as

$$
\overline{L}_p^{\text{trig}}(k)F_k(\lambda, s) = p(s)F_k(\lambda, s).
$$

Let $\rho$ be the weight $\rho = (\frac{N-1}{2}, \ldots, \frac{1-N}{2})$. The following characterization was given of certain joint eigenfunctions of this system known as Heckman-Opdam hypergeometric functions.

**Theorem 2.1.1** ([61, 87]). The hypergeometric system (2.1.3) has a unique symmetric real-analytic solution $F_k(\lambda, s)$ normalized so that the leading term of its series expansion in $\lambda$ is

$$
\frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N} \prod_{i<j} \prod_{a=0}^{k-1} (s_i - s_j + a)^{-1} e^{(\lambda, s - k\rho)}.
$$

This $F_k(\lambda, s)$ extends to a holomorphic function of $\lambda$ on a symmetric tubular neighborhood of $\mathbb{R}^n \subset \mathbb{C}^n$.

The corresponding rational degenerations are a family of symmetric real-analytic joint eigenfunctions $B_k(\lambda, s)$ of $L_p(k)$ satisfying

$$
\overline{L}_p(k)B_k(\lambda, s) = p(s)B_k(\lambda, s)
$$

and normalized so that $B_k(\lambda, 0) = 1$. They are known as multivariate Bessel functions and have been studied in [18, 16, 86, 82, 54, 51].
2.1.2 Poisson-Lie group structure on $u_N$ and $U_N$

The Lie algebra $\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{C})$ has real Iwasawa decomposition $\mathfrak{gl}_N = u_N \oplus b_N$ with $b_N \cong u_N^*$. Let $t_N \subset u_N$ be the Cartan subalgebra. We identify $u_N$ with $\mathfrak{p}_N$, the trivial Lie algebra of $N \times N$ Hermitian matrices by the map $x \mapsto \frac{1}{2}(x + x^*)$. Equip $\mathfrak{p}_N$ with the Kirillov-Kostant-Souriau Poisson structure, and denote the coadjoint orbit of a diagonal matrix $\lambda \in \mathfrak{p}_N$ by $O_\lambda$. We will use $\lambda$ interchangeably for the diagonal matrix and its sequence of diagonal entries. Denote the symplectic form and Liouville measure on $O_\lambda$ by $\omega_\lambda$ and $d\mu_\lambda$, respectively, and let $\mathbb{C}[b_N]$ be the corresponding Poisson algebra.

In the corresponding Iwasawa decomposition $GL_N = U_N B_N$ for the group, give $U_N$ the Lu-Weinstein Poisson-Lie structure (see [74]) so that $B_N$ is the dual Poisson-Lie group to $U_N$. Let $T_N \subset U_N$ denote the diagonal torus. Identify $B_N$ with the Poisson manifold $P^+_N$ of $N \times N$ positive definite Hermitian matrices via $\text{sym}(b) = (b^*b)^1/2$ so that $\text{sym}$ intertwines the dressing and conjugation actions of $U_N$ on $B_N$ and $P^+_N$. For $\Lambda = e^\lambda \in P^+_N$, denote by $O_\Lambda$, $\omega_\Lambda$, and $d\mu_\Lambda$ the dressing orbit containing $\Lambda$, its symplectic form, and its Liouville measure. Let $\mathbb{C}[B_N]$ and $\mathbb{C}[O_\Lambda]$ denote the corresponding Poisson algebras; these algebras possess a $*$-structure given by complex conjugation on each matrix element.

2.1.3 The main results

Restrict now to the case of positive integer $k$. Let $W_{k-1}$ denote the $U_N$-representation

$$L_{((k-1)(N-1),-(k-1),\ldots,-(k-1))} = \text{Sym}^{(k-1)N} \mathbb{C}^N \otimes (\det)^{-(k-1)},$$

and choose an isomorphism $W_{k-1}[0] \cong \mathbb{C} \cdot w_{k-1}$ for some $w_{k-1} \in W_{k-1}[0]$ which spans the 1-dimensional zero weight space $W_{k-1}[0]$. Let $f_{k-1}: O_\Lambda \to W_{k-1}$ and $F_{k-1}: O_\Lambda \to W_{k-1}$ denote the unique $U_N$-equivariant maps such that $f_{k-1}(\Lambda) = F_{k-1}(\Lambda) = w_{k-1}$. Our main results are Theorems 2.4.1 and 2.5.1, which realize the multivariate Bessel functions and Heckman-Opdam hypergeometric functions as representation-valued integrals over coadjoint and dressing orbits under the identification of $W_{k-1}[0] \cong \mathbb{C} \cdot w_{k-1}$ with $\mathbb{C}$.

**Theorem 2.4.1.** The multivariate Bessel function $B_k(\lambda, s)$ admits the integral representation

$$B_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N \prod_{i<j}(\lambda_i - \lambda_j)^k \prod_{i<j}(s_i - s_j)^k} \int_{X \in O_\Lambda} f_{k-1}(X) e^{\sum_{i=1}^N s_i X_i} d\mu_\lambda.$$ 

**Theorem 2.5.1.** The Heckman-Opdam hypergeometric function $F_k(\lambda, s)$ admits the
integral representation

\[
F_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N \prod_{1<i<j}(e^{\lambda_i-\lambda_j} - e^{\lambda_j-\lambda_i})} \prod_{i=1}^{k-1} \prod_{i<j} (s_i - s_j - a) \int_{X \in \mathcal{C}_A} F_{k-1}(X) \prod_{l=1}^{N} \left( \frac{\det(X_i)}{\det(X_{l-1})} \right)^{s_l} d\mu_A,
\]

where \(X_i\) is the principal \(l \times l\) submatrix of \(X\).

**Remark.** The \(k = 1\) case of the integral of Theorem 2.4.1 is the HCIZ integral of [56, 57, 62]. It also generalizes the construction of [54], where a similar construction is made for \(k = 1, 2\).

### 2.1.4 Existing integral formulas and connection to \(\beta\)-Jacobi corners ensemble

Scalings of Heckman-Opdam functions appeared in the work [9] of Borodin-Gorin on the \(\beta\)-Jacobi corners ensemble, where they were obtained as a certain scaling limit of the Macdonald polynomials \(P_\mu(x; q, t)\). For \(\lambda_1 \geq \cdots \geq \lambda_N \in \mathbb{R}^N\), define the Gelfand-Tsetlin polytope to be

\[
\text{GT}_\lambda := \{(\mu_i^l)_{1 \leq i \leq l \leq N} | \mu_i^l \geq \mu_i^{l+1}\},
\]

where we take \(\mu_i^N = \lambda_i\). A point \(\{\mu_i^l\}\) in \(\text{GT}_\lambda\) is called a Gelfand-Tsetlin pattern. To state the result of [9], we define the integral formulas

\[
\phi_k(\lambda, s) = \Gamma(k)^{-\frac{N(N-1)}{2}} \int_{\mu \in \text{GT}_\lambda} e^{\sum_{i=1}^{N} s_i \left( \sum_{i=1}^{l} \mu_i^l - \Sigma_{i=1}^{l-1} \mu_i^{l-1} \right)} \prod_{i=1}^{N-1} \prod_{i<j} \left| \mu_i^l - \mu_j^{l+1} \right|^{k-1} \prod_{i=1}^{l} \prod_{i<j} \left| \mu_i^{l+1} - \mu_j^{l+1} \right|^{k-1} \prod_{i=1}^{l} d\mu_i^l \tag{2.1.5}
\]

and

\[
\Phi_k(\lambda, s) = \Gamma(k)^{-\frac{N(N-1)}{2}} \int_{\mu \in \text{GT}_\lambda} e^{\sum_{i=1}^{N} s_i \left( \sum_{i=1}^{l} \mu_i^l - \Sigma_{i=1}^{l-1} \mu_i^{l-1} \right)} \prod_{i=1}^{N-1} \prod_{i<j} \left| \mu_i^l - \mu_j^{l+1} \right|^{k-1} \prod_{i=1}^{l} \prod_{i<j} \left| \mu_i^{l+1} - \mu_j^{l+1} \right|^{k-1} \prod_{i=1}^{l} d\mu_i^l \tag{2.1.6}
\]

where (2.1.5) is a rational degeneration of (2.1.6). In [54], the formula (2.1.5) was related to the multivariate Bessel functions as follows; a related approach was given for \(k = 1/2, 1, 2\) in [51, Appendix C].

**Theorem 2.1.2** ([54, Section V]). For positive real \(k > 0\) and \(\lambda_1 > \cdots > \lambda_N\), the
multivariate Bessel function is given by

\[ B_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N} \frac{\phi_k(\lambda, s)}{\prod_{i<j}(\lambda_i - \lambda_j)^k}. \]

**Remark.** We have adjusted the normalization of \( B_k(\lambda, s) \) in Theorem 2.1.2 from [54] so that \( B_k(\lambda, 0) = 1 \).

In the trigonometric setting, the integral formula of (2.1.6) was realized by Borodin-Gorin as a scaling limit of Macdonald polynomials. Applying this scaling to the eigenfunction relation for Macdonald polynomials, they showed that \( \Phi_k(\lambda, s) \) was an eigenfunction of the quadratic Calogero-Moser Hamiltonian \( L_{\text{trig}}^{(k-1)} \). Together with some arguments which we detail in Subsection 2.5.1 for \( k \) a positive integer, this relates \( \Phi_k(\lambda, s) \) to \( \mathcal{F}_k(\lambda, s) \).

**Theorem 2.1.3** ([9, Proposition 6.2]). For any positive real \( k > 0 \), \( \Phi_k(\lambda, s) \) is the following scaling limit of Macdonald polynomials

\[ \Phi_k(\lambda, s) = \lim_{\varepsilon \to 0} \varepsilon^{kN(N-1)/2} P_{[\varepsilon^{-1}\lambda]}(e^{\varepsilon s}, e^{-\varepsilon}, e^{-k\varepsilon}). \]

**Theorem 2.1.4** ([9, Definition 6.1 and Proposition 6.3]). For any positive real \( k > 0 \) and \( \lambda_1 > \cdots > \lambda_N \), the Heckman-Opdam hypergeometric function is given by

\[ \mathcal{F}_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N} \frac{\Phi_k(\lambda, s)}{\prod_{i<j}(e^{\lambda_i - \lambda_j}/2 - e^{-\lambda_i - \lambda_j}/2)^k}. \]

**Remark.** The integral formulas of Theorems 2.1.2 and 2.1.4 are stated only for \( \lambda_1 > \cdots > \lambda_N \). We may extend them to \( \{\lambda_i \neq \lambda_j\} \) by imposing that \( \mathcal{F}_k(\lambda, s) \) and \( B_k(\lambda, s) \) are symmetric in \( \lambda \). Under this extension, by taking limits of relevant normalizations of (2.1.5) and (2.1.6) we may show that the expressions of Theorems 2.1.2 and 2.1.4 extend to \( \lambda \in \mathbb{R}^N \). We give such arguments for the trigonometric case when \( k > 0 \) is a positive integer in Subsection 2.5.1.

**Remark.** The main result of [68, Theorem 6.3] gives for each Weyl chamber a contour integral formula for a solution to the hypergeometric system (2.1.3) holomorphic in that Weyl chamber. These formulas have the same integrand as the integral of Theorem 2.1.4 but contours which are different for each Weyl chamber.

### 2.1.5 Realization via quasi-classical limit of quantum group intertwiners

The formula of Theorem 2.5.1 is the quasi-classical limit of the trace of an intertwiner of quantum group representations. We will give a second approach to its proof using this theory; when combined with our first proof of Theorem 2.5.1, this provides a new proof of Theorem 2.1.4 from [9]. Our approach proceeds via the degeneration of \( U_q(\mathfrak{sl}_N) \)-representations; we summarize the main idea in this subsection and give full details in Section 2.3.
For a dominant integral weight $\lambda$, let $L_{\lambda}$ denote the corresponding highest weight irreducible representation of $U_q(\mathfrak{g}_N)$. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ be half the sum of the positive roots. In [26], it was shown that there exists a unique intertwiner $\Phi^N_{\lambda} : L_{\lambda+(k-1)\rho} \to L_{\lambda+(k-1)\rho} \otimes W_{k-1}$ of $U_q(\mathfrak{g}_N)$-representations such that the highest weight vector $v_{\lambda+(k-1)\rho} \in L_{\lambda+(k-1)\rho}$ is mapped to

$$\Phi^N_{\lambda}(v_{\lambda+(k-1)\rho}) = v_{\lambda+(k-1)\rho} \otimes w_{k-1} + \text{(lower order terms)},$$

where the lower order terms have weight less than $\lambda + (k-1)\rho$ in the $L_{\lambda+(k-1)\rho}$ tensor factor. They expressed Macdonald polynomials in terms of these intertwiners in the following theorem.

**Theorem 2.1.5** ([26, Theorem 1]). The Macdonald polynomial $P_\lambda(x; q^2, q^{2k})$ is given by

$$P_\lambda(x; q^2, q^{2k}) = \frac{\operatorname{Tr}(\Phi^N_{\lambda} x^k)}{\operatorname{Tr}(\Phi^N_{\lambda} x^k)}.$$  \hfill (2.1.7)

We characterize both sides of (2.1.7) under the quasi-classical limit transition of [9] in the following two results. Corollary 2.3.10 converts traces of quantum group representations to integrals over dressing orbits to yield an integral expression for the limit. Theorem 2.3.14 uses the fact that the Macdonald difference operators diagonalize both sides of (2.1.7) to show that this limiting integral is diagonalized by the quadratic trigonometric Calogero-Moser Hamiltonian.

**Corollary 2.3.10.** For sequences of dominant integral signatures $\{\lambda_m\}$ and real quantization parameters $\{q_m\}$ so that $\lim_{m \to \infty} q_m \to 1$ and $\lim_{m \to \infty} -2 \log(q_m) \lambda_m = \lambda$ is dominant regular, we have

$$\lim_{m \to \infty} (-2 \log(q_m))^{kN(N-1)/2} P_{\lambda_m}(q_m^{-2s} ; q_m^2, q_m^{2k}) = \frac{\int_{[\mathfrak{g}_N]} F_{k-1}(X) \prod_{i=1}^N \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{s_i} d\mu_{\lambda}}{\prod_{i=1}^{k-1} \prod_{i < j} (s_i - s_j - a)}.$$  

**Theorem 2.3.14.** The trigonometric Calogero-Moser Hamiltonian $L_{p^2}^{(\text{trig})}(k)$ is diagonalized on

$$\frac{1}{\prod_{i < j} (x_i - x_j - a)^k \prod_{i=1}^{k-1} \prod_{i < j} (s_i - s_j - a)} \int_{[\mathfrak{g}_N]} F_{k-1}(X) \prod_{i=1}^N \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{s_i} d\mu_{\lambda}$$

with eigenvalue $\sum_i s_i^2$.

**Remark.** Combining these two results and our first proof of Theorem 2.5.1 yields a new proof of Theorem 2.1.4 which is independent of the results of [9].

**Remark.** In Chapter 3 of this thesis, which is based on [95], we give a representation theoretic proof of Macdonald's branching rule using a quantum analogue of the results of the present work. In particular, we identify diagonal matrix elements of $\Phi^N_{\lambda}$ in the Gelfand-Tsetlin basis with the application of higher Macdonald-Ruijsenaars Hamiltonians to a kernel. We then apply adjunction to the Etingof-Kirillov Jr. trace
formula to recover the branching rule. The link established in this chapter between
the expressions given in Theorem 2.5.1 and [9] for the Heckman-Opdam hypergeo-
metric functions is the quasiclassical limit of this argument and inspired the approach
of Chapter 3.

2.1.6 Outline of method and organization

We outline our approach. We first show that the quasi-classical limit of the Etingof-
Kirillov Jr. construction of Macdonald polynomials as traces of $U_q(g(N))$-intertwiners
corresponds to integrals over dressing orbits of $B_N$ in Corollary 2.3.10 and that these
integrals diagonalize the quadratic Calogero-Moser Hamiltonian in Theorem 2.3.14.
The Gelfand-Tsetlin action on these dressing orbits then defines a classical integrable
system whose moment map is the logarithmic Gelfand-Tsetlin map $GT$ of [50, 1].
Integration over the Liouville tori reduces the integral of Theorem 2.5.1 to an integral
with respect to the Duistermaat-Heckman measure $GT_{\lambda}(d\mu_{\lambda})$ on $GT_{\lambda}$, which is the
Lebesgue measure. This yields an integral expression for $\Phi_{\lambda}(\lambda, s)$ over $GT_{\lambda}$. The new
integrand differs from that of Theorem 2.1.4, but we show equality of the integrals
by applying adjunction for higher Calogero-Moser Hamiltonians.

The remainder of this chapter is organized as follows. In Section 2.2, we give
the geometric setup for our integral formulas. In Section 2.3, we prove Corollary
2.3.10 and Theorem 2.3.14 by taking the quasi-classical limit of the quantum group
setting. In Section 2.4, we prove Theorem 2.4.1 in the rational setting, establishing
in particular the key Proposition 2.4.5. In Section 2.5, we use Proposition 2.4.5 to
give another proof of Theorem 2.5.1 in the trigonometric setting via the formula of
[9]. In Section 2.6, we provide proofs for some technical lemmas whose proofs were
deferred.

2.2 Geometric setup

2.2.1 Notations

For sets of variables $\{x_i\}$ and $\{y_j\}$, we denote the Vandermonde determinant by
$\Delta(x) = \Pi_{i<j}(x_i - x_j)$, and the product of differences by $\Delta(x, y) = \Pi_{i,j}(x_i - y_j)$. Denote also the trigonometric Vandermonde by $\Delta^{\text{trig}}(x) = \Pi_{i<j}(e^{\frac{x_i - x_j}{2}} - e^{-\frac{x_i - x_j}{2}})$.

2.2.2 Gelfand-Tsetlin coordinates

Define the Gelfand-Tsetlin map $gt : O_{\lambda} \to GT_{\lambda}$ by
$$gt(X) = \{\lambda_i(X_i)\}_{1 \leq i \leq 1, 1 \leq i < N},$$
where $X_i$ is the principal $l \times l$ submatrix of $X$, and $\lambda_1(X_1) \geq \ldots \geq \lambda_l(X_l)$ are its
eigenvalues. Define the logarithmic Gelfand-Tsetlin map $GT : O_{\lambda} \to GT_{\lambda}$ by
$$GT(X) = \{\log(\lambda_i(X_i))\}_{1 \leq i \leq 1, 1 \leq i < N}.$$
By a theorem of Ginzburg and Weinstein (see [53]), the Poisson structures we have described on $b_N$ and $B_N$ make them isomorphic as Poisson manifolds. By [1], there exists a Ginzburg-Weinstein isomorphism $b_N \to B_N$ which intertwines the logarithmic and ordinary Gelfand-Tsetlin maps. In particular, this map restricts to a symplectomorphism $O_\lambda \to O_\Lambda$.

### 2.2.3 Gelfand-Tsetlin integrable system

Let $T := T_1 \times \cdots \times T_{N-1}$ be a torus of dimension $\frac{N(N-1)}{2}$, where $\dim T_l = l$. For $t_l \in T_l$ and $X$ in $O_\lambda$ or $O_\Lambda$ whose principal $l \times l$ submatrix $X_l$ is diagonalized by $X_l = U_l \Lambda_l U_l^*$, the *Gelfand-Tsetlin action* of $t_l$ on $X$ is defined as

$$t_l \cdot X = \text{Ad}_{U_l \Lambda_l U_l^*}(X),$$

where for $Y_l \in U(l)$, the matrix $\overline{Y}_l \in U_N$ is defined to be the square block matrix

$$\overline{Y}_l = \begin{pmatrix} Y_l & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & cI_{N-1} \end{pmatrix},$$

where $c$ is chosen so that $\overline{Y}_l \in U_N$. The actions of $T_l$ preserve $l \times l$ principal submatrices and pairwise commute, giving actions of $T$ on $O_\lambda$ and $O_\Lambda$. These actions are Hamiltonian with moment maps $\mu_T$ and $\mu_{GT}$, respectively, and the corresponding classical integrable system is known as the Gelfand-Tsetlin integrable system (see [1, 55, 50] for more about this integrable system).

We may use the Gelfand-Tsetlin action to write any $X_0$ in $\mu^{-1}(\mu)$ or $\mu_{GT}^{-1}(\mu)$ in a special form. Write $X_0$ as either $u_N \lambda u_N^*$ or $u_N \Lambda u_N^*$ for some unitary matrix $u_N$ and decompose $u_N$ as

$$u_N = \overline{u}_1 (\overline{u}_2^* \cdots (\overline{u}_{N-1}^* u_N)$$

for $u_m \in U(m)$ and $v_m := \overline{u}_{m-1}^* u_m$ satisfying either

$$(v_m u_m^* v_m^*)_{m-1} = \mu^{-1}$$

or

$$(v_m e_{m-1}^* u_m^*)_{m-1} = e^{\mu^{-1}},$$

where $(M)_{m-1}$ denotes the principal $(m-1) \times (m-1)$ submatrix of a matrix $M$.

Lemma 2.2.1 gives a compatibility property between this decomposition and the Gelfand-Tsetlin action.

**Lemma 2.2.1.** For any $l \leq m$ and $t_m \in T_m$, we have

$$t_m \cdot \text{ad}_{\overline{u}_l \cdots \overline{u}_N}(\lambda) = \text{ad}_{\overline{u}_l \cdots \overline{u}_m}(t_m \cdot \text{ad}_{\overline{u}_{m+1} \cdots \overline{u}_N}(\lambda)), \text{ and}$$

$$t_m \cdot \text{ad}_{\overline{u}_l \cdots \overline{u}_N}(\Lambda) = \text{ad}_{\overline{u}_l \cdots \overline{u}_m}(t_m \cdot \text{ad}_{\overline{u}_{m+1} \cdots \overline{u}_N}(\Lambda)).$$

**Proof.** By construction, the principal $m \times m$ submatrix of $\text{ad}_{\overline{u}_{m+1} \cdots \overline{u}_N}(\lambda)$ is diagonal,
implying that
\[ t_m \cdot \text{ad}_{\tilde{v}_1 \cdots \tilde{v}_N}(\lambda) = \text{ad}_{\text{ad}_{\tilde{v}_1 \cdots \tilde{v}_m}(t_m)}(\text{ad}_{\tilde{v}_1 \cdots \tilde{v}_N}(\lambda)) = \text{ad}_{\tilde{v}_1 \cdots \tilde{v}_m}(t_m \cdot \text{ad}_{\tilde{v}_{m+1} \cdots \tilde{v}_N}(\lambda)). \]

An analogous proof yields the lemma for \( \Lambda \) in place of \( \lambda \).

\[ \square \]

### 2.2.4 Duistermaat-Heckman measures

The pushforwards \( g_\ast(d\mu_\lambda) \) and \( \Gamma_\ast_T(d\mu_\lambda) \) of the Liouville measures on \( \mathcal{O}_\lambda \) and \( \mathcal{O}_\Lambda \) to \( \Gamma_T \) are called Duistermaat-Heckman measures. Because the Ginzburg-Weinstein isomorphism intertwines the two Gelfand-Tsetlin maps, the two Duistermaat-Heckman measures on \( \Gamma_T \) coincide. It is known (see [52, 7, 2, Section 5.6]) that the Duistermaat-Heckman measure for the coadjoint orbit \( \mathcal{O}_\lambda \) is proportional to the Lebesgue measure on the Gelfand-Tsetlin polytope. To compute the normalization constant, we recall Harish-Chandra’s formula (see [69, Theorem 3, Section 3])

\[ \int_{\mathcal{O}_\lambda} e^{(b,x)} d\mu_\lambda = \frac{\sum_{w \in \mathcal{W}} (-1)^w e^{(w,\lambda,x)}}{\prod_{i<j} (x_i - x_j)}, \tag{2.2.1} \]

which upon taking \( x \to 0 \) (via \( x = \epsilon \cdot \rho \) and \( \epsilon \to 0 \)) shows that

\[ \text{Vol}(\mathcal{O}_\lambda) = \frac{\prod_{i<j} (\lambda_i - \lambda_j)}{(N-1)! \cdots 1!}. \]

On the other hand, it is known (see [83, Corollary 3.2]) that \( \text{Vol}(\Gamma_T) = \frac{\prod_{i<j} (\lambda_i - \lambda_j)}{(N-1)! \cdots 1!} \), meaning that \( g_\ast(d\mu_\lambda) = 1_{\Gamma_T} \cdot dx \). This discussion establishes the following Proposition 2.2.2.

**Proposition 2.2.2.** The Duistermaat-Heckman measures \( g_\ast(d\mu_\lambda) = \Gamma_\ast_T(d\mu_\lambda) \) are equal to the Lebesgue measure \( dx \) on the Gelfand-Tsetlin polytope. Explicitly, we have

\[ g_\ast(d\mu_\lambda) = \Gamma_\ast_T(d\mu_\lambda) = 1_{\Gamma_T} dx. \]
2.3 Quasi-classical limits of quantum group intertwiners

2.3.1 Finite-type quantum group

Let $U_q(gl_N)$ be the associative algebra over $\mathbb{C}(q^{1/2})$ with generators $e_i, f_i$ for $i = 1, \ldots, N - 1$ and $q^{\pm h_i}$ for $i = 1, \ldots, N$ and relations

$$
q^{\frac{h}{2}} e_i q^{-\frac{h}{2}} = q^{\frac{1}{2}} e_i, \quad q^{\frac{h}{2}} e_{i-1} q^{-\frac{h}{2}} = q^{-\frac{1}{2}} e_{i-1}, \quad q^{\frac{h}{2}} f_i q^{-\frac{h}{2}} = q^{-\frac{1}{2}} f_i,
$$

$$
q^{\frac{h}{2}} f_{i-1} q^{-\frac{h}{2}} = q^{\frac{1}{2}} f_{i-1}, \quad [q^{\frac{h}{2}}, e_j] = [q^{\frac{h}{2}}, f_j] = 0 \text{ for } j \neq i, i - 1,
$$

$$
[e_i, f_j] = \delta_{ij} \frac{g_{hi-h_{i+1}} - g_{h_{i+1}-h_i}}{q - q^{-1}}, \quad [e_i, e_j] = [f_i, f_j] = 0 \text{ for } |i - j| > 1,
$$

$$
q^{\frac{h}{2}} \cdot q^{-\frac{h}{2}} = 1, \quad e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0,
$$

$$
f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \text{ for } |i - j| = 1.
$$

We take the coproduct on $U_q(gl_N)$ defined by

$$
\Delta(e_i) = e_i \otimes q^{\frac{h_{i+1}-h_i}{2}} + q^{\frac{h_i-h_{i+1}}{2}} \otimes e_i,
$$

$$
\Delta(f_i) = f_i \otimes q^{-\frac{h_{i+1}-h_i}{2}} + q^{-\frac{h_i-h_{i+1}}{2}} \otimes f_i,
$$

$$
\Delta(q^{\frac{h}{2}}) = q^{\frac{h}{2}} \otimes q^{\frac{h}{2}}
$$

and the antipode given by

$$
S(e_i) = -e_i q^{-1}, \quad S(f_i) = -f_i q, \quad S(q^{h_i}) = q^{-h_i}.
$$

Taking the $*$-structure on $U_q(gl_N)$ given by

$$
e_i^* = f_i \quad \text{and} \quad f_i^* = e_i \quad \text{and} \quad (q^{h_i/2})^* = q^{h_i/2}
$$

yields the $*$-Hopf algebra $U_q(u_N)$. Its restriction to the algebra span of $q^{h_i/2}$ is the $*$-Hopf algebra $U_q(t_N)$.

2.3.2 Macdonald polynomials and Etingof-Kirillov Jr. construction

Let $\rho = \binom{\frac{N-1}{2}, \ldots, \frac{1-N}{2}}{N}$ and let $e_\rho$ denote the elementary symmetric polynomial. For a partition $\lambda$, the Macdonald polynomial $P_\lambda(x; q^2, t^2)$ is the joint polynomial eigenfunction with leading term $x^\lambda$ and eigenvalue $e_\rho(q^{2\lambda t^2})$ of the operators

$$
D_{N,x}(q^2, t^2) = x^{(r-N)} \sum_{|l|=r} \prod_{i \in I, j \notin I} \frac{t^2 x_i - x_j}{x_i - x_j} T_{q^2, l},
$$

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where $T_{q^2,i} = \prod_{i \in I} T_{q^2,i}$ and $T_{q^2,i} f(x_1, \ldots, x_n) = f(x_1, \ldots, q^2 x_i, \ldots, x_N)$ so that we have

$$D_{N,x}(q^2, t^2) P_\lambda(x; q^2, t^2) = e_r(q^2 \lambda r^2) P_\lambda(x; q^2, t^2).$$

Note that our normalization of $D_{N,x}(q^2, t^2)$ differs from that of [76]. In [26], Etingof and Kirillov Jr. gave an interpretation of Macdonald polynomials in terms of representation-valued traces of $U_q(g_N)$. For a signature $\lambda$, there exists a unique intertwiner

$$\Phi^N_\lambda : L_{\lambda+(k-1)\rho} \rightarrow L_{\lambda+(k-1)\rho} \otimes W_{k-1}$$

normalized to send the highest weight vector $v_{\lambda+(k-1)\rho}$ in $L_{\lambda+(k-1)\rho}$ to

$$v_{\lambda+(k-1)\rho} \otimes w_{k-1} + \text{(lower order terms)},$$

where (lower order terms) denotes terms of weight lower than $\lambda + (k-1)\rho$ in the first tensor coordinate. As shown in [26, Theorem 1] (reproduced as Theorem 2.1.5), traces of these intertwiners lie in $W_{k-1}[0] = C \cdot w_{k-1}$ and yield Macdonald polynomials when interpreted as scalar functions via the identification $w_{k-1} \mapsto 1$. The denominator also admits the following explicit form.

**Proposition 2.3.1 ([26, Main Lemma]).** On $L_{(k-1)\rho}$, the trace may be expressed explicitly as

$$\text{Tr}(\Phi^N_0 x^h) = (x_1 \cdots x_N)^{-\frac{1}{2}(k-1)(N-1)} \prod_{a=1}^{k-1} \prod_{1 \leq i < j} (x_i - q^{2a} x_j).$$

**Remark.** Our notation for Macdonald polynomials is related to that of [26] via $P^K_\lambda(x; q, t) = P_\lambda(x; q^2, t^2)$.

### 2.3.3 Braid group action, PBW theorem, and integral forms

In this section, we define an integral form $U'_q(g_N) \subset U_q(g_N)$ which will allow us to realize it as a quantum deformation of the Poisson algebra $C[B_N]$ in the sense of [15, Section 11]. For this, we require Lusztig's braid group action on $U_q(g_N)$. Following [75], the braid group $B_N = \langle T_1, \ldots, T_{N-1} | T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rangle$ of type $A_{N-1}$ acts via algebra automorphisms on $U_q(g_N)$ by

$$
T_i(e_i) = -f_i q^{h_i-h_{i+1}} e_i, \quad T_i(e_{i \pm 1}) = q^{-1} e_{i \pm 1} e_i - e_i e_{i \pm 1}, \quad T_i(e_j) = e_j \text{ for } |i - j| > 1
$$

$$
T_i(f_i) = -q^{h_i+h_{i+1}} e_i, \quad T_i(f_{i \pm 1}) = f_{i \pm 1} f_i - f_i f_{i \pm 1}, \quad T_i(f_j) = f_j \text{ for } |i - j| > 1
$$

$$
T_i(q^{h_i/2}) = q^{h_i/2}, \quad T_i(q^{h_{i+1}/2}) = q^{h_{i+1}/2}, \quad T_i(q^{h_{i+1}/2}) = q^{h_{i+1}/2} \text{ for } j \neq i, i + 1.
$$

Let $U'_q(g_N)$ be the smallest $C[q^{\pm 1/2}]$-subalgebra of $U_q(g_N)$ containing

$$
\bar{e}_i = (q - q^{-1}) e_i, \quad \bar{f}_i = (q - q^{-1}) f_i, \quad q^{h_i/2}
$$

and stable under the action of $B_N$ described above. For a choice of simple roots $\{\alpha_1, \ldots, \alpha_{N-1}\}$ and a fixed decomposition $w_0 = s_{i_1} \cdots s_{i_M}$ of the longest word $w_0$ in
$S_N$, let $\beta_i = s_{i_1} \cdots s_{i_{t-1}}(\alpha_i)$ and define
\[ \bar{\mathcal{e}}_{\beta_i} = (q - q^{-1}) T_{i_1} \cdots T_{i_{t-1}}(e_i) \text{ and } \bar{f}_{\beta_i} = (q - q^{-1}) T_{i_1} \cdots T_{i_{t-1}}(f_i). \]
By the PBW theorem, $U'_q(\mathfrak{g}_N)$ has a $\mathbb{C}[q^{1/2}]$-basis given by monomials
\[ \bar{\mathcal{e}}^{k_1}_{\beta_1} \cdots \bar{\mathcal{e}}^{k_M}_{\beta_M} q^{\mathcal{f}^{l_1}_{\beta_1}} \cdots \mathcal{f}^{l_t}_{\beta_t}. \]
Following [15, Section 10], assign such a monomial a degree of
\[ \deg \left( \bar{\mathcal{e}}^{k_1}_{\beta_1} \cdots \bar{\mathcal{e}}^{k_M}_{\beta_M} q^{\mathcal{f}^{l_1}_{\beta_1}} \cdots \mathcal{f}^{l_t}_{\beta_t} \right) = \left( k_M, \ldots, k_1, l_1, \ldots, l_M, \sum_{i=1}^{M} (k_i + l_i), \text{ht(}\beta_i) \right) \in \mathbb{Z}^{2M+1}_{\geq 0}, \]
where if $\beta = \sum c_i \alpha_i$ as the sum of simple roots, its height is $\text{ht(}\beta) = \sum c_i$. The algebra $U'_q(\mathfrak{g}_N)$ is a $\mathbb{Z}^{2M+1}_{\geq 0}$-filtered algebra under the degree filtration, known as the de Concini-Kac filtration.

**Proposition 2.3.2** ([15, Section 10]). The associated graded of $U'_q(\mathfrak{g}_N)$ under the de Concini-Kac filtration is generated by $\bar{\mathcal{e}}_{\beta_i}, \mathcal{f}_{\beta_i}, q^{\mathcal{h}^{1/2}}$ subject to the relations
\[ [q^{\mathcal{h}^{1/2}} q^{\mathcal{h}^{1/2}_j}], q^{\mathcal{h}^{1/2}} \bar{\mathcal{e}}_{\beta_i}, q^{\mathcal{h}^{1/2}_j} \bar{\mathcal{e}}_{\beta_i} = q^{\mathcal{h}^{1/2} \mathcal{f}_{\beta_i}} q^{\mathcal{h}^{1/2}_j} \mathcal{f}_{\beta_i}, \mathcal{f}_{\beta_i} \mathcal{f}_{\beta_i} \text{ for } i > j. \]

**2.3.4 Infinitesimal dressing action and Poisson bracket**

In what follows, we will consider functions on $B_N$ pulled back from matrix elements of $P_N^+$ via the map $\text{sym} : B_N \to P_N^+$ as in the statement of Theorem 2.5.1. The derivative of the dressing action of $U_N$ on $B_N$ yields a map of vector fields $\delta_r : u_N \to \text{Vect}(B_N)$ called the infinitesimal dressing action. Let $\delta : \mathbb{C}[B_N] \to \mathbb{C}[B_N] \otimes \mathbb{C}[B_N]$ and $S : \mathbb{C}[B_N] \to \mathbb{C}[B_N]$ denote the coproduct and antipode on $\mathbb{C}[B_N]$. In [73], it is shown that the infinitesimal $u_N$-action may be realized via the Poisson bracket.

**Proposition 2.3.3** ([73, Theorem 3.10]). For $f \in \mathbb{C}[B_N]$ with $\delta(f) = \sum_i f_i^{(1)} \otimes f_i^{(2)}$, the infinitesimal dressing action of $df|_e \in T^*_e(B_N) \simeq u_N$ on $\mathbb{C}[B_N]$ is implemented via the vector field
\[ \sigma_f := - \sum_i S(f_i^{(2)}) \{ f_i^{(1)}, - \}. \]

**2.3.5 Degeneration of $U'_q(\mathfrak{g}_N)$**

It is shown in [15, Section 12] that $U'_q(\mathfrak{g}_N)$ is a quantum deformation of $\mathbb{C}[B_N]$. To interpret this statement, let $GL_N^*$, the Poisson-Lie group dual to $GL_N$, be given explicitly by
\[ GL_N^* = \left\{ (g, f) \mid g, f \in GL_N, \text{ g lower triangular, } f \text{ upper triangular}, g_{ii} = f_{ii}^{-1} \right\}. \]
Taking the real form $f^* = g^{-1}$ on $GL^*_N$ yields $\mathbb{C}[B_N]$ as the corresponding $\ast$-Poisson Hopf algebra. Under this identification, we have the following result of [14].

**Theorem 2.3.4** ([14, Theorem 7.6 and Remark 7.7(c)]). The algebra $U'_q(\mathfrak{g}_N)$ satisfies:

1. $U'_q(\mathfrak{g}_N)$ is flat over $\mathbb{C}[q^{\pm 1/2}]$;
2. we have an isomorphism $U'_q(\mathfrak{g}_N) \otimes \mathbb{C}(q^{1/2}) \simeq U_q(\mathfrak{g}_N)$;
3. $U'_q(\mathfrak{g}_N)/(q^{1/2} - 1)U'_q(\mathfrak{g}_N)$ is commutative;
4. there is an isomorphism of Hopf algebras
   \[
   \pi : U'_q(\mathfrak{g}_N)/(q^{1/2} - 1)U'_q(\mathfrak{g}_N) \to \mathbb{C}[B_N]
   \]
   which satisfies
   \[
   \pi \left( (4(q^{1/2} - 1))^{-1}[x, y]\right) = \{\pi(x), \pi(y)\};
   \]
5. $\pi$ takes the special value $\pi(q^{h_1}) = \left(\frac{\det(X_i)}{\det(X_{i-1})}\right)^{1/2}$.

**Remark.** Note that $(4(q^{1/2} - 1))^{-1}[x, y]$ is a well-defined element of $U'_q(\mathfrak{g}_N)$ by Theorem 2.3.4(c).

For $r$ which is not a root of unity, define $\tilde{U}_r(\mathfrak{g}_N)$ to be the corresponding numerical specialization of $U'_q(\mathfrak{g}_N)$. Denote the specialization map by $\pi_r : U'_q(\mathfrak{g}_N) \to \tilde{U}_r(\mathfrak{g}_N)$.

Define also the map of $\mathbb{C}$-algebras $\varphi : U'_q(\mathfrak{g}_N) \to U'_q(\mathfrak{g}_N)$ by

\[
\varphi(\bar{e}_i) = \bar{e}_i, \quad \varphi(f_i) = f_i, \quad \varphi(q^{h_1}) = q^{-h_1}, \quad \varphi(q) = q^{-1}. \quad (2.3.1)
\]

**Theorem 2.3.5.** Fix $z \in U'_q(\mathfrak{g}_N)$. For sequences of dominant integral signatures \(\{\lambda_m\}\) and real quantization parameters \(\{q_m\}\) for which we have \(\lim_{m \to \infty} q_m = 1\) and \(\lim_{m \to \infty} -2 \log(q_m)\lambda_m = \lambda\) dominant regular, we have

\[
\lim_{m \to \infty} (-2 \log(q_m))^{N(N-1)/2} \text{Tr}_{L_{\lambda_m}}(\pi_{q_m}(z) q_m^{-2s(h)}) = \int_{\mathcal{O}_A} \pi(\varphi(z)) \prod_{l=1}^N \left(\frac{\det(X_l)}{\det(X_{l-1})}\right)^{a_l} d\mu_A,
\]

where we consider $L_{\lambda_m}$ as a representation of $\tilde{U}_{q_m}(\mathfrak{g}_N)$ and $\det(X_l)$ as a function on $B_N$ via composition with $\text{sym} : B_N \to P_N^+$ and where $X_l$ is the principal $l \times l$ submatrix of $X \in \mathcal{O}_A \subset P_N^+$.

**Proof.** It suffices to consider monomials $z$, for which we induct on degree. For the base case, monomials of degree 0 lie in the Cartan subalgebra, so we have $z = q^{\sum_1^N 2a_l h_i}$. 

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for some $c_i$. In this case, we have

$$\lim_{m \to \infty} (-2 \log(q_m))^N(N-1)/2 \text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(z) \cdot q_m^{-2(s,h)})$$

$$= \lim_{m \to \infty} (-2 \log(q_m))^N(N-1)/2 \text{Tr}|_{L_{\lambda_m}}(e^{-2 \log(q)} \sum_i (-c_i + s_i) h_i)$$

$$= \lim_{m \to \infty} (-2 \log(q_m))^N(N-1)/2 \frac{\prod_{i<j}(-c_i + s_i + c_j - s_j)}{m} \frac{\prod_{i<j}(e^{-c_i + s_i + c_j - s_j}/2m - e^{-c_i + s_i + c_j - s_j}/2m)}{\text{Tr} (e^{-2 \log(q)} \sum_i (-c_i + s_i) X_i) d\mu_{\lambda_m + \rho}}$$

$$= \int_{C_{\lambda_m + \rho}} e^{-2 \log(q)} \sum_i (-c_i + s_i) X_i d\mu_{\lambda_m + \rho}$$

$$= \int_{C_{\lambda_m + \rho}} \prod_{i=1}^N \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{-c_i + s_i} d\mu_{\lambda},$$

where the second equality follows from Kirillov's character formula, the third from a change of variables and (2.2.1), and the last by the Ginzburg-Weinstein isomorphism.

The fact that

$$\pi(\varphi(q \sum_i 2 c_i h_i)) = \pi(q^{-2} \sum_i 2 c_i h_i) = \prod_{i=1}^N \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{-c_i}$$

by Theorem 2.3.4 completes the base case.

Suppose that $z = \prod_i \bar{e}_{\beta_i}^1 q^{h_i} \prod_i \bar{f}_{\beta_i}^1$ is a PBW monomial of non-zero degree and the claim holds for all monomials of smaller degree. If all $k_i$ are 0, not all $l_i$ can be 0, so the limiting trace is 0; similarly, $\pi(z)$ is not invariant under the torus action in this case, so the integral is also 0. Otherwise, let $i^*$ be minimal so that $k_{i^*} > 0$, and write $z = abc$ with $a = \bar{e}_{\beta_{i^*}}$, $b = \bar{e}_{\beta_{i^*}}^{-1} \prod_i \bar{e}_{\beta_i}^{k_i} q^{h_i}$, and $c = \prod_i \bar{f}_{\beta_i}^{l_i}$. We then have that

$$\text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(z) q_m^{-2(s,h)}) = \text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(bc) q_m^{-2(s,h)} \pi_{q_m}(a))$$

$$= \text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(bca) q_m^{-2(s,b_i^*)} q_m^{-2(s,h)})$$

$$= q_m^{-2(s,b_i^*)} \text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(abc + [b, a] c + b[c, a]) q_m^{-2(s,h)}).$$

By the relations in Proposition 2.3.2, we see that

$$[b, a] = (q^{f(b, a)} - 1) ab + (\text{terms of lower degree})$$

for some function $f(b, a)$. This means that $[b, a] - (q^{f(b, a)} - 1) ab$ lies in a lower degree of the filtration than $ab$. Solving for the new trace in the rewritten equation

$$\text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(z) q_m^{-2(s,h)})$$

$$= q_m^{-2(s,b_i^*)} \text{Tr}|_{L_{\lambda_m}}(\pi_{q_m}(q^{f(b, a)} z + ([b, a] c - (q^{f(b, a)} - 1) abc) + b[c, a]) q_m^{-2(s,h)}).$$
yields the solution
\[ \text{Tr}|_{\mathcal{L}_m}(\pi_{\text{qm}}(z)q^{-2(s,h)}) = \frac{q^{-2(s,\beta_\ast)}(1 - q^{1/2})}{1 - q^{-2(s,\beta_\ast) + f(b,a)}} \text{Tr}|_{\mathcal{L}_m}(\pi_{\text{qm}}(\frac{([b,a]c - (q^{f(b,a)} - 1)abc + b[c,a])}{4(1 - q^{1/2})})q^{-2(s,h)}). \]

Using the notation \( \pi_a := \pi(\varphi(a)), \pi_b := \pi(\varphi(b)), \) and \( \pi_c := \pi(\varphi(c)) \), notice that
\[ \pi \left( \varphi\left(\frac{([b,a]c - (q^{f(b,a)} - 1)abc + b[c,a])}{4(1 - q^{1/2})}\right) \right) = \{\pi_b, \pi_a\} \pi_c + \frac{1}{2} f(b,a) \pi_a \pi_b \pi_c + \pi_b \{\pi_c, \pi_a\}. \]

Because \( ([b,a]c - (q^{f(b,a)} - 1)abc + b[c,a] \) lies in a lower degree of the filtration than \( abc \), we conclude by the inductive hypothesis that
\[
\lim_{m \to \infty} (-2 \log(q_m))^{N(N-1)/2} \text{Tr}|_{\mathcal{L}_m}(\pi_{\text{qm}}(z)q^{-2(s,h)})
= \frac{1}{-(s, \beta_\ast) + f(b,a)/2} \int_{\mathcal{O}_\Lambda} \left( \{\pi_b, \pi_a\} \pi_c + \frac{1}{2} f(b,a) \pi_a \pi_b \pi_c + \pi_b \{\pi_c, \pi_a\} \right)
= \prod_{l=1}^{N} \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_\Lambda. 
\]

On the other hand, because integrating against Liouville measure kills Poisson brackets and
\[
\left\{ \prod_{l=1}^{N} \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l}, \pi(\varphi_\ast) \right\} = (s, \beta_\ast) \prod_{l=1}^{N} \frac{\det(X_l)}{\det(X_{l-1})} \pi(\varphi_\ast),
\]
we have that
\[
0 = \int_{\mathcal{O}_\Lambda} \left\{ \pi_b \pi_c \prod_{l=1}^{N} \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l}, \pi_a \right\} d\mu_\Lambda
= \int_{\mathcal{O}_\Lambda} \left\{ \{\pi_b, \pi_a\} \pi_c + \pi_b \{\pi_c, \pi_a\} + (s, \beta_\ast) \pi_a \pi_b \pi_c \right\}
\prod_{l=1}^{N} \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_\Lambda,
\]
which implies that
\[
\int_{\mathcal{O}_\Lambda} \left( \{\pi_b, \pi_a\} \pi_c + \pi_b \{\pi_c, \pi_a\} \right) \prod_{l=1}^{N} \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_\Lambda
= \int_{\mathcal{O}_\Lambda} (s, \beta_\ast) \pi_a \pi_b \pi_c \prod_{l=1}^{N} \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_\Lambda.
\]
Substituting this into (2.3.2) completes the induction by yielding the desired
\[
\lim_{m \to \infty} (-2 \log(q_m))^N/(N-1/2) \text{Tr} L_{\lambda m}(\pi_{q_m}(z)q_m^{-2(s_A)})
\]
\[
= \int_{\mathcal{O}_\lambda} \pi_a \pi_b \pi_c \prod_{l=1}^N \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{q_l} d\mu_A. \quad \square
\]

2.3.6 Degenerations of intertwiners

We now degenerate $\Phi_\lambda^N$ to $F_{k-1}$, for which we wish to represent $\Phi_\lambda^N$ as the evaluation of an element of $U'_q(\mathfrak{gl}_N) \otimes W_{k-1}$ under the map $\text{ev} : U'_q(\mathfrak{gl}_N) \to \text{End}_C(L_{\lambda+(k-1)p}, L_{\lambda+(k-1)p})$.

Consider the space of invariants $(U'_q(\mathfrak{gl}_N) \otimes W_{k-1})^{U_q(\mathfrak{gl}_N)}$, where the action is given by
\[
x \cdot (u \otimes w) = \sum x(1)uS(x(3)) \otimes x(2)w
\]
in the Sweedler notation
\[
\Delta^{(3)}(x) = \sum x(1) \otimes x(2) \otimes x(3).
\]
We first show that this space of invariants maps to the space of intertwiners under evaluation.

**Lemma 2.3.6.** The action of the first tensor factor on $L_{\lambda+(k-1)p}$ sends $(U'_q(\mathfrak{gl}_N) \otimes W_{k-1})^{U_q(\mathfrak{gl}_N)}$ to an intertwiner $L_{\lambda+(k-1)p} \to L_{\lambda+(k-1)p} \otimes W_{k-1}$.

**Proof.** Let $z = \sum_i x_i \otimes w_i$ be an element of $(U'_q(\mathfrak{gl}_N) \otimes W_{k-1})^{U_q(\mathfrak{gl}_N)}$. By invariance under $U'_q(\mathfrak{gl}_N)$, the action of $q^{h_j}$ satisfies
\[
z = q^{\pm h_j} \cdot z = \sum_i q^{\pm h_j} x_i q^{\mp h_j} \otimes q^{\pm h_j} w_i,
\]
which implies that
\[
\sum_i x_i q^{\mp h_j} \otimes w_i = \sum_i q^{\pm h_j} x_i \otimes q^{\pm h_j} w_i.
\]
The action of $\bar{e}_j$ satisfies
\[
0 = \bar{e}_j \cdot z = \sum_i \left( \bar{e}_j x_i q^{(h_j-h_{j+1})/2} \otimes q^{-(h_j-h_{j+1})/2} w_i + q^{(h_j-h_{j+1})/2} x_i q^{(h_j-h_{j+1})/2} \otimes \bar{e}_j w_i - q^{-1}q^{(h_j-h_{j+1})/2} x_i \bar{e}_j \otimes q^{(h_j-h_{j+1})/2} w_i \right),
\]

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which upon noting that $q^{h_j}x_iq^{-h_j} \otimes w_i = x_i \otimes q^{-h_j}w_i$ implies that

$$
\sum_i x_i \bar{e}_j \otimes w_i = \sum_i \left( q^{-\frac{(h_j-h_{j+1})}{2}} \bar{e}_j x_i q^{-\frac{(h_j-h_{j+1})}{2}} \otimes q^{-\frac{(h_j-h_{j+1})}{2}}w_i \\
+ qx_iq^{-\frac{(h_j-h_{j+1})}{2}} \otimes q^{-\frac{(h_j-h_{j+1})}{2}} e_j w_i \right) \\
= \sum_i \left( \bar{e}_j x_i \otimes q^{-\frac{(h_j-h_{j+1})}{2}} w_i + q^{\frac{(h_j-h_{j+1})}{2}} x_i \otimes e_j w_i \right) = \Delta(\bar{e}_j)z.
$$

A similar computation for $\bar{f}_j$ yields that $\sum_i x_i \bar{f}_j \otimes w_i = \Delta(\bar{f}_j)z$, so $z$ gives the desired intertwiner. \qed

The degeneration $\pi : U'_q(\mathfrak{g}l_N) \to \mathbb{C}[B_N]$ and the automorphism $\varphi$ of (2.3.1) give rise to a map

$$(\pi \circ \varphi) \otimes 1 : (U'_q(\mathfrak{g}l_N) \otimes W_{k-1}) \rightarrow \mathbb{C}[B_N] \otimes W_{k-1}.$$ 

The left dressing action on the first tensor factor gives a $U(\mathfrak{u}_n)$ action on $\mathbb{C}[B_N] \otimes W_{k-1}$; we now show that $(\pi \circ \varphi) \otimes 1$ lands in the space of invariants for this action.

Lemma 2.3.7. The image of $(U'_q(\mathfrak{g}l_N) \otimes W_{k-1})$ under $(\pi \circ \varphi) \otimes 1$ lies in $(\mathbb{C}[B_N] \otimes W_{k-1})^{U(\mathfrak{u}_n)}$.

Proof. Let $z$ be an element of $(U'_q(\mathfrak{g}l_N) \otimes W_{k-1})^{U'_q(\mathfrak{g}l_N)}$, and let $z' = ((\pi \circ \varphi) \otimes 1)(z)$. Write $z = \sum_i x_i \otimes w_i$ and $z' = \sum_i x'_i \otimes w_i$ for $x_i \in U'_q(\mathfrak{g}l_N)$, $x'_i = \pi(\varphi(x_i)) \in \mathbb{C}[B_N]$, and $w_i \in W_{k-1}$. By invariance, $z$ lies in the zero weight space, so $z'$ lies in the zero weight space of $\mathbb{C}[B_N] \otimes W_{k-1}$. By definition of the action of $\bar{e}_j - \bar{f}_j \in U'_q(\mathfrak{g}l_N)$ on $z$ and the fact that $z$ has weight 0, we have

$$0 = \sum_i \left( \bar{e}_j x_i q^{\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes q^{-\frac{(h_j-h_{j+1})}{2}} w_i - q^{-\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes q^{-\frac{(h_j-h_{j+1})}{2}} w_i \right)$$

$$- \sum_i \left( \bar{f}_j x_i q^{\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes q^{-\frac{(h_j-h_{j+1})}{2}} w_i - q^{-\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes q^{-\frac{(h_j-h_{j+1})}{2}} w_i \right)$$

$$+ \sum_i \left( q^{\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes \bar{e}_j w_i - q^{\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes \bar{f}_j w_i \right)$$

$$= \sum_i \left( \bar{e}_j q^{\frac{(h_j-h_{j+1})}{2}} x_i \otimes w_i - x_i \bar{e}_j q^{\frac{(h_j-h_{j+1})}{2}} \otimes w_i \right)$$

$$- \sum_i \left( \bar{f}_j q^{\frac{(h_j-h_{j+1})}{2}} x_i \otimes w_i - x_i \bar{f}_j q^{\frac{(h_j-h_{j+1})}{2}} \otimes w_i \right)$$

$$+ \sum_i \left( q^{\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes \bar{e}_j w_i - q^{\frac{(h_j-h_{j+1})}{2}} q^{\frac{(h_{j+1}-h_j)}{2}} \otimes \bar{f}_j w_i \right).$$

Dividing this equality by $4(q^{1/2} - 1)$, noting that for any $x$ we have

$$[\bar{e}_j q^{\frac{(h_j-h_{j+1})}{2}}, x] = \bar{e}_j [q^{\frac{(h_j-h_{j+1})}{2}}, x] + [\bar{e}_j, x] q^{\frac{(h_j-h_{j+1})}{2}}$$

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and
\[ [\bar{f}_j q^{(h_j - h_{j+1})/2}, x] = \bar{f}_j [q^{(h_j - h_{j+1})/2}, x] + [\bar{f}_j, x] q^{(h_j - h_{j+1})/2}, \]
applying \((\pi \circ \varphi) \otimes 1\), and multiplying by \(\pi(q^{h_j - h_{j+1}})\), we find that
\[
0 = \sum_l \left( \pi(q^{h_j - h_{j+1}}) \pi(\bar{e}_j - \bar{f}_j) \{\pi(q^{-h_j - h_{j+1}}/2), x'_l\} 
+ \pi(q^{h_j - h_{j+1}}/2) \{\pi(\bar{e}_j - \bar{f}_j), x'_l\} \right) \otimes w_l + \sum_l x'_l \otimes (E_{j,j+1} - E_{j+1,j}) \cdot w_l 
= \sum_l \left( \pi(q^{h_j - h_{j+1}}/2) \{\pi(\bar{e}_j - \bar{f}_j), x'_l\} - \pi(\bar{e}_j - \bar{f}_j) \{\pi(q^{h_j - h_{j+1}}/2), x'_l\} \right) \otimes w_l 
+ \sum_l x'_l \otimes (E_{j,j+1} - E_{j+1,j}) \cdot w_l,
\]
where \((E_{j,j+1} - E_{j+1,j}) \cdot w_l\) denotes the action of \(E_{j,j+1} - E_{j+1,j} \in u_N\) on \(w_l \in W_{k-1}\). By Proposition 2.3.3, the dressing action of \(d\pi(\bar{e}_j - \bar{f}_j)|_e\) is implemented by the vector field
\[
\sigma_{\pi(\bar{e}_j - \bar{f}_j)} = -\pi(q^{-h_j - h_{j+1}}/2) \{\pi(\bar{e}_j - \bar{f}_j), -\} + \pi(\bar{e}_j - \bar{f}_j) \{\pi(q^{h_j - h_{j+1}}/2), -\},
\]
which means that
\[
\pi(q^{h_j - h_{j+1}}/2) \{\pi(\bar{e}_j - \bar{f}_j), x'_l\} - \pi(\bar{e}_j - \bar{f}_j) \{\pi(q^{h_j - h_{j+1}}/2), x'_l\} = -\sigma_{\pi(\bar{e}_j - \bar{f}_j)}(x'_l)
\]
and hence that
\[
\sum_l \sigma_{\pi(\bar{e}_j - \bar{f}_j)}(x'_l) \otimes w_l = \sum_l x'_l \otimes (E_{j,j+1} - E_{j+1,j}) \cdot w_l.
\]
Under the identification \(T^*_e B_N \simeq u_N\), we have \(d\pi(\bar{e}_j - \bar{f}_j)|_e = E_{j,j+1} - E_{j+1,j}\) in \(u_N\) by [14, Theorem 7.6(b)], so we conclude that \(z'\) is invariant under the action of \(E_{j,j+1} - E_{j+1,j} \in U(u_N)\). A similar argument yields invariance under the action of \(iE_{j,j+1} + iE_{j+1,j}\), completing the proof. \(\square\)

**Lemma 2.3.8.** For any \(k\), there exists an element \(c_k \in U'_q(\mathfrak{g}_N) \otimes W_{k-1}\) so that
\[
((\pi \circ \varphi) \otimes 1)(c_k)|_{\mathcal{O}_\lambda} = F_{k-1}
\]
and the intertwiner \(\Phi^N_\lambda\) is implemented by \(c_k|_{L_{\lambda+k-1}\rho}\).

**Proof.** We show that \((U'_q(\mathfrak{g}_N) \otimes W_{k-1})^{U'_q(\mathfrak{g}_N)}\) is non-zero under the action of (2.3.3) and then normalize an element of it appropriately. For this, we first show that it is non-zero under the adjoint action
\[
x \cdot (u \otimes w) = \sum x_{(1)} u S(x_{(2)}) \otimes x_{(3)} w.
\]
1. **Showing the space of invariants for the action (2.3.4) is non-zero:** Following [64], let \(\mathcal{F}(U_q(\mathfrak{gl}_N))\) denote the locally finite part of \(U_q(\mathfrak{gl}_N)\) under the adjoint action. By
[64, Theorem 7.4], there is an isomorphism

$$\mathcal{F}(U_q(\mathfrak{g}^N)) \simeq Z(U_q(\mathfrak{g}^N)) \otimes H_q$$

for $Z(U_q(\mathfrak{g}^N))$ the center of $U_q(\mathfrak{g}^N)$ and $H_q$ a $U_q(\mathfrak{g}^N)$-submodule of $\mathcal{F}(U_q(\mathfrak{g}^N))$ under the adjoint action which is a direct sum of $\dim V[0]$ copies of each finite dimensional representation $V$ of $U_q(\mathfrak{g}^N)$. Because $W_{k-1}^*$ has a one-dimensional zero weight space, there exists an embedding $W_{k-1}^* \to U_q(\mathfrak{g}^N)$ of $U_q(\mathfrak{g}^N)$-representations and therefore a non-zero invariant element under the action of (2.3.4).

2. Showing the space of invariants for the action (2.3.3) is non-zero: Let $\mathcal{P}$ denote the transposition map and $\mathcal{R}$ the universal $\mathcal{R}$-matrix of $U_q'(\mathfrak{g}^N)$ and let $\mathcal{P}_{23}$ and $\mathcal{R}_{23}$ denote their application in the second and third tensor factor. Consider the diagram of maps of $U_q'(\mathfrak{g}^N)$-representations

$$
\begin{array}{ccc}
U_q'(\mathfrak{g}^N) \otimes W_{k-1} & \longrightarrow & U_q'(\mathfrak{g}^N) \otimes W_{k-1} \\
(m_{13} \circ S_3) \otimes 1 & \downarrow & (m_{12} \circ S_2) \otimes 1 \\
(U_q'(\mathfrak{g}^N) \otimes W_{k-1})_{(2.3.3)} & \longrightarrow & (U_q'(\mathfrak{g}^N) \otimes W_{k-1})_{(2.3.4)}
\end{array}
$$

where the $U_q'(\mathfrak{g}^N)$-actions on $U_q'(\mathfrak{g}^N) \otimes W_{k-1}$ are given by (2.3.3) and (2.3.4) as specified. We claim that $\mathcal{P}_{23} \mathcal{R}_{23}$ maps the kernels $K_3$ and $K_2$ of $(m_{13} \circ S_3) \otimes 1$ and $(m_{12} \circ S_2) \otimes 1$ to each other. Indeed, if $\sum_i u_i \otimes v_i \otimes w_i$ is in $K_2$, then writing

$$E = a_3 \sum_j b_j,$$

we see that

$$((m_{13} \circ S_3) \otimes 1) \mathcal{P}_{23} \mathcal{R}_{23} \left( \sum_i u_i \otimes v_i \otimes w_i \right) = 0,$$

where we note that

$$((m_{12} \circ S_2) \otimes 1) \left( \sum_i u_i \otimes v_i \otimes w_i \right) = 0.$$

A similar argument shows that $(\mathcal{P}_{23} \mathcal{R}_{23})^{-1}$ maps $K_3$ to $K_2$. Now, we showed that $(U_q'(\mathfrak{g}^N) \otimes W_{k-1})_{(2.3.4)}$ is non-zero. Choose a one-dimensional space of such invariants and let its preimage under $(m_{12} \circ S_2) \otimes 1$ be $V \subset U_q'(\mathfrak{g}^N) \otimes U_q'(\mathfrak{g}^N) \otimes W_{k-1}$ so that $V/K_3 \simeq \mathbb{C}$ as $U_q'(\mathfrak{g}^N)$-representations. We conclude that $\mathcal{P}_{23} \mathcal{R}_{23}(V)/K_2 \simeq \mathbb{C}$, implying that $(U_q'(\mathfrak{g}^N) \otimes W_{k-1})_{(2.3.3)}$ is non-zero.

3. Choosing a normalized invariant: Choose a non-zero element $c_k \in (U_q'(\mathfrak{g}^N) \otimes W_{k-1})_{(2.3.3)}$, normalized so that by Lemma 2.3.6, we have $\Phi_{\lambda}^N = c_k |_{L^{(k-1)}_{+}}$. Now, by Lemma 2.3.7, the image of $c_k$ under $((\pi \circ \varphi) \otimes 1)$ lies in $(\mathbb{C}[\mathfrak{B}_N] \otimes W_{k-1})^{(v_{\lambda})}$. On
the other hand, because \( \dim W^*_k[0] = 1 \), by [89, Theorem A], \( W^*_k \) has multiplicity 1 as a \( U(u_N) \)-representation in \( \mathbb{C}[B_N] \), so \( (\mathbb{C}[B_N] \otimes W_{k-1})^{U(u_N)} \) has dimension 1. In particular, this means that \( ((\pi \circ \varphi) \otimes 1)(c_k) \) restricts to a non-zero multiple of \( F_{k-1} \) in \( (\mathbb{C}[O_A] \otimes W_{k-1})^{U(u_N)} \). Because the normalization of \( c_k \) agrees with that of \( \Phi^N_{\lambda_m} \), the projection of \( c_k \) to \( U'(t_N) \otimes w_{k-1} \) must be \( 1 \otimes w_{k-1} \), which implies that the restriction of the \( w_{k-1} \)-component of \((\pi \circ \varphi) \otimes 1)(c_k) \) to \( \mathbb{C}[T_N] \) is 1 and hence that

\[
((\pi \circ \varphi) \otimes 1)(c_k)|_{\mathcal{O}_A} = F_{k-1}.
\]

**Corollary 2.3.9.** For sequences of dominant integral signatures \( \{\lambda_m\} \) and real quantization parameters \( \{q_m\} \) so that \( \lim_{m \to \infty} q_m = 1 \) and \( \lim_{m \to \infty} -2 \log(q_m)\lambda_m = \lambda \) is dominant regular, we have

\[
\lim_{m \to \infty} (-2 \log(q_m))^{N(N-1)/2} \text{Tr}_{L_{\lambda_m+(k-1)\rho}} (\pi_{q_m}(\Phi^N_{\lambda_m}) \cdot q_m^{-2(s,h)}) = \int_{\mathcal{O}_A} F_{k-1}(X) \prod_{i=1}^N \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{s_i} d\mu_\lambda.
\]

**Proof.** This follows by combining Theorem 2.3.5 and Lemma 2.3.8. \( \square \)

**Corollary 2.3.10.** For sequences of dominant integral signatures \( \{\lambda_m\} \) and real quantization parameters \( \{q_m\} \) so that \( \lim_{m \to \infty} q_m = 1 \) and \( \lim_{m \to \infty} -2 \log(q_m)\lambda_m = \lambda \) is dominant regular, we have

\[
\lim_{m \to \infty} (-2 \log(q_m))^{kN(N-1)/2} P_{\lambda_m}(q_m^{-2s}, q_m^2, q_m^{2k}) = \int_{\mathcal{O}_A} F_{k-1}(X) \prod_{i=1}^N \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{s_i} d\mu_\lambda \prod_{i=1}^{k-1} \prod_{1<i}(s_i - s_j - a).
\]

**Proof.** Set \( \lambda_m = m\lambda + (k-1)\rho \) in Corollary 2.3.9 and explicitly take the limit in Proposition 2.3.1. \( \square \)

### 2.3.7 Degeneration of Macdonald operators

We now put everything together to show that the limiting integral expression satisfies a differential equation in the indices. This differential equation will be a scaling limit of the difference equations satisfied as a result of the Macdonald symmetry identity, recalled below. For this, we abuse notation to write \( D_{N,q}^{2\lambda+2k\rho} \) for difference operators acting on additive indices \( \lambda \) as well as multiplicative variables \( q^{2\lambda+2k\rho} \). Denote also by \( [a]_q \) the \( q \)-number \( [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}} \) and \( [a]_{q,l} \) the falling \( q \)-factorial \( [a]_{q,l} := [a]_q \cdots [a - l + 1]_q \).

**Proposition 2.3.11** (Macdonald symmetry identity). We have

\[
P_\lambda(q^{2\mu+2k\rho}, q^2, q^{2k}) = \prod_{i<j} [\lambda_i - \lambda_j + k(j - i) + k - 1]_{q,k} \prod_{i} [\mu_i - \mu_j + k(j - i) + k - 1]_{q,k} P_\mu(q^{2\lambda+2k\rho}, q^2, q^{2k}).
\]

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Proposition 2.3.12. The operator
\[ \overline{D}_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k}) = \prod_{i<j} [\lambda_i - \lambda_j + k(j-i) + k - 1]_{q,k}^{-1} \]
satisfies
\[ \overline{D}_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k}) = \sum_{|I|=r} \prod_{i<j} [\lambda_i - \lambda_j + k(j-i) + k - 1]_{q,k}^{-1} T_{q^2,I} \]
and
\[ \overline{D}_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k}) P(x; q^2, q^{2k}) = e_r(x) P_\lambda(x; q^2, q^{2k}). \]

Proof. The expression for \( \overline{D}_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k}) \) follows by direct computation, and the eigenvalue identity from the Macdonald symmetry identity.

Consider now the operator
\[ D_\lambda(q) = D^1_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k})^2 - 2D^2_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k}) - 2D^1_{N,q^{2\lambda+2\lambda}}(q^2, q^{2k}) + 1. \]
By Proposition 2.3.12, \( D_\lambda(q) \) acts by \( \sum (x_i - 1)^2 \) on
\[ \prod_{i<j} [\lambda_m,i - \lambda_m,j + k(j-i) + k - 1]_{q,k}^{-1} P_\lambda(x; q^2, q^{2k}). \]
We characterize the scaling limit of \( D_\lambda(q) \) as a second-order differential operator in the following lemma, whose proof is computational and deferred to Subsection 2.6.1

Lemma 2.3.13. Suppose that \( \{f_m\} \) is a sequence of functions so that for \( \{q_m\} \) and \( \{\lambda_m\} \) with \( \lim_{m \to \infty} q_m = 1 \) and \( \lim_{m \to \infty} -2\log(q_m)\lambda_m = \lambda \), we have \( \lim_{m \to \infty} f_m(\lambda_m; q_m) = f(\lambda) \) for a twice-differentiable function \( f \). Then we have
\[ \lim_{m \to \infty} (-2\log(q_m))^{-2} D_\lambda(q_m) f_m(\lambda_m; q_m) = T^\text{trig}(k) f(\lambda). \]
Combining Lemma 2.3.13 and our results on the degeneration of Macdonald polynomials implies that our representation-valued integrals are diagonalized by the trigonometric Calogero-Moser Hamiltonian.

Theorem 2.3.14. The trigonometric Calogero-Moser Hamiltonian \( T^\text{trig}(k) \) is diagonalized on
\[ \frac{1}{\prod_{i<j} (e^\lambda_i - e^\lambda_j)} \prod_{i<j} (s_i - s_j - a) \int_{\mathcal{A}} F_{k-1}(X) \prod_{l=1}^N \left( \frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_A \]
with eigenvalue \( \sum_i s_i^2 \).
Proof. Take any sequence \( \{q_m\} \) and \( \{\lambda_m\} \) for which we have \( \lim_{m \to \infty} q_m = 1 \) and \( \lim_{m \to \infty} -2 \log(q_m) \lambda_m = \lambda \); for instance, we may take \( q_m = e^{-1/2m} \) and \( \lambda_m = \lfloor m \lambda \rfloor \).

Notice that
\[
\lim_{m \to \infty} (2 \log(q_m))^{k(N-1)/2} \prod_{i < j} [\lambda_{m,i} - \lambda_{m,j} + k(j - i) + k - 1]_{q_m,k} = \prod_{i < j} (e^{\frac{\lambda_{i,j}}{2}} - e^{-\frac{\lambda_{i,j}}{2}})^k
\]
so that by Corollary 2.3.10 we have
\[
\lim_{m \to \infty} (-1)^{k(N-1)/2} \prod_{i < j} [\lambda_{m,i} - \lambda_{m,j} + k(j - i) + k - 1]_{q_m,k}^{-1} \frac{\int_{O_{\lambda} \setminus \{0\}} F_{k-1}(X) \prod_{i=1}^{N} \left( \frac{\det(X_{i})}{\det(X_{i-1})} \right) s_i}{\prod_{i \neq j} (e^{\frac{\lambda_{i,j}}{2}} - e^{-\frac{\lambda_{i,j}}{2}})^k} d\mu_{\lambda} = \lim_{m \to \infty} \left( -1 \right)^{k(N-1)/2} (-2 \log(q_m))^{-2} D_{\lambda_m}(q_m)
\]
where \( \lim_{m \to \infty} (-2 \log(q_m))^{-2} \prod_{i} (q_m^{-2s_i} - 1)^2 = \prod_{i} s_i^2 \). Therefore, by Lemma 2.3.13, we have
\[
\mathcal{T}_{\text{trig}}(k) = \left( \sum_{i} s_i^2 \right) \frac{\int_{O_{\lambda} \setminus \{0\}} F_{k-1}(X) \prod_{i=1}^{N} \left( \frac{\det(X_{i})}{\det(X_{i-1})} \right) s_i}{\prod_{i \neq j} (e^{\frac{\lambda_{i,j}}{2}} - e^{-\frac{\lambda_{i,j}}{2}})^k} d\mu_{\lambda}
\]
Note now that \( D_{\lambda_m}(q_m) \) acts by \( \sum_{i} (x_i - 1)^2 = \sum_{i} (q_m^{-2s_i} - 1)^2 \) on
\[
\prod_{i < j} [\lambda_{m,i} - \lambda_{m,j} + k(j - i) + k - 1]_{q_m,k}^{-1} \frac{\int_{O_{\lambda} \setminus \{0\}} F_{k-1}(X) \prod_{i=1}^{N} \left( \frac{\det(X_{i})}{\det(X_{i-1})} \right) s_i}{\prod_{i \neq j} (e^{\frac{\lambda_{i,j}}{2}} - e^{-\frac{\lambda_{i,j}}{2}})^k} d\mu_{\lambda} = \lim_{m \to \infty} \left( -1 \right)^{k(N-1)/2} (-2 \log(q_m))^{-2} D_{\lambda_m}(q_m)
\]
where \( \lim_{m \to \infty} (2 \log(q_m))^{k(N-1)/2} \prod_{i < j} [\lambda_{m,i} - \lambda_{m,j} + k(j - i) + k - 1]_{q_m,k}^{-1} \frac{\int_{O_{\lambda} \setminus \{0\}} F_{k-1}(X) \prod_{i=1}^{N} \left( \frac{\det(X_{i})}{\det(X_{i-1})} \right) s_i}{\prod_{i \neq j} (e^{\frac{\lambda_{i,j}}{2}} - e^{-\frac{\lambda_{i,j}}{2}})^k} d\mu_{\lambda} = \lim_{m \to \infty} \left( -1 \right)^{k(N-1)/2} (-2 \log(q_m))^{-2} D_{\lambda_m}(q_m)
\]

\section{2.4 The rational case}

\subsection{2.4.1 Statement of the result}
Recall that \( f_{k-1} : O_{\lambda} \to W_{k-1} \) is the unique \( U_{N} \)-equivariant map so that \( f_{k-1}(\lambda) = w_{k-1} \). Define the representation-valued integral
\[
\psi_k(\lambda, s) = \int_{X \in O_{\lambda}} f_{k-1}(X) e^{\sum_{i=1}^{N} s_i X_{ii}} d\mu_{\lambda}
\]
over the coadjoint orbit \( O_{\lambda} \). The integrand and Liouville measure are invariant under the action of the maximal torus of \( U_{N} \), so \( \psi_k(\lambda, s) \) lies in \( W_{k-1}[0] = \mathbb{C} \cdot w_{k-1} \). We interpret the integrals \( \psi_k(\lambda, s) \) as complex-valued functions by identifying \( \mathbb{C} \cdot w_{k-1} \) with \( \mathbb{C} \). Our first result relates these integrals to the multivariate Bessel functions.
**Theorem 2.4.1.** The multivariate Bessel function $B_k(\lambda, s)$ admits the integral representation
\[
B_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N} \prod_{i<j}(\lambda_i - \lambda_j)^k \prod_{i<j}(s_i - s_j)^{k-1} \int_{X \in C_\lambda} f_{k-1}(X)e^{\sum_{i=1}^N s_i X_i} d\mu.
\]

**2.4.2 Adjoint of rational Calogero-Moser operators**

The rational Dunkl operators in variables $\mu_i$ are
\[
D_{\mu_i}(k) := \partial_i + k \sum_{i \neq j} \frac{1}{\mu_i - \mu_j} (1 - s_{ij}),
\]
where $s_{ij}$ exchanges $\mu_i$ and $\mu_j$. Let $m$ denote the restriction of a differential-difference operator to its differential part. For a symmetric polynomial $p$, recall that
\[
m(p(D_{\mu_i}(k))) = \overline{L}_p(k),
\]
for $\overline{L}_p(k)$ defined in (2.1.1) as a conjugate of the rational Calogero-Moser Hamiltonian corresponding to $p$. We now compute the adjoint of the differential operator $\overline{L}_p(k)$.

**Proposition 2.4.2.** For a homogeneous symmetric polynomial $p$, the adjoint of $\overline{L}_p(k)$ as a differential operator is
\[
\overline{L}_p(k) = (-1)^{\deg p} \Delta(\mu)^{2k} \circ \overline{L}_p(k) \circ \Delta(\mu)^{-2k}.
\]

*Proof.* The desired is an equality of symmetric differential operators, so it suffices to verify that for smooth compactly supported symmetric functions $f(\mu)$ and $g(\mu)$ we have
\[
\int_{\mathbb{R}^N-1} f(\mu)\overline{L}_p(k)g(\mu)d\mu = \int_{\mathbb{R}^N-1} (-1)^{\deg p}[\Delta(\mu)^{2k}\overline{L}_p(k)\Delta(\mu)^{-2k}f(\mu)]g(\mu)d\mu.
\]
In this case, we have $\overline{L}_p(k) = m(p(D_{\mu_i}(k)))$, so it suffices to check that for smooth compactly supported functions $f(\mu)$ and $g(\mu)$ we have
\[
\int_{\mathbb{R}^N-1} f(\mu)D_{\mu_i}(k)g(\mu)d\mu = -\int_{\mathbb{R}^N-1} [\Delta(\mu)^{2k}D_{\mu_i}(k)\Delta(\mu)^{-2k}f(\mu)]g(\mu)d\mu.
\]
For this, notice that
\[
\int_{R^{N-1}} f(\mu)D_{\mu_i}(k)g(\mu)d\mu = \int_{R^{N-1}} f(\mu)\left(\partial_ig(\mu) + k\sum_{j\neq i} \frac{g(\mu) - s_{ij}g(\mu)}{\mu_i - \mu_j}\right)d\mu
\]
\[
= -\int_{R^{N-1}} \partial_i f(\mu)g(\mu)d\mu + k\sum_{j\neq i} \int_{R^{N-1}} \frac{1}{\mu_i - \mu_j}f(\mu)g(\mu)d\mu
\]
\[
+ k\sum_{j\neq i} \int_{R^{N-1}} s_{ij}f(\mu)g(\mu)\frac{1}{\mu_i - \mu_j}d\mu
\]
\[
= -\int_{R^{N-1}} \left(\partial_i - k\sum_{j\neq i} \frac{1 + s_{ij}}{\mu_i - \mu_j}\right)f(\mu)g(\mu)d\mu
\]
and that
\[
\Delta(\mu)^{2k} \circ D_{\mu_i}(k) \circ \Delta(\mu)^{-2k}
\]
\[
= \partial_i - 2k\sum_{j\neq i} \frac{1}{\mu_i - \mu_j} + k\sum_{j\neq i} \frac{1 - s_{ij}}{\mu_i - \mu_j} = \partial_i - k\sum_{j\neq i} \frac{1 + s_{ij}}{\mu_i - \mu_j}. \quad \Box
\]

We now make precise the adjunction when integrated against a specific domain. For multi-indices \(\alpha = (\alpha_i)\) and \(\beta = (\beta_i)\), write \(\beta \leq \alpha\) if \(\alpha_i \leq \beta_i\) for all \(i\).

**Proposition 2.4.3.** Let \(A\) be a rectangular domain. Let \(p = \sum_\alpha c_\alpha \mu^\alpha\) be a symmetric homogeneous function. If for each non-zero monomial \(\mu^\alpha\) appearing in \(p\), \(\partial_\mu^\beta f\) vanishes on the boundary of \(A\) for any \(\beta \leq \alpha\), then we have the adjunction relation
\[
\int_A (L_p(f(\mu))) g(\mu)d\mu = \int_A f(\mu) L_p^\dagger(g(\mu))d\mu.
\]

**Proof.** Applying repeated integration by parts with Proposition 2.4.2 shows the two sides of the desired relation differ by the sum of several terms, each of which contains a factor which is the evaluation of \(\partial_\mu^\beta(f)\) on a point of the boundary of \(A\) for some \(\beta \leq \alpha\) with \(\mu^\alpha\) appearing in \(p\). These terms vanish, giving the lemma. \( \Box \)

### 2.4.3 A matrix element computation

Recall that sequences \(\{\lambda_i\}_{1 \leq i \leq N}\) and \(\{\mu_i\}_{1 \leq i \leq N-1}\) *interlace* if
\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N,
\]
which we denote by \(\mu < \lambda\). Define the real matrix \(u(\mu, \lambda)\) by
\[
u(\mu, \lambda)_{ij} = \begin{cases} \left(\frac{\prod_{j \neq i}(\mu_j - \lambda_i)}{\prod_{j \neq i}(\lambda_j - \lambda_i)}\right)^{1/2} & i = N \\ (\lambda_j - \mu_i)^{-1} \left(\frac{\prod_{j \neq i}(\mu_j - \lambda_i)}{\prod_{j \neq i}(\lambda_j - \lambda_i)}\right)^{1/2} & 0 < N, & \end{cases}
\]
where each square root is applied to a non-negative real number because \( \mu < \lambda \), and we take the non-negative branch. The following lemma, whose proof is given in Section 2.6.2, shows that \( u(\mu, \lambda) \) conjugates a diagonal matrix to a matrix with diagonal principal submatrix.

**Lemma 2.4.4.** The matrix \( u(\mu, \lambda) \) is unitary, and the \((N - 1) \times (N - 1)\) principal submatrix of

\[
u(\mu, \lambda) \text{diag}(\lambda_1, \ldots, \lambda_N) u(\mu, \lambda)^*\]

is \( \text{diag}(\mu_1, \ldots, \mu_{N-1}) \).

We would like to understand a specific matrix element of \( u(\mu, \lambda) \) in \( W_{k-1} \). For this, notice that \( W_{k-1} \cong \text{Sym}^{(k-1)N} \mathbb{C}^N \) as an \( SU_N \)-representation via an isomorphism sending \( w_{k-1} \) to \((x_1 \cdots x_N)^{k-1}\). We now compute an auxiliary quantity. Let \( Z_k(\mu, \lambda) \) denote the coefficient of \((x_1 \cdots x_j)^k\) in the polynomial

\[
\frac{1}{(l - N + 1)!} \prod_{j=1}^{l} \left( \sum_{i=1}^{N-1} \frac{x_i}{\mu_\lambda - \lambda_j} + x_N + \cdots + x_l \right).
\]

By Proposition 2.4.5, we may express \( Z_k(\mu, \lambda) \) via a conjugated Calogero-Moser Hamiltonian, where we recall that \( L_p(\kappa) \) was defined in (2.1.1); we defer the proof to Section 2.6.3. The computation of the desired matrix element of \( u(\mu, \lambda) \) is an easy consequence.

**Proposition 2.4.5.** We may write

\[
Z_k(\mu, \lambda) = \frac{k!}{(N-1)!} \Delta(\mu, \lambda)^{-k} L_{\mu_N \cdots \mu_1} (\mu, \lambda)^k.
\]

**Remark.** It is convenient for us to formulate and prove Proposition 2.4.5 for general \( l \). However, in our main application Lemma 2.4.6, it will be only be used with \( l = N \).

**Lemma 2.4.6.** The coefficient of \((x_1 \cdots x_{N-1})^{k-1}\) in \( u(\mu, \lambda) \cdot (x_1 \cdots x_{N-1})^{k-1} \) is

\[
(-1)^{(k-1)(N+2)(N-1)/2} (k - 1)! (N-1)! \Delta(\mu)^1 \Delta(\lambda)^{1-k} (L_{\mu_1 \cdots \mu_{N-1}}(1 - k))^{k-1} \Delta(\mu, \lambda)^{k-1}.
\]

**Proof.** By Lemma 2.4.4, the desired coefficient is given by

\[
(-1)^{(k-1)(N+2)(N-1)/2} \frac{\Delta(\mu, \lambda)^{k-1}}{\Delta(\mu)^{k-1} \Delta(\lambda)^{k-1}} Z_{k-1}(\mu, \lambda),
\]

which is equal to the desired by Proposition 2.4.5.

**2.4.4 Proof of Theorem 2.4.1**

Integrating over Liouville tori of the Gelfand-Tsetlin integrable system on \( \mathcal{O}_\lambda \) yields the expression

\[
\psi_k(\lambda, s) = \int_{\mu \in \Gamma_{\lambda}} \int_{t \in \Gamma, X_0 \in \Gamma^{-1}(\mu)} f_{k-1}(t \cdot X_0) dt \epsilon^{\sum_{j=1}^{N} \delta_i (\sum_{i=1}^{N} \mu_i^j - \sum_{i=1}^{N} \mu_i^{j-1})} g_T(d \mu_\lambda),
\]

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where \( dt \) is an invariant probability measure on \( T \) and \( \mu_i \) are the Gelfand-Tsetlin coordinates. Recall that \( g_t \cdot (d\mu) \) is equal to Lebesgue measure on \( GT_\lambda \) by Proposition 2.2.2. Adopting the notations of Section 2.2.3, by repeated application of Lemma 2.2.1 we have for \( t = t_1 \cdots t_{N-1} \) in the Gelfand-Tsetlin torus that

\[
t \cdot X_0 = \text{ad}(\overline{v}_1) t_1 \cdot \text{ad}(\overline{v}_2) \cdots t_{N-1} \cdot \text{ad}(\overline{v}_N) \cdot \lambda.
\]

On the other hand, if \( w \in W_{k-1} \) lies in \( \mathbb{C}[x_1, \ldots, x_l](x_{l+1} \cdots x_N)^{k-1} \) under the identification of \( W_{k-1} \cong \text{Sym}^{(k-1)} N \mathbb{C}^N \) of \( \text{SU}_N \)-representations, then

\[
\int_{T_i} t_i \cdot w \ dt_i = \{ \text{coefficient of } (x_1 \cdots x_N)^{k-1} \text{ in } w \}. \tag{2.4.2}
\]

Together, these imply that

\[
\int_{t \in T, X_0 \in g(T^{-1}(\mu))} f_{k-1}(t \cdot X_0) \ dt = \prod_{m=1}^{N-1} W_m,
\]

where \( W_m \) denotes the coefficient of \( (x_1 \cdots x_m)^{k-1} \) in \( v_m \cdot (x_1 \cdots x_m)^{k-1} \). Substituting in this result, inducting on \( N \), applying the integral formula (2.1.5), and applying the shift formula

\[
e^{c \sum_i \mu_i} \phi_k(\mu, s) = \phi_k(\mu, s_1 + c, \ldots, s_{N-1} + c) \tag{2.4.2}
\]

for the integral expressions (2.1.5) in \( N - 1 \) variables with \( c = -s_N \), we obtain

\[
\psi_k(\lambda, s) = \int_{\mu \in GT_\lambda} \prod_{m=1}^{N-1} W_m e^{sN\sum_i \mu_i - sN \sum_i \mu_i - (N-1) \sum_i \mu_i} \ dt_i
\]

\[
= \int_{\mu \notin \lambda} W_{N-1} e^{sN\sum_i \mu_i - sN \sum_i \mu_i} \prod_{1 \leq i < j \leq N-1} (s_i - s_j)^{k-1} \phi_k(\mu, s) \prod_i \ dt_i
\]

\[
e^{sN\sum_i \mu_i} \prod_{1 \leq i < j \leq N-1} (s_i - s_j)^{k-1} \int_{\mu \notin \lambda} W_{N-1} \phi_k(\mu, s') \prod_i \ dt_i,
\]

where \( s' = (s_1 - s_N, \ldots, s_{N-1} - s_N) \). Recall now that \( v_m \) was chosen so that \( (v_m \text{diag}(\mu^{m+1}) v_m^*)_m = \text{diag}(\mu^m) \), meaning by Lemma 2.4.6 that

\[
W_m = \frac{(-1)^{(k-1)(m+3)(m+1)/2}(k-1)!}{\Delta(\mu^m)^{k-1}\Delta(\mu^{m+1})^{k-1}} (L_{\mu_1} \cdots L_{\mu_m})(1 - k)^{k-1} \Delta(\mu, \mu^{m+1})^{k-1}.
\]

Substituting this into the previous expression, we find that

\[
\psi_k(\lambda, s) = (-1)^{(k-1)(N^2 + N - 1)/2} e^{sN\sum_i \lambda_i \Gamma(k)} (N-1) \prod_{1 \leq i < j \leq N-1} (s_i - s_j)^{k-1} \Delta(\lambda)^{1-k}
\]

\[
\int_{\mu \notin \lambda} \Delta(\mu)^{1-k} \phi_k(\mu, s') (L_{\mu_1} \cdots L_{\mu_{N-1}})(1 - k)^{k-1} \Delta(\mu, \lambda)^{k-1} \prod_i \ dt_i.
\]
Recall that $L_p(k) = L_p(1 - k)$ for any symmetric polynomial $p$, which by (2.1.1) implies that
\[ L_p(k) = \Delta(\mu)^{1-2k} \circ L_p(1 - k) \circ \Delta(\mu)^{2k-1}. \] (2.4.3)

Recall also that
\[ L_{\mu_1 \ldots \mu_{N-1}}(1 - k) = (-1)^{N-1} \Delta(\mu)^{2-2k} \circ L_{\mu_1 \ldots \mu_{N-1}}(1 - k) \circ \Delta(\mu)^{2k-2} \]
by Proposition 2.4.2. Applying Proposition 2.4.3, we have that
\[
\int_{\mu<\lambda} \Delta(\mu)^{1-k} \phi_k(\mu, s') L_{\mu_1 \ldots \mu_{N-1}}(1 - k)^{k-1} \Delta(\mu, \lambda)^{k-1} \prod_i d\mu_i
\]
\[
= (-1)^{(N-1)(k-1)} \int_{\mu<\lambda} \Delta(\mu, \lambda)^{k-1} \Delta(\mu)^{2-2k} 
\]
\[
L_{\mu_1 \ldots \mu_{N-1}}(1 - k)^{k-1} \Delta(\mu)^{k-1} \phi_k(\mu, s') \prod_i d\mu_i
\]
\[
= (-1)^{(N-1)(k-1)} \int_{\mu<\lambda} \Delta(\mu, \lambda)^{k-1} \Delta(\mu)^{k-1} \phi_k(\mu, s') \prod_i d\mu_i
\]
\[
= (-1)^{(N-1)(k-1)} \prod_{i=1}^{N-1} (s_i - s_N)^{k-1} \int_{\mu<\lambda} \frac{\Delta(\mu, \lambda)^{k-1}}{\Delta(\mu)^{k-1}} \phi_k(\mu, s') \prod_i d\mu_i,
\]
where in the first equality we apply adjunction, in the second equality we apply (2.4.3), and in the third equality we apply the inductive hypothesis and (2.1.4). We conclude that
\[
\psi_k(\lambda, s) = (-1)^{(k-1)(N-1)} \frac{e^{N} \sum_i \lambda_i \Gamma(k)^{-N-1}}{2}
\]
\[
\prod_{1 \leq i < j \leq N} (s_i - s_j)^{k-1} \int_{\mu<\lambda} \frac{\Delta(\mu, \lambda)^{k-1}}{\Delta(\mu)^{k-1}} \phi_k(\mu, s') \prod_i d\mu_i
\]
\[
= \prod_{1 \leq i < j \leq N} (s_i - s_j)^{k-1} \phi_k(\lambda, s),
\]
which implies the result by Theorem 2.1.2.

### 2.5 The trigonometric case

#### 2.5.1 Identifying $\Phi_k(\lambda, s)$ with the Heckman-Opdam hypergeometric functions

In this subsection, we provide details of how to relate the integral formula (2.1.6) for $\Phi_k(\lambda, s)$ to the Heckman-Opdam hypergeometric function $F_k(\lambda, s)$ for the case where $k > 0$ is a positive integer. We use the characterization of Theorem 2.1.1. First, we claim the symmetric extension of $\Delta^{\text{trig}}(\lambda)^{-k} \Phi_k(\lambda, s)$ extends to a holomorphic function of $\lambda$ on a symmetric tubular neighborhood of $\mathbb{R}^N$. Observe that (2.1.6) has
recursive structure

\[
\frac{\Phi_k(\lambda, s)}{\Delta(e^\lambda)^k} = \frac{(-1)^{(k-1)N(N-1)}}{\Gamma(k)^N N^N} \int_{\mu < \lambda} e^{sN(\sum_i \lambda_i - \sum_i \mu_i)-(k-1)\sum_i \mu_i} \frac{\Delta(e^\mu, e^\lambda)^{k-1}}{\Delta(e^\lambda)^{2k-1}} \frac{\Delta(e^\mu)^k}{\Delta(e^\lambda)^k} d\mu. \tag{2.5.1}
\]

We induct on \(N\) with trivial base case. For the inductive step, change variables to \(r_i = \frac{\mu_i - \lambda_i + 1}{\lambda_i - \lambda_{i+1}}\). We obtain

\[
\frac{\Phi_k(\lambda, s)}{\Delta(e^\lambda)^k} = \frac{(-1)^{(k-1)N(N-1)}}{\Gamma(k)^N N^N} \int_{[0,1]^N} \prod_i (\lambda_i - \lambda_{i+1}) e^{sN(\sum_i \lambda_i - \sum_i \mu_i)-(k-1)\sum_i \mu_i} \frac{\Delta(e^\mu, e^\lambda)^{k-1}}{\Delta(e^\lambda)^{2k-1}} \frac{\Delta(e^\mu)^k}{\Delta(e^\lambda)^k} d\tau,
\]

where we view \(\mu\) as a function of \(\tau\) and \(\lambda\) in the integrand. As a function of \(\lambda\), the integrand is meromorphic with poles away from the set \(\{\lambda_i = \lambda_{i+1}\}\). It is easy to check that there are no poles on the subsets of hyperplanes \(\lambda_i = \lambda_{i+1}\) where no other coordinates are equal, so by Hartog's theorem, the integrand is holomorphic in \(\lambda\). By the induction hypothesis, it is also holomorphic in \(\tau\), hence the result is holomorphic in \(\lambda\) and admits the claimed extension.

We now claim that \(\frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k}\) satisfies the hypergeometric system; it suffices to show

\[
\mathcal{L}_p^{\text{trig}}(k) \frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k} = p(s) \frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k}
\]

for any symmetric \(p\). It was shown in [9, Proposition 6.3] that

\[
\mathcal{L}_p^{\text{trig}}(k) \frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k} = p_2(s) \frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k}
\]

for \(\lambda_1 > \cdots > \lambda_N\). Suppose now that \(s\) is not integral and dominant (if \(s\) is not dominant, choose \(w \in S_N\) so that \(ws\) is dominant and replace \(s\) by \(ws\)). We claim by induction that \(\frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k}\) admits a series expansion of the form

\[
\frac{\Phi_k(\lambda, s)}{\Delta^{\text{trig}}(\lambda)^k} = \prod_{a=0}^{k-1} \prod_{i<j} (s_i - s_j + a)^{-1} c^{(s-kp, \alpha)} + (\text{l.o.t.}),
\]

where we use \(\text{l.o.t.}\) to denote terms of the form \(c_\alpha e^{(s-kp, \alpha)}\) for \(\alpha\) in the positive weight lattice. By (2.5.1), the leading monomial is the same as the leading monomial
\[e^{(s_N + \frac{k(N-1)}{2})|\lambda|}e^{(k-1)(\rho, \lambda)} + \frac{(N-3)(k-1)}{2}|\lambda| + (k-1)\lambda_N \Gamma(k)^{(N-1)} \prod_{a=0}^{k-1} \prod_{1 \leq i < j < N} (s_i - s_j + a) \Delta(e^\lambda)^{2k-1} \int_{\mu = \lambda}^{N-1} \prod_{i=1}^{N-1} (e^{\lambda_i} - e^{\mu_i})^{k-1} e^{(\mu, \sigma')} \prod_i d\mu_i, \]

where \( \sigma' = (s_1 - s_N, \ldots, s_{N-1} - s_N) \). Note that for \( b_2 < b_1 \), the expression

\[
\int_{b_2}^{b_1} (e^{b_1} - e^{x})^{k-1} e^{-x} dx = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} e^{x(s+i)} e^{b_1(k-1-i)} dx
\]

has leading monomial \( e^{b_1(k-1+s)} \) in \( b \) with coefficient \( \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(-1)^i}{i} \binom{k-1}{i} = \frac{\Gamma(k)}{\prod_{a=0}^{k-1} (s+a)} \).

Therefore, the leading monomial in (2.5.2) is

\[
e^{(s_N + \frac{k(N-1)}{2})|\lambda|}e^{(k-1)(\rho, \lambda)} + \frac{(N-3)(k-1)}{2}|\lambda| + (k-1)\lambda_N \Gamma(k)^{(N-1)} \prod_{a=0}^{k-1} \prod_{1 \leq i < j < N} (s_i - s_j + a) e^{((2k-1)\rho, \lambda)} + \frac{(2k-1)(N-1)}{2}|\lambda|
\]

as claimed. By the results of [61, Section 4.2], for such \( s \), any analytic symmetric eigenfunction of \( \overline{T_{\text{trig}}(k)} \) with leading monomial \( e^{(s-k, \rho, \lambda)} \) diagonalizes \( \overline{T_{\text{trig}}(k)} \) for any \( p \). Therefore, \( \Phi_k(\lambda, s) \) satisfies the full hypergeometric system, and our computation of its leading monomial coefficient implies that

\[
\mathcal{F}_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k) \Phi_k(\lambda, s)}{\Delta(e^\lambda)^k}
\]

for non-integral \( s \). Both sides of the expression are holomorphic functions of \( s \), so this continues to hold for non-generic \( s \), yielding Theorem 2.1.4.

### 2.5.2 Statement of the result

Let \( F_{k-1} : \mathcal{O}_\Lambda \rightarrow W_{k-1} \) be the unique \( U_N \)-equivariant map so that \( F_{k-1}(\Lambda) = w_{k-1} \). Define the representation-valued integral

\[
\Psi_k(\lambda, s) = \int_{X \in \mathcal{O}_\Lambda} F_{k-1}(X) \prod_{i=1}^{N} \left( \frac{\det(X_i)}{\det(X_{i-1})} \right)^{s_i} d\mu_\Lambda,
\]

where \( X_i \) denotes the principal \( l \times l \) submatrix of \( X \). As in the rational case, the integrand and Liouville measure in the definition of \( \Psi_k(\lambda, s) \) are invariant under the action of the maximal torus of \( U_N \), so \( \Psi_k(\lambda, s) \) lies in \( W_{k-1}[0] = \mathbb{C} \cdot w_{k-1} \). We will again interpret it as a complex-valued function via the identification of \( \mathbb{C} \cdot w_{k-1} \)
with C. Our result in the trigonometric setting uses these integrals to express the Heckman-Opdam hypergeometric functions.

**Theorem 2.5.1.** The Heckman-Opdam hypergeometric function $F_k(\lambda, s)$ admits the integral representation

$$F_k(\lambda, s) = \frac{\Gamma(Nk) \cdots \Gamma(k)}{\Gamma(k)^N \prod_{i<j} \left(e^{-\lambda - \lambda j} - e^{-\lambda_i - \lambda_j}\right) \prod_{a=1}^{k-1} \prod_{i<j} (s_i - s_j - a)}$$

$$\int_{X \in \mathcal{O}_\Lambda} F_{k-1}(X) \prod_{l=1}^{N} \left(\frac{\det(X_l)}{\det(X_{l-1})}\right)^{s_l} d\mu,$$

where $X_l$ is the principal $l \times l$ submatrix of $X$.

### 2.5.3 Adjoint of trigonometric Calogero-Moser operators

The trigonometric Dunkl operators in variables $\mu_i$ are defined by

$$T_{\mu_i}(k) := \partial_i + k \sum_{\alpha>0} (\alpha, \mu_i) \frac{1}{1 - e^{-\alpha}} (1 - s) - k(\alpha, \mu_i).$$

For a symmetric polynomial $p$, $m(p(T_{\mu_i}(k))) = T_{p}^{\text{trig}}(k)$ is the conjugate (2.1.2) of the trigonometric Calogero-Moser Hamiltonian corresponding to $p$. Let $w_0 \in S_{N-1}$ be the permutation sending $i$ to $N - i$. We now compute the adjoint of the differential operator $T_{p}^{\text{trig}}(k)$ via an analogue of [87, Lemma 7.8].

**Proposition 2.5.2.** For a homogeneous symmetric polynomial $p$, the adjoint of $T_{p}^{\text{trig}}(k)$ as a differential operator is

$$T_{p}^{\text{trig}}(k)^\dagger = (-1)^{\deg p} \Delta^{\text{trig}}(\mu)^{2k} \circ T_{p}^{\text{trig}}(k) \circ \Delta^{\text{trig}}(\mu)^{-2k}.$$

**Proof.** As in the proof of Proposition 2.4.2, it suffices to check that for smooth compactly supported functions $f(\mu)$ and $g(\mu)$ we have

$$\int_{\mathbb{R}^{N-1}} f(\mu) T_{\mu}(k) g(\mu) d\mu = -\int_{\mathbb{R}^{N-1}} [\Delta^{\text{trig}}(\mu)^{2k} \circ w_0 \circ T_{\mu N^{-1}}(k) \circ w_0 \circ \Delta^{\text{trig}}(\mu)^{-2k} f(\mu)] g(\mu) d\mu.$$
For this, we compare both sides by computing

$$
\int_{\mathbb{R}^{N-1}} f(\mu) T_{\mu}(k) g(\mu) d\mu \\
= \int_{\mathbb{R}^{N-1}} f(\mu) \left( \partial_i - k \sum_{j<i} \frac{1-s_{ij}}{1-e^{\mu_i-\mu_j}} + k \sum_{j>i} \frac{1-s_{ij}}{1-e^{\mu_i-\mu_j}} - k \frac{N-2i}{2} \right) g(\mu) d\mu \\
= -\int_{\mathbb{R}^{N-1}} \partial_i f(\mu) g(\mu) d\mu - k \sum_{j<i} \int_{\mathbb{R}^{N-1}} \left( \frac{1}{1-e^{\mu_i-\mu_j}} - \frac{s_{ij}}{1-e^{\mu_i-\mu_j}} \right) f(\mu) g(\mu) d\mu \\
+ k \sum_{j>i} \int_{\mathbb{R}^{N-1}} \left( \frac{1}{1-e^{\mu_i-\mu_j}} - \frac{s_{ij}}{1-e^{\mu_i-\mu_j}} \right) f(\mu) g(\mu) d\mu - k \frac{N-2i}{2} \int_{\mathbb{R}^{N-1}} f(\mu) g(\mu) d\mu \\
= -\int_{\mathbb{R}^{N-1}} \left( \partial_i - k \sum_{j<i} \frac{s_{ij} + e^{\mu_i-\mu_j}}{1-e^{\mu_i-\mu_j}} + k \sum_{j>i} \frac{s_{ij} + e^{\mu_i-\mu_j}}{1-e^{\mu_i-\mu_j}} + k \frac{N-2i}{2} \right) f(\mu) g(\mu) d\mu
$$

and

$$
\Delta_{\text{trig}}^{2k} \circ w_0 \circ T_{\mu_{N-1}}(k) \circ w_0 \circ \Delta_{\text{trig}}(\mu)^{-2k} \\
= \Delta_{\text{trig}}^{2k} \circ \left( \partial_i - k \sum_{j<i} \frac{1-s_{ij}}{1-e^{\mu_i-\mu_j}} + k \sum_{j<i} \frac{1-s_{ij}}{1-e^{\mu_i-\mu_j}} + k \frac{N-2i}{2} \right) \circ \Delta_{\text{trig}}(\mu)^{-2k} \\
= \partial_i - k \sum_{j>i} \frac{1-s_{ij}}{1-e^{\mu_i-\mu_j}} + k \sum_{j<i} \frac{1-s_{ij}}{1-e^{\mu_i-\mu_j}} + k \frac{N-2i}{2} + k \sum_{j \neq i} \frac{1+e^{\mu_i-\mu_j}}{1-e^{\mu_i-\mu_j}} \\
= \partial_i + k \sum_{j>i} \frac{e^{\mu_i-\mu_j} + s_{ij}}{1-e^{\mu_i-\mu_j}} - k \sum_{j<i} \frac{e^{\mu_i-\mu_j} + s_{ij}}{1-e^{\mu_i-\mu_j}} + k \frac{N-2i}{2}.
$$

We again make precise the adjunction when integrated against a specific domain.

**Proposition 2.5.3.** Let $A$ be a rectangular domain. Let $p = \sum_{\alpha} c_\alpha \mu^\alpha$ be a symmetric function. If for each non-zero monomial $\mu^\alpha$ appearing in $p$, $\partial_\mu^\beta f$ vanishes on the boundary of $A$ for any $\beta \leq \alpha$, then we have the adjunction relation

$$
\int_A (T_p^{\text{trig}}(k) f(\mu)) g(\mu) d\mu = \int_A f(\mu) T_p^{\text{trig}}(k) g(\mu) d\mu.
$$

**Proof.** The proof is the same as for Proposition 2.4.3. \qed

### 2.5.4 Matrix elements in the trigonometric case

Take $l \geq N - 1$ and consider variables $\lambda_1, \ldots, \lambda_l$ and $\mu_1, \ldots, \mu_{N-1}$. Recall that $Z_k(e^\mu, e^\lambda)$ denotes the coefficient of $(x_1 \cdots x_l)^k$ in the polynomial

$$
\frac{1}{(l-N+1)!} \prod_{j=1}^l \left( \sum_{i=1}^{N-1} \frac{x_i}{e^{\mu_i} - e^{\lambda_j}} + x_N + \cdots + x_l \right)^k.
$$

We express $Z_k(e^\mu, e^\lambda)$ via trigonometric Calogero-Moser Hamiltonians in Proposition 2.5.4.
**Proposition 2.5.4.** We have the identity

\[
Z_k(e^\mu, e^\lambda) = k!^{-(N-1)} \Delta(e^\mu, e^\lambda)^{-k} \left( e^{-\sum_i \mu_i \bar{L}^{\text{trig}}_m (\mu_i - k \frac{N-2}{2})} \right)^k \Delta(e^\mu, e^\lambda)^k.
\]

**Proof.** We use the result in the rational case. By Proposition 2.4.5 and (2.1.2), we have that

\[
Z_k(e^\mu, e^\lambda) = k!^{-(N-1)} \Delta(e^\mu, e^\lambda)^{-k} \left( D_{\varphi N-1}(-k) \cdots D_{\varphi 1}(-k) \right)^k \Delta(e^\mu, e^\lambda)^k.
\]

It therefore suffices to check that

\[
D_{\varphi N-1}(-k) \cdots D_{\varphi 1}(-k) = e^{-\sum_i \mu_i \left( T_{\mu N-1}(-k) - k \frac{N-2}{2} \right)} \cdots \left( T_{\mu 1}(-k) - k \frac{N-2}{2} \right)
\]

on \( \mathbb{C}[e^{\mu_i}]^{S_{N-1}} \). We may rewrite \( T_{\mu_i}(-k) \) in the form

\[
T_{\mu_i}(-k) = \partial_{\mu_i} - k \sum_{j \neq i} \frac{e^{\mu_i}}{e^{\mu_i} - e^{\mu_j}} (1 - s_{ij}) - k \sum_{j < i} s_{ij} + k \frac{N-2}{2}
\]

\[
= e^{\mu_i} D_{\varphi i}(-k) - k \sum_{j < i} s_{ij} + k \frac{N-2}{2}, \quad (2.5.3)
\]

where \( D_{\varphi i}(-k) \) is the rational Dunkl operator in the exponential variables \( e^{\mu_i} \), which implies that

\[
D_{\varphi i}(-k) = e^{-\mu_i} \left( T_{\mu_i}(-k) + k \sum_{j < i} s_{ij} - k \frac{N-2}{2} \right). \quad (2.5.4)
\]

Further, we may check that for \( i > j \), we have \( (T_{\mu_i}(-k) + ks_{ij}) e^{-\mu_j} = e^{-\mu_j} T_{\mu_i}(-k) \), so substituting (2.5.4) and shifting each \( e^{-\mu_i} \) term to the beginning of the expression, we see that

\[
D_{\varphi N-1}(-k) \cdots D_{\varphi 1}(-k) = e^{-\sum_i \mu_i} \prod_{i=1}^{N-1} \left( T_{\mu_i}(-k) - k \frac{N-2}{2} \right). \quad \square
\]

**2.5.5 Proof of Theorem 2.5.1**

We again compute \( \Psi_k(\lambda, s) \) by integrating over the Liouville tori given by the Gelfand-Tsetlin coordinates. We may write

\[
\Psi_k(\lambda, s) = \int_{\mu \in \text{GT}_\Lambda} \int_{t \in T, X_0 \in \text{GT}^{-1}(\mu)} F_{k-1}(t : X_0) dt e^{\sum_i \mu_i \left( \sum_{i} \mu_i - \sum_{i} \mu_i^{-1} \right) \text{GT}_s (d\mu_\Lambda)}, \quad (2.5.5)
\]

where \( dt \) is the invariant probability measure on the torus, and \( \mu_i \) are the logarithmic Gelfand-Tsetlin coordinates. As in the rational case, by Lemma 2.2.1, we have

\[
\int_{t \in T, X_0 \in \text{GT}^{-1}(\mu)} F_{k-1}(t : X_0) dt = \prod_{m=1}^{N-1} W_m,
\]

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where \( W_m \) denotes the coefficient of \((x_1 \cdots x_m)^{k-1}\) in \( v_m \cdot (x_1 \cdots x_m)^{k-1} \). Noting that \( GT_*(d\mu_A) = 1_{GT_1} \cdot dx \) by Proposition 2.2.2 and inducting on \( N \), we transform (2.5.5) to

\[
\Psi_k(\lambda, s) = \int_{\mu < \lambda} \prod_{i=1}^{N-1} W_m e^{\sum_{i=1}^N s_i (\sum_i \mu_i - \sum_i \mu_i - 1)} \prod_i d\mu_i
\]

\[
= \int_{\mu < \lambda} W_{N-1} e^{s_N \left( \sum_i \mu_i \cdot (1 - k) \right)} \prod_{i=1}^{k-1} \prod_{1 \leq i < j \leq N-1} (s_i - s_j - a) \Phi_k(\mu, s) \prod_i d\mu_i
\]

\[
= \prod_{1 \leq i < j \leq N-1} (s_i - s_j - a) e^{s_N \sum_i \lambda_i \int_{\mu < \lambda} W_{N-1} \Phi_k(\mu, s') \prod_i d\mu_i},
\]

where \( s' = (s_1 - s_N, \ldots, s_{N-1} - s_N) \) and the last equality follows from the \( c = -s_N \) case of the shift identity

\[
e^{s_N \sum_i \lambda_i \Phi_k(\lambda, s)} = \Phi_k(\lambda, s_1 + c, \ldots, s_N + c). \tag{2.5.6}
\]

Notice that \((v_m \text{diag}(e^{\mu_{N+1}}))_m = \text{diag}(e^{\mu_m})\), so writing \( \mu := \mu^{N-1} \) and \( \lambda := \mu^N \), we have by Lemma 2.4.4 and Proposition 2.5.4 that

\[
W_{N-1} = (-1)^{(k-1)(N+2)(N-1)/2} \frac{\Delta(e^\mu, e^\lambda)^{k-1}}{\Delta(e^\mu)^{k-1} \Delta(e^\lambda)^{k-1}} Z_{k-1}(e^\mu, e^\lambda)
\]

\[
= (-1)^{(k-1)(N+2)(N-1)/2} \frac{\Gamma(k) - (N-1)}{\Delta^{(k-1)}(\mu) \Delta^{(k-1)}(\lambda)} \frac{\left(e^{s_N \sum_i \mu_i (1 - k)}\right)^{k-1}}{\Delta(e^\mu, e^\lambda)^{k-1}}
\]

\[
= (-1)^{(k-1)(N+2)(N-1)/2} \frac{\Gamma(k) - (N-1)}{\Delta^{(k-1)}(\mu) \Delta^{(k-1)}(\lambda)} \left(e^{s_N \sum_i \mu_i (1 - k)}\right)^{k-1} e^{-\frac{(k-1)(N-1)}{2} |\mu| |\mu| \Delta(e^\mu, e^\lambda)^{k-1}},
\]

where in the last line we use that \( e^{-|\mu| |\mu|} T_{\mu}(k) e^{-|\mu| |\mu|} = T_{\mu}(k) + c \). We conclude that

\[
\Psi_k(\lambda, s) = (-1)^{(k-1)(N+2)(N-1)/2} \Gamma(k)^{-1} \prod_{1 \leq i < j \leq N-1} (s_i - s_j - a) e^{s_N \sum_i \lambda_i \Delta(e^\mu, e^\lambda)^{1-k}} \prod_i \Phi_k(\mu, s')
\]

Recall that \( L_p^{\text{trig}}(k) = L_p^{\text{trig}}(1 - k) \) for any symmetric polynomial \( p \), which by (2.1.2) implies that

\[
L_p^{\text{trig}}(k) = \Delta^{\text{trig}}(\mu)^{1-2k} \circ L_p^{\text{trig}}(1 - k) \circ \Delta^{\text{trig}}(\mu)^{2k-1}. \tag{2.5.7}
\]

In addition, by Proposition 2.5.2, we have that

\[
L_{\mu_1 \cdots \mu_{N-1}}^{\text{trig}}(1 - k)^t = (-1)^{N-1} \Delta^{\text{trig}}(\mu)^{2-2k} L_{\mu_1 \cdots \mu_{N-1}}^{\text{trig}}(k - 1) \circ \Delta^{\text{trig}}(\mu)^{2k-2}. \tag{2.5.8}
\]
Applying Proposition 2.5.3, we have that
\[
\int_{\mu<\lambda} \frac{\Phi_k(\mu, s')}{\Delta^{\text{trig}}(\mu)^{k-1}} \left( e^{-\sum_i \mu_i} L_{\mu_1 \cdots \mu_{N-1}}^{\text{trig}} (1-k) \right) e^{-\frac{(k-1)(N-1)}{2} |\mu|} \Delta(e^\mu, e^\lambda)^{k-1} \prod_i d\mu_i \\
= (-1)^{(N-1)(k-1)} \int_{\mu<\lambda} e^{-\frac{(k-1)(N-1)}{2} |\mu|} \Delta(e^\mu, e^\lambda)^{k-1} \Delta^{\text{trig}}(\mu)^{2-2k} \\
\left( L_{\mu_1 \cdots \mu_{N-1}}^{\text{trig}} (1-k) e^{-\sum_i \mu_i} \right) e^{-\sum_i \mu_i} \Delta^{\text{trig}}(\mu)^{k-1} \Phi_k(\mu, s') \prod_i d\mu_i \\
= (-1)^{(N-1)(k-1)} \prod_{a=1}^{k-1} \prod_{i=1}^{N-1} (s_i - s_N - a) \int_{\mu<\lambda} \frac{\Delta(e^\mu, e^\lambda)^{k-1}}{\Delta(e^\mu, e^\lambda)^{k-1}} \Phi_k(\mu, s') e^{-\frac{(k-1)}{2} |\mu|} \prod_i d\mu_i,
\]
where we apply adjunction and (2.5.8) in the first equality, (2.5.7) in the second equality, and the inductive hypothesis and (2.5.6) in the last equality. Substituting into our previous expression, we obtain
\[
\Psi_k(\lambda, s) = (-1)^{(k-1)N(N-1)/2} \Gamma(k)^{(N-1)} \prod_{a=1}^{k-1} \prod_{1 \leq i < j \leq N} (s_i - s_j - a) \\
\int_{\mu<\lambda} \frac{\Delta(e^\mu, e^\lambda)^{k-1}}{\Delta(e^\mu, e^\lambda)^{k-1}} e^{\sigma_N((\lambda^-|\mu|)} \Phi_k(\mu, s) e^{-\frac{(k-1)}{2} |\mu|} \prod_i d\mu_i \\
= \prod_{a=1}^{k-1} \prod_{1 \leq i < j \leq N} (s_i - s_j - a) \Phi_k(\lambda, s),
\]
which implies the theorem by normalizing via Theorem 2.1.4.

### 2.6 Proofs of some technical lemmas

#### 2.6.1 Proof of Lemma 2.3.13

For a subset $I$ of indices, denote by $1_I$ and $2_I$ the vectors with 1 and 2 in the indices of $I$ and 0 elsewhere. We first expand the Macdonald difference operators in $\log(q_m)$,
yielding

\[ D'_{N,q_m}^{2\lambda_m+2k\rho}(q_m^2, q_m^{2k}) f_m(\lambda_m; q_m) \]

\[ = q_m^{2r(r-n)k} \sum_{|l|=r} \prod_{i \in I} q_m^{q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))}} - q_m^{-k} f_m(\lambda_m + 1_i; q_m) \]

\[ = \sum_{|l|=r} \prod_{i \in I, j \notin I} \left( 1 + (1 - q_m^k) \frac{q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))} + q_m^{-k}}{1 - q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))}} \right) f_m(\lambda_m + 1_I; q_m) \]

\[ = \sum_{|l|=r} \left( 1 + \sum_{i \in I, j \notin I} (1 - q_m^k) \frac{q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))} + q_m^{-k}}{1 - q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))}} + C_r(\lambda_m, q_m \log(q_m)^2) \right) f_m(\lambda_m + 1_I; q_m) + O(\log(q_m)^3) \]

for some functions \( C_r(\lambda_m, q_m) = o(\log(q_m)^{-1}) \). Specializing this, we see that

\[ D'_{N,q_m}^{2\lambda_m+2k\rho}(q_m^2, q_m^{2k}) f_m(\lambda_m; q_m) \]

\[ = \sum_{i=1}^N \left( 1 + \sum_{j \neq i} (1 - q_m^k) \frac{q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))} + q_m^{-k}}{1 - q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))}} + C_1(\lambda_m, q_m \log(q_m)^2) \right) f_m(\lambda_m + 1_i; q_m) + O(\log(q_m)^3) \]

and

\[ D'_{N,q_m}^{2\lambda_m+2k\rho}(q_m^2, q_m^{2k})^2 f_m(\lambda_m; q_m) \]

\[ = \sum_{i=1}^N \left( 1 + S_1(\lambda_m, q_m \log(q_m)^2) \right) f_m(\lambda_m + 2_i; q_m) \]

\[ + \sum_{i_1 \neq i_2} \left( 1 + S_2(\lambda_m, q_m \log(q_m)^2) \right) f_m(\lambda_m + 1_{i_1,i_2}; q_m) + O(\log(q_m)^2) \]

\[ + (1 - q_m^k) \sum_{i=1}^N \sum_{j \neq i} \left( \frac{q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))}}{1 - q_m^{2(\lambda_m,i-\lambda_m,j+k(j-i))}} + \frac{q_m^{2(\lambda_m,i+1-\lambda_m,j+k(j-i))} + q_m^{-k}}{1 - q_m^{2(\lambda_m,i+1-\lambda_m,j+k(j-i))}} \right) \]

\[ f_m(\lambda_m + 2_i; q_m) \]

\[ + (1 - q_m^k) \sum_{i_1 \neq i_2} \sum_{j \neq i_1,i_2} \left( \frac{q_m^{2(\lambda_m,i_1-\lambda_m,j+k(j-i_2))}}{1 - q_m^{2(\lambda_m,i_1-\lambda_m,j+k(j-i_2))}} + \frac{q_m^{2(\lambda_m,i_1+1-\lambda_m,j+k(j-i_2))} + q_m^{-k}}{1 - q_m^{2(\lambda_m,i_1+1-\lambda_m,j+k(j-i_2))}} \right) \]

\[ f_m(\lambda_m + 1_{i_1,i_2}; q_m) \]

\[ + (1 - q_m^k) \sum_{i_1 \neq i_2} \left( \frac{q_m^{2(\lambda_m,i_2-\lambda_m,i_1+k(i_2-i_1))} + q_m^k}{1 - q_m^{2(\lambda_m,i_2-\lambda_m,i_1+k(i_2-i_1))}} + \frac{q_m^{2(\lambda_m,i_2+1-\lambda_m,i_1+k(i_2-i_1))} + q_m^{-k}}{1 - q_m^{2(\lambda_m,i_2+1-\lambda_m,i_1+k(i_2-i_1))}} \right) \]

\[ f_m(\lambda_m + 1_{i_1,i_2}; q_m) \]

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for some functions $S_1(\lambda_m, q_m)$ and $S_2(\lambda_m, q_m)$, both of which are $o(\log(q_m)^{-1})$. We define

$$A_{i_1, i_2}(\lambda_m, q_m) = \frac{1}{1 - q_m^2} \left( \frac{2(\lambda_m, i_2 - \lambda_m, i_1 + k(i_1 - i_2)) + q_m^{-k}}{2(\lambda_m, i_2 - \lambda_m, i_1 + k(i_1 - i_2)) + q_m^{-k}} + \frac{2(\lambda_m, i_1 - \lambda_m, i_2 - 1 + k(i_2 - i_1)) + q_m^{-k}}{1 - q_m} \right)$$

$$B_{i, j}(\lambda_m, q_m) = \frac{2(\lambda_m, i - \lambda_m, j + k(j - i)) + q_m^{-k}}{2(\lambda_m, i - \lambda_m, j + k(j - i)) + q_m^{-k}} + \frac{2(\lambda_m, i + 1 - \lambda_m, j + k(j - i)) + q_m^{-k}}{1 - q_m^{2(\lambda_m, i + 1 - \lambda_m, j + k(j - i))}}$$

so that

$$\sum_{i_1 \neq i_2} \left( \frac{q_m}{1 - q_m} \frac{2(\lambda_m, i_2 - \lambda_m, i_1 + k(i_1 - i_2)) + q_m^{-k}}{2(\lambda_m, i_2 - \lambda_m, i_1 + k(i_1 - i_2)) + q_m^{-k}} + \frac{2(\lambda_m, i_1 - \lambda_m, i_2 - 1 + k(i_2 - i_1)) + q_m^{-k}}{1 - q_m} \right) f_m(\lambda_m + 1, i_1, i_2; q_m)
\quad
= (1 - q_m^2) \sum_{i_1 \neq i_2} A_{i_1, i_2}(\lambda_m, q_m) f_m(\lambda_m + 1, i_1, i_2; q_m) + O(\log(q_m)^2)$$

and

$$\sum_{j \neq i} \left( \frac{q_m}{1 - q_m} \frac{2(\lambda_m, i - \lambda_m, j + k(j - i)) + q_m^{-k}}{2(\lambda_m, i - \lambda_m, j + k(j - i)) + q_m^{-k}} + \frac{2(\lambda_m, i + 1 - \lambda_m, j + k(j - i)) + q_m^{-k}}{1 - q_m^{2(\lambda_m, i + 1 - \lambda_m, j + k(j - i))}} \right) f_m(\lambda_m + 2i; q_m)
\quad
= \sum_{j \neq i} B_{i, j}(\lambda_m, q_m) f_m(\lambda_m + 2i; q_m).$$

Notice that

$$\lim_{m \to \infty} A_{i_1, i_2}(\lambda_m, q_m) = \frac{ke^{2\lambda_1 - 2\lambda_2} - 2(k - 2)e^{\lambda_1 - \lambda_2} + k}{(1 - e^{\lambda_1 - \lambda_2})^2}$$

and

$$\lim_{m \to \infty} B_{i, j}(\lambda_m, q_m) = \frac{2(1 + e^{\lambda_1 - \lambda_j})}{1 - e^{\lambda_1 - \lambda_j}}.$$

We have also that

$$D^2_{N, q_m^{2\lambda_m + 2kP}}(\lambda_m, q_m^{2k}) f_m(\lambda_m; q_m)
\quad
= \sum_{i_1 \neq i_2} \left( 1 + C(\lambda_m, q_m) \log(q_m)^2 \right) f_m(\lambda_m + 1, i_1, i_2; q_m) + O(\log(q_m)^2)
\quad
+ (1 - q_m^k) \sum_{i_1 \neq i_2} \sum_{j \neq i_1, i_2} \left( \frac{q_m}{1 - q_m} \frac{2(\lambda_m, i_2 - \lambda_m, j + k(j - i_1)) + q_m^{-k}}{2(\lambda_m, i_2 - \lambda_m, j + k(j - i_1)) + q_m^{-k}} + \frac{2(\lambda_m, i_1 - \lambda_m, i_2 - 1 + k(j_2 - i_1)) + q_m^{-k}}{1 - q_m} \right) f_m(\lambda_m + 1, i_1, i_2; q_m)
\quad
+ (1 - q_m^k) \sum_{i_1 \neq i_2} \sum_{j \neq i_1, i_2} \left( \frac{q_m}{1 - q_m} \frac{2(\lambda_m, i_2 - \lambda_m, j + k(j - i_2)) + q_m^{-k}}{2(\lambda_m, i_2 - \lambda_m, j + k(j - i_2)) + q_m^{-k}} + \frac{2(\lambda_m, i_1 - \lambda_m, i_2 - 1 + k(j_2 - i_1)) + q_m^{-k}}{1 - q_m} \right) f_m(\lambda_m + 1, i_1, i_2; q_m).$$
Together, these imply that
\[
D_{\lambda_m}(q_m) f_m(\lambda_m; q_m)
= \sum_{i=1}^{N} \left( 1 + (1 - q_m^k) \sum_{j \neq i} B_{i,j}(\lambda_m, q_m) + S_1(\lambda_m, q_m) \log(q_m)^2 \right) f_m(\lambda_m + 2i; q_m)
- 2 \sum_{i=1}^{N} \left( 1 + \sum_{j \neq i} (1 - q_m^k) \frac{q_m^{2(\lambda_m, i = \lambda_m-j+k(j-i))} + q_m^{-k}}{1 - q_m^{2(\lambda_m, i = \lambda_m-j+k(j-i))}} \right) f_m(\lambda_m + 1i; q_m)
+ (1 - q_m^k)(1 - q_m^2) \sum_{A_{i_1,i_2}} A_{i_1,i_2}(\lambda_m, q_m) f_m(\lambda_m + 1i_1,i_2; q_m)
+ (C_2(\lambda_m, q_m) - S_2(\lambda_m, q_m)) \log(q_m)^2 f_m(\lambda_m + 1i_1,i_2; q_m)
+ N f_m(\lambda_m; q_m) + O(\log(q_m)^2).
\]

Taking limits in the previous expression yields that
\[
\lim_{m \to \infty} (-2 \log(q_m))^{-2} D_{\lambda_m}(q_m) f_m(\lambda_m; q_m)
= \Delta f(\lambda) - k^2 \sum_{i \neq j} \frac{1 + e^{\lambda_i - \lambda_j}}{1 - e^{\lambda_i - \lambda_j}} \partial_i f(\lambda) + R(\lambda) f(\lambda)
= \left( \Delta - k \sum_{i < j} \frac{1 + e^{\lambda_i - \lambda_j}}{1 - e^{\lambda_i - \lambda_j}} (\partial_i - \partial_j) + R(\lambda) \right) f(\lambda)
\]
for some function \( R(\lambda) \). Note that \( f_m(\lambda_m) \equiv 1 \) is the Macdonald polynomial in \( q^{2\lambda_m} \) corresponding to the empty partition, hence we conclude that
\[
D_\lambda(q) \cdot 1 = p_2(q^{2\rho}) - 2p_1(q^{2\rho}) + N = \sum_i (q^{2\rho_i} - 1)^2,
\]
which implies that
\[
\lim_{m \to \infty} (-2 \log(q_m))^{-2} D_{\lambda_m}(q_m) \cdot 1 = k^2(\rho, \rho),
\]
hence \( R(\lambda) \equiv k^2(\rho, \rho) \). We conclude that
\[
\lim_{m \to \infty} (-2 \log(q_m))^{-2} D_{\lambda_m}(q_m) f_m(\lambda_m; q_m) = \left( \Delta - k \sum_{i < j} \frac{1 + e^{\lambda_i - \lambda_j}}{1 - e^{\lambda_i - \lambda_j}} (\partial_i - \partial_j) + k^2(\rho, \rho) \right) f(\lambda),
\]
where we note that by [61, Theorem 2.1.1], we have
\[
T_{p_2}^{\text{trig}}(k) = \Delta - k \sum_{i < j} \frac{1 + e^{\lambda_i - \lambda_j}}{1 - e^{\lambda_i - \lambda_j}} (\partial_i - \partial_j) + k^2(\rho, \rho).
\]
2.6.2 Proof of Lemma 2.4.4

We verify the statement by direct computation. Use the notations \( u = u(\mu, \lambda) \) and \( \lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \). Define the non-negative real numbers \( x_1, \ldots, x_{N-1} \) by

\[
x_i^2 = -\frac{\prod_j (\lambda_j - \mu_i)}{\prod_{j \neq i} (\mu_j - \mu_i)},
\]

where we note that the right side of the definition is non-negative because \( \lambda \) and \( \mu \) interlace. Define \( y = \sum_i \lambda_i - \sum_i \mu_i \). For \( i < N \), our definition of \( u \) implies that

\[
u_{ij} = \frac{x_i}{\lambda_j - \mu_i} u_{Nj}.
\]

We first claim that \( uA = \mu' u \) for the matrix

\[
\mu' = \begin{pmatrix}
\mu_1 & \cdots & x_1 \\
\mu_2 & \cdots & x_2 \\
\vdots & \ddots & \vdots \\
\mu_{N-2} & \cdots & x_{N-2} \\
x_1 & \cdots & x_{N-2} & x_{N-1} & y
\end{pmatrix}.
\]

For \( i < N \), this holds for each element of row \( i \) by the equality

\[
\lambda_j u_{ij} = \mu_j u_{ij} + x_i u_{Nj}
\]

implied by (2.6.1). For row \( N \), we must check that

\[
\lambda_j u_{Nj} = \sum_{i=1}^{N-1} x_i u_{ij} + y u_{Nj} = \left( y + \sum_{i=1}^{N-1} \frac{x_i^2}{\lambda_j - \mu_i} \right) u_{Nj},
\]

for which it suffices to check that

\[
\sum_{i=1}^{N-1} \frac{\prod_{j \neq i} (\lambda_i - \mu_i)}{\prod_{j \neq i} (\mu_i - \mu_j)} = \sum_{i=1}^{N-1} \sum_{i \neq j} \lambda_i - \sum_{i} \mu_i.
\]

The left side of (2.6.2) is a symmetric rational function in the \( \mu_i \) which may be expressed as a quotient

\[
P(\mu) = \frac{P(\mu)}{\prod_{i < j} (\mu_i - \mu_j)},
\]

whose numerator \( P(\mu) \) has degree at most \( N(N-1)/2 + 1 \) in the \( \mu \)-variables. Therefore, \( P(\mu) \) is antisymmetric, meaning the quotient is symmetric of degree at most 1. In particular, it takes the form \( C_1 + C_2 \sum_i \mu_i \) for \( C_1 \) and \( C_2 \) constant in \( \mu \). Noting that the coefficient of \( \mu_1^{N-1} \mu_2^{N-3} \mu_3^{N-4} \cdots \mu_{N-2} \) in \( P(\mu) \) is \(-1\) shows that \( C_2 = -1 \). Finally,
$C_1$ is a polynomial of degree 1 in $\lambda$, so it is given by

$$C_1 = \sum_{i} \frac{\mu_i^{N-2}(-1)^{N-2} \sum_{l \neq j} \lambda_l}{\prod_{l \neq i}(\mu_l - \mu_i)} = \sum_{i} \frac{\mu_i^{N-2}}{\prod_{l \neq i}(\mu_l - \mu_i)} \cdot \left(\sum_{l \neq i} \lambda_l\right) = \sum_{i \neq j} \lambda_i,$$

where the last equality follows by noting that $\sum_{i} \frac{\mu_i^{N-2}}{\prod_{l \neq i}(\mu_l - \mu_i)}$ is symmetric of degree 0 in $\mu$ and a rational function whose denominator is $\prod_{i \neq j}(\mu_i - \mu_j)$ and whose numerator contains $\mu_1^{N-2} \mu_2^{N-3} \cdots \mu_{N-2}$ with coefficient 1. This establishes (2.6.2).

It remains to check that $u$ is unitary. For this, we check that the columns of $u$ are orthonormal. Choose any $1 \leq a < b \leq N$. We have that

$$\sum_{i} u_{ia} u_{ib} = \left(\sum_{i} \frac{x_i^2}{(\lambda_a - \mu_i)(\lambda_b - \mu_i)} + 1\right) u_{Na} u_{Nb} = \left(1 - \sum_{i} \frac{\prod_{j \neq a,b}(\lambda_j - \mu_i)}{\prod_{j \neq i}(\mu_j - \mu_i)}\right) u_{Na} u_{Nb}.$$

Observe that $\sum_{i} \frac{\prod_{j \neq a,b}(\lambda_j - \mu_i)}{\prod_{j \neq i}(\mu_j - \mu_i)}$ is symmetric in the $\mu_i$ and may be expressed as a rational function with denominator $\prod_{i < j}(\mu_i - \mu_j)$ and numerator of degree at most $\frac{N(N-1)}{2}$ in $\mu$. Further, the coefficient of $\mu_1^{N-2} \mu_2^{N-3} \cdots \mu_{N-2}$ in the numerator is 1, so we conclude that

$$1 - \sum_{i} \frac{\prod_{j \neq a,b}(\lambda_j - \mu_i)}{\prod_{j \neq i}(\mu_j - \mu_i)} = 0,$$

hence $\sum_{i} u_{ia} u_{ib} = 0$. It remains only to show that

$$1 = \sum_{i} u_{ia}^2 = \left(1 + \sum_{i} \frac{x_i^2}{(\lambda_a - \mu_i)^2}\right) u_{Na}^2,$$

for which we must check that

$$\frac{\prod_{l \neq a}(\lambda_l - \lambda_a)}{\prod_l(\mu_l - \lambda_a)} = 1 - \sum_{i} \frac{\prod_{j \neq a}(\lambda_j - \mu_i)}{(\lambda_a - \mu_i) \prod_{j \neq i}(\mu_j - \mu_i)},$$

which is equivalent to

$$\prod_{l \neq a}(\lambda_l - \lambda_a) = \prod_l(\mu_l - \lambda_a) \left(1 - \sum_{i} \frac{\prod_{j \neq a}(\lambda_j - \mu_i)}{(\lambda_a - \mu_i) \prod_{j \neq i}(\mu_j - \mu_i)}\right).$$

View both sides of (2.6.4) as polynomials in $\lambda_a$. If $\lambda_a = \lambda_b$ for $b \neq a$, the right side becomes

$$1 - \sum_{i} \frac{\prod_{j \neq a,b}(\lambda_j - \mu_i)}{\prod_{j \neq i}(\mu_j - \mu_i)} = 0$$

by (2.6.3). Therefore, both sides of (2.6.4) are polynomials in $\lambda_a$ of the same degree with the same roots and the same leading coefficient $(-1)^{N-1}$, so they are equal,
completing the proof.

Remark. The expressions above for $x_i^2$ and $y$ appeared previously in [81]. Similar computations appeared also in [54, 51].

### 2.6.3 Proof of Proposition 2.4.5

Before beginning the proof, we outline our approach. We first obtain an alternate expression for $Z_1(\mu, \lambda)$ in Lemma 2.6.1. We then observe that $Z_k(\mu, \lambda)$ is a constant multiple of $Z_1(\mu', \lambda')$ for sets of variables $\mu'$ and $\lambda'$ which contain $k$ duplicate copies of each value of $\mu$ and $\lambda$. Relating Calogero-Moser Hamiltonians at different values of $k$ in Lemma 2.6.2 leads to the result. Recall here that $D_{\mu_1}(\kappa)$ denotes the rational Dunkl operator of (2.4.1).

**Lemma 2.6.1.** For any $\kappa \in \mathbb{C}$, we have

$$\Delta(\mu, \lambda)^{-\kappa} D_{\mu_{N-1}}(-\kappa) \cdots D_{\mu_1}(-\kappa) \Delta(\mu, \lambda)^\kappa = \kappa^{N-1} Z_1(\mu, \lambda).$$

**Proof.** We first claim that

$$\Delta(\mu, \lambda)^{-\kappa} D_{\mu_a}(-\kappa) \cdots D_{\mu_1}(-\kappa) \Delta(\mu, \lambda)^\kappa = \kappa^a \sum_{\sigma \in \{1, \ldots, a\} \rightarrow \{1, \ldots, d\}} \prod_{i=1}^{a} (\mu_i - \lambda_{\sigma(i)})^{-1}. \quad (2.6.5)$$

Taking $a = N - 1$ in (2.6.5) and expanding the product in the definition of $Z_1(\mu, \lambda)$ then completes the proof. We prove (2.6.5) by induction on $a$. The base case $a = 1$ holds because $D_{\mu_1}(-\kappa)$ acts by $\partial_1$ on the symmetric function $\Delta(\mu, \lambda)^\kappa$ in $\mu$. For the inductive step, note that $D_{\mu_a}(-\kappa) \cdots D_{\mu_1}(-\kappa) \Delta(\mu, \lambda)^\kappa$ is symmetric in $\mu_{a+1}, \ldots \mu_{N-1}$ by the inductive hypothesis. Applying $D_{\mu_{a+1}}(-\kappa)$, we see that

$$\Delta(\mu, \lambda)^{\kappa} D_{\mu_{a+1}}(-\kappa)(D_{\mu_a}(-\kappa) \cdots D_{\mu_1}(-\kappa) \Delta(\mu, \lambda)^\kappa)$$

$$= \kappa^{a+1} \sum_{j=1}^{a+1} (\mu_{a+1} - \lambda_j)^{-1} \sum_{\sigma \in \{1, \ldots, a\} \rightarrow \{1, \ldots, d\}} \prod_{i=1}^{a} (\mu_i - \lambda_{\sigma(i)})^{-1}$$

$$- \kappa^{a+1} \sum_{\sigma \in \{1, \ldots, a\} \rightarrow \{1, \ldots, d\}} \prod_{i=1}^{a} (\mu_i - \lambda_{\sigma(i)})^{-1} \sum_{i=1}^{a+1} (\mu_{a+1} - \lambda_{\sigma(i)})^{-1}$$

$$= \kappa^{a+1} \sum_{\sigma \in \{1, \ldots, a+1\} \rightarrow \{1, \ldots, d\}} \prod_{i=1}^{a+1} (\mu_i - \lambda_{\sigma(i)})^{-1},$$

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where we repeatedly make use of the identity
\[ \frac{1}{\mu_{a+1} - \mu_i} \left( (\mu_{a+1} - \lambda_j) - (\mu_i - \lambda_j) \right) = 1. \]

**Proof of Proposition 2.4.5.** Replace \( l \) by \( kl \) and apply Lemma 2.6.1 with \( \kappa = \frac{1}{k}, k \) copies of each \( \lambda_j \), and \( k(N - 1) \) different variables \( \mu_1^k, \ldots, \mu_{N-1}^k, \ldots, \mu_{N-1}^k \). We obtain
\[
\Delta((\mu_j^k), \{\lambda_i\}) D_{\mu_{N-1}^k}(-1/k) \cdots D_{\mu_1^k}(-1/k) \Delta((\mu_j^k), \{\lambda_i\}) = k^{-(N-1)k} Z_k((\mu_j^k), \{\lambda_i^k\}).
\]

(2.6.6)

Now, make the specialization \( \mu_1^k = \cdots = \mu_N^k = \mu_1, \ldots, \mu_{N-1}^k = \cdots = \mu_{N-1}^k = \mu_{N-1} \).

We first claim that
\[
Z_k((\mu_j^k), \{\lambda_i^k\}) = k!^{(N-1)} Z_k((\mu_i), \{\lambda_i\})
\]
under this specialization. Indeed, we see that
\[
Z_k((\mu_j^k), \{\lambda_i^k\}) = \sum_{|\sigma|=k} \prod_{i,j} (\mu_i - \lambda_j)^{-1}
\]
\[
= k!^{(N-1)} \prod_{i,j} (\mu_i - \lambda_j)^{-1}
\]
which is a direct expansion of \( Z_k((\mu_i), \{\lambda_i\}) \). The conclusion will now follow from Lemma 2.6.2, which describes what occurs under specialization to the other side of Lemma 2.6.1. Indeed, applying Lemma 2.6.2 for \( p(y) = y_1^k \cdots y_{N-1}^k \) to (2.6.6), we see that
\[
Z_k((\mu_i), \{\lambda_i\}) = k!^{-(N-1)} k^{-(N-1)k} \Delta((\mu_i), \{\lambda_i\})^k D_{\mu_{N-1}}(-k) \cdots D_{\mu_1}(-k) \Delta((\mu_i), \{\lambda_i\})^k
\]
\[
= k!^{-(N-1)} \Delta((\mu_i), \{\lambda_i\})^k D_{\mu_{N-1}}(-k) \cdots D_{\mu_1}(-k) \Delta((\mu_i), \{\lambda_i\})^k.
\]

\( \square \)

**Lemma 2.6.2.** Let \( p \in C[y_1^k, \ldots, y_{N-1}^k]^{S_k(N-1)} \) be a symmetric polynomial. Then the map \( \text{Res}_k : C[\mu_i] \to C[\mu_i] \) given by \( \mu_i^k \mapsto \mu_i \) satisfies
\[
\text{Res}_k \circ p(D_{\mu_i^k}(-k), \ldots, D_{\mu_{N-1}^k}(-k)) = p\left( \frac{1}{k} D_{\mu_1}(-k), \ldots, \frac{1}{k} D_{\mu_1}(-k), \ldots, \frac{1}{k} D_{\mu_1}(-k), \frac{1}{k} D_{\mu_{N-1}}(-k) \right) \circ \text{Res}_k.
\]

**Proof.** For any \( c \) and \( n \), let \( H_{c,n} \) denote the rational Cherednik algebra associated to
$S_n$ with parameter $c$, given in terms of generators and relations by

$$H_{c,n} := \left\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid [x_i, x_j] = [y_i, y_j] = 0, [y_i, x_i] = \delta_{ij} - c \sum_{j \neq i} s_{ij}, [y_i, x_j] = cs_{ij} \right\rangle.$$

Let $H_{1/k,(N-1)k}$ and $H_{k,N-1}$ denote the rational Cherednik algebras of $S_{(N-1)k}$ and $S_{N-1}$, respectively. Within $H_{1/k,(N-1)k}$ and $H_{k,N-1}$, denote the power sums $p_a(x) = \sum_{i,j}(x_i^a)^a$ and $p'_a(x) = \sum_i x_i^a$, and define $p_a(y), p'_a(y)$ similarly. Write $\Theta_{1/k,(N-1)k} : H_{1/k,(N-1)k} \to \text{End}(\mathbb{C}[\mu_i])$ and $\Theta_{k,N-1} : H_{k,N-1} \to \text{End}(\mathbb{C}[\mu_i])$ for the Dunkl embeddings induced by $\Theta_{1/k,(N-1)k}(x_i^j) = \mu_i^j, \Theta_{1/k,(N-1)k}(y_i^j) = D_{\mu_i}(-1/k), \Theta_{k,N-1}(x_i) = kx_i$, and $\Theta_{k,N-1}(y_i) = \frac{1}{k} D_{\mu_i}(-k)$. In this language, we wish to show that

$$\text{Res}_k \circ \Theta_{1/k,(N-1)k}(p_a(y)) = \Theta_{k,N-1}(p'_a(y)) \circ \text{Res}_k. \quad (2.6.7)$$

Suppose first that the statement held for $p_2(y)$. Then, we have for any $a$ that

$$\text{Res}_k \circ \Theta_{1/k,(N-1)k}(\text{ad}^a_{p_2(y)} p_a(x)) = \Theta_{k,N-1}(\text{ad}^a_{p'_2(y)} p'_a(x)) \circ \text{Res}_k \quad (2.6.8)$$

Recall that for $h = \frac{1}{2} \sum_{i,j}(x_i y_i x_i y_i x_i + y_i x_i y_i x_i)$ and $h' = \frac{1}{2} \sum_i(x_i y_i + y_i x_i)$, the triples

$$(f, e, h) = \left( \frac{1}{2} p_2(y), -\frac{1}{2} p_2(x), h \right) \quad \text{and} \quad (f', e', h') = \left( \frac{1}{2} p'_2(y), -\frac{1}{2} p'_2(x), h' \right)$$

are copies of $\mathfrak{sl}_2$ inside $H_{1/k,(N-1)k}$ and $H_{k,N-1}$ corresponding to the $SL_2(\mathbb{C})$-actions given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_i = ax_i + by_i, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} y_i = cx_i + dy_i,$$

and similar formulas for $x_i^j, y_i^j$. In particular, $p_a(x)$ and $p'_a(x)$ are highest weight vectors of weight $a$ for these representations, so $\text{ad}^a_{p_2(y)/2} p_a(x)$ and $\text{ad}^a_{p'_2(y)/2} p'_a(x)$ are the same fixed constant multiple of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p_a(x) = p_a(y) \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p'_a(x) = p'_a(y),$$

respectively. Combining with (2.6.8) and canceling common constant factors yields the desired relation (2.6.7).

It remains to check the statement for $p_2(y)$ directly. Observe that

$$\text{Res}_k \circ \sum_j \partial_{\mu_i^j} = \partial_{\mu_i} \circ \text{Res}_k,$$

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which implies that
\[
\text{Res}_k \left( \sum_{j_1, j_2} \frac{\partial \mu_{j_1} - \partial \mu_{j_2}}{\mu_{j_1} - \mu_{j_2}} f \right) = k \frac{\partial \mu_{i_1} - \partial \mu_{i_2}}{\mu_{i_1} - \mu_{i_2}} \text{Res}_k(f). \tag{2.6.9}
\]

For a partition \( \tau \) with at most \( k \) parts, let \( m_\tau(\mu_i^j) \) be the monomial symmetric function in \( \mu_i^1, \ldots, \mu_i^k \). Then we see that
\[
\text{Res}_k \left( \left( \sum_j \frac{\partial^2 \mu_i^j}{\mu_i^j} - \frac{2}{k} \sum_{j_1 < j_2} \frac{\partial \mu_{i_1} - \partial \mu_{i_2}}{\mu_{i_1} - \mu_{i_2}} \right) m_\tau(\mu_i^j) \right)
\]
\[
= \left( \sum_j \tau_j (\tau_j - 1) - \frac{2}{k} \sum_{j_1 < j_2} \frac{1}{2} \left( \tau_{j_1} (\tau_{j_1} - 1 - \tau_{j_2}) + \tau_{j_2} (\tau_{j_2} - 1 - \tau_{j_1}) \right) \right) k! \mu_i^{\left| \tau \right| - 2}
\]
\[
= \frac{1}{k} \sum_i \tau_i (\tau_i - 1) + \frac{1}{k} \sum_{j_1 < j_2} \tau_{j_1} \tau_{j_2} \left( \mu_i^j \right)^{-2} \text{Res}_k(\mu_\lambda(\mu_i^j))
\]
\[
= \frac{1}{k} \text{Res}_k(m_\tau(\mu_i^j)). \tag{2.6.10}
\]

Combining (2.6.9) and (2.6.10), the statement for \( p_2(y) \) follows by computing
\[
\text{Res}_k \circ \mathcal{L}_{p_2}(-1/k) = \text{Res}_k \circ \left( \sum_i \frac{\partial^2 \mu_i}{\mu_i} - \frac{2}{k} \sum_{(j_1, j_2) < (j_2, j_2)} \frac{\partial \mu_{j_1} - \partial \mu_{j_2}}{\mu_{j_1} - \mu_{j_2}} \right)
\]
\[
= \text{Res}_k \circ \left( \sum_i \left( \sum_j \frac{\partial^2 \mu_i^j}{\mu_i^j} - \frac{2}{k} \sum_{j_1 < j_2} \frac{\partial \mu_{i_1} - \partial \mu_{i_2}}{\mu_{i_1} - \mu_{i_2}} \right) - \frac{2}{k} \sum_{i_1 \neq i_2} \sum_{j_1, j_2} \frac{\partial \mu_{j_1} - \partial \mu_{j_2}}{\mu_{j_1} - \mu_{j_2}} \right)
\]
\[
= \frac{1}{k} \left( \sum_i \frac{\partial^2 \mu_i}{\mu_i} - 2k \sum_{i_1 \neq i_2} \frac{\partial \mu_{i_1} - \partial \mu_{i_2}}{\mu_{i_1} - \mu_{i_2}} \right) \circ \text{Res}_k
\]
\[
= \frac{1}{k} \mathcal{L}_{p_2}(-k) \circ \text{Res}_k. \quad \square
\]

**Remark.** Lemma 2.6.2 may be extracted from [11, Proposition 9.5(ii)] on representations of the rational Cherednik algebras \( H_{1/k,(N-1)_k} \) and \( H_{k,N-1} \). We give a proof to keep the exposition self-contained.
Chapter 3

Representation-theoretic proof of the Macdonald branching rule

3.1 Introduction

The Macdonald polynomials $P_\lambda(x; q, t)$ are a two-parameter family of symmetric functions indexed by partitions $\lambda$ which form an orthogonal basis for the ring of symmetric functions with respect to a $(q,t)$-deformation of the standard inner product. They were originally introduced by Macdonald (see [76]) as a generalization of many known families of special functions, including Schur functions, Jack and Hall-Littlewood polynomials, and Heckman-Opdam hypergeometric functions. Macdonald proved a branching rule for the $P_\lambda(x; q, t)$ and conjectured three additional symmetry, evaluation, and norm identities collectively known as Macdonald's conjectures. These conjectures were proven by Cherednik using techniques from double affine Hecke algebras in [12]. Etingof and Kirillov Jr. realized the Macdonald polynomials in [26] in terms of traces of intertwiners of the quantum group $U_q(\mathfrak{gl}_n)$; using this interpretation, they gave new proofs of Macdonald's conjectures in [29].

The purpose of this chapter is to give a representation-theoretic proof and interpretation of Macdonald's branching rule from the perspective of quantum groups. We give a new expression for diagonal matrix elements of $U_q(\mathfrak{gl}_n)$-intertwiners in the Gelfand-Tsetlin basis as the application of Macdonald's difference operators to a simple kernel. We then show that the resulting summation expression for $P_\lambda(x; q, t)$ becomes Macdonald's branching rule after a summation by parts procedure. A key ingredient which is of independent interest is the construction of a map $\text{Res}_s$ between spherical parts of double affine Hecke algebras of different ranks. Our construction makes essential use of the Dunkl-Kasatani conjecture stated in [19, 66] and proven in [20, 34] and is compatible with Cherednik's $SL_2(\mathbb{Z})$-action on spherical DAHA.

In the remainder of the introduction, we summarize our motivations, give precise statements of our results, and explain how they relate to other recent work. This chapter is based on the paper [95].
3.1.1 Macdonald polynomials

Let $\rho = \left( \frac{n-1}{2}, \ldots, \frac{1-n}{2} \right)$ and let $e_r$ denote the elementary symmetric polynomial. For a partition $\lambda$, the Macdonald polynomial $P_\lambda(x; q^2, t^2)$ is the joint polynomial eigenfunction with leading term $x^\lambda$ and eigenvalue $e_r(q^{2\lambda}t^{2\rho})$ of the operators

$$D_{n,x}^r(q^2, t^2) = t^{r(r-n)} \sum_{|I|=r} \prod_{i \in I, j \notin I} \frac{t^2 x_i - x_j}{x_i - x_j} T_{q^2, I},$$

where $T_{q^2, I} = \prod_{i \in I} T_{q^2, i}$ and $T_{q^2, i} f(x_1, \ldots, x_n) = f(x_1, \ldots, q^2 x_i, \ldots, x_n)$ so that we have

$$D_{n,x}^r(q^2, t^2) P_\lambda(x; q^2, t^2) = e_r(q^{2\lambda}t^{2\rho}) P_\lambda(x; q^2, t^2).$$

Note that our normalization of $D_{n,x}^r(q^2, t^2)$ differs from that of [76]. An integral signature $\lambda$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i - \lambda_j \in \mathbb{Z}$, and it is dominant if $\lambda_i \geq \lambda_{i+1}$. We extend the definition of Macdonald polynomials to arbitrary signatures by setting

$$P_{(\lambda_1+c, \ldots, \lambda_n+c)}(x; q^2, t^2) = (x_1 \cdots x_n)^c P_\lambda(x; q^2, t^2).$$

We say that dominant integral signatures $\mu = (\mu_1 \geq \cdots \geq \mu_{n-1})$ and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ interlace if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$

Denote interlacing by $\mu \prec \lambda$ and write $|\lambda| = \sum \lambda_i$. A Gelfand-Tsetlin pattern subordinate to $\lambda$ is an interlacing sequence

$$\mu = \{ \mu^i \}_{1 \leq i \leq n} = \{ \mu^1 \prec \mu^2 \prec \cdots \prec \mu^{n-1} \prec \mu^n = \lambda \}$$

ending in $\lambda$. Define the $q$-Pochhammer symbol by

$$(u; q)_\infty = \prod_{n \geq 0} (1 - uq^n).$$

In [76], Macdonald showed that $P_\lambda(x; q, t)$ satisfies the following branching rule, which yields an explicit summation expression for $P_\lambda(x; q, t)$ over Gelfand-Tsetlin patterns subordinate to $\lambda$.

**Theorem 3.1.1 ([76, VI.7.13])**. The Macdonald polynomials satisfy the branching rule

$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\mu \prec \lambda} \psi_{\lambda/\mu}(q, t) P_\mu(x_1, \ldots, x_{n-1}; q, t) x_n^{|\lambda| - |\mu|},$$

where the branching coefficient is

$$\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{(q^{\mu_i - \mu_j + j - i + 1}; q)_\infty}{(q^{\mu_i - \mu_j + 1}; q)_\infty} \frac{(q^{\lambda_i - \mu_j + j - i}; q)_\infty}{(q^{\lambda_i - \mu_j + 1}; q)_\infty} \frac{(q^{\mu_i - \lambda_j + 1}; q)_\infty}{(q^{\mu_i - \lambda_j + 1}; q)_\infty}.$$
Corollary 3.1.2. The Macdonald polynomials admit the summation formula
\[ P_\lambda(x, q, t) = \sum_{\mu_1 \leq \cdots \leq \mu_n = \lambda} \prod_{i=1}^{n} \psi_{[\mu_i]_t}^{-1}(q, t) \prod_{i=1}^{n} x_i^{[\mu_i]_t - [\mu_{i-1}]_t}. \]

3.1.2 The quantum group \( U_q(\mathfrak{gl}_n) \)

For a generic value of \( q^{1/2} \), let \( U_q(\mathfrak{gl}_n) \) be the associative algebra with generators \( e_i, f_i \) for \( i = 1, \ldots, n-1 \) and \( q^{\pm \frac{b_i}{2}} \) for \( i = 1, \ldots, n \) and relations
\[ q^{\frac{b_i}{2}} e_i q^{-\frac{b_i}{2}} = q^{\frac{1}{2}} e_i, \quad q^{\frac{b_i}{2}} e_{i-1} q^{-\frac{b_i}{2}} = q^{-\frac{1}{2}} e_{i-1}, \quad q^{\frac{b_i}{2}} f_i q^{-\frac{b_i}{2}} = q^{-\frac{3}{2}} f_i, \]
\[ q^{\frac{b_i}{2}} f_{i-1} q^{-\frac{b_i}{2}} = q^{\frac{1}{2}} f_{i-1}, \quad [q^{\frac{b_i}{2}}, e_j] = [q^{\frac{b_i}{2}}, f_j] = 0 \text{ for } j \neq i, i-1, \]
\[ [e_i, f_j] = \delta_{ij} \left( q^{h_i - h_{i+1}} - q^{h_{i+1} - h_i} \right), \quad [e_i, e_j] = [f_i, f_j] = 0 \text{ for } |i-j| > 1, \]
\[ q^{\frac{b_i}{2}} \cdot q^{-\frac{b_i}{2}} = 1, \quad e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0, \]
\[ f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \text{ for } |i-j| = 1. \]

We take the coproduct on \( U_q(\mathfrak{gl}_n) \) defined by
\[ \Delta(e_i) = e_i \otimes q^{\frac{h_{i+1} - h_i}{2}} + q^{-\frac{h_{i+1} - h_i}{2}} \otimes e_i, \]
\[ \Delta(f_i) = f_i \otimes q^{\frac{h_{i+1} - h_i}{2}} + q^{-\frac{h_{i+1} - h_i}{2}} \otimes f_i, \]
\[ \Delta(q^{\frac{b_i}{2}}) = q^{\frac{b_i}{2}} \otimes q^{\frac{b_i}{2}}. \]

Denote the subalgebra generated by \( f_i \) and \( q^{\frac{b_i}{2}} \) by \( U_q(\mathfrak{b}_-) \). For each \( r < n \), the subalgebra generated by \( e_1, \ldots, e_{r-1}, f_1, \ldots, f_{r-1} \), and \( q^{\frac{b_1}{2}}, \ldots, q^{\frac{b_r}{2}} \) forms a copy of \( U_q(\mathfrak{gl}_r) \) within \( U_q(\mathfrak{gl}_n) \). Finally, we denote the finite dimensional irreducible \( U_q(\mathfrak{gl}_n) \)-representation corresponding to a dominant integral signature \( \lambda \) by \( L_\lambda \).

3.1.3 Etingof-Kirillov Jr. approach to Macdonald polynomials

In [26], Etingof and Kirillov Jr. gave an interpretation of Macdonald polynomials in terms of representation-valued traces of \( U_q(\mathfrak{gl}_n) \). Let \( W_{k-1} \) denote the \( U_q(\mathfrak{gl}_n) \)-representation
\[ L_{((k-1)(n-1), -(k-1), \ldots, -(k-1))} = \text{Sym}^{(k-1)n}(\mathbb{C}^n) \otimes (\text{det})^{-(k-1)}, \]
and choose an isomorphism \( W_{k-1}[0] \simeq \mathbb{C} \cdot w_{k-1} \) for some \( w_{k-1} \in W_{k-1}[0] \) which spans the 1-dimensional zero weight space \( W_{k-1}[0] \). Define the weight \( \rho_n = \left( \frac{n-1}{2}, \ldots, \frac{1-n}{2} \right) \). Writing \( \rho \) for \( \rho_n \), for a signature \( \lambda \), there exists a unique intertwiner
\[ \Phi_\lambda^n : L_{\lambda + (k-1)\rho} \to L_{\lambda + (k-1)\rho} \otimes W_{k-1} \]
normalized to send the highest weight vector $v_{\lambda+(k-1)\rho}$ in $L_{\lambda+(k-1)\rho}$ to

$$v_{\lambda+(k-1)\rho} \otimes w_{k-1} + \text{(lower order terms)},$$

where (lower order terms) denotes terms of weight lower than $\lambda+(k-1)\rho$ in the first tensor coordinate. Traces of these intertwiners lie in $W_{k-1}[0] = \mathbb{C} \cdot w_{k-1}$ and yield Macdonald polynomials when interpreted as scalar functions via the identification $w_{k-1} \leftrightarrow 1$. Write $x^h$ for $x^h = x_1^{h_1} \cdots x_n^{h_n}$, where in any $U_q(\mathfrak{g}_n)$-representation we interpret $x_i^{h_i}$ as acting on the $\mu$ weight space by $x_i^{h_i}$.

**Theorem 3.1.3** ([26, Theorem 1]). The Macdonald polynomial $P_\lambda(x; q^2, q^{2k})$ is given by

$$P_\lambda(x; q^2, q^{2k}) = \frac{\text{Tr}(\Phi_0^\mu x^h)}{\text{Tr}(\Phi_0^\mu x^h)}.$$

**Proposition 3.1.4** ([26, Main Lemma]). On $L_{(k-1)\rho}$, the trace may be expressed explicitly as

$$\text{Tr}(\Phi_0^\mu x^h) = (x_1 \cdots x_n)^{-\tfrac{1}{2}(k-1)(n-1)} \prod_{s=1}^{k-1} \prod_{i<j} (x_i - q^2 x_j).$$

**Remark.** Our notation for Macdonald polynomials is related to that of [26] via $P_\lambda^{EK}(x; q, t) = P_\lambda(x; q^2, t^2)$.

### 3.1.4 Gelfand-Tsetlin basis

The representation $L_\lambda$ of $U_q(\mathfrak{gl}_n)$ admits a basis $\{v_\mu\}$ indexed by Gelfand-Tsetlin patterns $\mu$ subordinate to $\lambda$. The weight of a basis vector $v_\mu$ is

$$\text{wt}(v_\mu) = \left(|\mu_1^n| - |\mu_1^{n-1}|, \ldots, |\mu_1^2| - |\mu_1|, |\mu_1|\right).$$

It was shown in [98] that these basis vectors may be expressed in terms of lowering operators $d_{r,i}$ in $U_q(\mathfrak{g}_n) \cap U_q(\mathfrak{b}_-)$ applied to the highest weight vector $v_\lambda$. More precisely, we have the following.

**Proposition 3.1.5** ([98, Theorem 2.9]). There exist lowering operators $d_{r,i} \in U_q(\mathfrak{g}_n) \cap U_q(\mathfrak{b}_-)$ so that the Gelfand-Tsetlin basis vectors are given by

$$v_\mu = d_1^{\mu_1} d_2^{\mu_2-\mu_1} \cdots d_n^{\mu_n-\mu_1-\cdots-\mu_2} v_\lambda,$$

where $d_r = d_{r,1}^{\tau_1} \cdots d_{r,r}^{\tau_r}$ for a partition $\tau$.

### 3.1.5 Statement of the main results

Computing the trace of $U_q(\mathfrak{gl}_n)$-intertwiners in Theorem 3.1.3 in the Gelfand-Tsetlin basis of $L_{\lambda+(k-1)\rho}$ yields an expression for $P_\lambda(x; q^2, t^2)$ as a summation over Gelfand-Tsetlin patterns subordinate to $\lambda+(k-1)\rho$. Our main result shows that diagonal
matrix elements of these intertwiners are given by application of Macdonald’s operators to a simple kernel.

**Theorem 3.4.4.** In the Gelfand-Tsetlin basis, the diagonal matrix element of $\Phi^\mu_\lambda$ on the basis vector corresponding to the Gelfand-Tsetlin pattern

$$\{\sigma^1 < \cdots < \sigma^{n-1} < \lambda + (k-1)\rho\}$$

with $\sigma^i_l = \mu_i + (k - 1) \frac{n+1-2i}{2}$ for all $l$ is given by

$$c(\mu, \lambda) = \prod_{i,j} [\mu_i - \mu_j + k(j - i) + k - 1]^{-1}_{k-1} \prod_{i,j} [\lambda_i - \lambda_j + k(j - i) - 1]^{-1}_{k-1}$$

$$\prod_{a=1}^{k-1} D_{n-1,q^2}^a(q^{2\alpha}; q^{-2}, q^{2(k-1)}) \prod_{i,j} [\lambda_i - \mu_j + k(j - i)]^{-1}_{k-1} \prod_{i,j} [\mu_i - \lambda_j + k(j - i) + k - 2]^{-1}_{k-1},$$

where $\bar{\mu}_i = \mu_i - k(i - 1)$, $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$, $[m]_k = [m] \cdots [m - k + 1]$, and

$$D_{n-1,q^2}(u; q^2, t^2) = \sum_{r=0}^{n-1} (-1)^{n-r} u^{n-r} D_{n-1,q^2}^r(q^2, t^2).$$

Using Theorem 3.4.4, we give a new representation-theoretic proof of Macdonald’s branching rule.

**Theorem 3.5.1.** At $t = q^k$ for positive integer $k$, we have

$$P_\lambda(x_1, \ldots, x_n; q^2, q^{2k}) = \sum_{\mu < \lambda} x_n^{[\lambda]_{[\mu]}} P_\mu(x_1, \ldots, x_{n-1}; q^2, q^{2k}) \psi_{\lambda/\mu}(q^2, q^{2k})$$

with

$$\psi_{\lambda/\mu}(q^2, q^{2k}) = \frac{\prod_{i,j} [\lambda_i - \mu_j + k(j - i) + k - 1]^{-1}_{k-1} \prod_{i,j} [\mu_i - \lambda_j + k(j - i) - 1]^{-1}_{k-1}}{\prod_{i,j} [\mu_i - \mu_j + k(j - i) + k - 1]^{-1}_{k-1} \prod_{i,j} [\lambda_i - \lambda_j + k(j - i) - 1]^{-1}_{k-1}}.$$
proof is a map
\[ \text{Res}_i(q^2) : e\mathcal{H}_n(q^{-2}, q^2) e \to e\mathcal{H}_n(q^{-2}, q^{2l}) e \]

between spherical DAHA's of different ranks which results from the Dunkl-Kasatani conjecture of [19, 20, 34, 66]. We show in Theorem 3.3.7 and Corollary 3.3.8 that \( \text{Res}_i(q^2) \) commutes with Cherednik's \( SL_2(\mathbb{Z}) \)-action on DAHA and that it intertwines the map \( \text{Res}_i(q^2) : \mathbb{C}[X_i^\pm, X_i^\pm]^{S_n} \to \mathbb{C}[X_i^\pm, X_i^\pm]^{S_n} \) of spherical polynomial representations given by
\[ \text{Res}_i(q^2) : X_i^a \mapsto q^{2i-t+2a} X_i. \]

**Remark.** Such maps were considered in the rational limit in [11], [25, Theorem 7.11], and [93].

### 3.1.7 Degenerations of our results and connections to recent work

Considering our results under the many degenerations of Macdonald polynomials to other special functions yields some connections to recent literature and some interpretations of independent interest. In this section, we discuss the Heckman-Opdam, Jack, and Hall-Littlewood limits and a generalization to the Macdonald functions of [36].

- In the quasi-classical limit \( q = e^\varepsilon, t = q^k, \lambda = [\varepsilon^{-1}\Lambda], x = e^{\varepsilon X}, \) and \( \varepsilon \to 0 \), the Macdonald polynomials become the Heckman-Opdam hypergeometric functions introduced in [60, 59, 84, 85]. These functions were recently realized as integrals over Gelfand-Tsetlin polytopes in [9] by taking a scaling limit of Corollary 3.1.2. In Chapter 2, the expression of [9] was lifted to an integral over dressing orbits of a Poisson-Lie group by integration over the Liouville tori and an adjunction procedure involving Calogero-Sutherland Hamiltonians. The techniques of this chapter degenerate to the techniques used in Chapter 2 under the degeneration from Macdonald-Ruijsenaars to Calogero-Sutherland integrable systems.

- The Jack polynomials are a scaling limit of Macdonald polynomials under the specialization \( t = q^k \) and the limit \( q \to 1 \) and have a similar branching rule. They were given in [23] as traces of intertwiners of \( U(g\mathfrak{t}_n) \)-modules using a degeneration of the Etingof-Kirillov Jr. construction, under which our methods degenerate to a representation-theoretic proof of the Jack branching rule.

- In the specialization \( q = 0 \), the Macdonald polynomials become the Hall-Littlewood polynomials. In [99], a summation expression was given for matrix elements of the \( U_q(g\mathfrak{t}_n) \)-intertwiners \( \Phi^n_i \) in the Gelfand-Tsetlin basis; this expression factors and becomes particularly simple in the Hall-Littlewood limit. In the notation of [99], \( \Omega_{\beta/\mu}(q^2, q^{2k}) \) can be non-zero only if \( \mu_i - \beta_i \leq \mu_t + (k - 1) \), meaning that the prelimit expression of [99, Theorem 1.3] is a sum over an index set similar to that which appears in Proposition 3.4.3. It would be interesting to understand if the factorization which results from degenerating [99, Theorem 1.3] may be obtained by degenerating our Theorem 3.4.4.
Replacing the finite dimensional module $L_{\lambda+(k-1)\rho}$ by the Verma module $M_\lambda$ in the Etingof-Kirillov Jr. construction yields the Macdonald functions of [36]. In particular, for a (possibly non-integral) $\lambda$, for the normalization factor
\[
\chi_{k-1}(\lambda) = \prod_{a=1}^{k-1} \prod_{1 < j} (1 - q^{-2(\lambda_i - \lambda_j + j - i + 2a)}
\]
and $\tilde{\Psi}_\lambda : M_\lambda \to M_\lambda \otimes W_{k-1}$ the unique intertwiner so that
\[
\tilde{\Psi}_\lambda(v_\lambda) = v_\lambda \otimes \chi_{k-1}(\lambda)w_{k-1} + \text{(lower order terms)},
\]
the Macdonald function is the joint eigenfunction of the $D_{n,e}(q^2, q^{2k})$ given by
\[
\varphi(\lambda, x) = \frac{q^{2(k-1)(\rho, \lambda - \rho)} \text{Tr}(\tilde{\Psi}_{\lambda-\rho}x^L)}{\prod_{1 < j} \prod_{a=1}^{k-1} (q^a e(x_i-x_j)/2 - q^{-a} e(x_j-x_i)/2)}.
\]

Note that our notation is related to that of [36] by the substitution $k \mapsto k + 1$. For a dominant integral signature $\lambda$ and $\tau$ a dominant weight in the root lattice, if $\lambda_i - \lambda_{i+1} \geq l$ for some $l > 0$, the quotient map $M_\lambda \to L_\lambda$ is an isomorphism in the $(\lambda - \tau)$-weight space. The fact from [36] that the branching coefficient for Macdonald functions is a rational function in $q^{\lambda_i}$ and $q^{\mu_i}$ therefore implies the branching rule
\[
\varphi(\lambda, x) = \sum_{\mu_1 \in \lambda_1 - \mathbb{Z} \geq 0} \cdots \sum_{\mu_n \in \lambda_n - \mathbb{Z} \geq 0} \tilde{\psi}_{\lambda/\mu}^k(q^2) \varphi(\mu, x_1, \ldots, x_n) e_n((\lambda|-|\mu))
\]
with branching coefficient given by
\[
\tilde{\psi}_{\lambda/\mu}^k(q^2) = q^{-(n-1)k(k-1)/2} q^{2(k-1)((\rho, \lambda - \rho) - (\rho, \mu - \rho))} \frac{\chi_{k-1}(\lambda)}{\chi_{k-1}(\mu)} \varphi_{\lambda-k\rho/\mu-k\rho}(q^2, q^{2k})
\]
\[
= (q - q^{-1})^{(k-1)(n-1)} \prod_{1 < j} [\lambda_i - \lambda_j + j - i - 1]_{k-1} \prod_{1 < j} [\mu_i - \mu_j + j - i - 1]_{k-1} \prod_{1 < j} [\lambda_i - \mu_j + k - 1]_{k-1} \prod_{1 < j} [\mu_i - \lambda_j + 1]_{k-1} \prod_{1 < j} [\lambda_i - \lambda_j + 1]_{k-1} \prod_{1 < j} [\mu_i - \mu_j + 1]_{k-1} \prod_{1 < j} [\lambda_i - \mu_j + 1]_{k-1} \prod_{1 < j} [\mu_i - \lambda_j + 1]_{k-1} \prod_{1 < j} [\lambda_i - \mu_j + 1]_{k-1} \prod_{1 < j} [\mu_i - \lambda_j + 1]_{k-1}
\]

Our techniques apply to this setting. For any $M > 0$, there is some $l > 0$ so that if $|\tau| < M$ and $\lambda_i - \lambda_{i+1} \geq l$, the matrix elements of $\tilde{\Psi}_{\lambda-\rho}$ on the Gelfand-Tsetlin basis elements of $M_{\lambda-\rho}$ of weight $\lambda - \rho - \tau$ coincide with those of $\chi_{k-1}(\lambda)\Phi_{\lambda-k\rho}^2$. As shown in [36], the matrix elements are rational functions, hence coincide with the expression of Theorem 3.4.4 for all (possibly non-integral) $\lambda$. Applying a adjunction argument similar to that of the polynomial case yields the branching rule for Macdonald functions.
3.1.8 Outline of method and organization

We briefly outline our method. Our main technical result is Theorem 3.3.7, which constructs and characterizes a map $\text{Res}_t(q^2)$ between spherical DAHA's of rank $nl$ and $n$. We use Theorem 3.3.7 to relate Macdonald difference operators in $n$ variables at $t = q^l$ to Macdonald difference operators in $nl$ variables at $t = q^{1/l}$. Combining this with an explicit summation expression for $U_q(g_{n,l})$ matrix elements given in [3], we obtain in Theorem 3.4.4 a new expression for diagonal $U_q(g_{n,l})$ matrix elements as the application of Macdonald difference operators to an explicit kernel.

To obtain Macdonald's branching rule, we interpret the Etingof-Kirillov Jr. expression for the Macdonald polynomial $P_{\lambda}(x; q^2, q^{2k})$ as a summation formula over Gelfand-Tsetlin patterns subordinate to $\lambda + (k-1)p$. Applying Theorem 3.4.4, the symmetry identity, and summation by parts reduces this to the summation over Gelfand-Tsetlin patterns subordinate to $\lambda$ found in the branching rule.

The remainder of this chapter is organized as follows. In Section 3.2, we give some necessary background on Macdonald polynomials and reformulate the results in a convenient form. In Section 3.3, we define a map $\text{Res}_t(q^2)$ between spherical double affine Hecke algebras of different rank and prove the key Theorem 3.3.7 which allows us to compute the image of a certain Macdonald operator in Lemma 3.3.10. In Section 3.4, we prove the main Theorem 3.4.4 on matrix elements of $U_q(g_{n,l})$-intertwiners by applying the technique developed in Section 3.3 and a formula from [3]. In Section 3.5, we put everything together to derive a new proof of Macdonald's branching rule. Section 3.6 contains some technical manipulations of the result of [3] postponed from Section 3.4.

3.2 Quantum groups and Macdonald polynomials

3.2.1 Notations

We will frequently need to consider expressions involving a signature and various shifts; we collect here the conventions we use to denote these. Set $\rho_{n,i} = \frac{n+1}{2} - i$ and $1 = (1, \ldots, 1)$. For any set of indices $I$, let $1_I$ denote the vector with 1's in those indices and 0's elsewhere. Define $\rho_n = \rho_n - \frac{n+1}{2}1$ so that $\rho_{n,i} = -(i-1)$ and $\rho_{n-1,i} = \rho_{n,i}$. For any signature $\lambda$, define the shifts $\bar{\lambda} = \lambda + (k-1)p$ and $\bar{\lambda} = \lambda + k\bar{p}$ so that $\bar{\lambda}_i = \lambda_i - (k-1)(i-1)$ and $\bar{\lambda}_i = \lambda_i - k(i-1)$. Finally, denote by $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$ the $q$-number, $[a]! = [a] \cdot [a-1] \cdots [1]$ the $q$-factorial, and $[a]_m = [a] \cdot [a-1] \cdots [a-m+1]$ the falling $q$-factorial.

3.2.2 Macdonald symmetry identity

In this subsection, we state the Macdonald symmetry identity and use it to produce conjugates of the Macdonald difference operators acting diagonally on the Macdonald polynomials via their index.
Proposition 3.2.1 (Macdonald symmetry identity). We have

\[ P_{\lambda}(q^{2\mu+2kp}, q^2, q^{2k}) = \prod_{i<j} \left[ \lambda_i - \lambda_j + k(j - i) + k - 1 \right] P_{\mu}(q^{2\lambda+2kp}, q^2, q^{2k}). \]

We would like now to produce Macdonald operators acting on indices of Macdonald polynomials. For this, we write \( D_{n-1,q^{2\mu}} \) for Macdonald difference operators acting on the variables \( q^{2\mu} \).

Proposition 3.2.2. The operator

\[ \tilde{D}_{n-1,q^{2\mu}}(q^2, q^{2k}) = \prod_{i<j} (\mu_i - \mu_j + k - 1) \circ D_{n-1,q^{2\mu}}(q^2, q^{2k}) \circ \prod_{i<j} (\mu_i - \mu_j + k - 1)^{-1} \]

satisfies

\[ \tilde{D}_{n-1,q^{2\mu}}(q^2, q^{2k}) = \sum_{|l|=r} \prod_{i,j,l,i>j} \frac{[\mu_i - \mu_j + k]}{[\mu_i - \mu_j]} T_{q^2,l} \]

and

\[ \tilde{D}_{n-1,q^{2\mu}}(q^2, q^{2k}) P_{\mu}(x; q^2, q^{2k}) = e_r(x) P_{\mu}(x; q^2, q^{2k}). \]

Proof. The expression for \( \tilde{D}_{n-1,q^{2\mu}}(q^2, q^{2k}) \) follows by direct computation, and the eigenvalue identity from the Macdonald symmetry identity.

\[ \square \]

3.2.3 Adjoints of Macdonald difference operators

We would like now to consider adjoints of Macdonald operators with respect to a Jackson-type inner product. Fix lower and upper limits \( \zeta = (\zeta^-, \zeta^+) \) with \( \zeta^- = (\zeta_1^-, \ldots, \zeta_{n-1}^-) \), \( \zeta^+ = (\zeta_1^+, \ldots, \zeta_{n-1}^+) \), and \( \zeta_i^+ - \zeta_i^- \in \mathbb{Z}_{\geq 0} \). Define the inner product

\[ \langle f, g \rangle_{\zeta} := \sum_{\mu = \zeta^-}^{\zeta^+} f(q^{2\mu}) g(q^{2\mu}), \]

where we define the iterated summation symbol by

\[ \sum_{\mu = \zeta^-}^{\zeta^+} := \sum_{\mu_i = \zeta_i^-}^{\zeta_i^+} \cdots \sum_{\mu_{n-1} = \zeta_{n-1}^-}^{\zeta_{n-1}^+}. \] (3.2.1)

We will consider situations where \( g \) vanishes along a border of the region of summation. In particular, we say that the function \( g(q^{2\mu}) \) is \((\zeta, l)\)-adapted if \( g(q^{2\mu}) = 0 \) on the set

\[ \{ \mu \mid \zeta_i^+ < \mu_i \leq \zeta_i^+ + l \text{ or } \zeta_i^- - l \leq \mu_i < \zeta_i^- \text{ for any } i \}. \]

We now characterize adjoints with respect to \( \langle \cdot, \cdot \rangle_{\zeta} \) when applied to an \((\zeta, l)\)-adapted function.
Proposition 3.2.3. If $f(q^{2\mu})$ is $(\zeta, l)$-adapted, we have for any $g$ that

$$
\left\langle \prod_{i=1}^{l} \overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger f, g \right\rangle_{(\zeta^-, \zeta^+ + 1)} = \left\langle f, \prod_{i=1}^{l} \overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger g \right\rangle_{\zeta},
$$

where

$$
\overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger = \prod_{i<j} [\mu_i - \mu_j + k - 1]_{k-1} \circ D_{n-1,q^{2\mu}}(q^{-2}, q^{2(k-1)}) \circ \prod_{i<j} [\mu_i - \mu_j + k - 1]_{k-1}.
$$

Proof. First, we check by a direct computation that

$$
\overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger = \sum_{|I|=r} \prod_{i,j \notin I, i > j} \frac{[\mu_i - \mu_j + k - 1][\mu_i - \mu_j - k]}{[\mu_i - \mu_j - 1][\mu_i - \mu_j]} T_{q^{-2}, I}.
$$

Now, for any subset of indices $I$, we have

$$
\langle f, T_{q^{2}, I} g \rangle_{\zeta} = \sum_{\mu=\zeta}^{\zeta^+} f(q^{2\mu}) g(q^{2(\mu+1)})
$$

$$
= \sum_{\mu=\zeta^-}^{\zeta^+ + 1} f(q^{2(\mu-1)}) g(q^{\mu}) = \langle T_{q^{-2}, I} f, g \rangle_{(\zeta^- + 1, \zeta^+ + 1)}.
$$

(3.2.2)

Using this, we induct on $l$. For $l = 1$, we have

$$
\left\langle \prod_{i=1}^{l} \overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger f, g \right\rangle_{(\zeta^-, \zeta^+ + 1)}
$$

$$
= \sum_{\mu=\zeta^-}^{\zeta^+ + 1} \sum_{|I|=r} \prod_{i,j \notin I, i > j} \frac{[\mu_i - \mu_j + k - 1][\mu_i - \mu_j - k]}{[\mu_i - \mu_j - 1][\mu_i - \mu_j]} \prod_{i,j \notin I, i > j} f(q^{2(\mu-1)}) g(q^{2\mu})
$$

$$
= \sum_{|I|=r} \sum_{\mu=\zeta^-}^{\zeta^+} f(q^{2\mu}) T_{q^{2}, I} \left( \prod_{i,j \notin I, i > j} \frac{[\mu_i - \mu_j + k - 1][\mu_i - \mu_j - k]}{[\mu_i - \mu_j - 1][\mu_i - \mu_j]} \right) g(q^{2(\mu+1)})
$$

$$
= \sum_{|I|=r} \sum_{\mu=\zeta^-}^{\zeta^+} f(q^{2\mu}) \prod_{i,j \notin I, i > j} \frac{[\mu_i - \mu_j + k][\mu_i - \mu_j - k + 1]}{[\mu_i - \mu_j][\mu_i - \mu_j + 1]} g(q^{2(\mu+1)})
$$

$$
= \left\langle f, \overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger g \right\rangle_{\zeta},
$$

where the second equality follows from (3.2.2), the third follows because $f$ is $(\zeta, 1)$-adapted, and the last equality follows from Proposition 3.2.2. Now suppose the claim holds for $l - 1$. If $f$ is $(\zeta, l)$-adapted, then $\overline{D}_{n-1,q^{2\mu}}(q^{2}, q^{2k})^\dagger f$ is $(\zeta^-, \zeta^+ + 1, l - 1)$-
adapted, so we have by applying the cases of \( l - 1 \) and then 1 that

\[
\left\langle \prod_{i=1}^{l} \widetilde{D}_{n-i,q^{2k}}^\dagger(q^2, q^{2k}) f, g \right\rangle_{(\zeta^- \zeta^+ + 1)}
= \left\langle \prod_{i=1}^{l-1} \widetilde{D}_{n-i,q^{2k}}^\dagger(q^2, q^{2k}) f, \prod_{i=1}^{l-1} \widetilde{D}_{n-i,q^{2k}}^\dagger(q^2, q^{2k}) g \right\rangle_{(\zeta^- \zeta^+ + 1)}
= \left\langle f, \prod_{i=1}^{l} \widetilde{D}_{n-i,q^{2k}}^\dagger(q^2, q^{2k}) g \right\rangle_{\zeta}.
\]

### 3.2.4 Reformulating the Etingof-Kirillov Jr. construction

In this subsection we shift the weights of the representations used in the Etingof-Kirillov Jr. construction to make restriction from \( U_q(\mathfrak{gl}_n) \) to \( U_q(\mathfrak{gl}_{n-1}) \) more notationally convenient. For a partition \( \lambda \), define the intertwiner

\[
\tilde{\Phi}^n_\lambda : L_{\lambda+(k-1)\rho} \rightarrow L_{\lambda+(k-1)\rho} \otimes W_{k-1}
\]

to be \( \tilde{\Phi}^n_\lambda = \Phi^n_\lambda \otimes \text{id}_{(\text{det})^{-\frac{(k-1)(n-1)}{2}}} \). We now rephrase Theorem 3.1.3 in terms of the intertwiners \( \tilde{\Phi}^n_\lambda \).

**Corollary 3.2.4.** The Macdonald polynomial \( P_\lambda(x; q^2, q^{2k}) \) is given by

\[
P_\lambda(x; q^2, q^{2k}) = \frac{\text{Tr}(\tilde{\Phi}^n_\lambda x^h)}{\text{Tr}(\tilde{\Phi}^n_0 x^h)}.
\]

**Proof.** This follows from Theorem 3.1.3 and the relation

\[
\text{Tr}(\tilde{\Phi}^n_\lambda x^h) = \text{Tr}(\Phi^n_\lambda x^h)(x_1 \cdots x_n)^{\frac{(k-1)(n-1)}{2}}.
\]

**Corollary 3.2.5.** The denominator in Corollary 3.2.4 is given by

\[
\text{Tr}(\tilde{\Phi}^n_0 x^h) = (x_1 \cdots x_n)^{-\frac{(k-1)(n-1)}{2}} \prod_{s=1}^{k-1} \prod_{i < j} (x_i - q^{2s} x_j).
\]

**Proof.** This follows from Proposition 3.1.4 and the definition of \( \tilde{\Phi}^n_0 \).

### 3.3 Spherical subalgebras of double affine Hecke algebras of different ranks

#### 3.3.1 Double affine Hecke algebras

Let \( H_n(q, t) \) denote the double affine Hecke algebra (DAHA) of \( GL_n \) defined by [12]. Following [90], it is defined as the associative algebra generated by invertible elements
$X_1^{\pm 1}, \ldots, X_n^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$, and $T_1^{\pm 1}, \ldots, T_n^{\pm 1}$ subject to the relations

- $(T_i - t^{1/2})(T_i + t^{-1/2}) = 0$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $[T_i, T_j] = 0$ for $|i - j| \neq 1$;
- $T_i X_i T_i = X_{i+1}$, $T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$, and $[T_i, X_j] = [T_i, Y_j] = 0$ for $|i - j| > 1$;
- $[X_i, X_j] = 0$, $[Y_i, Y_j] = 0$, $Y_1 X_1 \cdots X_n = q X_1 \cdots X_n Y_1$, and $X_1^{-1} Y_2 = Y_2 X_1^{-1} T_1^{-2}$.

Note that $\{T_i\}$ generate a copy of the finite-type Hecke algebra, and $\{T_i, X_j\}$ and $\{T_i, Y_j\}$ generate copies of the affine Hecke algebra. For $\sigma = s_{i_1} \cdots s_{i_l}$ a reduced decomposition in $S_n$, let $T_\sigma = T_{i_1} \cdots T_{i_l}$. Define the idempotent

$$e = \frac{(1 - t)^n}{(t; t)_n} \sum_{\sigma \in S_n} t^{\ell(\sigma)/2} T_\sigma.$$

The spherical DAHA is defined to be the subalgebra $e\mathcal{H}_n(q, t)e$. From the results of [6, 90, 10, 91] surveyed in [80] on maps between the Drinfeld double of the elliptic Hall algebra and the spherical DAHA, we may extract the following small set of generators.

**Lemma 3.3.1** ([80, Section 2.4]). Let $p_1(X) = X_1 + \cdots + X_n$ and $p_{-1}(X) = X_1^{-1} + \cdots + X_n^{-1}$, and define $p_1(Y)$ and $p_{-1}(Y)$ similarly. The elements $ep_1(Y)e$, $ep_{-1}(Y)e$, $ep_1(X)e$, and $ep_{-1}(X)e$ generate $e\mathcal{H}_n(q, t)e$.

### 3.3.2 Polynomial representation of DAHA and Macdonald operators

The double affine Hecke algebra admits a faithful polynomial representation $\rho$ on $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ given by

$$\rho(X_i) = X_i \cdot -$$

$$\rho(T_i) = t^{1/2} s_i + \frac{t^{1/2} - t^{-1/2}}{X_i / X_{i+1} - 1} (s_i - 1)$$

$$\rho(Y_i) = \rho(T_i) \cdots \rho(T_{i-1}) s_{i-1} \cdots s_1 T q X_i \rho(T_1^{-1}) \cdots \rho(T_{i-1}^{-1}),$$

where $X_i \cdot -$ denotes multiplication by $X_i$, $s_i$ exchanges $X_i$ and $X_{i+1}$ and $T q X_i$ is the $q$-shift operator in $X_1$. The action of elements of $e\mathcal{H}_n(q, t)e$ on the symmetric part of the polynomial representation yields the Macdonald operators.

**Proposition 3.3.2.** When restricted to $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}$, the action of $e \cdot e_r(Y_1, \ldots, Y_n) \cdot e$ is given by Macdonald’s operator

$$\rho(e \cdot e_r(Y_1, \ldots, Y_n) \cdot e) = \rho(e_r(Y_1, \ldots, Y_n)) = D_{n,X}^r(q, t).$$

In particular, for any $n$-variable symmetric polynomial $f$, the operator

$$L_f = f(Y_1, \ldots, Y_n)$$

is diagonalized on $P_{n}(X; q, t)$ with eigenvalue $f(q^a t^b)$. 

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Proposition 3.3.3. When restricted to $C[X_1^\pm, \ldots, X_n^\pm]^{S_n}$, the action of $e \cdot p_1(Y^{-1}) \cdot e$ is given by

$$D_{n,X}(q,t) \circ D_{n,Y}(q,t)^{-1} = t^{-n^2} \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{x_j - x_i}{x_j - x_i} T_{q^{-1},i}.$$

Proof. This is a consequence of Proposition 3.3.2 and the relation $e p_1(Y^{-1}) e = e \cdot e_{n-1}(Y) e_n(Y^{-1}) \cdot e$.

Remark. By faithfulness, we will refer interchangeably to elements of the DAHA and spherical DAHA and their images under the polynomial representation in what follows.

### 3.3.3 $SL_2(Z)$-action on DAHA

Define the isomorphisms $\epsilon(q,t) : \mathcal{H}_n(q,t) \rightarrow \mathcal{H}_n(q^{-1},t^{-1})$ given by

$$\epsilon(q,t) : X_i \mapsto Y_i, Y_i \mapsto X_i, T_i \mapsto T_i^{-1}, q \mapsto q^{-1}, t \mapsto t^{-1}$$

and $\tau_+(q,t) : \mathcal{H}_n(q,t) \rightarrow \mathcal{H}_n(q,t)$ given by

$$\tau_+ : X_i \mapsto X_i, T_i \mapsto T_i, Y_1 \cdots Y_r \mapsto q^{-r/2}X_1 \cdots X_r Y_1 \cdots Y_r.$$

Define also the composition $\tau_- = \epsilon \tau_+ \epsilon$.

Proposition 3.3.4 ([13, Section 3.2.2]). The map

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto \tau_-, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \tau_+$$

defines an action of $SL_2(Z)$ on $\mathcal{H}_n(q,t)$ which preserves $e \mathcal{H}_n(q,t) e$.

The action of $\tau_+$ in the polynomial representation is realized via conjugation by the Gaussian

$$\gamma_n(q) = q^{\sum_i \bar{x}_i^2 / 2},$$

where $\bar{x}_i = X_i$. Here, we view $\gamma_n(q)$ as an element in the completion of $\mathcal{H}_n(q,t)$ by degree of $X$.

Proposition 3.3.5 ([13, Section 3.7]). When evaluated in the polynomial representation, the action of $\tau_+$ on $\mathcal{H}_n(q,t)$ is given by conjugation by $\gamma_n(q)$. That is, we have the equality

$$\rho(\tau_+(f)) = \gamma_n(q) \rho(f) \gamma_n(q)^{-1}.$$

### 3.3.4 Multiwheel condition and the restriction map

Following the generalization in [66] of the original wheel condition of [40, 41], we say that $(X_1^0, \ldots, X_n^{l-1}) \in C^n$ satisfies the **multiwheel condition** if the indices may be
permuted so that
\[ X_i^a = X_i^0 t^{a-1} \text{ for } 1 \leq i \leq n \text{ and } 0 \leq a \leq l - 1. \]

Define the ideal \( I_{nl}(t) \subset \mathbb{C}[(X_i^a)^\pm] \) by
\[ I_{nl}(t) = \{ f \mid f(X) = 0 \text{ if } X \text{ satisfy the multiwheel condition} \}. \]

In [66], this ideal was characterized as a \( \mathcal{H}_{nl}(q, t) \)-submodule.

**Proposition 3.3.6** ([66, Theorem 6.3] and [34, Theorem 5.10]). The subspace \( I_{nl}(t) \subset \mathbb{C}[(X_i^a)^\pm] \) is a \( \mathcal{H}_{nl}(q, t) \)-submodule and \( \mathbb{C}[(X_i^a)^\pm]/I_{nl}(t) \) is irreducible.

**Remark.** Along with some finer statements about the structure of \( I_{nl}(t) \) and other submodules defined by similar multiwheel conditions, Proposition 3.3.6 was conjectured in [66, Conjecture 6.4] and in the rational limit in [19]. These statements are known as the Dunkl-Kasatani conjecture and were later proven in [20] for generic values of parameters and for all values of parameters in [34, Theorem 5.10].

Define the map \( \text{Res}_l(q^2) : C[(X_i^a)^\pm]_{Sn_i} \to C[X_i^\pm]_{Sn} \) by
\[ \text{Res}_l(q^2)(X_i^a) = q^{1-l}e^{a}X_i. \]

The kernel of \( \text{Res}_l(q^2) \) is \( I_{nl}^{Sn_i}(q^2) \), so \( \text{Res}_l(q^2) \) induces by Proposition 3.3.6 an action of \( e\mathcal{H}_{nl}(q^{-2l}, q^2)e \) on \( C[X_i^\pm]_{Sn} \), giving a map
\[ \text{Res}_l(q^2) : e\mathcal{H}_{nl}(q^{-2l}, q^2)e \to \text{End}(C[X_i^\pm]_{Sn}). \]

We claim that this map factors through the polynomial representation
\[ e\mathcal{H}_n(q^{-2}, q^{2l})e \to \text{End}(C[X_i^\pm]_{Sn}) \]
via a map of algebras \( \text{Res}_l(q^2) : e\mathcal{H}_{nl}(q^{-2l}, q^2)e \to e\mathcal{H}_n(q^{-2}, q^{2l})e \).

**Theorem 3.3.7.** The map \( \text{Res}_l(q^2) : e\mathcal{H}_{nl}(q^{-2l}, q^2)e \to e\mathcal{H}_n(q^{-2}, q^{2l})e \) defined by
\[ \text{Res}_l(q^2)(ep(X_i^a)e) = ep(q^{l-1}X_1, \ldots, q^{-1}X_1, \ldots, q^{l-1}X_n, \ldots, q^{-1}X_n)e \text{ and} \]
\[ \text{Res}_l(q^2)(ep(Y_i^a)e) = ep(q^{l-1}Y_1, \ldots, q^{l-1}Y_1, \ldots, q^{l-1}Y_n, \ldots, q^{l-1}Y_n)e \]
for \( p \in \mathbb{C}[(X_i^a)^\pm]_{Sn_i} \) is well defined and satisfies
(a) for any \( h \in e\mathcal{H}_{nl}(q^{-2l}, q^2)e \), as operators on \( C[(X_i^a)^\pm]_{Sn_i} \) we have
\[ \text{Res}_l(q^2) \circ h = \text{Res}_l(h) \circ \text{Res}_l(q^2); \]
(b) as operators on \( e\mathcal{H}_{nl}(q^{-2l}, q^2)e \), we have
\[ \text{Res}_l(q^{-2}) \circ \varepsilon_{nl}(q^{-2l}, q^2) = \varepsilon_n(q^{-2}, q^{2l}) \circ \text{Res}_l(q^2); \]

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(c) as operators on $eH_n(q^{-2l}, q^2)e$, we have
\[ \text{Res}_l(q^2) \circ \tau_+ = \tau_+ \circ \text{Res}_l(q^2). \]

**Proof.** We first check that (a) holds on the generating set of Lemma 3.3.1. This is evident for $h \in eC[(X_i^q)]^{\pm 1}e$. For $h = e_p(Y^a)e$, we compute

\[
\text{Res}_l(q^2)D_{nl,X}^1(q^{-2l}, q^2)f(X_1, \ldots, X_n) = q^{1-nl} \sum_{i=1}^n \sum_{a=0}^{l-1} \prod_{i=1}^n \frac{q^{2a}X_i - q^{2b}X_j}{q^{2a}X_i - q^{2b}X_j}
\]

\[
f(q^{1-l}X_1, \ldots, q^{1-l}X_1, \ldots, q^{1-l+2a-2l}X_i, \ldots, q^{1-l}X_n)
\]

\[
= q^{1-nl} \sum_{i=1}^n \prod_{i=1}^n \frac{q^{2l}X_i - q^{2b}X_j}{q^{2l-2}X_i - q^{2b}X_j}
\]

\[
f(q^{1-l}X_1, \ldots, q^{1-l}X_1, \ldots, q^{1-l}X_i, \ldots, q^{1-l}X_1, \ldots, q^{1-l}X_n)
\]

\[
= q^{1-nl} \sum_{i=1}^n \prod_{i=1}^n \frac{q^{2l}X_i - q^{2b}X_j}{q^{2l-2}X_i - q^{2b}X_j}
\]

\[
\tau_{-2}, X_i \text{Res}_l(q^2)f(X_1, \ldots, X_n)
\]

\[
= q^{1-nl} \sum_{i=1}^n \prod_{i=1}^n \frac{q^{2l}X_i - q^{2b}X_j}{q^{2l-2}X_i - q^{2b}X_j}
\]

\[
\tau_{-2}, X_i \text{Res}_l(q^2)f(X_1, \ldots, X_n)
\]

which shows that (a) holds for $h = e_p(Y^a)e$. A similar computation using the expression of Proposition 3.3.3 yields (a) for $h = e_p((Y^a)^{-1})e$. We conclude that (a) holds for $h$ in the generating set $e_p(X)e$, $e_p(X^{-1})e$, $e_p(Y)e$, $e_p(Y^{-1})e$. Therefore, the stated values of $\text{Res}_l(q^2)$ extend to a well-defined map satisfying (a).

We now use (a) and the value of $\text{Res}_l(q^2)$ on the generators to prove (b) and (c). For (b), by Lemma 3.3.1, it suffices to check on $e_p(Y^\pm 1)e$ and $e_p((Y^\pm 1)^{-1})e$. We give the computations for $e_p(X)e$ and $e_p(Y)e$; the checks for $e_p(X^{-1})e$ and $e_p(Y^{-1})e$ are analogous. For the first check, note that

\[
\text{Res}_l(q^{-2})(\varepsilon_n(q^{-2}, q^2)(D_{nl,X}^1(q^{-2l}, q^2))) = \text{Res}_l(q^{-2})(p_{l,n}(X)) = [l] p_{l,n}(X),
\]

and

\[
\varepsilon_n(q^{-2}, q^2)(\text{Res}_l(q^2)(D_{nl,X}^1(q^{-2l}, q^2))) = \varepsilon_n(q^{-2}, q^2)([l] D_{nl,X}^1(q^{-2}, q^2)) = [l] p_{l,n}(X).
\]

For the second check, note that

\[
\text{Res}_l(q^{-2})(\varepsilon_n(q^{-2l}, q^2)(p_1(X^a))) = \text{Res}_l(q^{-2})(D_{nl,X}^1(q^{2l}, q^{-2})) = [l] D_{nl,X}^1(q^2, q^{-2l})
\]

\[
\varepsilon_n(q^{-2}, q^2)(\text{Res}_l(q^2)(p_1(X^a))) = \varepsilon_n(q^{-2}, q^2)([l] p_1(X^a)) = [l] D_{nl,X}^1(q^2, q^{-2l}),
\]

where we apply the fact from (a) that

\[
\text{Res}_l(q^2)D_{nl,X}^1(q^{-2l}, q^2) = [l] D_{nl,X}^1(q^2, q^2)
\]
with $q$ and $q^{-1}$ interchanged. This completes the proof of (b).

For (c), note that in $\mathcal{H}_n(q^{-2l}; q^2)$, we have

$$\text{Res}_l(q^2)(q^{-2i+2}) = \text{Res}_l(q^2)(X_i^a) = X_i q^{2(1-l+2a)} = q^{-2i} + 2(1-l+2a).$$

This implies that

$$\text{Res}_l(q^2)(\gamma_{nl}(q^{-2l})) = \text{Res}_l(q^2)(q^{-l \sum j \alpha(j^2)}) = q^{-l \sum j \alpha(j^2)} \gamma_{nl}(q^{-2}),$$

which yields the desired by Proposition 3.3.5.

Finally, to obtain the claimed values on $ep(Y)e$ for all $p$, we note by (b) that

$$\text{Res}_l(ep(Y)e) = \text{Res}_l(q^2)(ep(q^{2l}, q^{-2})(ep(X_i^a)e)) = ep(q^{-1}Y_1, \ldots, q^{1-l}Y_1, \ldots, q^{1-l}Y_n, \ldots, q^{-l}Y_n).$$

**Corollary 3.3.8.** The map $\text{Res}_l(q^2)$ commutes with the action of $SL_2(\mathbb{Z})$ on the spherical DAHA.

**Proof.** By Theorem 3.3.7(bc) and the fact that the $SL_2(\mathbb{Z})$-action is implemented via $\varepsilon_\pm$ and $\tau_\pm$. □

### 3.3.5 Extending the restriction map

In our application, we must extend the restriction map slightly. The assignment

$$\text{Res}_l(q^2)((X_i^a)^{1/2}) = q^{a-(l-1)/2}X_i^{1/2}$$

extends $\text{Res}_l(q^2)$ to an operator $\mathbb{C}[\{(X_i^a)^{\pm1/2}\}]_{\text{Sn}^l} \to \mathbb{C}[X_i^{1/2}]_{\text{Sn}^l}$. If we identify elements of the spherical DAHA with difference operators, they define valid operators on the subspace

$$\prod_{i,a} (X_i^a)^{1/2} \cdot \mathbb{C}[\{(X_i^a)^{\pm1}\}]_{\text{Sn}^l} \subset \mathbb{C}[\{(X_i^a)^{\pm1/2}\}]_{\text{Sn}^l},$$

though they do not in general satisfy the spherical DAHA relations. We see that Theorem 3.3.7(a) continues to hold in this setting.

**Corollary 3.3.9.** For any $h \in e\mathcal{H}_nl(q^{-2l}; q^2)e$, as operators on $\prod_{i,a} (X_i^a)^{1/2} \cdot \mathbb{C}[\{(X_i^a)^{\pm1}\}]_{\text{Sn}^l}$ we have

$$\text{Res}_l(q^2) \circ h = \text{Res}_l(h) \circ \text{Res}_l(q^2).$$

**Proof.** We interpret both sides as operators

$$\prod_{i,a} (X_i^a)^{1/2} \cdot \mathbb{C}[\{(X_i^a)^{\pm1}\}]_{\text{Sn}^l} \to \prod_{i} X_i^{1/2} \cdot \mathbb{C}[X_i^{\pm1}]_{\text{Sn}}$$

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and identify $\Pi_{i,a}(X_i^a)^{1/2} \cdot \mathbb{C}[(X_i^a)^{\pm 1}]^{S_{nl}}$ with $\mathbb{C}[(X_i^a)^{\pm 1}]^{S_{nl}}$ and $\Pi_{i,a^{1/2}} \cdot \mathbb{C}[X_i^{\pm 1}]^{S_{nl}}$. For $h \in \mathbb{C}[(X_i^a)^{\pm 1}]e$, both sides yield the map of Theorem 3.3.7. For $h = e p(Y_i^a)e$ with $p$ a degree $r$ homogeneous symmetric polynomial, both sides are equal to the map of Theorem 3.3.7 multiplied by $q^{-lr}$. Together, these give the claim. 

### 3.3.6 Computing $\text{Res}_i(q^2)$ on a specific operator

Define the operator

$$D_{n,X}(u; q, t) = \sum_r (-1)^{n-r} u^{n-r} D_{n,X}^r(q, t). \quad (3.3.1)$$

Identify $eH_{nl}(q^{-2l}, q^2)e$ with its image under the polynomial representation; in this identification, we now compute the image of a specific operator under $\text{Res}_i$.

**Lemma 3.3.10.** We have the relation

$$\text{Res}_i(q^2)(D_{nl,X}(q^{l+1}, q^{-2l}, q^2)) = \prod_{a=1}^l D_{n,X}(q^{2a}; q^{-2}, q^{2l}).$$

**Proof.** Observe that

$$D_{nl,X}(u; q^{-2l}, q^2) = \sum_r (-1)^{nl-r} u^{nl-r} D_{nl,X}^r(q^{-2l}, q^2)$$

$$= \epsilon_{nl}(q^2, q^{-2l}) \sum_r (-1)^{nl-r} u^{nl-r} e_r(X_i^a) = \epsilon_{nl}(q^2, q^{-2l}) \prod_i (X_i^a - \bar{u}),$$

where $\bar{u} = \epsilon_{nl}(q^2, q^{-2})(u)$. Therefore, by Theorem 3.3.7(b) with $q$ and $q^{-1}$ interchanged we find that

$$\text{Res}_i(q^2)D_{nl,X}(u; q^{-2l}, q^2) = \text{Res}_i(q^2)\epsilon_{nl}(q^2, q^{-2}) \prod_{i,a} (X_i^a - \bar{u})$$

$$= \epsilon_{n}(q^2, q^{-2l}) \text{Res}_i(q^{-2}) \prod_{i,a} (X_i^a - \bar{u})$$

$$= \epsilon_{n}(q^2, q^{-2l}) \prod_{i,a} (q^{-2a+1} X_i - \bar{u})$$

$$= \epsilon_{n}(q^2, q^{-2l}) \prod_{i,a=0}^{l-1} \left( \sum_r (-1)^{n-r} (q^{2a-1+l} \bar{u})^{n-r} e_r(X_i) \right)$$

$$= \prod_{a=0}^{l-1} \left( \sum_r (-1)^{n-r} (q^{l-1-2a} u)^{n-r} D_{n,X}^r(q^{-2}, q^{2l}) \right)$$

$$= \prod_{a=0}^{l-1} D_{n,X}(u q^{l-1-2a}; q^{-2}, q^{2l}).$$
Setting $u = q^{i+1}$ implies the desired

$$\text{Res}_t(q^2)(D_{n,\lambda}(q^{i+1}; q^{-2}, q^2)) = \prod_{a=1}^{l} D_{n,\lambda}(q^{2a}; q^{-2}, q^{2l}).$$

\[ \Box \]

3.4 Diagonal matrix elements in the Gelfand-Tsetlin basis

3.4.1 Factorization of matrix elements

For a choice of $\mu^1, \ldots, \mu^n = \lambda$ so that $\tilde{\mu}^1 \prec \cdots \prec \tilde{\mu}^n = \tilde{\lambda}$ forms a Gelfand-Tsetlin pattern subordinate to $\lambda$, denote the pattern by $\tilde{\mu}$. Let $c(\tilde{\mu}, \lambda)$ denote the diagonal matrix coefficient of $v_{\tilde{\mu}}$ in $\Phi_\lambda$. For a signature $\mu \prec \lambda$, let the pattern $\text{gt}(\mu)$ be defined by

$$\text{gt}(\mu)_i = \mu_i \text{ for } 1 < n.$$  \hspace{1cm} (3.4.1)

Define $c(\mu, \lambda)$ to be the diagonal matrix coefficient $c(\text{gt}(\tilde{\mu}), \lambda)$ of $v_{\text{gt}(\tilde{\mu})}$ in $\Phi_\lambda$.

We show that $\Phi_\lambda$ has non-zero diagonal matrix elements only on basis vectors indexed by patterns of the form $\tilde{\mu}$ and that these elements admit a level-by-level factorization.

**Lemma 3.4.1.** If $v_\mu$ is not of the form $v_{\tilde{\mu}}$, then $v_\mu$ has zero diagonal matrix element in $\Phi_\lambda$.

**Proof.** For some $r < n$, we cannot write $\mu^r = \tilde{\tau}$ for any $\tau$. Let $U \subset W_{k-1}$ be the $U_q(\mathfrak{gl}_r)$-submodule consisting of vectors of weight 0 for $q^{h+1}, \ldots, q^{hn}$ so that $U \simeq L_{(k-1)(r-1, \ldots, -1)}$ as a $U_q(\mathfrak{gl}_r)$-module. Let $\tilde{\mu}^r$ denote the truncation of $\mu^r$ so that $\mu^r_i = \mu^r_i$. Consider the Gelfand-Tsetlin pattern $\xi$ given by

$$\xi = \{\text{gt}(\mu^r) \prec \mu^{r+1} \prec \cdots \prec \mu^{n-1} \prec \lambda\}.$$  \hspace{1cm} (3.4.1)

Let $L_{\mu^r} \subset L_{\lambda}$ be the $U_q(\mathfrak{gl}_r)$-submodule with highest weight $\mu^r$ generated by $v_\xi$. By Proposition 3.1.5, the diagonal matrix element of $v_\mu$ lies in $L_{\mu^r} \otimes U$, hence is a multiple of the matrix element of $v_\mu$ in the induced $U_q(\mathfrak{gl}_r)$-intertwiner

$$L_{\mu^r} \to L_{\lambda} \to L_{\lambda} \otimes W_{k-1} \to L_{\mu^r} \otimes U$$

given by projection onto $L_{\mu^r} \otimes U$. This intertwiner is zero because $\mu^r$ is not of the form $\mu^r = \tilde{\tau}$ for some $\tau$, giving the claim. \[ \Box \]

**Proposition 3.4.2.** For any Gelfand-Tsetlin pattern

$$\tilde{\mu} = \{\tilde{\mu}^1 \prec \tilde{\mu}^2 \prec \cdots \prec \tilde{\mu}^n = \tilde{\lambda}\}$$
subordinate to \( \bar{\lambda} \), we have the factorization

\[
c(\bar{\mu}, \lambda) = \prod_{i=1}^{n-1} c(\mu^i, \mu^{i+1}).
\]

**Proof.** By induction on \( n \), it suffices to check that

\[
c(\bar{\mu}, \lambda) = c(\mu, \lambda)c(\{\mu^1 < \cdots < \mu^{n-1}\}, \mu^{n-1}).
\]

Let \( \mu = \mu^{n-1} \). By Proposition 3.1.5, the basis vector \( v_{\bar{\mu}} \) lies in the \( U_q(g_{l_{n-1}}) \) submodule \( L_{\bar{\mu}} \subset L_{\bar{\lambda}} \) with highest weight vector \( v_{\bar{g}(\bar{\mu})} \). Let \( U \subset W_{k-1} \) be the \( U_q(g_{l_{n-1}}) \) submodule consisting of elements of weight 0 under \( q^{l_n} \). Consider the \( U_q(g_{l_{n-1}}) \)-intertwiner

\[
\phi : L_{\bar{\mu}} \rightarrow L_{\bar{\mu}} \otimes U
\]
given by composing \( \Phi_{\lambda} \) with the projection onto \( L_{\bar{\mu}} \otimes U \). The matrix element \( c(\bar{\mu}, \lambda) \) lies in \( U \), hence is the matrix element of \( v_{\bar{\mu}} \) in \( \phi \). Notice that \( \phi \) maps the \( U_q(g_{l_{n-1}}) \)-highest weight vector \( v_{\bar{g}(\bar{\mu})} \) to

\[
c(\mu, \lambda)v_{\bar{g}(\bar{\mu})} \otimes w_{k-1} + \text{(l.o.t.)}
\]

so that \( \phi = c(\mu, \lambda)\Phi_{\mu} \) and the matrix element of \( v_{\bar{\mu}} \) is the desired

\[
c(\mu, \lambda)c(\{\mu^1 < \cdots < \mu^{n-1}\}, \mu^{n-1}). \quad \square
\]

**3.4.2 Matrix elements as applications of Macdonald difference operators**

Our main technical result expresses matrix elements of \( U_q(g_{l_n}) \)-intertwiners as the application of Macdonald difference operators to an explicit kernel. Define the elements \( \Delta_1^{k-1}(\mu) \) and \( \Delta_2^{k-1}(\mu) \) by

\[
\Delta_1^{k-1}(\mu) = \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j + (k - 1)]_{k-1} \quad \Delta_2^{k-1}(\mu) = \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j - 1]_{k-1} \quad (3.4.2)
\]

and the element \( \Delta^{k-1}(\mu, \lambda) \) by

\[
\Delta^{k-1}(\mu, \lambda) = \prod_{i \leq j} [\lambda_i - \mu_j + k(j - i) + k - 1]_{k-1} \prod_{i > j} [\mu_i - \lambda_j + k(j - i) - 1]_{k-1}. \quad (3.4.3)
\]

We use Theorem 3.6.1 to compute the diagonal matrix elements of \( \Phi_{\lambda} \) in terms of these elements, resulting in the following expression after manipulation. We defer the proof of Proposition 3.4.3 to Section 3.6.

**Proposition 3.4.3.** Let \( \mu' = \mu + (k - 1)1 \), and \( \nu' = \nu + (k - 1)1 \). Then \( c(\mu, \lambda) \) is
given by
\[
c(\mu, \lambda) = \frac{(-1)^{(n-1)(k-1)}q^{(n-1)k(k-1)}}{\Delta^k_{n-1}(\lambda)\Delta^k_{n-1}(\mu)} \sum_{\nu' \in \tilde{\mu}^{(k-1)}(\lambda)} (-1)^{||\nu'||-||\tilde{\mu}'||} q^k(||\nu'||-||\tilde{\mu}'||)
\prod_{i \leq \tilde{\mu}_i} \frac{1}{[\tilde{\mu}'_i - \tilde{\mu}'_i + (k - 1)] [\tilde{\mu}'_i - \tilde{\mu}'_i]}
\prod_{i < j} [\tilde{\lambda}_i - \tilde{\lambda}_j + (k - 1)]\prod_{i < j} [\tilde{\nu}_i' - \tilde{\nu}_j']
\prod_{i \leq j} [\lambda_i - \lambda_j + (k - 1)]\prod_{i < j} [\nu_i' - \lambda_j - 1]_{k-1}.
\]

In this form, we can now identify the matrix element with an application of Macdonald difference operators.

**Theorem 3.4.4.** Let \( \mu' = \mu + (k - 1)1 \). The matrix element \( c(\mu, \lambda) \) is given by
\[
c(\mu, \lambda) = \frac{\prod_{a=1}^{k-1} D_{n-1,q^2}(q^{2a}; q^{-2}, q^{2(k-1)})\Delta^k_{n-1}(\mu', \lambda)}{\Delta^k_{n-1}(\mu)\Delta^k_{n-1}(\lambda)}
\]
where \( D_{n-1,q^2}(q^{2a}; q^{-2}, q^{2(k-1)}) \) was defined in (3.3.1).

**Proof.** For an expression \( E \), let \( 1_E = 1 \) if \( E \) holds and \( 1_E = 0 \) otherwise. Interpreting \( \text{Res}_\{q^2\} \) in the sense of Subsection 3.3.5, notice that
\[
\text{Res}_\{q^2\} D_{n-1,q^2}(q^{1+1}; q^{-2}, q^2) \prod_{a=0}^{l-1} \prod_{i \leq j} [\tilde{\lambda}_i - \tilde{\mu}_j^a + k/2] \prod_{i < j} [\tilde{\mu}_i^a - \tilde{\lambda}_j - k/2]
= \text{Res}_\{q^2\} \left( \sum_I (-1)^{(n-1)(l-|I|)}q^k((n-1)(l-|I|)) \prod_{(i,a) \in \tilde{I}} [\tilde{\mu}_i^a - \tilde{\mu}_j^b + 1] \prod_{i < j,a} [\tilde{\mu}_i^a - \tilde{\lambda}_j - l1_{(i,a) \in I} + k/2] \prod_{i < j,a} [\tilde{\mu}_j^a - \tilde{\lambda}_i - l1_{(i,a) \in I} - k/2] \right)
= \sum_I (-1)^{(n-1)(l-|I|)}q^k((n-1)(l-|I|)) \prod_{(i,a) \in \tilde{I}} [\tilde{\mu}_i - \tilde{\mu}_j + a-b+1] \prod_{i < j,a} [\tilde{\mu}_i - \tilde{\mu}_j + a - b] \prod_{i \leq j,a} [\tilde{\lambda}_i - \tilde{\mu}_j + a + l1_{(i,a) \in I} + l] \prod_{i < j,a} [\tilde{\mu}_i + a - \tilde{\lambda}_j - l1_{(i,a) \in I} - l],
\]
where both sums are over subsets \( I \subset \{(i,a) \mid 1 \leq i \leq n - 1, 0 \leq a \leq l - 1 \} \). If \( (i,a) \in I \) and \( (i,a + 1) \notin I \), then the corresponding term is zero, so the only subsets \( I \) which contribute to the sum are those of the form
\[I = \{(i,a) \mid a \geq s_i \} \text{ for some } s_1, \ldots, s_{n-1}.
\]
We rewrite the sum in these terms as

\[ \text{Res}_i(q^2)D_{(n-1)\Delta q^2}(q^{l+1}; q^{-2l},q^2) \prod_{a=0}^{l-1} \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + k/2] \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - k/2] \]

\[ = \sum_{s_1,\ldots,s_{n-1}=0} (-1)^{s_1} q^{s_1} \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + 2l - s_i] \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - l + s_i - 1] \]

\[ \prod_{i,j} \prod_{a=s_i}^{l-1} \prod_{b=0}^{s_j-1} [\bar{\mu}_i - \bar{\mu}_j + a - b + 1] \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j + a - b] \cdot \prod_{i<j} [\bar{\lambda}_i - \bar{\lambda}_j + a - b - 1] \]

Observe now that

\[ \prod_{i,j} \prod_{a=s_i}^{l-1} \prod_{b=0}^{s_j-1} [\bar{\mu}_i - \bar{\mu}_j + l]_{s_j} = \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j + l]_{s_j} = \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j + s_i + s_j] \]

Substituting this into the previous expression and changing variables to \( r_i = l - s_i \), we obtain

\[ \text{Res}_i(q^2)D_{(n-1)\Delta q^2}(q^{l+1}; q^{-2l},q^2) \prod_{a=0}^{l-1} \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + k/2] \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - k/2] \]

\[ = q^{kl(n-1)} (-1)^{(n-1)} \sum_{r_1,\ldots,r_{n-1}=0} (-1)^{\sum_i r_i} q^{-k \sum_i r_i} \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + r_j + l]_{i j} \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - r_i - 1]_{i j} \]

\[ \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j + r_j + l]_{i j} \prod_{i<j} [\bar{\lambda}_i - \bar{\lambda}_j + r_i + l]_{i j} \prod_{i<j} \left[ \prod_{i<j} [\tilde{\lambda}_i - \bar{\mu}_j + r_i + l]_{i j} \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - r_i - 1]_{i j} \right] \]

On the other hand, we have that

\[ \text{Res}_i(q^2) \left( \prod_{a=0}^{l-1} \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + k/2] \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - k/2] \right) \]

\[ = \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + k - 1]_{k-1} \prod_{i<j} [\tilde{\lambda}_i - \bar{\lambda}_j - 1]_{k-1} \]

Therefore, by Lemma 3.3.10 and Corollary 3.3.9 with \( l = k - 1 \) and \( X_i^a = q^{2 \bar{\mu}_i^a} \), we
conclude that

\[ \prod_{a=1}^{l} D_{n-1,q^a}(q^{2a}; q^{-2}, q^{2l}) \prod_{i \leq j} [\lambda_i - \mu_j + k - 1]_{k-1} \prod_{i < j} [\lambda_i - \lambda_j - 1]_{k-1} \]

\[ = q^{k(l(n-1))} (-1)^{l(n-1)} \sum_{r_1, \ldots, r_{n-1}=0}^{l} (-1)^{i} q^{|r|} \prod_{i \leq j} [\lambda_i - \mu_j + r_j + l]_{i} \prod_{i < j} [\mu_i - \mu_j + r_j]_{i+1} [\lambda_i - \lambda_j - r_i + l]_{i+1} \prod_{i} \frac{[l]!}{[r_i]![l-r_i]!}. \]

Dividing both sides by \( \prod_{i \leq j} [\lambda_i - \lambda_j - 1]_{i} \prod_{i < j} [\mu_i - \mu_j + l]_{i} \), we obtain

\[ \prod_{a=1}^{l} D_{n-1,q^a}(q^{2a}; q^{-2}, q^{2l}) \prod_{i \leq j} [\lambda_i - \mu_j + k - 1]_{k-1} \prod_{i < j} [\mu_i - \lambda_j - 1]_{k-1} \]

\[ = \frac{q^{k(l(n-1))} (-1)^{l(n-1)}}{\prod_{i \leq j} [\lambda_i - \lambda_j - 1]_{i} \prod_{i < j} [\mu_i - \mu_j + l]_{i}} \sum_{r_1, \ldots, r_{n-1}=0}^{l} (-1)^{i} q^{|r|} \prod_{i \leq j} [\lambda_i - \mu_j + r_j + l]_{i} \prod_{i < j} [\mu_i - \mu_j + r_j]_{i+1} [\lambda_i - \lambda_j - r_i + l]_{i+1} \prod_{i} \frac{1}{[r_i]![l-r_i]!}. \]

where the second expression is equal to \( c(\mu - (k-1)1, \lambda) \) by Proposition 3.4.3. Replacing \( \mu \) by its shift \( \mu' = \mu + (k-1)1 \) and recalling the definitions of \( \Delta_{k-1}^{k-1}(\mu) \), \( \Delta_{k}^{k-1}(\lambda) \), and \( \Delta^{k-1}(\mu, \lambda) \) yields the claimed expression

\[ c(\mu, \lambda) = \prod_{i \leq j} [\mu'_i - \mu'_j + k - 1]_{k-1} \prod_{i < j} [\lambda_i - \lambda_j - 1]_{k-1} \]

\[ \prod_{a=1}^{k-1} D_{n-1,q^a}(q^{2a}; q^{-2}, q^{2(k-1)}) \prod_{i \leq j} [\lambda_i - \lambda_j - 1]_{k-1} \prod_{i < j} [\mu_i - \lambda_j - 1]_{k-1}. \]

\[ \square \]

### 3.5 Proving Macdonald’s branching rule

We now put everything together to give a new proof of Macdonald’s branching rule, which we reformulate for \( t = q^k \) with \( k \) a positive integer.

**Theorem 3.5.1.** At \( t = q^k \) for positive integer \( k \), we have

\[ P_{\lambda}(x_1, \ldots, x_n; q^2, q^{2k}) = \sum_{\mu < \lambda} x^{\lambda}_{n-\lambda} P_{\lambda}(x_1, \ldots, x_{n-1}; q^2, q^{2k}) \psi_{\lambda/\mu}(q^2, q^{2k}) \]

with

\[ \psi_{\lambda/\mu}(q^2, q^{2k}) = \frac{\Delta_{k-1}(\mu, \lambda)}{\Delta_{k-1}(\mu)\Delta_{k-1}(\lambda)}. \]

**Proof.** We induct on \( n \). The base case is trivial because \( P_{\lambda}(x_1; q^2, q^{2k}) = x^{\lambda}_{1} \). For the inductive step, by Lemma 3.4.1, it is enough to consider matrix elements for basis
vectors of the form $v_{\mu}$. By Proposition 3.4.2 and the inductive hypothesis, we thus have

$$
\text{Tr}(\Phi_{\mu}^{n-1}x^h) = \sum_{\mu' < \mu \leq \lambda} c(\mu', \lambda) x_n^{[\mu'] - [\mu - (k-1)]) \prod_{i=1}^{n-1} x_i^{[\mu']_i - [\mu - (k-1)]}
$$

$$
= \sum_{\mu < \lambda} c(\mu, \lambda) x_n^{[\mu] - [\mu - (k-1)]} \prod_{i=1}^{n-1} x_i^{[\mu]_i - [\mu - (k-1)]} \text{Tr}(\Phi_{\mu}^{n-1}x^h)
$$

$$
= \sum_{\mu < \lambda} c(\mu, \lambda) x_n^{[\mu] - [\mu - (k-1)]} \text{Tr}(\Phi_{\mu}^{n-1}x^h)
$$

where $x = (x_1, \ldots, x_{n-1})$. By Corollary 3.2.5, we have that

$$
\frac{\text{Tr}(\Phi_{\mu}^{n-1}x^h)}{\text{Tr}(\Phi_{\mu}^{n-1}x^h)} = (x_1 \cdots x_{n-1})^{k-1} \prod_{s=1}^{n-1} (x_i - q^{2s}x_n)^{-1} \prod_{\mu < \lambda} c(\mu, \lambda) x_n^{[\mu]_i - [\mu - (k-1)]}.
$$

We conclude that

$$
\frac{\text{Tr}(\Phi_{\mu}^{n}x^h)}{\text{Tr}(\Phi_{\mu}^{n-1}x^h)} = (x_1 \cdots x_{n-1})^{k-1} \prod_{s=1}^{n-1} (x_i - q^{2s}x_n)^{-1} \sum_{\mu < \lambda} c(\mu, \lambda) x_n^{[\mu]_i - [\mu - (k-1)]} \prod_{s=1}^{n-1} (x_i - q^{2s}x_n)^{-1} \prod_{\mu < \lambda} c(\mu', \lambda') x_n^{[\mu']_i - [\mu' - (k-1)]}.
$$

where $\lambda_1 = (\lambda_2, \ldots, \lambda_n)$ and $\lambda^+ = (\lambda_1, \ldots, \lambda_{n-1})$ are vectors of lower and upper indices for $\mu$ so that $\sum_{\mu < \lambda} = \sum_{\mu < \lambda^+}$ in the notation of (3.2.1). Note that $\mu < \lambda$ if and only if $\lambda_i \geq \mu_i \geq \lambda_{i+1} - (k-1)$. By the expression for $c(\mu' - (k-1)\lambda', \lambda)$ given in Theorem 3.4.4, we obtain

$$
P_\lambda(x; q^2, q^{2k}) = \frac{\text{Tr}(\Phi_{\mu}^{n}x^h)}{\text{Tr}(\Phi_{\mu}^{n-1}x^h)}
$$

$$
= x_n^{(k-1)(n-1)} \prod_{s=1}^{n-1} (x_i - q^{2s}x_n)^{-1} \sum_{\mu' < \lambda^+} x_n^{[\mu']_i - [\mu' - (k-1)]} \prod_{s=1}^{n-1} (x_i - q^{2s}x_n)^{-1} \prod_{\mu' < \lambda^+} x_n^{[\mu']_i - [\mu' - (k-1)]}.
$$

$$
\prod_{s=1}^{n-1} D_{n-1, q^{2s}}(q^{2s} - q^{-2}, q^{2(k-1)} - q^{-2}) \prod_{i \leq j} [\lambda_i - \bar{\mu}_j + k - 1]_{k-1} \prod_{s=1}^{n-1} (x_i - q^{2s}x_n)^{-1} \prod_{\mu' < \lambda^+} x_n^{[\mu']_i - [\mu' - (k-1)]}.
$$
Define the operator
\[
\tilde{D}_{n-1,q^{2k}}(q^{2a}; q^2, q^{2k}) = \sum_r (-1)^{n-1-r} q^{2a(n-1-r)} \tilde{D}_{n-1,q^{2k}}(q^2, q^{2k}),
\]
and note that it is diagonalized on \( P_{\mu'}(\tilde{x}; q^2, q^{2k}) \) by Proposition 3.2.2. Notice now that the function
\[
\prod_{i<j}[\tilde{\lambda}_i - \tilde{\mu}_j + k - 1]_{k-1} \prod_{i<j}[\tilde{\mu}_i - \tilde{\lambda}_j - 1]_{k-1}
\]
is 0 for \( \lambda_{i+1} - (k - 1) \leq \mu'_i < \lambda_{i+1} \) and \( \lambda_i < \mu'_j \leq \lambda_i + (k - 1) \), so it is \((\lambda_1, \lambda^\dagger, k - 1)\)-adapted. Applying Proposition 3.2.3 to this function yields
\[
P_{\lambda}(x; q^2, q^{2k})
\]
\[
= x_n^{(k-1)(n-1)} \prod_{a=1}^{k-1} \prod_{i=1}^{n} \left( x_i - q^{2s} x_n \right)^{-1} \sum_{\lambda' = \lambda_1}^{\lambda^\dagger} x_n^{\lambda'} \prod_{a=1}^{k-1} \tilde{D}_{n-1,q^{2k}}(q^{2a}; q^2, q^{2k}) P_{\mu'}(x/x_n; q^2, q^{2k})
\]
\[
= x_n^{(k-1)(n-1)} \prod_{a=1}^{k-1} \prod_{i=1}^{n} \left( x_i - q^{2s} x_n \right)^{\lambda'} \sum_{\mu' = \lambda_1}^{\lambda^\dagger} x_n^{\lambda - |\mu'|} P_{\mu'}(x/x_n; q^2, q^{2k}) \prod_{i\leq j}[\tilde{\lambda}_i - \tilde{\mu}_j + k - 1]_{k-1} \prod_{i<j}[\tilde{\lambda}_i - \tilde{\lambda}_j - 1]_{k-1}
\]
\[
= \sum_{\mu' \geq \lambda} x_n^{\lambda - |\mu'|} P_{\mu'}(x/x_n; q^2, q^{2k}) \prod_{i\leq j}[\tilde{\lambda}_i - \tilde{\mu}_j + k - 1]_{k-1} \prod_{i<j}[\tilde{\lambda}_i - \tilde{\lambda}_j - 1]_{k-1},
\]
which is the desired result.

3.6 Specializing the expression for diagonal matrix elements

We will prove Proposition 3.4.3 using a result of [3] on reduced Clebsch-Gordan coefficients. We normalize and translate this result to matrix elements to obtain our desired expression. We first modify the intertwiner slightly. Consider the composition
\[
\tilde{\Psi}_\lambda : L_{\lambda+(k-1)\tilde{\rho}} \to L_{\lambda+(k-1)\tilde{\rho}} \otimes W_{k-1} \simeq L_{\lambda+(k-1)\tilde{\rho}-(k-1)1} \otimes \text{Sym}^{(k-1)n} \mathbb{C}^n.
\]
The diagonal matrix element \( c(\mu, \lambda) \) of \( v_{\tilde{\Phi}_\lambda}(\tilde{\mu}) \) in \( \tilde{\Phi}_\lambda \) is equal to the matrix element from \( v_{\tilde{\Phi}(\tilde{\mu})} \) to \( v_{\tilde{\Phi}((\tilde{\mu})-(k-1)1)} \) in \( \tilde{\Psi}_\lambda \). We will compute this matrix element instead.
3.6.1 The expression of [3] for reduced Clebsch-Gordan coefficients

The $U_q(\mathfrak{g}_n)$-representation $L_\tau \otimes \text{Sym}^p \mathbb{C}^n$ contains each irreducible with multiplicity at most one, meaning that for any $\tau'$, there is a one-dimensional family of intertwiners $L_{\tau'} \to L_\tau \otimes \text{Sym}^p \mathbb{C}^n$.

In [3], a general formula for the matrix coefficients in the Gelfand-Tsetlin basis of one such map is given. Such matrix coefficients are known as Clebsch-Gordan coefficients.

Remark. Note that [3] uses the coproduct $\Delta_{AS} = \Delta_{21}$. As bialgebras, $U_{q^{-1}}(\mathfrak{g}_n)$ equipped with $\Delta_{AS}$ and $U_q(\mathfrak{g}_n)$ equipped with $\Delta$ are isomorphic, so we state and apply here the formulas of [3] with $q$ and $q^{-1}$ exchanged.

Note that a Gelfand-Tsetlin basis vector $v_\xi$ for $\text{Sym}^p \mathbb{C}^n$ takes the form $\xi^i = (\xi^i, 0, \ldots, 0)$, so we will denote this by $\xi^i = \xi^i$. For basis vectors $v_\sigma \in L_{\tau'}$ and $v_\eta \otimes v_\xi \in L_\tau \otimes \text{Sym}^p \mathbb{C}^n$, it is shown in [3] that the corresponding Clebsch-Gordan coefficient is given by a product

$$C \left[ \begin{array}{ccc} \tau & p & \tau' \\ \eta & \xi & \sigma \end{array} \right] = \prod_{i=1}^{n-1} C \left[ \begin{array}{ccc} \sigma_{i+1} & \xi_{i+1} & \eta_{i+1} \\ \sigma_i & \xi_i & \eta_i \end{array} \right],$$

where product is over reduced Clebsch-Gordan coefficients whose values are given by the following.

Theorem 3.6.1 ([3, Equation (3.4)]). The reduced Clebsch-Gordan coefficient of the map $L_{\tau'} \to L_\tau \otimes \text{Sym}^p \mathbb{C}^n$ is given by

$$C \left[ \begin{array}{ccc} \tau & p & \tau' \\ \eta & \eta' \end{array} \right] = q^{-\frac{1}{2}} \frac{S(\eta', \eta)S(\tau, \eta)S(\tau', \tau')S(\eta, \eta)}{S(\tau', \tau)S(\tau', \eta')} [p - \tau]!^{1/2}$$

$$\sum_\sigma (-1)^{||\sigma||} q^{(p-r+1)(||\sigma||-||\eta||)} \frac{S(\sigma, \sigma)^2 S(\tau', \sigma)^2}{S(\sigma, \eta)^2 S(\eta', \sigma)^2 S(\tau, \sigma)^2},$$

where the sum is over $\sigma$ of length $n - 1$ satisfying

$$\max\{\eta_i, \tau_{i+1}'\} \leq \sigma_i \leq \min\{\eta_i', \tau_i\},$$

and where $b$ and $S$ are given by

$$b = \sum_{i<j} (\tau_i' - \tau_i)(\tau_j' - \tau_j) - \sum_{i<j} (\eta_i' - \eta_i)(\eta_j' - \eta_j)$$

$$+ \sum_i (\eta_i - \eta_i)(\eta_i - i + 1) - \sum_i (\tau_i' - \tau_i)(\tau_i - i + 1) + (p - \tau)(|\tau| - |\eta|)$$

and

$$S = \sum_{i<j} (\tau_i' - \tau_i)(\tau_j' - \tau_j) + \sum_i (\tau_i' - \tau_i)(\tau_i - i + 1) = (p - \tau)(|\tau| - |\eta|).$$
and
\[ S(a, b)^2 = \frac{\prod_{i<j}|a_i - b_j + j - i|!}{\prod_{i<j}|b_i - a_j + j - i - 1|!}. \]

### 3.6.2 Specializing the reduced Clebsch-Gordan coefficient

We restrict now to our case of \( \tilde{\Psi}_\lambda \). The relevant parameters are
\[
\tau' = \tilde{\lambda} \quad \tau = \tilde{\lambda} - (k - 1)1 \quad \eta' = \tilde{\mu} \quad \eta = \tilde{\mu} - (k - 1)1 \quad p = n(k - 1) \quad r = (n - 1)(k - 1).
\]

In this case, we see that
\[
b = \frac{n(n - 1)}{2}(k - 1)^2 - \frac{(n - 1)(n - 2)}{2}(k - 1)^2 + (k - 1)|\tilde{\mu}|
- \frac{(n - 1)(n - 2)}{2}(k - 1) - (k - 1)|\tilde{\lambda}| + \frac{n(n - 1)}{2}(k - 1) + (k - 1)(|\tilde{\lambda}| - |\tilde{\mu}|)
= (n - 1)(k - 1)^2 + (n - 1)(k - 1)\]
= \( (n - 1)k(k - 1) \).

Further, the constraint on \( \sigma \) takes the form
\[
\max\{\mu_i - (k - 1) - (k - 1)i, \lambda_{i+1} - (k - 1)(i + 1)\} \leq \sigma_i
\leq \min\{\mu_i - (k - 1)i, \lambda_i - (k - 1) - (k - 1)i\},
\]
so if \( \sigma = \nu + (k - 1)\tilde{\nu} \), we have
\[
\max\{\mu_i - (k - 1), \lambda_{i+1} - (k - 1)\} \leq \nu_i \leq \min\{\mu_i, \lambda_i - (k - 1)\}.
\]

Translating and canceling a factor, we have that
\[
C \begin{bmatrix} \tilde{\lambda} - (k - 1)1 & n(k - 1) & \tilde{\lambda} \\ \tilde{\mu} - (k - 1)1 & (n - 1)(k - 1) & \tilde{\mu} \end{bmatrix}
= q^{-\frac{(n-1)(k-1)}{2}} [k - 1]^{\frac{1}{2}} S(\tilde{\mu}, \tilde{\mu} - (k - 1)1) S(\tilde{\lambda}, \tilde{\lambda}) S(\tilde{\mu}, \tilde{\mu})
\]
\[ \sum_{\nu} (-1)^{|\nu| - |\mu|} q^{k(|\nu| - |\mu|)} S(\tilde{\nu}, \tilde{\nu})^2 S(\tilde{\lambda}, \tilde{\nu})^2 \]
\[ S(\tilde{\nu}, \tilde{\mu} - (k - 1)1)^2 S(\tilde{\mu}, \tilde{\nu})^2 S(\tilde{\lambda} - (k - 1)1, \tilde{\nu})^2. \]

Denote the latter sum by \( \Sigma(\mu, \lambda) \) and the prefactor by \( B(\mu, \lambda) \).
3.6.3 Computing the normalization factor

Write \( \overline{\lambda} \) for the truncation of \( \lambda \), and note that both interpretations of \( \overline{\lambda} \) are equal. Further, denote by \( \text{sgt}(r) \) the pattern

\[
\{(k - 1) \prec 2(k - 1) \prec \cdots \prec r(k - 1)\},
\]

where for \( 1 \leq i \leq r \), \( i(k - 1) \) is identified with the length \( i \) signature \((i(k - 1), 0, \ldots, 0)\); note that \( v_{\text{sgt}(n)} \) has weight \((k - 1)1\) in \( \text{Sym}^{n(k - 1)} \mathbb{C}^n \).

We now consider the special case where \( \mu = \overline{\lambda} \), which will allow us to translate between normalization factors for the Clebsch-Gordan coefficients. In this case, the constraint on \( \nu \) implies the sum is over the single term \( \nu = \mu - (k - 1)1 = \overline{\lambda} - (k - 1)1 \), and the matrix coefficient is

\[
C \begin{bmatrix} \overline{\lambda} - (k - 1)1 & n(k - 1) \\ \overline{\lambda} - (k - 1)1 & (n - 1)(k - 1) \end{bmatrix} \overline{\lambda} = q^{-\frac{3(n-1)(k-1)}{2}} \left[ k - 1 \right]^{1/2} (-1)^{(n-1)(k-1)} q^{-k(k-1)(n-1)} \frac{S(\overline{\lambda}, \overline{\lambda} - (k - 1)1) S(\overline{\lambda}, \overline{\lambda}) S(\overline{\lambda}, \overline{\lambda} - (k - 1)1)^2}{S(\lambda, \lambda - (k - 1)1) S(\lambda, \lambda) S(\lambda, \lambda - (k - 1)1)^2}.
\]

Notice now that

\[
\left( \frac{S(\overline{\lambda}, \overline{\tau}) S(\overline{\lambda}, \overline{\tau})}{S(\overline{\lambda}, \overline{\tau})^2} \right)^2 = \prod_{1 \leq i < j \leq n-1} \frac{[\overline{\lambda}_i - \overline{\tau}_j]^2}{[\overline{\lambda}_i - \overline{\tau}_j]^2} \prod_{1 \leq i < j \leq n-1} \frac{[\overline{\tau}_i - \overline{\lambda}_j]^2}{[\overline{\tau}_i - \overline{\lambda}_j]^2} \prod_{1 \leq i \leq n} [\overline{\tau}_i - \overline{\lambda}_n]^2 \prod_{1 \leq i \leq n} [\overline{\lambda}_i - \overline{\lambda}_n]^2.
\]

Applying this twice, we conclude that

\[
C \begin{bmatrix} \overline{\lambda} - (k - 1)1 & n(k - 1) \\ \overline{\lambda} - (k - 1)1 & (n - 1)(k - 1) \end{bmatrix} \overline{\lambda} = (-1)^{(n-1)(k-1)} q^{-\frac{3(n-1)(k-1)}{2}} \left( \prod_{i=1}^{n-1} \frac{[\overline{\lambda}_i - \overline{\lambda}_n][\overline{\lambda}_i - \overline{\lambda}_n - 1]}{[\overline{\lambda}_i - \overline{\lambda}_n + (k - 1)][\overline{\lambda}_i - \overline{\lambda}_n - (k - 1)]} \right)^{1/2}.
\]

Iterating this, we find that the diagonal Clebsch-Gordan coefficient of the highest
weight vector is
\[
C \begin{bmatrix}
\tilde{\lambda} - (k - 1) \mathbf{1} & n(k - 1) & \tilde{\lambda} \\
\gt(\tilde{\lambda} - (k - 1) \mathbf{1}) & \sgt(n - 1) & \gt(\tilde{\lambda})
\end{bmatrix}
= (-1)^{(n-1)(k-1)/2} q^{-\frac{3}{4}n(n-1)k(k-1)} \left( \prod_{i<j} \frac{[\tilde{\lambda}_i - \tilde{\lambda}_j - 1]_{k-1}}{[\tilde{\lambda}_i - \tilde{\lambda}_j + (k - 1)]_{k-1}} \right)^{1/2},
\]
where we recall that \( \gt \) was defined in (3.4.1).

### 3.6.4 Proof of Proposition 3.4.3

We now put everything together to prove Proposition 3.4.3. The diagonal Clebsch-Gordan coefficient of each highest weight vector for \( U_q(\mathfrak{g}_{n-1}) \) in the Gelfand-Tsetlin basis is
\[
C \begin{bmatrix}
\tilde{\lambda} - (k - 1) \mathbf{1} & n(k - 1) & \tilde{\lambda} \\
\gt(\tilde{\lambda} - (k - 1) \mathbf{1}) & \sgt(n - 1) & \gt(\tilde{\lambda})
\end{bmatrix}
= B(\mu, \lambda) \Sigma(\mu, \lambda)(-1)^{(n-1)(k-1)/2} q^{-\frac{3}{4}n(n-1)k(k-1)} \prod_{i<j} \frac{[\tilde{\mu}_i - \tilde{\mu}_j - 1]_{k-1}^{1/2}}{[\tilde{\mu}_i - \tilde{\mu}_j + (k - 1)]_{k-1}^{1/2}}.
\]

In terms of \( \Delta_1^{k-1} \) and \( \Delta_2^{k-1} \) from (3.4.2), the matrix element of \( \tilde{\Psi}_\lambda \) and hence \( \tilde{\Phi}_\lambda \) we are interested in is
\[
c(\mu, \lambda) = C \begin{bmatrix}
\tilde{\lambda} - (k - 1) \mathbf{1} & n(k - 1) & \tilde{\lambda} \\
\gt(\tilde{\lambda} - (k - 1) \mathbf{1}) & \sgt(n - 1) & \gt(\tilde{\lambda})
\end{bmatrix}^{-1}
= B(\mu, \lambda) \Sigma(\mu, \lambda)(-1)^{(n-1)(k-1)/2} q^{\frac{3}{4}n(n-1)k(k-1)} \frac{\Delta_1^{k-1}(\lambda)^{1/2}\Delta_2^{k-1}(\mu)^{1/2}}{\Delta_1^{k-1}(\lambda)^{1/2}\Delta_2^{k-1}(\mu)^{1/2}}
= (-1)^{(n-1)(k-1)} q^{(n-1)k(k-1)[k-1]^{1/2}} \frac{\Delta_1^{k-1}(\lambda)^{1/2}\Delta_2^{k-1}(\mu)^{1/2}}{\Delta_1^{k-1}(\lambda)^{1/2}\Delta_2^{k-1}(\mu)^{1/2}}
\frac{S(\tilde{\mu}, \tilde{\mu} - (k - 1)1)S(\tilde{\lambda}, \tilde{\lambda})S(\tilde{\mu}, \tilde{\mu})}{S(\tilde{\lambda}, \tilde{\lambda} - (k - 1)1)} \Sigma(\mu, \lambda),
\]
where
\[
\Sigma(\mu, \lambda) = \sum_{\nu} (-1)^{|\nu| - |\mu|} q^{k(|\nu| - |\mu|)} X(\nu, \mu, \lambda)
\]
with
\[
X(\nu, \mu, \lambda) = \frac{S(\tilde{\nu}, \tilde{\nu})^2 S(\tilde{\lambda}, \tilde{\nu})^2}{S(\tilde{\nu}, \tilde{\mu} - (k - 1)1)^2 S(\tilde{\mu}, \tilde{\nu})^2 S(\tilde{\lambda} - (k - 1)1, \tilde{\nu})^2}.
\]

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Observe that
\[
\frac{S(\tilde{\lambda}, \tilde{\lambda})}{S(\tilde{\lambda}, \tilde{\lambda} - (k - 1)1)} = \left( \prod_{i \leq j} \frac{[\tilde{\lambda}_i - \tilde{\lambda}_j]!}{[\tilde{\lambda}_i - \tilde{\lambda}_j + (k - 1)]!} \prod_{i < j} \frac{[\tilde{\lambda}_i - \tilde{\lambda}_j - k]!}{[\tilde{\lambda}_i - \tilde{\lambda}_j - 1]!} \right)^{1/2}
\]
and
\[
S(\bar{\mu}, \bar{\mu} - (k - 1)1)S(\bar{\mu}, \bar{\mu}) = \left( \prod_{i < j} \frac{[\bar{\mu}_i - \bar{\mu}_j + (k - 1)]!}{[\bar{\mu}_i - \bar{\mu}_j - k]!} \right)^{1/2} \prod_{i < j} [\bar{\mu}_i - \bar{\mu}_j] \Delta_1^{k-1}(\mu) \Delta_2^{k-1}(\mu)^{1/2}.
\]

We conclude that
\[
c(\mu, \lambda) = (-1)^{(n-1)(k-1)}q^{(n-1)k(k-1)} \prod_{i < j} [\bar{\mu}_i - \bar{\mu}_j] \Delta_2^{k-1}(\mu) \Delta_2^{k-1}(\lambda) \Sigma(\mu, \lambda).
\]

We now notice that
\[
S(\bar{\nu}, \bar{\mu} - (k - 1)1)^2S(\bar{\mu}, \bar{\nu})^2 = \frac{\prod_{i < j} [\bar{\nu}_i - \bar{\mu}_j + (k - 1)]! [\bar{\mu}_i - \bar{\mu}_j]!}{\prod_{i < j} [\bar{\nu}_i - \bar{\mu}_j - k]! [\bar{\mu}_i - \bar{\mu}_j - 1]!} \prod_{i} [\bar{\nu}_i - \bar{\mu}_i]! [\bar{\mu}_i - \bar{\nu}_i]! \prod_{i < j} [\bar{\nu}_i - \bar{\mu}_j + (k - 1)]k [\bar{\mu}_i - \bar{\nu}_j]_k
\]
and that
\[
\frac{S(\tilde{\lambda}, \tilde{\nu})^2}{S(\tilde{\lambda} - (k - 1)1, \tilde{\nu})^2} = \prod_{i \leq j} \frac{[\tilde{\lambda}_i - \tilde{\nu}_j]!}{[\tilde{\lambda}_i - \tilde{\nu}_j - (k - 1)]!} \prod_{i < j} \frac{[\tilde{\nu}_i - \tilde{\lambda}_j + k - 2]!}{[\tilde{\nu}_i - \tilde{\lambda}_j - 1]!} \prod_{i} [\tilde{\lambda}_i - \tilde{\nu}_i]_{k-1} \prod_{i < j} [\tilde{\lambda}_i - \tilde{\nu}_j]_{k-1} [\tilde{\nu}_i - \tilde{\lambda}_j + (k - 2)]_{k-1}.
\]

We conclude that
\[
X(\nu, \mu, \lambda) = \frac{S(\bar{\nu}, \bar{\mu})^2S(\tilde{\lambda}, \tilde{\nu})^2}{S(\bar{\nu}, \bar{\mu} - (k - 1)1)^2S(\tilde{\lambda}, \tilde{\nu})^2S(\tilde{\lambda} - (k - 1)1, \bar{\nu})^2} = \prod_{i < j} [\tilde{\nu}_i - \tilde{\nu}_j] \prod_i \frac{[\tilde{\lambda}_i - \tilde{\nu}_i]_{k-1}}{[\bar{\nu}_i - \bar{\mu}_i]! [\bar{\mu}_i - \bar{\nu}_i]!} \prod_{i < j} \frac{[\tilde{\nu}_i - \tilde{\lambda}_j]_{k-1}}{[\bar{\nu}_i - \bar{\lambda}_j + (k - 2)]_{k-1}}.
\]
Putting everything together, the expression we obtain for the diagonal matrix element is

\[
c(\mu, \lambda) = (-1)^{(n-1)(k-1)} q^{(n-1)k(k-1)} \prod_{i<j} [\bar{\mu}_i - \bar{\mu}_j] \frac{\Delta_2^{k-1}(\mu)}{\Delta_2^{k-1}(\lambda)} \sum_{\nu' = \mu'} (-1)^{|\nu'| - |\mu'|} q^{k(|\nu'| - |\mu'|)} \\
\prod_{i<j} [\bar{\lambda}_i - \bar{\nu}_j] \prod_{i} \frac{[\bar{\lambda}_i - \bar{\nu}_j]_{k-1}}{[\bar{\nu}_i - \bar{\mu}_i]_{k-1}} \prod_{i<j} \frac{[\bar{\lambda}_i - \bar{\nu}_j]_{k-1}[\bar{\nu}_i - \bar{\lambda}_j + (k - 2)]_{k-1}}{[\bar{\nu}_i - \bar{\mu}_j + (k - 1)]_{k-1}[\bar{\mu}_i - \bar{\nu}_j]_{k-1}}.
\]

In terms of \(\mu'\) and \(\nu'\), this is the desired

\[
c(\mu, \lambda) = (-1)^{(n-1)(k-1)} q^{(n-1)k(k-1)} \prod_{i<j} [\bar{\mu}_i' - \bar{\mu}_j'] \frac{\Delta_2^{k-1}(\mu')}{\Delta_2^{k-1}(\lambda)} \\
\sum_{\nu' = \mu'} (-1)^{|\nu'| - |\mu'|} q^{k(|\nu'| - |\mu'|)} \\
\prod_{i<j} [\bar{\lambda}_i - \bar{\nu}_j' + (k - 1)]_{k-1} \prod_{i} \frac{[\bar{\lambda}_i - \bar{\nu}_j' + (k - 1)]_{k-1}[\bar{\nu}_i' - \bar{\lambda}_j + 1]_{k-1}}{[\bar{\nu}_i' - \bar{\mu}_j + (k - 1)]_{k-1}[\bar{\mu}_i' - \bar{\nu}_j']_{k-1}} \\
= \frac{(-1)^{(n-1)(k-1)} q^{(n-1)(k-1)}}{\Delta_2^{k-1}(\lambda)\Delta_1^{k-1}(\mu')} \sum_{\nu' = \mu'} (-1)^{|\nu'| - |\mu'|} q^{k(|\nu'| - |\mu'|)} \\
\prod_{i<j} [\bar{\nu}_i' - \bar{\mu}_j' + k - 1]_{2k-1} \prod_{i<j} [\bar{\nu}_i' - \bar{\nu}_j'] \\
\prod_{i<j} [\bar{\lambda}_i - \bar{\nu}_j' + (k - 1)]_{k-1} \prod_{i<j} [\bar{\mu}_i' - \bar{\nu}_j']_{k-1}.
\]
Chapter 4

Traces of intertwiners for $U_q(\widehat{\mathfrak{sl}_2})$ and Felder-Varchenko functions

4.1 Introduction

The present chapter connects two approaches for studying a family of special functions occurring in the study of the $q$-KZB heat equation, one originating in the representation theory of quantum affine algebras and one in the theory of theta hypergeometric integrals. In [33], Etingof-Schiffmann-Varchenko showed that certain generalized traces for $U_q(\widehat{\mathfrak{g}})$-representations solve four commuting systems of difference equations: the $q$-KZB, dual $q$-KZB, Macdonald-Ruijsenaars, and dual Macdonald-Ruijsenaars systems. In [44, 45], Felder-Tarasov-Varchenko constructed solutions to the $q$-KZB and dual $q$-KZB systems in terms of certain theta hypergeometric integrals which we term Felder-Varchenko functions. The general philosophy of KZ-type equations predicts that these two families of solutions should be related by a simple renormalization, and this was conjectured by Etingof-Varchenko in [37, 33, 39]. In the trigonometric and classical limits, the Etingof-Varchenko conjecture was verified for the $\mathfrak{sl}_2$-case in [27, 37, 32, 39, 92].

In this chapter, we show that the traces of $U_q(\widehat{\mathfrak{sl}_2})$-intertwiners of [33] converge in a certain region of parameters and give a representation-theoretic construction of the Felder-Varchenko functions for the three-dimensional evaluation representation of $U_q(\widehat{\mathfrak{sl}_2})$. This gives the first proof that such a trace function converges and resolves the first case of the Etingof-Varchenko conjecture. As applications, we prove a symmetry property for traces of intertwiners and prove Felder-Varchenko's conjecture in [49] that their elliptic Macdonald polynomials are related to the affine Macdonald polynomials defined as traces over irreducible integrable $U_q(\widehat{\mathfrak{sl}_2})$-modules in [28]. We take the trigonometric and classical limits explicitly and recover results of [27, 37].

Our method relies on an interplay between the free field realization of the $q$-Wakimoto module of [77], convergence properties given by the theta hypergeometric integrals of [48], and rationality properties originating from the representation-theoretic definition of the trace function. We apply the method of coherent states to the $q$-vertex operator expression for the $U_q(\widehat{\mathfrak{sl}_2})$-intertwiner to express its trace as
a formal Jackson integral of iterated contour integrals. Evaluation and a non-trivial manipulation of the resulting integrals allows us to identify the Jackson integral with a renormalization of the Felder-Varchenko function and therefore prove that the trace function converges.

Our work is motivated by Felder-Varchenko’s conjecture in [47, 48] that their functions satisfy the $q$-KZB heat equation, an integral equation which endows them with a $SL(3, \mathbb{Z})$ modular symmetry. Felder-Varchenko proved their conjecture by hypergeometric integral computations only in two special cases. In future work, we hope to apply and extend the representation-theoretic understanding of the Felder-Varchenko functions provided by this work to show that all Felder-Varchenko functions satisfy the $q$-KZB heat equation and prove the Felder-Varchenko conjecture.

In the remainder of this introduction, we state our results more precisely and give some additional motivation and background. For convenience, all notations will be reintroduced in full detail in later sections. This chapter is based on the paper [96].

### 4.1.1 Trace functions for $\mathfrak{U}_q(\widehat{\mathfrak{sl}_2})$

We define now the trace function for $\mathfrak{U}_q(\widehat{\mathfrak{sl}_2})$ valued in the irreducible three-dimensional representation which appears in our main results. A more general definition appears in Section 4.2. For a weight $\mu$ and level $k$, let $M_{\mu,k}$ be the Verma module for $\mathfrak{U}_q(\widehat{\mathfrak{sl}_2})$ with highest weight $\mu \rho + k \Lambda_0$ and highest weight vector $v_{\mu,k}$. Let $\mu$ and $k$ be generic and $L_2(z)$ be the evaluation representation of $\mathfrak{U}_q(\widehat{\mathfrak{sl}_2})$ corresponding to the 3-dimensional irreducible representation of $\mathfrak{U}_q(\mathfrak{sl}_2)$. For $w_0 \in L_2[0]$, by [24, Theorem 9.3.1] there is a unique $\mathfrak{U}_q(\widehat{\mathfrak{sl}_2})$-intertwiner

$$\Phi^{w_0}_{\mu,k}(z) : M_{\mu,k} \rightarrow M_{\mu,k} \otimes L_2(z)$$

which satisfies

$$\Phi^{w_0}_{\mu,k}(z)v_{\mu,k} = v_{\mu,k} \otimes w_0 + (\text{l.o.t.})$$

where (l.o.t.) denotes terms of lower weight in the first tensor factor. Define the trace function by

$$T^{w_0}(q, \lambda, \omega, \mu, k) := \text{Tr}_{M_{(\mu-1)\rho+(k-2)\Lambda_0}} \left( \Phi^{w_0}_{\mu-1,k-2}(z) q^{2\lambda \rho + 2\omega \delta} \right),$$

where we treat the trace as a $\mathbb{C}$-valued function via the identification $L_2[0] \simeq \mathbb{C} \cdot w_0$ and we remark that the expression is independent of $z$.

In [33], it was shown that $T^{w_0}(q, \lambda, \omega, \mu, k)$ satisfies two commuting systems of infinite difference equations in $(\lambda, \omega)$ and $(\mu, k)$, the Macdonald-Ruijsenaars and dual Macdonald-Ruijsenaars equations. It was further shown that a generalization of $T^{w_0}(q, \lambda, \omega, \mu, k)$ valued in the tensor product of multiple representations satisfies the $q$-KZB and dual $q$-KZB equations, which are difference equations in $\lambda$ and $\mu$. 

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4.1.2 Felder-Varchenko functions

The other key object in the present work is the Felder-Varchenko function, which we again define for the three-dimensional representation here, leaving a more general definition to Section 4.3. In the region of parameters

\[ |q|, |q^{-2k}|, |q^{-2w}| < 1, \]

define the Felder-Varchenko function as the theta hypergeometric integral given by

\[
 u(q, \lambda, \omega, \mu, k) := q^{-\lambda \mu - \lambda - 2 - \mu - 2} \int_{C_t} \frac{dt}{2 \pi i t} \Omega_q(t; q^{-2w}, q^{-2k}) \frac{\theta_0(tq^{2\lambda}; q^{-2w})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{-2}; q^{-2w})}{\theta_0(tq^{-2}; q^{-2k})},
\]

where the phase function is

\[
 \Omega_q(z; r, p) := \frac{(za^{-1}; r, p)(z^{-1}a^{-1}rp; r, p)}{(za; r, p)(z^{-1}arp; r, p)},
\]

the theta function is \( \theta_0(u; q) := (u; q)(u^{-1}; q) \), the single and double q-Pochhammer symbols are \( (u; q) := \prod_{n \geq 0} (1 - uq^n) \) and \( (u; q, r) := \prod_{n, m \geq 0} (1 - uq^n r^m) \), and the contour \( C_t \) is the unit circle. Away from this region, \( u \) is defined by analytic continuation and extends to a function on the region of parameters satisfying \( |q^{-2w}| \neq 1 \) and \( |q^{-2k}| \neq 1 \).

In [48], the function \( u \) was shown to be a projective solution to the q-KZB heat equation, an integral equation which is the difference analogue of the KZB heat equation and endows the functions \( u \) with a \( SL(3, \mathbb{Z}) \)-modular symmetry. In [44, 45], generalizations of \( u \) valued in the tensor product of multiple representations were shown to satisfy the q-KZB and dual q-KZB equations.

4.1.3 Statement of the main results

Our main result links the trace function \( T^{w_0}(q, \lambda, \omega, \mu, k) \) to the Felder-Varchenko function in a certain region of the parameters \( q, q^{-2\mu}, q^{-2\lambda}, q^{-2w}, q^{-2k} \). We term this region the *good region of parameters* and define it as the region where the parameters satisfy

\[ 0 < |q^{-2\omega}| \ll |q^{-2\mu}| \ll |q^{-2\lambda}| \ll |q^{-2k}| \ll |q|, |q|^{-1}. \quad (4.5.1) \]

We show that the trace function converges and admits an integral formula on this region of parameters.

**Theorem 4.5.1.** For \( q^{-2\mu} \) and then \( q^{-2w} \) sufficiently close to 0 in the good region of
parameters (4.5.1), the trace function converges and has value

\[ T^{\omega_0}(q, \lambda, \omega, \mu, k) = \frac{q^{\lambda u - \lambda + 2}(q^{-4}; q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2}; q^{-2\omega})(q^{-2\lambda}; q^{-2\omega})} \frac{(q^{-2k}; q^{-2k})(q^{4} q^{-2k}; q^{-2k})}{(q^{-2\mu + 2}; q^{-2\mu + 2}; q^{-2k})} \frac{(q^{-2w+2}; q^{-2w}; q^{-2k})^2}{(q^{-2w-2}; q^{-2w}; q^{-2k})^2} \int_{C_t} \frac{dt}{2\pi i} \Omega q^2(t; q^{-2w}, q^{-2k}) \frac{\theta_0(t q^{-2\mu}; q^{-2k})}{\theta_0(t q^{-2}; q^{-2k})} \theta_0(t q^{2\lambda}; q^{-2\omega}) \]

where the integration cycle \( C_t \) is the unit circle.

**Corollary 4.5.12.** For \( q^{-2\mu} \) and then \( q^{-2\omega} \) sufficiently close to 0 in the good region of parameters (4.5.1), the trace \( T^{\omega_0}(q, \lambda, \omega, \mu, k) \) is related to the Felder-Varchenko function by

\[ T^{\omega_0}(q, \lambda, \omega, \mu, k) = \frac{q^{-\mu+4}(q^{-4}; q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2}; q^{-2\omega})(q^{-2\lambda}; q^{-2\omega})} \frac{(q^{-2\mu+2}; q^{-2\omega}; q^{-2k})^2}{(q^{-2\omega-2}; q^{-2\omega}; q^{-2k})^2} \frac{(q^{-2k}; q^{-2k})(q^{4} q^{-2k}; q^{-2k})}{(q^{-2\mu+2}; q^{-2\mu+2}; q^{-2k})} u(q, \lambda, \omega, -\mu, k). \]

**Remark.** Together Theorem 4.5.1 and Corollary 4.5.12 form the quantum affine generalization of [37, 33, 39, Theorem 8.1] and resolve the first case of Etingof-Varchenko’s conjecture from [37]. Further, Theorem 4.5.1 gives the first proof that a trace function for the quantum affine algebra is analytic, as conjectured in [33, Remark 1].

We obtain also a symmetry property for a renormalization of the trace function motivated by representation theory. Define a normalized trace by

\[ \tilde{T}^{\omega_0}(q, \lambda, \omega, \mu, k) := \text{Tr}|_{M_{-\lambda-2\omega}}(q^{2\lambda' + 2\omega'})^{-1} T^{\omega_0}(q^{-1}, -\lambda, -\omega, \mu, k), \]

where we interpret \( T^{\omega_0}(q^{-1}, -\lambda, -\omega, \mu, k) \) via quasi-analytic continuation to the region \( |q^{-2\omega}| < 1 \) and \( |q^{-2\mu}| > 1 \) of \( T^{\omega_0}(q, \lambda, \omega, \mu, k) \). By this, we mean that the coefficients of \( T^{\omega_0}(q, \lambda, \omega, \mu, k) \) as a formal series in \( q^{-2\omega} \) converge to rational functions in \( q^{-2\mu} \) and \( q^{-2k} \) for \( |q^{-2k}| < 1 \) and \( |q^{-2\mu}| < 1 \), and \( T^{\omega_0}(q^{-1}, -\lambda, -\omega, \mu, k) \) is interpreted by evaluating the same rational function coefficients in the region \( |q^{-2\omega}| > 1 \) and \( |q^{-2\mu}| < 1 \). We show the following symmetry property for the renormalized trace.

**Theorem 4.8.1.** The function \( \tilde{T}^{\omega_0}(q, \lambda, \omega, \mu, k) \) is symmetric under interchange of \( (\lambda, \omega) \) and \( (\mu, k) \).

**Remark.** Theorem 4.8.1 is the generalization to the quantum affine case of the symmetry of [37, Proposition 6.3]. In our setting, an additional difficulty which does not arise in the trigonometric limit is the necessity of introducing quasi-analytic continuation. We expect the other symmetry of [37, Theorem 1.5] to generalize to our setting as well.

\[ \text{Integral formulas for traces of screened vertex operators are given in [70, 71] as noted in [33, Remark 1], but there has been no prior analysis of the convergence of the Jackson integrals.} \]
4.1.4 Relation to elliptic and affine Macdonald polynomials

In [49], Felder-Varchenko introduced the elliptic Macdonald polynomials $\tilde{J}_{\mu,k}(q, \lambda, \omega)$ as hypergeometric theta functions in terms of their eponymous functions. In the trigonometric limit, they showed that these functions were related to Macdonald polynomials, and they conjectured that in the general case they were related to the affine Macdonald polynomials $J_{\mu,k,2}(q, \lambda, \omega)$ of [28]. We prove this conjecture in Theorem 4.9.9.

**Theorem 4.9.9.** Let $k = k + 4$. For $q^{-2\mu}$ and then $q^{-2\omega}$ sufficiently close to 0 in the good region of parameters (4.5.1), the elliptic and affine Macdonald polynomials are related by

$$J_{\mu,k,2}(q, \lambda, \omega) = \frac{\tilde{J}_{\mu,k}(q, \lambda, \omega) (q^{-4}; q^{2\omega}) (q^{-2\omega}, q^{-2\omega})^3 (q^{-2\omega+2}; q^{-2\omega}, q^{-2\omega})^2}{f(q, q^{-2\omega}) (q^{-4}; q^{-2\omega}) (q^{-2\omega}; q^{-2\omega}) (q^{-2\omega+2}; q^{-2\omega}, q^{-2\omega})^2}$$

$$q^{\mu+4}(q^{-2\mu-6}; q^{-2\omega}) (q^{2\mu+2}, q^{-2\omega}) (q^{-2\omega+2}; q^{-2\omega}, q^{-2\omega}),$$

where $f(q, q^{-2\omega})$ is the normalizing function of Proposition 4.9.8.

**Remark.** This theorem shows that $\tilde{J}_{\mu,k}(q, \lambda, \omega)$ and $J_{\mu,k,2}(q, \lambda, \omega)$ are proportional with constant of proportionality explicit aside from dependence on the normalization constant $f(q, q^{-2\omega})$ for the denominator of the affine Macdonald polynomial left undetermined in [28]. In future work, we plan to use the results of the present chapter to explicitly evaluate $f(q, q^{-2\omega})$.

4.1.5 Relation to the Felder-Varchenko conjecture

One of our motivations for studying the trace function $T(q, \lambda, \omega, \mu, k)$ originates in two streams of prior work. In [33], it was shown that traces of intertwining operators for quantum affine algebras satisfy the Macdonald-Ruijsenaars, dual Macdonald-Ruijsenaars, q-KZB, and dual q-KZB equations, giving four commuting systems of difference equations. On the other hand, in [44, 45], it was shown that the Felder-Varchenko functions satisfy the q-KZB and dual q-KZB equations, and in [47, 48] it was conjectured that they also satisfy the q-KZB heat equation, which is non-trivial in the case considered in this chapter.

In [37, 33, 39], Etingof-Varchenko posed the natural conjecture that the resulting functions are related by a simple renormalization, and in the trigonometric and classical limits proofs of this conjecture for the $\mathfrak{sl}_2$ case were given in [37, 27, 92]. Corollary 4.5.12 resolves this conjecture in the case of functions valued in the three dimensional representation. This gives a representation-theoretic interpretation of the Felder-Varchenko function and provides the first proof that the trace function for $U_q(\mathfrak{sl}_2)$ converges for parameters lying in a certain region.

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2In this chapter, we consider only traces valued in the dimension 3 irreducible representation, for which the q-KZB and dual q-KZB equations are trivial.
In future work, we aim to use this interpretation to study the Felder-Varchenko functions using the representation theory of quantum affine algebras. In particular, we hope to prove symmetry properties similar to that of Theorem 4.8.1 and to approach the Felder-Varchenko conjecture that the Felder-Varchenko function satisfies the $q$-KZB heat equation. This would yield generalizations of the corresponding theorems shown in the trigonometric limit in [39].

### 4.1.6 Relation to geometry of Laumon spaces

In [78, 79], Negut realizes certain intertwiners for $\mathfrak{sl}_n$ and $\hat{\mathfrak{sl}}_n$ via geometric actions of these algebras on the equivariant cohomology of certain moduli spaces known as (affine) Laumon spaces. He interprets traces of these intertwiners as generating functions for the integrals of Chern polynomials of tangent bundles of the Laumon spaces, thereby relating the generating functions to Calogero-Moser systems via results of [23]. This picture is expected to admit quantization, giving a relation between $U_q(\mathfrak{sl}_n)$-intertwiners and the $K$-theory of the affine Laumon space; for instance, an action of the quantum loop algebra $U_q(L\mathfrak{sl}_n)$ on the $K$-theory of the affine Laumon space was constructed in [97]. Under this expected correspondence, our trace function would thereby encode certain intersection-theoretic computations in $K$-theory giving enumerative information about the affine Laumon spaces.

### 4.1.7 Outline of method and organization

We briefly outline our method. The main technical inputs to our computation of the trace function are the free field construction of the $q$-Wakimoto module of [77] and the method of coherent states for one loop correlation functions. Combining these tools, choosing contours carefully to ensure convergence, and performing some intricate integral manipulations yields the Jackson integral expression in Proposition 4.5.3 for the trace function. In the good region of parameters, we then check that this expression agrees with the formal expansion of the Felder-Varchenko function to deduce that the trace function converges and has the integral form of Theorem 4.5.1.

In the remainder of the chapter, we take degenerations and give some applications of Theorem 4.5.1. In Theorems 4.6.9 and 4.7.4, we take the classical and trigonometric limits of our expression and show that they reproduce prior results from [28] and [37]. In Theorem 4.8.1, we show that a normalized quasi-analytic continuation of our trace function has a symmetry property predicted by representation theory. Finally, in Theorem 4.9.9, we use the BGG resolution for irreducible integrable modules to compute the affine Macdonald polynomial in terms of our trace functions. The resulting expression takes the form of the hypergeometric theta functions posed as elliptic Macdonald polynomials by Felder-Varchenko in [49], resolving their conjecture on the connection between the two.

The remainder of this chapter is organized as follows. The technical heart of the chapter is in Sections 4.2 to 4.5. In Section 4.2, we define our notations for $U_q(\mathfrak{sl}_2)$ and for trace functions for $U_q(\hat{\mathfrak{sl}}_2)$-intertwiners. In particular, we fix a coproduct which agrees with [77] and is the opposite of that of [33]. In Section 4.3, we fix
notation for the Felder-Varchenko function and give a formal series expansion and quasi-analytic continuation for it. In Section 4.4, we fix notations for the free field realization of $U_q(\widehat{\mathfrak{sl}_2})$-modules given in [77] and compute normalizations of $q$-vertex operator expressions for intertwiners. In Section 4.5, we apply the method of coherent states to the $q$-vertex operators from Section 4.4 to obtain a contour integral formula for the trace function in Theorem 4.5.1 and identify it with the Felder-Varchenko function in Corollary 4.5.12.

In the remaining sections, we apply and take degenerations of Theorem 4.5.1. In Sections 4.6 and Sections 4.7, we verify in Theorems 4.6.9 and 4.7.4 that our computations are consistent with existing computations in the classical and trigonometric limits. In Section 4.8, we show that a renormalization of the trace function satisfies a symmetry property motivated by representation theory. In Section 4.9, we relate the affine and elliptic Macdonald polynomials to our trace functions using the BGG resolution and dynamical Weyl group for $U_q(\widehat{\mathfrak{sl}_2})$-modules and prove Felder-Varchenko's conjecture on their relation. Appendices 4.10 and 4.11 contain notations and estimates for elliptic functions and computations of OPE's and one loop correlation functions which occur in the method of coherent states.

### 4.2 The trace function for $U_q(\widehat{\mathfrak{sl}_2})$-intertwiners

In this section, we give our notations and conventions for $U_q(\widehat{\mathfrak{sl}_2})$ and define the trace functions for $U_q(\widehat{\mathfrak{sl}_2})$-intertwiners which will be the focus of this work. Our coproduct is the opposite of that in [33] and therefore some of our variable shifts are also different.

#### 4.2.1 The Cartan subalgebra of $\widehat{\mathfrak{sl}_2}$ and $\widehat{\mathfrak{sl}_2}$

Denote by $\widehat{\mathfrak{sl}_2}$ the affinization of $\mathfrak{sl}_2$ and by $\mathfrak{sl}_2$ its central extension. The Cartan subalgebra of $\mathfrak{sl}_2$ and its dual are given by

$$\mathfrak{h} = \mathbb{C}a \oplus \mathbb{C}c \oplus \mathbb{C}d \text{ and } \mathfrak{h}^* = \mathbb{C}a \oplus \mathbb{C}a_0 \oplus \mathbb{C}\delta,$$

where $\Lambda_0 = c^*$ and $\delta = d^*$. Define $\rho = \frac{1}{2}a$, and recall that $\mathfrak{sl}_2$ has dual Coxeter number $h = 1 + (\theta, \rho) = 2$, where $\theta = a$ is the highest root. Define $\alpha_1 := a$, $\alpha_0 := \delta - a \in \mathfrak{h}^*$, and $\tilde{\rho} := \rho + 2\Lambda_0$.

The algebra $\widehat{\mathfrak{sl}_2}$ admits a non-degenerate invariant form $(-,-)$ whose restriction to $\mathfrak{h}$ has non-trivial values

$$(\alpha, \alpha) = 2, \quad (c, d) = 1, \quad (d, d) = 0$$

and agrees with the form on $\mathfrak{h}$. This defines the identification $\widetilde{\mathfrak{h}} \simeq \mathfrak{h}^*$ given by

$$\alpha \mapsto \alpha, \quad c \mapsto \delta, \quad d \mapsto \Lambda_0.$$
Transporting the form to \( \mathfrak{h}^* \) yields the non-trivial values
\[
(\alpha, \alpha) = 2, \quad (\delta, \Lambda_0) = 1, \quad (\Lambda_0, \Lambda_0) = 0.
\]
The Cartan matrix \( A = (a_{ij}) \) for \( \widehat{\mathfrak{sl}}_2 \) is defined by \( a_{ij} := (\alpha_i, \alpha_j) \).

4.2.2 The algebras \( U_q(\widehat{\mathfrak{sl}}_2) \) and \( U_q(\widehat{\mathfrak{sl}}_2) \)

Let \( q \) be a non-zero complex number transcendental over \( \mathbb{Q} \), and for an integer \( n \) define the \( q \)-number by \( [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \), \( q \)-factorial by \( [n]! = [n] \cdots [1] \), \( q \)-falling factorial by \( [n]_l = [n] \cdots [n-l+1] \), and \( q \)-binomial coefficient by \( \binom{n}{m}_q = \frac{[n]!}{[m]! [n-m]!} \). The quantum affine algebra \( U_q(\widehat{\mathfrak{sl}}_2) \) is the Hopf algebra generated as an algebra by \( e_i, f_i, q^{\pm h_i} \) for \( i = 0, 1 \) with relations
\[
[q^{h_i}, q^{h_j}] = 0, \quad q^{h_i} e_j q^{-h_j} = q^{(h_i, \alpha_j)} e_j, \quad q^{h_i} f_j q^{-h_j} = q^{-(h_i, \alpha_j)} f_j,
\]
\[
[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_q e_i^{1-a_{ij}-k} e_j e_i^k = 0,
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_q f_i^{1-a_{ij}-k} f_j f_i^k = 0.
\]

The coproduct, antipode, and counit of \( U_q(\widehat{\mathfrak{sl}}_2) \) are
\[
\Delta(e_i) = e_i \otimes 1 + q^{h_i} e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \quad \Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i};
\]
\[
S(e_i) = -q^{-h_i} e_i, \quad S(f_i) = -f_i q^{h_i}, \quad S(q^{h_i}) = q^{-h_i};
\]
\[
\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(q^{h_i}) = 1.
\]

We centrally extend \( U_q(\widehat{\mathfrak{sl}}_2) \) to \( U_q(\widehat{\mathfrak{sl}}_2) \) by adding a generator \( q^d \) which commutes with \( q^{h_i} \) and whose commutators with \( e_i \) and \( f_i \) are
\[
[q^d, e_i] = [q^d, f_i] = 0 \text{ for } i \neq 0, \quad q^d e_0 q^{-d} = q e_0, \quad q^d f_0 q^{-d} = q^{-1} f_0
\]
and on which the coproduct, antipode, and counit are
\[
\Delta(q^d) = q^d \otimes q^d, \quad S(q^d) = q^{-d}, \quad \varepsilon(q^d) = 1.
\]

Define the subalgebras \( U_q(\widehat{\mathfrak{b}}_+) = \langle e_i, q^{\pm h_i} \rangle \), \( U_q(\widehat{\mathfrak{b}}_-) = \langle f_i, q^{\pm h_i} \rangle \), \( U_q(\widehat{\mathfrak{n}}_+) = \langle e_i \rangle \), and \( U_q(\widehat{\mathfrak{n}}_-) = \langle f_i \rangle \) of \( U_q(\widehat{\mathfrak{sl}}_2) \).

Remark. This coproduct agrees with that of [77, 71] and is opposite to that of [33].

4.2.3 Verma modules for \( U_q(\widehat{\mathfrak{sl}}_2) \)

We denote by \( M_{\mu, k} := M_{\mu h + k \Lambda_0} \) the Verma module for \( U_q(\widehat{\mathfrak{sl}}_2) \) with highest weight \( \mu h + k \Lambda_0 \) and by \( v_{\mu, k} \in M_{\mu, k} \) a canonically chosen highest weight vector; this module
is of level \(k\). We extend it to a \(U_q(\widehat{\mathfrak{s}l}_2)\)-module by letting \(q^d\) act by 1 on \(v_{\mu,k}\). Define the restricted dual of \(M_{\mu,k}\) by

\[ M'_{\mu,k} := \bigoplus_{m \geq 0} M_{\mu,k}[-m\delta]^*, \]

where the action of \(U_q(\widehat{\mathfrak{s}l}_2)\) is given by \((u \cdot \phi)(m) := \phi(S(u)m)\). Let \(v_{\mu,k}^*\) be the dual vector to the highest weight vector \(v_{\mu,k}\). For generic \((\mu, k)\), the Verma module \(M_{\mu,k}\) is irreducible; we will make this assumption throughout the chapter outside of Section 4.9.

**Remark.** In the notation of [77], \(M_{\mu,k}\) is the Verma module of spin \(\frac{\mu}{2}\) at level \(k\). In the notation of [33], \(M_{\mu,k}\) is equal to \(M^{ESV}_{\mu-k/2,k}\).

Define the algebra anti-automorphism \(\omega : U_q(\widehat{\mathfrak{s}l}_2) \to U_q(\widehat{\mathfrak{s}l}_2)\) by \(\omega(e_i) = f_i\), \(\omega(f_i) = e_i\), and \(\omega(q^h) = q^h\). Define a symmetric form on \(M_{\mu,k}\) by

\[ \mathcal{F}(av_{\mu,k}, bv_{\mu,k}) = \langle \omega(a)v_{\mu,k}^*, bv_{\mu,k}\rangle, \]

where \(a, b \in U_q(\widehat{\mathfrak{n}}_-)\) and \(\omega(a)v_{\mu,k}^* \in M'_{\mu,k}\). On \(M_{\mu,k}[\mu\rho + k\Lambda_0 - \lambda\rho - a\delta]\), it is related to the Shapovalov \(F(-,-)\) form of [63, Theorem 4.1.16] by

\[ \mathcal{F}(-,-) = C\lambda q^{-\mu\rho + k\Lambda_0, \lambda\rho + a\delta} F(-,-) \]

for some constant \(C\). Noting that by [63, Lemma 3.4.8] the Kostant partition function satisfies

\[ P(\lambda\rho + a\delta)(\lambda\rho + a\delta) = \sum_{\beta > 0} \sum_{n \geq 1} P(\lambda\rho + a\delta - n\beta)\beta, \]

we obtain the following scaling of the Kac-Kazhdan determinant formula for \(\mathcal{F}\).

**Proposition 4.2.1** ([63, Theorem 4.1.16]). When restricted to \(M_{\mu,k}[\mu\rho + k\Lambda_0 - \lambda\rho - a\delta]\), the form \(\mathcal{F}\) has determinant

\[ C \prod_{\beta > 0} \prod_{n \geq 1} (1 - q^{-2(\beta, \mu\rho + k\Lambda_0 + \rho) + n(\beta, \beta)} P(\lambda\rho + a\delta - n\beta)), \]

where \(C\) is a non-zero constant depending on choice of basis, \(\beta\) ranges over positive roots \(\{\alpha, \pm\alpha + m\delta, m\delta \mid m > 0\}\), and \(P\) denotes the Kostant partition function.

**Remark.** Proposition 4.2.1 is the quantum analogue of the Kac-Kazhdan determinant formula of [65].

### 4.2.4 Evaluation modules for \(U_q(\widehat{\mathfrak{s}l}_2)\)

Let \(M'_{\mu}\) denote the restricted dual Verma module of lowest weight \(-\mu\) for \(U_q(\mathfrak{s}l_2)\); if \(\mu\) is a positive integer, let \(L_\mu \subset M'_{\mu}\) be the corresponding irreducible finite dimensional module. Let \(M'_{\mu}(z)\) and \(L_\mu(z)\) denote the corresponding affinizations. We choose an
explicit basis $w_{2\mu}, \ldots, w_{-2\mu}$ for $L_{2\mu}(z)$ so that $U_q(\hat{\mathfrak{sl}}_2)$ acts by

$$e_1w_{2m} \otimes z^n = [\mu - m]w_{2m+2} \otimes z^n$$
$$e_0w_{2m} \otimes z^n = [\mu + m]w_{2m-2} \otimes z^{n+1}$$
$$f_1w_{2m} \otimes z^n = [\mu + m]w_{2m-2} \otimes z^n$$
$$f_0w_{2m} \otimes z^n = [\mu - m]w_{2m+2} \otimes z^{n-1}$$
$$q^{h_1}w_{2m} \otimes z^n = q^{2m}w_{2m} \otimes z^n$$
$$q^{h_0}w_{2m} \otimes z^n = q^{-2m}w_{2m} \otimes z^n$$
$$q^{d}w_{2m} \otimes z^n = q^{3}w_{2m} \otimes z^n.$$

Remark. This basis is related to the basis $v_{\mu,m}$ in [77] by $w_{2m} = v_{\mu,-m}$.

### 4.2.5 Completed tensor product and intertwiners

Let $V(z)$ be a finite-dimensional evaluation representation for $U_q(\hat{\mathfrak{sl}}_2)$. Define the completed tensor product by

$$M_{\mu,k} \hat{\otimes} V(z) := (M^\vee_{\mu,k})^* \otimes V(z) \simeq \text{Hom}_C(M^\vee_{\mu,k}, V(z)),$$

where by $(M^\vee_{\mu,k})^*$ we mean the full linear dual, the $U_q(\hat{\mathfrak{sl}}_2)$-action on $\text{Hom}_C(M^\vee_{\mu,k}, V(z))$ is given in Sweedler notation by $(u \cdot \rho)(m) = u(z)\rho(S(u_{(1)})m)$, and the isomorphism is given by $\phi^* \otimes v \mapsto \left( \psi \mapsto \phi^*(q^{-2\partial}\psi)v \right)$. As a vector space we have an isomorphism

$$M_{\mu,k} \hat{\otimes} V(z) = (M^\vee_{\mu,k})^* \otimes V(z) \simeq \prod_{a \geq 0} \left( M_{\mu,k}[-a\delta] \otimes V(z) \right)$$

under which elements of $M_{\mu,k} \hat{\otimes} V(z)$ are sums $\sum_{i=0}^\infty m_i \otimes v_i$ with $m_i, v_i$ homogeneous and $\lim_{i \to \infty} \deg(m_i) = \infty$. For $v \in V[\tau]$, if $M^\vee_{\mu,k}$ is irreducible, by [24, Theorem 9.3.1] there is a unique $U_q(\hat{\mathfrak{sl}}_2)$-intertwiner

$$\Phi^\vee_{\mu,k}(z) : M_{\mu,k} \to M_{\mu,\tau,k} \hat{\otimes} V(z)$$

which satisfies

$$\Phi^\vee_{\mu,k}(z)v_{\mu,k} = v_{\mu,k} \otimes v + (\text{l.o.t.}),$$

where (l.o.t.) denotes terms of lower weight in the first tensor factor. If $V_1, \ldots, V_n$ are finite-dimensional representations, $v_i \in V_i[\tau_i]$, define the iterated intertwiner by the composition of intertwiners

$$\Phi^{v_1,\ldots,v_n}_{\mu,k}(z_1, \ldots, z_n) : M_{\mu,k} \to M_{\mu,\tau,n,k} \hat{\otimes} V_n(z_n) \to \cdots \to M_{\mu,\tau_1,\ldots,\tau_n,k} \hat{\otimes} V_1(z_1) \hat{\otimes} \cdots \hat{\otimes} V_n(z_n).$$

Remark. These intertwiners are known as Type I intertwiners. Due to our coproduct convention, they correspond to Type II intertwiners in the coproduct of [33].
4.2.6 Definition of the trace function

Define the trace function \( \Psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) as the formal power series in \( q^{-2w} \) given by

\[
\Psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) := \text{Tr}_{M_{\mu,k}} \left( \Phi^{v_1, \ldots, v_n}_{\mu,k}(z_1, \ldots, z_n)q^{2\lambda \rho + 2\omega d} \right).
\]

In the case \( n = 1 \), the trace function \( \Psi^{v}(z; \lambda, \omega, \mu, k) \) is independent of \( z \), and we use the notation

\[
T^{v}(q, \lambda, \omega, \mu, k) := \Psi^{v}(z; \lambda, \omega, \mu - 1, k - 2)
\]

\[
= \text{Tr}_{M(\mu-1, \rho+(k-2)\omega)} \left( \Phi^{v}_{\mu-1,k-2}(z)q^{2\lambda \rho + 2\omega d} \right). \tag{4.2.1}
\]

If \( v = w_0 \in L_2m[0] \), then \( T^{w_0}(q, \lambda, \omega, \mu, k) \) lies in \( L_2m[0] \simeq \mathbb{C} \cdot w_0 \), and we interpret it as a \( \mathbb{C} \)-valued function via the identification \( \mathbb{C} \cdot w_0 \simeq \mathbb{C} \).

**Remark.** Our notation for \( \Psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) is related to that of [33] by the use of the opposite coproduct and the variable shifts

\[
\Psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \Psi^{v_1, \ldots, v_n}_{ESV}(z_1, \ldots, z_n; \lambda - \omega/2, \omega, \mu - k/2, k).
\]

We choose our conventions to make the classical and trigonometric limits more transparent.

**Remark.** As defined, all trace functions we consider are formal power series in \( q^{-2w} \). Our main result Theorem 4.5.1 shows that for \( w_0 \in L_2 \) the formal power series for \( T^{w_0}(q, \lambda, \omega, \mu, k) \) converges in a certain region of parameters, meaning it is analytic.

4.2.7 Rationality properties of the trace function

The goal of this subsection is to prove Proposition 4.2.6, which characterizes the coefficient of the power series expansion of the trace function in \( q^{-2w} \). We begin by giving an explicit expression for the intertwiner in terms of the form \( \mathcal{F} \) on \( M_{\mu,k} \). Choose a homogeneous basis \( \{g^\lambda_i\} \) for \( U_q(\mathfrak{n}_-) \) so that \( \text{wt}(g^\lambda_i) = -\lambda \rho - a\delta \), and define the matrix elements

\[
\mathcal{F}_{\lambda,a}(\mu, k)_{ij} = \mathcal{F}(g^\lambda_i v_{\mu,k}, g^\lambda_j v_{\mu,k})
\]

of the form and the elements \( \mathcal{F}^{-1}_{\lambda,a}(\mu, k)_{ij} \) of its inverse matrix.

**Lemma 4.2.2.** For \( v \in V(z) \), the vector

\[
\phi_{\mu,k,v} = \sum_{\lambda,a} \sum_{i,j} \mathcal{F}^{-1}_{\lambda,a}(\mu, k)_{ij} (q^{2\delta} g^\lambda_i v_{\mu,k}) \otimes \omega(g^\lambda_j v_{\mu,k}) v
\]

in \( M_{\mu,k} \otimes V(z) \) is singular.
Proof. Notice that $\phi_{\mu,k,v}$ is singular if and only if the composition

$$\psi : M_{\mu,k}^\vee \otimes_{\phi_{\mu,k,v}} M_{\mu,k} \otimes (M_{\mu,k})^* \otimes V(z) \to V(z)$$

induced by $\phi_{\mu,k,v}$ and $m \otimes f \mapsto f(q^{-2}m)$ is a map of $U_q(\widehat{\mathfrak{n}_+})$-modules. This holds since

$$\psi(\omega(g_{\mu,k,v}^\vee b)v_{\mu,k}^\vee) = \sum_{i,j} \sum_{\lambda,a} F_{\lambda,a}^{-1}(\mu, k)_{ij} \omega(g_{\mu,k}^\vee a)v_{\mu,k}^\vee \omega(g_{\mu,k}^\vee a)v = \sum_{i,j} F_{\mu,k}^{-1}(\mu, k)_{ij} \omega(g_{\mu,k}^\vee b)v = \omega(g_{\mu,k}^\vee b)v.$$

Lemma 4.2.3. The matrix elements $F_{\lambda,a}^{-1}(\mu, k)_{ij}$ of the inverse of $F$ are rational functions in $q^{-2\mu}$ and $q^{-2k}$ with at most simple poles whose denominators are products of linear terms of the form

$$(1 - q^{-2(\mu+1) - 2m(k+2) + 2n}), \quad (1 - q^{2(\mu+1) - 2(m+1)(k+2) + 2n}),$$

$$(1 - q^{-2(m+1)(k+2)}) \quad \text{for } m \geq 0, n \geq 1.$$

Proof. This is proven in the same way as [35, Lemma 3.2], noting that these linear terms are exactly those appearing in the Kac-Kazhdan determinant of Proposition 4.2.1.

Lemma 4.2.4. If $v \in V[0]$, the only linear terms from Lemma 4.2.3 which appear in coefficients $F_{\lambda,a}^{-1}(\mu, k)_{ij}$ of $\phi_{\mu,k,v}$ are those for which $V[n\alpha] \neq 0$ or $V[-n\alpha] \neq 0$.

Proof. This is proven in the same way as [36, Proposition 2.2].

Lemma 4.2.5. For $w_0 \in L_2[0]$, as functions of $q^{-2\mu}$ and $q^{-2k}$, matrix elements of $\Phi_{\mu,k}^w(z)$ in the PBW basis may be written in the form $P_{ij}(q^{-2\mu}, q^{-2k})Q_{ij}(q^{-2\mu}, q^{-2k})$ where

- $P_{ij}(q^{-2\mu}, q^{-2k})$ is a polynomial in $q^{-2\mu}$ and Laurent polynomial in $q^{-2k}$;
- $Q_{ij}(q^{-2\mu}, q^{-2k})$ is a polynomial in $q^{-2\mu}$ and Laurent polynomial in $q^{-2k}$ given by a product of linear terms of the form

$$(1 - q^{-2(\mu+1) - 2m(k+2) + \pm 2}), \quad (1 - q^{2(\mu+1) - 2(m+1)(k+2) + \pm 2}),$$

$$(1 - q^{-2(m+1)(k+2)}) \quad \text{for } m \geq 0;$$

- in each graded piece $M_{\mu,k}[-a\delta]$, the $P_{ij}$ contain a finite number of distinct non-zero monomials and the $Q_{ij}$ contain a finite number of linear factors.

Proof. This follows for $\Phi_{\mu,k,w_0}^w = \Phi_{\mu,k}^w(z)v_{\mu,k}$ by Lemma 4.2.4. Each other matrix element of $\Phi_{\mu,k}^w(z)$ is computed by acting on $\phi_{\mu,k,w_0}$ by monomials composed of products of $\Delta(f_i)$ for different $i$. Therefore, matrix elements in $M_{\mu,k}[-a\delta]$ for $a \leq A$ are finite linear combinations of the coefficients of $\phi_{\mu,k,w_0}$ in degree at least $(-A)$ scaled by constants independent of $\mu$ and $k$, yielding the desired for all matrix elements.\qed
We now constrain the structure of poles of the trace function. This result is similar to that of [36, Proposition 3.1]. However, in the quantum affine setting, it is possible for the trace function to have an infinite number of poles, but as a formal power series in $q^{-2\omega}$ each coefficient does not.

**Proposition 4.2.6.** For $w_0 \in L_2[0]$, the function $\Psi^{w_0}(z; \lambda, \omega, \mu, k)$ may be written in the form

$$\Psi^{w_0}(z; \lambda, \omega, \mu, k) = q^{\lambda \mu} \sum_{n \geq 0} q^{-2\omega n} \tilde{\Psi}^{w_0}_n(z; \lambda, \mu, k)$$

as a formal power series in $q^{-2\omega}$ where $\tilde{\Psi}^{w_0}_n(z; \lambda, \mu, k)$ takes the form $\frac{P_n(q^{-2\mu}, q^{-2k})}{Q_n(q^{-2\mu}, q^{-2k})}$ with $P_n(q^{-2\mu}, q^{-2k})$ polynomial in $q^{-2\mu}$ and Laurent polynomial in $q^{-2k}$ and $Q_n(q^{-2\mu}, q^{-2k})$ a polynomial given by a finite product of linear terms of the form

$$(1 - q^{-2(\mu+1)-2m(k+2)\pm2}), \quad (1 - q^{2(\mu+1)-2(m+1)(k+2)\pm2}),$$

$$(1 - q^{-2(m+1)(k+2)}) \quad \text{for} \ m \geq 0.$$

**Proof.** This follows from Lemma 4.2.5 in the same way as [36, Proposition 3.1] follows from [36, Proposition 2.2].

### 4.3 The Felder-Varchenko function

In this section, we introduce the Felder-Varchenko functions, specialize their definition to the three-dimensional irreducible representation of $U_q(sl_2)$, and derive some of their properties which we will use later. In particular, we give a series expansion and quasi-analytic continuation.

#### 4.3.1 Definition of the Felder-Varchenko function

For $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$, let

$$L_\Lambda = L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$$

be the tensor product of finite dimensional irreducible representations of $U_q(sl_2)$ with highest weights $\Lambda_1, \ldots, \Lambda_n$ and highest weight vectors $v_{\Lambda_1}, \ldots, v_{\Lambda_n}$. In [47], Felder-Varchenko defined the $L_\Lambda[0] \otimes L_\Lambda[0]$-valued functions

$$u(z_1, \ldots, z_n, q, \lambda, \omega, \mu, k, q) = q^{-\lambda \mu} \int \prod_i \Omega_{q^{\Lambda_i}} \left( \frac{t_i}{z_i}; q^{-2\omega}, q^{-2k} \right)$$

$$\prod_{i < j} \Omega_{q^{-2}} \left( \frac{t_i}{t_j}; q^{-2\omega}, q^{-2k} \right) \sum_{i,j} \omega_i(t_1, \ldots, t_m, z_1, \ldots, z_n, \lambda, q^{-2\omega}, \Lambda)$$

$$\omega_j(t_1, \ldots, t_m, z_1, \ldots, z_n, \mu, q^{-2k}, \Lambda) \frac{dt_1 \cdots dt_m}{(2\pi i)^{m} t_1 \cdots t_m} e_I \otimes e_J. \quad (4.3.1)$$

The notations in this definition are as follows. The summation is over $I$ and $J$ so that for $e_I = e^{i_1} v_{\Lambda_1} \otimes \cdots \otimes e^{i_n} v_{\Lambda_n}$ and $e_J = e^{j_1} v_{\Lambda_1} \otimes \cdots \otimes e^{j_n} v_{\Lambda_n}$, $e_I$ and $e_J$ form bases
of $L_\Lambda[0]$. The phase function $\Omega$ is defined in (4.10.2) by
\[
\Omega_a(z; r, p) := \frac{(za^{-1}; r, p)(z^{-1}a^{-1}rp; r, p)}{(za; r, p)(z^{-1}arp; r, p)},
\]
where $(u; r, p)$ is the double $q$-Pochhammer symbol of Subsection 4.10.1. The weight function $\omega_I$ is defined by
\[
\omega_{(t_1, \ldots, t_m, z_1, \ldots, z_n, \lambda, r, \Lambda)} = \prod_{i<j} \frac{\theta(t_{i,j}; r)}{\theta(t_{i,j} q^2; r)} \prod_{i=1}^n \prod_{h \in I_i} \frac{\prod_{j \neq h} \theta(t_{i,j} q^\Lambda; r)}{\prod_{j \in I_i} \theta(t_{i,j} q^{-\Lambda}; r)} \prod_{h \in I_i \setminus I_{j \in I_{I_i}}} \frac{\theta(t_{i,j} q^{2\lambda - \Lambda h + 2h - 2} \sum_{l=1}^{h-1} (\Lambda_l - 2i_l); r)}{\theta(t_{i,j} q^{-\Lambda h}; r)},
\]
where $\theta(-)$ is Jacobi's first theta function as defined in (4.10.1) and the summation is over disjoint subsets $I_1, \ldots, I_n$ of $\{1, \ldots, m\}$ so that $|I_h| = i_h$. The mirror weight function $\omega_I^\gamma$ is defined by
\[
\omega_{(t_1, \ldots, t_m, z_1, \ldots, z_n, \mu, p, \Lambda_1, \ldots, \Lambda_n)} = \omega_{(t_1, \ldots, t_m, z_1, \ldots, z_n, \mu, p, \Lambda_1, \ldots, \Lambda_n)}
\]
Finally, the cycle of integration $\gamma$ is defined by analytic continuation from the domain on which $|q|, |q^{-2k}|, |q^{-2\omega}|, |q^{\Lambda_j}| < 1$, where it is the $m$-fold product of unit circles. We refer to $u(z_1, \ldots, z_n, \lambda, \omega, \mu, k)$ as Felder-Varchenko functions.

**Remark.** We have adopted different notations from those used in [47]; in particular, we use multiplicative notation. For $q = e^{2\pi i \eta}$, we have
\[
u(q, \lambda, \omega, \mu, k) = u^{\text{FV}}(2\lambda \eta, 2\mu \eta, -2\omega \eta, -2k \eta, \eta),
\]
where $u^{\text{FV}}(\lambda, \mu, \tau, p, \eta)$ denotes the function from [47].

Let $n = 1$ and $\Lambda = (2)$, so that $L_\Lambda = L_2$, where $L_2$ is the 3-dimensional irreducible representation of $U_q(sl_2)$. Notice then that $L_2[0] \otimes L_2[0]$ is 1-dimensional, so we may consider $u(z, \lambda, \mu, \tau, p, \eta)$ as a numerical function. In this case, we have
\[
\omega_1(t, z, \lambda, r) = \frac{\theta(t \zeta q^{2\lambda}; r)}{\theta(t \zeta q^{-2}; r)} \quad \text{and} \quad \omega_1^\gamma(t, z, \mu, p) = \frac{\theta(t \zeta q^{2\mu}; p)}{\theta(t \zeta q^{-2}; p)}.
\]
Specializing (4.3.1) and noting that the result is independent of $z$ yields
\[
u(q, \lambda, \omega, \mu, k)
\]
\[
= q^{-\lambda \mu - \lambda - \mu - 2} \int_{\mathbb{C}^n} \frac{dt}{2\pi i \tau} \Omega(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(t \zeta q^{2\mu}; q^{-2k}) \theta_0(t \zeta q^{2\lambda}; q^{-2\omega})}{\theta_0(t \zeta q^{-2}; q^{-2k}) \theta_0(t \zeta q^{-2}; q^{-2\omega})},
\]
(4.3.3)
where for \(|q|, |q^{-2k}|, |q^{-2\omega}| < 1\) the contour \(C_t\) is the unit circle and we omit the \(z\) argument. The remainder of this section will be devoted to proving some properties of the Felder-Varchenko function.

### 4.3.2 Series expansion of the Felder-Varchenko function

In this section, we give formal power series expansions for the Felder-Varchenko function in formal neighborhoods of 0 and \(\infty\) in \(q^{-2\mu}\). Recall that the terminating \(q\)-Pochhammer symbol is \((u; q)_m = \frac{(u)_m}{(q)_m}\) as in Subsection 4.10.1.

**Proposition 4.3.1.** As a formal power series in \(q^{-2\mu}\), we have in a formal neighborhood of 0 in \(q^{-2}\) that

\[
q^{-\lambda} u(q, \lambda, \omega, \mu, k) \frac{1}{\theta_0(q^{2\mu+2}; q^{-2k})} = -q^{-\lambda-\mu-2} \left(\frac{q^{-4} q^{-2\omega}; q^{-2k}, q^{-2\omega}}{q^{-4} q^{-2k}; q^{-2k}, q^{-2\omega}}\right) \frac{1}{(q^{-4} q^{-2k}; q^{-2k})} \sum_{n \geq 0} q^{-(2\mu+2)n} \frac{\theta_0(q^{2\lambda+2}; q^{-2kn}; q^{-2\omega})}{\theta_0(q^{4} q^{-2kn}; q^{-2\omega})} \prod_{l=1}^{n} \theta_0(q^{4} q^{-2kl}; q^{-2\omega})
\]

and in a formal neighborhood of \(\infty\) in \(q^{-2\mu}\) that

\[
q^{\lambda+\mu} u(q, \lambda, \omega, \mu, k) \frac{1}{\theta_0(q^{2\mu-2}; q^{-2k})} = q^{-\lambda-2} \left(\frac{q^{-4} q^{-2\omega}; q^{-2k}, q^{-2\omega}}{q^{-4} q^{-2k}; q^{-2k}, q^{-2\omega}}\right) \frac{1}{(q^{-4} q^{-2k}; q^{-2k})} \sum_{n \geq 0} q^{(2\mu+2)n} \frac{\theta_0(q^{2\lambda-2}; q^{-2k}; q^{-2\omega})}{\theta_0(q^{4} q^{-2k}; q^{-2\omega})} \prod_{l=1}^{n} \theta_0(q^{-4} q^{-2k}; q^{-2\omega}).
\]

**Proof.** Denote the integrand of the Felder-Varchenko function by

\[
v(t, q, \lambda, \omega, \mu, k) = \Omega_t^2(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{-2}; q^{-2\omega})},
\]

and take the formal expansion

\[
v(t, q, \lambda, \omega, \mu, k) = \sum_{n \geq 0} v_n(t, q, \lambda, \mu, k) q^{-2\omega n}.
\]

We may write \(v(t, q, \lambda, \omega, \mu, k) = v_0(t, q, \lambda, \mu, k) \tilde{v}(t, q, \lambda, \omega, \mu, k)\), where \(\tilde{v}\) has power series expansion

\[
\tilde{v}(t, q, \lambda, \omega, \mu, k) = 1 + \sum_{n \geq 0} \tilde{v}_n(t, q, \lambda, \mu, k) q^{-2\omega n}
\]

with each \(\tilde{v}_n\) a Laurent polynomial in \(t\) of degree at most \(n\). In particular, all poles of \(v_n \in \mathbb{C}^*\) are those of \(v_0\). For small enough \(\varepsilon > 0\), we may choose contours for \(t\) with \(|t| \to 0\) satisfying the hypotheses of Corollary 4.10.2 for \(\varepsilon\). On such contours,
we have the estimate
\[
\left| \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} t^n \right| \leq D_1(q^{-2k}, \varepsilon)|q|^{n+1} |t^2q^{2\mu-2}|^{\frac{n+1}{2k}} |t|^n \leq C |t|^\frac{n+1}{k+n} \quad (4.3.4)
\]
for a constant \(C\) is independent of \(t\).

On a formal neighborhood of 0 in \(q^{-2\mu}\), because \(\tilde{v}_n\) is a Laurent polynomial in \(t\) of degree at most \(n\), by (4.3.4) we may compute \(u(q, \lambda, \omega, \mu, k)\) by deforming \(C_t\) to 0 and summing residues at \(t = q^2q^{-2k}\). As a power series in \(q^{-2\omega}\), we obtain
\[
q^{\lambda+\mu} \frac{u(q, \lambda, \omega, \mu, k)}{\theta_0(q^{2\mu+2}; q^{-2k})} = \frac{q^{-\lambda-\mu-2}}{\theta_0(q^{2\mu+2}; q^{-2k})} \sum_{n \geq 0} \text{Res}_{t=q^2q^{-2k}} \left( \frac{dt}{2\pi it} \Omega_q^2(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \right)
\]
\[
= -q^{-\lambda-\mu-2} \sum_{n \geq 0} \frac{(q^{-2k}(n+1); q^{-2k})_{n-1}(q^{-2kn}; q^{-2k})}{(q^{-2k}; q^{-2k})(q^{2k(n-1)}; q^{-2k})_{n-1}(q^{-2kn}; q^{-2k})} \frac{1}{(q^{-4}; q^{-2\omega})(q^{-4}; q^{-2k})(q^{-2k}; q^{-2\omega})} \sum_{n \geq 0} q^{-(2\mu+2)n} \frac{\theta_0(q^{2\mu+2}q^{-2kn}; q^{-2k})}{\theta_0(q^4q^{-2kn}; q^{-2\omega})} \prod_{l=1}^n \frac{\theta_0(q^{4q^{2kln}}; q^{-2\omega})}{\theta_0(q^{2kln}; q^{-2\omega})}.
\]
On a formal neighborhood of \(\infty\) in \(q^{-2\mu}\), we may similarly compute \(u(q, \lambda, \omega, \mu, k)\) by deforming \(C_t\) to \(\infty\) and summing residues at \(t = q^2q^{-2k}\), yielding
\[
q^{\lambda+\mu} \frac{u(q, \lambda, \omega, \mu, k)}{\theta_0(q^{2\mu-2}; q^{-2k})} = \frac{q^{-\lambda-2}}{\theta_0(q^{2\mu-2}; q^{-2k})} \sum_{n \geq 0} \text{Res}_{t=q^{-2}q^{-2k}} \left( \frac{dt}{2\pi it} \Omega_q^2(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \right)
\]
\[
= -q^{-\lambda-2} \sum_{n \geq 0} \frac{(q^{-4q^{-2\omega}}(q^{-2\omega}; q^{-2k})_n(q^{-2kn}; q^{-2k})(q^{-2k}; q^{-2\omega})}{(q^{-4}; q^{-2\omega})(q^{-4}; q^{-2k})(q^{-2k}; q^{-2\omega})} \sum_{n \geq 0} q^{-(2\mu+2)n} \frac{\theta_0(q^{2\mu+2}q^{-2kn}; q^{-2k})}{\theta_0(q^4q^{-2kn}; q^{-2\omega})} \prod_{l=1}^n \frac{\theta_0(q^{4q^{2kln}}; q^{-2\omega})}{\theta_0(q^{2kln}; q^{-2\omega})}.
\]
4.3.3 Quasi-analytic continuation of the Felder-Varchenko function

In what follows, we require the concept of quasi-analytic continuation of a formal power series in multiple variables. Let \( f^1(z, w_1, \ldots, w_m) \) and \( f^2(z, w_1, \ldots, w_m) \) be formal power series of the form

\[
f^i(z, w_1, \ldots, w_m) = \sum_{n \geq 0} f^i_n(w_1, \ldots, w_m) z^n.
\]

Suppose that for \(|z| < 1\), \( f^1(z, w_1, \ldots, w_m) \) and \( f^2(z, w_1, \ldots, w_m) \) converge on possibly disjoint regions \( U_1 \) and \( U_2 \) for \( w = (w_1, \ldots, w_m) \) and that each \( f^i_n(w_1, \ldots, w_m) \) admits analytic continuation to a rational function of \( w \). We say that \( f^1 \) is a quasi-analytic continuation of \( f^2 \) if for each \( n \) we have an equality of analytic continuations \( f^i_n(w_1, \ldots, w_m) = f^2_n(w_1, \ldots, w_m) \). We denote this by \( f^1(z, w_1, \ldots, w_m) \equiv f^2(z, w_1, \ldots, w_m) \). As an example, we compute a quasi-analytic continuation we will use later.

Lemma 4.3.2. As quasi-analytic continuations in the formal variable \( p \) from \(|r| < 1\) to \(|r| > 1\), we have

\[
(p; r) \equiv (r^{-1} p; r^{-1})^{-1} \quad \text{and} \quad (ap; p, r) \equiv (ar^{-1} p; p, r^{-1})^{-1}.
\]

Proof. For the first claim, notice that

\[
(p; r) = \prod_{n \geq 0} (1 - pr^n) = \sum_{k \geq 0} (-1)^k p^k r^{k(2)} \prod_{n \geq 0} \frac{(1 - r^{-1}) \cdots (1 - r^{-k})}{(1 - r) \cdots (1 - r^k)} = (r^{-1} p; r^{-1})^{-1}.
\]

For the second claim, by the first claim, we have

\[
(ap; p, r) = \prod_{n \geq 1} (ap^n; r) \equiv \prod_{n \geq 1} (ar^{-1} p^n; r^{-1})^{-1} = (ar^{-1} p; p, r^{-1})^{-1}.
\]

Remark. The domains \( U_1 \) and \( U_2 \) on which \( f^1 \) and \( f^2 \) converge in \( w \) are allowed not only to be disjoint but also to be separated by a dense set of singularities, as occurs in Lemma 4.3.2. This is forbidden when analytic continuation is considered instead of quasi-analytic continuation.

In what follows, we consider quasi-analytic continuation in the formal variable \( q^{-2\omega} \). We give a quasi-analytic continuation of a normalization of the Felder-Varchenko function to the region \(|q| > 1, |q^{-2k}| > 1\). Note that the integrand of \( u(q, \lambda, \omega, \mu, k) \) does not admit quasi-analytic continuation in \( q^{-2k} \) when \( q^{-2\omega} \) is considered as a formal variable, but the normalized result of the integration does admit such a continuation. Recall the terminating \( q \)-Pochhammer symbol \( (u; q)_m \) of Subsection 4.10.1.
Lemma 4.3.3. For $|q| < 1$, $|q^{-2k}| < 1$, and $\frac{2(\mu+1)}{k} > m$, we have the integral

$$I_m = \int_{|t|=1} \frac{dt}{2\pi i t} \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \left(1 - tq^{2\lambda} t^m\right)$$

$$= -q^{2m} \frac{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k}) (q^{-2\mu+2-2k}; q^{-2k})_m}{(q^{-2k}; q^{-2k})(q^4; q^{-2k})} \left(\frac{1}{1 - q^{2\lambda+2} - q^{-2\mu+2-2km} + q^{2\lambda-2\mu-2km}}\right).$$

Proof. By Corollary 4.10.2, we have for

$$\min_n \left|\log |tq^{2\mu}q^{-2kn}|\right| > \varepsilon \quad \text{and} \quad \min_n \left|\log |tq^{-2}q^{-2kn}|\right| > \varepsilon \quad (4.3.5)$$

that there is some $D_1(q, \varepsilon)$ for which

$$\left|\frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})}\right| \leq D_1(q, \varepsilon) |t|^{2(\mu+1)} = D_1(q, \varepsilon) |t|^{\frac{2(\mu+1)}{k}}.$$ 

Because all other terms converge to 1 as $t \to 0$, for $\frac{2(\mu+1)}{k} > m$ the integral vanishes after deforming the $t$-contour to a contour near 0 which satisfies (4.3.5). Therefore, the value of $I_m$ is the sum of residues near 0, which occur at the poles $t = q^2q^{-2nk}$ for $n \geq 0$. To compute the sum of residues, notice that

$$I_m = \int_{|t|=1} \frac{dt}{2\pi i t} \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \left(1 - tq^{2\lambda} t^m\right)$$

$$= -q^{2m} \int_{|t|=1} \frac{dt}{2\pi i t} \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} (1 - tq^{2\lambda}) t^{m-1}$$

$$= -q^{2m} \sum_{n \geq 0} \frac{\theta_0(q^{2\mu+2}; q^{-2k})}{(q^{-2k}; q^{-2k})^{n+1}} (1 - q^{2\lambda+2} q^{-2kn}) q^{-2(m-1)kn}$$

$$= -q^{2m} \frac{\theta_0(q^{2\mu+2}; q^{-2k})}{(q^{-2k}; q^{-2k})^{n+1}} (1 - q^{2\lambda+2} q^{-2kn}) q^{-2(m-1)kn}$$

$$= -q^{2m} \frac{\theta_0(q^{2\mu+2}; q^{-2k})}{(q^{-2k}; q^{-2k})^{n+1}} (1 - q^{2\lambda+2} q^{-2kn}) q^{-2(m-1)kn}$$

$$= -q^{2m} \frac{\theta_0(q^{2\mu+2}; q^{-2k})}{(q^{-2k}; q^{-2k})^{n+1}} (1 - q^{2\lambda+2} q^{-2kn}) q^{-2(m-1)kn}$$

$$= -q^{2m} \frac{\theta_0(q^{2\mu+2}; q^{-2k})}{(q^{-2k}; q^{-2k})^{n+1}} (1 - q^{2\lambda+2} q^{-2kn}) q^{-2(m-1)kn}$$

$$= -q^{2m} \frac{\theta_0(q^{2\mu+2}; q^{-2k})}{(q^{-2k}; q^{-2k})^{n+1}} (1 - q^{2\lambda+2} q^{-2kn}) q^{-2(m-1)kn}.$$
Lemma 4.3.4. For $|q| < 1$, $|q^{2k}| < 1$, and $\frac{2(\mu + 1)}{k} > m$, we have the integral

$$I'_m = \int_{C_t} \frac{dt}{2\pi it} \frac{(t^q - q^{2k}) \theta_0(tq^{-2\mu}; q^{2k})}{\theta_0(tq^2; q^{2k})} \frac{1 - tq^{2\lambda}}{1 - tq^2 t^m}$$

$$= -q^{-2m} \frac{(q^{-2\mu - 2}; q^{2k})(q^{2\mu - 2 + 2k}; q^{2k}) \cdot (q^{2\mu + 2 + 2k}; q^{2k})}{(q^{-4}; q^{2k})(q^{2k}; q^{2k})} \cdot \frac{1 - q^{2\lambda - 2} - q^{2\mu - 2 + 2km} + q^{2\lambda + 2\mu + 2km}}{1 - q^{2\mu + 2 + 2km}},$$

where $C_t$ is a contour containing the poles inside $|t| = 1$ except $t = q^2$ and excluding the poles outside $|t| = 1$ except $t = q^{-2}$.

Proof. By Corollary 4.10.2, we have for

$$\min_n \left| \log |tq^{-2\mu}q^{2kn}| \right| > \varepsilon \quad \text{and} \quad \min_n \left| \log |tq^2q^{2kn}| \right| > \varepsilon \quad (4.3.6)$$

that there is some $D_1(q, \varepsilon)$ for which

$$\left| \frac{\theta_0(tq^{-2\mu}; q^{2k})}{\theta_0(tq^2; q^{2k})} \right| \leq D_1(q, \varepsilon)|q|^{-\mu - 1}|t^2q^{2\mu + 2}|^{\frac{2(\mu + 1)}{k}}.$$

All other terms in the integrand converge to 1 as $t \to 0$, so for $\frac{2(\mu + 1)}{k} > m$, the integral vanishes after deforming the $t$-contour to a contour near 0 which satisfies (4.3.6). Therefore, the value of $I'_m$ is the sum of residues which are picked up when deforming the contour to 0, which occur at the poles $t = q^{-2}q^{2kn}$ for $n \geq 0$. We deform the contour and pick up residues to find

$$I'_m = \int_{C_t} \frac{dt}{2\pi it} \frac{(t^q - q^{2k}) \theta_0(tq^{-2\mu}; q^{2k})}{\theta_0(tq^2; q^{2k})} \frac{1 - tq^{2\lambda}}{1 - tq^2 t^m}$$

$$= -q^{-2m} \frac{(q^{-2\mu - 2}; q^{2k})(q^{2\mu - 2 + 2km}; q^{2k}) \cdot (q^{2\mu + 2 + 2km}; q^{2k})}{(q^{-4}; q^{2k})(q^{2k}; q^{2k})} \cdot \frac{1 - q^{2\lambda - 2} - q^{2\mu - 2 + 2km} + q^{2\lambda + 2\mu + 2km}}{1 - q^{2\mu + 2 + 2km}}.$$
Remark. In Lemma 4.3.3, the quantity

\[
\frac{(q^{-2k}; q^{-2k})(q^4; q^{-2k})}{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k})} I_m
\]

\[
= -q^{2m} \frac{q^{-2\mu-2-2k}; q^{-2k}, \ldots, q^{-2(k+2)\mu+2-2k}; q^{-2k}}{(q^{-2\mu+2-2k}; q^{-2k})} \frac{1-q^{2\lambda+2} - q^{-2\mu+2-2k m} + q^{2\lambda-2\mu-2km}}{1-q^{-2\mu-2-2km}}
\]

is a rational function in all variables. This suggests the correct function to quasi-analytically continue is

\[
\frac{(q^{-2k}; q^{-2k})(q^4; q^{-2k})}{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k})} u(q, \lambda, \omega, \mu, k).
\]

Define the function

\[
f(t, q, \lambda, \omega, k) = \frac{(tq^{-2}q^{-2\omega}; q^{-2\omega}, q^{-2k})(t^{-1}q^{-2}q^{-2\omega}q^{-2k}; q^{-2\omega}, q^{-2k})}{(tq^2q^{-2\omega}; q^{-2\omega}, q^{-2k})(t^{-1}q^2q^{-2\omega}q^{-2k}; q^{-2\omega}, q^{-2k})}
\]

so that

\[
\Omega_{\gamma}(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{2\mu}; q^{-2k}) \theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{-2}; q^{-2\omega})} \frac{1-tq^{2\lambda}}{1-tq^{-2}} f(t, q, \lambda, \omega, k).
\]

Note that \(f(t, q, \lambda, \omega, k)\) admits a formal expansion

\[
f(t, q, \lambda, \omega, k) = \sum_{n \geq 0} \sum_{m=-n}^{n} q^{-2\omega n t^m} f_{n,m}(q, \lambda, k)
\]

for some rational functions \(f_{n,m}(q, \lambda, k)\) of \(q, q^{2\lambda}, \text{and } q^{-2k}\) with \(f_{0,0}(q, \lambda, k) = 1\). We now find a quasi-analytic continuation of \(f(t, q, \lambda, \omega, k)\) and then use it in conjunction with our computations in Lemmas 4.3.3 and 4.3.4 to obtain the desired continuation of the Felder-Varchenko function in Proposition 4.3.7.

Lemma 4.3.5. The quasi-analytic continuation of \(f(t, q, \lambda, \omega, k)\) to \(|q^{-2k}| > 1\) is

\[
f(t, q, \lambda, \omega, k) = \frac{(tq^2q^{-2\omega}q^{-2k}; q^{-2\omega}, q^{-2k})(t^{-1}q^2q^{-2\omega}q^{-2k}; q^{-2\omega}, q^{-2k})}{(tq^{-2}q^{-2k}; q^{-2\omega}, q^{-2k})(t^{-1}q^{-2}q^{-2k}; q^{-2\omega}, q^{-2k})}
\]

\[
\times \frac{(tq^{2\lambda}q^{-2\omega}; q^{-2\omega})(t^{-1}q^{2\lambda}q^{-2\omega}; q^{-2\omega})}{(tq^{-2}q^{-2\omega}; q^{-2\omega})(t^{-1}q^{2}q^{-2\omega}; q^{-2\omega})}.
\]

Proof. This follows by applying the computation of Lemma 4.3.2 repeatedly. \(\square\)
Proposition 4.3.6. The quasi-analytic continuation of

$$\frac{(q^{-2k}; q^{-2k})(q^4; q^{-2k})}{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k})} u(q, \lambda, \omega, \mu, k)$$

to $|q| < 1$, $|q^{-2\omega}| < 1$, and $|q^{-2k}| > 1$ is

$$q^{-\lambda \mu - \lambda - \mu + 2} \frac{(q^{-4}; q^{2k})(q^{2k}; q^{2k})}{(q^{-2\mu-2}; q^{2k})(q^{2\mu-2+2k}; q^{2k})}$$

$$\int_{C_t} \frac{dt}{2\pi i t} \frac{\Omega_{q-2}(t; q^{-2\omega}, q^{2k}) \theta_0(tq^{-2\mu}; q^{2k}) \theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^2; q^{2k}) \theta_0(tq^{2}; q^{-2\omega})},$$

where $C_t$ contains poles inside $|t| = 1$ except $t = q^2$ and excludes poles outside $|t| = 1$ except $t = q^{-2}$.

Proof. Define $\tilde{u}(q, \lambda, \omega, \mu, k) = q^{\lambda \mu + \lambda + \mu + 2} u(q, \lambda, \omega, \mu, k)$. By definition, we have

$$\tilde{u}(q, \lambda, \omega, \mu, k) =$$

$$\int_{|t|=1} \frac{dt}{2\pi i t} \frac{(tq^2; q^{2k})}{(tq^{-2}; q^{2k})} \frac{\theta_0(tq^{-2\mu}; q^{2k}) \theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^2; q^{2k}) \theta_0(tq^{2}; q^{-2\omega})} f_{n,m}(q, \lambda, k).$$

By Lemmas 4.3.3 and 4.3.4, we have on a formal neighborhood of 0 in $q^{-2\mu}$ that

$$\frac{(q^{-2k}; q^{-2k})(q^4; q^{-2k})}{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k})}$$

$$= - \sum_{n \geq 0} q^{-2n\omega} \sum_{m=-n}^{n} q^{2m} \frac{(q^{2\mu-2-2k}; q^{-2k})_m}{(q^{-2\mu-2+2k}; q^{-2k})_m} \frac{1 - q^{2\lambda + 2} - q^{-2\mu + 2 - 2km} + q^{2\lambda - 2\mu - 2km}}{1 - q^{-2\mu + 2 - 2km}} f_{n,m}(q, \lambda, k)$$

$$= q^4 \sum_{n \geq 0} q^{-2n\omega} \sum_{m=-n}^{n} q^{-2m} \frac{(q^{2\mu+2+2k}; q^{2k})_m}{(q^{2\mu-2+2k}; q^{2k})_m} \frac{q^{2\mu-2+2km} - q^{2\lambda + 2 + 2km} - 1 + q^{2\lambda + 2}}{1 - q^{2\mu+2+2km}} f_{n,m}(q, \lambda, k)$$

$$= q^4 \frac{(q^{-4}; q^{2k})(q^{2k}; q^{2k})}{(q^{-2\mu-2}; q^{2k})(q^{2\mu-2+2k}; q^{2k})} \sum_{n \geq 0} q^{-2n\omega} \sum_{m=-n}^{n} f_{n,m}(q, \lambda, k) I'_m$$

$$= q^4 \frac{(q^{-4}; q^{2k})(q^{2k}; q^{2k})}{(q^{-2\mu-2}; q^{2k})(q^{2\mu-2+2k}; q^{2k})} \sum_{n \geq 0} q^{-2n\omega} \sum_{m=-n}^{n} f_{n,m}(q, \lambda, k)$$

$$\int_{C_t} \frac{dt}{2\pi i t} \frac{(tq^2; q^{2k})}{(tq^{-2}; q^{2k})} \frac{\theta_0(tq^{-2\mu}; q^{2k}) \theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^2; q^{2k}) \theta_0(tq^{2}; q^{-2\omega})} f(t, q, \lambda, \omega, k).$$
Thus, the coefficients of each formal expansion in $q^{2\omega}$ have the same analytic continuations as formal series in $q^{-2\mu}$ and $q^{-2k}$ for the first and formal series in $q^{2\mu}$ and $q^{2k}$ for the second, so the above equality holds at the level of formal series in $q^{2\omega}$ with coefficients which are rational functions in $q^{2\mu}$ and $q^{2k}$. Substituting the quasi-analytic continuation of $f(t, q, \lambda, \omega, k)$ from Lemma 4.3.5, we conclude that

$$\frac{(q^{-2k}; q^{-2k})(q^{4k}; q^{-2k})}{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k})} \tilde{u}(q, \lambda, \omega, \mu, k) = q^4 \frac{q^{4k}}{(q^{-2\mu-2}; q^{2k})(q^{2\mu-2+2k}; q^{2k})} \oint_{C} \frac{dt}{2\pi it} \frac{(tq^2; q^{2k})\theta_0(tq^{-2\mu}; q^{2k})}{(tq^2; q^{2\mu})\theta_0(tq^{-2\mu}; q^{2k})} \frac{1 - tq^{2\lambda}}{1 - tq^2} \Omega_{q^2}(t; q^{-2\omega}, q^{2\omega}) \frac{\theta_0(tq^{-2\mu}; q^{2k})\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^2; q^{2\omega})},$$

which implies the desired. \(\square\)

**Proposition 4.3.7.** The quasi-analytic continuation of

$$\frac{(q^{-2k}; q^{-2k})(q^{4k}; q^{-2k})}{(q^{2\mu+2}; q^{-2k})(q^{-2\mu+2-2k}; q^{-2k})} \tilde{u}(q, \lambda, \omega, \mu, k)$$

to $|q| > 1$, $|q^{-2\omega}| < 1$, and $|q^{-2k}| > 1$ is

$$q^{-\lambda\mu - \lambda + 2} \frac{(q^{-4k}; q^{2k})(q^{2k}; q^{2k})}{(q^{-2\mu-2}; q^{2k})(q^{2\mu-2+2k}; q^{2k})} \oint_{|t|=1} \frac{dt}{2\pi it} \Omega_{q^2}(t; q^{-2\omega}, q^{2\omega}) \frac{\theta_0(tq^{-2\mu}; q^{2k})\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^2; q^{2\omega})}.$$

**Proof.** If we view $q$, $q^{-2\omega}$, $q^{-2k}$, $q^{2\lambda}$, and $q^{2\mu}$ as algebraically independent variables, the integrand of Proposition 4.3.6 depends meromorphically on $q$ and $t$ and verifies the conditions of [88, Theorem 10.2], hence our modification of the integration cycle yields the desired meromorphic extension of the quasi-analytic continuation of Proposition 4.3.6 to $|q| > 1$. \(\square\)

### 4.4 Free field realization and $q$-Wakimoto modules for $U_q(\widehat{sl}_2)$

Our approach to computing the traces of intertwiners is to realize Verma modules for $U_q(\widehat{sl}_2)$ as $q$-Wakimoto modules. In the free-field realization of [77], we apply the method of coherent states to the expression for intertwiners given by [71] to obtain contour integral formulas for the traces. In this section we formulate the necessary vertex operators for this free field realization.
4.4.1 Fock modules for $U_q(\mathfrak{sl}_2)$

Fix a level $k$. For $\star \in \{\alpha, \bar{\alpha}, \beta\}$, define the Heisenberg algebras $H_{\star}$ to be generated by $\{\star_n \mid n \in \mathbb{Z}\}$ with $\star_0$ central and relations

$$[\alpha_m, \alpha_n] = \delta_{n+m,0} \frac{[2m][km]}{m}, \quad [\bar{\alpha}_m, \bar{\alpha}_n] = -\delta_{n+m,0} \frac{[2m][km]}{m},$$

$$[\beta_m, \beta_n] = \delta_{n+m,0} \frac{[2m][(k+2)m]}{m}.$$  

Each Heisenberg algebra is the direct sum of the subalgebras $H_{\star,n}$ generated by $\star_{-n}$ and $\star_n$ for $n \geq 0$. Define the bosonic Fock space $F_{\star,a}$ to be the highest weight $H_{\star}$-module generated by a vector $v_{\star,a}$ so that

$$\star_0 v_{\star,a} = 2av_{\star,a} \text{ for } \star \in \{\alpha, \beta\} \quad \text{and} \quad \bar{\alpha}_0 v_{\bar{\alpha},a} = -2av_{\bar{\alpha},a}.$$  

We define also the action of $q^\delta$ so that

$$q^\delta \star_m q^{-\delta} = q^m \star_m \text{ for } \star \in \{\alpha, \bar{\alpha}, \beta\}$$

and $q^\delta v_{\star,a} = v_{\star,a}$ for all $a$. Each bosonic Fock space admits the tensor decomposition

$$F_{\star,a} = \bigotimes_{n\geq 0} F_{\star,a,n},$$

where $F_{\star,a,0} = \mathbb{C}v_{\star,a}$ and $F_{\star,a,n}$ for $n > 0$ is the Fock space for $H_{\star,n}$. Define tensor products of these Fock spaces by

$$F_{a,b_1,b_2} := F_{\beta,a} \otimes F_{\alpha,b_1} \otimes F_{\bar{\alpha},b_2} \quad \text{and} \quad F_{a,b} := F_{a,b,b}.$$  

4.4.2 $q$-Wakimoto modules for $U_q(\mathfrak{sl}_2)$ and the free field construction

The $q$-Wakimoto module $W_{\mu,k}$ of level $k$ is defined by

$$W_{\mu,k} := \ker \left( \eta_0 : \bigoplus_s F_{\mu,s} \to \bigoplus_s F_{\mu,s+1} \right),$$

where $\eta_0$ is the zero mode of the operator $\eta(w_0) = \sum_{m \in \mathbb{Z}} \eta_m w_0^{-m-1}$ of Subsection 4.11.1. In [77], a free field representation of $U_q(\mathfrak{sl}_2)$ on $W_{\mu,k}$ was given in terms of the vertex operators listed in Subsection 4.11.1, and $W_{\mu,k}$ was identified with a Verma module for $U_q(\mathfrak{sl}_2)$ at generic $\mu$ and $k$.

**Proposition 4.4.1** ([77, Proposition 3.1]). There is a $U_q(\mathfrak{sl}_2)$-action $\pi'_\mu$ on $W_{\mu,k}$ given
by
\[ \begin{align*}
\pi'_\mu(e_1) &= x^+_0, \\
\pi'_\mu(f_1) &= x^-_0, \\
\pi'_\mu(q^{h_1}) &= q^{\alpha_0}, \\
\pi'_\mu(e_0) &= x^-_1 q^{-\alpha_0}, \\
\pi'_\mu(f_0) &= q^{\alpha_0} x^+_1, \\
\pi'_\mu(q^{h_0}) &= q^k q^{-\alpha_0}, \\
\pi'_\mu(q^d) &= q^\delta.
\end{align*} \]

**Proposition 4.4.2** ([77, Corollary 4.4]). For generic \( \mu \) and \( k \), we have an isomorphism of \( U_q(\mathfrak{sl}_2) \) modules

\[ W_{\mu,k} \simeq M_{2\mu + k\Lambda_0} = M_{2\mu,k}. \]

The following \( q \)-Sugawara construction for \( L_0 \) gives the action of \( q^d \) on Fock space in terms of the free field construction.

**Proposition 4.4.3** ([77, Equation 3.13]). For the operator \( L_0 \) defined by

\[ L_0 = \sum_{m > 0} \frac{m^2}{[2m][km]} (\alpha_m \alpha_m - \bar{\alpha}_m \bar{\alpha}_m) + \frac{1}{4k} (\alpha_0^2 - \bar{\alpha}_0^2), \]

\[ + \sum_{m > 0} \frac{m^2}{[2m][(k + 2)m]} \beta_m \beta_m + \frac{1}{4(k + 2)} (\beta_0^2 + 2\beta_0), \]

we have \( \delta + L_0 = \frac{\mu(\mu + 1)}{k + 2} \) on \( F_{\mu,m_1,m_2} \).

Finally, to compute traces in the free field realization over \( W_{\mu,k} \), we require the following reduction to a trace over the big Fock space.

**Proposition 4.4.4** ([70, Section 6]). For an operator \( \psi : \bigoplus_s F_{\mu,s} \to \bigoplus_s F_{\mu,s} \) which preserves \( W_{\mu,k} \), we have \( \text{Tr}_{W_{\mu,k}}(\psi) = \text{Tr}_{\bigoplus_s F_{\mu,s}}(\mathcal{P} \circ \psi) \), where \( \mathcal{P} \) is the projection onto \( W_{\mu,k} \) defined by

\[ \mathcal{P} := \int \int \frac{dw dz}{(2\pi i)^2} \frac{1}{z} \eta(w) \xi(z), \]

where the contours are loops enclosing \( w = 0 \) and \( z = 0 \).

### 4.4.3 Vertex operators and Jackson integrals for intertwiners

We use the vertex operator expression for intertwiners of [77], which we summarize here. For the map \( \phi_j(z) : F_{r,m_1,m_2} \to F_{r+j,m_1+j,m_2+j} \) defined by (4.11.1), define maps \( \phi_{j,m}(z) \) by \( \phi_{j,j}(z) = \phi_j(z) \) and

\[ \phi_{j,m-1}(z) := \frac{1}{[j - m + 1]} [\phi_{j,m}(z), x^+_0] q^{2m}. \]

Let also \( \Delta_{\mu} = \frac{\mu(\mu + 1)}{k + 2} \). The Jackson integral of a function \( f(t) \) for a period \( p \) on the cycle \( s \) is the formal sum

\[ \int_0^{s \cdot \infty} f(t) \frac{dt}{t} := \sum_{n \in \mathbb{Z}} f(s^n p^n). \]
Recall that a quasi-meromorphic function is a function of the form $f(t) = t^a g(t)$ for some $a \in \mathbb{C}$; if $g(t)$ is defined on an open domain and $\bar{g}(t)$ is a meromorphic continuation of $g(t)$, outside that domain, we say that $\bar{f}(t) = t^a \bar{g}(t)$ is the meromorphic continuation$^3$ of $f(t)$. We will sometimes consider the Jackson integral of a quasi-meromorphic function $f(t)$ which is defined only on a subset of the range of the Jackson cycle. In these cases, we interpret this notation with $f(t)$ replaced by its meromorphic continuation to a branch of $t^a$ along the Jackson cycle.

**Proposition 4.4.5 ([77, Theorem 5.4]).** For $p = q^{2k+4}$, any $\tau \leq \nu$, and Jackson cycles $s_1, \ldots, s_r$ for which matrix elements on $\mathcal{M}_{2\mu, k}[a\delta]$ of the Jackson integral expression

$$
\tilde{\Phi}_{\mu, \nu}^{\nu-\tau}(z) := z^{\Delta_\mu + \Delta_\nu - \Delta_{\mu + \nu - \tau}} \int_{0}^{s_1} \cdots \int_{0}^{s_r} \sum_{m \geq 0} S(t_1) \cdots S(t_r) \phi_{\nu, \nu - m}(z) \otimes w_{2m - 2\nu} \, d_{p_1} \cdots d_{p_r}
$$

converge for $a \leq A$, they coincide with matrix elements of an intertwining operator $M_{2\mu, k} \to M_{2\mu + 2\nu - 2r, k} \otimes \mathcal{L} \mathcal{L}_{2\nu}(z)$.

**Remark.** In [77, Theorem 5.4], the content of Proposition 4.4.5 is stated with $M_{2\nu}^r(z)$ in place of $L_{2\nu}(z)$ and with $2\nu$ not an integer. To obtain Proposition 4.4.5, note first that the proof of [77] via [77, Proposition 5.3] applies for $M_{2\nu}^r(z)$ for all values of $2\nu$, meaning that matrix elements of $\tilde{\Phi}_{\mu, \nu}^{\nu-\tau}(z)$ coincide with matrix elements of $M_{2\mu, k} \to M_{2\mu + 2\nu - 2r, k} \otimes M_{2\nu}^r(z)$ when they converge. The degree zero part therefore consists of matrix elements of an intertwiner

$$
\psi : M_{2\mu} \to M_{2\mu + 2\nu - 2r} \otimes M_{2\nu}^r
$$

of $U_q(\mathfrak{sl}_2)$-modules. Now, we may find another intertwiner

$$
\psi' : M_{2\mu} \to M_{2\mu + 2\nu - 2r} \otimes \mathcal{L} \mathcal{L}_{2\nu} \leftarrow M_{2\mu + 2\nu - 2r} \otimes M_{2\nu}^r
$$

with the same highest weight matrix element as $\psi$. We then have $\psi' = \psi$ as intertwiners $M_{2\mu} \to M_{2\mu + 2\nu - 2r} \otimes M_{2\nu}^r$, hence matrix elements of $\tilde{\Phi}_{\mu, \nu}^{\nu-\tau}(z)$ coincide with those of the intertwiner associated to $\psi'$ by [24, Theorem 9.3.1]. In particular, it factors through the submodule $L_{2\nu}(z)$ of $M_{2\nu}^r(z)$, yielding Proposition 4.4.5 as stated here.

**Remark.** In [77, Theorem 5.4], it is shown that the operator $S(t_1) \cdots S(t_r) \phi_{\nu, \nu - m}(z)$ converges and commutes with the operators implementing the $U_q(\mathfrak{sl}_2)$-action of Proposition 4.4.1 up to total $q^{2k+4}$-difference in an open region for $t_1, \ldots, t_r$. This implies that the same is true of its meromorphic continuation along the full Jackson cycle, allow us to interpret Proposition 4.4.1 with this meromorphic continuation.

$^3$We use the term meromorphic continuation instead of quasi-meromorphic continuation to avoid confusion with quasi-analytic continuation.
4.4.4 Convergence and normalization of vertex operator expression for intertwiners

We now show that the Jackson integral in the vertex operator expression for the intertwiner of Proposition 4.4.5 converges in a certain region of parameters when \( \nu = \tau = 1 \) and the Jackson cycle is chosen to be \( s_1 = zq^{-2} \). Consider the parameter region

\[
0 < |q^{-2\mu}| \ll |q^{-2k}| \ll |q|, |q|^{-1}.
\]  

(4.4.1)

For the rest of the chapter, we use \( \Phi_{\mu,1}(z) \) to denote the intertwiner of Proposition 4.4.5 with this Jackson cycle. We first show in Proposition 4.4.6 that the matrix element of the highest weight vector converges and compute its value. For convenience, we use the notation \( \kappa := k + 2 \).

**Proposition 4.4.6.** In the region of parameters (4.4.1), if \( \nu = \tau = 1 \) and \( s_1 = zq^{-2} \), the diagonal matrix element of the Jackson integral for \( \Phi_{\mu,1}(z) \) on the highest weight vector of \( M_{2\mu,k} \) converges and equals

\[
C_{\mu,1} := -(1 + q^2)q^{-2\mu-3} q^{2\mu+4} \left( q^{-2\mu}; q^{-2\mu} \right) \left( q^{-4\mu}; q^{-2\mu} \right) \left( q^{-2k}; q^{-2k} \right) \left( q^2; q^{-2k} \right).
\]

**Proof.** For \( |t| > |zq^{-4k}| \), we may choose contours \( C_{w,c} \) around \( w = 0 \) so that for \( w \in C_{w,c} \), we have \( |t| > |wq^{2c}| \) for all \( c, |w| < |zq^k|, |zq^{k+4}| \) for \( c = 1 \), and \( |w| > |zq^k|, |zq^{k+4}| \) for \( c = -1 \). Define also a cycle \( C_w = C_{w,-1} - C_{w,1} \) around \( zq^k \) and \( zq^{k+4} \) but not \( 0 \).

Applying Proposition 4.4.1 and substituting (4.11.2), we find for \( p = q^{2k+4} \) that

\[
C_{\mu,1} = \left< v_{\mu,\mu}^*, \Phi_{\mu,1}(z) v_{\mu,\mu} \right> \\
= - \frac{z^{\Delta_1}}{(q - q^{-1})} \sum_{b \in \{\pm 1\}} (-1)^{\frac{1-b}{2}} \int_0^{zq^{-2} - \infty} dt \int_{C_{w,c}} \frac{dw}{2\pi i w} \frac{wq^{-2c} - zq^{k+2}}{w - zq^{k+2 + 2b}} \\
\left< v_{\mu,\mu}^*, S(t) : \phi_1(z) X_b^-(w) : v_{\mu,\mu} \right> \\
= - \frac{z^{\Delta_1}}{(q - q^{-1})} \sum_{b \in \{\pm 1\}} (-1)^{\frac{1-b}{2}} \int_0^{zq^{-2} - \infty} dt \int_{C_w} \frac{dw}{2\pi i w} q^{2b} wq^{-2} - zq^{k+2} \\
\left< v_{\mu,\mu}^*, S(t) : \phi_1(z) X_b^-(w) : v_{\mu,\mu} \right> \\
+ \int_{C_{w,1}} \frac{dw}{2\pi i w} q^{2b} w(q^{-2} - q^2) w - zq^{k+2 + 2b} \left< v_{\mu,\mu}^*, S(t) : \phi_1(z) X_b^-(w) : v_{\mu,\mu} \right>.
\]

Denote the first term by \( C_{\mu,1,1} \) and the second by \( C_{\mu,1,2} \). Noting the inequalities \( |t| > |zq^2|, |zq^{-2}|, |wq^{2b}| \) for all \( w \in C_w \), we have by the OPE's in Subsection 4.11.2 that

\[
\left< v_{\mu,\mu}^*, S(t) : \phi_1(z) X_b^-(w) : v_{\mu,\mu} \right> \\
= \frac{t^{-\frac{1}{2}} (zt^{-1}q^{-2}; q^{-2k}) q^{t - wq^{2k+1} b - a} - q^{2b} wq^{-2} - zq^k}{t - wq^{2k} - q^{2b} wq^{-2} - zq^{k+2 + 2b}} \left< v_{\mu,\mu}^*, S(t) : \phi_1(z) X_b^-(w) : v_{\mu,\mu} \right>.
\]
Applying meromorphic continuation to all $t$ on the Jackson cycle, we may substitute and take the poles at $w = zq^\kappa + 2b$ to find that

$$C_{\mu,1,1} = -\frac{z^{2\mu+2}}{(q - q^{-1})^2} \sum_{a,b \in \{\pm 1\}} (-1)^{2-a-b} \int_0^{zq^{-2\kappa} \infty} \frac{dp}{\int_{C_{w,1}} \frac{dw}{2\pi i} t^{2\mu+2}} a^{a+b+2b} \mu$$

$$\frac{\left(zt^{-1}q^{-2}; q^{-2\kappa}\right)}{t - wq^{(k+1)b-a}} \frac{wq^2 - zq^\kappa}{t - wq^b w - zq^{\kappa+2b}}$$

$$= -\frac{z^{2\mu+2}}{q - q^{-1}} \sum_{b \in \{\pm 1\}} (-1)^{2-a} \int_0^{zq^{-2\kappa} \infty} \frac{dp}{\int_{C_{w,1}} \frac{dw}{2\pi i} t^{2\mu+2}} a^{a+b+2b} \mu$$

$$\frac{\left(zt^{-1}q^{-2}; q^{-2\kappa}\right)}{t - wq^{(k+1)b-a}} \frac{wq^2 - zq^\kappa}{t - wq^b w - zq^{\kappa+2b}}$$

where convergence holds because we are in the region (4.4.1). For $C_{\mu,1,2}$, the contour $C_{w,1}$ contains only poles at $w = 0$, which vanish by the manipulations

$$C_{\mu,1,2} = -\frac{z^{2\mu+2}}{(q - q^{-1})^2} \sum_{a,b \in \{\pm 1\}} (-1)^{2-a-b} \int_0^{zq^{-2\kappa} \infty} \frac{dp}{\int_{C_{w,1}} \frac{dw}{2\pi i} t^{2\mu+2}} a^{a+b+2b} \mu$$

$$\frac{\left(zt^{-1}q^{-2}; q^{-2\kappa}\right)}{t - wq^{(k+1)b-a}} \frac{wq^2 - zq^\kappa}{t - wq^b w - zq^{\kappa+2b}}$$

The result follows because $C_{\mu,1} = C_{\mu,1,1} + C_{\mu,1,2} = C_{\mu,1,1}$. □

We would like now to relate $\bar{\Phi}_{\mu,1}^\mu(z)$ to a multiple of $\Phi_{2\mu,k}^{w_0}(z)$. Because the space of intertwiners has dimension 1, the constant of proportionality is constrained to be $C_{\mu,1}$. Unfortunately, it is not true that all matrix elements of $\bar{\Phi}_{\mu,1}^\mu(z)$ converge simultaneously in any open neighborhood of 0 in $q^{-2\mu}$. Instead, we show in Proposition 4.4.7 that in each degree, matrix elements of the the Jackson integral converge on a open
neighborhood of 0 in \( q^{-2\mu} \) dependent on degree and coincide with matrix elements of the intertwiner. This will suffice for our later computations.

**Proposition 4.4.7.** If \( \nu = \tau = 1 \) and \( s_1 = zq^{-2} \), for each \( A \geq 0 \), there exists an open neighborhood of 0 in \( q^{-2\mu} \) so that in the region of parameters (4.4.1), matrix elements of the operator \( \Phi_{\mu,1}(z) \) on the space

\[
\bigoplus_{a \leq A} M_{\mu,k}[-a\delta]
\]

of vectors of degree at least \(-A\) converge and equal matrix elements of \( C_{\mu,1} \Phi_{2\mu,k}^{bo}(z) \).

**Proof.** By Propositions 4.4.1 and 4.4.2, every matrix element is a finite linear combination of expressions of the form

\[
\int_0^{q^{-2}} \frac{d\alpha}{t} \left( v_{\mu,\mu,\mu} \prod_{i=1}^m x_{a_i}^\alpha \right) S(t)\phi_1(z), x_0^\alpha \right] q^2 \prod_{j=1}^l x_{b_j}^\alpha v_{\mu,\mu,\mu}.
\]  

(4.4.2)

for some choice of \( m, l, \) and \( a_i, b_j, c_i, d_j \). Notice that

\[
S(t)\phi_1(z), x_0^- q^2 = S(t)\phi_1(z) x_0^- q^2 S(t) x_0^- \phi_1(z) = S(t)\phi_1(z) x_0^- q^2 x_0^- S(t) \phi_1(z) - q^2 [x_0^-, S(t)] \phi_1(z).
\]

By the OPE’s of Subsection 4.11.2, we have

\[
S(t)\phi_1(z) = t^{-\frac{2}{q} \left( q^{-2} - q^{-2\kappa} \right)} : S(t)\phi_1(z) : \quad |t| > |zq^{-2}|, |zq^{-2}|
\]

and by Lemma 4.11.1, we have

\[
: [x_0^-, S(t)] \phi_1(z) := -\frac{1}{(q - q^{-1})t} \left( t^{-\frac{2}{q} \left( q^{-2} - q^{-2\kappa} \right)} \right) q^2 U(t) Y(t^{-\frac{2}{q} \left( q^{-2} - q^{-2\kappa} \right)}) q^{-2} : U(t) Y(t^{-\frac{2}{q} \left( q^{-2} - q^{-2\kappa} \right)}) q^{-2} \phi_1(z) :
\]

These computations imply that \( \prod_{i=1}^m x_{a_i}^\alpha S(t)[\phi_1(z), x_0^-] q^2 \prod_{j=1}^l x_{b_j}^\alpha \) is a linear combination of expressions of the form

\[
t^{-\frac{2}{q} \left( q^{-2} - q^{-2\kappa} \right)} \prod_{i=1}^m x_{a_i}^\alpha U(t) F(t) \phi_1(z) : \prod_{j=1}^l x_{b_j}^\alpha
\]

for \( s \geq 0 \) and \( F(t) \) a \( q \)-vertex operator whose degree 0 term does not contain \( t^{\beta_0}, t^{\alpha_0} \),
or $t^{E_0}$. Observe now that

$$
\langle \psi_{\mu, \nu, \mu, \nu}^\ast, t^{-\frac{2}{\alpha}} \left( \frac{z}{q^2 q^{2\omega}}, q^{-2\kappa} \right) \prod_{i=1}^{I} \prod_{j=1}^{I} x_{ij}^\ast \right| U(t) F(t) \phi_1(z) : \prod_{j=1}^{I} x_{ij}^j \psi_{\mu, \nu, \mu, \nu} \rangle = t^{-2\mu} \left( \frac{z}{q^2 q^{2\omega}}, q^{-2\kappa} \right) g(t)
$$

for some Laurent polynomial $g(t)$. The Jackson integral of this expression converges on a neighborhood of 0 in $q^{-2\mu}$ depending on the maximum degree of $g(t)$. By Proposition 4.4.1, matrix elements in degree at least $-A$ are computed with $\prod_{j=1}^{I} x_{ij}^j \psi_{\mu, \nu, \mu, \nu}$ of degree at most $A + 1$ in $\mathcal{F}_{\mu}$, meaning that in degree $A$, we have $\deg g \leq A + 1$. This implies that $\deg g(t)$ is bounded for matrix elements of degree bounded below, hence the neighborhood of 0 for $q^{-2\mu}$ may be chosen uniformly in degree.

Because the space of intertwiners is one-dimensional, the matrix elements of $\Phi_{\mu, k}(z)$ are determined as the unique solutions to a system of linear equations expressing the intertwining relations. Therefore, if matrix elements of $C_{\mu, 1}^{-1} \Phi_{\mu, 1}(z)$ of at least a fixed degree converge in some region, they coincide with the corresponding matrix elements of $\Phi_{\mu, k}(z)$, giving the claim. □

4.5 Contour integral formula for the trace in the three-dimensional representation

In this section, we combine the tools we have assembled to give a contour integral formula for the trace function in Theorem 4.5.1. First, we use the free field realization of Section 4.4 to represent the trace function as a Jackson integral of an iterated contour integral via the method of coherent states. We then simplify the integral and identify the formal expansion of the Jackson integral with the expansion of a renormalization of the Felder-Varchenko function computed in Section 4.3. The contour integral formula for the Felder-Varchenko function then yields convergence of the trace function and our desired integral formula.

4.5.1 Statement of the result

We will compute the trace $T^{w_0}(q, \lambda, \omega, \mu, k)$ of (4.2.1) when $V = L_2(z)$ is the irreducible three-dimensional evaluation representation of $U_q(\widehat{sl}_2)$. Recall it is defined by

$$
T^{w_0}(q, \lambda, \omega, \mu, k) := \text{Tr} |_{M_{(\mu-1)\alpha + (k-2)\lambda}} \left( \Phi_{w_0}^{w_0}(q, \lambda, \omega, \mu, k, 2) q^{2\lambda} q^{2\omega} \right).
$$

Define the good region of parameters to be the region with

$$
0 < |q^{-2\omega}| \ll |q^{-2\mu}| \ll |q^{-2\lambda}| \ll |q^{-2k}| \ll |q|, |q|^{-1}. \quad (4.5.1)
$$

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Notice that this region includes the region (4.4.1) on which we have shown convergence of the vertex operator expression for the intertwiner. Our main result is the following computation of the trace function.

**Theorem 4.5.1.** For $q^{-2\mu}$ and then $q^{-2\omega}$ sufficiently close to 0 in the good region of parameters (4.5.1), the trace function converges and has value

$$T^{\omega\omega}(q, \lambda, \omega, \mu, k) = \frac{q^{\lambda u - \lambda + 2}(q^{-4}; q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2 q^{-2\omega}}; q^{-2\omega})(q^{-2\lambda - 2}; q^{-2\omega})} \frac{(q^{-2k}; q^{-2k})(q^{4} - q^{-2k}; q^{-2k})}{(q^{-2\mu + 2}; q^{-2k})(q^{2\mu + 2} q^{-2k}; q^{-2k})}$$

$$\int_{C_t} \frac{dt}{2\pi i t} \Omega(q^2(t; q^{-2\omega}, q^{-2k}) \theta_0(t q^{2\mu}; q^{-2k}) \theta_0(t q^{2\lambda}; q^{-2\omega})},$$

where the integration cycle $C_t$ is the unit circle.

### 4.5.2 Structure of the proof of Theorem 4.5.1

We now outline the proof of Theorem 4.5.1, which occupies the remainder of Section 4.5. We will compute formally the trace of the free field intertwiner $\Phi_{\mu,1}^\mu(z)$ in two steps. First, we compute it as a formal series in $q^{-2\omega}$ and then $q^{-2\mu}$ in Proposition 4.5.3, for which we work in the doubly formal good region of parameters

$$0 < |q^{-2\lambda}| \ll |q^{-2k}| \ll |q|, |q|^{-1}. \quad (4.5.2)$$

This computation proceeds through the method of coherent states applied to the free field construction summarized in Section 4.4 and some intricate contour integral manipulations which hold for parameter values in (4.5.2). We then show in Proposition 4.5.11 that the result of Proposition 4.5.3 converges as a formal series in $q^{-2\omega}$ and then $q^{-2\mu}$. The proof of Theorem 4.5.1 follows by matching this expansion with that of the Felder-Varchenko function in Proposition 4.3.1 and then applying Proposition 4.2.6 and the integral expression for the Felder-Varchenko function to show that the resulting equality holds for small numerical values of $q^{-2\mu}$ and $q^{-2\omega}$ in the good region (4.5.1).

Define the trace of the Jackson integral by

$$\Xi(q, \lambda, \omega, \mu, k) := \text{Tr}|_{M_{2\mu + 2\lambda_0}}(\Phi_{\mu,1}^\mu(z)q^{2\lambda u + 2\omega d}), \quad (4.5.3)$$

where the quantity on the right does not depend on $z$. It is related to the trace function in the doubly formal good region as follows.

**Lemma 4.5.2.** In the doubly formal region, as formal power series in $q^{-2\omega}$ and then $q^{-2\mu}$, we have the equality

$$T^{\omega\omega}(q, \lambda, \omega, \mu, k) = C_{\mu,1}^{-1, \lambda} \Xi \left( q, \lambda, \omega, \frac{\mu - 1}{2}, k - 2 \right),$$

where $C_{\mu,1}$ is the normalizing constant of Proposition 4.4.6.
Proof. This follows from the identification $M_{2\mu \nu + k \Lambda_0} = W_{\mu, k}$ between the $q$-Wakimoto module and the Verma module and the fact that Proposition 4.4.7 shows $\Phi_{\mu, 1}(z) = C_{\mu, 1} \Phi_{2\mu, k}(z)$ as formal series in $q^{-2 \omega}$ and then $q^{-2 \mu}$ in the doubly formal good region.

We now state and give integration cycles for Proposition 4.5.3, which gives a formal computation of the trace function. Its proof occupies Subsections 4.5.3 to 4.5.9. Recall the notation $\kappa = k + 2$.

**Proposition 4.5.3.** In the doubly formal good region of parameters (4.5.2), as a formal series in $q^{-2 \omega}$, the trace function $\Xi(q, \lambda, \omega, \mu, k)$ has formal Jackson integral expansion

$$
\Xi(q, \lambda, \omega, \mu, k) = C(\lambda, \mu) \int_0^{q^{-2 \omega} \infty} \frac{dx}{x} t^{-2 \omega} \Omega_2(t; q^{-2 \omega}, q^{-2 \omega}) \theta_0(t q^2; q^{-2 \omega}) \theta_0(t q^{-2}; q^{-2 \omega}) \theta_0(t q^{2 \lambda}; q^{-2 \omega}) t^{-2 \omega} \Omega_2(t q^{2 \lambda}; q^{-2 \omega}) \theta_0(t q^{-2}; q^{-2 \omega}) \theta_0(t q^{-2}; q^{-2 \omega})
$$

for $p = q^{2 \lambda}$ and

$$
C(\lambda, \mu) = \frac{1}{q - q^{-1}} \frac{q^{2 \lambda - 2 \mu - 2}(q^{-4}; q^{-2 \omega})(q^{-2 \omega + 2}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})(q^{-2 \lambda - 2}; q^{-2 \omega})(q^{-2 \omega - 2}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})}{q^{2 \lambda - 2 q^{-2 \omega}}(q^{2 \lambda}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})(q^{-2 \omega}; q^{-2 \omega})}.
$$

Remark. Although the Jackson integral involves summation of the integrand at values of $t$ where the one loop correlation functions from the method of coherent states may not converge, we may replace them by their meromorphic continuations in these regions.

### 4.5.3 Definition of integration cycles

We now define integration cycles for parameters in the doubly formal good region of parameters (4.5.2). First, for numerical $q^{-2 \omega}$, define the **good spectral region** $S_c$ by

$$
|w_0| \gg |z_0| \gg |t q^k|, |t q^{-k}| \gg |z q^k|, |z q^{k+4}|, |w| > |z| \gg |w_0 q^{-2 \omega}|
$$

and

$$
|w| < |z q^k|, |z q^{k+4}| = \begin{cases} 1 & c = 1 \\ |w| > |z q^k|, |z q^{k+4}| & c = -1. \end{cases}
$$

By $a \gg b$, we mean that $\frac{a}{b} > |q^{100 \lambda}|, |q^{100 k}|, |q^{100}|$ so that for each instance of $q^{C \lambda}$, $q^{C k}$, or $q^C$ which appears in our formulas, we have $\frac{a}{b} > |q^{C \lambda}|, |q^{C k}|, |q^C|$. Notice that if $(t, z_0, w_0, w)$ lies in the good spectral region, then all requirements for convergence of one loop correlation functions in Table 4.1 are satisfied. Now, define the **formal good spectral region** $S_{c}$ to be the region of spectral parameters satisfying (4.5.5) and

$$
|w_0| \gg |z_0| \gg |t q^k|, |t q^{-k}| \gg |z q^k|, |z q^{k+4}|, |w| > |z|,
$$

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where we use $\gg$ in the same way as in (4.5.4). If $(t, z_0, w_0, w)$ lies in the formal good spectral region $\mathcal{S}_c$, then for $q^{-2\omega}$ sufficiently close to 0 it lies in the good spectral region $\mathcal{S}_c$. Therefore, on a formal neighborhood of 0 in $q^{-2\omega}$, if the spectral parameters lie in the formal good spectral region, then the requirements for convergence of one loop correlation functions in Table 4.1 are satisfied.

**Lemma 4.5.4.** In the formal good spectral region and the doubly formal good region of parameters, we have

$$|z_0| > |w q^{b(k+1)}|, |t q^{-a}| \quad \text{and} \quad |w_0 q^{-2\lambda-b+a}| > |w q^{b(k+1)}|.$$  

**Proof.** This follows from (4.5.6) and the definition of $\gg$. □

**Lemma 4.5.5.** There is some $R > 0$, so that for $t$ lying in the open region $C_t = \{ |t| > R|z| \}$ and any $q, q^{-2\lambda}, q^{-2k}$ in the doubly formal region of parameters (4.5.2), we may choose circular contours $C_{w_0}, C_{z_0},$ and $C_{w,c}$ enclosing 0 so that for $z_0 \in C_{z_0}, w_0 \in C_{w_0}$, and $w \in C_{w,c}$, the spectral parameters $(t, z_0, w_0, w)$ lie in the formal good spectral region.

**Proof.** We take the contours to be nested circles around the origin with radii obeying (4.5.6). The only obstacle is that the relations $|t q^k|, |t q^{-k}| \gg |w|$ and $|w| > |z q^k|, |z q^{-k+4}|$ must simultaneously hold on $C_{w,-1}$, which is possible for $|t/z|$ sufficiently large, as needed. □

**Remark.** For the rest of the proof of Proposition 4.5.3, we will compute the integrand of the Jackson integral by meromorphic continuation from the region $C_t$.

### 4.5.4 Reducing to a trace over Fock space

Assume now that all parameters lie in the doubly formal good region of parameters. For $t$ in the region $C_t$ of Lemma 4.5.5, fix spectral variables on the cycles $C_{w_0}, C_{z_0},$ and $C_{w,c}$ of Lemma 4.5.5 so that they lie in the formal good spectral region. By Propositions 4.4.2 and 4.4.4, we have

$$\Xi(q, \lambda, \omega, \mu, k) = \sum_{s \in \mu + Z} \text{Tr}|_{\mathcal{F}_\mu,s}\left( \int_{C_{w_0}} \int_{C_{z_0}} \frac{d w_0 d z_0}{(2\pi i)^2} \frac{1}{z_0} \eta(w_0) \xi(z_0) \Phi_{\mu,1}^s(z) q^{2\lambda \rho + 2\omega d} \right)$$

$$= \sum_{s \in \mu + Z} \int_{C_{w_0}} \int_{C_{z_0}} \frac{d w_0 d z_0}{(2\pi i)^2} \frac{1}{z_0} \text{Tr}|_{\mathcal{F}_\mu,s}\left( \eta(w_0) \xi(z_0) \Phi_{\mu,1}^s(z) q^{2\lambda \rho + 2\omega d} \right)$$

since cycles $C_{w_0}$ and $C_{z_0}$ enclose $w_0 = 0$ and $z_0 = 0$.

**Remark.** The interchange of sum and integral is valid when

$$\text{Tr}|_{\mathcal{F}_\mu,s}\left( \eta(w_0) \xi(z_0) \Phi_{\mu,1}^s(z) q^{2\lambda \rho + 2\omega d} \right)$$

is convergent, which holds for our choice of contours.
Recall that we identified $C \cdot w_0 \simeq \mathbb{C}$. Using $S_{\pm 1}$ and $X_{\pm 1}$ to denote $S_ \pm$ and $X_ \pm$, we obtain by the free field realization of Proposition 4.4.1 that for $p = q^2 \lambda$, we have

$$\text{Tr}_{\mathcal{F}_{\mu,s}} \left( \eta(w_0) \xi(z_0) \tilde{\Phi}_{\mu,1}^\mu(z) q^{2\lambda \rho+2w_0} \right) = \frac{z^\Delta_1}{(q-q^{-1})^2} \sum_{a,b \in \{\pm 1\}} (-1)^{a-b-c} \int_0^{\frac{q^{-2} - \infty}{t}} \frac{dz}{2\pi i w} \frac{dw}{w} \text{Tr}_{\mathcal{F}_{\mu,s}} \left( \eta(w_0) \xi(z_0) S_a(t) \phi_1(z) X_b(w) \right) q^{2\lambda \rho + 2w_0},$$

where $C_{w,1}$ applies for $\phi_1(z) X_b^-(w)$ and $C_{w,-1}$ applies for $X_b^-(w) \phi_1(z)$. Putting these observations together and noting that (4.11.2) applies for parameters on the chosen contours, we obtain

$$\Xi(q, \lambda, \omega, \mu, k) = \frac{z^{\Delta_1}}{(q-q^{-1})^2} \sum_{a,b,c \in \{\pm 1\}} (-1)^{a-b-c} \int_{\mathfrak{m}_+} \frac{dz}{2\pi i w} \frac{dw}{w} \text{Tr}_{\mathcal{F}_{\mu,s}} \left( \eta(w_0) \xi(z_0) S_a(t) : \phi_1(z) X_b^-(w) : q^{2\lambda \rho + 2w_0} \right),$$

where the OPE for $\phi_1(z)$ and $X_b^-(w)$ is valid on a formal neighborhood of 0 in $q^{-2\omega}$ because the parameters lie in the formal good spectral region.

### 4.5.5 Applying the method of coherent states

We now use the method of coherent states as applied in [72] to compute

$$T_{a,b} := \text{Tr}_{\mathcal{F}_{\mu,s}} \left( \eta(w_0) \xi(z_0) S_a(t) : \phi_1(z) X_b^-(w) : q^{2\lambda \rho + 2w_0} \right).$$

Note that each Fock space takes the form

$$\mathcal{F}_{\mu,s} = \mathcal{F}_{\phi,\mu} \otimes \mathcal{F}_{\alpha,\beta} \otimes \mathcal{F}_{\phi,\gamma} = \bigotimes_{m \geq 0} \mathcal{F}_{\phi,\mu,m} \otimes \mathcal{F}_{\alpha,\beta,m} \otimes \mathcal{F}_{\phi,\gamma,m}$$

and that each mode involved in our vertex operators acts only in a single tensor component. Define

$$\mathcal{F}_{\mu,s}^0 := \mathcal{F}_{\phi,\mu,0} \otimes \mathcal{F}_{\alpha,\beta,0} \otimes \mathcal{F}_{\phi,\gamma,0} \quad \text{and} \quad \mathcal{F}_{\mu,s}^{>0} := \bigotimes_{m > 0} \mathcal{F}_{\phi,\mu,m} \otimes \mathcal{F}_{\alpha,\beta,m} \otimes \mathcal{F}_{\phi,\gamma,m}$$

so that $\mathcal{F}_{\mu,s} = \mathcal{F}_{\mu,s}^0 \otimes \mathcal{F}_{\mu,s}^{>0}$ and $T_{a,b}$ is the product of traces over $\mathcal{F}_{\mu,s}^0$ and $\mathcal{F}_{\mu,s}^{>0}$. We compute the trace $\text{Tr}_{\mathcal{F}_{\mu,s}^0}$ over $\mathcal{F}_{\mu,s}^0$ in Subsection 4.11.3. By Corollary 4.11.4, the trace over $\mathcal{F}_{\mu,s}^{>0}$ is a product of one loop correlation functions

$$T_{P,Q} := (q^{-2\omega} ; q^{-2\omega}) \prod_{m \geq 1} \text{Tr}_{\mathcal{F}_{\phi,\mu,m} \otimes \mathcal{F}_{\alpha,\beta,m} \otimes \mathcal{F}_{\phi,\gamma,m}} \left( P(z) Q(w) q^{2w_0} \right).$$
for each pair of vertex operators $P(z)$ and $Q(w)$ appearing in the free field realization and a global factor $T_g = (q^{-2w}; q^{-2w})^{-3}$ coming from collecting the prefactors in Corollary 4.11.4 over $m \geq 1$ for each of $\alpha, \tilde{\alpha},$ and $\beta$. In Subsection 4.11.4, we compute these one loop correlation functions and their regions of convergence, with results recorded in Table 4.1. From these computations, we conclude that $T_{a,b}$ is convergent with value

$$T_{a,b} = \text{Tr}_{\bar{F}_{\mu,s}} T_s T_{\eta S} T_{a} T_{\bar{a}} T_{X^{-}} X^{-} T_{\eta S} T_{s} T_{\bar{a}} T_{X^{-}} T_{a} T_{X^{-}}$$

for $w_0, z_0, w$ lying on the specified cycles and $t \in C_t$. Substituting the values of the degree 0 trace and the one loop correlation functions from Subsections 4.11.3 and 4.11.4 to compute $T_{a,b}$, we obtain

$$\Xi(q, \lambda, \omega, \mu, k) = \frac{z^{A_1}}{(q - q^{-1})^2} \sum_{a,b,c} (-1)^{\frac{3-a-b-c}{2}} \sum_{s \in \mu + Z} \int_0^{\infty} \frac{d \mu}{t} \int_{C_{a},c} d\omega d\sigma d\tau dq dw d\theta$$

$$\theta_0\left(\frac{z_0}{w_0} ; q^{-2\omega}\right)^{-1} \text{Tr}_{\bar{F}_{\mu,s}} T_s T_{\eta S} T_{a} T_{\bar{a}} T_{X^{-}} X^{-} T_{\eta S} T_{s} T_{\bar{a}} T_{X^{-}} T_{a} T_{X^{-}}$$

$$\theta_0\left(\frac{z_0}{w_0} ; q^{-2\omega}\right)^{-1} \text{Tr}_{\bar{F}_{\mu,s}} T_s T_{\eta S} T_{a} T_{\bar{a}} T_{X^{-}} X^{-} T_{\eta S} T_{s} T_{\bar{a}} T_{X^{-}} T_{a} T_{X^{-}}$$

$$\frac{\theta_0\left(\frac{z_0}{w_0} ; q^{-2\omega}\right)}{\theta_0\left(\frac{z_0}{w_0} ; q^{-2\omega}\right)} \begin{cases} a = b & \left(\frac{z_0}{w_0} ; q^{-2\omega}\right) = 1 \quad w_0 - zq^{k+2} \\ a \neq b & \left(\frac{z_0}{w_0} ; q^{-2\omega}\right) = -1 \quad w - zq^{k+2}\end{cases}$$

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Table 4.1: Values and regions of convergence of one loop correlation functions

<table>
<thead>
<tr>
<th>Value</th>
<th>Region of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{\eta} ) ( \theta_0 \left( \frac{z_0}{w_0}; q^{-2\omega} \right)^{-1} )</td>
<td>(</td>
</tr>
<tr>
<td>( T_{\eta S_0} ) ( \theta_0 \left( \frac{t}{w_0} q^{-a}; q^{-2\omega} \right)^{-1} )</td>
<td>(</td>
</tr>
<tr>
<td>( T_{\eta} \phi )</td>
<td>all</td>
</tr>
<tr>
<td>( T_{\eta X_5} ) ( \theta_0 \left( \frac{w_0}{q^{-1} q^{(k+1)}}; q^{-2\omega} \right) )</td>
<td>(</td>
</tr>
<tr>
<td>( T_{\xi S_0} ) ( \theta_0 \left( \frac{t}{z_0} q^{-a}; q^{-2\omega} \right) )</td>
<td>(</td>
</tr>
<tr>
<td>( T_{\xi} \phi )</td>
<td>all</td>
</tr>
<tr>
<td>( T_{\xi X_5} ) ( \theta_0 \left( \frac{w_0}{q^{-1} q^{(k+1)}}; q^{-2\omega} \right)^{-1} )</td>
<td>(</td>
</tr>
<tr>
<td>( T_{\xi S_0} \phi ) ( \frac{\phi_2}{2} \sum_{\delta_2} \theta_0 \left( \frac{t}{q^{-1} q^{(k+1)}}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{q^{k-2} q^{-2\omega}}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{q^{-1} q^{-2\omega}}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{q^{k-2} q^{-2\omega}}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{q^{k-2} q^{-2\omega}}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{q^{k-2} q^{-2\omega}}; q^{-2\omega} \right) )</td>
<td>(</td>
</tr>
<tr>
<td>( T_{\xi S_0} )</td>
<td>all</td>
</tr>
<tr>
<td>( T_{\xi S_0} \phi )</td>
<td>all</td>
</tr>
</tbody>
</table>

Changing the index of summation to \( r = \mu - s \), we can simplify this expression to

\[
\Xi(q, \lambda, \omega, \mu, k) = \frac{1}{(q - q^{-1})^2 (q^{-2\omega - 2}; q^{-2\omega}; q^{-2\omega})^2} \int_{q^{-2\omega}; q^{-2\omega}; q^{-2\omega}} d\omega dzo dw q^{2\mu a + 2b + 2 - 2(\mu + 1)\pi}
\]

\[
\sum_{a,b,c,d,e,f \in \{1\}} (-1)^{n-s-b-c} \int_{r \in \mathbb{C}} \int_{z \in \mathbb{C}} \int_{w \in \mathbb{C}} d\omega dzo dw q^{2\mu a + 2b + 2 - 2(\mu + 1)\pi}
\]

\[
q^{-2\omega - 2}(z_0 r w_0^{-1}) \theta_0 \left( \frac{z_0}{w_0}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{w_0} q^{-a}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{w_0} q^{-a}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{w_0} q^{-a}; q^{-2\omega} \right) \theta_0 \left( \frac{t}{w_0} q^{-a}; q^{-2\omega} \right)
\]

We now simplify the expression by performing the integrals over \( z_0, w_0 \), and \( w \) in that order.
4.5.6 Computing the $z_0$ contour integral

We apply the following complex analysis lemma.

**Lemma 4.5.6.** Let $f(z)$ be a meromorphic function of $z$ with annulus of convergence $r < |z| < R$ near 0 for some $0 < r < R$. For $r < r_0, |w| < R$ we have that

$$
\sum_{m \in \mathbb{Z}} \int_{|z| = r_0} \frac{w^m}{z^{m+1}} f(z) \frac{dz}{2\pi i} = f(w).
$$

**Proof.** Because $r < r_0 < R$, the Laurent series expansion of $f(z)$ at 0 converges uniformly on the compact contour $|z| = r_0$. Therefore, we may take the term-by-term residue expansion of the integral to obtain a Laurent series expansion which converges to $f(w)$ because $r < |w| < R$. \qed

In the formal good spectral region, we have $|w_0| > |z_0| > |w q^{b(k+1)}|$, which implies that the $q^{-2w}$-coefficients of the formal series expansion of

$$
\frac{\theta_0(\frac{w}{w_0} q^{2\lambda - b - 2a}; q^{-2w})}{\theta_0(\frac{w_0}{w}; q^{-2w}) \theta_0(\frac{w_0 q^{b(k+1)}; q^{-2w}}{z_0})}
$$

admit convergent Laurent series expansion in $z_0$. In the formal good spectral region and in the doubly formal good region of parameters, we have $|w_0| > |w q^{2\lambda - b + a}| > |w q^{b(k+1)}|$ by Lemma 4.5.4, so we may apply Lemma 4.5.6 to each coefficient to obtain

$$
\sum_{r \in \mathbb{Z}} \int_{C_{r_0}} \frac{z_{0}^{r-1} w_{0}^{r-1} q^{-(2\lambda - b + a) r} \theta_0(\frac{w}{w_0} q^{a}; q^{-2w})}{\theta_0(\frac{w}{w_0}; q^{-2w}) \theta_0(\frac{w_0 q^{b(k+1)}; q^{-2w}}{z_0})} dz_{0} \frac{1}{2\pi i} \frac{\theta_0(\frac{w}{w_0} q^{2\lambda + b - 2a}, q^{-2w})}{w_0 \theta_0(\frac{w_0}{w} q^{2\lambda + b - a}, q^{-2w}) \theta_0(\frac{w_0 q^{b(k+1)}; q^{-2w}}{z_0})}
$$

as formal series in $q^{-2w}$. Substitution into the original expression yields

$$
\Xi(q, \lambda, \omega, \mu, k) = \frac{1}{(q - q^{-1})^2 (q^{-2w - 2}, q^{-2w}, q^{-2\kappa})^2 (q^{-2w}, q^{-2w}) (q^{-2w} - w q^{-2w})} q^{2a\lambda z^{2\mu + 2}}
$$

$$
\sum_{a,b,c \in \{\pm 1\}} (-1)^{3 - a - b - c} \int_0^{\infty} \int_{C_{w_0}} \int_{C_{w_0}} \int_0^{(2\pi i)^2 w_0 z w_0 \theta_0(q^{-2b + a}, q^{-2w})} \frac{q^{2b\mu + a - 2b \ell + 2(\mu + 1)}}{w_0 q^{2a + b - 2w} q^{-2w}} \theta_0(\frac{w}{w_0} q^{2\lambda + b - 2a}, q^{-2w}) \theta_0(\frac{w}{w_0} q^{2\lambda + b - a}, q^{-2w}) \Omega(q^2; q^{-2w}) \theta_0(\frac{q^2}{z^2}; q^{-2w}) \theta_0(\frac{z^2}{q^2}; q^{-2w})
$$

$$
\theta_0(\frac{q^2}{z^2}; q^{-2w}) \left( \theta_0(\frac{w q^{2\lambda + b - 2a}}{q^2}, q^{-2w}) \theta_0(\frac{w q^2}{z^2}, q^{-2w}) \right) a = b \begin{cases}
1 & \text{if } a \neq b \\
(\frac{q^2}{z^2} q^{-2w}) & \text{if } a = b
\end{cases}
$$

$$
\text{if } b = 1 \text{ and } w q^{-2c} - z q^{k+2} 
\text{if } b = -1 \text{ and } z q^{k+2 + 2b}.
$$
4.5.7 Computing the $w_0$ contour integral

We compute the integral of the convergent formal series in $q^{-2\omega}$ by computing it for sufficiently small $q^{-2\omega}$. In this subsection, take $q^{-2\omega}$ sufficiently small so that the spectral parameters lie in the good spectral region $S_c$. We first compute the integral of an elliptic function.

**Lemma 4.5.7.** For constants $c_1, c_2, c_3$ so that $|c_1|, |c_2|, |c_3|/|c_3|, |c_1|/|c_3| \neq |q|^{-2\omega n}$ for any $n$ and a contour $|w_0| = r$ satisfying $r|q^{-2\omega}| < |c_2|$, we have

$$
\int_{|w_0|=r} \frac{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_2}{c_3}; q^{-2\omega})}{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_2}{c_3}; q^{-2\omega})} \frac{dw_0}{2\pi i w_0} = \frac{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_1}{c_3}; q^{-2\omega})}{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_2}{c_3}; q^{-2\omega})} \left(2\eta_1(C_1 - C_2 - C_3) - \zeta(C_1 - C_2) + \zeta(C_3)\right),
$$

where $C_1 = \frac{1}{2\pi i} \log(c_1)$, $\zeta$ and $\sigma$ are the Weierstrass zeta and sigma functions with periods 1 and $\frac{1}{2\pi i} \log(q^{-2\omega})$, nome $q^{-\omega}$, and $\eta_1 = \zeta(\frac{1}{2})$.

**Proof.** Fix a branch of $\log(-)$ and change variables to the additive coordinate $W_0 = \frac{1}{2\pi i} \log w_0$. Define $R = \frac{1}{2\pi i} \log r$. By matching zeroes, poles, and residues of functions elliptic in $W_0$, the integrand is given by

$$
\frac{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_2}{c_3}; q^{-2\omega})}{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_1}{c_3}; q^{-2\omega})} = \frac{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_1}{c_3}; q^{-2\omega})}{\theta_0(\frac{c_1}{c_2}; q^{-2\omega})\theta_0(\frac{c_2}{c_3}; q^{-2\omega})} \left(\zeta(W_0 - C_2 + R) - \zeta(W_0 - C_1 + C_3 + R) - \zeta(C_1 - C_2) + \zeta(C_3)\right).
$$

Because $r|q^{-2\omega}| < |c_2|, |c_1|/|c_3| < r$, we have

$$
-\text{Im} \left(\frac{1}{2\pi i} \log q^{-2\omega}\right) < \text{Im}(R - C_2), \text{Im}(R - C_1 + C_3) < 0,
$$

which means that

$$
\int_0^1 \left(\zeta(W_0 - C_2 + R) - \zeta(W_0 - C_1 + C_3 + R)\right) dW_0 = \log \frac{\sigma(1 - C_2 + R)}{\sigma(-C_2 + R)} - \log \frac{\sigma(1 - C_1 + C_3 + R)}{\sigma(-C_1 + C_3)}
$$

$$
= \frac{\sigma(1 - C_2 + R)}{\sigma(-C_2 + R)} - \frac{\sigma(1 - C_1 + C_3 + R)}{\sigma(-C_1 + C_3)}
$$

$$
= \left(\eta_1 + 2\eta_1(-C_2 + R)\right) - \left(\eta_1 + 2\eta_1(-C_1 + C_3 + R)\right)
$$

$$
= 2\eta_1(C_1 - C_2 - C_3),
$$

where we may take the same branch of $\log(\sigma(-))$ in both terms of the first equality by (4.5.8) and we apply $\frac{\sigma(1+a)}{\sigma(a)} = -e^{\eta_1 + 2\eta_1 a}$ for the second. Noting that $\int_{|w_0|=r} f(w_0) \frac{d\arg w_0}{2\pi i w_0} = \int_0^1 f(\rho e^{2\pi i W_0}) dW_0$, integrating each term in (4.5.7) separately, and substituting, we
obtain the desired integral value of

\[ \frac{\theta_0 \left( \frac{c_1}{c_2}; q^{-2\omega} \right) \theta_0 \left( \frac{1}{c_3}; q^{-2\omega} \right)}{\theta_0 \left( \frac{c_1}{c_2 c_3}; q^{-2\omega} \right) \left( q^{-2\omega}; q^{-2\omega} \right)^2} \left( 2\eta_1 (C_1 - C_2 - C_3) - \zeta(C_1 - C_2) + \zeta(C_3) \right). \]

On our choice of contours, the integral over \( w_0 \) satisfies the hypotheses of Lemma 4.5.7 with \( c_1 = w q^b(k+1) \), \( c_2 = t q^{-a} \), and \( c_3 = q^{-2\lambda-b+a} \) for \( q^{-2\omega} \) sufficiently close to 0 by Lemma 4.5.4, yielding

\[
\int_{C_{w_0}} \frac{dw_0}{2\pi i w_0} \frac{\theta_0 \left( \frac{t}{w_0} q^{2\lambda+b-2a}, q^{-2\omega} \right) \theta_0 \left( \frac{w}{w_0} q^{b(k+1)}, q^{-2\omega} \right)}{\theta_0 \left( \frac{w}{w_0} q^{2\lambda+b-k}, q^{-2\omega} \right) \theta_0 \left( \frac{w}{w_0} q^{2\lambda+b-k}, q^{-2\omega} \right)} = \frac{\theta_0 \left( \frac{w}{t} q^{2\lambda+b-k}, q^{-2\omega} \right) \theta_0 \left( \frac{w}{t} q^{2\lambda+b-k}, q^{-2\omega} \right) \left( q^{-2\omega}; q^{-2\omega} \right)^2}{2\eta_1 \frac{w}{t} q^{2\lambda+b-k} - \zeta \left( \log \left( \frac{w}{t} q^{b(k+1)+a} \right) \right) + \zeta \left( \log (q^{2\lambda-b+a}) \right)}.
\]

Substituting in the result and noting that \( (q^{-2\omega} - 2q^{-2\omega})(q^{-2\omega+2} - q^{-2\omega}) = \frac{\theta_0 \left( q^2 q^{-2\omega} \right)}{1-q^2} \), we find that

\[
\Xi(q, \lambda, \omega, \mu, k) = -\frac{1}{q - q^{-1}} \frac{\left( q^{-2\omega+2}; q^{-2\omega}, q^{-2\omega} \right)^2 1}{\left( q^{-2\omega}; q^{-2\omega}, q^{-2\omega} \right)^2 \theta_0 \left( q^2; q^{-2\omega} \right) \theta_0 \left( q^{-2\omega}; q^{-2\omega} \right) q^{2\lambda(\mu+1)+1} z^{2k+2}}
\sum_{a,b,c \in \{\pm 1\}} (-1)^{\frac{a-b-c}{2}} \int_{C_{w_0}} \frac{dw}{2\pi i w} \int_{C_{u,c}} \frac{dt}{2\pi i w} \frac{dw}{w^{2\mu+3\lambda t - 2(\mu+1)}} \frac{\theta_0 \left( \frac{w}{t} q^{b(k+1)}, q^{-2\omega} \right)}{\theta_0 \left( \frac{w}{t} q^{2\lambda+b-k+2}, q^{-2\omega} \right)} \left( 2\eta_1 \frac{w}{t} q^{2\lambda+b-k} - \zeta \left( \log \left( \frac{w}{t} q^{b(k+1)+a} \right) \right) + \zeta \left( \log (q^{2\lambda-b+a}) \right) \right) \frac{\theta_0 \left( \frac{w}{t} q^{2\lambda+b-k}, q^{-2\omega} \right) \theta_0 \left( \frac{w}{t} q^{2\lambda+b-k}, q^{-2\omega} \right) \left( q^{-2\omega}; q^{-2\omega} \right)^2}{\theta_0 \left( \frac{w}{t} q^{2\lambda+b-k}, q^{-2\omega} \right) \theta_0 \left( \frac{w}{t} q^{2\lambda+b-k}, q^{-2\omega} \right) \left( q^{-2\omega}; q^{-2\omega} \right)^2}
\Omega_{q^2} \left( \frac{t}{z} q^{-2\omega}, q^{-2\omega} \right) \left( \frac{z}{t} q^2 q^{-2\omega} \right) \left( \frac{z}{t} q^2 q^{-2\omega} \right) \left( \frac{z}{t} q^2 q^{-2\omega} \right) \left( \frac{z}{t} q^2 q^{-2\omega} \right)

\begin{cases}
(\frac{w q^{-2\omega} g^{-2\omega}}{q^{-2\omega}}) & b = 1 \quad w q^{-2\omega} - z q^{k+2} \\
(\frac{w q^{-2\omega} g^{-2\omega}}{q^{-2\omega}}) & b = -1 \quad w - z q^{k+2+2b}.
\end{cases}
\]

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Summing over \( a \in \{\pm 1\} \), we obtain

\[
\Xi(q, \lambda, \omega, \mu, k) = -\frac{1}{q - q^{-1}} \sum_{b,c \in \{\pm 1\}} (-1)^{2\frac{b+c}{2}} \int_{0}^{\infty} d\mu t \int_{C_{w,c}}^{\delta} \frac{dw}{2\pi i z w} q^{2b\mu + 3b t - 2(b+1)} \frac{\theta_{0}(q^{b}; q^{-2\omega})}{\theta_{0}(q^{b}; q^{-2\omega})} \frac{1}{q^{2}\mu + \lambda b + k; q^{-2\omega}} \left( \zeta(\log(\frac{W}{t}q^{b(k+1)-1})) - \zeta(\log(\frac{W}{t}q^{b(k+1)+1})) + \zeta(\log(q^{-2\lambda-b+1})) - \zeta(\log(q^{-2\lambda-b-1})) \right)
\]

\[
\Omega_{q^{2}}(\frac{t}{z}; q^{2-\omega}, q^{-2\omega}) = \begin{cases} 
\frac{q^{k}q^{-2\omega}(q^{2-\omega})}{(2q^{2}q^{-2\omega})} b = 1 \quad wq^{2c - zwk^{2}} \\
\frac{q^{-k}q^{-2\omega}(q^{2-\omega})}{(2q^{-2}q^{2\omega})} b = -1 \quad w - zwk^{2} + 2b 
\end{cases}
\]

By matching zeroes, poles, and residues of elliptic functions in \( \log t \), we see that

\[
\Omega_{q^{2}}(\frac{t}{z}; q^{2-\omega}, q^{-2\omega}) = \begin{cases} 
\frac{q^{k}q^{-2\omega}(q^{2-\omega})}{(2q^{2}q^{-2\omega})} b = 1 \quad wq^{2c - zwk^{2}} \\
\frac{q^{-k}q^{-2\omega}(q^{2-\omega})}{(2q^{-2}q^{2\omega})} b = -1 \quad w - zwk^{2} + 2b 
\end{cases}
\]

where we may again interpret the expression as a formal series in \( q^{-2\omega} \).

4.5.8 Computing the \( w \) contour integrals

We now evaluate the contour integrals over \( w \). We again will compute the integral for numerical \( q^{-2\omega} \) sufficiently close to 0 that the parameters lie in the good spectral region. Define

\[
I_{w,1,c}(t) := \int_{C_{w,c}}^{\delta} \frac{dw}{2\pi i w} \frac{\theta_{0}(q^{b}; q^{-2\omega})}{\theta_{0}(q^{b}; q^{-2\omega})} \frac{1}{q^{k}q^{-2\omega}(q^{2\omega})(q^{2c} - zwq^{k+2})}
\]
and

\[
I_{w-1,c}(t) := \oint_{C_{w,c}} \frac{dw}{2\pi i w} \frac{\theta_0(q^{k-2\lambda}; q^{-2\omega}) \theta_0(q^{k-4}; q^{-2\omega}) \theta_0(w q^{-k}; q^{-2\omega})}{\theta_0(w q^{-k}; q^{-2\omega})} (q^2 - wz^{-1} q^{-k-2c}).
\]

**Lemma 4.5.8.** As formal series in \(q^{-2\omega}\), the differences of integrals \(I_{w,1}(t) := I_{w,1,1}(t) - I_{w,1,-1}(t)\) and \(I_{w,-1}(t) := I_{w,-1,1}(t) - I_{w,-1,-1}(t)\) are given by

\[
I_{w,1}(t) = I_{w,1}^1(t) + I_{w,1}^2(t) \quad \text{and} \quad I_{w,-1}(t) = I_{w,-1}^1(t),
\]

where

\[
I_{w,1}^1(t) = (q^2 - q^{-2\lambda}) \frac{\theta_0(q^{2-2\lambda} q^{-2\omega}; q^{-2\omega}) \theta_0(q^{2k+2}; q^{-2\omega})}{\theta_0(q^{-2\omega}; q^{-2\omega})^2} \left( \sum_{\ell = 0}^{2\lambda} \frac{(z q^{2k+6}; q^{-2\omega})}{(z q^{2k+6}; q^{-2\omega})} \right)
\]

and

\[
I_{w,1}^2(t) = (q^{2\lambda} - q^2) \frac{\theta_0(q^{2k+6}; q^{-2\omega})}{\theta_0(q^{2k+6}; q^{-2\omega})^2} \left( \sum_{\ell = 0}^{2\lambda} \frac{(z q^{2k+6}; q^{-2\omega})}{(z q^{2k+6}; q^{-2\omega})} \right)
\]

and

\[
I_{w,-1}^1(t) = (q^2 - q^{-2\lambda}) \frac{\theta_0(q^{2-2\lambda} q^{-2\omega}; q^{-2\omega}) \theta_0(q^{2k+2}; q^{-2\omega})}{\theta_0(q^{-2\omega}; q^{-2\omega})^2} \left( \sum_{\ell = 0}^{2\lambda} \frac{(z q^{2k+6}; q^{-2\omega})}{(z q^{2k+6}; q^{-2\omega})} \right)
\]

and \(2\phi_1(a_1, a_2; b_1; q, z)\) denotes the \(q\)-hypergeometric function

\[
2\phi_1(a_1, a_2; b_1; q, z) := \sum_{n \geq 0} \frac{(a_1; q)_n (a_2; q)_n}{(b_1; q)_n (q; q)_n} z^n.
\]

**Proof.** We compute the integrals by deforming contours to 0. By Corollary 4.10.2, we have the estimates

\[
\left| \frac{\theta_0(q^{k-2\lambda}; q^{-2\omega})}{\theta_0(q^{k+2}; q^{-2\omega})} \right| \leq D_1(q^{-2\omega}, \epsilon) |q|^{-\frac{3\lambda - 1}{2\omega}} |q^{-2\omega} q^{-2\lambda + 2}|^{-\frac{\lambda + 1}{2\omega}}
\]

and

\[
\left| \frac{\theta_0(q^{k-2\lambda}; q^{-2\omega})}{\theta_0(q^{k-2\lambda}; q^{-2\omega})} \right| \leq D_1(q^{-2\omega}, \epsilon) |q|^{-\frac{3\lambda - 1}{2\omega}} |q^{-2\omega} q^{-2\lambda - 2}|^{-\frac{\lambda + 1}{2\omega}}.
\]

For \(q^{-2\omega}\) sufficiently small, we see that \(\frac{\lambda + 1}{2\omega} < \frac{1}{2}\), meaning that we may compute
$I_{w,1,c}(t)$ and $I_{w,1,-c}(t)$ by deforming $C_{w,c}$ to 0. We now perform the deformations one by one.

**Computing $I_{w,1,1}(t)$:** For $I_{w,1,1}(t)$, we have $|w| < |zq^k|$, $|zq^{k+4}|$, $|tq^{-k-2}|$, so we wish to sum residues at $w = tq^{-k-2}q^{-2n(n+1)}$ and $w = zq^{k+4}q^{-2n(n+1)}$. The first set has residues

$$q^{-2} \frac{\theta_0(q^{-2-2\lambda}q^{-2w(n+1)}; q^{-2w}) (z \frac{q^{2k+2}q^{2w(n+1)}; q^{-2w}}{q^{-2w(n+1)}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w(n+1)}; q^{-2w}}{q^{-2w}; q^{-2w}})} = q^{-2}q^{(2\lambda-2)n} \frac{\theta_0(q^{-2-2\lambda}q^{-2w}; q^{-2w}) (z \frac{q^{2k+2}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}.$$ 

The second set has residues

$$q^{-2} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w(n+1)}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w(n+1)}; q^{-2w}}{q^{-2w}; q^{-2w}})} = q^{-2}q^{(2\lambda-2)n} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}.$$ 

Summing over these residues yields

$$I_{w,1,1}(t) = q^{-2}q^{(2\lambda-2)n} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}.$$ 

**Computing $I_{w,1,-1}(t)$:** We have $|tq^{-k-2}| > |w| > |zq^k|$, $|zq^{k+4}|$, so we wish to sum residues at $w = tq^{-k-2}q^{-2n(n+1)}$ and $w = zq^{k+4}q^{-2n}$. The first set has residues

$$q^2 \frac{\theta_0(q^{-2-2\lambda}q^{-2w(n+1)}; q^{-2w}) (z \frac{q^{2k+2}q^{2w(n+1)}; q^{-2w}}{q^{-2w(n+1)}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w(n+1)}; q^{-2w}}{q^{-2w}; q^{-2w}})} = q^2 q^{(2\lambda-2)n} \frac{\theta_0(q^{-2-2\lambda}q^{-2w}; q^{-2w}) (z \frac{q^{2k+2}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}.$$ 

The second set has residues

$$q^2 \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})} = q^2 q^{(2\lambda-2)n} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})} \frac{\theta_0(z \frac{q^{2k+4-2\lambda}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}{(z \frac{q^{2k+6}q^{-2w}; q^{-2w}}{q^{-2w}; q^{-2w}})}.$$ 

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Summing over these residues yields

\[ I_{w,1,1}(t) = q^{-2\lambda} \theta_0(q^{2-2\lambda}q^{-2\omega}; q^{-2\omega}) \left( \frac{\xi^2 q^{2k+2}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

\[ \times _2 \phi_1 \left( \frac{t}{z} q^{-2k-2}q^{-2\omega}, q^{-2w}; \frac{t}{z} q^{-2k-6}q^{-2\omega}, q^{-2w}, q^{2\lambda-2} \right) \]

\[ + q^{-2\lambda} \theta_0 \left( \frac{\xi^{2k+4-2\lambda}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) + (q^{2\lambda} - q^2) \theta_0 \left( \frac{\xi^{2k+4-2\lambda}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right). \]

Computing \( I_{w,1}(t) \): Combining these terms, we find that

\[ I_{w,1}(t) = (q^{-2} - q^{-2\lambda}) \theta_0(q^{2-2\lambda}q^{-2\omega}; q^{-2\omega}) \left( \frac{\xi^2 q^{2k+2}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

\[ \times _2 \phi_1 \left( \frac{t}{z} q^{-2k-2}q^{-2\omega}, q^{-2w}; \frac{t}{z} q^{-2k-6}q^{-2\omega}, q^{-2w}, q^{2\lambda-2} \right) \]

\[ + q^{-2\lambda} \theta_0\left( \frac{\xi^{2k+2}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

\[ + (q^{2\lambda} - q^2) \theta_0 \left( \frac{\xi^{2k+4-2\lambda}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

Computing \( I_{w,-1,1}(t) \): In this case, we have |\( w \)| \( < |zq^k|, |zq^{k+4}|, |tq^{k+2}| \) on \( C_{w,1} \), so it suffices to consider the poles at \( w = tq^{k+2}q^{-2\omega(n+1)} \) for \( n \geq 0 \). They have residues

\[ q^2 \theta_0(q^{2-2\lambda}q^{-2\omega}; q^{-2\omega}) \left( \frac{\xi^2 q^{2k+2}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

\[ \times _2 \phi_1 \left( \frac{t}{z} q^{-2k-2}q^{-2\omega}, q^{-2w}; \frac{t}{z} q^{-2k-6}q^{-2\omega}, q^{-2w}, q^{2\lambda-2} \right) \]

Summing over these yields

\[ I_{w,-1,1}(t) = q^2 \theta_0(q^{2-2\lambda}q^{-2\omega}; q^{-2\omega}) \left( \frac{\xi^2 q^{2k+2}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

\[ \times _2 \phi_1 \left( \frac{t}{z} q^{-2k-2}q^{-2\omega}, q^{-2w}; \frac{t}{z} q^{-2k-6}q^{-2\omega}, q^{-2w}, q^{2\lambda-2} \right) \]

Computing \( I_{w,-1,-1}(t) \): In this case, we have |\( w \)| \( > |zq^k|, |zq^{k+4}| \) and \( w \leq |tq^{k+2}| \) on \( C_{w,-1} \), so it suffices to consider the poles at \( w = tq^{k+2}q^{-2\omega(n+1)} \) for \( n \geq 0 \). They have residues

\[ q^2 \theta_0(q^{2-2\lambda}q^{-2\omega}; q^{-2\omega}) \left( \frac{\xi^2 q^{2k+2}; q^{-2w}}{(q^{-2w}; q^{-2\omega})^2 (\xi^2 q^{2k+6}; q^{-2w})} \right) \]

\[ \times _2 \phi_1 \left( \frac{t}{z} q^{-2k-2}q^{-2\omega}, q^{-2w}; \frac{t}{z} q^{-2k-6}q^{-2\omega}, q^{-2w}, q^{2\lambda-2} \right) \]
Summing over these residues yields

\[
I_{w,-1}(t) = q^{4-2\lambda} \theta_0(q^{2-2\lambda} q^{-2w}; q^{-2\omega}) \left( \frac{t}{z} q^{-2} q^{-2w}; q^{-2\omega} \right) \left( \frac{q}{z} q^{2} q^{-2w}; q^{-2\omega} \right) (2\phi_1(\frac{t}{z} q^{-2w}, q^{-2\omega}; \frac{t}{z} q^{-2} q^{-2w}, q^{-2\omega}) - 1). 
\]

**Computing \( I_{w,-1}(t) \):** Combining these terms, we find that

\[
I_{w,-1}(t) = (q^2 - q^{4-2\lambda}) \theta_0(q^{2-2\lambda} q^{-2w}; q^{-2\omega}) \left( \frac{t}{z} q^{-2} q^{-2w}; q^{-2\omega} \right) \left( \frac{q}{z} q^{2} q^{-2w}; q^{-2\omega} \right) (2\phi_1(\frac{t}{z} q^{-2w}, q^{-2\omega}; \frac{t}{z} q^{-2} q^{-2w}, q^{-2\omega}) + q^{4-2\lambda} \theta_0(q^{2-2\lambda} q^{-2w}; q^{-2\omega}) \left( \frac{t}{z} q^{-2} q^{-2w}; q^{-2\omega} \right) \left( \frac{q}{z} q^{2} q^{-2w}; q^{-2\omega} \right). \]

\[ \square \]

4.5.9 **Completing the proof of Proposition 4.5.3**

We now rearrange the results of our integrals to obtain the desired result. All expressions are now formal series in \( q^{-2\omega} \) and then \( q^{-2\mu} \) and meromorphically continued in \( t \) to the full Jackson cycle from \( \mathcal{C}_t \). For \( i \in \{1, 2\} \), define \( K_i(t) := L(t) J_{w,b}(t) \) with

\[
J_{w,b}(t) = q^{2b\mu+3b} \frac{I_{w,b}}{\theta_0(q^{-b+1-2\lambda}; q^{-2\omega})} \theta_0(q^{b+1+2\lambda}; q^{-2\omega})
\]

and

\[
L(t) = t^{-2(q+4) \frac{\Omega (t/z; q^{-2w}, q^{-2k-4}) \theta_0(t/zq^2; q^{-2k-4}) \theta_0(z/tq^2; q^{-2w})}{\theta_0(t/zq^{-2}; q^{-2k+4}) \theta_0(z/tq^2; q^{-2w})}. 
\]

In this notation, we have

\[
\Xi(q, \lambda, \omega, \mu, k) = -\frac{1}{q - q^{-1}} \frac{(q^{-2\omega+2}; q^{-2w}, q^{-2k-4})^2}{(q^{-2\omega-2}; q^{-2w}, q^{-2k-4})^2} (q^{-2\omega}, q^{-2\omega}) q^{2\lambda(\mu+1)+1} \int_0^{\infty} \frac{dt}{t} (K_1^1(t) + K_1^2(t) - K_1^1(t)). \tag{4.5.9}
\]

**Lemma 4.5.9.** We have \( K_1^i(pt) = K_1^1(t) \).

**Proof.** By Lemma 4.10.3, we have

\[
\Omega_q^2(p; q^{-2w}, q^{-2\kappa}) = \frac{\theta_0(t; q^{2\kappa-2}; q^{-2\omega})}{\theta_0(t; q^{2\kappa+2}; q^{-2\omega})} \Omega_q^2(\frac{t}{z}; q^{-2w}, q^{-2\kappa}),
\]

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and we also have that
\[
\frac{\theta_0(p_{\frac{t}{z}} q^2; q^{-2\kappa})\theta_0(p_{\frac{t}{z}} q^{-2}; q^{-2\kappa})}{\theta_0(p_{\frac{t}{z}} q^{-2}; q^{-2\kappa})\theta_0(p_{\frac{t}{z}} q^{-1} q^{-2}; q^{-2\omega})} = \frac{\theta_0(p_{\frac{t}{z}} q^2; q^{-2\kappa})\theta_0(p_{\frac{t}{z}} q^{-2} q^{-2}\omega; q^{-2\omega})}{\theta_0(p_{\frac{t}{z}} q^{-2}; q^{-2\kappa})\theta_0(p_{\frac{t}{z}} q^{-2} q^{-2}\omega; q^{-2\omega})}.
\]
Together, these imply that
\[
L(pt) = q^{-4\mu-4} \frac{\theta_0(p_{\frac{t}{z}} q^2; q^{-2\omega})}{\theta_0(p_{\frac{t}{z}} q^{-2}; q^{-2\omega})} L(t)
\]
and therefore by Lemma 4.5.8 and the definition of $J_{w,\pm}^1(t)$ we have
\[
J_{w,1}^1(pt) = - \frac{q^{2\mu+2\lambda+3}(q^2 - q^{-2\lambda})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{-2}\omega; q^{-2\omega})^2} \frac{(q^{2}\omega; q^{-2\omega})}{(q^{2}\omega; q^{-2\omega})^2} (\frac{t}{z} q^{-2}; q^{-2\omega}) \frac{2\phi_1\left(\frac{t}{z} q^{-2\omega}, q^{-2\omega}; q^{-2\omega}, q^{-2\omega}, q^{2\lambda}-2\right)}{2\phi_1\left(\frac{t}{z} q^{-2}; q^{-2\omega}; q^{-2\omega}, q^{-2\omega}, q^{2\lambda}-2\right)}
\]
and
\[
J_{w,-1}^1(t) = - \frac{q^{-2\mu+2\lambda-1}(q^2 - q^{-2\lambda})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{-2}\omega; q^{-2\omega})^2} \frac{(q^{2}\omega; q^{-2\omega})}{(q^{2}\omega; q^{-2\omega})^2} (\frac{t}{z} q^{-2}; q^{-2\omega}) \frac{2\phi_1\left(\frac{t}{z} q^{-2\omega}, q^{-2\omega}; q^{-2\omega}, q^{-2\omega}, q^{2\lambda}-2\right)}{2\phi_1\left(\frac{t}{z} q^{-2}; q^{-2\omega}; q^{-2\omega}, q^{-2\omega}, q^{2\lambda}-2\right)}
\]
We conclude that
\[
K_1^1(pt) = L(t)q^{-4\mu-4} \frac{\theta_0(p_{\frac{t}{z}} q^2; q^{-2\omega})}{\theta_0(p_{\frac{t}{z}} q^{-2}; q^{-2\omega})} J_{w,1}^1(pt) = L(t)J_{w,-1}^1(t) = K_{-1}^1(t). \quad \square
\]

We are now ready to deduce Proposition 4.5.3. By (4.5.9), invariance of the formal
Jackson integral under $p$-shifts, and Lemma 4.5.9, we obtain

$$\Xi(q, \lambda, \omega, \mu, k) = -\frac{1}{q - q^{-1}} \left( \frac{q^{2w+2-2\omega}}{q^{2w-2}q^{-2\omega}q^{-2\kappa}} \right)^2 \left( \frac{q^{-2\omega}}{q^{-2\lambda}q^{-2w}q^{-2\omega}} \right)^{2\lambda(\mu+1)+1} \int_0^{zq^{-2}\infty} \frac{d\rho}{\rho} \left( K_1^1(\rho) + K_1^2(\rho) - K_{-1}^1(\rho) \right)$$

Now, by Lemma 4.5.8, we find that

$$J_{w, 1}(pt) = -\frac{q^{2w+1}(q\lambda - q^2)}{\theta_0(q^{-2w})(q^{2w-2}q^{-2w})(q^{-2\lambda-2}(q^{-2w}q^{-2\omega})(q^{-2w}q^{-2\omega})(q^{-2w}q^{-2\omega}))}$$

and therefore that

$$K_1^2(t) = -C_1(\lambda, \mu) t^{-\frac{2(\mu+1)}{z}} \Omega_{q^2}(z^tq^{-2w}, q^{-2\kappa}) \frac{\theta_0(\frac{t}{z}q^2; q^{-2w})}{\theta_0(\frac{t}{z}q^{-2w}; q^{-2\kappa})}$$

for the constant

$$C_1(\lambda, \mu) = \frac{q^{-2\mu-1}}{\theta_0(q^{2\lambda}; q^{-2w})(q^{-2\lambda-2}q^{-2w})(q^{-2\lambda-2}q^{-2w})(q^{-2w}q^{-2\omega})} \frac{(q^{-4}; q^{-2w})}{(q^{-4}; q^{-2w})}$$

Substituting back into our computation yields

$$\Xi(q, \lambda, \omega, \mu, k) = \frac{1}{q - q^{-1}} \left( \frac{q^{2w+2-2\omega}}{q^{2w-2}q^{-2\omega}q^{-2\kappa}} \right)^2 \left( \frac{q^{-2\omega}}{q^{-2\lambda}q^{-2w}q^{-2\omega}} \right)^{2\lambda(\mu+1)+1} \int_0^{zq^{-2}\infty} \frac{d\rho}{\rho} \left( K_1^1(\rho) + K_1^2(\rho) - K_{-1}^1(\rho) \right)$$

which upon noting that

$$C(\lambda, \mu) = \frac{1}{q - q^{-1}} \left( \frac{q^{2w+2-2\omega}}{q^{2w-2}q^{-2\omega}q^{-2\kappa}} \right)^2 \left( \frac{q^{-2\omega}}{q^{-2\lambda}q^{-2w}q^{-2\omega}} \right)^{2\lambda(\mu+1)} C_1(\lambda, \mu)$$

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simplifies to

$$
\Xi(q, \lambda, \omega, \mu, k) = q^{2\lambda+2} C(\lambda, \mu) z^{2\mu+2 \frac{\lambda}{2}} \int_0^{2q^{-2}} \frac{d_q t}{t} Q q^2 \left( \frac{t}{z}; q^{-2\omega}, q^{-2\omega} \right) \frac{\theta_0 \left( \frac{t}{z}; q^{-2\omega}, q^{-2\omega} \right)}{\theta_0 \left( \frac{t}{z}; q^{-2\omega}, q^{-2\omega} \right)} 
$$

where in the last step we change variables to eliminate $z$. This completes the proof of Proposition 4.5.3.

### 4.5.10 Jackson integral expression for the trace function

We now apply Lemma 4.5.2 to obtain a Jackson integral expression for $T^{\omega_0}(q, \lambda, \omega, \mu, k)$.

**Proposition 4.5.10.** In the doubly formal good region of parameters (4.5.2), as a formal Jackson integral and a formal series in $q^{-2\omega}$ we have

$$
T^{\omega_0}(q, \lambda, \omega, \mu, k) = D(\lambda, \mu) \int_0^{2q^{-2}} \frac{d_q t}{t} Q q^2 \left( t; q^{-2\omega}, q^{-2\omega} \right) \frac{\theta_0 \left( t q^2; q^{-2\omega} \right)}{\theta_0 \left( t q^2; q^{-2\omega} \right)}
$$

where the constant $D(\lambda, \mu)$ is defined by

$$
D(\lambda, \mu) = \frac{(q^2 q^{-2k}; q^{-2k})(q^4, q^{-2\omega}) (q^{-2\omega}+2, q^{-2\omega}, q^{-2k})^2}{(q^{-2k}; q^{-2k})(q^{-2\omega}-2, q^{-2\omega}, q^{-2k})^2} q^{\lambda \mu - \lambda + 2 - 2k} 
$$

**Remark.** In the statement of Proposition 4.5.10, the functions

$$
f(t) = t^{-\mu+1 \frac{\lambda}{k}} \frac{\theta_0 \left( t q^2; q^{-2k} \right)}{\theta_0 \left( t q^2; q^{-2k} \right)} \quad \text{and} \quad g(t) = \frac{\theta_0 \left( t q^{-2\mu}; q^{-2k} \right)}{\theta_0 \left( t q^{-2\mu}; q^{-2k} \right)}
$$

have the same transformation properties under $q^{-2k}$-shifts, meaning that $f(q^{-2k} t) = q^{2k} g(t)$.

**Proof of Proposition 4.5.10.** By Lemma 4.5.2, we have

$$
T^{\omega_0}(q, \lambda, \omega, \mu, k) = C_{-1}^{-1} \Xi \left( q, \lambda, \omega, \frac{\mu-1}{2}, k - 2 \right). \quad (4.5.10)
$$
By Proposition 4.5.3, we find that

\[
\Xi(q, \lambda, \omega, \frac{\mu - 1}{2}, k - 2) = D_1(\lambda, \mu) \int_0^{q^{-2-\infty}} \frac{d_{q^{-2}k t}}{t} \Omega^2(t; q^{-2\omega}, q^{-2k}) t^{-\frac{\mu + 1}{k}} \frac{\theta_0(tq^2; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \theta_0(tq^2; q^{-2\omega}),(3.5.1)
\]

where the constant \( D_1(\lambda, \mu) \) is defined by

\[
D_1(\lambda, \mu) := C(\lambda, \frac{\mu - 1}{2}) \frac{q^\lambda - q^{\lambda - \mu - 1}(q^{-4}; q^{-2\omega})}{q - q^{-1} \theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2q^{-2\omega}; q^{-2\omega}})(q^{-2\lambda - 2q^{-2\omega}; q^{-2\omega}})} \frac{(q^{-2\omega + 2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2\omega - 2}; q^{-2\omega}, q^{-2k})^2}.
\]

Further, by Proposition 4.4.6, we have that

\[
C_{\mu + 1} = -(1 + q^2)q^{2\mu + 2 - 2\mu} \frac{(q^{-2k}; q^{-2k})}{(q^{-2\mu - 2}; q^{-2k})} \frac{(q^{-2\mu + 2}; q^{-2k})}{(q^4; q^{-2k})}.
\]

Substituting these into (4.5.10) yields the desired.

\[\square\]

### 4.5.11 Relating the Jackson integral to the Felder-Varchenko function

In this section, we convert the Jackson integral expression of Proposition 4.5.10 to a contour integral expression to give a proof of Theorem 4.5.1. For this, we show in Proposition 4.5.11 that when formally expanded in \( q^{-2\omega} \), the coefficients of both the Jackson and contour integrals converge in a formal neighborhood of 0 as functions of \( q^{-2\mu} \).

**Proposition 4.5.11.** In the doubly formal good region of parameters (4.5.2), as a formal power series in \( q^{-2\omega} \), the Jackson integral for \( T^{wo}(q, \lambda, \omega, \mu, k) \) converges in a formal neighborhood of 0 in the variable \( q^{-2\mu} \) and equals

\[
T^{wo}(q, \lambda, \omega, \mu, k) = \frac{(q^{-4}; q^{-2\omega})}{(q^{-2\omega}; q^{-2\omega})} \frac{(q^{-2\omega + 2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2\omega - 2}; q^{-2\omega}, q^{-2k})^2} \frac{(q^{-4q^{-2\omega}; q^{-2k}; q^{-2k}})}{(q^{-2\mu - 2}; q^{-2k})^2} \frac{q^{\lambda - \mu + 2}}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2q^{-2\omega}; q^{-2\omega}})(q^{-2\lambda - 2q^{-2\omega}; q^{-2\omega}})} \frac{(q^{-2\mu + 2}; q^{-2k})}{\theta_0(q^{-4q^{2k}; q^{-2\omega}})} \frac{(q^{2k}; q^{-2\omega})}{\prod_{l=1}^{n} \theta_0(q^{2kl}; q^{-2\omega})}.
\]

**Proof.** Define the integral expression

\[
I = \int_0^{q^{-2-\infty}} \frac{d_{q^{-2}k t}}{t} \Omega^2(t; q^{-2\omega}, q^{-2k}) t^{-\frac{\mu + 1}{k}} \frac{\theta_0(tq^2; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^2; q^{-2\omega})}{\theta_0(tq^{-2}; q^{-2k})} \theta_0(tq^2; q^{-2\omega}).
\]
so that \( T^{\omega_0}(q, \lambda, \omega, \mu, k) = D(\lambda, \mu)I \) by Proposition 4.5.10. Denote the integrand by

\[
J(t) = \Omega q^2(t; q^{-2\omega}, q^{-2k}) t^{-\frac{\mu+1}{k}} \frac{\theta_0(tq^2; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{2\lambda}; q^{-2\omega})}
\]

and let its formal power series expansion in \( q^{-2\omega} \) be \( J(t) = \sum_{n \geq 0} J_n(t) q^{-2\omega n} \). We have that

\[
J_0(t) = \left( \frac{tq^{-2}; q^{-2k}}{(tq^2; q^{-2k})} \right)^{\frac{\mu+1}{k}} \frac{\theta_0(tq^2; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{2\lambda}; q^{-2\omega})} = t^{-\frac{\mu+1}{k}} \left( \frac{t^{-1}q^{-2}; q^{-2k}}{(t^{-1}q^{-2}; q^{-2k})} \right)^{\frac{\mu+1}{k}} \frac{(1-tq^{2\lambda})}{(1-tq^{-2})} \frac{(1-tq^{-2})}{(1-tq^{-2})}
\]

We conclude that \( \int_0^{q^{-2\omega}} \frac{J_0(t)^n}{t^n} \) converges for \( \frac{n+1}{k} > n \), which holds in the doubly formal good region of parameters (4.5.2) on a formal neighborhood of 0 in \( q^{-2\mu} \). Defining the formal power series \( \tilde{J}(t) \) so that \( J(t) = J_0(t) \tilde{J}(t) \), we see that \( \tilde{J}(t) \) has expansion of the form

\[
\tilde{J}(t) = 1 + \sum_{n > 0} \tilde{J}_n(t) q^{-2\omega n},
\]

where \( \tilde{J}_n(t) \) is a Laurent series in \( t \) of degree at most \( n \). This implies that for each \( n \) the quantity \( \int_0^{q^{-2\omega}} \frac{J_n(t)^n}{t^n} \) converges. We conclude that each coefficient in the formal power series expansion of \( I = \int_0^{q^{-2\omega}} \frac{J_n(t)}{t} \) converges in a formal neighborhood of 0, so as a formal power series in \( q^{-2\omega} \) it equals

\[
I = \sum_{n \geq 0} J(q^{-2}q^{2kn})
\]

\[
= \sum_{n \geq 0} (q^{-2}q^{2kn})^{-\frac{\mu+1}{k}} \left( \frac{q^{-4}q^{2kn}; q^{-2\omega}, q^{-2k}}{(q^{2kn}; q^{-2\omega}, q^{-2k})} \frac{q^{-2}(q^{-2}q^{2kn}; q^{-2\omega})}{(q^{-2}q^{2kn}; q^{-2\omega})} \frac{\theta_0(q^{-4}q^{2kn}; q^{-2\omega})}{\theta_0(q^{-4}q^{2kn}; q^{-2\omega})} \frac{\theta_0(q^{2\lambda-2}q^{2kn}; q^{-2\omega})}{\theta_0(q^{2\lambda-2}q^{2kn}; q^{-2\omega})}
\]

\[
= q^{\frac{2\mu+2}{k}} ((q^{-4}; q^{-2\omega}, q^{-2k})(q^{-4}q^{-2}q^{2k}; q^{-2\omega}, q^{-2k}) \theta_0(q^{-4}; q^{-2k}) \theta_0(q^{2\lambda-2}q^{2kn}; q^{-2\omega}) \theta_0(q^{-4}q^{2kn}; q^{-2\omega}) \theta_0(q^{2\lambda-2}q^{2kn}; q^{-2\omega}) \theta_0(q^{-4}q^{2kn}; q^{-2\omega}) \prod_{i=1}^{n} \theta_0(q^{-4}q^{2k}; q^{-2\omega}) \theta_0(q^{2k}; q^{-2\omega})
\]

which yields the desired upon substitution of \( D(\lambda, \mu) \).

We now prove Theorem 4.5.1 and thus connect the trace function and the Felder-Varchenko function.

Proof of Theorem 4.5.1. Comparing Proposition 4.3.1 in the formal neighborhood of \( \infty \) for \( q^{-2\mu} \) and Proposition 4.5.11 in the formal neighborhood of 0 for \( q^{-2\mu} \), we find as formal power series in \( q^{-2\omega} \) and then \( q^{-2\mu} \), in the doubly formal good region of
parameters (4.5.2) we have

\[ T^{\nu_0}(q, \lambda, \omega, \mu, k) = \frac{q^{-\mu+4}(q^{-4}, q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda-2}q^{-2\omega}; q^{-2\omega})(q^{-2\omega-2}; q^{-2\omega})^{2} (q^{-2k}; q^{-2k})(q^{4}q^{-2k}; q^{-2k})(q^{2\mu+2}q^{-2k}; q^{-2k})} \frac{u(q, \lambda, \omega, -\mu, k)}{(q^{-2w+2}; q^{-2w}, q^{-2k})^2} \]

Recalling the definition (4.3.3) of the Felder-Varchenko function, we obtain

\[ T^{\nu_0}(q, \lambda, \omega, \mu, k) = \frac{(q^{-4}, q^{-2\omega})(q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2w-2}; q^{-2w}, q^{-2k})^2} \frac{q^{\mu-\lambda+2}}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda-2}q^{-2\omega}; q^{-2\omega})(q^{-2\omega-2}; q^{-2\omega})^{2} (q^{-2k}; q^{-2k})(q^{4}q^{-2k}; q^{-2k})(q^{2\mu+2}q^{-2k}; q^{-2k})} \int_{C_t} \frac{dt}{2\pi i t} \Omega^{q^2}(t; q^{-2w}, q^{-2k}) \theta_0(tq^{-2\mu}; q^{-2k}) \theta_0(tq^{2\lambda}; q^{-2\omega}) \theta_0(tq^{-2}; q^{-2k}) \theta_0(tq^{-2}; q^{-2\omega}). \]

By Proposition 4.2.6, each coefficient of \( q^{-\lambda \mu} T^{\nu_0}(q, \lambda, \omega, \mu, k) \) as a formal power series in \( q^{-2\omega} \) is a rational function in \( q^{-2\mu} \) and \( q^{-2k} \). Therefore, the coefficients of the formal power series in \( q^{-2\omega} \) in (4.5.11) are equal as rational functions because they are equal on a formal neighborhood of 0 in \( q^{-2\mu} \) by (4.5.11). Define now the formal good region of parameters

\[ 0 < |q^{-2\mu}| \ll |q^{-2\lambda}| \ll |q^{-2k}| \ll |q|, |q|^{-1}. \]  

We conclude that (4.5.11) holds in the formal good region of parameters at the level of formal power series in \( q^{-2\omega} \) for numerical \( q^{-2\mu} \) sufficiently close to 0.

It remains only to check that this formal series in \( q^{-2\omega} \) converges. Because the poles of the integrand in the integral expression are bounded uniformly away from \( C_t \) as \( q^{-2\omega} \to 0 \), the formal power series expansion of the integrand converges uniformly on the compact cycle \( C_t \) to the integrand. We conclude that the integral may be integrated term-wise and therefore that its formal expansion in \( q^{-2\omega} \) converges. Because \( T^{\nu_0}(q, \lambda, \omega, \mu, k) \) shares a formal expansion with the integral, it also converges for numerical \( q^{-2\omega} \) sufficiently close to 0. We conclude that the desired equality holds for numerical parameters in the good region (4.5.1) with \( q^{-2\mu} \) and then \( q^{-2\omega} \) sufficiently close to 0, completing the proof.

**Corollary 4.5.12.** For \( q^{-2\mu} \) and then \( q^{-2\omega} \) sufficiently close to 0 in the good region of parameters (4.5.1), the trace \( T^{\nu_0}(q, \lambda, \omega, \mu, k) \) is related to the Felder-Varchenko

\[ \text{Corollary 4.5.12. For } q^{-2\mu} \text{ and then } q^{-2\omega} \text{ sufficiently close to 0 in the good region of parameters (4.5.1), the trace } T^{\nu_0}(q, \lambda, \omega, \mu, k) \text{ is related to the Felder-Varchenko} \]

\[ \text{Corollary 4.5.12. For } q^{-2\mu} \text{ and then } q^{-2\omega} \text{ sufficiently close to 0 in the good region of parameters (4.5.1), the trace } T^{\nu_0}(q, \lambda, \omega, \mu, k) \text{ is related to the Felder-Varchenko} \]

\[ \text{Corollary 4.5.12. For } q^{-2\mu} \text{ and then } q^{-2\omega} \text{ sufficiently close to 0 in the good region of parameters (4.5.1), the trace } T^{\nu_0}(q, \lambda, \omega, \mu, k) \text{ is related to the Felder-Varchenko} \]

\[ \text{Corollary 4.5.12. For } q^{-2\mu} \text{ and then } q^{-2\omega} \text{ sufficiently close to 0 in the good region of parameters (4.5.1), the trace } T^{\nu_0}(q, \lambda, \omega, \mu, k) \text{ is related to the Felder-Varchenko} \]
function by

\[
T_{\mu \omega}(q, \lambda, \omega, \mu, k) = \frac{q^{\mu+4}(q^{-4}, q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda-2q-2\omega}; q^{-2\omega})(q^{-2\lambda-2q-2\omega}; q^{-2\omega})}
\]

\[
\frac{(q^{2\omega+2}; q^{-2\omega}, q^{-2k})^2 (q^{-2k}; q^{-2k})(q^{-2k}; q^{-2k})}{(q^{-2\omega-2}; q^{-2\omega}, q^{-2k})^2 (q^{-2\omega+2}; q^{-2\omega}, q^{-2k})(q^{2\omega+2q-2\omega}; q^{-2\omega})} u(q, \lambda, \omega, -\mu, k).
\]

**Proof.** This follows by combining Theorem 4.5.1 and the formal equality (4.5.11). 

### 4.6 The classical limit

In this section, we take the classical limit of our expression for the trace of a \(U_q(\mathfrak{sl}_2)\)-intertwiner and recover the expression for the trace of a \(U(\mathfrak{sl}_2)\)-intertwiner given in [27].

#### 4.6.1 Verma modules, evaluation modules, and intertwiners

Let \(M_{\mu, k}\) denote the Verma module for \(U(\mathfrak{sl}_2)\) with highest weight \(\mu \rho + k \Lambda_0\), and let \(L_\mu(z)\) denote the finite-dimensional evaluation module with highest weight \(\mu\). The module \(L_\mu(z)\) has a basis \(w_\mu, w_{\mu-2}, \ldots, w_{-\mu}\) which coincides with the basis for \(L_\mu(z)\) as a \(U_q(\mathfrak{sl}_2)\)-module. In particular, it satisfies

\[
ei w_m \otimes z^n = \frac{\mu - m}{2} w_{m+2} \otimes z^n.
\]

For \(v \in L_\mu(z)[0]\), denote by

\[
\Phi^{v,cl}_{\mu, k}(z) : M_{\mu, k} \to M_{\mu, k} \otimes L_\mu(z)
\]

the corresponding \(U(\mathfrak{sl}_2)\)-intertwiner, normalized so that

\[
\Phi^{v,cl}_{\mu, k}(z) \cdot v_{\mu, k} = v_{\mu, k} \otimes v + \text{(l.o.t.)}.
\]

#### 4.6.2 Wakimoto realization of intertwiners

In [27, Section 5], a different normalization of this intertwiner is considered via the Wakimoto realization. Let \(A\) be the algebra generated by \(\alpha_n, \beta_n, \gamma_n\) for \(n \in \mathbb{Z}\) subject to the relations

\[
[\alpha_n, \alpha_m] = 2n \delta_{n+m,0} \quad \text{and} \quad [\beta_n, \gamma_m] = \delta_{n+m,0}
\]

with all other commutators zero. Define also the bosonic fields

\[
\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}, \quad \beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n}.
\]

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Define the Fock module $H_{\lambda,k}$ to be generated by the vacuum vector $v_{\lambda,k}$ with properties

$$\alpha_n v_{\lambda,k} = 0 \text{ for } n > 0 \quad \beta_n v_{\lambda,k} = \gamma_{n+1} v_{\lambda,k} = 0 \text{ for } n \geq 0 \quad \alpha_0 v_{\lambda,k} = \frac{\lambda}{\sqrt{k+2}} v_{\lambda,k}.$$ 

A free field realization of the Verma module was given in terms of these Fock modules by Wakimoto in [100]. In our notation, this realization is given as follows.

**Proposition 4.6.1** ([27, Equation 5.7]). The assignments

$$J_<(z) = \beta(z)$$
$$J_>(z) = -2 : \beta(z) \gamma(z) : + \sqrt{\kappa} \alpha(z)$$
$$J_f(z) = - : \gamma(z)^2 \beta(z) : + \sqrt{\kappa} \alpha(z) \gamma(z) + (\kappa - 2) \gamma'(z)$$

define an action of $U(\hat{sl}_2)$ on $H_{\lambda,k}$. For generic $\lambda$ and $k$, the resulting module is isomorphic to the Verma module $M_{\lambda,k}$ with $v_{\lambda,k}$ the highest weight vector.

Define now the classical vertex operator

$$X(\alpha, z) = \exp \left( c \sum_{n<0} \frac{\alpha_n}{-n} z^{-n} \right) \exp \left( c \sum_{n>0} \frac{\alpha_n}{n} z^{-n} \right) e^{\alpha z} e^{ca}$$

and the screening operator

$$U(t) = \beta(t) X(-\alpha/\sqrt{\kappa}, t).$$

These two operators define the intertwining operator which appears in [27, Section 5].

**Proposition 4.6.2** ([27, Equation 5.8]). For any cycle $\Delta$ on which log($t_1$), log($t_i-t_j$), and log($t_i-z$) are well-defined, the operator $\Phi_\mu(z) : M_{\mu,k} \to M_{\mu,k} \otimes L_{2m}(z)$ given by

$$\Phi_\mu(z)v = z^{\frac{m(m+1)}{k+2}} \sum_{n=0}^{2m+1} \left( \int_{\Delta} X \left( \frac{m}{\sqrt{k+2}} \alpha(z) (-\gamma(z))^n U(t_1) \cdots U(t_m) dt_1 \cdots dt_m \right) v \otimes e^{n \alpha} e^{ca} \right)$$

is an intertwining operator of $U(\hat{sl}_2)$-representations.

**Remark.** Proposition 4.6.2 has a different normalization from [27, Section 5] because we take $d$ to act by 0 on $M_{\lambda,k}$. 

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4.6.3 The Etingof-Kirillov Jr. expression for the classical trace function

We now apply the computation of [27, Theorem 5.1] to compute in Corollary 4.6.8 the classical trace function

\[ \text{Tr}|_{M_{\mu-1,k-2}}(\Phi_{\mu-1,k-2}^{\text{uncl}} e^{2\pi i \lambda h}) \]

We begin by defining the normalized traces which appear in [27, Theorem 5.1]. For \( q = e^{\pi i r} \), define the function

\[ F(\lambda, \tau, \mu, k) = \frac{\text{Tr}|_{M_{\mu,k}}(\hat{\Phi}_{\mu-1} q^{-2d-\frac{1}{2}} e^{-\pi i \lambda h})}{\text{Tr}|_{M_{\frac{1}{2},k}}(q^{-2d-\frac{1}{2}} e^{-\pi i \lambda h})} \]

and its normalized version given by

\[ \tilde{F}(\lambda, \tau, \mu, k) := F(-\lambda + \tau/2, -\tau, \mu - 1, k - 2) = \frac{\text{Tr}|_{M_{\mu-1,k-2}}(\hat{\Phi}_{\mu-1} q^{2d} e^{\pi i \lambda h})}{\text{Tr}|_{M_{\frac{1}{2},k-2}}(q^{2d} e^{\pi i \lambda h})}. \]

An integral expression was given in [27] for \( F(\lambda, \tau, \mu, k) \) using the additive notation

\[ \theta_1(\zeta | \tau) = 2e^{\pi i r/4} \sin(\zeta) \prod_{n \geq 1} (1 - e^{2\pi i n r} e^{2i \zeta})(1 - e^{2\pi i n r} e^{-2i \zeta})(1 - e^{2\pi i n r}) = \theta(e^{2i \zeta}; e^{2\pi i r}) \]

for the first Jacobi theta function.

**Theorem 4.6.3 ([27, Theorem 5.1]).** For \( \mu \in \mathbb{C} \) and \( \kappa = k+2 \) with \( \kappa \neq 0, \Im(\tau) > 0, \Im(\lambda) > 0, \) and \( m \geq 0 \), we have

\[ F(\lambda, \tau, \mu, k) = e^{-\pi i \lambda \mu(\mu-\frac{1}{2})z^{\frac{m(m-\mu)}{\kappa}}} \int_{\Delta} \prod_{i=1}^{m} (-1-m)^{-\mu} i^{1-\mu} \prod_{i=1}^{m} \left( \frac{\theta_1(\tau \zeta_i - \zeta_i | \tau)}{\theta_1(0 | \tau)} \right)^{-2m/\kappa} \prod_{i<j} \left( \frac{\theta_1(\tau \zeta_i - \zeta_j | \tau)}{\theta_1(0 | \tau)} \right)^{2/\kappa} m! \prod_{i=1}^{m} G(z, t_i | \tau, \lambda + \tau/2) dt_1 \cdots dt_m, \]

where \( \Delta \) is a cycle chosen so that \( \log t_i, \log(t_i - t_j), \) and \( \log(t_i - z) \) have single-valued branches on \( \Delta \), the function \( G \) is defined by

\[ G(1, e^{2\pi i \zeta} | \tau, \lambda) = \frac{i}{2} e^{-2\pi i \zeta} \frac{\theta_1(\tau(\lambda - \zeta) | \tau) \theta_1(0 | \tau)}{\theta_1(\tau \lambda) \theta_1(\tau \zeta | \tau)}, \]

and we take the coordinates \( z = e^{2\pi i \zeta_0}, t_i = e^{2\pi i \zeta_i}, \) and \( q = e^{\pi i r}. \)

**Remark.** In [27, Theorem 5.1], \( d \) is normalized to act on \( v_{\mu,k} \) by \(-\frac{\mu^2}{2(k+2)}\). In this work we normalize \( d \) to act on \( v_{\mu,k} \) by 0, hence the statement of Theorem 4.6.3 differs
in normalization from the statement in [27, Theorem 5.1].

We now specialize to \( m = 1 \) and set \( z = 1 \) (since \( F(\lambda, \tau, \mu, k) \) is independent of \( z \)). After computing all relevant normalizations, we obtain in Corollary 4.6.8, the classical limit of \( T^{\text{lim}}(q, \lambda, \omega, \mu, k) \) from Theorem 4.5.1.

**Corollary 4.6.4.** For \( \mu, \kappa \in \mathbb{C}, \kappa \neq 0, |q| > 1, \text{Im}(\lambda) < 0, \) and \( m = 1 \), we have

\[
\bar{F}(\lambda, \tau, \mu, k) = 2^2 e^{-\pi i \lambda (\mu - \frac{1}{2})} \frac{\theta_1(0 | \tau)}{\theta_1(0; q^2)} \int_{\Delta} t^{-\frac{\mu + 1}{\kappa} - 1} \frac{\theta_1(\pi(\lambda + \tau/2 - \zeta_1) | \tau)}{\theta_1(\pi(\lambda + \tau/2) | \tau) \theta_1(\pi \zeta_1 | \tau)} dt,
\]

where \( \Delta \) is a Pochhammer loop around 0 and 1.

**Proof.** Direct substitution and setting \( z = 1 \) in Theorem 4.6.3 allows us to choose \( \Delta \) to be a Pochhammer loop around 0 and 1 and yields

\[
F(\lambda, \tau, \mu, k) = e^{-\pi i \lambda (\mu - \frac{1}{2})} \int_{\Delta} t^{-\frac{\mu + 1}{\kappa} - 1} \frac{\theta_1(\pi(\lambda + \tau/2 - \zeta_1) | \tau)}{\theta_1(\pi(\lambda + \tau/2) | \tau) \theta_1(\pi \zeta_1 | \tau)} dt.
\]

We now compute the derivative

\[
\theta_1(0 | \tau) = -2 e^{\pi i r/4} (q^2; q^2) \theta_1(0; q^2) = 2 e^{\pi i r/4} (q^2; q^2)^3.
\]

Substituting and applying (4.6.1) and (4.10.1), we conclude

\[
\bar{F}(\lambda, \tau, \mu, k) = e^{-2 \pi i \lambda (\mu - \frac{1}{2})} \frac{\theta_1(0 | \tau)}{\theta_1(\pi \zeta_1 | \tau)} \int_{\Delta} t^{-\frac{\mu + 1}{\kappa} - 1} \frac{\theta_1(\pi(\lambda - \zeta_1) | \tau)}{\theta_1(\pi(\lambda - \zeta_1) | \tau)} dt.
\]

We now compute the derivative

\[
\theta_1(0 | \tau) = -2 e^{\pi i r/4} (q^2; q^2) \theta_1(0; q^2) = 2 e^{\pi i r/4} (q^2; q^2)^3.
\]

Substituting and applying (4.6.1) and (4.10.1), we conclude

\[
\bar{F}(\lambda, \tau, \mu, k) = e^{-2 \pi i \lambda (\mu - \frac{1}{2})} \frac{\theta_1(0 | \tau)}{\theta_1(\pi(\lambda - \zeta_1) | \tau)} \int_{\Delta} t^{-\frac{\mu + 1}{\kappa} - 1} \frac{\theta_1(\pi(\lambda - \zeta_1) | \tau)}{\theta_1(\pi(\lambda - \zeta_1) | \tau)} dt.
\]

**Lemma 4.6.5.** For \(|q| > 1\) and \( \text{Im}(\lambda) < 0 \), we have that

\[
\text{Tr}_{M_{\frac{1}{2}} \lambda}(q^{2d} e^{\pi i \lambda \eta}) = e^{\frac{\pi i \lambda}{2} (q^2; q^2)^{-1}} \theta_1(\eta, q^2; q^2)^{-1}.
\]

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Proof. Since negative roots take the form $-\alpha, \pm \alpha + m\delta, m\delta$ for $m < 0$, we obtain

$$\text{Tr}|_{M_{\frac{1}{2},k}}(q^{2d}e^{\pi i\lambda h}) = e^{\pi i(\lambda h + 2\pi \delta, \frac{h}{k} + k\lambda_0)} \prod_{\beta < 0} (1 - e^{\pi i(\lambda h + 2\pi \delta, \beta)})^{-1} = e^{\frac{\pi i\lambda h}{2}}(q^{-2}; q^{-2})^{-1} \theta_0(e^{-2\pi i\lambda}; q^{-2})^{-1}.$$  

Corollary 4.6.6. For $|q| > 1$ and $\text{Im}(\lambda) < 0$, we have that

$$\text{Tr}|_{M_{\mu-1,k-2}}(\tilde{\Phi}_{\mu-1}q^{2d}e^{\pi i\lambda h})$$

$$= -2^\\frac{d}{k}e^{-\\frac{3\pi i}{k}e^{\pi i\lambda(\mu+1)}}(q^{-2}; q^{-2})^\frac{k+1}{k} \theta_0(e^{2\pi i\lambda}; q^{-2})^2 \int_\Delta t^{-\frac{\mu-1}{k}-1} \theta_0(te^{2\pi i\lambda}; q^{-2}) \theta_0(t; q^{-2})^\frac{k+1}{k} dt,$$

where $\Delta$ is a Pochhammer loop around 0 and 1.

Proof. By Corollary 4.6.4 and Lemma 4.6.5, we may compute

$$\text{Tr}|_{M_{\mu-1,k-2}}(\tilde{\Phi}_{\mu-1}q^{2d}e^{\pi i\lambda h}) = \tilde{F}(\lambda, \tau, \mu, k)\text{Tr}|_{M_{\frac{1}{2},k-2}}(q^{2d}e^{\pi i\lambda h})$$

$$= 2^\frac{d}{k}e^{-\\frac{3\pi i}{k}e^{\pi i\lambda(\mu+1)}}(q^{-2}; q^{-2})^\frac{k+1}{k} \theta_0(e^{2\pi i\lambda}; q^{-2})^2 \theta_0(e^{-2\pi i\lambda}; q^{-2}) \int_\Delta t^{-\frac{\mu-1}{k}-1} \theta_0(te^{2\pi i\lambda}; q^{-2}) \theta_0(t; q^{-2})^\frac{k+1}{k} dt$$

$$= -2^\frac{d}{k}e^{-\\frac{3\pi i}{k}e^{\pi i\lambda(\mu+1)}}(q^{-2}; q^{-2})^\frac{k+1}{k} \theta_0(e^{2\pi i\lambda}; q^{-2})^2 \int_\Delta t^{-\frac{\mu-1}{k}-1} \theta_0(te^{2\pi i\lambda}; q^{-2}) \theta_0(t; q^{-2})^\frac{k+1}{k} dt.$$  

Lemma 4.6.7 (Beta integral). If $\Delta$ is a Pochhammer loop around 0 and 1, then

$$\int_\Delta t^{\alpha-1}(1 - t)^{\beta-1} dt = (1 - e^{2\pi i\alpha})(1 - e^{2\pi i\beta}) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$  

Corollary 4.6.8. For $|q| > 1$ and $\text{Im}(\lambda) < 0$, we have that

$$\text{Tr}|_{M_{\mu-1,k-2}}(\tilde{\Phi}_{\mu-0,\mu-1}q^{2d}e^{\pi i\lambda h})$$

$$= -\frac{e^{\pi i\lambda(\mu+1)}e^{\pi i\mu-1}}{1 - e^{-\frac{4\pi i}{k}}} \frac{\Gamma(\mu-1)}{\Gamma(-\frac{2}{k})} \frac{\Gamma(k-\frac{\mu-1}{k})}{\theta_0(e^{2\pi i\lambda}; q^{-2})^2} \int_\Delta 2\pi it^{-\frac{\mu-1}{k}-1} \theta_0(te^{2\pi i\lambda}; q^{-2}) \theta_0(t; q^{-2})^\frac{k+1}{k} dt,$$

where $\Delta$ is a Pochhammer loop around 0 and 1.

Proof. Because both $\tilde{\Phi}_{\mu-1}(z)$ and $\tilde{\Phi}_{\mu-0,\mu-1}(z)$ are intertwiners $M_{\mu-1,k-2} \rightarrow M_{\mu-1,k-2} \hat{\otimes} L_{2m}(z)$, $\tilde{\Phi}_{\mu-1}(z)$ is a scalar multiple of $\tilde{\Phi}_{\mu-0,\mu-1}(z)$. Applying Lemma 4.6.7, the constant of pro-
portionality is
\[
\langle v_{\mu-1,k-2}^{*}, \Phi_{\mu-1}(z) v_{\mu-1,k-2} \rangle = \lim_{q \to \infty} \lim_{\mu \to \infty} e^{-\pi i \lambda (\mu - 1)} \text{Tr}|_{M_{\mu-1,k-2}} \left( \Phi_{\mu-1} q^{2d} e^{\pi i \lambda} \right)
\]
\[
= 2 \frac{e^{-\pi i \frac{1}{\mu}}}{\frac{1}{\mu} \Gamma(-\frac{2}{\mu})} \int_{\Delta} t^{-\frac{\mu-1}{k}} (1 - t)^{-\frac{k+2}{2}} dt
\]
\[
= 2 \frac{e^{-\pi i \frac{1}{\mu}}}{\frac{1}{\mu} \Gamma(-\frac{2}{\mu})} (1 - e^{-2\pi i \frac{1}{k}})(1 - e^{-\frac{4\pi i}{k}}) \frac{\Gamma\left(\frac{k-\mu+1}{k}\right) \Gamma\left(\frac{-2}{k}\right)}{\Gamma\left(\frac{k-\mu-1}{k}\right)}
\]
Normalizing the result of Corollary 4.6.6 we conclude
\[
\text{Tr}|_{M_{\mu-1,k-2}} \left( \Phi_{\mu-1,k-2}^{\text{cl}} q^{2d} e^{\pi i \lambda} \right) = \frac{\text{Tr}|_{M_{\mu-1,k-2}} \left( \Phi_{\mu-1} q^{2d} e^{\pi i \lambda} \right)}{2 \frac{e^{-\pi i \frac{1}{\mu}}}{\frac{1}{\mu} \Gamma(-\frac{2}{\mu})} \int_{\Delta} t^{-\frac{\mu-1}{k}} (1 - t)^{-\frac{k+2}{2}} dt}
\]
\[
= - \frac{e^{\pi i \lambda (\mu + 1)} e^{-\pi i \frac{1}{\mu}}}{1 - e^{-\frac{4\pi i}{k}}} \frac{\Gamma\left(\frac{\mu-1}{k}\right) \Gamma\left(\frac{k-\mu-1}{k}\right) (q^2 - q^{-2})^{\frac{k+4}{2}}}{\Gamma\left(-\frac{2}{k}\right)} \int_{\Delta} \frac{1}{2\pi i t} e^{-\frac{\mu-1}{k} \theta_0(t e^{2\pi i \lambda}; q^2)} \theta_0(t; q^{-2})^{\frac{k+4}{2}} dt.
\]

4.6.4 The classical limit of the trace function

We now obtain the result of Corollary 4.6.8 as a consequence of Theorem 4.5.1. For this, we take the classical limit of \(T_{\mu_0}(q, \lambda, \omega, \mu, k)\), given by \(\mu\) and \(k\) fixed and
\[
q = e^\varepsilon \quad \omega = e^{-1} \Omega \quad \lambda = e^{-1} \Lambda
\]
as \(\varepsilon \to 0\). Define the resulting limit of \(T_{\mu_0}(q, \lambda, \omega, \mu, k)\) by
\[
t(\Lambda, \Omega, \mu, k) := \lim_{\varepsilon \to 0} T_{\mu_0}(e^\varepsilon, e^{-1} \Lambda, e^{-1} \Omega, \mu, k).
\]

**Theorem 4.6.9.** If \(\Lambda, \Omega, \mu, k\) are real with \(-1 < \mu < 1\), the classical limit of \(T_{\mu_0}(q, \lambda, \omega, \mu, k)\) is given by
\[
t(\Lambda, \Omega, \mu, k) = e^{\mu \Lambda + \Lambda} e^{\pi i \frac{\mu-1}{k}} (e^{-2\Omega}; e^{-2\Omega})^{\frac{k+4}{2}} \frac{\Gamma\left(\frac{\mu-1}{k}\right) \Gamma\left(\frac{k-\mu-1}{k}\right) \theta_0(t e^{2\Lambda}; e^{-2\Omega})}{\Gamma\left(-\frac{2}{k}\right) \int_{\Delta} 2\pi i t \theta_0(t; e^{-2\Omega})^{\frac{k+4}{2}} t^{-\frac{\mu-1}{k}} dt}.
\]
where \(\Delta\) is a Pochhammer loop around 0 and 1.

**Remark.** Theorem 4.6.9 computes the classical trace function
\[
\text{Tr}|_{M_{\mu-1,k-2}} \left( \Phi_{\mu-1,k-2}^{\text{cl}}(z) e^{2\Omega d} e^{2\Lambda \rho} \right)
\]
and agrees with Corollary 4.6.8 under the variable substitutions \(q = e^\Omega\) and \(\lambda = \frac{\Delta}{\pi i}\) as expected.

We require some elementary asymptotic lemmas for the proof of Theorem 4.6.9.
Lemma 4.6.10 ([4, Corollary 10.3.4]). For \( x \in \mathbb{C} - \{0, -1, \ldots\} \), we have
\[
\lim_{p \to 1^-} (1 - p)^{1-x} \frac{[p; p]}{[p^x; p]} = \Gamma(x).
\]

Lemma 4.6.11. We have
\[
\lim_{p \to 1^-} \frac{(p^a; p)}{(p^b; p)} (1 - p)^{a-b} = \frac{\Gamma(b)}{\Gamma(a)}
\]

Proof. This follows by applying Lemma 4.6.10 twice. \( \square \)

Lemma 4.6.12. If \( u \) and \( v \) are complex with \( |u|, |v| < 1 \), we have
\[
\lim_{p \to 1^-} (p^a u; v) = (u; v)
\]
uniformly on compact subsets of \( |u| < 1 \).

Proof. We evaluate the limit as
\[
\lim_{p \to 1^-} (p^a u; v) = \lim_{p \to 1} \exp \left( - \sum_{m>0} \sum_{n>0} \frac{p^{am} u^m v^m}{m} \right) = \exp \left( - \sum_{m>0} \sum_{n>0} \frac{u^m v^m}{m} \right) = (u; v). \quad \square
\]

Lemma 4.6.13 ([4, Theorem 10.2.4]). For complex \( u \) with \( |u| \leq 1 \), if \( b \geq a \) and \( a + b \geq 1 \), we have
\[
\lim_{p \to 1^-} \frac{(p^a u; p)}{(p^b u; p)} = (1 - u)^{b-a}
\]
uniformly on compact subsets of \( |u| \leq 1 \). If \( a + b \geq 1 \) does not hold, the convergence is uniform on compact subsets of \( \{|u| \leq 1\} \) avoiding 1.

Lemma 4.6.14. For complex \( u \), if \( b \geq a \) and \( b \not\in \{0, -1, \ldots\} \), uniformly in \( p < 1 \) near 1 and \( |u| = 1 \) near 1, we have
\[
\frac{(p^a u; p)}{(p^b u; p)} \leq C(1 - u)^{b-a}
\]
for some constant \( C \) uniform in \( u \) and \( p \).

Proof. Choose \( m \geq 0 \) so that \( a + b + 2m \geq 1 \). For this \( m \), we have
\[
\frac{(p^a u; p)}{(p^b u; p)} = \frac{(p^a u; p)_m (p^{a+m} u; p)}{(p^b u; p)_m (p^{b+m} u; p)},
\]
so by Lemma 4.6.13 it suffices to show that \( \frac{1-p^{a+l}}{1-p^{b+l}} \) is uniformly bounded for \( 0 \leq l \leq m - 1 \). Let \( p = e^\epsilon \) and \( u = e^{2\pi is} \) and notice that for some constant \( C_1 \) and \( p, u \)
sufficiently close to 1 that

\[
\frac{1 - p^{a+1}u}{1 - p^{b+1}u} = 1 + \frac{u^{p^a}(p^a - p^b)}{1 - p^{b+1}u} \leq 1 + \frac{|p^a - p^b|}{1 - p^{b+1}u} \leq 1 + C_1 \frac{|b - a|\varepsilon}{(b + I)\varepsilon + 2\pi i\delta} \leq 1 + C_1 \frac{|b - a|}{|b + I|},
\]

which is uniformly bounded, as needed.

\[\square\]

**Proof of Theorem 4.6.9.** Recall from Theorem 4.5.1 that

\[
T_{\mu_0}(q, \lambda, \omega, \mu, k) = \frac{q^{\lambda\mu - \lambda + 2}(q^{-4}; q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2q^{-2\omega}}; q^{-2\omega})(q^{-2\lambda - 2}; q^{-2\omega})} \frac{(q^{-2k}; q^{-2k})(q^4q^{-2k}; q^{-2k})}{(q^{-2\mu+2}; q^{-2k})(q^{2\mu+2}q^{-2k}; q^{-2k})}
\]

\[
\times \frac{(q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2\omega-2}; q^{-2\omega}, q^{-2k})^2} \int_{C_t} \Omega_q(t; q^{-2\omega}, q^{-2k}) \theta_0(tq^{-2\mu}; q^{-2k}) \theta_0(tq^{2\lambda}; q^{-2\omega}) \theta_0(tq^{-2}; q^{-2k}) \theta_0(tq^{2}; q^{-2\omega}),
\]

where \(C_t\) is the unit circle. Applying Lemmas 4.6.11, 4.6.12, and 4.6.13, we obtain that

\[
\lim_{\varepsilon \to 0} \frac{q^{\lambda\mu - \lambda + 2}(q^{-4}; q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda - 2q^{-2\omega}}; q^{-2\omega})(q^{-2\lambda - 2}; q^{-2\omega})} \frac{(q^{-2k}; q^{-2k})(q^4q^{-2k}; q^{-2k})}{(q^{-2\mu+2}; q^{-2k})(q^{2\mu+2}q^{-2k}; q^{-2k})}
\]

\[
\times \frac{(q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2\omega-2}; q^{-2\omega}, q^{-2k})^2} = \frac{e^\mu^{\lambda-\lambda}(e^{-2\Omega}; e^{-2\Omega})}{\theta_0(e^{2\lambda}; e^{-2\Omega})(e^{2\lambda-2\Omega}; e^{-2\Omega})(e^{-2\lambda}; e^{-2\Omega})} \Gamma(\mu^{\varepsilon-1})\Gamma(\frac{k-\mu-1}{k})
\]

and

\[
\lim_{\varepsilon \to 0} \frac{(q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2\omega-2}; q^{-2\omega}, q^{-2k})^2} = \lim_{\varepsilon \to 0} \prod_{n \geq 0} \frac{(q^{-2\omega(n+1)+2}; q^{-2k})^2}{(q^{-2\omega(n+1)-2}; q^{-2k})^2} = \prod_{n \geq 0} (1 - e^{-2\Omega(n+1)})^\frac{4}{k} = (e^{-2\Omega}; e^{-2\Omega})^\frac{4}{k}.
\]

Uniformly on compact subsets of \(C_t\), we have by Lemma 4.6.12 that

\[
\lim_{n \to 0} \frac{(tq^{2\lambda}q^{-2\omega}; q^{-2\omega})(t^{-1}q^{-2\lambda}q^{-2\omega}; q^{-2\omega})}{(tq^{-2\omega}; q^{-2\omega})(t^{-1}q^2q^{-2\omega}; q^{-2\omega})} = \frac{(te^{2\lambda}e^{-2\Omega}; e^{-2\Omega})(t^{-1}e^{-2\lambda}e^{-2\Omega}; e^{-2\Omega})}{(te^{-2\Omega}; e^{-2\Omega})(t^{-1}e^{-2\Omega}; e^{-2\Omega})}.
\]

and for sufficiently small \(q^{-2\omega}\) by Lemma 4.6.13 that

\[
\lim_{\varepsilon \to 0} \prod_{n \geq 0} \frac{(tq^{-2q^{2\omega(n+1)}}; q^{-2k})(t^{-1}q^{-2q^{2\omega(n+1)}}q^{-2k}, q^{-2k})}{(tq^{2q^{2\omega(n+1)}}; q^{-2k})(t^{-1}q^2q^{2\omega(n+1)}q^{-2k}, q^{-2k})} = \prod_{n \geq 0} (1 - te^{-2\Omega(n+1)})^{-\frac{2}{k}} (1 - t e^{-2\Omega(n+1)})^{-\frac{2}{k}}.
\]

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Finally, by Lemma 4.6.13, uniformly on compact subsets of $C_t$ avoiding 1 we have
\[
\lim_{\varepsilon \to 0} \frac{1 - t q^{2\lambda} (t q^{-2\mu}; q^{-2k})(t^{-1} q^{2\mu} q^{-2k}; q^{-2k})}{1 - t q^{-2} (t q^{-2}; q^{-2k})(t^{-1} q^{2} q^{-2k}; q^{-2k})} = (1 - t q^{2k})(1 - t) - \frac{\mu_{\lambda + 1}^k}{1 - t},
\]
where we note that for $-1 < \mu < 1$ both $-\mu_{\lambda + 1}^k$ and $\mu_{\lambda - 1}^k$ are positive because $k < 0$ for our choice of parameters. Combining the previous three limits and applying Lemma 4.6.14, on a small enough compact neighborhood of 1 in $C_t$, the integrand has asymptotics bounded uniformly in $\varepsilon$ by a constant multiple of $(1 - t)^{-\frac{k+1}{2}}$. Therefore, the limit of the integral may be computed by omitting a shrinking compact neighborhood of 1 in $C_t$, so we conclude by uniformity of the limits on compact subsets of $C_t$ avoiding 1 that
\[
\ell(\Lambda, \Omega, \mu, k) = -\frac{e^{\mu \Lambda - \lambda} (e^{-2\Omega}; e^{-2\Omega})}{\Theta_0(e^{2\Lambda}; e^{-2\Omega})(e^{2\lambda-2\Omega}; e^{-2\Omega})} \frac{1}{\Gamma(\frac{\mu}{k}) \Gamma(1 - \frac{\mu}{k})} \frac{1}{\Gamma(\frac{\lambda}{k}) \Gamma(1 - \frac{\lambda}{k})} (e^{-2\Omega}; e^{-2\Omega})^\frac{1}{k}
\]
where in the second equality we recall that $\Delta$ is a Pochhammer loop around 0 and 1, note that
\[
(1 - t)^{-\frac{\mu_{\lambda + 1}^k}{k}} (1 - t^{-1})^{-\frac{\mu_{\lambda - 1}^k}{k}} = e^{\pi i \frac{\mu_{\lambda + 1}^k}{k}} t^{-\frac{\mu_{\lambda - 1}^k}{k}} (1 - t)^{-\frac{k+1}{2}}
\]
uniformly in $t$ on the complement of the positive real axis, and note that the integral over $\Delta$ is related to the integral over $C_t$ by the monodromy around the branch point at $t = 1$.

4.7 The trigonometric limit

We show that the trigonometric limit of our computation recovers the trace function for $U_q(sl_2)$.  

4.7.1 Conventions for $U_q(sl_2)$

Recall that $U_q(sl_2)$ is the Hopf algebra generated by $e, f, q^h$ with relations
\[
q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f, \quad [e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}
\]
and coproduct and antipode
\[
\Delta(e) = e \otimes 1 + q^h \otimes e, \quad \Delta(f) = f \otimes q^{-h} + 1 \otimes f, \quad \Delta(q^h) = q^h \otimes q^h;
\]
\[
S(e) = -q^{-h} e, \quad S(f) = -f q^h, \quad S(q^h) = q^{-h}.
\]
As defined, $U_q(\mathfrak{sl}_2)$ is a Hopf subalgebra of $U_q(\mathfrak{sl}_2)$. Let $M_\mu$ denote the Verma module for $U_q(\mathfrak{sl}_2)$ with highest weight $\mu$, and let $v_\mu$ be its highest weight vector. It has a basis $\{f^j v_\mu\}_{j \geq 0}$, on which $U_q(\mathfrak{sl}_2)$ acts by

$$e \cdot f^j v_\mu = [\mu - j + 1][j] f^{j-1} v_\mu, \quad f \cdot f^j v_\mu = f^{j+1} v_\mu, \quad q^h \cdot f^j v_\mu = q^{\mu-2j} f^j v_\mu.$$ 

If $\mu$ is a non-negative integer, let $L_\mu$ denote the finite-dimensional irreducible module with highest weight $\mu$. We pick a basis $\{w_{2j}\}$ for $L_{2\mu}$ so that $U_q(\mathfrak{sl}_2)$ acts by

$$e \cdot w_{2j} = [\mu - j] w_{2j+2}, \quad f \cdot w_{2j} = [\mu + j] w_{2j-2}, \quad q^h \cdot w_{2j} = q^{2j} w_{2j}.$$ 

Note that this basis is compatible with the basis for the evaluation module $L_{2\mu}(z)$ for $U_q(\mathfrak{sl}_2)$.

### 4.7.2 Computing the trace function for $U_q(\mathfrak{sl}_2)$

For $w_0 \in L_{2m}$, let $\Phi_{\mu}^{w_0,\text{trig}}$ be the unique intertwiner

$$\Phi_{\mu}^{w_0,\text{trig}} : M_\mu \to M_\mu \otimes L_{2m}$$

for which $\Phi_{\mu}^{w_0,\text{trig}}(v_\mu) = v_\mu \otimes w_0 + (\text{t.o.t.})$. We compute the trace function associated to $\Phi_{\mu}^{w_0,\text{trig}}$ in a manner similar to that used in the opposite coproduct in [37, Section 7].

**Lemma 4.7.1.** For $w_0 \in L_{2m}$, the intertwiner $\Phi_{\mu}^{w_0,\text{trig}}$ acts on the highest weight vector $v_\mu \in M_\mu$ by

$$\Phi_{\mu}^{w_0,\text{trig}}(v_\mu) = \sum_{j=0}^{m} (-1)^j q^{\mu j - j(j - 1)} \frac{[m]}{[\mu j][j]} f^j v_\mu \otimes w_{2j}.$$ 

**Proof.** For some constants $c_j(\mu, m)$ with $c_0(\mu, m) = 1$, we have

$$\Phi_{\mu}^{w_0,\text{trig}}(v_\mu) = \sum_{j=0}^{m} c_j(\mu, m) f^j v_\mu \otimes w_{2j}.$$ 

Because $\Phi_{\mu}^{w_0,\text{trig}}(v_\mu)$ is killed by $\Delta(e)$, we find that

$$0 = \sum_{j=0}^{m} c_j(\mu, m) e f^j v_\mu \otimes w_{2j} + \sum_{j=0}^{m} c_j(\mu, m) q^h f^j v_\mu \otimes ew_{2j}$$

$$= \sum_{j=0}^{m} c_j(\mu, m) \mu - j + 1][j] f^{j-1} v_\mu \otimes w_{2j} + \sum_{j=0}^{m} c_j(\mu, m) q^{\mu-2j} m - j f^j v_\mu \otimes w_{2j+2}.$$ 

Taking the coefficient of $f^j v_\mu \otimes w_{2j+2}$ yields

$$c_j(\mu, m) q^{\mu-2j} m - j + c_{j+1}(\mu, m) \mu - j [j + 1] = 0$$
and therefore by induction we find that
\[ c_j(\mu, m) = (-1)^j q^{\mu j-j(j-1)} \frac{[m]_j}{[\mu]_j [j]_j}. \]

Proposition 4.7.2. If \(|q^{-2\lambda}| \ll 1\), the trace function for \(\Phi^{w, \text{trig}}_\mu\) converges and has value
\[
\text{Tr}|_{M_\mu}(\Phi^{w, \text{trig}}_\mu q^{2\lambda \rho})
= q^{\lambda \mu} \sum_{l=0}^{m} (-1)^l q^{-2\lambda l} q^{-l(l-1)/2} \frac{[m]_l [m + l]_l}{[\mu]_l [l]_l} \frac{(q - q^{-1})^l}{\prod_{i=0}^{l-1}(1 - q^{-2\mu+2i}) \prod_{i=0}^{l}(1 - q^{-2\lambda-2i})}.
\]

Proof. Applying Lemma 4.7.1 and the expansion
\[
\Delta(f^k) = \sum_{i=0}^{k} \prod_{l=i+1}^{k} \frac{(1 - q^{-2i})}{(1 - q^{-2l})} f^{k-l} \otimes q^{-(k-l)h} f^l,
\]
we obtain
\[
\Phi^{w, \text{trig}}_\mu(q^{2\lambda \rho} f^k \psi_\mu) = q^{\lambda \mu - 2\lambda k} \Delta(f^k) \Phi^{w, \text{trig}}_\mu(\psi_\mu)
= q^{\lambda \mu - 2\lambda k} \sum_{l=0}^{k} \frac{\prod_{i=k-l+1}^{k} (1 - q^{-2i})}{\prod_{i=1}^{l} (1 - q^{-2i})} \sum_{j=0}^{m} (-1)^j q^{\mu j-j(j-1)} \frac{[m]_j}{[\mu]_j [j]_j} f^j \psi_\mu \otimes \psi_{2j}
= q^{\lambda \mu - 2\lambda k} \sum_{l=0}^{k} \frac{\prod_{i=k-l+1}^{k} (1 - q^{-2i})}{\prod_{i=1}^{l} (1 - q^{-2i})} \sum_{j=0}^{m} (-1)^j q^{\mu j-j(j-1)} \frac{[m]_j}{[\mu]_j [j]_j} f^{k+j-l} \psi_\mu \otimes q^{-(k-l)h} f^l \psi_{2j}.
\]
The diagonal terms correspond to the \(j = l\) term of the above summation. Notice that \(q^{-(k-l)h} f^l \psi_{2j} = [m + l]_l \psi_0\). We conclude that
\[
\text{Tr}|_{M_\mu}(\Phi^{w, \text{trig}}_\mu q^{2\lambda \rho})
= q^{\lambda \mu} \sum_{l=0}^{k} \frac{\prod_{i=k-l+1}^{k} (1 - q^{-2i})}{\prod_{i=1}^{l} (1 - q^{-2i})} \sum_{l=0}^{k} \frac{q^{-2\lambda l} q^{-l(l-1)/2} (1 - q^{-2i})}{\prod_{i=0}^{l}(1 - q^{-2\mu+2i}) \prod_{i=0}^{l}(1 - q^{-2\lambda-2i})}
= q^{\lambda \mu} \sum_{l=0}^{k} \frac{[m]_l [m + l]_l}{[\mu]_l [l]_l} \frac{1}{\prod_{i=0}^{l}(1 - q^{-2\lambda-2i})}.
\]
We make our previous computation explicit in the special case \(m = 1\) correspond-
ing to the three-dimensional representation. For \( w_0 \in L_2 \), define
\[
T^{w_0, \text{tri}g}(q, \lambda, \mu) = \text{Tr}|_{M_{\nu - 1}}(\Phi^{w_0, \text{tri}g}_{\nu - 1} q^{2\lambda \rho}).
\]

Lemma 4.7.3. We have
\[
T^{w_0, \text{tri}g}(q, \lambda, \mu) = \frac{q^{\lambda \mu - \lambda}}{1 - q^{-2\lambda}} \left( \frac{1 - q^{-2\mu + 2} - q^{-2\lambda + 2} + q^{-2\mu - 2\lambda}}{(1 - q^{-2\mu + 2})(1 - q^{-2\lambda + 2})} \right).
\]

Proof. This follows specializing Proposition 4.7.2 to the case \( m = 1 \). \( \square \)

4.7.3 The trigonometric limit of the trace function

We check that the trigonometric limit \( q^{-2\omega} \to 0 \) of \( T^{w_0}(q, \lambda, \omega, \mu, k) \) corresponds with the trace function for \( U_q(s_2) \).

Theorem 4.7.4. We have that
\[
\lim_{q^{-2\omega} \to 0} T^{w_0}(q, \lambda, \omega, \mu, k) = T^{w_0, \text{tri}g}(q, \lambda, \mu).
\]

Proof. The limiting expression is the constant coefficient in \( q^{-2\omega} \) of \( T^{w_0}(q, \lambda, \omega, \mu, k) \), which by Proposition 4.5.11 and Lemma 4.7.3 has value
\[
\lim_{q^{-2\omega} \to 0} T^{w_0}(q, \lambda, \omega, \mu, k) = \frac{q^{\lambda \mu - \lambda + 2}(1 - q^{-4})(q^{2\lambda + 2}; q^{-2k})}{(1 - q^{2\lambda})(1 - q^{-2\lambda - 2})(q^{-2\mu + 2}; q^{-2k})} \sum_{n \geq 0} \frac{q^{-(2\mu + 2)n}(1 - q^{2\lambda - 2}q^{2kn}) (q^{-4}; q^{2kn}; q^{-2k})_n}{(1 - q^{-4}q^{2kn})(q^{2kn}; q^{-2k})_n}
\]
\[
= -\frac{q^{\lambda \mu - \lambda + 2}}{(1 - q^{2\lambda})(1 - q^{-2\lambda - 2})(q^{-2\mu + 2}; q^{-2k})(1 - q^{-4})} \sum_{n \geq 0} \frac{q^{-(2\mu - 2 - 2k)n}(1 - q^{2\lambda - 2}q^{2kn}) (q^{4}; q^{-2k})_n}{(q^{2k}; q^{-2k})_n}
\]
\[
= \frac{q^{\lambda \mu - \lambda + 2}}{(1 - q^{2\lambda})(1 - q^{-2\lambda - 2})(q^{-2\mu + 2}; q^{-2k})} \sum_{n \geq 0} \frac{q^{-(2\mu - 2 - 2k)n}(1 - q^{2\lambda - 2}q^{2kn}) (q^{-2\mu + 2 - 2k}; q^{-2k})_n (q^{2k}; q^{-2k})_n}{(q^{-2\mu + 2}; q^{-2k})_n}
\]
\[
= T^{w_0, \text{tri}g}(q, \lambda, \mu). \quad \square
\]

4.8 Symmetry of the trace function

In this section, we show that a certain normalization of the trace function is symmetric under interchanging \((\lambda, \omega)\) and \((\mu, k)\). To state the symmetry which we will show, define the Weyl denominator \( \delta_q(\lambda, \omega) \) by
\[
\delta_q(\lambda, \omega) := \text{Tr}|_{M_{\rho}}(q^{2\lambda \rho + 2\omega - 1})^{-1}
\]
and the normalized trace
\[ \bar{T}^{u_1}(q, \lambda, \omega, \mu, k) := \delta_q(\lambda, \omega)T^{u_1}(q^{-1}, -\lambda, -\omega, \mu, k), \]

where on the right we consider the quasi-analytic continuation of \( T^{u_1}(q, \lambda, \omega, \mu, k) \) to the region of parameters \(|q| > 1, |q^{-2\omega}| < 1, \text{ and } |q^{-2k}| > 1\).

**Theorem 4.8.1.** The function \( \bar{T}^{u_1}(q, \lambda, \omega, \mu, k) \) is symmetric under interchange of \((\lambda, \omega)\) and \((\mu, k)\).

For Theorem 4.8.1, we first compute the quasi-analytic continuation and the normalization factor \( \delta_q(\lambda, \omega) \).

**Lemma 4.8.2.** The quasi-analytic continuation of \( T^{u_1}(q, \lambda, \omega, \mu, k) \) to \(|q| > 1, |q^{-2\omega}| < 1, \text{ and } |q^{-2k}| > 1\) is
\[
T^{u_1}(q, \lambda, \omega, \mu, k) \equiv \frac{q^{\lambda+\lambda+2}(q^{-4}; q^{-2\omega})_1}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda-2q^{-2\omega}; q^{-2\omega}})(q^{-2\lambda-2q^{-2\omega}; q^{-2\omega}}) 1 - q^4}
\frac{(q^{-4}; q^{2k})_1(q^{2k}; q^{2k})}{(q^{-2\omega-22k}; q^{-2\omega})(q^{-2\omega+22k}; q^{-2\omega})(q^{-2\omega-2k}; q^{-2\omega})(q^{-2\omega+2k}; q^{-2\omega})}
\int_{|t|=1}^{2\pi} \Omega q^{-2}(t; q^{-2\omega}, q^{2k}) \frac{\theta_0(tq^{2\mu}; q^{2k})}{\theta_0(tq^{2}; q^{2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{2}; q^{-2\omega})} dt.
\]

**Proof.** This follows from Corollary 4.5.12 and Proposition 4.3.7. \( \square \)

**Proposition 4.8.3.** We have
\[ \delta_q(\lambda, \omega) = q^\lambda(q^{-2\omega}; q^{-2\omega}) \theta_0(q^{-2\lambda}; q^{-2\omega}). \]

**Proof.** Notice that
\[
\delta_q(\lambda, \omega) = \text{Tr}_M\left. q^{2\lambda+22\omega} \right|_{\lambda=0} = q^{(\alpha+2\lambda, 2\omega+2\omega) \prod_{\beta > 0} (1 - q^{-2\lambda+2\omega})} \frac{1}{\prod_{\beta > 0} (1 - q^{-2\lambda+2\omega})} \mu(\beta)
\]
\[
= q^\lambda(1 - q^{-2\lambda}) \prod_{m > 0} (1 - q^{-2\omega m})(1 - q^{-2\lambda-2\omega m})(1 - q^{-2\lambda+2\omega m})
\]
\[
= q^\lambda(q^{-2\omega}; q^{-2\omega}) \theta_0(q^{-2\lambda}; q^{-2\omega}), \tag{4.8.1}
\]

where we recall that the positive roots for \( U_q(\tilde{sl}_2) \) have multiplicity 1 and are given by
\[
\{\alpha, \pm \alpha + m\delta, m\delta \mid m > 0\}. \] \( \square \)

**Proof of Theorem 4.8.1.** By Lemma 4.8.2 and Proposition 4.8.3, after some cancellation it suffices to check that
\[
q^{-2\lambda} \int_{|t|=1}^{2\pi} \Omega q^2(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{-2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{2}; q^{-2\omega})} dt
\]
\[
= q^{-2\mu} \int_{|t|=1}^{2\pi} \Omega q^2(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{2\mu}; q^{-2k})}{\theta_0(tq^{-2}; q^{-2k})} \frac{\theta_0(tq^{-2\lambda}; q^{-2\omega})}{\theta_0(tq^{-2}; q^{-2\omega})} dt.
\]

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By Lemma 4.10.3, we have that
\[
\Omega_{q^2}(t; q^{-2\omega}, q^{-2k}) = \Omega_{q^2}(t^{-1}; q^{-2\omega}, q^{-2k}) \frac{\theta_0(t^{-1}q^2; q^{-2k})\theta_0(tq^{-2}; q^{-2\omega})}{\theta_0(tq^2; q^{-2k})\theta_0(tq^{-2}; q^{-2\omega})} = q^4 \Omega_{q^2}(t^{-1}; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{-2}; q^{-2k})\theta_0(tq^{-2}; q^{-2\omega})}{\theta_0(tq^2; q^{-2k})\theta_0(tq^{-2}; q^{-2\omega})}.
\]

Upon substitution, this implies that
\[
q^{-2\lambda} \int_{|t|=1} \frac{dt}{2\pi i t} \Omega_{q^2}(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{-2\mu}; q^{-2k})\theta_0(tq^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{-2}; q^{-2k})\theta_0(tq^{-2}; q^{-2\omega})} = q^{-2\lambda+4} \int_{|t|=1} \frac{dt}{2\pi i t} \Omega_{q^2}(t^{-1}; q^{-2\omega}, q^{-2k}) \frac{\theta_0(t^{-1}q^{2\mu}; q^{-2k})\theta_0(tq^{-2\lambda}; q^{-2\omega})}{\theta_0(t^{-1}q^{-2}; q^{-2k})\theta_0(t^{-1}q^{-2}; q^{-2\omega})} = q^{-2\mu} \int_{|t|=1} \frac{dt}{2\pi i t} \Omega_{q^2}(t; q^{-2\omega}, q^{-2k}) \frac{\theta_0(tq^{2\mu}; q^{-2k})\theta_0(t^{-1}q^{2\lambda}; q^{-2\omega})}{\theta_0(tq^{-2}; q^{-2k})\theta_0(t^{-1}q^{-2}; q^{-2\omega})},
\]
where in the last step we make the change of variables \( t \mapsto t^{-1} \).

\[\square\]

4.9 Application to affine Macdonald polynomials

In this section we explain how our trace functions relate to the affine Macdonald polynomials for \( \widehat{\mathfrak{sl}}_2 \) defined by Etingof-Kirillov Jr. in [28]. We use them to prove Felder-Varchenko’s conjecture that their definition of affine Macdonald polynomials via hypergeometric theta functions in [49] coincides with that of [28]. The classical degeneration of this section is related to the study of conformal blocks in [43].

4.9.1 Affine Macdonald polynomials as traces of intertwiners

Fix an integer \( k \geq 0 \). For integers \( k \geq \mu \geq 0 \), let \( L_{\mu, k} \) denote the irreducible integrable module for \( \mathcal{U}_q(\widehat{\mathfrak{sl}}_2) \) with highest weight \( \mu \rho + k\Lambda_0 \) and highest weight vector \( v_{\mu, k} \). For \( v \in \mathfrak{S}ym^{2(k-1)}L_1[0] \), we have an intertwiner
\[
\Upsilon_{\mu, k}(z) : L_{\mu \rho + k\Lambda_0 + (k-1)\rho} \rightarrow L_{\mu \rho + k\Lambda_0 + (k-1)\rho} \otimes \mathfrak{S}ym^{2(k-1)}L_1(z)
\]
such that \( \Upsilon_{\mu, k}(z)v_{\mu \rho + k\Lambda_0 + (k-1)\rho} = v_{\mu \rho + k\Lambda_0 + (k-1)\rho} \otimes v + (\text{l.o.t.}) \). Define the trace function
\[
\chi_{\mu, k}(q, \lambda, \omega) = \text{Tr}|_{L_{\mu \rho + k\Lambda_0 + (k-1)\rho}} \left( \Upsilon_{\mu, k}(z)q^{2\lambda}q^{2\omega} \right),
\]
where the trace is independent of \( z \). In [28], the affine Macdonald polynomial for \( \widehat{\mathfrak{sl}}_2 \) was defined to be
\[
J_{\mu, k}(q, \lambda, \omega) := \frac{\chi_{\mu, k}(q, \lambda, \omega)}{\chi_{0, 0, k}(q, \lambda, \omega)}.
\]

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It is a symmetric Laurent polynomial with highest term $e^{(\mu \rho + k \Lambda_0, 2 \lambda \rho + 2 \omega d)}$.

**Remark.** To avoid conflict with the use of $k$ for level, we use $k$ to denote the parameter of the Macdonald polynomial. This corresponds to the variable $k$ in [28] and $m$ in [49].

### 4.9.2 Elliptic Macdonald polynomials as hypergeometric theta functions

In [49], the elliptic Macdonald polynomial was defined by Felder-Varchenko in terms of hypergeometric theta functions. Fix parameters satisfying $|q| > 1$, $|q^{-2k}| < 1$, and $|q^{-2\omega}| < 1$. Define

$$Q(q, \mu, k) := -\frac{2\pi i q^{2\mu - 2} (q^{-2k}; q^{-2k})^2 \theta_0(q^4; q^{-2k})}{\theta_0(q^{2\mu - 2}; q^{-2k}) \theta_0(q^{2\mu + 2}; q^{-2k})}.$$

In terms of $Q$, define the $l$th non-symmetric hypergeometric theta function of level $\kappa + 2$ by the convergent series

$$\Delta_{\mu, \kappa}(q, \lambda, \omega) := q^{\frac{2\mu^2}{2k^2}} Q(q, \mu, \kappa) \sum_{j \in \mathbb{Z} + \mu} u(q, \lambda, \omega, j, \kappa) q^{-\frac{\omega^2 j^2}{2k^2}},$$

and the $\mu$th hypergeometric theta function of level $\kappa + 2$ by

$$\Delta_{\mu, \kappa}(q, \lambda, \omega) := \Delta_{\mu, \kappa}(q, \lambda, \omega) - \Delta_{\mu, \kappa}(q, -\lambda, \omega).$$

Then the elliptic Macdonald polynomial is given by

$$\tilde{J}_{\mu, \kappa}(q, \lambda, \omega) := i q^{\frac{\mu^2}{2k} + \frac{\mu^2}{2} + 3\lambda} \frac{\Delta_{\mu, \kappa+2}(q, \lambda, \omega)}{(q^{-2\omega}; q^{-2\omega})^3 \theta_0(q^{2\lambda - 2}; q^{-2\omega}) \theta_0(q^{2\lambda}; q^{-2\omega}) \theta_0(q^{2\lambda + 2}; q^{-2\omega})}.$$

In [49], Felder-Varchenko conjectured that $\tilde{J}_{\mu, \kappa}(q, \lambda, \omega)$ is related to the affine Macdonald polynomial of [28]. Define the quantity

$$\tilde{J}^0_{\mu, \kappa}(q, \lambda, \omega) := \frac{2\pi q^{2\mu + 3\lambda + 2}}{(q^{-2\omega}; q^{-2\omega})^3 \theta_0(q^{2\lambda - 2}; q^{-2\omega}) \theta_0(q^{2\lambda}; q^{-2\omega}) \theta_0(q^{2\lambda + 2}; q^{-2\omega})} \frac{(q^{-2\kappa}; q^{-2\kappa})^2 \theta_0(q^4; q^{-2\kappa})}{\theta_0(q^{2\mu + 2}; q^{-2\kappa}) \theta_0(q^{2\mu + 6}; q^{-2\kappa})}.$$

**Lemma 4.9.1.** For $q^{-2\mu}$ and then $q^{-2\omega}$ sufficiently close to 0 in the good region of parameters (4.5.1), the elliptic Macdonald polynomial may be expressed in the form

$$\tilde{J}_{\mu, \kappa}(q, \lambda, \omega) = \tilde{J}^0_{\mu, \kappa}(q, \lambda, \omega) \sum_{j \in \mathbb{Z}} q^{-2j(\kappa j + \mu + 2)(\omega + 2)} \left( u(q, \lambda, \omega, \mu + 2 + 2\kappa j, \kappa) - u(q, -\lambda, \omega, \mu + 2 + 2\kappa j, \kappa) \right).$$
Proof. Applying the definitions, we find that
\[
\tilde{J}_{\mu,k}(q,\lambda,\omega) = \frac{iq^{-\frac{(\mu+2)^2}{3}} + \frac{q}{2\epsilon}(\mu+2)^2 + 3\lambda}{(q^{-2\omega}; q^{-2\omega})^3 \theta_0(q^{2\lambda/2}; q^{-2\omega}) \theta_0(q^{2\lambda}; q^{-2\omega}) \theta_0(q^{2\lambda/2}; q^{-2\omega})} \sum_{j \in 2\kappa Z + \mu + 2} (u(q, \lambda, \omega, j, \kappa) - u(q, -\lambda, \omega, j, \kappa))q^{-\frac{\mu+2}{2\kappa}j^2} \]
\[
= \frac{2\pi q^{2\mu+2} q^{-\frac{(\mu+2)^2}{3}} + \frac{q}{2\epsilon}(\mu+2)^2 + 3\lambda}{(q^{-2\omega}; q^{-2\omega})^3 \theta_0(q^{2\lambda/2}; q^{-2\omega}) \theta_0(q^{2\lambda}; q^{-2\omega}) \theta_0(q^{2\lambda/2}; q^{-2\omega})} \sum_{j \in 2\kappa Z + \mu + 2} (u(q, \lambda, \omega, j, \kappa) - u(q, -\lambda, \omega, j, \kappa))q^{-\frac{\mu+2}{2\kappa}j^2} \]
\[
= \tilde{J}_{\mu,k}(q,\lambda,\omega) \sum_{j \in \mathbb{Z}} q^{-2j(\kappa j + \mu + 2)(\omega + 2)} (u(q, \lambda, \omega, \mu + 2 + 2\kappa j, \kappa) - u(q, -\lambda, \omega, \mu + 2 + 2\kappa j, \kappa)). \quad \Box
\]

4.9.3 BGG resolution for $U_q(\mathfrak{sl}_2)$-modules with integral highest weight

We introduce now the BGG resolution of $L_{\mu,k}$, which will allow us to compute the affine Macdonald polynomial in terms of our trace functions. Denote the affine Weyl group of $\mathfrak{sl}_2$ by

$$W^a = \langle s_0, s_1 | s_0^2 = s_1^2 = 1 \rangle.$$

It acts on $\tilde{h}$ via

$$s_1(\alpha) = -\alpha \quad s_1(c) = c \quad s_1(d) = d \quad s_0(\alpha) = 2c - \alpha \quad s_0(c) = c \quad s_0(d) = d + \alpha - c$$

and on $\tilde{h}^*$ via

$$s_1(\alpha) = -\alpha \quad s_1(\Lambda_0) = \Lambda_0 \quad s_1(\delta) = \delta \quad s_0(\alpha) = -\alpha + 2\delta \quad s_0(\Lambda_0) = \Lambda_0 + \alpha - \delta \quad s_0(\delta) = \delta. \quad (4.9.1)$$

Define the dotted action of $W^a$ on $\tilde{h}^*$ by $w \cdot \tilde{\mu} = w \cdot (\tilde{\mu} + \tilde{\rho}) - \tilde{\rho}$. For $l > 0$, denote the length $l$ elements of $W^a$ by $w_0^l := s_0s_1 \cdots$ and $w_1^l := s_1s_0 \cdots$, where $w_0^0$ and $w_1^0$ contain $l$ reflections. We compute the actions of $w_0^l$ and $w_1^l$ on $\mu \rho + k\Lambda_0$. 

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**Lemma 4.9.2.** The dotted action of $w_i^l$ on $\mu \rho + k\Lambda_0$ is given by

$$w_0^{2l} \cdot (\mu \rho + k\Lambda_0) = (\mu + 2l(k + 2))\rho + k\Lambda_0 + (-l(\mu + 1) - l^2(k + 2))\delta$$

$$w_1^{2l} \cdot (\mu \rho + k\Lambda_0) = (\mu - 2l(k + 2))\rho + k\Lambda_0 + (l(\mu + 1) - l^2(k + 2))\delta$$

$$w_0^{2l+1} \cdot (\mu \rho + k\Lambda_0) = (\mu - 2 + 2(l + 1)(k + 2))\rho + k\Lambda_0 + ((l + 1)(\mu + 1) - (l + 1)^2(k + 2))\delta$$

$$w_1^{2l+1} \cdot (\mu \rho + k\Lambda_0) = (\mu - 2 - 2l(k + 2))\rho + k\Lambda_0 + (-l(\mu + 1) - l^2(k + 2))\delta.$$ 

**Proof.** This follows from an induction using (4.9.1). □

For $w \in W^\alpha$ and a reduced decomposition $w = s_{i_1} \cdots s_{i_t}$, define $\alpha^j = \alpha_{i_j}$ and $\alpha^j = (s_{i_1} \cdots s_{i_{j-1}})(\alpha_{i_j})$. For a fixed weight $\mu \rho + k\Lambda_0$, define $n_{\mu,k,j} = \frac{2(\mu \rho + k\Lambda_0 + \alpha^j)}{\langle \alpha^j, \alpha^j \rangle}$. Recalling that $v_{\mu \rho + k\Lambda_0}$ is the highest weight vector in $M_{\mu \rho + k\Lambda_0}$, define the vector $v_{\mu \rho + k\Lambda_0}^{w \cdot (\mu \rho + k\Lambda_0)}$ by

$$v_{w \cdot (\mu \rho + k\Lambda_0)}^{\mu \rho + k\Lambda_0} := \frac{f_{i_1}^{n_{\mu,k,1}}}{[n_{\mu,k,1}]!} \cdots \frac{f_{i_j}^{n_{\mu,k,j}}}{[n_{\mu,k,j}]!} v_{\mu \rho + k\Lambda_0}.$$

By [38, Section 2.7], the vectors $v_{w \cdot (\mu \rho + k\Lambda_0)}^{\mu \rho + k\Lambda_0}$ are independent of the choice of reduced decomposition and span the space of singular vectors in $M_{\mu \rho + k\Lambda_0}$. They yield a BGG-type resolution for $L_{\mu,k}$ given by Verma modules indexed by $W^\alpha$.

**Proposition 4.9.3 ([58, Theorem 3.2]).** For $k \geq \mu \geq 0$, there is an exact complex of $U_q(\mathfrak{sl}_2)$-modules

$$\cdots \to C_2 \to C_1 \to M_{\mu \rho + k\Lambda_0} \to L_{\mu \rho + k\Lambda_0} \to 0$$

with each term in the resolution given by $C_0 = M_{\mu \rho + k\Lambda_0}$ and

$$C_l = M_{(w_0^l \cdot (\mu \rho + k\Lambda_0)} \oplus M_{(w_1^l \cdot (\mu \rho + k\Lambda_0)}$$

for $l > 0$.

### 4.9.4 Intertwiners for $U_q(\hat{\mathfrak{sl}_2})$-modules with integral highest weight

We now give a criterion for the existence of intertwiners involving Verma modules with dominant integral highest weight.

**Proposition 4.9.4.** For integers $k \geq \mu \geq 0$, let $n_i = (\mu \rho + k\Lambda_0, \alpha_i) + 1$. If $V[n_i \alpha_i] = 0$ for $i = 0, 1$, then for any $v \in V[0]$, there exists a unique intertwiner

$$\Phi_{\mu,k}^v(\mu \rho, k) : M_{\mu,k} \to M_{\mu,k} \otimes V(z)$$

which satisfies

$$\Phi_{\mu,k}^v(\mu \rho, k) v_{\mu,k} = v_{\mu,k} \otimes v + (\text{l.o.t.})$$

where (l.o.t.) denotes terms of lower weight in the first tensor factor.
Proof. The space of intertwiners $M_{\mu,k} \rightarrow M_{\mu,k} \otimes V(z)$ is given by
\[
\text{Hom}_{U_q(\widehat{sl}_2)}(M_{\mu,k}, M_{\mu,k} \otimes V(z)) = \text{Hom}_{U_q(\widehat{sl}_2)}(\text{Ind}_{U_q(\widehat{g}_+)}(M_{\mu,k}^\vee)^* \otimes V(z))
\]
\[
= \text{Hom}_{U_q(\widehat{g}_+)}(C_{\mu,k} \otimes_k M_{\mu,k}^\vee, V(z))
\]
\[
= \{ v \in V[0] \mid I_{\mu,k}v = 0 \},
\]
where $I_{\mu,k} := \{ u \in U_q(\widehat{sl}_2) \mid u \cdot v_{\mu,k}^* = 0 \}$ is the annihilator ideal of the lowest weight vector of $M_{\mu,k}^\vee$. By the identification of singular vectors in Proposition 4.9.3, the $U_q(\widehat{g}_+)$-submodule of $M_{\mu,k}^\vee$ generated by $v_{\mu,k}^*$ has relations $e_i^+ v_{\mu,k}^* = 0$ so that $I_{\mu,k}$ is generated by $e_i^+$, meaning that $I_{\mu,k}v = 0$ for any $v \in V[0]$ and completing the proof. \qed

Remark. By Lemma 4.2.5, for $(\mu, k)$ satisfying the conditions of Proposition 4.9.4, the matrix elements of $\Phi_{\mu,k}(z)$ are given by analytic continuation from the generic case. Therefore, trace functions for such integrable modules are analytic continuations of trace functions for generic highest weight and continue to be given by the expression of Theorem 4.5.1.

4.9.5 The affine dynamical Weyl group action

Let $(\mu, k)$ be chosen to satisfy the conditions of Proposition 4.9.4 with respect to $L_2$. In [38], for $w$ in the affine Weyl group $W^a$ and $w_0 \in L_2[0]$, it was shown that there exist dynamical Weyl group operators $A_{w,L_2}(\rho + k\Lambda_0)$ so that
\[
\Phi_{\mu,k}(z)v_{w-\rho + k\Lambda_0} = v_{w-\rho + k\Lambda_0} \otimes A_{w,L_2}(\rho + k\Lambda_0)w_0 + (\text{l.o.t.}).
\]
In particular, $\Phi_{\mu,k}(z)$ restricts to a multiple of an intertwiner
\[
M_{w-\rho + k\Lambda_0} \rightarrow M_{w-\rho + k\Lambda_0} \otimes L_2(z).
\]
For any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, we have
\[
A_{w,L_2}(\rho + k\Lambda_0) = A_{s_{i_l}} \left( (s_{i_2} \cdots s_{i_l})(\rho + k\Lambda_0) \right) \cdots A_{s_{i_1-1}} \left( s_{i_1}(\rho + k\Lambda_0) \right) A_{s_{i_1}}(\rho + k\Lambda_0).
\]
(4.9.2)
Because $L_2[0]$ is spanned by $w_0$, the dynamical Weyl group acts by a scalar on it, so we will treat it as a number. We now compute the action explicitly in Proposition 4.9.6 by reducing to the trigonometric limit.

Lemma 4.9.5. Let $\mu$ be a positive integer. For $w_0 \in L_2$, the diagonal matrix element of the singular vector $f^{\mu+1}v_{\mu} \in M_\mu$ in $\Phi_{\mu,w_0,\text{trig}}$ is $-\frac{[\mu+2]}{[\mu]}$. 157
Proof. By Lemma 4.7.1 and the expansion
\[
\Delta(f^{\mu+1}) = \sum_{l=0}^{\mu+1} \prod_{i=0}^{l} (1-q^{-2i})(1-q^{-2l}) f_{\mu+1-l} \otimes q^{-(\mu-l+1)h} ft,
\]
we find that the \( f^{\mu+1} v_\mu \) coefficient in \( \Phi_{\mu}^{w_0,\text{tri}(f^{\mu+1} v_\mu)} \) is given by
\[
1 - \frac{[\mu+1][2]}{[\mu]} = -\frac{[\mu+2]}{[\mu]}.
\]

Proposition 4.9.6. The dynamical Weyl group action on \( w_0 \in L_2[0] \) is given by
\[
\begin{align*}
A_{w_0^{2l+1}, L_2}(\mu \rho + k\Lambda_0)w_0 &= -q^{-4l-2}(q^{2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l+1} \\
&\quad \cdot (q^{2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l+1}
A_{w_0^{2l+1}, L_2}(\mu \rho + k\Lambda_0)w_0 &= -q^{-4l-2} \cdot \frac{(q^{2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l+1}}{(q^{-2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l+1}}
A_{w_0^{2l}, L_2}(\mu \rho + k\Lambda_0)w_0 &= q^{-4l} \cdot \frac{(q^{2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l}}{(q^{2\mu+4} q^{2l(2k+4)}; q^{-2k+4})_{2l}}
A_{w_0^{2l}, L_2}(\mu \rho + k\Lambda_0)w_0 &= q^{-4l} \cdot \frac{(q^{2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l}}{(q^{2\mu+4} q^{2l(2k+4)}; q^{-2k-4})_{2l}}
\end{align*}
\]

Proof. The first value for \( l = 0 \) follows from Lemma 4.9.5 and the fact that \( v_{s_1(\mu \rho + k\Lambda_0)} \) is a singular vector for the \( U_q(s_2) \)-submodule \( M_\mu \subset M_{\mu \rho + k\Lambda_0} \), so we may compute using the dynamical Weyl group \( U_q(s_2) \). The second for \( l = 0 \) follows by applying the symmetry between \( s_0 \) and \( s_1 \) and noting that \( \mu \rho + k\Lambda_0 = \mu\Lambda_1 + (k - \mu)\Lambda_0 \). The other values follow by applying Lemma 4.9.2 and (4.9.2).

4.9.6 Proof of Felder-Varchenko's conjecture
We specialize to \( k = 2 \), where [49] conjectured that elliptic and affine Macdonald polynomials are related. We have then that \( \text{Sym}^{2(k-1)} L_1(z) = L_2(z) \) and
\[
\chi_{\mu,k,2}(q, \lambda, \omega) = \text{Tr}|_{L_{\mu+1,k+2}} \left( \Phi_{\mu+1,k+2}(z) q^{2\lambda} q^{2\omega} \right).
\]
In this case, we relate the trace function to the affine Macdonald polynomial via the BGG resolution and apply Theorem 4.5.1 to express it as a hypergeometric theta function in the sense of [49]. The proof of Felder-Varchenko's conjecture in Theorem 4.9.9 then follows. Define the quantity
\[
\chi_{\mu,k}^0(q, \lambda, \omega) := \frac{q^{-\mu+2}(q^{-4}; q^{-2\omega})}{\theta_0(q^{2\lambda}; q^{-2\omega})(q^{2\lambda-2} q^{-2\omega}; q^{-2\omega})(q^{-2\lambda-2}; q^{-2\omega})}
\]
\[
\frac{(q^{-2\omega+2}; q^{-2\omega})^2 (q^{-2k}; q^{-2k})^2 (q^4 q^{-2k}; q^{-2k}) (q^{-2k}; q^{-2k})}{(q^{-2\omega-2}; q^{-2\omega}, q^{-2k}; q^{-2k})^2 (q^{2\mu+6} q^{-2k}; q^{-2k}) (q^{-2k}; q^{-2k})^2}
\]

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Theorem 4.9.7. For $q^{-2\mu}$ and then $q^{-2\omega}$ sufficiently close to 0 in the good region of parameters (4.5.1), the trace function $\chi_{\mu,k,2}(q, \lambda, \omega)$ is given by

$$\chi_{\mu,k,2}(q, \lambda, \omega) = \sum_{j \in \mathbb{Z}} q^{-2j(\omega+2)(\mu+2+j\kappa)} \left( u(q, \lambda, \omega, -\mu-2-2j\kappa, \kappa) - u(q, \lambda, \omega, \mu+2+2j\kappa, \kappa) \right),$$

where $\tilde{k} = k + 4$.

Proof. Define the function

$$T(q, \lambda, \omega, \mu \rho + k \Lambda_0 + \Delta \delta) := \text{Tr}_{M_{\mu + k \Lambda_0 + \Delta \delta}} \left( \Phi_{\mu,k}^w(z) q^{2\lambda - 2\omega} \right)$$

so that

$$T(q, \lambda, \omega, \mu \rho + k \Lambda_0 + \Delta \delta) = q^{2\omega}T_{\mu \rho}(q, \lambda, \omega, \mu + 1, k + 2).$$

Applying the BGG resolution of Proposition 4.9.3, we find that

$$\chi_{\mu,k,2}(q, \lambda, \omega) = \sum_{a=1}^{\infty} \sum_{i=0}^{1} (-1)^a A_{w_1^*,L_2}(\mu \rho + k \Lambda_0 + \bar{\rho}) T \left( q, \lambda, \omega, w_1^a \cdot (\mu \rho + k \Lambda_0 + \bar{\rho}) \right) \quad (4.9.3)$$

We compute the sum in (4.9.3) by dividing it into four pieces depending on the choice of $i$ and the parity of $a$. In each case, by Lemma 4.9.2 and Proposition 4.9.6, the summand is given by a multiple of a trace function for integral values of $\mu$ and $k$. By Lemma 4.2.5 and uniqueness of the intertwiner

$$\Phi_{\mu-1,k-2}^w(z) : M_{\mu-1,k-2} \rightarrow M_{\mu-1,k-2} \otimes L_2(z),$$

when such an intertwiner exists, its matrix elements are given by continuations of the matrix elements given in Lemma 4.2.5. Therefore, its trace function is given by continuation of the expression of Proposition 4.2.6 even for non-generic $(\mu, k)$. In particular, for all intertwiners involving the Verma modules which appear in the affine BGG resolution, the corresponding trace function $T_{\mu \rho}(q, \lambda, \omega, \mu, k)$ is given by the formula of Corollary 4.5.12. We now analyze each piece of (4.9.3) in turn. Setting $\tilde{k} = k + 4$, we have

$$A_{w_1^*,L_2}(\mu \rho + k \Lambda_0 + \bar{\rho}) T \left( q, \lambda, \omega, w_1^2 \cdot (\mu \rho + k \Lambda_0 + \bar{\rho}) \right)$$

$$= q^{2(\mu+2-\tilde{k})\omega-4t} (q^{-2\mu-2q\tilde{k}}; q^{-2k})_2 q^{-2t} T(q, \lambda, \omega, \mu + 2 - 2\tilde{k}, \tilde{k})$$

$$= \chi_{\mu,k}^0(q, \lambda, \omega) q^{2(\mu+2-\tilde{k})(\omega+2)}u(q, \lambda, \omega, -\mu - 2 + 2\tilde{k}, \tilde{k})$$

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and

$$A_{w_0^{2l}, L_2}(\mu \rho + k \Lambda_0 + \widetilde{\rho}) T\left(q, \lambda, \omega, w_0^{2l} \cdot (\mu \rho + k \Lambda_0 + \widetilde{\rho})\right)$$

$$= q^{-2l(\mu+2+k)\omega-4l} \left(\frac{q^{2\mu+6q^2(4l-2)\widetilde{k}}; q^{-2k}_{2l}}{q^{2\mu+2q^2(4l-2)\widetilde{k}; q^{-2k}_{2l}} T(q, \lambda, \omega, \mu + 2 + 2l\widetilde{k}, \widetilde{k})\right)$$

$$= \chi_{\mu, k}^0(q, \lambda, \omega) q^{-2l(\mu+2+k)\omega-4l} u(q, \lambda, \omega, -\mu - 2 - 2l\widetilde{k}, \widetilde{k})$$

and

$$-A_{w_0^{2l+1}, L_2}(\mu \rho + k \Lambda_0 + \widetilde{\rho}) T\left(q, \lambda, \omega, w_0^{2l+1} \cdot (\mu \rho + k \Lambda_0 + \widetilde{\rho})\right)$$

$$= q^{2(l+1)(\mu+2-(l+1)\widetilde{k})\omega-4l+2} \left(\frac{q^{-2\mu+2q^2(4l+2)\widetilde{k}}; q^{-2k}_{2l+1}}{q^{-2\mu+6q^2(4l+2)\widetilde{k}; q^{-2k}_{2l+1}} T(q, \lambda, \omega, -\mu - 2 + 2(l + 1)\widetilde{k}, \widetilde{k})\right)$$

$$= -\chi_{\mu, k}^0(q, \lambda, \omega) q^{2(l+1)(\mu+2-(l+1)\widetilde{k})\omega-4l+2} u(q, \lambda, \omega, \mu + 2 + 2l\widetilde{k}, \widetilde{k})$$

and

$$-A_{w_1^{2l+1}, L_2}(\mu \rho + k \Lambda_0 + \widetilde{\rho}) T\left(q, \lambda, \omega, w_1^{2l+1} \cdot (\mu \rho + k \Lambda_0 + \widetilde{\rho})\right)$$

$$= q^{-2(l+2+k)\omega-4l-2} \left(\frac{q^{2\mu+6q^2(4l+k)}; q^{-2k}_{2l+1}}{q^{2\mu+2q^2(4l+k); q^{-2k}_{2l+1}} T(q, \lambda, \omega, -\mu - 2 + 2l\widetilde{k}, \widetilde{k})\right)$$

$$= -\chi_{\mu, k}^0(q, \lambda, \omega) q^{-2(l+2+k)\omega-4l-2} u(q, \lambda, \omega, \mu + 2 + 2l\widetilde{k}, \widetilde{k}).$$

Combining our previous computations, we find that

$$\frac{\chi_{\mu, k, 2}(q, \lambda, \omega)}{\chi_{\mu, k}^0(q, \lambda, \omega)} = u(q, \lambda, \omega, -\mu - 2, \widetilde{k})$$

$$+ \sum_{l=1}^{\infty} \left( q^{2l(\mu+2-l\widetilde{k})\omega-4l} u(q, \lambda, \omega, -\mu - 2 + 2l\widetilde{k}, \widetilde{k}) + q^{-2l(\mu+2+l\widetilde{k})\omega-4l+2} u(q, \lambda, \omega, -\mu - 2 - 2l\widetilde{k}, \widetilde{k}) \right)$$

$$- \sum_{l=0}^{\infty} \left( q^{2(l+1)(\mu+2-(l+1)\widetilde{k})\omega-4l+2} u(q, \lambda, \omega, \mu + 2 + 2(l + 2)\widetilde{k}, \widetilde{k}) + q^{-2l(\mu+2+l\widetilde{k})\omega+4l+2} u(q, \lambda, \omega, \mu + 2 + 2l\widetilde{k}, \widetilde{k}) \right)$$

$$= \sum_{j \in \mathbb{Z}} q^{-2j\omega+2(\mu+2+j\widetilde{k})} \left( u(q, \lambda, \omega, -\mu - 2 - 2j\widetilde{k}, \widetilde{k}) - u(q, \lambda, \omega, \mu + 2 + 2j\widetilde{k}, \widetilde{k}) \right).$$

**Remark.** For Verma modules with dominant integral highest weight, the analogue of Proposition 4.4.2 may not hold, and the Verma module differs in general from the Wakimoto module. In the classical limit, this difficulty is addressed in [8] by realizing the irreducible integrable module via a BRST-type two-sided resolution via a complex
of Fock modules. Our proof of Theorem 4.9.7 avoids the use of this resolution, which to our knowledge is available in the literature for the quantum affine setting only in the bosonization of [70]. Instead, we apply the affine BGG resolution to relate trace functions for irreducible integrable modules to trace functions for Verma modules with non-generic highest weight, which we then compute by analytic continuation from the case of generic highest weight. Our approach should apply equally well in setting of the undeformed classical affine algebra.

Combining Theorem 4.9.7 with Proposition 4.9.8 on \( \chi_{0,0,2}(q, \lambda, \omega) \) from [28], we obtain Theorem 4.9.9 relating the elliptic and affine Macdonald polynomials.

**Proposition 4.9.8** ([28, Theorem 11.1]). There is a function \( f(q, q^{-2\omega}) \) with unit constant term whose formal power series expansion in \( q^{-2\omega} \) has rational function coefficients in \( q \) such that

\[
\chi_{0,0,2}(q, \lambda, \omega) = f(q, q^{-2\omega})q^{\lambda}(q^{-2\lambda+2}; q^{-2\omega})(q^{2\lambda+2}q^{-2\omega}; q^{-2\omega}).
\]

**Theorem 4.9.9.** Let \( \tilde{k} = k + 4 \). For \( q^{-2\mu} \) and then \( q^{-2\omega} \) sufficiently close to 0 in the good region of parameters (4.5.1), the elliptic and affine Macdonald polynomials are related by

\[
J_{\mu,k,2}(q, \lambda, \omega) = \frac{\widetilde{J}_{\mu,k}(q, \lambda, \omega)}{2\pi f(q, q^{-2\omega})} \frac{(q^{-4}; q^{-2\omega})(q^{-2\omega}; q^{-2\omega})^3 (q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2}{(q^{-2k}; q^{-2k})(q^{-4}; q^{-2k})^2 (q^{-2\omega-2}; q^{-2\omega}, q^{-2k})^2 (q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2} q^{\mu+4}(q^{-2\mu-6}; q^{-2k})(q^{2\mu+2} q^{-2k}; q^{-2k}),
\]

where \( f(q, q^{-2\omega}) \) is the normalizing function of Proposition 4.9.8.

**Proof.** By definition, we have that

\[
\frac{\chi_{0,k}(q, \lambda, \omega)}{J_{0,k}(q, \lambda, \omega)} = -\frac{q^{\mu+\lambda+4}(q^{-4}; q^{-2\omega})(q^{-2\omega}; q^{-2\omega})^3 (q^{-2\omega+2}; q^{-2\omega}, q^{-2k})^2}{2\pi(q^{-2k}; q^{-2k})(q^{-4}; q^{-2k})^2 (q^{-2\omega-2}; q^{-2\omega}, q^{-2k})^2} (q^{-2\lambda+2}; q^{-2\omega})(q^{2\lambda+2} q^{-2\omega}; q^{-2\omega})(q^{-2\mu-6}; q^{-2k})(q^{2\mu+2} q^{-2k}; q^{-2k}).
\]

By Lemma 4.9.1 and Theorem 4.9.7 and the fact that \( u(q, \lambda, \omega, \mu, k) = u(q, -\lambda, \omega, -\mu, k) \) from [49, Lemma 2.2], we see that

\[
\frac{\widetilde{J}_{\mu,k}(q, \lambda, \omega)}{J_{0,k}(q, \lambda, \omega)} = \frac{\chi_{\mu,k}(q, \lambda, \omega)}{\chi_{0,k}(q, \lambda, \omega)}.
\]
We conclude by Proposition 4.9.8 that

\[ J_{\mu,k,2}(q, \lambda, \omega) = \frac{X_{\mu,k,2}(q, \lambda, \omega)}{\chi_{0,0,2}(q, \lambda, \omega)} = -J_{\mu,k}^0(q, \lambda, \omega) \frac{\chi_{\mu,k}^0(q, \lambda, \omega)}{\chi_{0,0,2}^0(q, \lambda, \omega)} \]

\[ = \tilde{J}_{\mu,k}(q, \lambda, \omega) \frac{q^{\mu+1}(q^{-4}; q^{-2\omega})(q^{-2\omega}; q^{-2\omega})^3}{2\pi f(q, q^{-2\omega})(q^{-2k}; q^{-2k})(q^{-4}; q^{-2k})} \]

\[ \frac{(q^{-2\omega+2}; q^{-2\omega, q^{-2k}}^2)}{(q^{-2\omega-2}; q^{-2\omega, q^{-2k}}^2)^2(q^{-2\mu-6}; q^{-2k})(q^{2\mu+2}; q^{-2k})}. \]

\[ \square \]

4.10 Elliptic functions

In this appendix, we give our conventions on theta functions, elliptic gamma functions, and phase functions and provide some estimates for these functions. For a more detailed discussion of the elliptic gamma function, we refer the reader to [46]. We use multiplicative notation throughout the text.

4.10.1 Theta functions

We often use the single and double \( q \)-Pochhammer symbols

\[ (u; q) = \prod_{n \geq 0} (1 - uq^n) \quad \text{and} \quad (u; q, r) = \prod_{n,m \geq 0} (1 - uqn^m), \]

which are convergent for \(|q|, |r| < 1\). We also use the terminating \( q \)-Pochhammer symbol

\[ (u; q)_m = \frac{(u; q)}{(uq^m; q)} \]

Define now the theta function

\[ \theta_0(u; q) = (u; q)(qu^{-1}; q). \]

The theta function satisfies the transformation properties

\[ \theta_0(qu; q) = -u^{-1}\theta_0(u; q) \quad \text{and} \quad \theta_0(q^{-1}u; q) = -uq^{-1}\theta_0(u; q) \]

\[ \text{and} \quad \theta_0(u^{-1}; q) = -u^{-1}\theta_0(u; q). \]

We also use Jacobi’s first theta function \( \theta(u; q) \), given by

\[ \theta(u; q) = i\pi q^{i\pi(\tau/4-z)}(u; q)(qu^{-1}; q)(q; q) = i\pi q^{i\pi(\tau/4-z)}(q; q)\theta_0(u; q) \quad (4.10.1) \]

for \( q = e^{2\pi i\tau} \) and \( u = e^{2\pi iz} \). We have the following asymptotic estimates for \(|\theta_0(u; q)|\) which parallel those in [48, Appendix C].

**Lemma 4.10.1.** If \(|q| < 1\), we have the following estimates on \(|\theta_0(u; q)|\).
(a) There is a constant $C_1(q) > 0$ so that for any $u \neq 0$ we have

$$|\theta_0(u; q)| \leq C_1(q)|u|^{1/2} \exp\left(-\frac{(\log |u|)^2}{2 \log |q|}\right).$$

(b) For any $\epsilon > 0$, there is a constant $C_2(q, \epsilon) > 0$ so that for any $u \neq 0$ with

$$\min_n \left| \log |uq^n| \right| > \epsilon,$$

we have

$$|\theta_0(u; q)| \geq C_2(q, \epsilon)|u|^{1/2} \left(\frac{(\log |u|)^2}{2 \log |q|}\right).$$

**Proof.** For (a), define the constant $C'_1(q)$ by

$$C'_1(q) := \max_{|q| \leq |u| \leq 1} |\theta_0(u; q)|.$$

For any $u \neq 0 \in \mathbb{C}$, choose

$$n = -\left\lfloor \log |\log |q|\log |u| \right\rfloor$$

so that $|q| \leq |uq^n| < 1$. We then have

$$\theta_0(uq^n; q) = (-1)^n u^{-n} q^{-\frac{n(n-1)}{2}} \theta_0(u; q).$$

We then see that

$$|\theta_0(u; q)| = |u|^n |q|^{-\frac{n(n-1)}{2}} |\theta_0(uq^n; q)| \leq C_1(q) \exp\left(n \log |u| + \frac{n(n-1)}{2} \log |q|\right).$$

Notice that

$$n \log |u| \leq \begin{cases} \frac{(\log |u|)^2}{\log |q|} + \log |u| & |u| \geq 1 \\ \frac{(\log |u|)^2}{\log |q|} & |u| < 1 \end{cases}$$

and that

$$\frac{n(n-1)}{2} \log |q| \leq \frac{(\log |u|)^2}{2 \log |q|} + \begin{cases} -\frac{1}{2} \log |u| + 1 & |u| \geq 1 \\ \frac{1}{2} \log |u| + 1 & |u| < 1 \end{cases}.$$

Combining these, for $C_1(q) := \epsilon C'_1(q)$ we find that

$$|\theta_0(u; q)| \leq C_1(q)|u|^{1/2} \exp\left(-\frac{(\log |u|)^2}{2 \log |q|}\right).$$

For (b), define the constant $C'_2(q, \epsilon)$ by

$$C'_2(q, \epsilon) := \min_{|q| \leq |u| \leq 1} \frac{|\theta_0(u; q)|}{|\log |u| |^\epsilon, |\log |uq^{-1}| |^\epsilon}}.$$
If \( \min n \left| \log \left| uq^n \right| \right| > \varepsilon \), then we have that

\[
|\theta_0(u; q)| = |u|^n|q|^{n(n-1)\frac{1}{2}} |\theta_0(uq^n; q)| \geq C_2(q, \varepsilon) \exp \left( n \log |u| + \frac{n(n-1)}{2} \log |q| \right).
\]

Notice now that

\[
n \log |u| \geq \begin{cases} \frac{-\left( \log |u| \right)^2}{\log |q|} & |u| \geq 1 \\ \frac{-\left( \log |u| \right)^2}{\log |q|} + \log |u| & |u| < 1 \end{cases}
\]

and that

\[
\frac{n(n-1)}{2} \log |q| \geq \frac{\left( \log |u| \right)^2}{2 \log |q|} + \begin{cases} \frac{1}{2} \log |u| - 1 & |u| \geq 1 \\ -\frac{1}{2} \log |u| - 1 & |u| < 1 \end{cases}.
\]

This implies that for \( C_2(q, \varepsilon) := e^{-1}C'_2(q, \varepsilon) \), we have the desired bound

\[
|\theta_0(u; q)| \geq C_2(q, \varepsilon)|u|^{1/2} \varepsilon \left( -\frac{\left( \log |u| \right)^2}{2 \log |q|} \right).
\]

\[\square\]

**Corollary 4.10.2.** If \( |q| < 1 \), for any \( \varepsilon > 0 \) there are constants \( D_1(q, \varepsilon), D_2(q, \varepsilon) > 0 \) so that for \( a, b \) and \( z \neq 0 \) satisfying

\[
\min n \left| \log \left| zq^aq^n \right| \right| > \varepsilon \quad \text{and} \quad \min n \left| \log \left| zq^aq^n \right| \right| > \varepsilon,
\]

we have

\[
D_2(q, \varepsilon)q^{\frac{a-b}{2}} |z^2q^{a+b}|^{-\frac{a-b}{2}} \leq \left| \frac{\theta_0(zq^aq^n; q)}{\theta_0(zq^aq^n; q)} \right| \leq D_1(q, \varepsilon)q^{\frac{a-b}{2}} |z^2q^{a+b}|^{-\frac{a-b}{2}}.
\]

**Proof.** This follows from Lemma 4.10.1 and the factorization \( (\log |zq^aq^n|)^2 - (\log |zq^aq^n|)^2 = \log |z^2q^{a+b}| \log |q^{a-b}|. \) \[\square\]

### 4.10.2 Elliptic gamma function and phase function

For \( |r|, |p| < 1 \), define the elliptic gamma function \( \Gamma(z; r, p) \) by

\[
\Gamma(z; r, p) = \frac{(z^{-1}rp; r, p)}{(z; r, p)}.
\]

Define the phase function \( \Omega_a(t/z; r, p) \) by

\[
\Omega_a(z; r, p) = \frac{(za^{-1}; r, p)(z^{-1}a^{-1}rp; r, p)}{(za; r, p)(z^{-1}ar; r, p)} = \frac{\Gamma(za; r, p)}{\Gamma(za^{-1}; r, p)}.
\]

It has the following transformation properties.
Lemma 4.10.3. The phase function satisfies

\[ \Omega_a(z; r, p) = \Omega_a(z^{-1}; r, p) \frac{\theta_0(z^{-1}a; p)\theta_0(z^{-1}a^{-1}; r)}{\theta_0(z^{-1}a^{-1}; r)\theta_0(za; r)} \]
\[ \Omega_a(pz; r, p) = \frac{\theta_0(za; r)}{\theta_0(za^{-1}; r)} \Omega_a(z; r, p) \]
\[ \Omega_a(p^{-1}z; r, p) = \frac{\theta_0(za^{-1}p^{-1}; r)}{\theta_0(zap^{-1}; r)} \Omega_a(z; r, p). \]

Proof. Observe that

\[ \Omega_a(z; r, p) = \Omega_a(z^{-1}; r, p) \frac{(1 - za^{-1})(za^{-1}r; r)(za^{-1}p, p)}{(1 - za)(zar; r)(zap; p)} \frac{(1 - z^{-1}a)(z^{-1}ap; p)(z^{-1}ar; r)}{(1 - z^{-1}a^{-1})(z^{-1}a^{-1}p; p)(z^{-1}a^{-1}r; r)} \]
\[ = \Omega_a(z^{-1}; r, p) \frac{\theta_0(z^{-1}a; p)\theta_0(za^{-1}; r)}{\theta_0(z^{-1}a^{-1}; p)\theta_0(za; r)}. \]

In addition, we have that

\[ \Omega_a(pz; r, p) = \frac{(za; r)(z^{-1}a^{-1}r; r)}{(za^{-1}; r)(z^{-1}ar; r)} \Omega_a(z; r, p) = \frac{\theta_0(za; r)}{\theta_0(za^{-1}; r)} \Omega_a(z; r, p) \]

and that

\[ \Omega_a(p^{-1}z; r, p) = \frac{(za^{-1}p^{-1}; r)(z^{-1}arp; r)}{(zap^{-1}; r)(z^{-1}a^{-1}pr; r)} \Omega_a(z; r, p) = \frac{\theta_0(za^{-1}p^{-1}; r)}{\theta_0(zap^{-1}; r)} \Omega_a(z; r, p). \]

4.11 Computation of vertex operators, OPE's, and one loop correlation functions

We give some explicit computations of OPE's and one loop correlation functions in the method of coherent states. For each OPE, we write $\equiv$ to denote equality of analytic continuations outside of the specified domain.
4.11.1 Table of vertex operators

We give the vertex operators used in the free field construction of [77]. Define first $Y^\pm(z)$, $Z_\pm(z)$, $W_\pm(z)$, and $U(z)$ by

\[
Y^\pm(z) = \exp \left( \pm \sum_{m \geq 0} q^\frac{m}{2m} \left( \frac{z^m}{k_m} (\alpha_m + \bar{\alpha}_m) \right) e^{\pm 2(\alpha + \bar{\alpha})} z^{\pm \frac{1}{2}(\alpha_0 + \bar{\alpha}_0)} \right)
\]
\[
Z_\pm(z) = \exp \left( \mp (q - q^{-1}) \sum_{m > 0} \frac{z^m}{2m} \alpha_\pm \right) q^{\mp \alpha_0}
\]
\[
W_\pm(z) = \exp \left( \mp (q - q^{-1}) \sum_{m > 0} \frac{z^m}{2m} \beta_\pm \right) q^{\mp \frac{1}{2} \beta_0}
\]
\[
U(z) = \exp \left( - \sum_{m > 0} \frac{q^{-k^2/2} z^{-m}}{(k + 2)m} \beta_{-m} \right) e^{-2\beta} z^{-\frac{k^2}{4} \beta_0} \exp \left( \sum_{m > 0} \frac{q^{-k^2/2} z^{-m}}{(k + 2)m} \beta_m \right).
\]

In terms of these operators, we define

\[
X^+(z) := \frac{1}{(q - q^{-1}) z} (X_+^+(z) - X^-_+(z)) \quad \text{and} \quad X^-(z) := -\frac{1}{(q - q^{-1}) z} (X_+^-(z) - X^-_-(z))
\]

for

\[
X^+_+(z) := Y^+(z) Z_+(q^{-k^2/2} z) W_+(q^{-k^2/2} z) : \quad \text{and} \quad X^+_-(z) := Y^+(z) W_-(q^{k^2/2} z) Z_-(q^{k^2/2} z) :
\]
\[
X^-_+(z) := Y^-(z) Z_+(q^{k^2/2} z) W_+(q^{-k^2/2} z)^{-1} : \quad \text{and} \quad X^-_-(z) := Y^-(z) W_-(q^{k^2/2} z)^{-1} Z_-(q^{-k^2/2} z)^{-1} :
\]

Define also the screening operator

\[
S(z) := -\frac{1}{(q - q^{-1}) z} (S_+(z) - S_-(z))
\]

for

\[
S_+(z) := U(z) Z_+(q^{-k^2/2} z)^{-1} W_+(q^{-k^2/2} z)^{-1} :
\]

and

\[
S_-(z) := U(z) W_-(q^{k^2/2} z)^{-1} Z_-(q^{k^2/2} z)^{-1} :
\]
Finally, define the operators

\[ \eta(w_0) = \exp \left( \sum_{m>0} \frac{w_0^m}{[2m]} \left( q^{\frac{m}{2}} \beta_m + q^{\frac{k+2}{2}} \tilde{\alpha}_m \right) \right) e^{(k+2)\beta + k\tilde{\alpha} \frac{1}{2}(\delta_0 + \tilde{\alpha}_0)} \]

\[ \exp \left( -\sum_{m>0} \frac{w_0^{-m}}{[2m]} \left( q^{\frac{m}{2}} \beta_m + q^{\frac{k+2}{2}} \tilde{\alpha}_m \right) \right) \]

\[ \xi(z_0) = \exp \left( -\sum_{m>0} \frac{z_0^m}{[2m]} \left( q^{\frac{m}{2}} \beta_m + q^{\frac{k+2}{2}} \tilde{\alpha}_m \right) \right) e^{-\frac{1}{2} \beta_0 - \frac{1}{2} \tilde{\alpha}_0} \]

\[ \exp \left( \sum_{m>0} \frac{z_0^{-m}}{[2m]} \left( q^{\frac{m}{2}} \beta_m + q^{\frac{k+2}{2}} \tilde{\alpha}_m \right) \right). \]

and the intertwining vertex operator

\[ \phi_j(z) = \exp \left( \sum_{m>0} \frac{(q^{k+2}z)^m q^{\frac{km}{2}} [2jm]}{[m][2m]} \left( \alpha_m + \tilde{\alpha}_m \right) \right) e^{2j(\alpha + \tilde{\alpha}) z^{\frac{1}{2m}} (a_0 + \tilde{\alpha}_0) - \sum_{m>0} \frac{(q^{k+2}z)^{-m} q^{\frac{k+2}{2}m} [2jm]}{[m][2m]} \beta_m} \]

\[ e^{2j\beta z^{\frac{1}{2m}}} \exp \left( -\sum_{m>0} \frac{(q^{k+2}z)^{-m} q^{\frac{k+2}{2}m} [2jm]}{[m][2m]} \beta_m \right). \]  

(4.11.1)

We define the modes \( \{a_n\} \) of each vertex operator \( A(z) \) listed by

\[ A(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \]

with the exception of \( \xi(z) \), whose modes are defined by \( \xi(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n} \).

**Remark.** Our definition of \( \phi_j(z) \) corrects a typographical error in [77, Equation 5.1].

### 4.11.2 Operator product expansions

We give some OPE's which will be used in our computations. In each case, we mention the domain on which the relevant OPE converges. If no domain is specified, then the OPE converges for all values of the variables.
We record the OPE's between $S(t)$ and $X^-(w)$. We have that

\[
S_+(t)X^-(w) = \frac{qt - wq^{k+1}}{t - wq^{k+2}} : S_+(t)X^-(w) : |wq^{k+2}| < |t| \\
X^-(w)S_+(t) = \frac{qt - wq^{k+1}}{t - wq^{k+2}} : S_+(t)X^-(w) : |wq^{k+2}| > |t|
\]

\[
S_+(t)X^-(w) = q : S_+(t)X^-(w) :
\]

\[
X^-(w)S_+(t) = q : S_+(t)X^-(w) :
\]

\[
S_-(t)X^+(w) = q^{-1} : S_-(t)X^+(w) :
\]

\[
X^+(w)S_-(t) = q^{-1} : S_-(t)X^+(w) :
\]

\[
S_-(t)X^-(w) = \frac{tq^{k+1} - wq}{tq^{k+2} - w} : S_-(t)X^-(w) : |w| < |tq^{k+2}|
\]

\[
X^-(w)S_-(t) = \frac{tq^{k+1} - wq}{tq^{k+2} - w} : S_-(t)X^-(w) : |w| > |tq^{k+2}|.
\]

We will require the following computation for analysis of convergence of the Jackson integral.

**Lemma 4.11.1.** We have that

\[
[x_0^-, S(t)] = -\frac{1}{(q - q^{-1})t} \left( : U(t)Y^-(tq^{-k-2}W_+(tq^{-\frac{k}{2}-2})^{-1}W_+(tq^{-\frac{k}{2}})^{-1} : \\
- : U(t)Y^-(tq^{k+2})W_-(tq^{\frac{k}{2}+2})^{-1}W_-(tq^{\frac{k}{2}})^{-1} : \right).
\]

**Proof.** This follows from the OPE's between $X^\pm(w)$ and $S_\pm(t)$ computed above. □
**OPE's between \( S(t) \) and \( X^+(w) \)**

We record the OPE's between \( S(t) \) and \( X^+(w) \). We have that

\[
S_+(t)X^+(w) = \frac{t - wq^2}{q(t - w)} : S_+(t)X^+(w) : \quad |w| < |t|
\]

\[
X^+(w)S_+(t) = \frac{t - wq^2}{q(t - w)} : S_+(t)X^+(w) : \quad |w| > |t|
\]

\[
S_+(t)X^+(w) = q^{-1} : S_+(t)X^+(w) : \quad X^+(w)S_+(t) = q^{-1} : S_+(t)X^+(w) :
\]

\[
S_-(t)X^+(w) = q : S_-(t)X^+(w) : \quad X^+(w)S_-(t) = q : S_-(t)X^+(w) :
\]

\[
S_-(t)X^+(w) = \frac{q^2(t - w)}{q(t - w)} : S_-(t)X^+(w) : \quad |w| < |t|
\]

\[
X^+(w)S_-(t) = \frac{q^2(t - w)}{q(t - w)} : S_-(t)X^+(w) : \quad |w| > |t|.
\]

**OPE's between \( S(t) \) and \( \phi_1(z) \)**

We compute OPE's between \( S(t) \) and \( \phi_1(z) \). We obtain

\[
S_+(t)\phi_1(z) = t^{-z+2} \left( \frac{zt^{-1}q^{-2}; q^{-2k-4}}{zt^{-1}q^2; q^{-2k-4}} \right) : S_+(t)\phi_1(z) : \quad |t| > |zq^2|, |zq^{-2}|
\]

\[
\phi_1(z)S_+(t) = z^{-z+2} \left( \frac{zt^{-1}q^{-2k-6}; q^{-2k-4}}{zt^{-1}q^2; q^{-2k-4}} \right) : S_+(t)\phi_1(z) : \quad |t| < |zq^{2k+2}|, |zq^{2k+6}|
\]

\[
S_-(t)\phi_1(z) = t^{-z+2} \left( \frac{zt^{-1}q^{-2k-6}; q^{-2k-4}}{zt^{-1}q^2; q^{-2k-4}} \right) : S_-(t)\phi_1(z) : \quad |t| > |zq^{-2}|, |zq^2|
\]

\[
\phi_1(z)S_-(t) = z^{-z+2} \left( \frac{zt^{-1}q^{-2k-6}; q^{-2k-4}}{zt^{-1}q^2; q^{-2k-4}} \right) : S_-(t)\phi_1(z) : \quad |t| < |zq^{2k+6}|, |zq^{2k+2}|
\]

which we summarize as

\[
S_a(t)\phi_1(z) = t^{-z+2} \left( \frac{zt^{-1}q^{-2}; q^{-2k-4}}{zt^{-1}q^2; q^{-2k-4}} \right) : S_a(t)\phi_1(z) : \quad |t| > |zq^2|, |zq^{-2}|
\]

\[
\phi_1(z)S_a(t) = z^{-z+2} \left( \frac{zt^{-1}q^{-2k-6}; q^{-2k-4}}{zt^{-1}q^2; q^{-2k-4}} \right) : S_a(t)\phi_1(z) : \quad |t| < |zq^{2k+6}|, |zq^{2k+2}|.
\]
OPE’s between $\phi_j(z)$ and $X_\sigma^-(w)$

Computing OPE’s, we obtain

$$\phi_1(z)X^+_-(w) =: \phi_1(z)X^+_-(w) :$$

$$X^+_-(w)\phi_1(z) = \frac{q^2(w - zq^k)}{w - zq^{k+4}} : \phi_1(z)X^+_-(w) : \quad |w| > |zq^{k+4}|, |zq^k|$$

$$\phi_1(z)X^-_+(w) = \frac{zq^{k+4} - w}{zq^{k+4} - q^4w} : \phi_1(z)X^-_+(w) : \quad |w| < |zq^{k+4}|, |zq^k|$$

$$X^-_+(w)\phi_1(z) = q^{-2} : X^-_+(w)\phi_1(z) : .$$

We conclude that

$$\phi_1(z)X^+_b(w) = q^{2b-2} \frac{zq^{k+4} - w}{zq^{k+2b+2} - w} : \phi_1(z)X^+_b(w) : \quad |w| < |zq^{k+4}|, |zq^k|$$

$$X^+_b(w)\phi_1(z) = q^{2b} \frac{w - zq^k}{w - zq^{k+2+2b}} : \phi_1(z)X^+_b(w) : \quad |w| > |zq^{k+4}|, |zq^k|$$

and therefore that

$$[\phi_1(z), X^+_b(w)]q^2 = \sum_{c \epsilon \{1\}} (-1)^{c-1} q^{c}w^{2c} - zq^{k+2} w^{-zq^{k+2+2c}} : \phi_1(z)X^+_b(w) :, \quad (4.11.2)$$

where in (4.11.2) the analytic continuation holds in the region $|w| < |zq^k|, |zq^{k+4}|$ for $c = 1$ and $|w| > |zq^k|, |zq^{k+4}|$ for $c = -1$.

OPE’s between $X^\pm_\sigma(z)$ and $X^\pm_\delta(w)$

Computing OPE’s, we obtain

$$X^+_\sigma(z)X^+_\delta(w) = q \frac{1 - w^z}{1 - q^{2z^2}} : X^+_\sigma(z)X^+_\delta(w) : \quad |z| > |w|, |wq^2|$$

$$X^+_\sigma(z)X^-_\delta(w) = q \frac{1 - q^2w^z}{1 - q^2w^z} : X^+_\sigma(z)X^-_\delta(w) : \quad |z| > |wq^2|, |wq^2|$$

$$X^+_\sigma(z)X^-_\delta(w) = q^{-1} \frac{1 - q^{k+2}w^z}{1 - q^{k+2}w^z} : X^+_\sigma(z)X^-_\delta(w) : \quad |z| > |wq^k|, |wq^{k+2}|$$

$$X^+_\sigma(z)X^-_\delta(w) = q^{-1} : X^+_\sigma(z)X^-_\delta(w) : .$$
and
\[ X^+(z)X^+(w) = q^{-1} : X^+(z)X^+(w) : \]
\[ X^+(z)X^+(w) = q^{-1} \frac{1 - \frac{w}{z}}{1 - q^{2w}z} : X^+(z)X^+(w) : \quad |z| > |w|, |wq^2| \]
\[ X^+(z)X^-(w) = q : X^+(z)X^-(w) : \]
\[ X^+(z)X^-(w) = q^{-1} \frac{1 - q^{k+2w}z}{1 - q^{k+2w}z} : X^+(z)X^-(w) : \quad |z| > |wq^k|, |wq^{k+2}| \]

and
\[ X^+(z)X^+(w) = q^{-1} : X^+(z)X^+(w) : \]
\[ X^+(z)X^+(w) = q^{-1} \frac{1 - \frac{w}{z}}{1 - q^{-k-w}z} : X^+(z)X^+(w) : \quad |z| > |w|, |wq^{-k-2}| \]
\[ X^+(z)X^+(w) = q : X^+(z)X^+(w) : \]
\[ X^+(z)X^+(w) = q^{-1} \frac{1 - q^{-k-w}z}{1 - q^{-k-w}z} : X^+(z)X^+(w) : \quad |z| > |w|, |wq^{-2}| \]

and
\[ X^-(z)X^+(w) = q^{-1} : X^-(z)X^+(w) : \]
\[ X^-(z)X^+(w) = q^{-1} \frac{1 - q^{k+2w}z}{1 - q^{k+2w}z} : X^-(z)X^+(w) : \quad |z| > |wq^{k+2}|, |wq^k| \]
\[ X^-(z)X^+(w) = q : X^-(z)X^+(w) : \]
\[ X^-(z)X^+(w) = q^{-1} \frac{1 - \frac{w}{z}}{1 - q^{-w}z} : X^-(z)X^+(w) : \quad |z| > |w|, |wq^{-2}| \]

**OPE's between \( U(t) \), \( W_\pm(t) \), and \( \phi_1(z) \)**

We record OPE's between the constituents \( U(t) \) and \( W_\pm(t) \) of \( S(t) \) and \( \phi_1(z) \). We have that
\[ U(t)\phi_1(z) = t^{-k-2} \frac{(z, q^{-2}; q^{-2k-4})}{(t, q^2; q^{-2k-4})} : U(t)\phi_1(z) : \quad |t| > |zq^2|, |zq^{-2}| \]
\[ W_+(t)\phi_1(z) = q^{-1} \frac{1 - \frac{z}{t}q^{3k+4}}{1 - \frac{z}{t}q^{3k+2}} : W_+(t)\phi_1(z) : \quad |t| > |zq^{3k+2}|, |zq^{3k+4}| \]
\[ W_-(t)\phi_1(z) = q : W_-(t)\phi_1(z) : \]

**4.11.3 Trace of the degree 0 part**

In this section we compute the action of the intertwiner in the degree 0 part of Fock space.
Proposition 4.11.2. The trace in the degree 0 part of Fock space is

$$\text{Tr}|_{\mathcal{F}_{\mu,s}}(\eta(w_0)\xi(z_0)S_{\alpha}(t) : \phi_1(z)X_b^-(w) : q^{2\lambda p+2\omega d})$$

$$= q^{(2\lambda+b-a)s+(a+b)\mu+a} z^t - \frac{2(\mu+1)}{z_0} w_0^{-\mu-s} u_0^{\mu-s-1}.$$ 

Proof. Because the degree 0 part of Fock space has dimension 1, the trace is given by the action of the degree zero part of the intertwiner on the highest weight vector, which is given by

$$v_{\mu,s} \mapsto q^{2\lambda s} v_{\mu,s} \mapsto q^{2\lambda s} q^{b s} q^{b \mu} z^{\frac{2e}{s}} v_{\mu+1,s} \mapsto q^{2\lambda s} q^{b s} q^{b \mu} z^{\frac{2e}{s}} t^{-\frac{2(\mu+1)}{s}} q^{a(\mu+1)} q^{-a} v_{\mu,s}$$

$$\mapsto q^{2\lambda s} q^{b s} q^{b \mu} z^{\frac{2e}{s}} t^{-\frac{2(\mu+1)}{s}} q^{a(\mu+1)} q^{-as} z_0^{-\mu+s} v_{\mu-\frac{s}{2},s,-\frac{b}{2}}$$

$$\mapsto q^{2\lambda s} q^{b s} q^{b \mu} z^{\frac{2e}{s}} t^{-\frac{2(\mu+1)}{s}} q^{a(\mu+1)} q^{-as} z_0^{-\mu+s} w_0^{-s-1} v_{\mu,s},$$

yielding the result after simplification. \qed

4.11.4 One loop correlation functions

In this section we compute the one loop correlation functions

$$T_{PQ} := (q^{-2\omega}, q^{-2\omega})^3 \prod_{m \geq 1} \text{Tr}|_{\mathcal{F}_{\beta,\mu,m} \otimes \mathcal{F}_{\alpha,s,m} \otimes \mathcal{F}_{\alpha,m}}(P(z)Q(w) q^{2\omega d})$$

for each pair of the vertex operators which appear in

$$\text{Tr}|_{\mathcal{F}_{\mu,s}}(\eta(w_0)\xi(z_0)S_{\alpha}(t) : \phi_1(z)X_b^-(w) : q^{2\lambda p+2\omega d}).$$

We use a general computation of traces in Heisenberg algebras. Consider the algebra

$$\mathcal{A} = (\alpha_-, \alpha_+ | [\alpha_+, \alpha_-] = 1).$$

Let $\mathcal{F}$ be the Fock space for $\mathcal{A}$ with highest weight vector $v_{\mathcal{F}}$.

Lemma 4.11.3. If $e^z < 1$, the operator $\psi : \mathcal{F} \to \mathcal{F}$ defined by

$$\psi = \prod_{i=1}^{N} \exp(x_i \alpha_-) \exp(y_i \alpha_+) \cdot \exp(z \alpha_- \alpha_+)$$

has trace

$$\text{Tr}|_{\mathcal{F}}(\psi) = \frac{1}{1 - e^z} \exp \left( \sum_{i>j} \frac{x_i y_j}{1 - e^z} + \sum_{i \leq j} e^z x_i y_j \right).$$

Proof. Let $X = \sum_i x_i$ and $Y = \sum_i y_i$. Recalling that for operators $A$ and $B$ with
central we have $e^A e^B = e^B e^A e^{[A,B]}$, we obtain

$$\psi = \exp(Y \alpha_+) \exp(X \alpha_-) \exp(z \alpha_- \alpha_+) \exp\left(-\sum_{i \leq j} x_i y_j\right).$$

Applying [72, Equation B.9] to compute $\text{Tr}_F(\psi)$, we obtain

$$\text{Tr}_F(\psi) = \frac{1}{\pi} \int e^{2\lambda e^{-|\lambda|^2}} \sum_{n,m \geq 0} \left(\frac{\lambda + \bar{X}}{n!} \alpha_+^n v_+^m \exp\left(-\sum_{i \leq j} x_i y_j\right) \frac{(e^z \lambda + Y)^m}{m!} \alpha_-^m v_-ight)$$

$$= \frac{1}{\pi} e^{-\sum_{i \leq j} x_i y_i} \int e^{2X Y} \int_{\mathbb{R}} \exp \left((e^z - 1) a^2 + (Y + X e^z) a\right) da$$

$$\int_{\mathbb{R}} \exp \left((e^z - 1) b^2 + i(X e^z - Y) b\right) db$$

$$= \frac{1}{1 - e^z} \exp \left(\frac{X Y}{1 - e^z} - \sum_{i \leq j} x_i y_j\right)$$

$$= \frac{1}{1 - e^z} \exp \left(\sum_{i > j} x_i y_j + \sum_{i \leq j} e^z x_i y_j\right)$$

using the fact that $\int e^{-ax^2 + bx} dx = \frac{\sqrt{\pi} e^{b^2/4a}}{\sqrt{a}}$ for $a > 0$. \hfill \Box

**Corollary 4.11.4.** If $e^{z c} < 1$, if $[\beta_+, \beta_-] = c$, then

$$\text{Tr}_F(\prod_i \exp(x_i \beta_-) \exp(y_i \beta_+) \exp(z \beta_- \beta_+)) = \frac{1}{1 - e^{z c}} \exp\left(\sum_{i > j} \frac{c x_i y_j}{1 - e^{z c}} + \sum_{i \leq j} \frac{c x_i y_j e^{zc}}{1 - e^{zc}}\right).$$

**Proof.** The conclusion follows by taking $\alpha_+ = \frac{\beta_+}{\sqrt{c}}$ in Lemma 4.11.3. \hfill \Box

We may now compute each $T_{PP}$ by applying Corollary 4.11.4 on the Fock spaces for the Heisenberg algebra generated by $\star_{-m}$ and $\star_m$ for $m > 0$ and $\star \in \{\alpha, \alpha, \beta\}$ and multiplying the results. We state the results and the domains on which the relevant one loop correlation functions converge. These are also recorded in Table 4.1. We first record $T_{PP}$ for each choice of $P$. We have

$$T_{m} = T_{\xi \xi} = (q^{-2\omega}; q^{-2\omega})$$

$$T_{S_{-m} S_{m}} = \frac{(q^{-2\omega+2}; q^{-2\omega+2}; q^{-2\omega}; q^{-2\omega})}{(q^{-2\omega}; q^{-2\omega}; q^{-2\omega+2}; q^{-2\omega})}$$

$$T_{\phi \phi} = \frac{(q^{-2\omega+2}; q^{-2\omega}; q^{-2\omega})}{(q^{-2\omega}; q^{-2\omega}; q^{-2\omega})}$$

$$T_{X_{-} X_{-}} = \frac{(q^{-2\omega}; q^{-2\omega})}{(q^{-2\omega}; q^{-2\omega})}. $$

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We now record $T_{\eta Q}$ for $Q$ different from $\eta$. We have

$$T_{\eta \xi} = \theta_0 \left( \frac{z_0}{w_0}; q^{-2\omega} \right)^{-1} \quad |w_0q^{-2\omega}| < |z_0| < |w_0|$$
$$T_{\eta S_a} = \theta_0 \left( \frac{t}{w_0} q^{-a}; q^{-2\omega} \right)^{-1} \quad |w_0q^{-2\omega}| < |tq^{-a}| < |w_0|$$
$$T_{\eta \phi} = 1$$
$$T_{\eta X_b} = \theta_0 \left( \frac{w}{w_0} q^{b(k+1)}; q^{-2\omega} \right) \quad |w_0q^{-2\omega}| < |wq^{(k+1)b}| < |w_0|.$$

For any $P \neq \eta$, we have that $T_{\xi P}(z_0) = T_{\eta P}^{-1}(z_0)$, where we mean that $w_0$ is replaced by $z_0$ and the region of convergence has the same constraints. The other one loop correlation functions are given by

$$T_{S_a \phi} = \Omega_{q^2}(\frac{t}{z}; q^{-2\omega}, q^{-2\omega}) \frac{\theta_0(\frac{1}{z} q^{2}; q^{-2\omega}) \theta_0(\frac{1}{z} q^{-2}; q^{-2\omega})}{\theta_0(\frac{1}{z} q^{-2}; q^{-2\omega}) \theta_0(\frac{1}{z} q^{2}; q^{-2\omega})} \quad |tq^{-2\omega-2\omega}|, |tq^{-2\omega-2\omega-2}| < |z| < |tq^2|, |tq^2|$$

$$T_{S_a X_a^-} = \begin{cases} \frac{\theta_0(\frac{1}{z} q^{k+2}; q^{-2\omega})}{\theta_0(\frac{1}{z} q^{k+2}; q^{-2\omega})} & |tq^{-k} q^{-2\omega}|, |tq^{-k-2} q^{-2\omega}| < |w| < |tq^{-k}|, |tq^{-k-2}| \quad a = 1 \\ \frac{\theta_0(\frac{1}{z} q^{k}; q^{-2\omega})}{\theta_0(\frac{1}{z} q^{k}; q^{-2\omega})} & |tq^{k} q^{-2\omega}|, |tq^{k+2} q^{-2\omega}| < |w| < |tq^{k}|, |tq^{k+2}| \quad a = -1 \end{cases}$$
$$T_{S_a X_{-a}} = 1$$

$$T_{\phi X_b^-} = \begin{cases} \frac{\theta_0(\frac{1}{z} q^{k+2}; q^{-2\omega})}{\theta_0(\frac{1}{z} q^{k+2}; q^{-2\omega})} & |zq^{k} q^{-2\omega}|, |zq^{k+4} q^{-2\omega}| < |w| \quad b = 1 \\ \frac{\theta_0(\frac{1}{z} q^{k+2}; q^{-2\omega})}{\theta_0(\frac{1}{z} q^{k+2}; q^{-2\omega})} & |w| < |zq^{k} q^{-2\omega}|, |zq^{k+4} q^{-2\omega}| \quad b = -1 \end{cases}.$$
Chapter 5

Traces of quantum affine intertwiners and difference equations

5.1 Introduction

This chapter presents complete proofs for modifications of the results of Etingof-Schiffmann-Varchenko in [33] on traces of intertwiners of untwisted quantum affine algebras. In that work, certain renormalizations $F_{V_1,...,V_n}(z_1,...,z_n;\lambda,\omega,\mu,k)$ of generalized trace functions for $U_q(\hat{g})$ were shown to solve four commuting systems of $q$-difference equations: the Macdonald-Ruijsenaars, dual Macdonald-Ruijsenaars, $q$-KZB, and dual $q$-KZB equations. In addition, these renormalized trace functions were shown to satisfy a symmetry property. These results were generalizations to the quantum affine setting of the prior results [37, 34] of Etingof-Varchenko and Etingof-Schiffmann for finite-type quantum groups and [21, 31] of Etingof and Etingof-Schiffmann for classical affine algebras. The $q$-KZB equations which appear were previously studied by Felder-Tarasov-Varchenko in [44, 45, 48].

The purpose of the present chapter is twofold. First, we provide an exposition of the proofs omitted from [33]. Second, we modify the statements of [33] to (1) use the opposite coproduct and (2) use the standard grading instead of the principal grading. The modification (2) in particular allows trigonometric limits of the results to be easily taken to recover the results of [37] for finite-type quantum groups.

These modifications were motivated by the recent work [96] of the author, where a trace function for $U_q(\hat{\mathfrak{s}}_2)$ was explicitly computed and related to certain theta hypergeometric integrals appearing in [48] as part of Felder-Varchenko’s solutions to the $q$-KZB heat equation. In this work, the opposite coproduct to that of [33] was considered in order to use the bosonization of [77], and the standard grading was used both to compare with the Felder-Varchenko function and to enable comparison with the trigonometric limit. While the techniques used in this paper are similar to those sketched in [33], to ensure that the appropriate modifications to the corresponding $q$-difference equations are made, we have chosen to write complete proofs for these...
modified versions.

In the remainder of this introduction, we describe the organization of this chapter and provide the reader with a roadmap for the main results. This chapter is based on the paper [94].

5.1.1 Organization

The remainder of this chapter is organized as follows. In Section 5.2, we fix our conventions for the quantum affine algebra $U_q(\hat{g})$ and its universal $R$-matrix. In Section 5.3, we define and characterize intertwiners for $U_q(\hat{g})$. In Section 5.4, we define fusion and exchange operators in the quantum affine setting and relate their universal versions to their evaluations in representations. In Section 5.5, we define the normalized trace function. In Section 5.6, we prove the Macdonald-Ruijsenaars equations (Theorem 5.6.1) by computing difference operators originating as the radial part of certain central elements for $U_q(\hat{g})$. In Section 5.7, we prove the dual Macdonald-Ruijsenaars equations (Theorem 5.7.1) by computing an $U_q(\hat{g})$-intertwiner in two different ways. In Section 5.8, we use these two equations to prove a symmetry identity (Theorem 5.8.1) for renormalized trace functions. In Section 5.9, we prove the dual $q$-KZB equation (Theorem 5.9.2) and use the symmetry identity to deduce the $q$-KZB equation (Theorem 5.9.1).

5.2 Quantum affine algebras and $R$-matrices

In this section we fix our conventions on the quantum affine algebra $U_q(\hat{g})$ and its central extension $U_q(\hat{g})$. We recall also Drinfeld’s construction of a central element in a completion of $U_q(\hat{g})$. 

5.2.1 Cartan subalgebras

Let $g$ be a simple Lie algebra, $\hat{g}$ its affinization, and $\hat{g}$ the central extension. Let $\alpha_i$, $i = 1, \ldots, r$ be the simple roots, $r$ the rank of $g$, $\theta$ the highest root, $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$, and $h^\vee = 1 + (\theta, \rho)$ the dual Coxeter number. Let $A = (a_{ij})_{i,j=0}^r$ be the extended Cartan matrix of $\hat{g}$ and $d_i$ relatively prime positive integers so that $(d_i \alpha_{ij})$ is symmetric. Let the Cartan and dual Cartan algebras be

$$\tilde{h} = h \oplus \mathbb{C}e \oplus \mathbb{C}d \quad \text{and} \quad \tilde{h}^* = h^* \oplus \mathbb{C} \Lambda_0 \oplus \mathbb{C} \delta,$$

with $\Lambda_0 = c^*$ and $\delta = d^*$. Take $\alpha_0 := \delta - \theta \in \tilde{h}$. The algebra $\tilde{g}$ admits a non-degenerate invariant form $(-,-)$ whose restriction to $\tilde{h}$ has non-trivial values given by

$$(d,d) = 0 \quad (c,d) = 1 \quad (\alpha_i, \alpha_i) = 2 \text{ for } i > 0$$

and agrees with the standard non-degenerate form on $h$. Fix an orthonormal basis $\{x_i\}$ of $h$ under $(,)$. Define $\tilde{\rho} := \rho + h^\vee \Lambda_0$. 

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5.2.2 Quantum affine algebra

Let \( q \) be a non-zero complex number with \( |q| < 1 \). The quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) is the Hopf algebra generated as an algebra by \( e_i, f_i, q^{\pm h_i} \) for \( 0 \leq i \leq r \) with relations

\[
\begin{align*}
[q^{h_i}, q^{h_j}] &= 0 \\
q^{h_i} e_j q^{-h_j} &= q^{(h_i, h_j)} e_j \\
q^{h_i} f_j q^{-h_j} &= q^{-(h_i, h_j)} f_j \\
[e_i, f_j] &= \delta_{ij} q^{d_{h_i} - d_{h_j}} - \frac{q - q^{-1}}{q - q^{-1}} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k} q^{d_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k} q^{d_i} e_i^{1-a_{ij}-k} f_j e_i^k &= 0,
\end{align*}
\]

where we use the notations \( [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \), \( [n]! = [n] \cdots [1] \), and \( \binom{a}{b}_q = \frac{[a]_q! [a-b]_q!}{[b]_q! [a-b]_q!} \). The coproduct of \( U_q(\hat{\mathfrak{g}}) \) is

\[
\Delta(e_i) = e_i \otimes 1 + q^{d_{h_i}} \otimes e_i \\
\Delta(f_i) = f_i \otimes q^{-d_{h_i}} + 1 \otimes f_i \\
\Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i}.
\]

The antipode of \( U_q(\hat{\mathfrak{g}}) \) is

\[
S(e_i) = -q^{-d_{h_i}} e_i \\
S(f_i) = -f_i q^{d_{h_i}} \\
S(q^{h_i}) = q^{-h_i}
\]

and the counit is

\[
\epsilon(e_i) = \epsilon(f_i) = 0 \\
\epsilon(q^{h_i}) = 1.
\]

In what follows, we will often use the Sweedler notation

\[
\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}.
\]

When the context is clear, we will omit the summation sign, writing \( \Delta(x) = x^{(1)} \otimes x^{(2)} \) to denote an implicit summation over the pure tensor summands of the coproduct.

**Remark.** This coproduct is the opposite of the one in [50, 33] but agrees with those in the bosonizations of [67, 71, 77]. Our motivation for using it is to produce results compatible with the results of [96], which uses the bosonization and coproduct of [77].

Let the Hopf-subalgebras \( U_q(\hat{\mathfrak{g}}_+) \) and \( U_q(\hat{\mathfrak{g}}_-) \) of \( U_q(\hat{\mathfrak{g}}) \) be generated by \( \{e_i, q^{\pm h_i}\} \) and \( \{f_i, q^{\pm h_i}\} \), respectively. We centrally extend \( U_q(\hat{\mathfrak{g}}_+) \) to \( U_q(\hat{\mathfrak{g}}) \) by adding a generator \( q^d \) which commutes with \( q^{h_i} \) and interacts with \( e_i \) and \( f_i \) via

\[
[q^d, e_i] = [q^d, f_i] = 0 \\
q^d e_0 q^{-d} = q e_0 \\
q^d f_0 q^{-d} = q^{-1} f_0
\]

and on which the coproduct, antipode, and counit are

\[
\Delta(q^d) = q^d \otimes q^d \\
S(q^d) = q^{-d} \\
\epsilon(q^d) = 1.
\]

For \( z \in \mathbb{C}^\times \), define the automorphism \( D_z := \text{Ad}(z^d) \in \text{Aut}(U_q(\hat{\mathfrak{g}})) \). The action of \( d \) gives a grading on \( U_q(\hat{\mathfrak{g}}) \). In [17, Section 5], the square of the antipode is shown to act...
by conjugation by an explicit Cartan element.

**Lemma 5.2.1** ([17, Section 5]). For any \( x \in U_q(\mathfrak{g}) \), we have \( S^2(x) = q^{-2\tilde{\rho}} xq^{2\tilde{\rho}} \).

### 5.2.3 Universal \( \mathcal{R} \)-matrix and Drinfeld element for \( U_q(\mathfrak{g}) \)

Define \( \Omega = c \otimes d + d \otimes c + \sum_i x_i \otimes x_i \), and let \( \Omega^0 = c \otimes d + d \otimes c \) and \( \Omega^1 = \sum_i x_i \otimes x_i \).

By the general construction of [17], there is a universal \( \mathcal{R} \)-matrix \( \mathcal{R} \) for \( U_q(\mathfrak{g}) \) with

\[
\mathcal{R} = q^{-\Omega} \mathcal{R}_0 \quad \text{and} \quad \mathcal{R}_0 \in (1 + U_q(\hat{\mathfrak{g}}^+)) \otimes U_q(\hat{\mathfrak{g}}^-) < 0).
\]

It satisfies

\[
\Delta^{21}(x) \mathcal{R} = \mathcal{R} \Delta(x) \quad \text{for} \quad x \in U_q(\mathfrak{g}) \quad \quad (\Delta \otimes 1) \mathcal{R} = \mathcal{R}^{13} \mathcal{R}^{23} \quad \quad (1 \otimes \Delta) \mathcal{R} = \mathcal{R}^{13} \mathcal{R}^{12}.
\]

**Lemma 5.2.2** ([17, Proposition 3.1]). The universal \( \mathcal{R} \)-matrix satisfies:

(a) \( (S \otimes 1) \mathcal{R} = (1 \otimes S^{-1}) \mathcal{R} = \mathcal{R}^{-1} \);

(b) \( (S \otimes S) \mathcal{R} = \mathcal{R} \).

**Remark.** Note that the sign of \( \Omega \) is reversed from that of [33] because we use the opposite coproduct.

In [17, Section 5], the Drinfeld element \( u \) in a completion of \( U_q(\mathfrak{g}) \) is defined by

\[
u = m_{21} \left( (1 \otimes S) \mathcal{R} \right)
\]

and shown to satisfy the following properties.

**Lemma 5.2.3** ([17, Section 5]). The Drinfeld element \( u \) satisfies the following:

(a) \( u^{-1} = m_{21} \left( (1 \otimes S^{-1}) \mathcal{R}^{-1} \right) \)

(b) \( S^2(x) = uxu^{-1} \);

(c) \( q^{2\tilde{\rho}} u = uq^{2\tilde{\rho}} \) and \( S(u)q^{-2\tilde{\rho}} = q^{-2\tilde{\rho}} S(u) \) are central in different completions of \( U_q(\mathfrak{g}) \);

### 5.3 Representations and intertwiners for \( U_q(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \)

In this section we fix conventions for Verma and evaluation modules for \( U_q(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \) and characterize intertwiners between them, which will be the central object of study of this paper. Throughout the paper, we work with evaluation representations valued in the ring of Laurent polynomials or formal Laurent series.

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5.3.1 Evaluation modules

If $V$ is a $U_q(\hat{\mathfrak{g}})$-module, for $z \neq 0$ we define the $U_q(\hat{\mathfrak{g}})$-module $V(z)$ to be the vector space $V$ with action of $U_q(\hat{\mathfrak{g}})$ given by

$$\pi_{V(z)}(a) = \pi_V(D_z(a)).$$

In type $A$, if $V$ is a $U_q(\mathfrak{sl}_r)$-module treated as a $U_q(\hat{\mathfrak{sl}}_r)$-module via evaluation at 1, then $V(z)$ is the evaluation module at $z$.

If $V$ is a finite-dimensional or highest weight $U_q(\hat{\mathfrak{g}})$-module, define the $U_q(\mathfrak{b})$-modules $z^{-\Delta}V(z)$ and $z^{-\Delta}V((z))$ so that generators of $U_q(\mathfrak{b})$ act in the same way as on $V(z)$ and $d$ acts by $z\frac{\partial}{\partial z}$ if $V$ is finite-dimensional and by $z\frac{\partial}{\partial z} + d$ if $V$ is highest weight. Notice that both $z^{-\Delta}V[z, z^{-1}]$ and $z^{-\Delta}V((z))$ are infinite-dimensional as vector spaces over $\mathbb{C}$.

5.3.2 Verma modules

We denote by $M_{\mu,k}$ the Verma module for $U_q(\mathfrak{g})$ with highest weight $\mu + k\Lambda_0$ and by $v_{\mu,k} \in M_{\mu,k}$ a canonically chosen highest weight vector. Define the restricted dual of $M_{\mu,k}$ by

$$M_{\mu,k}^\vee := \bigoplus_{\tau,\alpha} M_{\mu,k}^\tau [\tau + k\Lambda_0 - a\delta]^*,$$

where the action of $U_q(\hat{\mathfrak{g}})$ is given by $(u-q)(m) := \phi(S(u)m)$. Define the representation $M_{\mu,k,a}$ to coincide with $M_{\mu,k}$ as a $U_q(\hat{\mathfrak{g}})$-representation, but with $U_q(\mathfrak{g})$-action given by letting $q^d$ act by $q^a$ on $v_{\mu,k}$. If $a = 0$, we write $M_{\mu,k}$ for $M_{\mu,k,0}$. Our convention is chosen to be consistent with the following explicit computation of the action of the Drinfeld element on $M_{\mu,k}$.

**Lemma 5.3.1** ([17, Section 5]). The action of $q^{2\delta\rho}$ on $M_{\mu,k}$ is given by $q^{(\mu + k\Lambda_0, \mu + k\Lambda_0 + 2\rho)}$.

For generic $(\mu, k)$, the Verma module $M_{\mu,k,a}$ is irreducible. On the other hand, if $\mu\rho + k\Lambda_0$ is dominant integral, then the singular vectors in $M_{\mu,k,a}$ may be determined explicitly.

**Proposition 5.3.2.** Suppose $\mu + k\Lambda_0$ is dominant integral. For a reduced decomposition $s_{i_j} \cdots s_{i_1}$ of $w \in \tilde{W}$, define $\alpha^j = \alpha_i$, $\alpha^i = (s_{i_j} \cdots s_{i_{j+1}})(\alpha_{i_j})$, and $n_j = 2 \frac{\mu + k\Lambda_0 + \rho \alpha^j}{(\alpha^j, \alpha^j)}$. Then the vectors

$$v_{\mu,k}^w := \frac{f_{i_1}^{n_1}}{[n_1]!} \cdots \frac{f_{i_j}^{n_j}}{[n_j]!} v_{\mu,k}$$

are the only singular vectors in $M_{\mu,k,a}$.

**Remark.** We give any highest weight $U_q(\hat{\mathfrak{g}})$-module $M$ the structure of a $U_q(\hat{\mathfrak{g}})$-module by imposing that $q^d$ acts by 1 on the highest weight vector. This convention differs from that of [50] but is consistent with our notation for $M_{\mu,k}$ above.
5.3.3 \( \mathcal{R} \)-matrices and intertwiners between representations

We will use the following intertwining property of the universal \( \mathcal{R} \)-matrix on evaluation representations of \( U_q(\mathfrak{g}) \). Let \( V_1, \ldots, V_n \) be finite-dimensional \( U_q(\mathfrak{g}) \)-representations, and let \( W \) be a \( U_q(\mathfrak{g}) \)-semisimple \( U_q(\mathfrak{g}) \)-representation. Define the tensor products \( V \) and \( \hat{V} \) of evaluation representations by

\[
V := V_1[z_1^{-1}] \otimes \cdots \otimes V_n[z_n^{-1}]
\quad \text{and} \quad
\hat{V} := V_1((z_1)) \otimes \cdots \otimes V_n((z_n)).
\]

**Lemma 5.3.3.** The operator \( P_{VW} \mathcal{R}_{WV} \) gives an intertwiner

\[
P_{VW} \mathcal{R}_{WV} : V_1[z_1^{-1}] \otimes \cdots \otimes V_n[z_n^{-1}] \otimes W \to W \otimes V_1((z_1)) \otimes \cdots \otimes V_n((z_n)).
\]

**Proof.** Because \( \mathcal{R} \in q^{-\Omega} \left(1 + U_q(\mathfrak{h}_+) \otimes U_q(\mathfrak{h}_-)\right)\), \( \mathcal{R}_{WV} \) defines a linear map \( V \otimes W \to \hat{V} \otimes W \). The composed map \( P_{VW} \mathcal{R}_{WV} \) is then an intertwiner of \( U_q(\mathfrak{g}) \)-representations by the property \( \Delta^{21}(x) \mathcal{R} = \mathcal{R} \Delta(x) \) for \( x \in U_q(\mathfrak{g}) \).

\[ \square \]

5.3.4 Intertwiners of \( U_q(\mathfrak{g}) \)-representations

For any \( U_q(\mathfrak{g}) \)-module \( W \) which is \( U_q(\mathfrak{g}) \)-semisimple, define the completed tensor product by

\[
M_{\mu,k,a} \hat{\otimes} W := \text{Hom}_C(M_{\mu,k,a}^\vee, W),
\]

where the \( U_q(\mathfrak{g}) \)-action is given by \( (u \cdot \phi)(m) = u^{(1)} \phi(S(u^{(2)})m) \). Elements of \( M_{\mu,k,a} \hat{\otimes} W \) are sums \( \sum_{i=0}^{\infty} m_i \otimes w_i \) with \( m_i, w_i \) homogeneous and \( \lim_{i \to \infty} \deg(m_i) = \infty \).

A key construction in this paper will be of intertwiners between a Verma module and its completed tensor product with either a finite-dimensional or integrable module. We begin by characterizing the space of such intertwiners when the weight is either generic or dominant integral. Denote the highest term of an intertwiner \( \Phi : M_{\mu_1,k_1,a_1} \to M_{\mu_2,k_2,a_2} \otimes W \) by

\[
\langle \Phi \rangle := \langle u_{\mu_2,k_2}^* \Phi u_{\mu_1,k_1} \rangle.
\]

**Proposition 5.3.4.** Let \( M_{\lambda,k,a} \) and \( M_{\mu,k-k',a} \) be Verma modules and \( W \) a \( U_q(\mathfrak{g}) \)-representation of level \( k' \) on which \( q^{h_i} \) and \( q^d \) act diagonally. Suppose that either (1) \( (\mu, k) \) is generic or (2) \( \mu + (k-k')\Lambda_0 \) is dominant integral and \( W[\lambda - \mu + n_i \alpha_i + k' \Lambda_0] = 0 \) for all \( i > 0 \). We have an isomorphism

\[
\text{Hom}_{U_q(\mathfrak{g})}(M_{\lambda,k,a}, M_{\mu,k-k',a} \hat{\otimes} W) \simeq W[\lambda - \mu + k' \Lambda_0]
\]

given by \( \Phi \mapsto \langle \Phi \rangle \).
Proof. The space of $U_q(\hat{\mathfrak{g}})$-intertwiners $M_{\lambda,k,a} \to M_{\mu,k-k',a} \otimes W$ is given by

\[
\text{Hom}_{U_q(\hat{\mathfrak{g}})}(M_{\lambda,k}, M_{\mu,k-k'} \otimes W) \cong \text{Hom}_{U_q(\hat{\mathfrak{g}})}(\text{Ind}_{U_q(\hat{\mathfrak{b}})} U_q(\hat{\mathfrak{g}}) \mathcal{C}_{\lambda,k}, \text{Hom}_C(M_{\lambda,k}^{\vee}, W)) \\
\cong \text{Hom}_{U_q(\hat{\mathfrak{g}})}(\mathcal{C}_{\lambda,k}, \text{Hom}_C(M_{\lambda,k}^{\vee}, W)) \\
\cong \text{Hom}_{U_q(\hat{\mathfrak{g}})}(\mathcal{C}_{\lambda,k} \otimes_k M_{\mu,k-k'}^{\vee}, W) \\
\cong \{v \in W[\lambda - \mu + k' \Lambda_0] \mid I_{\mu,k-k'} v = 0\},
\]

where $I_{\mu,k-k'} := \{u \in U_q(\hat{\mathfrak{g}}) \mid u \cdot v_{\mu,-k} = 0\}$ is the annihilator ideal of the lowest weight vector of $M_{\lambda,k}^{\vee}$. By Proposition 5.3.2, the $U_q(\hat{\mathfrak{b}}_+)$-submodule of $M_{\mu,k}^{\vee}$ generated by $v_{\mu,-k+k'}$ has relations

\[e_i^{n_i} v_{\mu,-k+k'} = 0\]

so that $I_{\mu,k-k'}$ is generated by $e_i^{n_i}$. The given condition ensures any element of $W[\lambda - \mu + k' \Lambda_0]$ yields a map of $U_q(\hat{\mathfrak{g}})$-modules in both cases (1) and (2). Computing the action of $q^d$ on the source and target and recalling our convention that $q^d$ acts by $q^a$ on the highest-weight vector shows that a valid $v$ must lie in the degree 0 part of $W$, completing the proof. \qed

Remark. In what follows, we will apply Proposition 5.3.4 for representations $W$ which are either integrable modules or tensor products of evaluation modules associated to finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules.

For a $U_q(\hat{\mathfrak{g}})$-semisimple representation $V$ of level $k'$ which is either highest weight or finite-dimensional, $v \in V[\tau]$ and $(\mu, k)$ so that the conditions of Proposition 5.3.4 hold, denote by

\[
\Phi_{\mu,k,a}^v(z) : M_{\mu,k,a} \to M_{\mu-k-k',a} \otimes V[z, z^{-1}]
\]

the unique corresponding $U_q(\hat{\mathfrak{g}})$-intertwiner. Each $z$-coefficient of $\Phi_{\mu,k,a}^v(z)$ lies in the tensor product $M_{\mu-k-k',a} \otimes V$ without completion. Similarly, if $W$ is a $U_q(\hat{\mathfrak{b}})$-semisimple $U_q(\hat{\mathfrak{g}})$-representation of level $k_W$, and $w \in W[\tau + k_W \Lambda_0]$, denote by

\[
\Phi_{\mu,k,a}^w : M_{\mu,k,a} \to M_{\mu-k-k_W,a} \otimes W
\]

the corresponding intertwiner given by Proposition 5.3.4. For notational convenience, we will sometimes denote this by $\Phi_{\mu,k,a}^w(1) := \Phi_{\mu,k,a}^w$. For $i = 1, \ldots, n$, let $V_i$ be a $U_q(\hat{\mathfrak{g}})$-representation of level $k_i'$ which is either finite-dimensional or a highest weight $U_q(\hat{\mathfrak{g}})$-representation. For $v_i \in V_i[\tau_i + k_i' \Lambda_0]$, define the iterated intertwiner

\[
\Phi_{\mu,k,a}^{v_1, \ldots, v_n}(z_1, \ldots, z_n) : M_{\mu,k,a} \to M_{\mu-k_1-k_2-\ldots-k_n,a} \otimes V_1[z_1^{\pm 1}] \otimes \cdots \otimes V_n[z_n^{\pm 1}]
\]

by the composition

\[
\Phi_{\mu,k,a}^{v_1, \ldots, v_n}(z_1, \ldots, z_n) = \Phi_{\mu-k_1-k_2-\ldots-k_n,a}^{v_1, \ldots, v_n}(z_1) \circ \cdots \circ \Phi_{\mu,k,a}^{v_n}(z_n), \tag{5.3.1}
\]

where we adopt the convention that $z_i \equiv 1$ and $V_i[z_i^{\pm 1}] \equiv V_i$ if $V_i$ is a highest weight.
If all $V_i$ are finite-dimensional, define also the universal intertwiner

$$
\Phi^{V_1,\ldots,V_n}_{\mu,k,a}(z_1,\ldots,z_n) := \sum_{v_1,\ldots,v_n} \Phi^{v_1,\ldots,v_n}_{\mu,k,a}(z_1,\ldots,z_n) \otimes v_n^* \otimes \cdots \otimes v_1^*,
$$

(5.3.2)

where in the sum $\{v_i\}$ and $\{v_i^*\}$ range over dual bases of $V_1,\ldots,V_n$ and $V_1^*\ldots,V_n^*$.

Finally, denote the single-step intertwiner associated to $v_1 \otimes \cdots \otimes v_n \in V_1[z_1^{\pm1}] \otimes \cdots \otimes V_n[z_n^{\pm1}]$ by Proposition 4.9.4 by

$$
\tilde{\Phi}^{v_1,\ldots,v_n}_{\mu,k,a}(z_1,\ldots,z_n) : M_{\mu,k,a}

\rightarrow M_{\mu-\tau_1-\cdots-\tau_n-k-k',\ldots,k_n,a} \hat{\otimes} V_1[z_1^{\pm1}] \otimes \cdots \otimes V_n[z_n^{\pm1}],
$$

(5.3.3)

with the same convention that $z_i \equiv 1$ and $V_i[z_i^{\pm1}] \equiv V_i$ if $V_i$ is a highest weight $U_q(\mathfrak{g})$-module. If all $V_i$ are finite-dimensional, let its universal version be

$$
\tilde{\Phi}^{V_1,\ldots,V_n}_{\mu,k,a}(z_1,\ldots,z_n) := \sum_{v_1,\ldots,v_n} \tilde{\Phi}^{v_1,\ldots,v_n}_{\mu,k,a}(z_1,\ldots,z_n) \otimes v_n^* \otimes \cdots \otimes v_1^*.
$$

(5.3.4)

Notice that $\Phi^V_{\mu,k,a}(z) = \tilde{\Phi}^V_{\mu,k,a}(z)$ and $\Phi^V_{\mu,k,a}(z) = \tilde{\Phi}^V_{\mu,k,a}(z)$. As maps of $U_q(\mathfrak{g})$-modules, these intertwiners are independent of $a$; we denote by $\Phi^{v_1,\ldots,v_n}_{\mu,k}(z_1,\ldots,z_n)$, $\Phi^{V_1,\ldots,V_n}_{\mu,k}(z_1,\ldots,z_n)$, $\tilde{\Phi}^{v_1,\ldots,v_n}_{\mu,k}(z_1,\ldots,z_n)$, and $\tilde{\Phi}^{V_1,\ldots,V_n}_{\mu,k}(z_1,\ldots,z_n)$ the corresponding intertwiners with $a = 0$.

**Remark.** As defined, the composed intertwiners $\Phi^{v_1,\ldots,v_n}_{\mu,k,a}(z_1,\ldots,z_n)$ are formal series in $z_1,\ldots,z_n$. It was shown in [24, 50] that the matrix elements of these formal series converge for $z_1 \gg \cdots \gg z_n$ and admit extension to meromorphic functions.

**Remark.** For finite-dimensional $V_1,\ldots,V_n$, the intertwiners $\Phi^{v_1,\ldots,v_n}_{\mu,k}(z_1,\ldots,z_n)$ appeared in [24, 50] with Weyl modules in the place of Verma modules. The two constructions coincide for generic weight.

## 5.4 Fusion and exchange operators and ABRR equation

In this section, we introduce the fusion and exchange operators as operators on representations via intertwiners and as universal elements via the ABRR equation. After characterizing their basic properties, we give normalizations of the parameters for which evaluation of the universal operators in tensor products of representations coincides with the construction via intertwining operators.

### 5.4.1 Dynamical notation

Throughout the following sections, we will use dynamical notation to express the action of certain operators on tensor products of $U_q(\mathfrak{g})$-modules. Suppose that $f(\mu, k)$:
The function $U_q(\mathfrak{g})$ is a function and $V_1 \otimes \cdots \otimes V_n \otimes W_1^* \otimes \cdots \otimes W_m^*$ a tensor product of $U_q(\mathfrak{g})$-representations. We denote by $f((\mu, k) + ah^{(j)} + bh^{(e)})$ the element of $\text{End}(V_1 \otimes \cdots \otimes V_n \otimes W_1^* \otimes \cdots \otimes W_m^*)$ acting by

$$f((\mu, k) + ah^{(j)} + bh^{(e)})(v_1 \otimes \cdots \otimes v_n \otimes w_1^* \otimes \cdots \otimes w_m^*)$$

for $v_j \in V_j[\mu_j]$ and $w_i^* \in W_i^*[\nu_i]$, where we use $(\mu, k)$ to denote the element $\mu + k\Lambda_0 \in \mathfrak{h}^*$. If $\mu_j, \nu_l \in \mathfrak{h}^*$, we will also denote this by the notation $f(\mu + ah^{(j)} + bh^{(e)}, k)$. We denote by $f((\mu, k) + a\mathcal{H}^{(j)})$ the element which acts by

$$f\tau((\mu, k) + a\mathcal{H}^{(j)})(v_1 \otimes \cdots \otimes v_n \otimes w_1^* \otimes \cdots \otimes w_m^*)$$

where $f\tau$ is the part of $f$ which shifts the weight in $V_j$ by $\tau$.

### 5.4.2 Fusion and exchange operators in representations

Let $V_1$ and $V_2$ be $U_q(\mathfrak{g})$-representations of level $k_1$ and $k_2$ which are either finite-dimensional or highest weight $U_q(\mathfrak{g})$-representations. As before, if $V$ is a highest weight $U_q(\mathfrak{g})$-representation, let $z_i \equiv 1$ and interpret $V_i[z_i^{\pm 1}] \equiv V_i$. Suppose also that at most one of $V_i$ is highest weight. The fusion operator $J_{V_1, V_2}(z_1, z_2; \mu, k)$:

$$J_{V_1, V_2}(z_1, z_2; \mu, k)(v_1 \otimes v_2)$$

on homogeneous $v_1 \otimes v_2 \in V_1 \otimes V_2$, where $v_{(\mu, k)-\text{wt}(v_2)-\text{wt}(v_1)}$ is the dual vector to the highest weight vector $v_{(\mu, k)-\text{wt}(v_2)-\text{wt}(v_1)}$ for $M_{(\mu, k)-\text{wt}(v_2)-\text{wt}(v_1)}$, and $\text{wt}(v_1)$ denotes the $\mathfrak{h}^*$-weight of $v_1$. As defined, it is a formal series in $z_1/z_2$, and it was shown in [24] that it converges to a meromorphic function on $z_1 \gg z_2$ if $V_1$ and $V_2$ are finite-dimensional.

For $U_q(\mathfrak{g})$ representations $V_i$ of level $k_i$ which are either finite-dimensional or highest weight with at most one highest weight, define the iterated fusion operator by

$$J_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \mu, k)(v_1 \otimes \cdots \otimes v_n)$$

Define also its multicomponent version by

$$\widetilde{J}_{V_1, \ldots, V_i; V_{i+1}, \ldots, V_n}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k)(v_1 \otimes v_{i+1} \otimes \cdots \otimes v_n)$$
where we note that \( \tilde{J}_{v_1,v_2}(z_1;z_2;\mu,k) = J_{v_1,v_2}(z_1,z_2;\mu,k) \). We may relate the two types of intertwiners via the multicomponent fusion operator.

**Lemma 5.4.1.** For homogeneous vectors \( v_1, \ldots, v_n \), we have that
\[
\Phi_{\mu,k}^{v_1,\ldots,v_n}(z_1,\ldots,z_n) = \Phi_{\mu,k}^{v_1,\ldots,v_n}(z_1,\ldots,z_n).
\]

**Proof.** The two intertwiners both have highest term
\[
J_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\mu,k)(v_1 \otimes \cdots \otimes v_n),
\]
hence they coincide by Proposition 5.3.4.

**Lemma 5.4.2.** The iterated fusion operator satisfies
\[
J_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\mu,k) = \tilde{J}_{v_1,v_2}(z_1;z_2;\mu,k) \cdots \tilde{J}_{v_{n-1},v_n}(z_{n-1};z_n;\mu,k).
\]

**Proof.** These are two different ways of expressing the highest term of the intertwiner \( \Phi_{\mu,k}^{v_1,\ldots,v_n}(z_1,\ldots,z_n) \), hence they are equal.

Let \( V_1, V_2 \) be \( U_q(\mathfrak{g}) \)-representations which are either finite-dimensional or highest weight \( U_q(\mathfrak{g}) \)-representations, with at most one being highest weight. The exchange operator \( R_{V_1,V_2}(z_1,z_2;\mu,k) : V_1[z_1^{-1}] \otimes V_2[z_2^{-1}] \rightarrow V_1((z_1)) \otimes V_2((z_2^{-1})) \) is defined as
\[
R_{V_1,V_2}(z_1,z_2;\mu,k) := J_{V_1,V_2}(z_1,z_2;\mu,k)^{-1} \tilde{R}_{V_1,V_2}^{21} \tilde{J}_{V_1,V_2}^{21}(z_2,z_1;\mu,k).
\]

**Remark.** Both the fusion and exchange operators depend only on the \( U_q(\mathfrak{g}) \)-structure of the Verma module \( M_{\mu,k,a} \), meaning that their definition is independent of the choice of normalization for the grading. Therefore, their value remains the same if \( M_{\mu,k}, \Phi_{\mu,k}^{v_1,\ldots,v_n}(z_1,\ldots,z_n) \), and \( \Phi_{\mu,k}^{v_1,\ldots,v_n}(z_1,\ldots,z_n) \) are replaced by \( M_{\mu,k,a}, \Phi_{\mu,k,a}^{v_1,\ldots,v_n}(z_1,\ldots,z_n) \), and \( \Phi_{\mu,k,a}^{v_1,\ldots,v_n}(z_1,\ldots,z_n) \) in their definitions.

### 5.4.3 Universal fusion operators

In [34], a universal fusion operator \( \mathcal{J}(\mu,k) \) living in a completion of \( U_q(\mathfrak{g}) \) under the principal grading is defined; when evaluated in finite dimensional representations, \( \mathcal{J}(\mu,k) \) yields the previously defined fusion operators. We modify this definition by using the ABRR equation for the opposite coproduct and using the standard grading instead of the principal grading. For this, we define the coefficient ring
\[
\mathcal{A}_{\mu,k} := \mathbb{C}[[q^{-2(\mu,\alpha_1)}, \ldots, q^{-2(\mu,\alpha_r)}, q^{-2k+2(\mu,\theta)}]]
\]
and work formally over \( \mathcal{A}_{\mu,k} \). We then have the following analogue of [34, Theorem 8.1].
Proposition 5.4.3. There is a unique element $J(\mu, k) \in 1 + (U_q(\mathfrak{g}_-))_{<0} \otimes U_q(\mathfrak{g}_+)^{>0} \otimes A_{\mu, k}$ satisfying the ABRR equation

$$R^{21} q_1^{2\mu + 2kd} J(\mu, k) = J(\mu, k) q_1^{2\mu + 2kd} q^{-\Omega}. \tag{5.4.1}$$

Moreover, the universal fusion operator $J(\mu, k)$ satisfies

$$J^{12,3}(\mu, k) J^{12}((\mu, k) - h(3)/2) = J^{1,23}(\mu, k) J^{23}((\mu, k) + h(1)/2). \tag{5.4.2}$$

Proof. Write $J(\mu, k) = \sum_{i,j \geq 0} J_{i,j}(\mu) q^{-2ki}$, where $J_{i,j}(\mu)$ consists of terms with degree $-j$ in the first tensor component with respect to the standard grading, and write $R = \sum_{l \geq 0} R_l$, where $R_l$ consists of terms of degree $l$ in the first tensor factor and $R_0 = R_{\text{trig}}$. The ABRR equation (5.4.1) may be rewritten as

$$\sum_{i,j,l} R_l^{21} q_1^{2\mu} J_{i,j}(\mu) q_1^{-2\mu} q^{\Omega} q^{-2k(i+j)} = \sum_{i,j} J_{i,j}(\mu) q^{-2ki}.$$

Matching degree $j$ coefficients of $q^{-2k}$, the ABRR equation is equivalent to

$$J_{i,j}(\mu) = \sum_{a=0}^{\min(i,j)} R_{j-a}^{21} q_1^{2\mu} J_{i-a,a}(\mu) q_1^{-2\mu} q^{\Omega}.$$

For $j = 0$, this yields

$$J_{i,0}(\mu) = R_{\text{trig}}^{21} q_1^{2\mu} J_{i,0}(\mu) q_1^{-2\mu} q^{\Omega},$$

which is the ABRR equation for $U_q(\mathfrak{g})$. By the existence and uniqueness of solutions to the ABRR equation for $U_q(\mathfrak{g})$ given by [5, Proposition 1], we find that $J_{i,0}(\mu) = J_{\text{trig}}(\mu)$ and $J_{i,0}(\mu) = 0$ for $i > 0$. For $j > 0$, we have a recursion relation expressing $J_{i,j}(\mu)$ in terms of elements with smaller $i$ or $j$, yielding existence and uniqueness.

To show (5.4.2), define the operators

$$A_L X = R^{21} R^{31} q_1^{2\mu} q^{\Omega_2} q^{\Omega_3} q_1^{-2\mu} X q_1^{2\mu} q^{\Omega_2} q^{\Omega_3},$$

$$A_R X = R^{32} R^{31} q_1^{-2\mu} q^{\Omega_2} q^{\Omega_3} q_3^{2\mu} X q_3^{-2\mu} q^{\Omega_2} q^{\Omega_3}.$$

First, we claim that $A_L$ and $A_R$ commute. This is equivalent to the identity

$$R^{21} R^{31} q_1^{2\mu + 2kd} R^{32} R^{31} q_3^{-2\mu - 2kd} = R^{32} R^{31} q_3^{-2\mu - 2kd} R^{21} R^{31} q_1^{2\mu + 2kd},$$

which follows from the Yang-Baxter equation after canceling $R^{31}$ and the factors of $q_1^{2\mu + 2kd}$ and $q_3^{-2\mu - 2kd}$.

Now, we claim that both sides are solutions to $A_L X = X$ and $A_R X = X$ which agree in degree 0 terms under the principal grading in the first and third components.
Because such solutions are unique, they must be equal. Notice first that
\[
A_L \mathcal{J}^{1,23}(\mu, k) \mathcal{J}^{23}(\mu, k + h^{(1)}/2) = \mathcal{J}^{1,23}(\mu, k) \mathcal{J}^{23}(\mu, k + h^{(1)}/2) q^{\Omega_{1,23}}
\]
\[
= \mathcal{J}^{1,23}(\mu, k) q^{-\Omega_{1,23}} \mathcal{J}^{23}(\mu, k + h^{(1)}/2) q^{\Omega_{1,23}}
\]
\[
= \mathcal{J}^{1,23}(\mu, k) \mathcal{J}^{23}(\mu, k + h^{(1)}/2).
\]

Because \(A_L\) and \(A_R\) commute, \(X = A_R \mathcal{J}^{1,23}(\mu, k) \mathcal{J}^{23}(\mu, k + h^{(1)}/2)\) satisfies \(A_L X = X\), so to check that \(X = \mathcal{J}^{1,23}(\mu, k) \mathcal{J}^{23}(\mu, k + h^{(1)}/2)\) under the principal grading is \(\mathcal{J}^{23}(\mu, k + h^{(1)}/2)\), the degree zero term of the first component of \(X\) under the principal grading is
\[
\mathcal{R}^{32} q^{-\Omega_{31}} q_3^{-2\mu-2kd} \mathcal{J}^{23}(\mu, k + h^{(1)}/2) q_3^{2\mu+2kd} q^{\Omega_{32}}
\]
\[
= \mathcal{R}^{23} \text{Ad}(q_3^{-2\mu-2kd-h^{(1)}}) \mathcal{J}^{23}(\mu, k + h^{(1)}/2) q^{\Omega_{23}}
\]
\[
= \mathcal{R}^{23} \text{Ad}(q_3^{2\mu+2kd+h^{(1)}}) \mathcal{J}^{23}(\mu, k + h^{(1)}/2) q^{\Omega_{23}},
\]
and that they are equal by the ABRR equation. A similar computation shows that \(\mathcal{J}^{12,3}(\mu, k) \mathcal{J}^{12}(\mu, k - h^{(3)}/2)\) is a solution. \(\square\)

We require also a renormalization of \(\mathcal{J}(\mu, k)\) which will have good convergence properties when evaluated on \(U_q(\mathfrak{g})\)-representations which are locally nilpotent with respect to the induced \(U_q(\mathfrak{g})\)-action. Define the renormalized universal fusion operator \(L(\mu, k) \in 1 + (U_q(\mathfrak{g})_{<0} \otimes U_q(\mathfrak{g})_{>0})^\mathbb{N} \otimes A_{\mu,k}\) by
\[
L(\mu, k) := (\mathcal{R}^{21})^{-1} \mathcal{J}(\mu, k).
\]

This renormalized operator satisfies the following formal properties.

**Proposition 5.4.4.** The following hold for \(L(\mu, k)\):

(a) the element \(L(\mu, k)\) satisfies the ABRR equation
\[
q_1^{2\mu+2kd} \mathcal{R}^{21} L(\mu, k) = L(\mu, k) q_1^{2\mu+2kd} q^{-\Omega};
\]
(b) each coefficient of \(L(\mu, k)\) as a power series in \(q^{-2k}\) has finite degree in the standard grading;

(c) the element \(L(\mu, k)\) satisfies the shifted 2-cocycle relation
\[
L^{21,3}(\mu, k) L^{12}(\mu, k - h^{(3)}/2) = L^{1,32}(\mu, k) L^{23}(\mu, k + h^{(1)}/2).
\]

**Proof.** Claim (a) follows directly from the ABRR equation (5.4.1) for \(\mathcal{J}(\mu, k)\). For
(b), let the power series expansion of \( \mathcal{L}(\mu, k) \) be

\[
\mathcal{L}(\mu, k) = \sum_{i,j \geq 0} \mathcal{L}_{i,j}(\mu) q^{-2ki},
\]

where \( \mathcal{L}_{i,j} \) has terms of degree \(-j\) in the first tensor factor, and write a series expansion

\[
\mathcal{R} = \sum_{l \geq 0} \mathcal{R}_l,
\]

where \( \mathcal{R}_l \) consists of terms of degree \( l \) in the first tensor factor and \( \mathcal{R}_0 = \mathcal{R}_{\text{trig}} \), the universal \( \mathcal{R} \)-matrix of \( U_q(\mathfrak{g}) \). The (renormalized) ABRR equation (5.4.3) yields

\[
\sum_{i,j \geq 0} \mathcal{L}_{i,j}(\mu) q^{-2ki} q^{-\Omega} = \sum_{i,j \geq 0} q_1^{2\mu+2kd} \mathcal{R}_i^{21} \mathcal{L}_{i,j}(\mu) q_1^{-2\mu-2kd} q^{-2ki}
\]

\[
= \sum_{i,j \geq 0} q_1^{2\mu} \mathcal{R}_i^{21} \mathcal{L}_{i,j}(\mu) q_1^{-2\mu} q^{-2k(i+j+l)}.
\]

This implies that the constant term \( \mathcal{L}_0(\mu) \) satisfies

\[
\mathcal{L}_0(\mu) q^{-\Omega} = q_1^{2\mu} \mathcal{R}_{\text{trig}}^{21} \mathcal{L}_0(\mu) q_1^{-2\mu}
\]

and is therefore equal to \( (\mathcal{R}_{\text{trig}}^{21})^{-1} \mathcal{J}_{\text{trig}}(\mu) \), the analogous quantity for \( U_q(\mathfrak{g}) \). For the higher terms, matching coefficients yields

\[
\sum_{j \geq 0} \mathcal{L}_{i,j}(\mu) q^{-\Omega} = \sum_{a+j+l=i} q_1^{2\mu} \mathcal{R}_i^{21} \mathcal{L}_{a,j}(\mu) q_1^{-2\mu}
\]

and therefore that

\[
\sum_{j \geq 0} \mathcal{L}_{i,j}(\mu) q^{-\Omega} - q_1^{2\mu} \mathcal{R}_0^{21} \mathcal{L}_{i,0}(\mu) q^{-2\mu} = \sum_{a+j+l=i} q_1^{2\mu} \mathcal{R}_i^{21} \mathcal{L}_{a,j}(\mu) q_1^{-2\mu}.
\]

Induction on the power of \( q^{-2k} \) yields (b). For (c), the 2-cocycle relation (5.4.2) for the fusion operator implies that

\[
\mathcal{J}^{12,3}(\mu, k) \mathcal{R}^{21} \mathcal{L}^{12}(\mu, k) - h^{(3)}/2) = \mathcal{J}^{1,23}(\mu, k) \mathcal{R}^{32} \mathcal{L}^{23}(\mu, k) + h^{(1)}/2.
\]

Recalling that \( \mathcal{J}^{12,3}(\mu, k) \mathcal{R}^{21} = \mathcal{R}^{21} \mathcal{J}^{21,3}(\mu, k) \mathcal{R}^{32} = \mathcal{R}^{32} \mathcal{J}^{1,32}(\mu, k) \) transforms this into

\[
\mathcal{R}^{21} \mathcal{R}^{3,21} \mathcal{L}^{21,3}(\mu, k) \mathcal{L}^{12}(\mu, k) - h^{(3)}/2) = \mathcal{R}^{32,1} \mathcal{R}^{32} \mathcal{J}^{1,32}(\mu, k) \mathcal{L}^{23}(\mu, k) + h^{(1)}/2.
\]

The result follows from noting that \( \mathcal{R}^{21} \mathcal{R}^{3,21} = \mathcal{R}^{32,1} \mathcal{R}^{32} \) by the Yang-Baxter equation. \( \square \)

**Remark.** A consequence of Proposition 5.4.4(b) is that \( \mathcal{L}(\mu, k) \) may be evaluated on the tensor product of any two representations which are locally nilpotent with respect to the action of \( U_q(\mathfrak{g}) \).
Define the shifted universal fusion operators

\[ J(\mu, k) := \mathcal{J} \left( (\mu, k) - h^{(1)} / 2 - h^{(2)} / 2 \right) \quad \text{and} \quad L(\mu, k) := \mathcal{L} \left( (\mu, k) - h^{(1)} / 2 - h^{(2)} / 2 \right) . \]

For finite-dimensional \( U_q(\mathfrak{g}) \)-representations \( V_1, \ldots, V_n \), consider the corresponding evaluation representations

\[ V := V_1[z_1^{\pm 1}] \otimes \cdots \otimes V_n[z_n^{\pm 1}] \quad \text{and} \quad \hat{V} := V_1((z_1) \otimes \cdots \otimes V_n((z_n)). \]

If \( W \) is a highest weight \( U_q(\mathfrak{g}) \)-representation, then evaluation of \( J(\mu, k) \) gives linear maps \( V \otimes W \to V \otimes W \) and \( W \otimes V \to W \otimes \hat{V} \), which we denote by \( J_{VW}(z_1, \ldots, z_n; 1; \mu, k) \) and \( J_{WV}(1; z_1, \ldots, z_n; \mu, k) \). Evaluation of \( L(\mu, k) \) gives linear maps \( V \otimes W \to V \otimes W \) and \( W \otimes V \to W \otimes V \), which we denote by \( L_{VW}(z_1, \ldots, z_n; 1; \mu, k) \) and \( L_{WV}(1; z_1, \ldots, z_n; \mu, k) \).

### 5.4.4 Universal exchange operators

Define the universal exchange operator in \( U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \) by

\[ R(\mu, k) := J(\mu, k)^{-1} R^{21} J^{21}(\mu, k) \] (5.4.5)

In terms of the renormalized universal fusion operator, we have

\[ R(\mu, k) = L(\mu, k)^{-1} R LL^{21}(\mu, k) . \] (5.4.6)

Evaluation of \( R(\mu, k) \) gives linear maps \( V \otimes W \to \hat{V} \otimes W \) and \( W \otimes V \to W \otimes V \), which we denote by \( R_{VW}(z_1, \ldots, z_n; 1; \mu, k) \) and \( R_{WV}(1; z_1, \ldots, z_n; \mu, k) \).

**Proposition 5.4.5.** The universal exchange operator satisfies the quantum dynamical Yang-Baxter equation

\[ R^{23}(\mu, k) R^{13}(\mu, k - h^{(2)}) R^{12}(\mu, k) = R^{12}(\mu, k - h^{(3)}) R^{13}(\mu, k) R^{23}(\mu, k - h^{(1)}). \]

**Proof.** By Proposition 5.4.3, we obtain that

\[ J^{13}(\mu, k - h^{(2)})^{-1} = J^{32}(\mu, k)^{-1} J^{1,32}(\mu, k)^{-1} J^{13,2}(\mu, k) \]
\[ J^{31}(\mu, k - h^{(2)}) = J^{31,2}(\mu, k)^{-1} J^{3,12}(\mu, k) J^{12}(\mu, k) . \]

Substituting these relations into the definition of the universal exchange operator, we
find that

\[ R^{23}(\mu, k)R^{13}((\mu, k) - h(3))R^{12}(\mu, k) \]
\[ = J^{23}(\mu, k)^{-1}R^{32}J^{32}(\mu, k)J^{13}((\mu, k) - h(2))^{-1} \]
\[ \times R^{31}J^{31}((\mu, k) - h(2))J^{12}(\mu, k)^{-1}R^{21}J^{21}(\mu, k) \]
\[ = J^{23}(\mu, k)^{-1}R^{32}J^{32}(\mu, k)J^{13,2}(\mu, k)R^{31}J^{31,2}(\mu, k)^{-1}J^{3,12}(\mu, k)R^{21}J^{21}(\mu, k) \]
\[ = J^{23}(\mu, k)^{-1}J^{3,12}(\mu, k)R^{32}R^{31}J^{32,21}(\mu, k)J^{21}(\mu, k) \]
\[ = J^{23}(\mu, k)^{-1}J^{3,12}(\mu, k)^{-1}R^{21}R^{31}J^{32,21}(\mu, k)J^{21}(\mu, k). \]

On the other hand, using the relations

\[ J^{12}((\mu, k) - h(3))^{-1} = J^{23}(\mu, k)^{-1}J^{1,123}(\mu, k)^{-1}J^{12,3}(\mu, k) \]
\[ J^{21}((\mu, k) - h(3)) = J^{21,3}(\mu, k)^{-1}J^{2,13}(\mu, k)J^{13}(\mu, k) \]
\[ J^{23}(\mu, k) - h(1))^{-1} = J^{31}(\mu, k)^{-1}J^{23,1}(\mu, k)^{-1}J^{23,1}(\mu, k) \]
\[ J^{32}(\mu, k) - h(1))^{-1} = J^{32,1}(\mu, k)^{-1}J^{3,21}(\mu, k)J^{21}(\mu, k) \]

from Proposition 5.4.3, we obtain

\[ R^{12}((\mu, k) - h(3))R^{13}(\mu, k)R^{23}(\mu, k) - h(1)) \]
\[ = J^{12}((\mu, k) - h(3))^{-1}R^{21}J^{21}((\mu, k) - h(3))J^{13}(\mu, k)^{-1} \]
\[ \times R^{31}J^{31}((\mu, k) - h(1))^{-1}R^{32}J^{32}(\mu, k)^{-1} \]
\[ = J^{23}(\mu, k)^{-1}J^{3,12}(\mu, k)^{-1}R^{32}R^{31}J^{32,21}(\mu, k)J^{21}(\mu, k) \]
\[ = J^{23}(\mu, k)^{-1}J^{3,12}(\mu, k)^{-1}R^{21}R^{31}J^{32,21}(\mu, k)J^{21}(\mu, k), \]

which yields the desired. \(\square\)

5.4.5 Evaluation of universal fusion operators

Let \( V_1, \ldots, V_n \) be \( U_q(\widehat{\mathfrak{g}}) \)-representations which are either finite-dimensional or highest weight \( U_q(\widehat{\mathfrak{g}}) \)-representations. Let \( z_i \) be a variable if \( V_i \) is finite-dimensional or 1 otherwise, and let \( V_i[z_{i+1}] \) be an evaluation representation of \( U_q(\widehat{\mathfrak{g}}) \) if \( V_i \) is finite-dimensional or \( V_i \) itself otherwise. We now relate the multicomponent fusion operators \( \mathcal{J}_{V_1, \ldots, V_i; V_{i+1}, \ldots, V_n}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k) \) to the shifted universal fusion operator \( \mathcal{J}(\mu + \rho, k + h^\vee) \) evaluated in a representation. Define the representations

\[ W_1 = V_1[z_1^{\pm 1}] \otimes \cdots \otimes V_i[z_i^{\pm 1}] \quad \text{and} \quad W_2 = V_{i+1}[z_{i+1}^{\pm 1}] \otimes \cdots \otimes V_n[z_n^{\pm 1}], \]

and denote by \( \mathcal{J}_{W_1, W_2}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu + \rho, k + h^\vee) \) the evaluation of \( \mathcal{J}(\mu, k) \) on \( W_1 \otimes W_2 \).

**Proposition 5.4.6.** If at most one \( V_i \) is a highest weight \( U_q(\widehat{\mathfrak{g}}) \)-representation, we
have that
\[ \mathcal{J}w_1w_2(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu + \rho, k + h^\vee) = \mathcal{J}v_1, \ldots, V_i; V_{i+1}, \ldots, V_n(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k). \]

**Proof.** By Lemmas 5.2.3 and 5.3.1, the element \( C = e^{2\rho u} \) is central and satisfies \( C|_{M_{\lambda,k}} = q^{(\lambda + k\Lambda_0, \lambda + k\Lambda_0 + 2\rho)} \cdot \text{id} \). Consider a series expansion
\[ \mathcal{R} = \sum_i a_i \otimes b_i \]
so that \( u = \sum_i S(b_i)a_i \). Choose homogeneous vectors \( w_1 \in W_1 \) and \( w_2 \in W_2 \), and define the quantity
\[ L = \left( v_{(\mu,k) - \text{wt}(w_1) - \text{wt}(w_2)}^* \right) \circ \Phi^{w_1}_{(\mu,k) - \text{wt}(w_1)}(z_1, \ldots, z_i) \circ u|_{M_{(\mu,k) - \text{wt}(w_2)} \circ \Phi^{w_2}_{\mu,k}(z_{i+1}, \ldots, z_n)v_{\mu,k} \rangle, \tag{5.4.7} \]
where \( \text{wt}(w_i) \) denotes the weight of \( w_i \). We compute \( L \) in two different ways. First, computing the action of \( C \) on \( M_{\mu,k} \), we find for weight vectors \( v \in V \) and \( w \in W \) that
\[ L = q^{(\mu + k\Lambda_0 - \text{wt}(w_2), \mu + k\Lambda_0 - \text{wt}(w_2) + 2\rho)} \cdot \mathcal{J}v_1, \ldots, V_i; V_{i+1}, \ldots, V_n(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k)(w_1 \otimes w_2) \]
\[ = q^{(\mu + k\Lambda_0 - \text{wt}(w_2), \mu + k\Lambda_0 - \text{wt}(w_2) + 2\rho)} \cdot \mathcal{J}v_1, \ldots, V_i; V_{i+1}, \ldots, V_n(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k)(w_1 \otimes w_2). \]
For the second way, we have
\[ \Delta_2(S_2(\mathcal{R})) = (1 \otimes S \otimes S)\Delta_2^1(\mathcal{R}) = (1 \otimes S \otimes S)(\mathcal{R}^{12} \mathcal{R}^{13}), \]
which implies that
\[ \sum_i a_i \otimes S(b_i)_{(1)} \otimes S(b_i)_{(2)} = \sum_{i,j} a_i a_j \otimes S(b_i) \otimes S(b_j). \]
We conclude that
\[ L = \sum_i \left( v_{(\mu,k) - \text{wt}(w_1) - \text{wt}(w_2)}^* \right) \circ \Phi^{w_1}_{(\mu,k) - \text{wt}(w_1)}(z_1, \ldots, z_i) \circ S(b_i)a_i \circ \Phi^{w_2}_{\mu,k}(z_{i+1}, \ldots, z_n)v_{\mu,k} \right), \]
\[ \text{so that} \ u = \sum_i S(b_i)a_i. \]
Choose homogeneous vectors \( w_1 \in W_1 \) and \( w_2 \in W_2 \), and define the quantity
\[ L = \left( v_{(\mu,k) - \text{wt}(w_1) - \text{wt}(w_2)}^* \right) \circ \Phi^{w_1}_{(\mu,k) - \text{wt}(w_1)}(z_1, \ldots, z_i) \circ u|_{M_{(\mu,k) - \text{wt}(w_2)} \circ \Phi^{w_2}_{\mu,k}(z_{i+1}, \ldots, z_n)v_{\mu,k} \rangle, \tag{5.4.7} \]
where \( \text{wt}(w_i) \) denotes the weight of \( w_i \). We compute \( L \) in two different ways. First, computing the action of \( C \) on \( M_{\mu,k} \), we find for weight vectors \( v \in V \) and \( w \in W \) that
Recall now that \( R \in q^{-\Omega}(1 + U_q(\hat{h}_+)_{>0} \otimes U_q(\hat{b}_-)_{>0}) \), meaning that all \( i \)-indexed terms aside from the ones corresponding to \( q^\pm \) are zero in the sum. This implies that

\[
L = \sum_j S(b_j)|W_1 \left( v_{(\mu,k)}^{*} - \text{wt}(w_1) - \text{wt}(w_2) \right),
\]

\[
((m_{31} \otimes 1)(\Delta_1 \otimes S)(q^{-\Omega})) \Phi_{(\mu,k)}^{w_1} - \text{wt}(w_1)(z_1, \ldots, z_i) a_j \Phi_{\mu,k}^{w_2}(z_{i+1}, \ldots, z_n) v_{\mu,k}
\]

\[
= q^{\mu - \text{wt}(w_1) - \text{wt}(w_2)} \sum_j (S(b_j)q^{\mu - \text{wt}(w_1) - \text{wt}(w_2)})|W_1
\]

\[
\left\langle v_{(\mu,k)}^{*} - \text{wt}(w_1) - \text{wt}(w_2), \Phi_{(\mu,k)}^{w_1} - \text{wt}(w_1)(z_1, \ldots, z_i) a_j \Phi_{\mu,k}^{w_2}(z_{i+1}, \ldots, z_n) v_{\mu,k} \right\rangle,
\]

where we note that

\[
(m_{31} \otimes 1) o ((\Delta \otimes S)q^{-\Omega}) = q^{\sum_i z_i^2 + 2cd} q^\Omega.
\]

Observe now that

\[
S_3(\Delta_1(R)) = S_3(R^{13}R^{23})
\]

\[
= \sum_{i,j} a_i \otimes a_j \otimes S(b_j) S(b_i) = \left( \sum_j 1 \otimes a_j \otimes S(b_j) \right) \left( \sum_i a_i \otimes 1 \otimes S(b_i) \right),
\]

which we may rearrange to obtain

\[
\sum_i a_i \otimes 1 \otimes S(b_i) = \left( \sum_j 1 \otimes a_j \otimes S(b_j) \right)^{-1} S_3(\Delta_1(R)).
\]

This implies that

\[
\sum_j (S(b_j)q^{\mu - \text{wt}(w_1) - \text{wt}(w_2)})|W_1
\]

\[
\left\langle v_{(\mu,k)}^{*} - \text{wt}(w_1) - \text{wt}(w_2), \Phi_{(\mu,k)}^{w_1} - \text{wt}(w_1)(z_1, \ldots, z_i) a_j \Phi_{\mu,k}^{w_2}(z_{i+1}, \ldots, z_n) v_{\mu,k} \right\rangle
\]

\[
= \left( \sum_j S(b_j) \otimes a_j \right)^{-1} \sum_j S(b_j)|W_1 q^{\mu - \text{wt}(w_1) - \text{wt}(w_2)}
\]

\[
\left\langle v_{(\mu,k)}^{*} - \text{wt}(w_1) - \text{wt}(w_2), \Phi_{(\mu,k)}^{w_1} - \text{wt}(w_1)(z_1, \ldots, z_i) a_j \Phi_{\mu,k}^{w_2}(z_{i+1}, \ldots, z_n) v_{\mu,k} \right\rangle
\]

\[
= \left( \sum_j S(b_j) \otimes a_j \right)^{-1} q^{\mu - \text{wt}(w_1) - \text{wt}(w_2)}
\]

\[
\left\langle v_{(\mu,k)}^{*} - \text{wt}(w_1) - \text{wt}(w_2), \Phi_{(\mu,k)}^{w_1} - \text{wt}(w_1)(z_1, \ldots, z_i) \Phi_{\mu,k}^{w_2}(z_{i+1}, \ldots, z_n) v_{\mu,k} \right\rangle
\]

\[
= \left( \sum_j S(b_j) \otimes a_j \right)^{-1} q^{\mu - \text{wt}(w_1) - \text{wt}(w_2)}
\]

\[
\tilde{J}_{V_1, \ldots, V_i, \ldots, z_i, z_{i+1}, \ldots, z_n, \mu, k}(w_1 \otimes w_2),
\]

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where in the second equality we notice that all terms in the sum over \( j \) aside from those of \( q^{-\Omega} \) vanish. Now by Lemma 5.2.2 we have that

\[
S_2(\mathcal{R}) = q_2^{-2\rho} \mathcal{R}^{-1} q_2^{2\rho},
\]

meaning that

\[
\left( \sum_j S(b_j) \otimes a_j \right)^{-1} = q_2^{-2\rho} \mathcal{R}^{21}_{W_1 W_2} q_2^{2\rho}.
\]

We conclude that

\[
L = q^{(\mu - \text{wt}(w_1) - \text{wt}(w_2))} q_{W_1}^{2\rho} \mathcal{R}^{21}_{W_1 W_2} q_{W_1}^{2\rho} \mathcal{J}_{V_1, V_{i+1}, \ldots, V_n}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k)(w_1 \otimes w_2).
\]

Combined with our previous expression, we find that

\[
\mathcal{R}^{21}_{W_1 W_2} q_{W_1}^{2\rho} \mathcal{J}_{V_1, V_{i+1}, \ldots, V_n}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k)(w_1 \otimes w_2) = q^{(\mu - \text{wt}(w_1) - \text{wt}(w_2))} q_{W_1}^{2\rho} \mathcal{J}_{V_1, V_{i+1}, \ldots, V_n}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k)(w_1 \otimes w_2)
\]

This coincides with the evaluation of the ABRR equation

\[
\mathcal{R}^{21}_{W_1 W_2} q_{W_1}^{2\rho} \mathcal{J}(\mu + \rho, k + h^\vee) = q(\mu + \rho, k + h^\vee) q_{W_1}^{2\rho} \mathcal{J}(\mu + \rho, k + h^\vee) q_{W_1}^{2\rho}
\]

for \( \mathcal{J}(\mu + \rho, k + h^\vee) \) in \( W_1 \otimes W_2 \). Since the solution of the ABRR equation in \( \text{End}(W_1 \otimes W_2) \) is unique, we conclude the desired equality

\[
\mathcal{J}_{V_1, V_{i+1}, \ldots, V_n}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu, k) = \mathcal{J}_{W_1 W_2}(z_1, \ldots, z_i; z_{i+1}, \ldots, z_n; \mu + \rho, k + h^\vee).
\]

**Corollary 5.4.7.** We have that

\[
\mathcal{J}_{V_1, V_n}(z_1, \ldots, z_n; \mu, k) = \mathcal{J}_{V_1[z_1^{z_1}], V_2[z_2^{z_2}] \otimes \cdots \otimes V_n[z_n^{z_n}]}(z_1, \ldots, z_n; \mu + \rho, k + h^\vee) \cdots \mathcal{J}_{V_{n-1}[z_{n-1}^{z_{n-1}}], V_n[z_n^{z_n}]}(z_{n-1}, z_n; \mu + \rho, k + h^\vee).
\]

**Proof.** This follows by repeatedly applying Proposition 5.4.6 and Lemma 5.4.2. \( \square \)

### 5.4.6 Adjoint of fusion and exchange operators

In what follows, we will often consider adjoints of fusion and exchange operators evaluated in representations. For vector spaces \( W_1, \ldots, W_m \) and an operator

\[
T \in \text{End}(W_1) \otimes \text{End}(W_m) \otimes \mathbb{C}(\{z_2/z_1, \ldots, z_n/z_{n-1}\}),
\]

\[
T \in \text{End}(W_1) \otimes \text{End}(W_m) \otimes \mathbb{C}(\{z_2/z_1, \ldots, z_n/z_{n-1}\}),
\]
we denote by

\[ T^{*W_i} \in \text{End}(W_1) \otimes \cdots \otimes \text{End}(W^*_i) \otimes \cdots \otimes \text{End}(W_m) \otimes \mathbb{C}((z_2/z_1, \ldots, z_n/z_{n-1})) \]

the operators with adjoints taken in the corresponding vector spaces. We will use the notation \( T^* := T^{*W_1 \cdots W_m} \) for the case where adjoints are taken in all vector spaces. For example, the adjoint of the iterated fusion operator is denoted by

\[ J_{V_1, \ldots, V_n}(z_1, \ldots, z_n；μ, k)^* \in \text{End}(V^*_1) \otimes \cdots \otimes \text{End}(V^*_n) \otimes \mathbb{C}((z_1/z_2, \ldots, z_{n-1}/z_n)) \]

and the adjoint in \( V \) of the universal exchange operator evaluated in \( W \otimes V \) is denoted by

\[ R_{WV}(1; z_1, \ldots, z_n；μ, k)^* \in \text{End}(W) \otimes \text{End}(V^*) \otimes \mathbb{C}((z_1/z_2, \ldots, z_{n-1}/z_n)). \]

5.5 Normalized trace functions

In this section, we define the trace function for \( U_q(\mathfrak{g}) \) and give it a normalization under which it will satisfy four systems of \( q \)-difference equations. We note that the parameter shifts in the normalization differ from those of \cite{33} due to our different choice of coproduct.

5.5.1 Unnormalized traces

Let \( V_1, \ldots, V_n \) be finite dimensional \( U_q(\mathfrak{g}) \)-representations. For \( v_i \in V_i[\tau_i] \), define the trace function \( \psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n；λ, ω, μ, k) \) by

\[ \psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n；λ, ω, μ, k) := \text{Tr}|_{M_{μ, k}}(\Phi^{v_1, \ldots, v_n}(z_1, \ldots, z_n)q^{2λ+2ωd}). \]

Define the universal trace function \( \psi^{V_1, \ldots, V_n}(z_1, \ldots, z_n；λ, ω, μ, k) \) with values in

\[ q^{2(λ, ω)}A_{μ, k} \otimes A_{λ, ω} \otimes \mathbb{C}((z_1/z_2, \ldots, z_{n-1}/z_n)) \otimes (V_1 \otimes \cdots \otimes V_n)[0] \otimes (V^*_1 \otimes \cdots \otimes V^*_n)[0] \]

by the expression

\[ \psi^{V_1, \ldots, V_n}(z_1, \ldots, z_n；λ, ω, μ, k) := \sum_{v_i \in V_i} \psi^{v_1, \ldots, v_n}(z_1, \ldots, z_n；λ, ω, μ, k) \otimes (v^*_n \otimes \cdots \otimes v^*_1), \]

where \( v_i \) and \( v_i^* \) range over dual bases of \( V_i \). Define also the single-step versions

\[ \tilde{\psi}^{v_1, \ldots, v_n}(z_1, \ldots, z_n；λ, ω, μ, k) := \text{Tr}|_{M_{μ, k}}(\tilde{\Phi}^{v_1, \ldots, v_n}(z_1, \ldots, z_n)q^{2λ+2ωd}). \]
and

$$\tilde{\Psi}^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) := \text{Tr}_{M_{\mu, k}}(\tilde{\Phi}^{V_1, \ldots, V_n}(z_1, \ldots, z_n)q^{2\lambda + 2\omega d})$$

The two versions of the universal trace function are related by the adjoint of the iterated fusion operator.

**Lemma 5.5.1.** We have that

$$\psi^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = J_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \mu, k)^* \tilde{\Psi}^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k).$$

**Proof.** This is a consequence of Lemma 5.4.1. \qed

### 5.5.2 Normalization factors

To obtain interesting difference equations on traces, it will be convenient to introduce the normalization factor

$$Q(\mu, k) := m_{21}\left(S_2(\mathcal{L}(\mu, k))\right).$$

Consider the series expansions

$$\mathcal{R} = \sum_i a_i \otimes b_i, \quad \mathcal{R}^{-1} = \sum_i a'_i \otimes b'_i,$$

$$\mathcal{J}(\mu, k) = \sum_i c_i \otimes d_i(\mu, k), \quad \mathcal{J}(\mu, k)^{-1} = \sum_i c'_i \otimes d'_i(\mu, k),$$

$$\mathcal{L}(\mu, k) = \sum_i e_i \otimes f_i(\mu, k), \quad \mathcal{L}(\mu, k)^{-1} = \sum_i e'_i \otimes f'_i(\mu, k). \quad (5.5.1)$$

Notice that

$$Q(\mu, k) = \sum_i S(f_i(\mu, k))e_i \quad \text{and} \quad S(Q(\mu, k)) = \sum_i S(e_i)S^2(f_i(\mu, k)).$$

Recall by Proposition 5.4.4 that $\mathcal{L}(\mu, k)$ is a power series in $q^{-2k}$ whose coefficients have finite degree in the standard grading, which implies that $Q(\mu, k)$ may be evaluated in any representation which is locally nilpotent with respect to the action of $U_q(\mathfrak{g})$.

**Remark.** As noted in [33, Remark 2], the definition of $Q(\mu, k)$ differs from that used in [37, 34] and is related to the definition of [34] by viewing $Q(\mu, k)$ as a renormalization of the product $S(u)^{-1}Q^{[34]}(\mu, k)$ which involves divergent sums.

### 5.5.3 Weyl denominator

Define the Weyl denominator $\delta_q(\lambda, \omega)$ by

$$\delta_q(\lambda, \omega) := \text{Tr}_{M_{-p}}(q^{2\lambda + 2\omega d})^{-1}.$$
In [31, Lemma 3.3], an explicit product formula is given for \( \delta_q(\lambda, \omega) \). In our notation, it is given by

\[
\delta_q(\lambda, \omega) = e^{(\alpha, 2\lambda + 2\omega d)} \prod_{\alpha > 0} (1 - e^{-(\alpha, 2\lambda + 2\omega d)})^{-1},
\]

(5.5.2)

where the product is over positive roots \( \alpha \).

### 5.5.4 Combined fusion operator

Define the combined fusion operator by

\[
\mathcal{J}^{1-n}(\mu, k) := \mathcal{J}^{1,2-n}(\mu, k) \cdots \mathcal{J}^{n-1,n}(\mu, k)
\]

and denote its evaluation in \( V = V_1[z_1^{\pm 1}] \otimes \cdots \otimes V_n[z_n^{\pm 1}] \) by \( \mathcal{J}^{1-n}(z_1, \ldots, z_n; \mu, k) \). These combined fusion operators commute with exchange operators in the following way.

**Lemma 5.5.2.** We have that

\[
\mathcal{R}^{0,1-n}(\mu, k)\mathcal{J}^{1-n}(\mu, k) - h(0)) = \mathcal{J}^{1-n}(\mu, k)\mathcal{R}^{01}(\mu, k) - h(2^n)) \cdots \mathcal{R}^{0n}(\mu, k).
\]

**Proof.** We induct on \( n \). The base case \( n = 1 \) is trivial. For the inductive step, consider \( V_{n-1} \otimes V_n \) as one representation to obtain by the inductive hypothesis that

\[
\mathcal{R}^{0,1-n}(\mu, k)\mathcal{J}^{1-n}(\mu, k) - h(0))
\]

\[
= \mathcal{R}^{0,1-n}(\mu, k)\mathcal{J}^{1,2-n}(\mu, k) - h(0)) \cdots \\
\mathcal{J}^{n-2,(n-1)-n}(\mu, k) - h(0))\mathcal{J}^{n-1,n}(\mu, k) - h(0))
\]

\[
= \mathcal{J}^{1-2,n}(\mu, k) \cdots \mathcal{J}^{n-2,(n-1)-n}(\mu, k)\mathcal{R}^{01}(\mu, k) - h(2^n)) \cdots \\
\mathcal{R}^{0,n-1,n}(\mu, k)\mathcal{J}^{n-1,n}(\mu, k) - h(0)).
\]

Notice now that

\[
\mathcal{R}^{0,12}(\mu, k)\mathcal{J}^{12}(\mu, k) - h(0))
\]

\[
= \mathcal{J}^{12}(\mu, k) - h(2))\mathcal{J}^{01}(\mu, k) - h(0))\mathcal{J}^{12}(\mu, k) - h(0))
\]

\[
= \mathcal{J}^{12}(\mu, k) - h(2))\mathcal{J}^{01}(\mu, k) - h(0))\mathcal{J}^{12}(\mu, k) - h(0))
\]

\[
= \mathcal{J}^{12}(\mu, k) - h(2))\mathcal{J}^{01}(\mu, k) - h(0))\mathcal{J}^{12}(\mu, k) - h(0))
\]

We conclude that

\[
\mathcal{R}^{0,1-n}(\mu, k)\mathcal{J}^{1-n}(\mu, k) - h(0))
\]

\[
= \mathcal{J}^{1-n}(\mu, k)\mathcal{R}^{01}(\mu, k) - h(2^n)) \cdots \mathcal{R}^{0,n-1}(\mu, k) - h(n))\mathcal{R}^{0n}(\mu, k).
\]
5.5.5 Normalized trace function

We now put together the normalization factors to define the desired normalized trace function as

\[ F_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) := \mathcal{Q}_{V_1^*}((\mu, k) - h^{(1)}(z_1, \ldots, z_n; \lambda, \omega)^{-1} \mathcal{Q}_{V_n^*}((\mu, k) - h^{(n)}(z_1, \ldots, z_n; \lambda, \omega)^{-1} \psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu - \rho, k - h)) \delta_q(\lambda, \omega). \quad (5.5.3) \]

5.6 Macdonald-Ruijsenaars equations

In this section, we prove the Macdonald-Ruijsenaars equations for the trace function \( F_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) by explicitly computing the radial part of an explicit central element in a completion of \( U_q(\hat{g}) \). The bulk of this section is devoted to this computation.

5.6.1 The statement

Let \( W \) be an integrable lowest weight \( U_q(\hat{g}) \)-module of non-positive integer level \( k_W \), and let \( V_1, \ldots, V_n \) be finite-dimensional \( U_q(\hat{g}) \)-modules. Define the difference operator

\[ D_W(\omega, k) := \sum_{\nu \in \mathfrak{h}^*, a \in \mathbb{C}} \text{Tr}|_{W_{[\nu+a\delta+k\omega]_la0}}(\mathbb{R}_{W_{V_1}}(1, z_1; (\lambda, \omega) - h^{(2-n)}) \mathbb{R}_{W_{V_n}}(1, z_n; \lambda, \omega)) q^{-2kaT_{\nu, k\omega}^\lambda}. \]

where \( T_{\nu, k\omega}^\lambda f(\lambda, \omega) = f(\lambda - \nu, \omega - k\omega) \). Let this operator act on functions valued in

\[ (V_1[z_1^+] \otimes \cdots \otimes V_n[z_n^1]) \otimes (V_n^* \otimes \cdots \otimes V_1^*). \]

The Macdonald-Ruijsenaars equations, whose proof occupies the remainder of this section, state that \( D_W \) is diagonalized on the normalized trace functions.

**Theorem 5.6.1** (Macdonald-Ruijsenaars equation). For any integrable lowest weight representation \( W \) of non-positive integer level \( \eta \), we have

\[ D_W(\omega, k)F_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \chi_W(q^{-2\mu - 2kd})F_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k), \]

where \( \chi_W \) is the character of \( W \).
5.6.2 Difference equations from radial parts of central elements

The proof of Theorem 5.6.1 is based on the computation of radial parts of certain central elements of \( \mathfrak{U}_q(\mathfrak{g}) \). Their existence for quantum affine algebras was shown in [22]. Let \( W \) be an integrable lowest weight \( \mathfrak{U}_q(\mathfrak{g}) \)-representation of non-positive integer level \( k_W \). Consider the element

\[ C_W := (\text{id} \otimes \text{Tr}|_W)(\mathcal{R}^{21}\mathcal{R})(1 \otimes q^{-2\tilde{\rho}}). \]

It lies in a certain completion of \( \mathfrak{U}_q(\mathfrak{g}) \), and its properties were characterized in [22].

**Proposition 5.6.2** ([22, Theorem 2]). The following is true of the elements \( C_W \):

(a) \( C_W \) is central in (a completion of) \( \mathfrak{U}_q(\mathfrak{g}) \);

(b) \( C_W \) acts by \( \chi_W(q^{-2\mu - 2kd - 2\tilde{\rho}}) \) on \( M_{\mu,k} \).

**Remark.** The action given in Proposition 5.6.2(b) has a change in sign in the character because our coproduct and antipode are different from that of [22].

**Proposition 5.6.3** ([22, Theorem 5]). For any \( W \), there is a unique difference operator \( \mathcal{M}_W \) in \( \lambda \) so that

\[ \text{Tr}|_{M_{\mu,k}}(\tilde{\Phi}_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n)C_Wq^{2\lambda + 2\omega d}) = \mathcal{M}_W \text{Tr}|_{M_{\mu,k}}(\tilde{\Phi}_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n)q^{2\lambda + 2\omega d}). \]

**Remark.** The result of [22, Theorem 5] does not have a spectral parameter and is stated for a single evaluation representation, but the statement of Proposition 5.6.3 follows from the same argument.

**Proposition 5.6.4.** The operators \( \mathcal{M}_W \) satisfy:

(a) for \( W, W' \), we have \( [\mathcal{M}_W, \mathcal{M}_{W'}] = 0 \);

(b) we have

\[ \mathcal{M}_W \tilde{\Phi}_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda, \omega, \mu, k) = \chi_W(q^{-2\mu - 2kd - 2\tilde{\rho}})\tilde{\Phi}_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda, \omega, \mu, k). \]

To prove Theorem 5.6.1, we now identify the operators \( \mathcal{M}_W \) and \( \mathcal{D}_W \) via the following steps:

1. compute universal versions of the operators \( \mathcal{M}_W \);
2. express \( \mathcal{M}_W \) in terms of an unknown coefficient \( \mathcal{G}(\lambda, \omega) \) by evaluating this universal computation;
3. characterize \( \mathcal{G}(\lambda, \omega) \) as the solution to a coproduct identity;
4. solve the coproduct identity;
5. conclude the Macdonald-Ruijsenaars equations.

We carry out these steps in the following subsections.
5.6.3 Computing universal versions of the operators $M_W$

In this subsection, we express the trace function

$$\text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) C_W q^{2\lambda + 2\omega d} \right)$$

in terms of the evaluation of a certain universal expression involving fusion matrices. Define the representations

$$V := V_1[z_1^\pm 1] \otimes \ldots \otimes V_n[z_n^\pm 1] \quad \text{and} \quad V^* := V_1^* \otimes \ldots \otimes V_n^*.$$ 

Label tensor factors of the tensor product $M_{\mu,k} \otimes V \otimes V^* \otimes U_q(\mathfrak{g}) \otimes U_q(\tilde{\mathfrak{g}})$ by 0, 1, 1*, 2, and 3 in that order.

**Lemma 5.6.5.** The trace function is given by

$$\text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) C_W q^{2\lambda + 2\omega d} \right) = \text{Tr}|_W \circ m_{23} \left( S_3 \left( \text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} (\mathcal{R}^{03})^{-1} q_0^{2\lambda + 2\omega d} \right) \right) q_3^{-2\bar{p}}. \right)$$

**Proof.** By definition the trace function is given by the trace over $M_{\mu,k} \otimes W$ of the composition

$$M_{\mu,k} \otimes W q^{2\lambda + 2\omega d} \otimes q^{-2\bar{p}} \quad \rightarrow \quad M_{\mu,k} \otimes W \mathcal{R}^{20} \mathcal{R}^{02} \quad \rightarrow \quad M_{\mu,k} \otimes W \quad \rightarrow \quad M_{\mu,k} \otimes V(z) \otimes W.$$ 

The composition above is given by

$$\pi_{W,3} \circ m_{23} \circ \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} \mathcal{R}^{03} q_3^{-2\bar{p}} q_0^{2\lambda + 2\omega d} \right),$$

so we conclude that

$$\text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) C_W q^{2\lambda + 2\omega d} \right) = \text{Tr}|_{M_{\mu,k}} \text{Tr}|_W \left( m_{23} \circ \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} \mathcal{R}^{03} q_3^{-2\bar{p}} q_0^{2\lambda + 2\omega d} \right) \right)$$

$$= \text{Tr}|_{M_{\mu,k}} \text{Tr}|_W \left( m_{23} \circ \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} \mathcal{R}^{03} q_3^{-2\bar{p}} q_0^{2\lambda + 2\omega d} \right) \right)$$

$$= \text{Tr}|_W \circ m_{23} \left( S_3 \left( \text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} (\mathcal{R}^{03})^{-1} q_0^{2\lambda + 2\omega d} \right) \right) q_3^{-2\bar{p}}. \right).$$

We will compute universal versions of the outcome of Lemma 5.6.5 in two steps. Define the quantities

$$Z_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) := \text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} q_0^{2\lambda + 2\omega d} \right)$$

$$X_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) := \text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k} V_1 \ldots V_n(z_1, \ldots, z_n) \mathcal{R}^{20} (\mathcal{R}^{03})^{-1} q_0^{2\lambda + 2\omega d} \right),$$

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and notice that \( X_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = Z_V(z_1, \ldots, z_n; (\lambda, \omega) - h(3)/2, \mu, k) + (\text{I.O.T.}) \).

We compute these quantities in the following two lemmas, whose proofs are deferred to Subsection 5.6.8.

**Lemma 5.6.6.** We have the identity
\[
Z_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \mathcal{J}^{12}(\lambda, \omega) q_2^{-kd} \tilde{\Psi}_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) - h(2)/2, \mu, k).
\]

**Lemma 5.6.7.** We have the identity
\[
X_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = q_3^{2\lambda + 2\omega d} \mathcal{J}^{12,3}(\lambda, \omega) \mathcal{J}^{12}((\lambda, \omega) + h(3)/2) q_2^{-kd} q_3^{kd} \tilde{\Psi}_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) + h(3)/2 - h(2)/2, \mu, k) \mathcal{J}^{32}(\lambda, \omega)^{-1} q_3^{-2\lambda - 2\omega d}.
\]

### 5.6.4 Evaluating the universal computations

Define the quantity
\[
G(\lambda, \omega) := \mathcal{Q}((\lambda, \omega) - h(1))^{-1} S(\mathcal{Q}(\lambda, \omega)) q^{-2\rho}.
\]

The goal of this subsection is to evaluate the result of Lemma 5.6.5 using Lemma 5.6.7 to obtain the following expression for the trace function in terms of \( G(\lambda, \omega) \).

**Proposition 5.6.8.** We have the identity
\[
\text{Tr}_{\mathcal{M}_\mu, k} \left( \tilde{\Psi}_{V_1, \ldots, V_n}(z_1, \ldots, z_n) C_W q^{2\lambda + 2\omega d} \right) = \sum_{\nu \in \mathcal{H}^*, \alpha \in \mathcal{C}} \text{Tr}_{W_{[\nu + a\delta + k_W, \lambda_W]}} \left( q^{-2k\alpha - 2ha} G(\lambda, \omega) \mathcal{R}_{WV} \left( q^{2ka - 2ha} G(\lambda, \omega) \right) \tilde{\Psi}_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda - \nu, \omega - k_W, \mu, k) \right). \tag{5.6.1}
\]

**Proof.** Recall the notations of (5.5.1) for expansions of \( \mathcal{R}, \mathcal{L}(\lambda, \omega) \), and their inverses. We rewrite Lemma 5.6.7 in terms of the renormalized fusion operator as
\[
X_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = q_3^{2\lambda + 2\omega d} \mathcal{R}^{12,3}(\lambda, \omega) \mathcal{L}^{32}(\lambda, \omega) \mathcal{R}^{21}((\lambda, \omega) + h(3)/2) q_2^{-kd} q_3^{kd} \tilde{\Psi}_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) + h(3)/2 - h(2)/2, \mu, k) \mathcal{L}^{32}(\lambda, \omega)^{-1} (\mathcal{R}^{23})^{-1} q_3^{-2\lambda - 2\omega d}.
\]

In coordinates, this means that
\[
X_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \sum_{i, j, m, n, r, s} \left( a^{(1)}_i f^{(1)}_j (\lambda, \omega) b_m e_n \otimes a^{(2)}_i f^{(2)}_j (\lambda, \omega) a_m f_n ((\lambda, \omega) + h(3)/2) q_2^{-kd} \otimes q^{2\lambda + 2\omega d} b_{i'} e_{j'} q_3^{kd} \right) \tilde{\Psi}_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) + h(3)/2 - h(2)/2, \mu, k) \left( 1 \otimes f'_r (\lambda, \omega) a'_s \otimes e'_r b'_s q^{-2\lambda - 2\omega d} \right).
\]

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Therefore, rewriting Lemma 5.6.5 as

$$\text{Tr}|_{M_{\mu,k}} \left( \Phi^{V_1,...,V_n}_{\mu,k} (z_1, \ldots, z_n) C_w q^{2\lambda + 2\omega d} \right) = \text{Tr}|_W \circ \text{m}_{23} \left( \left( S_3(X_V(z_1, \ldots, z_n; \lambda, \omega, \mu, k)) q^{-2} \right) \right),$$

we obtain by substitution and cyclic permutation of the trace that

$$\text{Tr}|_{M_{\mu,k}} \left( \Phi^{V_1,...,V_n}_{\mu,k} (z_1, \ldots, z_n) C_w q^{2\lambda + 2\omega d} \right) = \sum_{i,j,m,n,r,s} a_i^{(1)} f_j^{(1)} (\lambda, \omega) b_m e_n \text{Tr}|_W \left( a_i^{(2)} f_j^{(2)} (\lambda, \omega) a_m f_n ((\lambda, \omega) - h^{(2)}/2) \right)$$

$$= \sum_{i,j,m,n,r,s} a_i^{(1)} f_j^{(1)} (\lambda, \omega) b_m e_n \Phi^{V_1,...,V_n} (z_1, \ldots, z_n; (\lambda, \omega) - h^{(2)}, \mu, k) q^{-kd} f_i^{(2)} (\lambda, \omega) a_s q^{2\lambda + 2\omega d}$$

$$S(b'_s) S(e'_r) q^{-kd} S(e_j) S(b_i) q^{-2\lambda - 2\omega d - 2\rho}$$

We now evaluate the sum over $r, s$ explicitly in terms of $Q(\lambda, \omega)^{-1}$.

**Lemma 5.6.9.** We have the sum

$$\sum_{r,s} f_i^{(1)} (\lambda, \omega) a_s q^{2\lambda + 2\omega d} S(b'_s) S(e'_r) = Q((\lambda, \omega) - h^{(1)})^{-1} q^{2\lambda + 2\omega d} q^{-2cd} \sum_i x_i^2.$$

**Proof.** Apply $m_{21} \circ S_1$ to

$$L(\lambda, \omega)^{-1} (R^{21})^{-1} q^{-2\lambda - 2\omega d} = q^{\Omega} q_1^{-2\lambda - 2\omega d} L(\lambda, \omega)^{-1}$$

to obtain

$$\sum_{r,s} f_i^{(1)} (\lambda, \omega) a_s q^{2\lambda + 2\omega d} S(b'_s) S(e'_r) = \sum_i f_i^{(1)} (\lambda, \omega) S(e'_i) q^{2\lambda + 2\omega d} q^{-2cd} \sum_i x_i^2.$$

Now, apply $m_{321} \circ S_2$ to

$$L^{21,3}(\lambda, \omega) L^{12}((\lambda, \omega) - h^{(3)}/2) L^{23}((\lambda, \omega) + h^{(1)}/2)^{-1} = L^{1,32}(\lambda, \omega).$$

The right hand side yields 1, while the left yields

$$\sum_{i,j,l} f_i^{(1)} ((\lambda, \omega) + h^{(1)}/2) S(e'_i) S(f_j ((\lambda, \omega) - h^{(1)}/2)) S(e_l^{(1)}) e_j$$

$$= \sum_i f_i^{(1)} ((\lambda, \omega) + h^{(1)}/2) S(e'_i) Q((\lambda, \omega) - h^{(1)}/2).$$

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We conclude that
\[ \sum_i f_i^*(\lambda, \omega) S(e_i^*) = Q((\lambda, \omega) - h^{(1)})^{-1}, \]
which yields the desired upon substitution. \(\square\)

From Lemma 5.6.9, we conclude that
\[
\begin{align*}
\text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n) C_W q^{2\lambda+2\omega d} \right) &= \sum_{\nu,a} a_{i,j,m,n}^{(1)} f_{j}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k) \\
&= \sum_{\nu,a} a_{i,j,m,n}^{(1)} f_{j}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k) \\
&= \sum_{\nu,a} a_{i,j,m,n}^{(1)} f_{j}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k) \\
&= \sum_{\nu,a} a_{i,j,m,n}^{(1)} f_{j}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k)
\end{align*}
\]

We now simplify the sum over \(i,j\). Apply \(m_{32} \circ S_3\) to the identity
\[
q_3^{2\lambda+2\omega d+2\tilde{p}} R_{12,3}^{3,12}(\lambda, \omega) = q_3^{2\tilde{p}} L_{3,12}^{3,12}(\lambda, \omega) q_3^{2\lambda+2\omega d} q^{-\Omega_{23} - \Omega_{13}}
\]
from Proposition 5.4.4 to obtain
\[
\begin{align*}
\sum_{i,j} a_{i,j}^{(1)} f_{j}^{(1)}(\lambda, \omega) \otimes S(e_j) S(b_i) q^{-2\lambda - 2\omega d - 2\tilde{p}} a_i^{(2)} f_j^{(2)}(\lambda, \omega) \\
&= \sum_{i} q^0 \left( f_{i}^{(1)}(\lambda, \omega) \otimes q \sum_{i} x_i^2 + 2\omega d \right) q^{-2\lambda - 2\omega d} S(e_i) q^{-2\tilde{p}} f_i^{(2)}(\lambda, \omega)
\end{align*}
\]

We may substitute this into the previous expression to obtain
\[
\begin{align*}
\text{Tr}|_{M_{\mu,k}} \left( \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n) C_W q^{2\lambda+2\omega d} \right) &= \sum_{\nu,a} a_{i,m,n}^{(1)} f_{i}^{(1)}(\lambda, \omega) b_{m} e_{n} \\
&= \sum_{\nu,a} a_{i,m,n}^{(1)} f_{i}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k) \\
&= \sum_{\nu,a} a_{i,m,n}^{(1)} f_{i}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k) \\
&= \sum_{\nu,a} a_{i,m,n}^{(1)} f_{i}^{(1)}(\lambda, \omega) b_{m} e_{n} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k)
\end{align*}
\]

Apply \(P_{12} \circ m_{12} \circ S_1\) to the shifted 2-cocycle relation (5.4.4) rewritten in the form
\[
q_2^{-2\tilde{p}} L_{1,32}^{1,32}(\lambda, \omega) = q_2^{-2\tilde{p}} L_{21,3}^{21,3}(\lambda, \omega) L_{12}^{12}((\lambda, \omega) - h^{(3)}/2) L_{23}^{23}((\lambda, \omega) + h^{(1)}/2)^{-1}
\]

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to obtain
\[
\sum_i f_i^{(1)}(\lambda, \omega) \otimes S(e_i) q^{-2\rho} f_i^{(2)}(\lambda, \omega)
= \sum_{i,j,l} f_i(\lambda, \omega) f_j^l((\lambda, \omega) - h^{(2)}/2) \otimes S(e_j) S(e_i^{(2)}) q^{-2\rho} e_i^{(1)} f_j((\lambda, \omega) - h^{(1)}/2)e_i
= \sum_{i,j} f_i^l((\lambda, \omega) - h^{(2)}/2) \otimes S(e_j) q^{-2\rho} f_j((\lambda, \omega) - h^{(1)}/2)e_i.
\]
Substituting this into the previous expression, we obtain
\[
\text{Tr}_{M_{\mu,k}}(\Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n)C_W q^{2\lambda+2\omega d})
= \sum_{\nu,a} \sum_{i,j,m,n} f_i^l(\nu/2, \omega - k_W/2)b_m e_n \Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k)
\text{Tr}|_{W[\nu+k_W\Lambda_0+a\delta]}(q^{-2\rho} Q(\lambda - \nu, \omega - k_W)^{-1} S(e_j) q^{-2\rho} f_j((\lambda, \omega) - h^{(1)}/2)e_i ) \Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k)
= \sum_{\nu,a} \sum_{i,m,n} f_i^l(\nu/2, \omega - k_W/2)b_m e_n \text{Tr}|_{W[\nu+k_W\Lambda_0+a\delta]}(q^{-2\rho} Q(\lambda - \nu, \omega - k_W)^{-1} S(Q(\lambda, \omega)) q^{-2\rho} e_i a_m f_n(\lambda - \nu/2, \omega - k_W/2) \Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k)
= \sum_{\nu,a} \text{Tr}|_{W[\nu+k_W\Lambda_0+a\delta]}(q^{-2\rho} Q(\lambda - \nu, \omega - k_W)^{-1} S(Q(\lambda, \omega)) q^{-2\rho} L_W V(\lambda - \nu/2, \omega - k_W/2) R_W V(\lambda - \nu/2, \omega - k_W/2) \Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k)
= \sum_{\nu,a} q^{-2(\nu+k_W)} \text{Tr}|_{W[\nu+k_W\Lambda_0+a\delta]}(G(\lambda, \omega) R_W V(1; z_1,\ldots,z_n; \lambda, \omega) \Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda - \nu, \omega - k_W, \mu, k).
\]

5.6.5 The coproduct identity for $G(\lambda, \omega)$

The goal of this subsection is to show that $G(\lambda, \omega)$ satisfies a coproduct identity, meaning that it is related in a simple way to its coproduct. We first prove an analogous result for $Q(\lambda, \omega)$. 

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Lemma 5.6.10. We have
\[
\Delta(Q(\lambda, \omega)) = ((S \otimes S)\mathcal{L}(\lambda, \omega) + h^{(1)}/2 + h^{(2)}/2)^{-1}
\]
\[
\left( Q((\lambda, \omega) + h^{(2)}) \otimes Q(\lambda, \omega) \right) \mathcal{L}^{21}(\lambda, \omega) + h^{(1)}/2 + h^{(2)}/2)^{-1}.
\]

Proof. Apply \((m_{42} \otimes m_{31}) \circ S_4 \circ S_3\) to the shifted 2-cocycle relation
\[
\mathcal{L}^{21,3}(\lambda, \omega)\mathcal{L}^{12}(\lambda, \omega - h^{(3)}/2) = \mathcal{L}^{1,32}(\lambda, \omega)\mathcal{L}^{32}(\lambda, \omega + h^{(1)}/2)
\]
from Proposition 5.4.4(c). From the left hand side, we obtain
\[
\sum_{i,j} \left( S(f_i^{(2)}(\lambda, \omega))e_i^{(1)} \otimes S(f_j^{(1)}(\lambda, \omega))e_j^{(2)} \right) \left( f_j((\lambda, \omega) + h^{(1)}/2 + h^{(2)}/2) \otimes e_j \right)
\]
\[
= \Delta(Q(\lambda, \omega))\mathcal{L}^{21}(\lambda, \omega) + h^{(1)}/2 + h^{(2)}/2). \quad (5.6.2)
\]

From the right hand side, we obtain
\[
\sum_{i,j} S(f_j^{(2)}((\lambda, \omega) + h^{(2)}/2))S(f_i^{(1)(2)}(\lambda, \omega))f_i^{(2)}(\lambda, \omega)e_j \otimes
\]
\[
= \sum_{i,j} S(f_j^{(2)}((\lambda, \omega) + h^{(2)}/2))S(f_i^{(1)(2)}(\lambda, \omega))f_i^{(2)(2)}(\lambda, \omega)e_j \otimes
\]
\[
= \sum_{i,j} S(f_j^{(2)}((\lambda, \omega) + h^{(2)}/2))e_j \otimes S(f_j^{(1)(2)}((\lambda, \omega) + h^{(2)}/2))S(f_i(\lambda, \omega))e_i
\]
\[
= \sum_{i,j} S(f_j^{(2)}((\lambda, \omega) + h^{(2)}/2))e_j \otimes S(f_j^{(1)(2)}((\lambda, \omega) + h^{(2)}/2))Q(\lambda, \omega), \quad (5.6.3)
\]
where the first equality follows by coassociativity and the second by the relation
\(m_{12} \circ S_1 \circ \Delta = \eta \circ \varepsilon\) in a Hopf algebra. Now, applying \(m_{21} \circ S_2 \circ S_3\) to the cocycle relation, we find for the left hand side
\[
\sum_{i,j} S(f_j((\lambda, \omega) + h^{(2)}/2))S(e_i^{(1)})e_i^{(2)}e_j \otimes S(f_i(\lambda, \omega)) = Q((\lambda, \omega) + h^{(2)}/2) \otimes 1 \quad (5.6.4)
\]
and for the right hand side
\[
\sum_{i,j} S(e_j)S(f_j^{(2)}(\lambda, \omega))e_i \otimes S(f_j((\lambda, \omega) + h^{(1)}/2))S(f_i^{(1)}(\lambda, \omega))
\]
\[
= ((S \otimes S)\mathcal{L}(\lambda, \omega) + h^{(1)}/2) \sum_i S(f_i^{(2)}(\lambda, \omega))e_i \otimes S(f_i^{(1)}(\lambda, \omega)). \quad (5.6.5)
\]
Equating (5.6.4) and (5.6.5), we obtain

\[ \sum_i S(f^{(2)}_i(\lambda, \omega)) e_i \otimes S(f^{(1)}_i(\lambda, \omega)) \]
\[ = ((S \otimes S) \mathcal{L}((\lambda, \omega) + h^{(1)}/2))^{-1}(Q((\lambda, \omega) + h^{(2)}/2) \otimes 1). \]  \hspace{1cm} (5.6.6)

Equating (5.6.2) and (5.6.3) and substituting (5.6.6), we conclude that

\[ \Delta(Q(\lambda, \omega)) = ((S \otimes S) \mathcal{L}((\lambda, \omega) + h^{(1)}/2 + h^{(2)}/2)^{-1} \]
\[ \times \left( Q((\lambda, \omega) + h^{(2)}) \otimes Q(\lambda, \omega) \right) \mathcal{L}^{21}((\lambda, \omega) + h^{(1)}/2 + h^{(2)}/2)^{-1}. \]  \hspace{1cm} \Box

We are now ready to prove the following coproduct identity for \( \mathbb{G}(\lambda, \omega) \).

**Lemma 5.6.11.** We have

\[ \Delta(\mathbb{G}(\lambda, \omega)) = \mathbb{L}^{21}(\lambda, \omega) \left( \mathbb{G}(\lambda, \omega) \otimes \mathbb{G}((\lambda, \omega) - h^{(1)}) \right) \mathbb{L}^{21}(\lambda, \omega)^{-1}. \]

**Proof.** Recall that \( \mathbb{G}(\lambda, \omega) = Q(\lambda - h^{(1)}, \omega)^{-1} S(Q(\lambda, \omega)) q^{-2p} \), so we have that

\[ \Delta(\mathbb{G}(\lambda, \omega)) = \Delta(Q((\lambda, \omega) - h^{(1)})^{-1} \left( (S \otimes S)(\Delta^{21}(Q(\lambda, \omega))) \right)(q^{-2p} \otimes q^{-2p}). \]

By Lemma 5.6.10, we have that

\[ \Delta(Q((\lambda, \omega) - h^{(1)})^{-1} \]
\[ = \mathbb{L}^{21}((\lambda, \omega) - h^{(1)}/2 - h^{(2)}/2) \left( Q((\lambda, \omega) - h^{(1)})^{-1} \otimes Q((\lambda, \omega) - h^{(1)} - h^{(2)})^{-1} \right) \]
\[ ((S \otimes S) \mathcal{L}((\lambda, \omega) - h^{(1)}/2 - h^{(2)}/2) \]
\[ = \mathbb{L}^{21}(\lambda, \omega) \left( Q((\lambda, \omega) - h^{(1)})^{-1} \otimes Q((\lambda, \omega) - h^{(1)} - h^{(2)})^{-1} \right) ((S \otimes S) \mathbb{L}(\lambda, \omega)) \]

and that

\[ (S \otimes S)(\Delta^{21}(Q(\lambda, \omega))) \]
\[ = \left( (S \otimes S) \mathcal{L}((\lambda, \omega) - h^{(1)}/2 - h^{(2)}/2)^{-1} \right) \]
\[ \left( S(Q(\lambda, \omega)) \otimes S(Q((\lambda, \omega) - h^{(1)})) \right)(S^2 \otimes S^2) \mathcal{L}^{21}((\lambda, \omega) - h^{(1)}/2 - h^{(2)}/2)^{-1} \]
\[ = \left( (S \otimes S) \mathbb{L}(\lambda, \omega)^{-1} \right) \left( S(Q(\lambda, \omega)) \otimes S(Q((\lambda, \omega) - h^{(1)})) \right) \mathbb{L}^{21}(\lambda, \omega)^{-1}, \]

where we use that \( \mathbb{L}^{21}(\lambda, \omega)^{-1} \) has weight 0. We conclude that

\[ \Delta(\mathbb{G}(\lambda, \omega)) = \mathbb{L}^{21}(\lambda, \omega) \left( \mathbb{G}(\lambda, \omega) \otimes \mathbb{G}((\lambda, \omega) - h^{(1)}) \right) \mathbb{L}^{21}(\lambda, \omega)^{-1}. \]  \hspace{1cm} \Box

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5.6.6 Solving the coproduct identity

The goal of this subsection is to prove Lemma 5.6.15 on the value of the \( G(\lambda, \omega) \). Recall that \( G(\lambda, \omega) \) has weight 0 and may be evaluated on any representation which is locally nilpotent with respect to the \( U_q(\mathfrak{g}) \)-action. In particular, letting \( L_{\mu, k} \) denote the irreducible integrable representation of \( U_q(\mathfrak{g}) \) with highest weight \( \mu + k\Lambda_0 \), we may define the function \( \eta_{\mu, k}(\lambda, \omega) \) to be the eigenvalue of \( G(\lambda, \omega) \) on the highest weight vector of \( L_{\mu, k} \). Our method proceeds by finding \( \eta_{\mu, k}(\lambda, \omega) \) and showing that \( G(\lambda, \omega) \) is determined by it. First, we derive a compatibility relation for \( \eta_{\mu, k}(\lambda, \omega) \).

**Lemma 5.6.12.** For dominant integral weights \( \mu_1 + k_1 \Lambda_0 \) and \( \mu_2 + k_2 \Lambda_0 \), the function \( \eta_{\mu_1, k_1}(\lambda, \omega) \) satisfies the zero-curvature relation

\[
\eta_{\mu_1, k_1}(\lambda, \omega)\eta_{\mu_2, k_2}(\lambda - \mu_1, \omega - k_1) = \eta_{\mu_2, k_2}(\lambda, \omega)\eta_{\mu_1, k_1}(\lambda - \mu_2, \omega - k_2). \tag{5.6.7}
\]

**Proof.** By [75, Theorem 6.2.2], the category of integrable highest-weight representations of \( U_q(\mathfrak{g}) \) is semisimple, meaning in particular that \( L_{\mu_1 + \mu_2, k_1 + k_2} \) is the subrepresentation in \( L_{\mu_1, k_1} \otimes L_{\mu_2, k_2} \) and \( L_{\mu_2, k_2} \otimes L_{\mu_1, k_1} \) generated by the highest weight vector. Therefore, by Lemma 5.6.11 and the fact that \( L_{\mu_1, k_1}(\lambda, \omega) \in q^{\Omega} \left( 1 + U_q(\mathfrak{b}_-)_0 \otimes U_q(\mathfrak{b}_+)_0 \right) \), both sides of the desired are equal to \( \eta_{\mu_1 + \mu_2, k_1 + k_2}(\lambda, \omega) \), hence equal. \( \square \)

We now work with the formal expansion of \( G(\lambda, \omega) \). Let \( q = e^h \) for the formal parameter \( h \), and work over the ring \( \mathbb{C}[[h]] \). From the compatibility relation, we now constrain the formal expansion of

\[
\tilde{\eta}_{\mu, k}(\lambda, \omega) := \eta_{\mu, k}\left(\frac{\lambda}{h}, \frac{\omega}{h}\right) \in \mathbb{C}[[h]].
\]

Notice that \( \lim_{h \to 0} \tilde{\eta}_{\mu, k}(\lambda, \omega) = 1 \), since \( \lim_{h \to 0} G(\lambda, \omega) = 1 \). The proof of [30, Lemma 7.56] applies verbatim to the affine setting, hence Lemma 5.6.12 implies the following lemma, which allows us to constrain the form of \( G(\lambda, \omega) \).

**Lemma 5.6.13.** We may find some function \( f(\lambda, \omega) \in \mathbb{C}[[h]] \) so that

\[
\tilde{\eta}_{\mu, k}(\lambda, \omega) = \frac{f(\lambda - h\mu, \omega - hk)}{f(\lambda, \omega)}
\]

as a formal series in \( h \).

**Lemma 5.6.14.** As formal series in \( h \), we have

\[
G\left(\frac{\lambda}{h}, \frac{\omega}{h}\right) = \frac{f(\lambda, \omega) - hh^{(1)}}{f(\lambda, \omega)}.
\]

**Proof.** By Lemma 5.6.13, the renormalized element

\[
\tilde{G}(\lambda, \omega) := \tilde{G}\left(\frac{\lambda}{h}, \frac{\omega}{h}\right) = \frac{f(\lambda, \omega)}{f(\lambda, \omega) - hh^{(1)}}
\]

is a formal series in \( h \).
acts by 1 on the highest weight vector of any highest weight irreducible integrable representation. If \( \tilde{G}(\lambda, \omega) \neq 1 \), let its formal expansion take the form

\[
\tilde{G}(\lambda, \omega) = 1 + h^n g(\lambda, \omega) + O(h^{n+1})
\]

for some non-zero \( g(\lambda, \omega) \in U(\widehat{g}) \). By Lemma 5.6.11, we have that

\[
\Delta \left( 1 + h^n g(\lambda, \omega) + O(h^{n+1}) \right)
\]

\[
= L^{21} \left( \frac{\lambda}{h}, \frac{\omega}{h} \right) (1 + h^n g(\lambda, \omega) + O(h^{n+1})) \otimes (1 + h^n g(\lambda, \omega) + O(h^{n+1})) \big| L^{21} \left( \frac{\lambda}{h}, \frac{\omega}{h} \right)^{-1},
\]

hence canceling terms, dividing by \( h^n \), and looking modulo \( hU_q(\widehat{g}) \) yields

\[
\Delta_0 (g(\lambda, \omega)) = g(\lambda, \omega) \otimes 1 + 1 \otimes g(\lambda, \omega),
\]

where \( \Delta_0 \) is the coproduct of \( U(\widehat{g}) \) and both sides are considered modulo \( hU_q(\widehat{g}) \). Let \( \overline{g}(\lambda, \omega) \) be the class of \( g(\lambda, \omega) \) modulo \( hU_q(\widehat{g}) \); this implies that \( \overline{g}(\lambda, \omega) \in \widehat{g} \). On the other hand, \( g(\lambda, \omega) \) has zero weight, so \( \overline{g}(\lambda, \omega) \in \widehat{h} \), which implies that it is 0 since \( g(\lambda, \omega) \) and hence \( \overline{g}(\lambda, \omega) \) vanishes on all highest weight vectors of highest weight irreducible integrable representations. This is a contradiction, so \( \tilde{G}(\lambda, \omega) = 1 \), as desired.

We are finally ready to compute \( G(\lambda, \omega) \) by computing the value of \( f(\lambda, \omega) \).

**Lemma 5.6.15.** The value of \( G(\lambda, \omega) \) is

\[
G(\lambda, \omega) = \frac{\delta_q((\lambda, \omega) - h^{(1)})}{\delta_q(\lambda, \omega)}.
\]

**Proof.** It suffices to check that

\[
\frac{f((\lambda, \omega) - h^{(1)})}{f(\lambda, \omega)} \cdot \frac{\delta_q((\frac{\lambda}{h}, \frac{\omega}{h}) - h^{(1)})}{\delta_q(\frac{\lambda}{h}, \frac{\omega}{h})},
\]

as this formal equality implies the desired equality at \( h = 1 \). Applying Proposition 5.6.8 to \( V = \mathbb{C} \) and noting that \( C_W \) acts on \( M_{\mu,k} \) by \( \chi_W(q^{-2\mu - 2kd - 2\rho}) \) by Proposition 206.
5.6.2, we obtain as formal series in $\hbar$ that
\[
\chi_{W}(q^{-2\mu-2kd-2\bar{\rho}})\overline{\Psi}^{C}(z; \lambda/\hbar, \omega/\hbar, \mu, k)
\]
\[
= \sum_{\nu \in \hbar^*, a \in \mathbb{C}} \text{Tr}_{W[\nu + a\delta + kW\Lambda_0]} \left( q^{-2ka-2ha} G(\lambda/\hbar, \omega/\hbar) \overline{R_{W}}(1, z; \lambda/\hbar, \omega/\hbar) \right)
\]
\[
\overline{\Psi}^{C}(z; \lambda/\hbar - \nu, \omega/\hbar - kW, \mu, k)
\]
\[
= \sum_{\nu \in \hbar^*, a \in \mathbb{C}} \text{Tr}_{W[\nu + a\delta + kW\Lambda_0]} \left( q^{-2ka-2ha} f(\lambda - h\nu, \omega - hkW) \right)
\]
\[
\overline{\Psi}^{C}(z; \lambda/\hbar - \nu, \omega/\hbar - kW, \mu, k).
\]

Notice now that
\[
\overline{\Psi}^{C}(z; \lambda, \omega, \mu, k) = \text{Tr}_{M_{\mu,k}}(q^{2\lambda + 2\omega d}) = \frac{q^{2(\mu + kd + \bar{\rho})}}{\delta_q(\lambda, \omega)}
\]
and therefore that
\[
\chi_{W}(q^{-2\mu-2kd-2\bar{\rho}}) = \sum_{\nu \in \hbar^*, a \in \mathbb{C}} \dim W[\nu + a\delta + kW\Lambda_0] q^{-2(\mu + kd + \bar{\rho} + \nu + a\delta + kW\Lambda_0)}
\]
so equating coefficients of power series in $q^{-2\mu-2kd-2\bar{\rho}}$ yields the desired
\[
\frac{f(\lambda, \omega) - h^{(1)}}{f(\lambda, \omega)} = \frac{\delta_q \left( \left( \frac{\lambda}{\hbar}, \frac{\omega}{\hbar} \right), \frac{\lambda}{\hbar}, \frac{\omega}{\hbar} \right)}{\delta_q \left( \left( \frac{\lambda}{\hbar}, \frac{\omega}{\hbar} \right), \frac{\lambda}{\hbar}, \frac{\omega}{\hbar} \right)}.
\]

5.6.7 The proof of the Macdonald-Ruijsenaars equations

We are now ready to deduce the Macdonald-Ruijsenaars equations from Proposition 5.6.8 and Lemma 5.6.15. Recalling the action of $C_{W}$ on $M_{\mu,k}$ from Proposition 5.6.2, we find that
\[
\chi_{W}(q^{-2\mu-2kd-2\bar{\rho}})\overline{\Psi}^{V_1,...,V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \sum_{\nu \in \hbar^*, a \in \mathbb{C}} q^{-2(k+h)a} \delta_q (\lambda - \nu, \omega - kW) \overline{\Psi}^{V_1,...,V_n}(z_1, \ldots, z_n; \lambda - \nu, \omega - kW, \mu, k).
\]
By Lemma 5.5.2, we have that

\[
\mathbb{R}_{WV}(1; z_1, \ldots, z_n; \lambda, \omega) = J_{V_i[z_1^{\pm 1}], V_j[z_2^{\pm 1}]} \cdots V_n[z_n^{\pm 1}](z_1; z_2, \ldots, z_n; \lambda, \omega) \cdots J_{V_{n-1}[z_{n-1}^{\pm 1}], V_n[z_n^{\pm 1}]}(z_{n-1}; z_n; \lambda, \omega) \mathbb{R}_{WV_1}(1, z_1; \lambda, \omega - h(2 \cdots n)) \cdots \mathbb{R}_{WV_n}(1, z_n; \lambda, \omega - k_W)
\]

\[
\left( J_{V_i[z_1^{\pm 1}], V_j[z_2^{\pm 1}]} \cdots V_n[z_n^{\pm 1}](z_1; z_2, \ldots, z_n; \lambda - \nu, \omega - k_W) \right)^{-1} \cdots J_{V_{n-1}[z_{n-1}^{\pm 1}], V_n[z_n^{\pm 1}]}(z_{n-1}; z_n; \lambda - \nu, \omega - k_W) \mathbb{R}_{WV_n}(1, z_n; \lambda, \omega - k_W)
\]

Multiplying both sides by \( J_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \mu, k)^* \), applying Lemma 5.5.1, and substituting in, we obtain

\[
\chi_W(q^{-2} - 2kd - 2\omega) J^{1 \cdots n}(z_1, \ldots, z_n; \lambda; \omega)^{-1} \delta_q(\lambda; \omega) \Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \sum_{\nu \in \mathfrak{h}^*, \rho \in \mathfrak{c}} q^{-2(k+h)aV-2kd} \mathbb{R}_{WV_1}(1, z_1; \lambda, \omega - h(2 \cdots n)) \cdots \mathbb{R}_{WV_n}(1, z_n; \lambda, \omega - k_W) J^{1 \cdots n}(z_1, \ldots, z_n; \lambda - \nu, \omega - k_W)^{-1} \delta_q(\lambda - \nu, \omega - k_W) \Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda - \nu, \omega - k_W, \mu, k).
\]

Recalling the definition of the normalized trace \( F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) now yields

\[
D_W(\omega, k)F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \chi_W(q^{-2} - 2kd) F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k).
\]

### 5.6.8 Computations for the Macdonald-Ruijsenaars equations

In this subsection we give proofs of Lemmas 5.6.6 and 5.6.7.

**Proof of Lemma 5.6.6.** Label the tensor factors of \( M_{\mu, k} \otimes V \otimes V^* \otimes U_q(\tilde{g}) \otimes U_q(\tilde{g}) \) by 0, 1, 1*, 2, and 3 in that order. By moving \( R^{20} \) around the trace, we have

\[
Z_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \text{Tr}|_{M_{\mu, k}} \left( \widetilde{\Phi}_{\mu, k}^{V_1, \ldots, V_n}(z_1, \ldots, z_n) R^{20} q_0^{2\lambda + 2\omega d} \right)
\]

\[
= q_2^{2\lambda + 2\omega d} \text{Tr}|_{M_{\mu, k}} \left( \widetilde{\Phi}_{\mu, k}^{V_1, \ldots, V_n}(z_1, \ldots, z_n) q_0^{2\lambda + 2\omega d} R^{20} \right) q_2^{-2\lambda - 2\omega d}
\]

\[
= q_2^{2\lambda + 2\omega d} \text{Tr}|_{M_{\mu, k}} \left( R^{20} \widetilde{\Phi}_{\mu, k}^{V_1, \ldots, V_n}(z_1, \ldots, z_n) q_0^{2\lambda + 2\omega d} \right) q_2^{-2\lambda - 2\omega d}
\]

\[
= q_2^{2\lambda + 2\omega d} \text{Tr}|_{M_{\mu, k}} \left( (R^{21})^{-1} \widetilde{\Phi}_{\mu, k}^{V_1, \ldots, V_n}(z_1, \ldots, z_n) R^{20} q_0^{2\lambda + 2\omega d} \right) q_2^{2\lambda - 2\omega d}
\]

\[
= q_2^{2\lambda + 2\omega d} (R^{21})^{-1} Z_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) q_2^{2\lambda - 2\omega d}.
\]

where we note that \( (R^{21})^{-1} \widetilde{\Phi}_{\mu, k}^{V_1, \ldots, V_n}(z_1, \ldots, z_n) R^{20} = R^{20} \widetilde{\Phi}_{\mu, k}^{V_1, \ldots, V_n}(z_1, \ldots, z_n). \) Denote
the claimed expression for $Z_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$ by $Z'_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$. Notice that

$$q_2^{2\lambda+2\omega d} (R^{21})^{-1} Z'_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega,\mu,k) q_2^{-2\lambda-2\omega d} = q_2^{2\lambda+2\omega d} (R^{21})^{-1} J^{12} (\lambda,\omega) q_2 q_2^{-2\lambda-2\omega d}$$

$$= q_2^{2\lambda+2\omega d} q_1^{2\lambda+2\omega d} J^{12} (\lambda,\omega) q_1^{-2\lambda-2\omega d} q^{-2\lambda-2\omega d} q_2^{-2\lambda-2\omega d}$$

$$\tilde{\Psi}_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega) - h^{(2)}/2,\mu,k) q_2^{-2\lambda-2\omega d}$$

$$= J^{12} (\lambda,\omega) q^{-2\lambda-2\omega d} \tilde{\Psi}_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega) - h^{(2)}/2,\mu,k) q_2^{-2\lambda-2\omega d}$$

$$= Z'_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega,\mu,k),$$

where we used the ABRR equation and applied

$$q_1^{2\lambda+2\omega d} \tilde{\Psi}_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega) - h^{(2)}/2,\mu,k) = \tilde{\Psi}_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega) - h^{(2)}/2,\mu,k).$$

Notice now that the weight 0 term in tensor factor 2 for $Z_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$ is given by

$$\text{Tr} M_{\mu,k} \left( \tilde{\Phi}_{\mu,k,v_n}(z_1,\ldots,z_n) q_0^{-1} q_0^{-2\lambda-2\omega d} \right) = \tilde{\Psi}_{v_1,\ldots,v_n} (z_1,\ldots,z_n;\lambda,\omega) - h^{(2)}/2,\mu,k) q_2^{-2\lambda-2\omega d}.$$

We conclude that both $Z_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$ and $Z'_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$ are solutions to

$$q_2^{2\lambda+2\omega d} (R^{21})^{-1} Z = Z q_2^{2\lambda+2\omega d}$$

whose weight 0 term in tensor factor 2 is $\tilde{\Psi}_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega) - h^{(2)}/2,\mu,k) q_2^{-2\lambda-2\omega d}$, hence they are equal.\[\square\]

**Proof of Lemma 5.6.7.** By moving $(R^{03})^{-1}$ around the trace, we obtain

$$X_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$$

$$= q_3^{2\lambda+2\omega d} \text{Tr} M_{\mu,k} \left( \tilde{\Phi}_{\mu,k,v_n}(z_1,\ldots,z_n) R^{20} q_0^{2\lambda+2\omega d} (R^{03})^{-1} \right) q_3^{-2\lambda-2\omega d}$$

$$= q_3^{2\lambda+2\omega d} \text{Tr} M_{\mu,k} \left( R^{03} \tilde{\Phi}_{\mu,k,v_n}(z_1,\ldots,z_n) R^{20} q_0^{2\lambda+2\omega d} \right) q_3^{-2\lambda-2\omega d}$$

$$= q_3^{2\lambda+2\omega d} R^{13} X_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k) (R^{23})^{-1} q_3^{-2\lambda-2\omega d}$$

where we note that

$$(R^{03})^{-1} \tilde{\Phi}_{\mu,k,v_n}(z_1,\ldots,z_n) = R^{13} \tilde{\Phi}_{\mu,k,v_n}(z_1,\ldots,z_n) (R^{03})^{-1}$$

and

$$(R^{03})^{-1} R^{20} = R^{23} R^{03} (R^{23})^{-1}.$$

Now, define $X'_{v_1,\ldots,v_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$ to be the claimed expression for the quan-
We have that
\[ q_3^{2\lambda + 2\omega d}R^{13}R^{23} X'_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \] 
\[ = q_3^{2\lambda + 2\omega d}R^{12,3}R^{23} R^{3,12}((\lambda, \omega) + h^{(3)}/2)q_2^{kd}q_3^{kd} \]
\[ = q_3^{2\lambda + 2\omega d}R^{12}R^{23}((\lambda, \omega) + h^{(3)}/2)q_2^{kd}q_3^{kd} \]
\[ = q_3^{2\lambda + 2\omega d}R^{12}R^{23}((\lambda, \omega) + h^{(3)}/2)q_2^{kd}q_3^{kd} \]
\[ = q_3^{2\lambda + 2\omega d}R^{12}R^{23}((\lambda, \omega) + h^{(3)}/2)q_2^{kd}q_3^{kd} \]
\[ = q_3^{2\lambda + 2\omega d}R^{12}R^{23}((\lambda, \omega) + h^{(3)}/2)q_2^{kd}q_3^{kd} \]
\[ = X'_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k), \]
where we used that \( R^{13}R^{23} = R^{12,3} \) and
\[ q^{-\Omega_{13}} \Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) + (h^{(3)} - h^{(2)})/2, \mu, k) \]
\[ = \Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) + (h^{(3)} - h^{(2)})/2, \mu, k). \]

Therefore, both \( X_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) and \( X'_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) are solutions to
\[ q_3^{2\lambda + 2\omega d}R^{13}R^{23} Z = Z q_3^{2\lambda + 2\omega d}R^{23}. \]

Define the quantities
\[ Y_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \]
\[ = R^{3,12}((\lambda, \omega) - h^{(3)}/2, \mu, k)q_3^{2\lambda + 2\omega d} R^{32}((\lambda, \omega) - h^{(3)}/2, \mu, k) \]
\[ = q_3^{2\lambda + 2\omega d}R^{12}R^{23}((\lambda, \omega) - h^{(3)}/2, \mu, k) \]
so that both \( Y_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) and \( Y'_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) are solutions to
\[ q_3^{2\lambda + 2\omega d}q^{-\Omega_{13,3}} Z = Z q_3^{2\lambda + 2\omega d}q^{-\Omega_{13,3}} \]
with weight 0 term in the third tensor factor given by
\[ R^{12}((\lambda, \omega) + h^{(3)}/2)q_2^{kd}q_3^{kd} \Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; (\lambda, \omega) + (h^{(3)} - h^{(2)})/2, \mu, k) \]
by Lemma 5.6.6, yielding the conclusion. \( \square \)

### 5.7 Dual Macdonald-Ruijsenaars equations

In this section we prove the dual Macdonald-Ruijsenaars equations for the trace function \( F_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \). Our method proceeds by showing that two naturally defined intertwiners are related by an application of a dynamical R-matrix, after which the result follows by computing the trace of a single intertwiner in two
different ways.

5.7.1 The statement

Let $W$ be an integrable lowest-weight $U_q(\mathfrak{g})$-module of level $k_W$, and define the difference operator

$$D_W^\nu(\omega, k) = \sum_{\nu \in \mathbb{H}^+, a \in \mathbb{C}} \text{Tr}|W|_{\nu+a\delta+k_W\Lambda_0}\left(\mathbb{R}_{WV_\nu^*}(1, z_n; (\mu, k) - h^{(1,\ldots,n-1)})\right)\ldots$$

where $T_{\nu,k_W}^\mu f(\mu, k) = f(\mu - \nu, k - k_W)$. Let this operator act on functions valued in

$$(V_1[z_1^+]) \otimes \cdots \otimes V_n[z_n^+] \otimes (V_1^* \otimes \cdots \otimes V_n^*),$$

where we interpret $\mathbb{R}_{WV_\nu^*}(1, z_m; \mu, k)$ as the evaluation of the universal fusion matrix on $W \otimes V_m[z_m^+]$. The dual Macdonald-Ruijsenaars equations state that $D_W^\nu(\omega, k)$ are diagonalized on renormalized trace functions.

**Theorem 5.7.1 (dual Macdonald-Ruijsenaars equation).** For any integrable lowest weight representation $W$ of non-positive integer level $k_W$, we have

$$D_W^\nu(\omega, k) F_{V_1,\ldots,V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \chi_W(q^{-2\omega - 2\omega_d}) F_{V_1,\ldots,V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k),$$

where $\chi_W$ is the character of $W$.

5.7.2 Computing an intertwiner in two different ways

Suppose that $(\mu, k)$ is generic so that all Verma modules $M_{\lambda,k}$ with highest weight given by a shift of $\mu + k\Lambda_0$ by an integral weight are irreducible. Let $W$ be a highest weight irreducible integrable module of level $k_W$. By Proposition 5.3.4, we have an isomorphism of $U_q(\mathfrak{g})$-modules

$$\eta : \bigoplus_{\lambda,a} W[\lambda - \mu + k_W\Lambda_0 - a\delta] \otimes M_{\lambda+(k+k_W)\Lambda_0-a\delta} \rightarrow M_{\mu,k} \otimes W.$$

For finite-dimensional $U_q(\mathfrak{g})$-representations $V_1, \ldots, V_n$, consider the representations

$$V := V_1[z_1^+] \otimes \cdots \otimes V_n[z_n^+] \quad \text{and} \quad \bar{V} := V_1((z_1)) \otimes \cdots \otimes V_n((z_n)).$$

We abuse notation to also define the vector space

$$V^* := V_1^* \otimes \cdots \otimes V_n^*.$$

This isomorphism will provide two natural ways of constructing an intertwiner

$$M_{\mu,k} \otimes W \rightarrow M_{\mu,k} \otimes W \bar{\otimes} \bar{V} \otimes V^*,$$

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which will be related by the application of a dynamical $R$-matrix.

**Proposition 5.7.2.** For any finite-dimensional $U_q(\hat{\mathfrak{g}})$-representation $V$, the following diagram commutes.

\[
\begin{array}{c}
\oplus_{\nu,a} W[\nu - \mu + k_W \Lambda_0 - a\delta] \otimes M_{\nu,k+k_W,-a} \\
\oplus_{\nu,a} W[\nu - \mu + k_W \Lambda_0 - a\delta] \otimes M_{\nu,k+k_W,-a} \otimes \bar{V} \otimes V^* \\
\oplus_{\nu,a} W[\nu - \mu + k_W \Lambda_0 - a\delta] \otimes M_{\nu,k+k_W,-a} \otimes \bar{V} \otimes V^* \\
\end{array}
\]

\[
\begin{array}{c}
\Phi_{\mu,k} \otimes W \\
\Phi_{\mu,k} \otimes V \otimes W \otimes V^* \\
\Phi_{\mu,k} \otimes V \otimes W \otimes V^* \\
\end{array}
\]

\[
\begin{array}{c}
\oplus_{\nu,a} W[\nu - \mu + k_W \Lambda_0 - a\delta] \otimes M_{\nu,k+k_W,-a} \\
\oplus_{\nu,a} W[\nu - \mu + k_W \Lambda_0 - a\delta] \otimes M_{\nu,k+k_W,-a} \otimes \bar{V} \otimes V^* \\
\oplus_{\nu,a} W[\nu - \mu + k_W \Lambda_0 - a\delta] \otimes M_{\nu,k+k_W,-a} \otimes \bar{V} \otimes V^* \\
\end{array}
\]

\[
\begin{array}{c}
P_{\bar{V}W} \mathcal{R}_{VW} \\
P_{\bar{V}W} \mathcal{R}_{VW} \\
P_{\bar{V}W} \mathcal{R}_{VW} \\
\end{array}
\]

**Proof.** For each choice of $(\nu, a)$ and $w \in W[\nu - \mu + k_W \Lambda_0 - a\delta]$, restricting each branch of the diagram to $w \otimes M_{\nu,k+k_W,-a}$ gives two intertwiners $M_{\nu,k+k_W,-a} \to M_{\mu,k} \otimes W \otimes \bar{V} \otimes V^*$, where we note that the top branch is an intertwiner by Lemma 5.3.3. To check that these intertwiners are equal, it suffices by Proposition 5.3.4 to check that they have the same highest term. Suppose that

\[
\mathbb{R}_{WV}(1; z_1, \ldots, z_n; \nu + \rho, k + k_W + h^\vee)^{*V} = \sum_j p_j \otimes q_j,
\]

where $p_j \in \text{End}(W)$ and $q_j(z) \in \text{End}(V^*)((z_1, \ldots, z_n))$, and let \{\(v_i\)\} and \{\(v_i^*\)\} be dual bases of $V_1 \otimes \cdots \otimes V_n$ and $V^*$. Applying Proposition 5.4.6, the highest term of the top branch is given by

\[
\left\langle P_{VW} \mathcal{R}_{VW} \Phi_{\mu,k}^{V_1,\ldots,V_n}(z_1, \ldots, z_n) \Phi_{\nu,k+k_W,-a}^W \right\rangle \\
= \sum_i P_{VW} \mathcal{R}_{VW} \mathcal{J}_{VW}(z_1, \ldots, z_n; 1; \nu + \rho, k + k_W + h^\vee)(v_i \otimes w) \otimes v_i^* \\
= \sum_i \mathcal{R}_{WV}^{2i} \mathcal{J}_{WV}^{2i}(1; z_1, \ldots, z_n; \nu + \rho, k + k_W + h^\vee)(w \otimes v_i) \otimes v_i^*. \quad (5.7.1)
\]
On the other hand, the highest term of the top branch is given by
\[
\sum_{i,j} \langle \Phi_{\mu,k}^{\rho,w} \Phi_{\nu,k+kW,-a}(z_1, \ldots, z_n) \rangle \otimes g_j v_i^* \\
= \sum_j J_{WV}(1; z_1, \ldots, z_n; \nu + \rho, k + kW + h^\gamma) R_{WV}(1; z_1, \ldots, z_n; \nu, k + kW)(w \otimes v_i) \otimes v_i^* \\
= \sum_i R_{WV}^{21} J_{WV}^{21}(1; z_1, \ldots, z_n; \nu + \rho, k + kW + h^\gamma)(w \otimes v_i) \otimes v_i^*. 
\]  
(5.7.2)

Comparing (5.7.1) and (5.7.2) yields the desired.

5.7.3 Computing the double dual of the dynamical $R$-matrix

We require also the following computation of the double dual of the dynamical $R$-matrix.

Lemma 5.7.3. The value of $R_{WV}(1; z_1, \ldots, z_n; \mu, k)^{W \ast V}$ is given by
\[
R_{WV}(1; z_1, \ldots, z_n; \mu, k)^{W \ast V} = \left( Q(\mu, k) \otimes Q((\mu, k) + h^{(1)}) \right) \\
R_{W \ast V}(1; z_1, \ldots, z_n; (\mu, k) + h^{(1)} + h^{(2)})(Q((\mu, k) + h^{(2)})^{-1} \otimes Q(\mu, k)^{-1}).
\]

Proof. Recalling that
\[
R(\mu, k) = L(\mu, k)^{-1} R L^{21}(\mu, k),
\]
we see that
\[
R_{WV}(1; z_1, \ldots, z_n; \mu, k)^{W \ast V} = (S^{-1} \otimes S^{-1})(L^{21}(\mu, k))|_{W^\ast \otimes (V_{\ast}^{[x^+]} \otimes \cdots \otimes V_{\ast}^{[y^+]})} \\
(S^{-1} \otimes S^{-1})(R_{W \ast V})(S^{-1} \otimes S^{-1})(L(\mu, k))^{-1}|_{W^\ast \otimes (V_{\ast}^{[x^+]} \otimes \cdots \otimes V_{\ast}^{[y^+]})}.
\]

Because $L(\mu, k)$ and $R$ are weight zero, we may replace $S^{-1} \otimes S^{-1}$ with $S \otimes S$ in the equation above. By Lemma 5.6.10, we have that
\[
(S \otimes S)(L(\mu, k)) = (S \otimes S)(L((\mu, k) + h^{(1)}/2 + h^{(2)}/2)) \\
= \left( Q((\mu, k) + h^{(2)}) \otimes Q(\mu, k) \right) L((\mu, k) + h^{(1)/2} + h^{(2)/2})^{-1} \Delta_{21}^{21}(Q(\mu, k)^{-1}).
\]

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Substituting in, we conclude that
\[
R_{WV}(1; z_1, \ldots, z_n; \mu, k)^* R_{WV}^*
\]
\[
= \left( Q_{W^*}(\mu, k) \otimes Q_{V^*}(\mu, k) + h^{(1)} \right) L_{W^*V^*}(\mu, k) + h^{(1)} / 2 + h^{(2)} / 2)^{-1}
\]
\[
\Delta_{W^*V^*}(Q(\mu, k)) R_{W^*V^*}(Q(\mu, k))
\]
\[
L_{W^*V^*}^2((\mu, k) + h^{(1)} / 2 + h^{(2)} / 2) \left( Q_{W^*}(\mu, k) + h^{(2)} / 2 \right)^{-1} \otimes Q_{V^*}(\mu, k)^{-1}
\]
\[
= \left( Q_{W^*}(\mu, k) \otimes Q_{V^*}(\mu, k) + h^{(1)} \right) R_{W^*V^*}(1; z_1, \ldots, z_n; (\mu, k) + h^{(1)} + h^{(2)})
\]
\[
\left( Q_{W^*}(\mu, k) + h^{(2)} / 2 \right)^{-1} \otimes Q_{V^*}(\mu, k)^{-1}.
\]

5.7.4 Proof of the dual Macdonald-Ruijsenaars identities

We are now ready to prove Theorem 5.7.1. If \( W \) is a lowest weight integrable module of level \( k_w \), then \( W^* \) is a highest weight integrable module of level \( -k_w \), for which we may apply Proposition 5.7.2 to obtain the equality
\[
P_{VW^*} R_{VW^*} \tilde{\Phi}^V_{\mu,k}(z_1, \ldots, z_n) = \eta \circ R_{W^*V}(1; z_1, \ldots, z_n; \nu + \rho, k - k_w + h^\nu)^* \circ \tilde{\Phi}^V_{\nu-k_w,-a}(z_1, \ldots, z_n) \circ \eta^{-1} \quad (5.7.3)
\]
of intertwiners \( M_{\mu,k} \otimes W^* \rightarrow M_{\mu,k} \otimes W^* \tilde{\Phi} \otimes V^* \). Consider the trace of both sides of (5.7.3) precomposed with \( q^{2\lambda+2\omega} \) and postcomposed with \( q^{-\Omega_{W^*}} \). Computing using the left hand expression of (5.7.3), we note that only terms involving the diagonal term of \( R_{VW^*}(z) \) contribute; since this diagonal term is \( q^{-\Omega_{W^*}} \), we conclude the trace is equal to
\[
\chi_{W^*}(q^{2\lambda+2\omega}) \tilde{\Phi}^V(z_1, \ldots, z_n; \lambda, \omega, \mu, k).
\]
(5.7.4)

Computing using the right hand expression of (5.7.3), we note that the value of the trace has zero weight in \( \tilde{V} \), hence \( q^{-\Omega_{W^*}} \) evaluates to 1. Therefore, in computing the trace we may ignore both \( q^{-\Omega_{W^*}} \) and the conjugation by \( \eta \), obtaining
\[
\sum_{\nu,a} \text{Tr}[W^*|\nu-k_w\Lambda_0-a\delta] \left( R_{W^*V}(1; z_1, \ldots, z_n; \nu + \rho, k - k_w + h^\nu)^* V \right)
\]
\[
q^{-2\omega a} \tilde{\psi}^V(z_1, \ldots, z_n; \lambda, \omega, \nu, k - k_w)
\]
\[
= \sum_{\nu,a} \text{Tr}[W^*|\nu-k_w\Lambda_0-a\delta] \left( R_{W^*V}(1; z_1, \ldots, z_n; \mu + \nu, k - k_w + h^\nu)^* V \right)
\]
\[
q^{-2\omega a} \tilde{\psi}^V(z_1, \ldots, z_n; \lambda, \omega, \mu + \nu, k - k_w). \quad (5.7.5)
\]

Recall now that \( V \) is a tensor product of evaluation representations. Equating (5.7.4) and (5.7.5), multiplying on the left by \( J_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \mu, k)^* \), and applying Lemma
5.5.1, we find that
\[
\chi_W(q^{-2\lambda-2\omega d}) \psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda, \omega, \mu, k) = \sum_{\nu, \alpha} J_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \mu, k)^* \]
\[
\text{Tr}[W^{*}[-\nu+kW\Lambda_0-\alpha\delta] \left( R_{W^{*}V_1}(1; z_1,\ldots,z_n; \mu + \nu + \rho, k - kW + h^\nu)^*V \right) q^{-2\omega a} \bar{\psi}^V(z_1,\ldots,z_n; \lambda, \omega, \mu + \nu, k - kW).}
\]

Now, by Corollary 5.4.7, we see that
\[
J_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \mu, k)^* = J_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \mu + \nu + \rho, k - kW + h^\nu)^*V \]
and therefore by Lemma 5.5.2 that on $W^{*}[\nu - kW\Lambda_0 - \alpha\delta]$ we have
\[
J_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \mu, k)^* R_{W^{*}V_1}(1; z_1,\ldots,z_n; \mu + \nu + \rho, k - kW + h^\nu)^*V
\]
\[
= R_{W^{*}V_1}(1, z_1; (\mu + \nu + \rho, k - kW + h^\nu) - h^{(2\cdots n)})^*V_i
\]
\[
J_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \mu, k - kW)^*.
\]

We conclude that
\[
\chi_W(q^{-2\lambda-2\omega d}) \psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda, \omega, \mu, k)
\]
\[
= \sum_{\nu, \alpha} \text{Tr}[W^{*}[-\nu+kW\Lambda_0+\alpha\delta] \left( R_{W^{*}V_1}(1, z_1; (\mu + \nu + \rho, k - kW + h^\nu) - h^{(2\cdots n)})^*V_i \right)
\]
\[
q^{-2\omega a} J_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \mu + \nu + k - kW)^* \bar{\psi}^V(z_1,\ldots,z_n; \lambda, \omega, \mu + \nu, k - kW)
\]
\[
= \sum_{\nu, \alpha} \text{Tr}[W^{*}[-\nu+kW\Lambda_0+\alpha\delta] \left( R_{W^{*}V_1}(1, z_1; (\mu + \nu + \rho, k - kW + h^\nu) - h^{(2\cdots n)})^{W^{*}V_i} \right)
\]
\[
q^{-2\omega a} \psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n; \lambda, \omega, \mu + \nu, k - kW).}
\]

By Lemma 5.7.3, we see that
\[
R_{W^{*}V_i}(1, z_1; (\mu + \nu + \rho, k - kW + h^\nu) - h^{(1+\cdots n)})^{W^{*}V_i}
\]
\[
= Q_{W^{*}}((\mu + \nu + \rho, k - kW + h^\nu) - h^{(1+\cdots n)}) Q_{V_i}^*(((\mu + \rho, k + h^\nu) - h^{(1+\cdots n)})
\]
\[
R_{W^{*}V_i}(1, z_1; (\mu + \rho, k + h^\nu) + h^{(\alpha)}) - h^{(1+\cdots n)}
\]
\[
Q_{W^{*}}((\mu + \nu + \rho, k - kW + h^\nu) + h^{(*)} - h^{(1+\cdots n)})^{-1}
\]
\[
Q_{V_i}^*(((\mu + \nu + \rho, k - kW + h^\nu) - h^{(1+\cdots n)})^{-1}.}
\]

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Substituting in, noting that $h^{(s)} = -h^{(i)}$ and $h^{(1-n)} = 0$, and canceling common terms in $W^*$, we obtain that

$$
\chi_W(q^{-2\lambda-2w})\Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k) = \sum_{\nu,a} \operatorname{Tr}|W|_{-\nu+kW,\Lambda_0+a\delta}(Q_{W^*}^\nu(\mu+\nu+k-kW+h^\nu)Q_{V_n^*}^\nu(\mu+k+h^\nu)) \ldots
$$

$$
Q_{V_1^*}^\nu((\mu+\nu+k+h^\nu)-h^{(1-n)})R_{W^*}^\nu(1,\mu+k+h^\nu+h^{(s)}-h^{(1-n)})
$$

$$
Q_{V_n^*}^\nu((\mu+\nu+k-kW+h^\nu)-h^{(1-n)}-1)
$$

$$
q^{-2\omega}q_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu+\nu,k-kW).
$$

Noting that $W^{**} \simeq W$ and that we may ignore the conjugation by the $Q_W$-term in computing the trace, we may simplify this to

$$
\chi_W(q^{-2\lambda-2w})\Psi_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k) = \sum_{\nu,a} \operatorname{Tr}|W|_{-\nu+kW,\Lambda_0+a\delta}(R_{WV_n^*}(1,\mu+k+h^\nu+h^{(s)})) \ldots
$$

$$
R_{WV_1^*}(1,\mu+k+h^\nu+h^{(s)}-1)
$$

$$
Q_{V_1^*}^\nu((\mu+\nu+k-kW+h^\nu)-h^{(1-n)}-1)
$$

$$
q^{-2\omega}q_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu+\nu,k-kW).
$$

Substituting in the definition of $F$, we find the desired

$$
\chi_W(q^{-2\lambda-2w})F_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k) = \sum_{\nu,a} \operatorname{Tr}|W|_{\nu+kW,\Lambda_0+a\delta}(R_{WV_n^*}(1,\mu+k+h^{(s)})) \ldots
$$

$$
R_{WV_1^*}(1,\mu+k+h^{(s)}-1)
$$

$$
q^{-2\omega}F_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu+\nu,k-kW).
$$

\section{5.8 Macdonald symmetry identity}

In this section we prove the Macdonald symmetry identity for the trace function $F_{V_1,\ldots,V_n}(z_1,\ldots,z_n;\lambda,\omega,\mu,k)$ under interchange of $(\lambda,\omega)$ and $(\mu,k)$. Our method uses the observation that the Macdonald-Ruijsenaars and dual Macdonald-Ruijsenaars operators $D_W(\lambda,\omega)$ and $D^*_W(\mu,k)$ are exchanged under this interchange and the fact...
that the Macdonald-Ruijsenaars equations admit a unique formal solution.

5.8.1 The statement

Theorems 5.6.1 and 5.7.1 show that \( F^{V_1, ..., V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) satisfies dual systems of difference equations. Define the function \( F^{V_1, ..., V_n}_{*, \ldots, V_1} \) to be the result of interchanging \( V_i \) and \( V_{n+1-i} \) in the definition of \( F^{V_1, ..., V_n} \). This section is devoted to proving the following symmetry relation.

**Theorem 5.8.1** (Macdonald symmetry identity). The functions \( F^{V_1, ..., V_n} \) and \( F^{V_1, ..., V_n}_{*, \ldots, V_1} \) satisfy the symmetry relation

\[
F^{V_1, ..., V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = F^{V_1, ..., V_n}_{*, \ldots, V_1}(z_n, \ldots, z_1; \mu, k, \lambda, \omega).
\]

Recall the coefficient rings

\[
\mathcal{A}_{\lambda, \omega} = \mathbb{C}[[q^{-2(\lambda, \alpha_1)}, \ldots, q^{-2(\lambda, \alpha_r)}, q^{-2\omega+2(\lambda, \theta)}]]
\]
and

\[
\mathcal{A}_{\mu, k} = \mathbb{C}[[q^{-2(\mu, \alpha_1)}, \ldots, q^{-2\mu + 2(\mu, \theta)}]]
\]

Our strategy will be to show that the Macdonald-Ruijsenaars equations admit unique formal solutions over \( \mathcal{A}_{\lambda, \omega} \) and \( \mathcal{A}_{\mu, k} \) with specified leading term. The fact that \( F^{V_1, ..., V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) satisfies these equations in both sets of variables will then give the conclusion.

5.8.2 Formal expansion properties of \( F^{V_1, ..., V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \)

In this subsection, we show that the renormalized trace functions admit a formal expansion in a certain coefficient ring.

**Lemma 5.8.2.** The renormalized trace function \( F^{V_1, ..., V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) has formal expansion lying in

\[
q^{2(\lambda, \omega)} \mathcal{A}_{\lambda, \omega} \otimes \mathcal{A}_{\mu, k} \otimes \mathbb{C}((z_2/z_1, \ldots, z_n/z_{n-1})) \otimes (V_1 \otimes \cdots \otimes V_n)[0] \otimes (V_1^* \otimes \cdots \otimes V_1^*)[0].
\]

**Proof.** For \( \Psi^{V_1, ..., V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \), this follows from an argument analogous to that of [96, Proposition 2.6]. Now, the normalization factors lie in

\[
\mathcal{A}_{\lambda, \omega} \otimes \mathcal{A}_{\mu, k} \otimes \mathbb{C}((z_2/z_1, \ldots, z_n/z_{n-1})) \otimes \text{End}((V_1 \otimes \cdots \otimes V_n)[0]) \otimes \text{End}((V_1^* \otimes \cdots \otimes V_1^*)[0])
\]
by definition, so combining these facts yields the desired. \( \square \)

5.8.3 Uniqueness of formal solutions to Macdonald-Ruijsenaars equations

We now prove a uniqueness property for formal solutions to the Macdonald-Ruijsenaars equations over \( \mathcal{A}_{\lambda, \omega} \).

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Lemma 5.8.3. For each \( v \in (V_1 \otimes \cdots \otimes V_n)[0] \), the system of equations

\[
D_W(\omega, k)F(\lambda, \omega) = \chi_W(q^{-2\mu-2kd})F(\lambda, \omega)
\]

for \( W \) ranging over all lowest weight integrable representations has a unique solution \( F(\lambda, \omega) \) valued in

\[
q^{2(\lambda, \mu)}A_{\lambda, \omega} \otimes A_{\mu, k} \otimes \mathbb{C}((z_2/z_1, \ldots, z_n/z_{n-1})) \otimes (V_1 \otimes \cdots \otimes V_n)[0]
\]

with leading term \( q^{2(\lambda, \mu)}v \) as a series in \( A_{\lambda, \omega} \).

Proof. Existence follows by taking \( (v^*, F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k)) \) and applying Lemma 5.8.2, where \( v^* \) is dual to \( v \). For uniqueness, suppose that \( F_1(\lambda, \omega) \) and \( F_2(\lambda, \omega) \) are two such solutions. If \( F'(\lambda, \omega) := F_1(\lambda, \omega) - F_2(\lambda, \omega) \) is non-zero, it is another solution which must contain a (possibly non-unique) leading monomial term of the form

\[
c q^{2(\lambda, \mu)}q^{-\sum_i 2n_i(\lambda, \alpha_i)}q^{-2m\omega + 2m(\lambda, \theta)}v' = c q^{(\lambda, 2\mu - 2 \sum_i n_i(\alpha_i + 2m\theta)) - 2m\omega}v'
\]

for some \( n_i \geq 0, m \geq 0 \) with at least one \( n_i, m \) non-zero and \( v' \in (V_1 \otimes \cdots \otimes V_n)[0] \). Notice that \( q^{(\lambda, 2\mu - 2 \sum_i n_i(\alpha_i + 2m\theta)) - 2m\omega}v' \) will again be a leading monomial term of \( D_W(\omega, k)F'(\lambda, \omega) \) with coefficient given by

\[
c \sum_{\nu, \alpha} \dim W[\nu + kW\Lambda_0 + a\alpha] q^{-2\kappa\alpha}q^{-(\nu, 2\mu - 2 \sum_i n_i(\alpha_i + 2m\theta))}q^{2\kappa\omega}
\]

\[
= c \chi_W(q^{-2\mu-2kd + 2 \sum_i n_i(\alpha_i + 2m\alpha_0)}).
\]

Since \( 2 \sum_i n_i(\alpha_i + 2m\alpha_0) \neq 0 \), the fact that this holds for all lowest weight integrable \( W \) contradicts the fact that \( D_W(\omega, k)F'(\lambda, \omega) = \chi_W(q^{-2\mu-2kd})F'(\lambda, \omega) \). We conclude that \( F'(\lambda, \omega) = 0 \) and the desired solution is unique.

5.8.4 Proof of the symmetry identity

We are now ready to prove Theorem 5.8.1. Notice that the Macdonald-Ruijsenaars equations for \( F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) correspond under variable exchange to the dual Macdonald-Ruijsenaars equations for \( F^{V_1^*, \ldots, V_n^*}(z_n, \ldots, z_1; \mu, k, \lambda, \omega) \). By Theorems 5.6.1 and 5.7.1, both of the trace functions \( F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) \) and \( F^{V_1^*, \ldots, V_n^*}(z_n, \ldots, z_1; \mu, k, \lambda, \omega) \) are solutions to

\[
D_W(\omega, k)F(\lambda, \omega) = \chi_W(q^{-2\mu-2kd})F(\lambda, \omega)
\]

for all lowest weight integrable \( W \). Now, their leading terms with respect to \( A_{\lambda, \omega} \) are related by an element \( M(\mu, k) \in A_{\mu, k} \otimes \text{End}(V_n^* \otimes \cdots \otimes V_1^*)[0] \), meaning that

\[
F^{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = M(\mu, k)F^{V_1^*, \ldots, V_n^*}(z_n, \ldots, z_1; \mu, k, \lambda, \omega).
\]
Repeating this argument with \((\mu, k)\), we find that
\[
F^V_{\nu_1^{\ast} \ldots \nu_1^1}(z_n, \ldots, z_1; \mu, k, \lambda, \omega) = M'(\lambda, \omega) F^V_{\nu_1^{\ast} \ldots \nu_1^1}(z_n, \ldots, z_1; \lambda, \omega, \mu, k)
\]
for some \(M'(\lambda, \omega) \in A_{\lambda, \omega} \otimes \text{End}(V_1 \otimes \cdots \otimes V_n)[0]\). This implies the equality \(M(\mu, k)M'(\lambda, \omega) = 1\), hence \(M(\mu, k)\) lies in \(\text{End}(V_1^{\ast} \otimes \cdots \otimes V_n^{\ast})[0]\). Now, comparing leading terms in \(A_{\lambda, \omega}\) in (5.8.1) implies that \(M(\mu, k) = 1\), yielding the desired.

5.9 \(q\)-KZB and dual \(q\)-KZB equations

In this section we prove the \(q\)-KZB and dual \(q\)-KZB equations describing the behavior of \(F^V_{\nu_1^{\ast} \ldots \nu_n^{\ast}}(z_1, \ldots, z_n; \lambda, \omega, \mu, k)\) under shifts of the spectral parameters by the modular parameters \(q^{-2\omega}\) and \(q^{-2k}\). We directly establish the dual \(q\)-KZB equations, after which the \(q\)-KZB equations follow by the symmetry relation of Theorem 5.8.1.

5.9.1 The statements

The \(q\)-KZB operators are defined by
\[
K_j(z_1, \ldots, z_n; \lambda, \omega, k) := R_{V_j}^V(z_j+1, q^{2k} z_j; \lambda, \omega) - h((j+2) \cdots n))^{-1} \ldots
\]
\[
R_{V_j}^V(z_n, q^{2k} z_j; \lambda, \omega)^{-1} \Gamma_j
\]
\[
R_{V_j}^V(z_j, z_1; \lambda, \omega) - h(2(j-1)) - h((j+1) \cdots n)) \ldots
\]
\[
R_{V_j}^V(z_j, z_{j-1}; \lambda, \omega) - h((j+1) \cdots n)) \ldots
\]
\[
D_j(\mu) := q^{-2\mu + \sum_i z_i^2 q^{\Omega_j \ast \ast (j-1)} \ldots q^{\Omega_j \ast \ast (n-1)}},
\]
where \(\Gamma_j f(\lambda, \omega) = f((\lambda, \omega) + h(j))\). The dual \(q\)-KZB operators are defined by
\[
K_j^\gamma(z_1, \ldots, z_n; \mu, k, \omega) := R_{V_j}^V(z_j-1, q^{2\omega} z_j; \mu, k) - h((1, * \cdots (j-2)))^{-1} \ldots
\]
\[
R_{V_j}^V(z_n, q^{2\omega} z_j; \mu, k)^{-1} \Gamma_j
\]
\[
R_{V_j}^V(z_j, z_n; \mu, k) - h((1, * \cdots (j-2))) - h((j+1) \cdots (n-1)) \ldots
\]
\[
R_{V_j}^V(z_j, z_{j+1}; \mu, k) - h((1, * \cdots (j-2)))
\]
\[
D_j^\gamma(\lambda) := q^{2\lambda + \sum_i z_i^2 q^{\Omega_j \ast (j+1) \ldots q^{\Omega_j \ast (n-1)}},
\]
where \(\Gamma_j f(\mu, k) = f((\mu, k) + h(j))\). Note that the operators \(K_j(z_1, \ldots, z_n; \lambda, \omega, k)\) and \(K_j^\gamma(z_1, \ldots, z_n; \mu, k, \omega)\) are difference operators in \((\lambda, \omega)\) and \((\mu, k)\) whose coefficients are linear operators on \(V\) and \(V^\ast\) and that \(D_j(\mu)\) and \(D_j^\gamma(\lambda)\) are linear operators on \(V^\ast\) and \(V\). It is known that the \(K_j(z_1, \ldots, z_n; \lambda, \omega, k)\) and the \(K_j^\gamma(z_1, \ldots, z_n; \mu, k, \omega)\) commute and form the \(q\)-KZB and dual \(q\)-KZB integrable systems.
The $q$-KZB equations relate a spectral shift by the modular parameter $q^{-2k}$ associated to $\mu$ to the action of the difference operator $K_j(z_1, \ldots, z_n; \lambda, \omega, k)$ in $\lambda$. Symmetrically, the dual $q$-KZB equations relates a spectral shift by the modular parameter $q^{-2\omega}$ associated to $\lambda$ to the action of the difference operator $K_j^\omega(z_1, \ldots, z_n; \mu, k, \omega)$ in $\mu$. In a different form, they were introduced by Felder in [42] and studied by Felder-Tarasov-Varchenko in [44, 45]. The remainder of this section will be devoted to the proof of these two equations, stated below. We will prove Theorem 5.9.2 directly, after which Theorem 5.9.1 follows from the symmetry property of Theorem 5.8.1.

**Theorem 5.9.1** ($q$-KZB equation). For $j = 1, \ldots, n$, we have

$$F^{V_1, \ldots, V_n}(z_1, \ldots, q^{2k}z_j, \ldots, z_n; \lambda, \omega, \mu, k) = \left(K_j(z_1, \ldots, z_n; \lambda, \omega, k) \otimes D_j(\mu)\right) F^{V_1, \ldots, V_n}(z_1, \ldots, z_j, \ldots, z_n; \lambda, \omega, \mu, k).$$

**Theorem 5.9.2** (dual $q$-KZB equation). For $j = 1, \ldots, n$, we have

$$F^{V_1, \ldots, V_n}(z_1, \ldots, q^{2\omega}z_j, \ldots, z_n; \lambda, \omega, \mu, k) = \left(D_j^\omega(\lambda) \otimes K_j^\omega(z_1, \ldots, z_n; \mu, k, \omega)\right) F^{V_1, \ldots, V_n}(z_1, \ldots, z_j, \ldots, z_n; \lambda, \omega, \mu, k).$$

### 5.9.2 Commutation relation for intertwiners

The fundamental operation in the proof of Theorem 5.9.2 is the application of the following commutation relation for intertwiners.

**Lemma 5.9.3.** For finite-dimensional $U_q(\hat{\mathfrak{g}})$-representations $V$ and $W$, we have the relation

$$P_{VW}R_{ VW} \Phi_{\mu,k}^{VW}(z_1, z_2) = R_{ VW}(z_2, z_1; \mu + \rho, k + h^\vee)^* \Phi_{\mu,k}^{WV}(z_2, z_1).$$

**Proof.** Both sides of the desired equality are intertwiners $M_{\mu,k} \rightarrow W[z_2^\pm 1] \otimes V[z_1^\pm 1] \otimes W^* \otimes V^*$. Let $\{v_i\}$ and $\{v_j\}$ be bases of $V, W$, and let $\{v_i^*\}, \{v_j^*\}$ be the dual bases. The highest term of the left side is given by

$$\sum_{i,j} v_j \otimes v_i \otimes J_{WV}^{2i}(z_1, z_2; \mu, k)^* (R_{WV}^{2i})^*(v_j^* \otimes v_i^*)$$

and the highest term of the right hand side is given by

$$\sum_{i,j} v_j \otimes v_i \otimes R_{WV}(z_2, z_1; \mu + \rho, k + h^\vee)^* J_{WV}(z_1, z_2; \mu, k)^* (v_j^* \otimes v_i^*),$$

so the result follows by noting that Proposition 5.4.6 implies these are equal. \[\square\]
5.9.3 Proof of the dual $q$-KZB equation

We are now ready to prove the dual $q$-KZB equation. Rewrite the conclusion of Lemma 5.9.3 as

$$\Phi^V_{(\mu, k)-h(w)}(z_1) \circ \Phi^W_{(\mu, k)}(z_2) = \mathcal{R}^{-1}_{VW} P_{VW} \mathcal{R}_{WV}(z_2, z_1; \mu + \rho, k + h V)^* \Phi^W_{(\mu, k)-h(v)}(z_2) \circ \Phi^V_{(\mu, k)}(z_1).$$

Swapping the roles of $V$ and $W$, we obtain also that

$$\Phi^V_{(\mu, k)-h(w)}(z_1) \circ \Phi^W_{(\mu, k)}(z_2) = \mathcal{R}_{VW}(z_1, z_2; \mu + \rho, k + h V)^* P_{VW} \mathcal{R}_{WV} \Phi^W_{(\mu, k)-h(v)}(z_2) \circ \Phi^V_{(\mu, k)}(z_1).$$

Apply these commutation relations to commute $\Phi^V_{(\mu, k)-h(j+1-n)}(z_j)$ to the left, apply the cyclic property of the trace to move it to the right, and then apply the commutation relations to commute it back to its original position in

$$\Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) := \text{Tr}_{M_{\mu,k}}(\Phi^V_{(\mu, k)-h(2-n)}(z_1) \cdots \Phi^V_{(\mu, k)}(z_n) q^{2\lambda + 2\omega})$$

to obtain

$$\Psi_{V_1, \ldots, V_n}(z_1, \ldots, z_n; \lambda, \omega, \mu, k) = \mathcal{R}_{V_j V_{j-1}}(z_{j-1}, z_j; (\mu + \rho, k + h V) - h(j+1-n))^* P_{V_j V_{j-1}} \mathcal{R}_{V_j V_{j-1}} \cdots$$

$$\mathcal{R}_{V_j V_{j-1}}(z_{j-1}, z_j; (\mu + \rho, k + h V) - h(j+1-n))^* P_{V_j V_{j-1}} \mathcal{R}_{V_j V_{j-1}} \cdots \mathcal{R}_{V_j V_{j+1}}$$

$$P_{V_{j+1} V_j} \mathcal{R}_{V_j V_{j+1}}(z_j, z_{j+1}; (\mu + \rho, k + h V) - h(j+1-n))^*$$

$$\mathcal{R}_{V_{j+1} V_j} \mathcal{R}_{V_j V_{j+1}} \mathcal{R}_{V_j V_{j+1}} \cdots \mathcal{R}_{V_{j+1} V_j} q^{2\lambda + 2\omega} \mathcal{R}_{V_{j+1} V_j} \mathcal{R}_{V_{j+1} V_j} \cdots \mathcal{R}_{V_{j+1} V_j} \mathcal{R}_{V_{j+1} V_j} \cdots$$

$$\mathcal{R}_{V_j V_{j-1}}(z_{j-1}, z_j; (\mu + \rho, k + h V) - h(j+1-n))^* \cdots$$

$$\mathcal{R}_{V_j V_{j-1}}(z_{j-1}, z_j; (\mu + \rho, k + h V) - h(j+1-n))^* \cdots \mathcal{R}_{V_j V_{j+1}}$$

$$\mathcal{R}_{V_j V_{j+1}}(q^{2\omega} z_j, z_{j+1}; (\mu + \rho, k + h V) - h(j+1-n))^*$$

$$\Psi_{V_1, \ldots, V_n}(z_1, \ldots, q^{2\omega} z_j, \ldots, z_n; \lambda, \omega, \mu, k).$$
Now, apply Lemma 5.7.3 and cancel terms to see that

\[
R_{V_{j-1}V_j}(z_{j-1}, z_j; (\mu + \rho, k + h^\nu) - h^{(j+1 \cdots n)} - \cdots
\]

\[
R_{V_jV_j}(z_1, z_j; (\mu + \rho, k + h^\nu) - h^{(2 \cdots j-1) + (j+1 \cdots n)} - \cdots \Gamma_{s_j}
\]

\[
R_{V_jV_n}(q^{-2\omega}z_j, z_n; \mu + \rho, k + h^\nu)^* \cdots R_{V_jV_{j+1}}(q^{-2\omega}z_j, z_{j+1}; (\mu + \rho, k + h^\nu) - h^{(j+2 \cdots n)}^*)
\]

\[
= \left( Q_{V_j}((\mu + \rho, k + h^\nu) + h^{(s(j+1) \cdots n)}) Q_{V_{j-1}}^*((\mu + \rho, k + h^\nu) + h^{(s(j) \cdots n)}) \cdots
\]

\[
Q_{V_j}((\mu + \rho, k + h^\nu) + h^{(s(j+2) \cdots n)}) Q_{V_n}^*(\mu + \rho, k + h^\nu) \cdots
\]

\[
= \left( Q_{V_{j-1}}((\mu + \rho, k + h^\nu) + h^{(s(j) \cdots n)})^{-1} \cdots Q_{V_j}^*(\mu + \rho, k + h^\nu) + h^{(s(j+2) \cdots n)} \right)^{-1}
\]

\[
Q_{V_n}^*(\mu + \rho, k + h^\nu)^{-1} \cdots Q_{V_{j+1}}^*((\mu + \rho, k + h^\nu) + h^{(s(j+2) \cdots n)})^{-1}
\]

\[
Q_{V_j}^*((\mu + \rho, k + h^\nu) + h^{(s(j+1) \cdots n)})^{-1}
\]

(5.9.1)

Finally, we claim that on \(V[0] \otimes V^*[0]\), we have

\[
J^{1 \cdots n}(\lambda, \omega)^{-1} \mathcal{R}^{j-j-1} \cdots \mathcal{R}^{j_1} q_j^{-2\lambda-2\omega d} (\mathcal{R}^{n j})^{-1} \cdots (\mathcal{R}^{j+1,j})^{-1} J^{1 \cdots n}(\lambda, \omega) q^{2\omega d}
\]

\[
= q_j^{-2\lambda-\sum_i z_i^2} q^{-\Omega_{j,j+1} \cdots \Omega_{n,n}} = D_j^\nu(\lambda). \quad (5.9.2)
\]

Notice that \(\mathcal{R}^{j-j-1} \cdots \mathcal{R}^{j_1} = (\mathcal{R}^{n j})^{-1} \cdots (\mathcal{R}^{j+1,j})^{-1} = (\mathcal{R}^{j+1 \cdots n,j})^{-1} \). Therefore, to prove (5.9.2), it suffices to check it for \(n = 3\) and \(n = 2\). For \(n = 3\), the product of both sides for \(j = 1, 2, 3\) is 1, hence it suffices to check for \(j = 1, 3\), in which case it reduces to the \(n = 2\) case. For \(n = 2\), (5.9.2) follows by rearranging the ABRR equation of Proposition 5.4.3. Substituting (5.9.1) and (5.9.2) into our previous relation and then applying the normalizations of (5.5.3) yields the result.
Bibliography


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