Antichains of Interval Orders and Semiorders, and Dilworth Lattices of Maximum Size Antichains

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2016

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Abstract

This thesis consists of two parts. In the first part we count antichains of interval orders and in particular semiorders. We associate a Dyck path to each interval order, and give a formula for the number of antichains of an interval order in terms of the corresponding Dyck path. We then use this formula to give a generating function for the total number of antichains of semiorders, enumerated by the sizes of the semiorders and the antichains.

In the second part we expand the work of Liu and Stanley on Dilworth lattices. Let L be a distributive lattice, let $\sigma(L)$ be the maximum number of elements covered by a single element in L, and let $K(L)$ be the subposet of L consisting of the elements that cover $\sigma(L)$ elements. By a result of Dilworth, $K(L)$ is also a distributive lattice. We compute $\sigma(L)$ and $K(L)$ for various lattices L that arise as the coordinate-wise partial ordering on certain sets of semistandard Young tableaux.

Thesis Supervisor: Richard P. Stanley
Title: Professor

Acknowledgments

Many people have contributed to my experience at MIT, both in the academic level and the personal level. First and foremost, I'd like to thank my advisor Richard Stanley for asking interesting questions, suggesting fruitful directions, and for being kind, patient and generous. I'd like to thank the other two members of my thesis committee, Alex Postnikov and Michelle Wachs, for taking the time to serve on the committee and for some important comments. I am also grateful to Doron Puder, a long time friend and mathematician, for reading the first part of my thesis draft and offering many helpful comments.

I'd like to thank the math professors from the Hebrew university in Jerusalem where I completed my undergraduate degree. I'd like to thank Gil Kalai – the beauty and amazement I have experienced in his undergraduate combinatorics class (with Eran Nevo as a TA) are probably what made me choose this field. I'd like to thank Nati Linial, Alex Lubotzky and others for their support and friendliness.

On the personal level, I am grateful to the many people who played a part in my life in the last few years. I enjoyed the continuing support of family and long time friends, and loved meeting new people from so many different backgrounds in the multicultural environment of MIT. Two social groups I was part of, the MIT Women's League Chorale and the MIT GradMoms group, were a source of encouragement and inspiration along the way.

I'd like to dedicate this thesis to my parents – I'm glad it makes you proud, to my husband – I wouldn't have been here if it weren't for you (literally), and to my son, who is responsible for both the hardest and the best moments I have had in the last 2+ years.

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Chapter 1

Introduction

The world of partially ordered sets (or *posets*) is vast and varied. So much so, that this thesis, concerned with certain enumerative aspects of posets, contains two very different chapters. Chapter 2 deals with antichains of interval orders and semiorders, while Chapter 3 deals with distributive lattices comprising of semistandard Young tableaux. We give a separate introduction for each of these chapters below, but first we start with some common basic definitions. For much more on poset theory, see for example [6, Chapter 3].

Let $(P, \leq)$ be a poset (in the rest of this thesis we usually omit the relation $\leq$ from the notation of a poset). An element $x \in P$ *covers* an element $y \in P$ if $x > y$ and there is no element $z \in P$ such that $x > z > y$. We represent posets graphically by a *Hasse diagram*: each element of P is represented by a point, and if x covers y we place the point representing x higher than the point representing y and draw a line between them (see Figure 1-1 for an example).

Two elements $x, y \in P$ are called *comparable* if $x \leq y$ or $y \leq x$, and *incomparable* otherwise. A *chain* of P is a set of elements of P in which any two elements are comparable, while an *antichain* is a set of elements of P in which no two elements are comparable.

An *induced subposet* of P is a poset Q whose elements are a subset of the elements of P, and $x, y \in Q$ satisfy $x \leq y$ if and only if $x \leq y$ in P (in other words, the relations among the elements of Q are precisely those inherited from P). Note that in Chapter

3 we use the term subposet to mean an induced subposet (this is common in the literature but not always the case).

1.1 Antichains of Interval Orders and Semiorders

In Chapter 2 we count antichains of (finite, unlabeled) interval orders, and in particular semiorders. We start our introduction by defining these concepts, as well as some other combinatorial concepts that will be used throughout that chapter.

An *interval order* is a poset P whose elements can be put in a one-to-one correspondence with nonempty open intervals in the real line $P \leftrightarrow \{I_p\}_{p \in P}$ such that if $I_p = (a_p, b_p)$ and $I_q = (a_q, b_q)$, then $p < q$ if and only if $b_p \leq a_q$.

A *semiorder* is an interval order P that can be represented by a set of intervals $\{I_p\}_{p \in P}$ in which all intervals have the same length (by convention, we assume all the intervals are unit intervals).

The boldface notation $\boldsymbol{n}$ signifies a chain with n elements, and if P and Q are two posets, $P + Q$ is their disjoint union (i.e., a poset whose elements are the disjoint union of the elements of P and Q and the only relations are the ones inherited from P and Q, respectively). We say a poset P *avoids* a poset Q if P does not contain Q as an induced subposet.

Interval orders and semiorders can be characterized as follows (see [6, exercise 3.15]):

- A poset P is an interval order if and only if it avoids $\mathbf{2} + \mathbf{2}$.
- A poset P is a semiorder if and only if it avoids $\mathbf{2} + \mathbf{2}$ and $\mathbf{3} + \mathbf{1}$.

Example Figure 1-1 shows an interval order (left) and a semiorder (right), both represented by a Hasse diagram and by a set of intervals (unit intervals for the semiorder). Note that the interval order contains a $\mathbf{3} + \mathbf{1}$ (highlighted) and therefore it is not a semiorder.

A lattice path in the (x, y) plane is a path consisting of straight line segments that start and end at points with integer coordinates. Sets of lattice paths can be

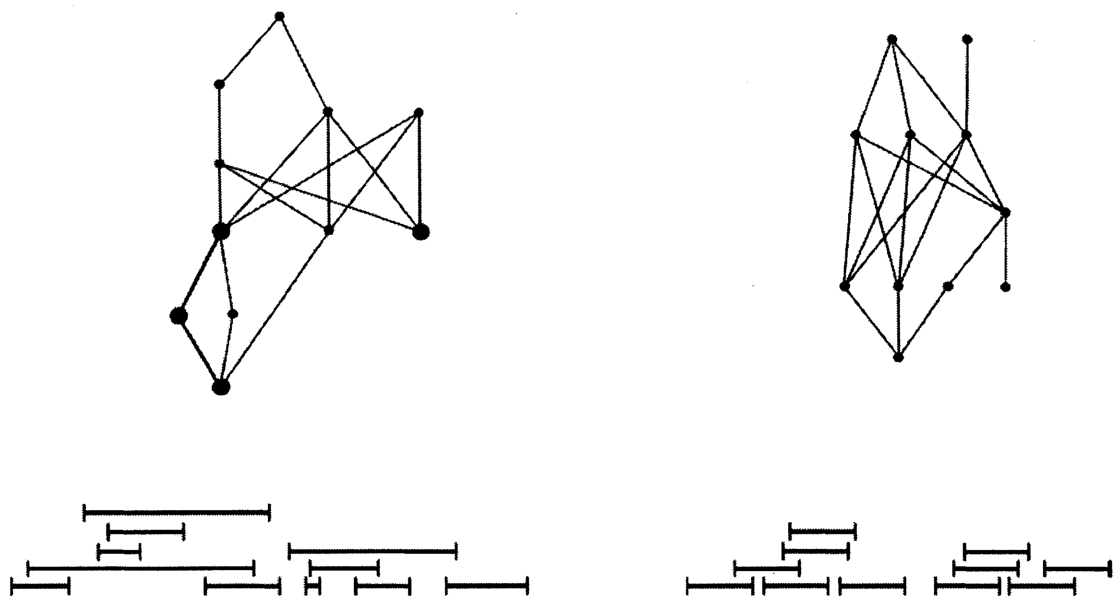


Figure 1-1: An interval order (left) and a semiorder (right)

described by a set of allowed steps, so that each lattice starts at some point and then proceeds by a sequence of allowed steps.

A lattice path in the (x, y) plane from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ ("up") and $(1,-1)$ ("down") that does not pass below the x axis is called a *Dyck path* of *semilength* n. An *ascent* (*descent*) of a Dyck path is a sequence of up steps (down steps) followed and preceded by a down step (up step) or an endpoint of the path. A *peak* (*valley*) of a Dyck path is a point where an ascent (descent) of the path ends and a descent (ascent) begins (note that we do not consider the endpoints of a Dyck path as valleys, and so the number of valleys of a Dyck path is one less than the number of peaks). See Figure 1-2 for an example of a Dyck path. We will denote the set of Dyck paths of semilength n by D_n.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of integers (usually these are the sizes of a sequence of sets $\{S_n\}_{n\in\mathbb{N}}$). Then the *ordinary generating function* of a_n is the formal power series $\sum_{n\geq 0} a_n x^n$. Similarly, the *bivariate ordinary generating function* of the integers $\{a_{n,k}\}_{n,k\in\mathbb{N}}$ is the formal power series $\sum_{n,k\geq 0} a_{n,k} x^n t^k$. Since all the generating functions we will use will be ordinary, from now on we omit the word ordinary.

Although we are not concerned with questions of convergence or divergence of

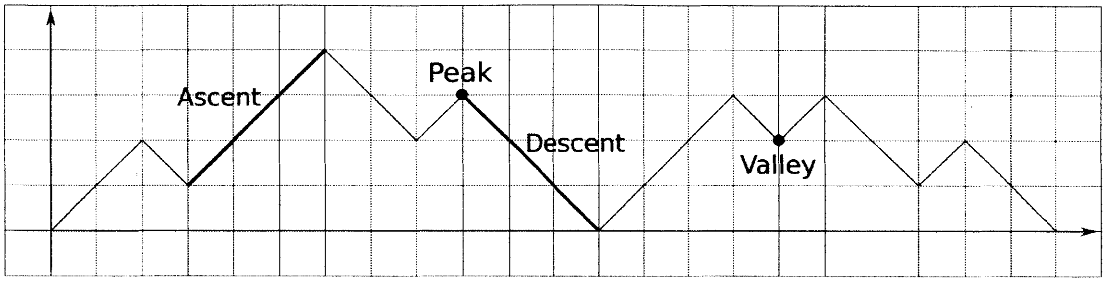


Figure 1-2: A Dyck path of semilength 11

generating functions, many natural operations on power series have a combinatorial significance, hence the usefulness of generating functions for counting combinatorial objects. Most notably for our purposes, one can multiply two generating functions according to the following rule:

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} c_n x^n$$

where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. For more details on generating functions, see [6, Section 1.1].

The *Catalan numbers* $C_n := \frac{1}{n+1}\binom{2n}{n}$ (Sequence A000108 of [3]) are one of the most ubiquitous sequences of numbers in mathematics (See [5], which details over 200 combinatorial objects counted by the Catalan numbers). Among the objects counted by C_n are both n-element semiorders (see [5, exercise 180]) and Dyck paths of semilength n (see [5, Theorem 1.5.1]). A generating function for the Catalan numbers is given in Theorem 1.1.1. We include a (somewhat abbreviated) standard proof of the formula, since the main ideas of the proof will be used again in the proofs in Chapter 2.

Theorem 1.1.1. *Let* $C(x) := \sum_{n\geq 0} C_n x^n$ *be the generating function for the Catalan numbers. Then* $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Proof. We start by establishing a basic recursive relation satisfied by the Catalan numbers $\{C_n\}_{n\geq 0}$. There are many ways to prove this relation using different interpretations of the Catalan numbers, and in this proof we choose to think of C_n as the size of D_n, the set of Dyck paths of semilength n.

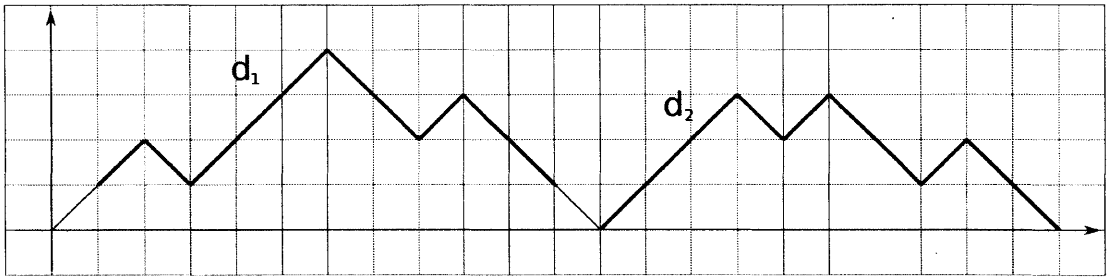


Figure 1-3: Splitting a Dyck path d into two shorter Dyck paths d_1 and d_2

Let d be a Dyck path of semilength $n+1$, and suppose $(2i+2,0)$ is the leftmost point where d touches the x-axis (except $(0,0)$ of course). Clearly the first step in d must be a step up, and the last step before reaching $(2i+2,0)$ must be a step down. Then the part of d from $(1,1)$ to $(2i+1,1)$, shifted left and down so that it starts at $(0,0)$, is a Dyck path of semilength i (denote it by d_1), and the part of d from $(2i+2,0)$ to $(2n+2,0)$, shifted left so that it starts at $(0,0)$, is a Dyck path of semilength $n-i$ (denote it by d_2). See Figure 1-3 for an illustration.

The map $d \mapsto (d_1, d_2)$ is easily seen to be a bijection $D_{n+1} \cong \cup_{i=0}^{n} D_i \times D_{n-i}$, so we get that

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \tag{1.1}$$

Now multiply both sides of equation (1.1) by x^n, and sum over all $n \geq 0$. The left-hand side becomes

$$\sum_{n\geq 0} C_{n+1} x^n = \frac{\sum_{n\geq 1} C_n x^n}{x} = \frac{C(x)-1}{x},$$

while the right-hand side becomes

$$\sum_{n\geq 0} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n = \left(\sum_{n\geq 0} C_n x^n \right) \left(\sum_{n\geq 0} C_n x^n \right) = C(x)^2.$$

Equating the two sides, we get the quadratic equation

$$xC(x)^2 - C(x) + 1 = 0.$$

Solving the equation and choosing the correct sign (see [5, Section 1.3] for more details), we get the result $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

□

The main goal of Chapter 2 is to be able to compute the average number of antichains in n-element semiorders (for all n). Along the way, we obtain more general results that apply to all interval orders, as well as stronger results for semiorders.

Section 2.1 contains some facts regarding interval orders that will be used in the following sections. These results appear (with some variations) in several references, and we follow their presentation in [9].

In Section 2.2 we present a formula (Theorem 1.1.2) to compute the number of antichains of any given size, which is valid for any interval order P. The formula is given in terms of a Dyck path $d(P)$ whose construction from P is described in that section.

Theorem 1.1.2. *Let P be an n-element interval order with a corresponding dyck path $d(P)$. Let* Peaks $= \{u_1, u_2, ..., u_s\}$ *be the set of peaks of $d(P)$, and let* Valleys $= \{v_1, v_2, ..., v_{s-1}\}$ *be the set of valleys of $d(P)$. Then the number of antichains of size k of P is equal to the coefficient of t^k in*

$$\sum_{j=1}^{s} (1+t)^{h(u_j)} - \sum_{j=1}^{s-1} (1+t)^{h(v_j)},$$

where h is the height function, i.e., $h((x,y)) = y$.

In Section 2.3 we proceed to use Theorem 1.1.2 to compute a bivariate generating function for the total number of antichains of semiorders, enumerated by the size of the semiorders and the size of the antichain:

Theorem 1.1.3. *Let $S_{n,k}$ be the total number of antichains of size k of all n-element semiorders, and set $S_{0,0} = 1$ (clearly, $S_{n,k} = 0$ for $k > n$). Let $S(x,t)$ be the bivariate generating function for the total number of antichains of semiorders, where the antichains are enumerated by the size of the antichain and the size of the semiorders,* *i.e.*, $S(x,t) := \sum_{n,k\geq 0} S_{n,k} t^k x^n$. *Then*

$$S(x,t) = \frac{C(x) - 2}{x(t+2)C(x) - 1}$$

where $C(x)$ *is the generating function of the Catalan numbers.*

Setting $t = 1$ in the above generating function results in a generating function for the total number of antichains of n-element semiorders, from which the average number of such antichains can be easily deduced if one so desires.

As an interesting byproduct of our formula, we also show in Section 2.3 that the numbers $S_{n,k}$ are equal to a number triangle that appears in the On-Line Encyclopedia of Integer Sequences (or OEIS, [3]) as the sequence A090285, thus giving the triangle a new interpretation. The triangle was originally contributed by Philippe Deléham, and it is described in terms of a recursive formula as well as a certain Riordan array.

Lastly, Section 2.4 details some curious additional results concerning subsets of minimal elements (or maximal elements) of n-element semiorders (A *minimal element* of a poset P is an element $p \in P$ such that no other element $q \in P$ satisfies $q < p$. *Maximal elements* are defined similarly).

1.2 Dilworth Lattices

In Chapter 3 we compute what we call the Dilworth lattice of certain lattices consisting of semistandard Young tableaux. We start our introduction with some background and definitions.

An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y < x$, then $y \in I$. The set of all order ideals of a poset P, ordered by inclusion, is a poset (in fact, a distributive lattice) denoted $J(P)$. There is an easy bijection between antichains and order ideals in P, namely, the elements of an antichain are the maximal elements of an order ideal. In this bijection, the size of an antichain is exactly the number of elements covered by the corresponding order ideal in the poset $J(P)$.

Let L be a finite distributive lattice. The *join-irreducibles* of L are those elements that cover exactly one element. Denote the subposet of join-irreducibles of L by P. The fundamental theorem for finite distributive lattices states that $L \cong J(P)$, and P is unique (up to isomorphism) in that sense.

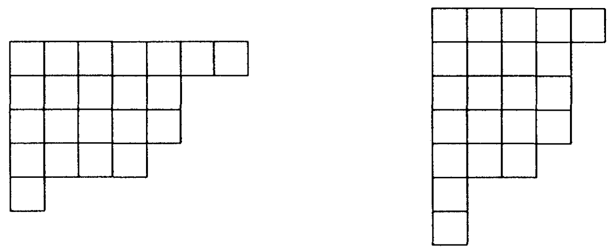
Let $L \cong J(P)$ be as above. Define $\sigma(L) := \max_{x \in L} \#\{y : x \text{ covers } y\}$, the maximum number of elements covered by an element in L. Define $K(L)$ to be the subposet of elements of L that cover exactly $\sigma(L)$ elements. By a result of Dilworth (see exercise 3.72(a) of [6]), $K(L)$ is a distributive lattice. Using the bijection between antichains and order ideals, $\sigma(L)$ is the size of the largest antichain in P, and the number of antichains of P of size $\sigma(L)$ is the size of $K(L)$.

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, ...)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ with a finite number of nonzero terms, called *parts.* When writing a partition we omit its trailing zeroes and write $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ if λ has k parts. The *Young diagram* D_λ of a partition λ is a left-justified array of squares with λ_i squares in the ith row from the top. The *conjugate partition* λ' of λ is the partition with $\lambda'_i = \max\{j : \lambda_j \geq i\}$. In terms of Young diagrams, the Young diagram $D_{\lambda'}$ of the conjugate partition λ' is the Young diagram D_λ reflected around its main diagonal. The number of parts in λ (which is the number of rows in D_λ) is denoted by $l(\lambda)$. Therefore the number of columns in D_λ is $l(\lambda')$.

A *Semistandard Young Tableau* (or *SSYT*) of shape λ is the Young diagram D_λ where each square is filled with a positive integer (called a *part*) such that the parts in each row are weakly increasing and the parts in each column are strictly increasing. For a SSYT T, we denote its entry in the ath row (from the top) and bth column (from the left) by $T_{a,b}$ (or we call it the (a, b)-entry of T). We use the term "entry" to mean either a box in a tableau or the integer (part) in it, and the meaning will be clear from the context. Very frequently, we use the convention $T_{0,b} = 0$ for all columns b and $T_{a,0} = a$ for all rows a, even though these entries are not part of T (note that with this convention a SSYT T is weakly increasing in rows and strictly increasing in columns, even including the 0'th row and column). For more information on partitions see for example [6, Chapter 1], and for more information on SSYT see [7, Chapter 7].

Example

1. Below are the Young diagram of shape $\lambda = (7, 5, 5, 4, 1)$ (left) and the Young diagram of shape $\lambda' = (5, 4, 4, 4, 3, 1, 1)$ (right).



2. Below is an example of a SSYT of shape $\lambda = (7, 5, 5, 4, 1)$.

1	1	3	4	4	6	9
2	2	5	6	6		
4	6	7	8	9		
5	8	9	9			
7						

In their recent paper [2], Liu and Stanley prove a conjecture of Elkies by considering M_p, the coordinate-wise partial ordering on SSYT of staircase shape $\delta_{p-1} := (p-2, p-3, ..., 1)$ with largest part at most $p-1$, which is easily seen to be a finite distributive lattice (they denote this lattice by M_n, but we denote it M_p here to avoid confusion with the rest of our notation). Elkies' conjecture can be transformed into the question of determining the number of antichains of maximal size in a poset Q_p, or equivalently, the size of the lattice $K(J(Q_p))$. Liu and Stanley first prove that $J(Q_p) \cong M_p$ (And compute the size of M_p, thus proving another conjecture of Elkies). They then analyze the structure of SSYT in $K(M_p)$ and determine its join-irreducibles, and this allows them to find a rank-generating function for $K(M_p)$ and to compute its size.

A question that naturally arises from Liu and Stanley's work is to try to describe the lattice $K(L)$ for other distributive lattices L. In Chapter 3 we consider distributive lattices that are the coordinate-wise partial ordering on the set of SSYT of some shape (a staircase, a rectangle, a double staircase $(2s, 2s-2, 2s-4, ..., 2)$ or a double staircase with one shorter row) with various bounds on the largest part. For each such lattice

L we compute $\sigma(L)$ and describe $K(L)$.

Suppose L is the coordinate-wise partial ordering on the set of SSYT of some shape λ with largest part at most n. In their paper [2, Section 2], Liu and Stanley define the notion of a reducible entry: an entry $T_{a,b}$ of a tableau $T \in L$ is called reducible if by replacing $T_{a,b}$ with $T_{a,b} - 1$ the result is another tableau in L. They note that the (a, b) entry of a SSYT T is reducible if and only if $T_{a,b} - T_{a,b-1} \geq 1$ and $T_{a,b} - T_{a-1,b} \geq 2$ (with the convention $T_{0,b} = 0$ and $T_{a,0} = a$). Moreover, the number of elements covered by $T \in L$ is the number of reducible entries in T. It follows that $\sigma(L)$ is the maximum number of reducible entries in a tableau $T \in L$, and $K(L)$ is the subposet of L consisting of the tableaux with $\sigma(L)$ reducible entries.

Let L_t^n be the coordinate-wise partial ordering on SSYT of staircase shape $\delta_t = (t-1, t-2, ..., 1)$ and largest part at most n. Using this notation, $M_p = L_{p-1}^{p-1}$. It turns out that Liu and Stanley's analysis of M_p generalizes very easily to L_t^n for any $n \geq t \geq 1$. Moreover, some key ideas in Liu and Stanley's paper can be used in the analysis of a variety of other lattices.

In Section 3.1 we state and prove some basic results based on those key ideas from Liu and Stanley's paper. These results will be used throughout the chapter. As an example for their use, we apply the results to the lattice L_t^n.

In Section 3.2 we show how to use Liu and Stanley's paper to compute $\sigma(L_t^n)$ and describe $K(L_t^n)$ for any $n \geq t \geq 1$. Since the analysis of the general case is so similar to that of M_p, we only quote the main results from Liu and Stanley's paper in their more general versions, without proofs. Let $m := n - (t-1)$. Section 3.2 shows that:

- For $m < t$,
  - if $t + m = 2l + 1$, we have $\sigma(L_t^n) = l^2 - \binom{m}{2}$ and $K(L_t^n) \cong L_{l-m+1}^l \times L_l^l$;
  - if $t + m = 2l$, we have $\sigma(L_t^n) = l(l-1) - \binom{m}{2}$. The description of $K(L_t^n)$ in this case is less compact, so we do not quote it here.
- For $m \geq t$ we have $\sigma(L_t^n) = \binom{t}{2}$ and $K(L_t^n) \cong L_t^m$.

In Section 3.3 we analyze lattices of SSYT of rectangular shape. Let $L_{r,c}^n$ be the coordinate-wise partial ordering on SSYT of rectangular shape with r rows and c

columns, with largest part at most n. Let $m := n - r$. We show that the lattice $L^n_{r,c}$ has three different behaviors as follows, depending on m. It turns out that for our purposes there is a symmetry between r and c, so we may assume $r \geq c$.

- For $m \leq r - c + 1$ we have $\sigma(L^n_{r,c}) = mc$ and $K(L^n_{r,c}) \cong L^{r-c+1}_{(r-c+1)-m,c}$.
- For $r - c + 1 < m < r + c - 1$,
  - if $m - (r - c + 1) = 2l$, then $\sigma(L^n_{r,c}) = mc - 2\binom{l+1}{2}$ and $K(L^n_{r,c})$ contains only one element;
  - if $m - (r - c + 1) = 2l - 1$, then $\sigma(L^n_{r,c}) = mc - l^2$ and $K(L^n_{r,c})$ is a chain of two elements.
- For $m \geq r + c - 1$ we have $\sigma(L^n_{r,c}) = rc$ and $K(L^n_{r,c}) \cong L^{n-(r+c-1)}_{r,c}$.

In Section 3.4 we analyze lattices of SSYT of the double staircase shape. Let L^n_s be the coordinate-wise partial ordering on SSYT of shape $(2s, 2s-2, 2s-4, ..., 2)$ with largest part at most n. Let $m := n - s$. Analyzing the SSYT in $K(L^n_s)$ we get the following behaviors, again depending on m.

- For $m = 1$ we have $\sigma(L^{s+1}_s) = \binom{s+1}{2}$ and $K(L^{s+1}_s) \cong J(A_s)$ where A_s is the poset of pairs $\{(x, y) \in \mathbb{P}^2 | x + y \leq s + 1\}$ ordered coordinate-wise.
- For $1 < m \leq s$ we have $\sigma(L^n_s) = \frac{1}{2}(s - m + 1)(s + 3m - 2) + m(m - 1)$ and there is exactly one element in $K(L^n_s)$.
- For $m > s$ we have $\sigma(L^n_s) = s^2 + s$ and $K(L^n_s) \cong P^{m-s}_s$, where P^{m-s}_s is the coordinate-wise partial ordering on reverse (nonstrict) plane partitions of shape $(2s, 2s-2, ..., 2)$ with largest part at most $m - s$. (A reverse (nonstrict) plane partition is a SSYT except both rows and columns are weakly increasing).

Lastly, let $L^{s+1}_{s,k}$ be the coordinate-wise partial ordering on SSYT of shape

$$(2s - 1, 2s - 3, ..., 2s - 2k + 1, 2s - 2k, 2s - 2k - 2, ..., 4, 2)$$

(like a double staircase, except the kth row is one box shorter) with largest part at most $n = s + 1$. In Section 3.5 we show that $\sigma(L_{s,k}^{s+1}) = \binom{s+1}{2}$ and $K(L_{s,k}^{s+1})$ has only one element.

Chapter 2

Antichains of Interval Orders and Semiorders

2.1 Background

The following definitions and results are quoted from [9] with some changes of notation.

Let P be a poset. For an element $p \in P$ let us denote

$$M_p := \{q \in P : q < p\} \text{ and } N_p := \{q \in P : q > p\}.$$

(In poset language, M_p is called the strict down-set generated by p and N_p is called the strict up-set generated by p). Moreover, let

$$\mathfrak{M} = \{M_p : p \in P\} \cup \{P\} \text{ and } \mathfrak{N} = \{N_p : p \in P\} \cup \{P\}.$$

Note that although different elements $p, q \in P$ can have $M_p = M_q$ or $N_p = N_q$, $\mathfrak{M}$ and $\mathfrak{N}$ are sets, not multisets, so they contain only one copy of each possible strict down-set or up-set. Interval orders can be characterized as follows:

Proposition 2.1.1 ([9, Theorem 1]). *For a poset P, the following conditions are equivalent:*

1. $(P, \leq)$ *is an interval order.*

2. $(P, \leq)$ *contains no induced* $\mathbf{2} + \mathbf{2}$.

3. $(\mathfrak{M}, \subseteq)$ *is a chain.*

4. $(\mathfrak{N}, \subseteq)$ *is a chain.*

Corollary 2.1.2 ([9, Corollary 1]). *If* P *is an interval order, then* $|\mathfrak{M}| = |\mathfrak{N}|$.

From now on, P denotes an interval order with n elements, unless otherwise stated. Let us denote $s = |\mathfrak{M}| - 1$ and assume

$$\mathfrak{M} = (M_0, M_1, ...M_{s-1}, M_s) \text{ where } \emptyset = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = P$$

and

$$\mathfrak{N} = (N_0, N_1, N_2, ..., N_s) \text{ where } P = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_s = \emptyset.$$

For every $p \in P$ let $l(p) \in \{0, 1, ..., s-1\}$ be the number such that $M_p = M_{l(p)}$ and let $r(p) \in \{1, 2, ..., s\}$ be the number such that $N_p = N_{r(p)}$.

Corollary 2.1.3 ([9, Corollary 2]). *The assignment* $P \to \{I_p = (l(p), r(p)) : p \in P\}$ *is an interval representation of* P.

We call $\{I_p\}_{p \in P}$ defined above the *canonical representation* of P.

Remark In general, a representation of an interval order by a collection of intervals in the real line is not unique. In particular, for a finite interval order one can always shrink an interval by some small amount without changing the order relations. Therefore, a multiset of intervals can also be considered a representation of an interval order, with the understanding that the intervals could be slightly changed to make it a set if one so desires. Note that the canonical representation $\{I_p = (l(p), r(p)) : p \in P\}$ of an interval order P defined in Corollary 2.1.3 can be a multiset.

Example Figure 2-1 shows an interval order P and its canonical representation. For the element marked by x, $M_x = \{p_1\}$ and $N_x = \{p_6, p_7, p_8, p_9, p_{10}\}$. $\mathfrak{M}$ and $\mathfrak{N}$ are of size 7 so $s = 6$, and $l(x) = 1$, $r(x) = 3$.

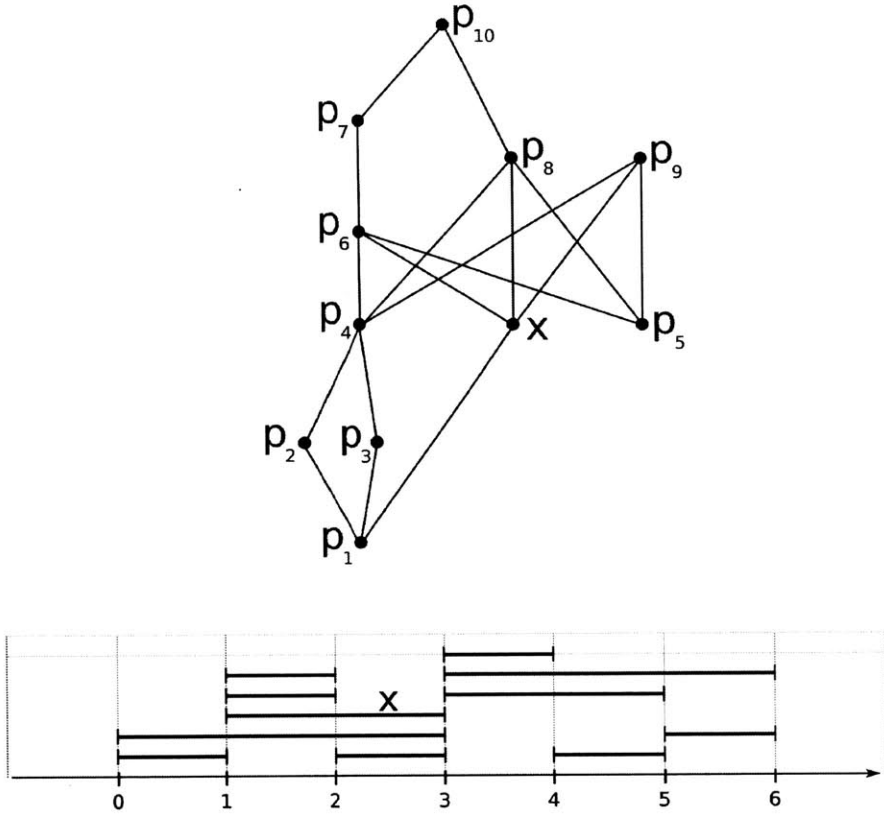


Figure 2-1: An interval order and its canonical representation

In this chapter we think of P mainly in terms of the multiset of intervals $\{I_p\}$, and we will use facts that follow from the validity of the construction (Corollary 2.1.3), for example:

- $l(p) < r(p)$ for all $p \in P$.
- $p_1 < p_2$ in P if and only if $r(p_1) \leq l(p_2)$.
- p_1 and p_2 are incomparable in P if and only if I_{p_1} and I_{p_2} intersect.

2.2 Antichains of Interval Orders

Definition 2.2.1. *Let P be an interval order with n elements. For $0 \leq i \leq s-1$ define*

$$E_i := \{p \in P : l(p) = i\} \text{ and } e_i := |E_i|.$$

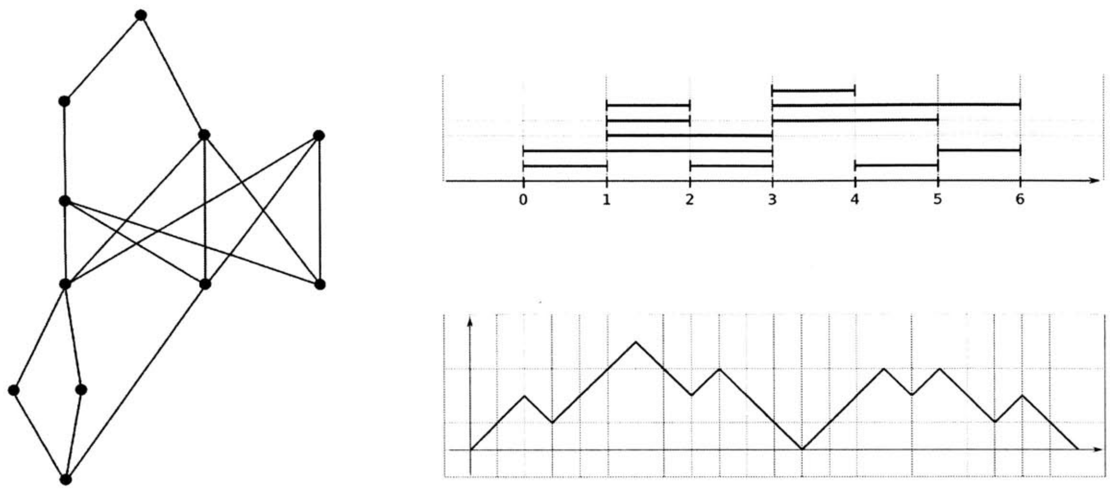


Figure 2-2: An interval order P, its canonical representation and the corresponding Dyck path $d(P)$

For $1 \leq i \leq s$ *define*

$$F_i := \{p \in P : r(p) = i\} \text{ and } f_i := |F_i|.$$

For ease of notation, set $E_s = F_0 = \emptyset$ *and* $e_s = f_0 = 0$ *(clearly, by definition, no other sets are empty).*

Let $d(P)$ *denote a lattice path in the* (x, y) *plane starting at* $(0, 0)$ *with steps* $(1, 1)$ *("up") and* $(1, -1)$ *("down") as follows:* e_0 *steps up,* f_1 *steps down,* e_1 *steps up,* f_2 *steps down, and so on.*

Example Figure 2-2 shows the interval order P of Figure 2-1, its canonical construction of intervals and the path $d(P)$. The first ascent of $d(P)$ is of length $e_0 = 2$ (there are two intervals of P starting at 0), and the first descent of $d(P)$ is of length $f_1 = 1$ (there is one interval of P ending at 1).

Consider the lattice path $d(P)$. It comprises of s ascents and s descents, therefore it has exactly s peaks and $s - 1$ valleys. The heights (y values) of the peaks of $d(P)$ are of the form $\sum_{i=0}^{j} e_i - \sum_{i=0}^{j} f_i$ for $0 \leq j \leq s - 1$. By definition, this is equal to $\sum_{i=0}^{j} |E_i| - \sum_{i=0}^{j} |F_i|$ and since the elements in $\bigcup_{i=0}^{j} F_i$ are contained in $\bigcup_{i=0}^{j} E_i$ (because $l(p) < r(p)$ for all $p \in P$), the last expression is equal to $|\bigcup_{i=0}^{j} E_i \setminus \bigcup_{i=0}^{j} F_i|$, which is the number of elements $p \in P$ with $l(p) \leq j$ and $r(p) > j$. Similarly, the heights of the valleys of $d(P)$ are of the form $\sum_{i=0}^{j-1} e_i - \sum_{i=0}^{j} f_i$ for $1 \leq j \leq s-1$, which is the number of elements $p \in P$ with $l(p) < j$ and $r(p) > j$. The following corollary is an easy consequence of these observations.

Corollary 2.2.2. *The lattice path $d(P)$ is a Dyck path of semilength n.*

Proof. Clearly, $\sum_{i=0}^{s-1} e_i = \sum_{i=1}^{s} f_i = n$, so $d(P)$ has n steps up and n steps down, hence it is a path from $(0,0)$ to $(2n,0)$. Since the heights of the valleys of the path are given by the numbers of elements $p \in P$ such that $l(p) < j$ and $r(p) > j$ for $1 \leq j \leq s-1$, these heights are nonnegative and thus the path does not pass below the x axis. □

We are now ready to prove our first main theorem, Theorem 1.1.2:

Theorem 1.1.2. Let P be an n-element interval order with a corresponding Dyck path $d(P)$. Let Peaks $= \{u_1, u_2, ..., u_s\}$ be the set of peaks of $d(P)$, and let Valleys $= \{v_1, v_2, ..., v_{s-1}\}$ be the set of valleys of $d(P)$. Then the number of antichains of size k of P is equal to the coefficient of t^k in

$$\sum_{j=1}^{s} (1+t)^{h(u_j)} - \sum_{j=1}^{s-1} (1+t)^{h(v_j)}.$$

where h is the height function, i.e., $h((x,y)) = y$.

Example For our running example (Figure 2-2), the heights of the peaks of $d(P)$ are $2,4,3,3,3,2$ and the heights of the valleys are $1,2,0,2,1$. We therefore have

$$\sum_{j=1}^{s} (1+t)^{h(u_j)} - \sum_{j=1}^{s-1} (1+t)^{h(v_j)} = 1 + 11t + 15t^2 + 7t^3 + t^4$$

Indeed, the interval order P has one 0-size antichain (the empty antichain), 11 antichains of size 1 (its 11 elements), 15 antichains of size 2, etc.

Proof. Let $A \subseteq P$ be a nonempty antichain, and let $j = max_{a \in A} l(a)$. We can split the elements of P into four types as follows.

1. Elements p with $l(p) = j$ (recall that these elements constitute the set E_j).

2. Elements p with $l(p) > j$.

3. Elements p with $l(p) < j$ and $r(p) > j$ (we denote the set of these elements by G_j, and set $G_0 = \emptyset$).

4. Elements p with $r(p) \leq j$.

By the definition of j, A must contain at least one element of type (1), and no element of type (2). Moreover, A cannot contain any element of type (4), since these elements are less than the elements of type (1). Hence any nonempty antichain of P gives rise to an index $0 \leq j \leq s-1$, a nonempty subset of E_j and a (possibly empty) subset of G_j. We claim that the converse is also true. For any choice of $0 \leq j \leq s-1$, the elements of types (1) and (3) are all incomparable (as intervals in the canonical representation of P, they all contain the segment $(j, j+1)$ which means they all intersect), so any choice of a nonempty subset of E_j and a (possibly empty) subset of G_j results in an antichain A of P with $max_{a \in A} l(a) = j$.

The number of ways to choose a nonempty subset of E_j and a (possibly empty) subset of G_j such that their sizes sum up to k is precisely the coefficient of t^k in

$$((1+t)^{|E_j|} - 1)(1+t)^{|G_j|}.$$

Summing up over all $0 \leq j \leq s-1$ and adding 1 for the empty antichain, we see that for any $k \geq 0$, the number of antichains of size k in P is the coefficient of t^k in

$$1 + \sum_{j=0}^{s-1} ((1+t)^{e_j} - 1)(1+t)^{|G_j|} = 1 + \sum_{j=0}^{s-1} (1+t)^{|G_j|+e_j} - (1+t)^{|G_j|}.$$

Now recall our observations regarding the Dyck path $d(P)$: for $j = 0$, $|G_0| = 0$ and $|G_0| + e_0 = e_0$ which is the height of the first peak u_1. For $0 < j \leq s-1$, the size of G_j is precisely the height of the valley v_j, and $|G_j| + e_j$ is the height of the adjacent peak u_{j+1}. The above sum now turns into

$$1+\sum_{j=0}^{s-1}(1+t)^{h(u_{j+1})}-[1+\sum_{j=1}^{s-1}(1+t)^{h(v_j)}]=\sum_{j=1}^{s}(1+t)^{h(u_j)}-\sum_{j=1}^{s-1}(1+t)^{h(v_j)}$$

and this completes the proof.

□

2.3 Antichains of Semiorders

Recall that a semiorder is an interval order P that has a representation by intervals where all the intervals are unit intervals (although the intervals of the canonical representation of a semiorder are not necessarily unit intervals). Equivalently, a semiorder is an interval order that contains no induced $\mathbf{3}+\mathbf{1}$ (or a poset that contains no induced $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$). Semiorders are Catalan objects, i.e., the number of n-element semiorders is the Catalan number C_n.

In this section we use Theorem 1.1.2 to compute a bivariate generating function for the total number of antichains of all finite semiorders, enumerated by the size of the semiorders and the size of the antichain. We start by proving some preliminary results.

Lemma 2.3.1. *Let P be an interval order. Then P is a semiorder if and only if the following condition holds:*

$$\textit{for any } p,q\in P,\ l(p)<l(q)\Rightarrow r(p)\leq r(q) \tag{2.1}$$

Proof. Assume $p,q\in P$ satisfy $l(p)<l(q)$ and $r(p)>r(q)$ (see Figure 2-3). We show that P contains an induced $\mathbf{3}+\mathbf{1}$, therefore it is not a semiorder. Recall the construction of $\mathfrak{M}$ and $\mathfrak{N}$ in Section 2.1: denote $l(q)=i$, then $0\leq l(p)<i<s$ and the (nonempty) set N_i must have come from at least one element $x\in P$ such that $N_i=N_x$, which means $r(x)=i=l(q)$. Similarly, there must be at least one element $y\in P$ such that $l(y)=r(q)$. Looking at the corresponding intervals in the canonical

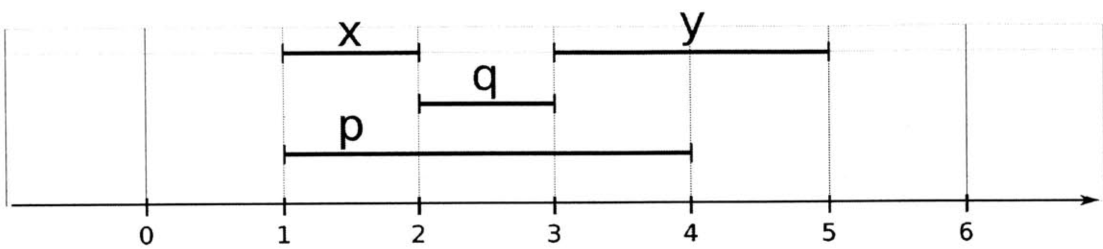


Figure 2-3: Illustration of the intervals in the first part of the proof of Lemma 2.3.1

representation of P one sees that $x < q < y$ is a chain of length three and all three elements are incomparable with p because they intersect it. So these four elements form an induced $\mathbf{3}+\mathbf{1}$ in P, which implies that P is not a semiorder.

Now suppose P is an interval order, and suppose the condition (2.1) holds. We show that P does not contain an induced $\mathbf{3}+\mathbf{1}$. Assume $x, q, y, p \in P$ are an induced $\mathbf{3}+\mathbf{1}$ in P, so $x < q < y$ and all three elements are incomparable with p. Looking at the corresponding intervals in the canonical representation of P, we see that $x < q$ implies $r(x) \leq l(q)$. Moreover, x and p are incomparable so they intersect, and in particular $l(p) < r(x)$. Hence $l(p) < l(q)$. Similarly, we have $r(q) \leq l(y)$ and $l(y) < r(p)$, hence $r(p) > r(q)$. But this is a contradiction to condition (2.1), so there is no induced $\mathbf{3}+\mathbf{1}$ in P, and therefore P is a semiorder. □

Proposition 2.3.2. *The map $P \to d(P)$ is a bijection between n-element semiorders and Dyck paths of semilength n.*

Proof. Let d be a Dyck path of semilength n. Identifying interval orders P with their canonical multiset of intervals $\{I_p\}_{p \in P}$, we try to "read" from d a multiset I of n intervals that is mapped to it via the map $P \to d(P)$, and is also a semiorder.

Let s be the number of peaks of d, then the intervals in I should have start and end points in $\{0, 1, ...s\}$. For $1 \leq i \leq s$ let e_{i-1}, f_i be the length of the i^{th} ascent or descent of d, respectively (so e_0 is the length of the first ascent, f_1 is the length of the first descent, and so on). In the map $P \to d(P)$, we know that $e_i := |\{p \in P : l(p) = i\}|$ and $f_i := |\{p \in P : r(p) = i\}|$. So we know how many intervals of I should start at each of the points $\{0, 1, ..., s-1\}$ and how many intervals should end at each of the points $\{1, 2, ..., s\}$, and all we have to decide is how to connect the start points to the end points. Clearly, any matching of the start points to the end points such that the start points are less than their respective end points results in a multiset of intervals that is the canonical representation of some interval order, and is mapped to d under the map $P \to d(P)$. We show that there is exactly one matching that results in a semiorder.

By Lemma 2.3.1, I is a multiset of intervals of a semiorder if and only if for any $0 \leq i \leq s-2$, any interval that starts at the point i ends at a point that is less than or equal to the end point of any interval that starts at the point $i+1$. In other words: denote by R_i the multiset of end points of the intervals of I that start at the point i. Then I is the multiset of intervals of a semiorder if and only if the elements of R_i are all less than or equal to the elements of R_{i+1}, for all $0 \leq i \leq s-2$. This condition is equivalent to the following simple greedy algorithm for matching start points and end points:

- Start by matching end points to the intervals starting at 0, then the intervals starting at 1 and so on up to $s-1$.
- When matching end points to the e_i intervals starting at i, the multiset R_i of end points consists of the e_i lowest "copies" of end points that haven't already been used in $R_0, ..., R_{i-1}$ (initially there are f_j copies of the end point j, for $1 \leq j \leq s$).

Note that since our posets and intervals are unlabeled, the multisets R_i suffice to define the matching of start points and end points. We give an example for the matching algorithm below.

It is easy to see that since d does not pass below the x axis, our matching algorithm indeed results with n valid intervals. Moreover, as discussed above, these intervals are the canonical representation of a semiorder and this semiorder is the only semiorder mapped to d under the map $P \to d(P)$. Hence the map $P \to d(P)$ is a bijection between n-element semiorders and Dyck paths of semilength n. □

Example Consider Figure 2-4. The Dyck path d at the top has $e_0 = 2, e_1 = 3, e_2 = 1$ etc. and $f_1 = 1, f_2 = 2, f_3 = 3$ etc. In order to build a semiorder that is mapped to d, we start with the two intervals starting at 0. One of them must end at 1, and since there is only one interval ending at 1 the second interval must end at 2 (Figure a). We now have three intervals starting at 1, only one more interval that should end at 2, and three intervals that should end at 3. So one of the intervals starting at 1 must end at 2, and the other two must end at 3 (Figure b). Continuing in this manner, we construct the intervals as shown in Figure c.

We can now prove the second main theorem of this chapter:

Theorem 1.1.3. Let $S_{n,k}$ be the total number of antichains of size k of all n-element semiorders, and set $S_{0,0} = 1$ (clearly, $S_{n,k} = 0$ for $k > n$). Let $S(x,t)$ be the bivariate generating function for the total number of antichains of semiorders, enumerated by the size of the antichain and the size of the semiorders, i.e., $S(x,t) := \sum_{n,k\geq 0} S_{n,k}t^k x^n$. Then

$$S(x,t) = \frac{C(x) - 2}{x(t+2)C(x) - 1}$$

where $C(x)$ is the generating function of the Catalan numbers.

Example There are five 3-element semiorders, as shown in Figure 2-5. For $n = 3$, we have $S_{3,0} = 5$ (each semiorder contributes one empty antichain), $S_{3,1} = 15$ (the total number of elements in all semiorders), $S_{3,2} = 7$ and $S_{3,3} = 1$ (the 3-element antichain of semiorder A).

Proof. Let $S_n(t) := \sum_{k\geq 0} S_{n,k}t^k$ be the total number of antichains of all n-element semiorders enumerated by size. For a Dyck path d define:

$$a_d(t) := \sum_{u\in \mathrm{Peaks}(d)} (1+t)^{h(u)} - \sum_{v\in \mathrm{Valleys}(d)} (1+t)^{h(v)}$$

where Peaks(d), Valleys(d) are the sets of peaks and valleys of d, respectively. By Theorem 1.1.2 and Proposition 2.3.2, for $n > 0$ we have $S_n(t) = \sum_{d\in D_n} a_d(t)$ where the summation is over all Dyck paths of semilength n. By definition, $S_0(t) = 1$.

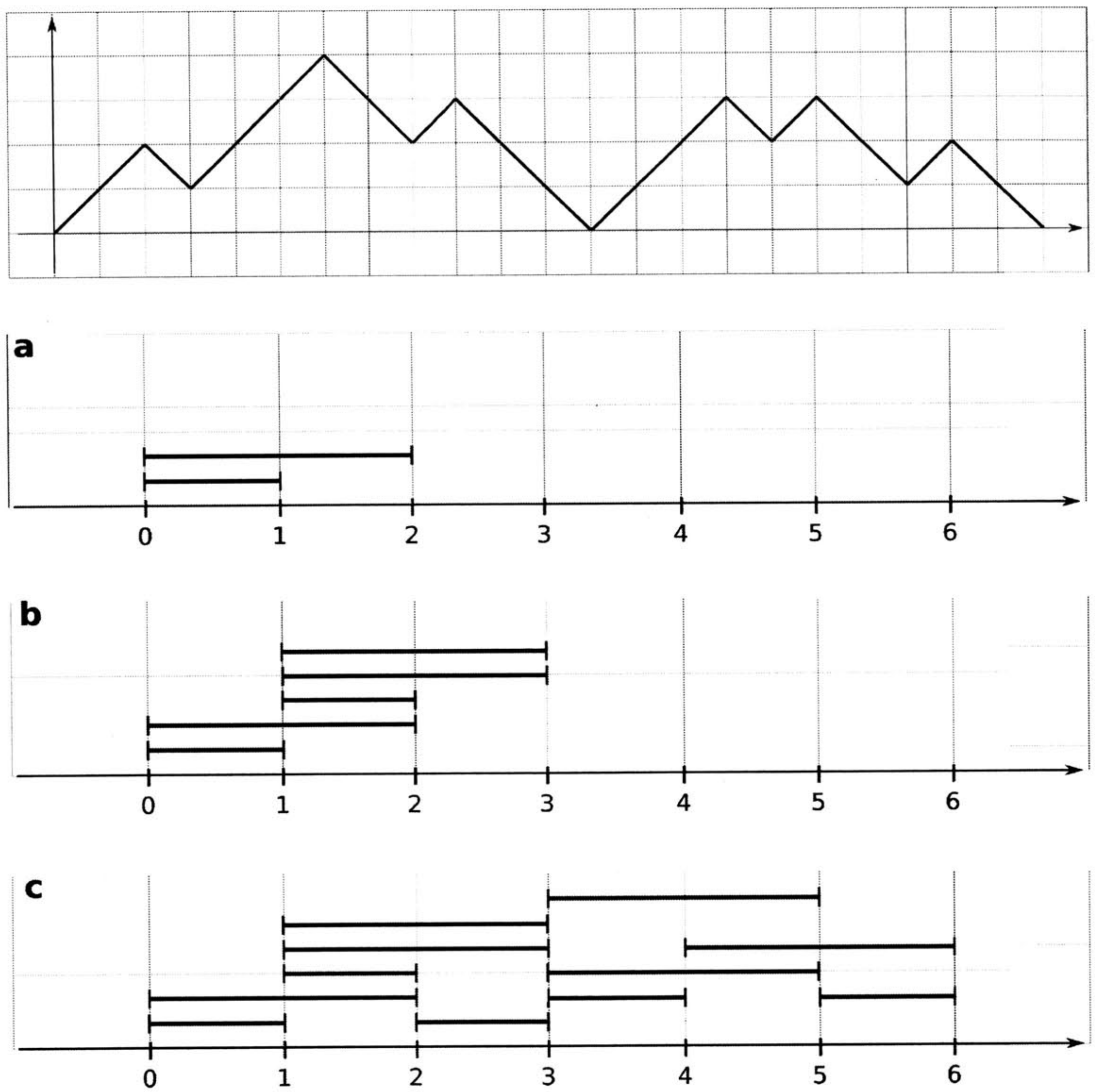


Figure 2-4: Reading the canonical representation of a semiorder from a Dyck path

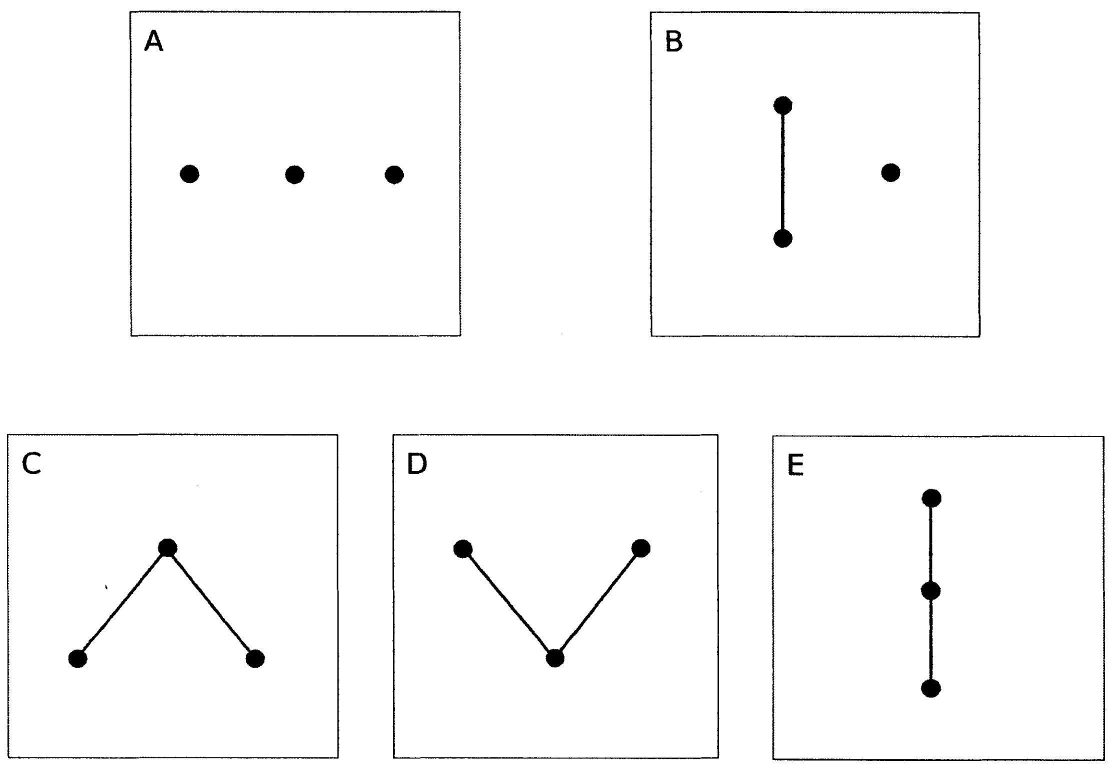


Figure 2-5: All 3-element semiorders

Lemma 2.3.3. *For all $n \geq 0$ we have:*

$$S_{n+1}(t) = (t+2)\sum_{j=0}^{n} S_j(t)C_{n-j} - C_{n+1} \tag{2.2}$$

We prove the lemma below. To finish the proof of the theorem, multiply both sides of equation (2.2) by x^n, and sum over $n \geq 0$. The left-hand side becomes

$$\sum_{n\geq 0} S_{n+1}(t)x^n = \frac{\sum_{n\geq 0} S_{n+1}(t)x^{n+1}}{x} = \frac{S(x,t) - S_0(t)}{x} = \frac{S(x,t)-1}{x},$$

while the right-hand side becomes

$$(t+2)\sum_{n\geq 0}\sum_{j=0}^{n} S_j(t)C_{n-j}x^n - \sum_{n\geq 0} C_{n+1}x^n = (t+2)S(x,t)C(x) - \frac{C(x)-1}{x}.$$

Equating the two expressions and simplifying, we get $S(x,t) = \frac{C(x)-2}{x(t+2)C(x)-1}$.

□

Proof of Lemma 2.3.3. We start by establishing a recursive relation satisfied by $a_d(t)$, using the same idea as in the proof of Theorem 1.1.1 of splitting a Dyck path d of semilength $n+1$ into two shorter Dyck paths d_1 and d_2 whose semilengths sum to n. Recall that we denote by $(2i+2,0)$ the leftmost point where d touches the x axis, and the paths d_1 and d_2 are the parts of d from $(1,1)$ to $(2i+1,1)$ and from $(2i+2,0)$ to $(2n,0)$ (shifted appropriately), respectively (see Figure 1-3). When splitting the Dyck path d in this manner the heights of the peaks of d are precisely the heights of the peaks of d_2, and the heights of the peaks of d_1 plus 1. The heights of the valleys of d are the heights of the valleys of d_2, the heights of the valleys of d_1 plus 1, and there is one additional valley of height 0: the point $(2i+2,0)$ which is not a valley of either d_1 or d_2. We get the relation

$$a_d(t) = (1+t)a_{d_1}(t) + a_{d_2}(t) - 1 \tag{2.3}$$

Note that when $i=0$ ($i=n$), we get that a_{d_1} (a_{d_2}) is empty. Setting $a_{d_0}(t) = 1$ for the empty Dyck path d_0 makes the relation hold for all $0 \leq i \leq n$, and is compatible with the equation $S_n(t) = \sum_{d \in D_n} a_d(t)$ since we have set $S_0(t) = 1$ (we think of D_0 as containing one Dyck path, d_0).

For $n \geq 0$ define $A_n(y,t) := \sum_{d \in D_n} y^{a_d(t)}$. Since the partition of d into d_1 and d_2 is a bijection $D_{n+1} \cong \bigcup_{i=0}^{n} D_i \times D_{n-i}$, by equation (2.3) we get:

$$\begin{aligned} A_{n+1}(y,t) &= \sum_{d \in D_{n+1}} y^{a_d(t)} = \sum_{i=0}^{n} \sum_{d_1 \in D_i} \sum_{d_2 \in D_{n-i}} y^{(1+t)a_{d_1}(t) + a_{d_2}(t) - 1} \\ &= \sum_{i=0}^{n} \left(\sum_{d_1 \in D_i} y^{(1+t)a_{d_1}(t)} \right) \left(\sum_{d_2 \in D_{n-i}} y^{a_{d_2}(t)} \right) y^{-1} \\ &= \sum_{i=0}^{n} A_i(y^{(1+t)}, t) \cdot A_{n-i}(y,t) \cdot y^{-1} \qquad (2.4) \end{aligned}$$

$A_n(y,t)$ is a polynomial in y, so let $A'_n(y,t)$ denote $A_n(y,t)$ differentiated with respect to y. Substituting $y = 1$ in these expressions we see that:

$$A_n(1,t) = \sum_{d \in D_n} 1^{a_d(t)} = |D_n| = C_n$$

and

$$A'_n(1,t) = \sum_{d \in D_n} (a_d(t))(1^{a_d(t)-1}) = \sum_{d \in D_n} a_d(t) = S_n(t)$$

We are now ready for the final steps of the proof of the lemma. Differentiate equation (2.4) with respect to y to get:

$$\begin{aligned} A'_{n+1}(y,t) &= \sum_{i=0}^{n} (1+t) y^t A'_i(y^{(1+t)}, t) A_{n-i}(y,t) y^{-1} \\ &\quad + A_i(y^{(1+t)}, t) A'_{n-i}(y,t) y^{-1} \\ &\quad - A_i(y^{(1+t)}, t) A_{n-i}(y,t) y^{-2} \qquad (2.5) \end{aligned}$$

Substitute $y = 1$ in both sides of equation (2.5) and use the Catalan numbers' recursion $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ to get:

$$\begin{aligned} S_{n+1}(t) &= \sum_{i=0}^{n} (1+t) S_i(t) C_{n-i} + C_i S_{n-i}(t) - C_i C_{n-i} \\ &= (t+2) \sum_{i=0}^{n} S_i(t) C_{n-i} - C_{n+1}. \end{aligned}$$

□

An interesting consequence of Theorem 1.1.3 is that the numbers $S_{n,k}$ are in fact equal to the number triangle which is the sequence A090285 in [3] (see Table 2.1 for the first five rows of the triangle).

Corollary 2.3.4. *Let $T_{n,k}$ be the number triangle which is sequence A090285 in [3] defined by:*

- $T_{n,k} := 0$ *if* $k > n$
- $T_{n,0} := C_n$
- $T_{n+1,k} := \sum_{j=0}^{n} T_{n-j,k-1} \binom{2j+1}{j+1}$ *for* $1 \le k \le n+1$

Then $T_{n,k} = S_{n,k}$ *for all* $n, k \ge 0$.

Apart from the formula for the numbers $T_{n,k}$ given in Corollary 2.3.4, it is mentioned in entry A090285 of [3] (without reference) that this number triangle is the Riordan array $(C(x), \frac{xC(x)^2}{1-xC(x)^2})$.

Let $T_n := \sum_{k=0}^{n} T_{n,k}$ be the sequence of row sums of the triangle $T_{n,k}$. It follows from Corollary 2.3.4 that T_n is the total number of anticahins of n-element semiorders. The sequence $\{T_n\}_{n\ge0}$ is the entry A090317 of [3], and other than being the row sums of $T_{n,k}$ it is mentioned in that entry that this sequence is achieved by applying the inverse of the Riordan array $(\frac{1}{1-x^2}, \frac{x}{(1+x)^2})$ to the sequence $\{2^n\}_{n\ge0}$. By [1, Proposition 7 and Table 2], the sequence $\{T_n\}_{n\ge0}$ is also the inverse generalized Ballot transform of the sequence $\{\frac{2^n+0^n}{2}\}_{n\ge0} = 1, 1, 2, 4, 8, 16, 32, ...$ (which is simply the sequence $\{2^n\}_{n\ge0}$ with an additional initial term 1). See [1] for further information on Riordan arrays.

$n \setminus k$	0	1	2	3	4
0	1				
1	1	1			
2	2	4	1		
3	5	15	7	1	
4	14	56	37	10	1

Table 2.1: The first five rows of [3, sequence A090285]

Proof of Corollary 2.3.4. We compute a bivariate generating function for $T_{n,k}$ that turns out to be equal to the bivariate generating function for $S_{n,k}$.

Let $T(x,t) := \sum_{n,k\geq 0} T_{n,k}t^k x^n$ be the generating function for $T_{n,k}$. By the definition of $T_{n,k}$

$$T(x,t) = \sum_{n\geq 0} C_n x^n + \sum_{n\geq 1}\sum_{k=1}^{n} T_{n,k}t^k x^n \tag{2.6}$$

Note that in the second sum n starts from 1 because for $n=0$, $T_{0,k}$ is nonzero only for $k=0$.

Multiply both sides of the definition $T_{n+1,k} := \sum_{j=0}^{n} T_{n-j,k-1}\binom{2j+1}{j+1}$ for $1 \leq k \leq n+1$ by $t^{k-1}x^n$ and sum over all $n \geq 0$ and $1 \leq k \leq n+1$. The left-hand side becomes:

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=1}^{n+1} T_{n+1,k}t^{k-1}x^n &= \frac{\sum_{n\geq 0}\sum_{k=1}^{n+1} T_{n+1,k}t^k x^{n+1}}{xt} \\ &= \frac{\sum_{n\geq 1}\sum_{k=1}^{n} T_{n,k}t^k x^n}{xt} = \frac{T(x,t)-C(x)}{xt}\end{aligned}$$

where the last equality follows from equation (2.6). Before computing the right-hand side of the equation, we mention the following computation of a generating function for the sequence $f_n = \binom{2n+1}{n+1}$:

$$\begin{aligned}\sum_{n\geq 0}\binom{2n+1}{n+1}x^n &= \sum_{n\geq 1}\binom{2n-1}{n}x^{n-1} = \frac{\sum_{n\geq 0}\binom{2n-1}{n}x^n - 1}{x} \\ &\overset{(*)}{=} \frac{\frac{1}{2}\sum_{n\geq 0}\binom{2n}{n}x^n + \frac{1}{2} - 1}{x} \overset{(*)}{=} \frac{\frac{1}{\sqrt{1-4x}} - 1}{2x} = \frac{2-C(x)}{1-4x}.\end{aligned}$$

See [6, exercise 1.8] for details on the equalities marked with $(*)$.

Back to our main computation, after multiplying the definition of $T_{n+1,k}$ by $t^{k-1}x^n$ and summing over all $n \geq 0$ and $1 \leq k \leq n+1$ the right-hand side becomes:

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{j=0}^{n}\binom{2j+1}{j+1}\left(\sum_{k=1}^{n+1}T_{n-j,k-1}t^{k-1}\right)x^n \\
=\ &\sum_{n\geq 0}\sum_{j=0}^{n}\binom{2j+1}{j+1}\left(\sum_{k=0}^{n-j}T_{n-j,k}t^{k}\right)x^n \\
=\ &\left(\sum_{n\geq 0}\binom{2n+1}{n+1}x^n\right)\left(\sum_{n\geq 0}\sum_{k=0}^{n}T_{n,k}t^k x^n\right) \\
=\ &\frac{2-C(x)}{1-4x}T(x,t)
\end{aligned}$$

Equating the two sides and simplifying, we get $T(x,t) = \frac{C(x)(1-4x)}{1-4x-xt(2-C(x))}$. Comparing this generating function to the generating function $S(x,t) = \frac{C(x)-2}{x(t+2)C(x)-1}$ of Theorem 1.1.3 (and using the formula for $C(x)$), one can see they are equal, which implies that for all $n, k \geq 0$ we have $T_{n,k} = S_{n,k}$. □

2.4 Additional Results

Theorem 2.4.1. *Let $B_{n,k}$ be the total number of k-element subsets of minimal elements of n-element semiorders, and let $B(x,t)$ be the corresponding bivariate generating function: $B(x,t) := \sum_{n,k\geq 0} B_{n,k}t^k x^n$. Then $B(x,t) = \frac{1}{1-(1+t)xC(x)}$.*

Example Consider the 3-element semiorders shown in Figure 2-5, and let us compute $B_{3,2}$. Semiorder A has three minimal elements, so it has three 2-element subsets of minimal elements. Semiorders B and C have two minimal elements each, so they each have one 2-element subset of minimal elements. Semiorders D and E have only one minimal element each, so they have no 2-element subsets of minimal elements. Therefore $B_{3,2} = 3 + 1 + 1 = 5$.

Proof. This proof uses once again the idea of splitting a Dyck path d into two shorter Dyck paths d_1 and d_2. An element p of a semiorder P is minimal if and only if $M_p = \emptyset$ (recall that M_p is the set of elements less than p), which is equivalent to the condition

$l(p) = 0$ or $p \in E_0$. So the number of minimal elements of P is the height e_0 of the first peak of $d(P)$. Denote by $h_0(d)$ the height of the first peak of a Dyck path d. Then the coefficient of t^k in the expression $(1+t)^{h_0(d(P))}$ equals the number of k-element subsets of minimal elements of P. Therefore $B_{n,k}$ is equal to the coefficient of t^k in $B_n(t) := \sum_{d \in D_n} (1+t)^{h_0(d)}$, and we have $B(x,t) = \sum_{n \geq 0} B_n(t) x^n$.

When partitioning the Dyck path d into the paths d_1 and d_2, recall that $h_0(d) = h_0(d_1) + 1$. If d starts with one up step and one down step so that d_1 is the empty Dyck path d_0, we want $h_0(d) = 1$, so we set $h_0(d_0) = 0$. Recall that we think of D_0 as containing the single Dyck path d_0, so $B_0(t) = 1$. Using the bijection $D_{n+1} \cong \bigcup_{i=0}^{n} D_i \times D_{n-i}$, we get:

$$\begin{aligned} B_{n+1}(t) &= \sum_{d \in D_{n+1}} (1+t)^{h_0(d)} = \sum_{i=0}^{n} \sum_{d_1 \in D_i} \sum_{d_2 \in D_{n-i}} (1+t) \cdot (1+t)^{h_0(d_1)} \\ &= \sum_{i=0}^{n} (1+t) \left(\sum_{d_1 \in D_i} (1+t)^{h_0(d_1)} \right) \left(\sum_{d_2 \in D_{n-i}} 1 \right) = \sum_{i=0}^{n} (1+t) B_i(t) C_{n-i} \end{aligned} \tag{2.7}$$

Multiply equation (2.7) by x^n and sum over all $n \geq 0$. The left-hand side becomes

$$\sum_{n \geq 0} B_{n+1}(t) x^n = \frac{\sum_{n \geq 0} B_n(t) x^n - 1}{x} = \frac{B(x,t) - 1}{x}$$

and the right-hand side becomes

$$\sum_{n \geq 0} \sum_{i=0}^{n} (1+t) B_i(t) C_{n-i} x^n = (1+t) B(x,t) C(x)$$

Equating the two expressions and simplifying, we get

$$B(x,t) = \frac{1}{1 - (1+t) x C(x)}$$

□

Corollary 2.4.2. *For $n \geq 0$, the total number of subsets of minimal elements of n-element semiorders equals $\binom{2n}{n} = (n+1) C_n$. Since these include precisely C_n empty* *subsets of minimal elements (one for each n-element semiorder), the total number of nonempty subsets of minimal elements of n-element semiorders is nC_n.*

Example The five 3-element semiorders of Figure 2-5 have five empty subsets of minimal elements (one for each semiorder), nine 1-element subsets of minimal elements (the total number of their minimal elements), five 2-element subsets of minimal elements and one 3-element subset of minimal elements. In total the 3-element semiorders have 20 subsets of minimal elements, which is equal to $\binom{6}{3}$. Subtracting the empty subsets, the 3-element semiorders have a total of 15 nonempty subsets of minimal elements, which is equal to $3C_3$.

Proof. By the definition of $B(x,t)$, substituting $t = 1$ yields a generating function for the total number of subsets of minimal elements of semiorders:

$$B(x,1) = \sum_{n\geq 0}\Big(\sum_{0\leq k\leq n} B_{n,k}\Big)x^n = \frac{1}{1-2xC(x)}.$$

By [6, exercise 1.8], a generating function for the sequence $f_n = \binom{2n}{n}$ is given by

$$D(x) := \sum_{n\geq 0}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}.$$

Comparing the two expressions one sees that $B(x,1) = D(x)$, and therefore for all $n \geq 0$, the total number of subsets of minimal elements of n-element semiorders equals $\binom{2n}{n}$. □

Remark By symmetry, Theorem 2.4.1 and corollary 2.4.2 also hold for subsets of maximal elements of semiorders.

Chapter 3

Dilworth lattices

3.1 Background and Main Ideas

Let L_t^n be the coordinate-wise partial ordering on SSYT of shape $\delta_t = (t-1, t-2, ..., 1)$ with largest part at most n. Recall that $\sigma(L_t^n)$ is the maximum possible number of reducible entries in a tableau in L_t^n, and $K(L_t^n)$ is the subposet of L_t^n that consists of the tableaux with $\sigma(L_t^n)$ reducible entries. The (a, b) entry in a SSYT T is reducible if and only if $T_{a,b} - T_{a,b-1} \geq 1$ and $T_{a,b} - T_{a-1,b} \geq 2$. In their paper [2], Liu and Stanley define M_p, the coordinate-wise partial order on SSYT of shape δ_{p-1} with largest part at most $p-1$, which in our notation is $M_p = L_{p-1}^{p-1}$. They then determine $\sigma(L_{p-1}^{p-1})$ and $K(L_{p-1}^{p-1})$ by analyzing the structure of tableaux in L_{p-1}^{p-1} that have the maximum possible number of reducible entries.

It turns out that the key ideas described in [2, Section 3] extend far and beyond the case L_{p-1}^{p-1}, and are useful not only for determining $\sigma(L_t^n)$ and $K(L_t^n)$ for all $n \geq t \geq 1$, but also for the analysis of σ and K for lattices that comprise of SSYT of other shapes.

In this section we state and prove some general lemmas and corollaries that will be used throughout this chapter. These results are based on results from [2, Section 3] and although Liu and Stanley only prove the "vertical" versions (the second parts of Lemma 3.1.1, Corollary 3.1.2 and Lemma 3.1.6, as well as Lemma 3.1.8) for the special case of the tableaux in L_{p-1}^{p-1}, the ideas of the proofs are evidently very similar.

In order to exemplify the use of the general results in this section, we apply them to the lattice L_t^n. We leave the rest of the analysis of the lattice L_t^n (which is more specific) to Section 3.2.

Lemma 3.1.1 (based on [2, Lemma 3.3]). *Let T be a SSYT of shape λ with largest part at most n.*

1. *For any $1 \leq a \leq l(\lambda)$,*

$$\#\textit{reducible entries in the ath row of } T \leq \min(n-a, \lambda_a).$$

Therefore,

$$\#\textit{ reducible entries in } T \leq \sum_{a=1}^{l(\lambda)} \min(n-a, \lambda_a).$$

2. *For any $1 \leq b \leq l(\lambda')$,*

$$\#\textit{reducible entries in the bth column of } T \leq \min(n-\lambda'_b, \lambda'_b).$$

Therefore,

$$\#\textit{ reducible entries in } T \leq \sum_{b=1}^{l(\lambda')} \min(n-\lambda'_b, \lambda'_b).$$

Proof. Clearly, the number of reducible entries in the ath row is at most the number of entries in the ath row, which is λ_a. Similarly, the number of reducible entries in the bth column is at most λ'_b.

1. Since the largest part of T is at most n, the last entry T_{a,λ_a} in the ath row satisfies

$$n \geq T_{a,\lambda_a} = T_{a,0} + \sum_{b=1}^{\lambda_a} (T_{a,b} - T_{a,b-1}). \tag{3.1}$$

Hence,

$$\begin{aligned} n &\geq T_{a,0} + 0 \times \#\text{irreducible entries in the } a\text{th row of } T + \\ &\quad 1 \times \#\text{reducible entries in the } a\text{th row of } T \\ &= a + \#\text{reducible entries in the } a\text{th row of } T \end{aligned} \tag{3.2}$$

Therefore the number of reducible entries in the ath row of T is at most $n - a$.

2. Similarly, the last entry $T_{\lambda'_b,b}$ in the bth column satisfies

$$n \geq T_{\lambda'_b,b} = T_{0,b} + \sum_{a=1}^{\lambda'_b}(T_{a,b} - T_{a-1,b}). \tag{3.3}$$

Hence,

$$\begin{aligned} n &\geq T_{0,b} + 1 \times \#\text{irreducible entries in the } b\text{th column of } T + \\ &\quad 2 \times \#\text{reducible entries in the } b\text{th column of } T \\ &= 0 + \#\text{entries in the } b\text{th column of } T + \\ &\quad \#\text{reducible entries in the } b\text{th column of } T \\ &= \lambda'_b + \#\text{reducible entries in the } b\text{th column of } T \end{aligned} \tag{3.4}$$

Therefore the number of reducible entries in the bth column of T is at most $n - \lambda'_b$.

□

Corollary 3.1.2 (based on [2, Corollary 3.7]). *Let T be a SSYT of shape λ with largest part at most n.*

1. *Suppose T has $\sum_{a=1}^{l(\lambda)} \min(n-a, \lambda_a)$ reducible entries (the maximum possible number of reducible entries by part (1) of Lemma 3.1.1). If the ath row of T satisfies $n - a \leq \lambda_a$, then:*

(a) *for any* $1 \leq b \leq \lambda_a$, *the* (a, b) *entry of* T *is reducible if and only if* $T_{a,b} - T_{a,b-1} = 1$, *and irreducible if and only if* $T_{a,b} - T_{a,b-1} = 0$;

(b) *the last entry of the ath row of* T *satisfies* $T_{a,\lambda_a} = n$.

If the ath row of T *satisfies* $n - a \geq \lambda_a$, *then all the entries of the ath row are reducible.*

2. *Suppose* T *has* $\sum_{b=1}^{l(\lambda')} \min(n - \lambda'_b, \lambda'_b)$ *reducible entries (the maximum possible number of reducible entries by part (2) of Lemma 3.1.1). If the bth column of* T *satisfies* $n - \lambda'_b \leq \lambda'_b$, *then:*

(a) *for any* $1 \leq a \leq \lambda'_b$, *the* (a, b) *entry of* T *is reducible if and only if* $T_{a,b} - T_{a-1,b} = 2$, *and irreducible if and only if* $T_{a,b} - T_{a-1,b} = 1$;

(b) *the last entry of the bth column of* T *satisfies* $T_{\lambda'_b,b} = n$.

If the bth column of T *satisfies* $n - \lambda'_b \geq \lambda'_b$, *then all the entries of the bth column are reducible.*

Proof. It follows from the proof of Lemma 3.1.1 that in order for the ath row of T to have $n - a$ reducible entries, we must have equalities in both equations (3.1) and (3.2), so (1)(a) and (1)(b) follow. The rest of part (1) is clear.

The proof for part (2) is similar to that of part (1), only using equations (3.3) and (3.4). □

The properties of rows and columns described in Corollary 3.1.2 are a key concept in the analysis of the various lattices in this chapter, so they deserve a definition.

Definition 3.1.3. *Suppose* T *is a SSYT of shape* λ.

1. *We say the ath row of* T *is strictly reducible if for any* $1 \leq b \leq \lambda_a$, *the* (a, b) *entry of* T *is reducible if and only if* $T_{a,b} - T_{a,b-1} = 1$, *and irreducible if and only if* $T_{a,b} - T_{a,b-1} = 0$.

2. *We say the bth column of* T *is strictly reducible if for any* $1 \leq a \leq \lambda'_b$, *the* (a, b) *entry of* T *is reducible if and only if* $T_{a,b} - T_{a-1,b} = 2$, *and irreducible if and only if* $T_{a,b} - T_{a-1,b} = 1$.

3. *If T has largest part at most n, we say the ath row (bth column) of T is maximally reducible if it is strictly reducible and its last entry satisfies $T_{a,\lambda_a} = n$ ($T_{\lambda'_b,b} = n$).*

We now apply Lemma 3.1.1 and Corollary 3.1.2 to L_t^n. Let $m := n - (t-1)$. For $\lambda = \delta_t$, the length of the bth column of λ is $\lambda'_b = t - b$, and $n - \lambda'_b \le \lambda'_b$ implies $b \le \frac{1}{2}(t - m + 1)$. By part(2) of Lemma 3.1.1, we see that $\sigma(L_t^n)$ is at most

$$\begin{aligned}\sum_{b=1}^{t-1} \min(n-(t-b), t-b) &= \sum_{b=1}^{t-1} \min(m+b-1, t-b) \\ &= \begin{cases} l(l-1) - \binom{m}{2} & \text{if } t+m = 2l; \\ l^2 - \binom{m}{2} & \text{if } t+m = 2l+1. \end{cases}\end{aligned}$$

Let K_t^n be the subposet of L_t^n consisting of the tableaux that have $\sum_{b=1}^{t-1} \min(m+b-1, t-b)$ reducible entries. Then if K_t^n is not empty, we have $\sigma(L_t^n) = \sum_{b=1}^{t-1} \min(m+b-1, t-b)$ and $K(L_t^n) = K_t^n$. The following corollary is a consequence of Lemma 3.1.1 and the above computation.

Corollary 3.1.4. *Suppose $T \in L_t^n$. Then $T \in K_t^n$ if and only if the following two conditions are satisfied.*

1. *For any $1 \le b \le \frac{1}{2}(t-m+1)$, the number of reducible entries in the bth column of T is $n - (t-b) = m + b - 1$.*

2. *For any $\frac{1}{2}(t-m+1) < b \le t-1$, all the entries in the bth column of T are reducible.*

Note that if $t \le m$, only condition (2) applies.

The following corollary now follows from Corollary 3.1.4 and part (2) of Corollary 3.1.2.

Corollary 3.1.5. *Suppose $T \in K_t^n$, and $t > m$. Then the first $\lfloor \frac{1}{2}(t-m+1) \rfloor$ columns of T are maximally reducible, and for $1 \le b \le \lfloor \frac{1}{2}(t-m+1) \rfloor$, the bth column of T has $m+b-1$ reducible entries.*

Therefore, among the $t-b$ entries $T_{a,b}$ in the bth column of T, there are $m+b-1$ entries satisfying $T_{a,b} - T_{a-1,b} = 2$, and the remaining $t-m-2b+1$ entries satisfy $T_{a,b} - T_{a-1,b} = 1$.

Most of the analysis in this work relies on tableaux having maximally reducible columns or rows. Intuitively, when a column or a row are maximally reducible it means they are "efficient" in using all the freedom given by the bound n to generate more reducible entries. The lemmas in the rest of this section demonstrate the power of strict and maximal reducibility, and will be used in the following sections.

Lemma 3.1.6 (based on [2, Corollary 3.8]). *Let T be a SSYT (of some shape).*

1. *Suppose $1 \leq a \leq l(\lambda) - 1$ and row $a+1$ of T is strictly reducible. Then for any b such that $T_{a+1,b+1}$ is an entry of T we have $T_{a+1,b} - T_{a,b+1} \geq 1$.*

2. *Suppose $1 \leq b \leq l(\lambda') - 1$ and column $b+1$ of T is strictly reducible. Then for any a such that $T_{a+1,b+1}$ is an entry of T we have $T_{a+1,b} - T_{a,b+1} \leq 1$.*

Proof. 1. If $T_{a+1,b+1}$ is reducible, then by strict reducibility $T_{a+1,b+1} - T_{a+1,b} = 1$ and by the property of reducible entries $T_{a+1,b+1} - T_{a,b+1} \geq 2$. If $T_{a+1,b+1}$ is irreducible, then by strict reducibility $T_{a+1,b+1} = T_{a+1,b}$ and by the definition of SSYT $T_{a+1,b+1} - T_{a,b+1} \geq 1$. In either case we get $T_{a+1,b} - T_{a,b+1} \geq 1$.

2. If $T_{a+1,b+1}$ is reducible, then by strict reducibility $T_{a+1,b+1} - T_{a,b+1} = 2$ and by the property of reducible entries $T_{a+1,b+1} - T_{a+1,b} \geq 1$. If $T_{a+1,b+1}$ is irreducible, then by strict reducibility $T_{a+1,b+1} - T_{a,b+1} = 1$ and by the definition of SSYT $T_{a+1,b+1} \geq T_{a+1,b}$. In either case we get $T_{a+1,b} - T_{a,b+1} \leq 1$.

□

In [2] Liu and Stanley call the property in part (2) of Lemma 3.1.6 *the diagonal property*. We follow their lead and call part (1) of the lemma *the row diagonal property*, and part (2) *the column diagonal property*. We now apply Lemma 3.1.6 to L_t^n.

Corollary 3.1.7. *Suppose $t > m$, and let $T \in K_t^n$. Then for any $1 \leq b \leq \lfloor \frac{1}{2}(t-m+1) \rfloor - 1$ and any $0 \leq a \leq t-b-1$ we have the column diagonal property $T_{a+1,b} - T_{a,b+1} \leq 1$.*

Proof. It follows from Corollary 3.1.5 and Lemma 3.1.6 that $T_{a+1,b} - T_{a,b+1} \leq 1$ for any $1 \leq b \leq \lfloor \frac{1}{2}(t-m+1) \rfloor - 1$ and any $0 \leq a \leq t-b-2$ (so that T has a $(a+1, b+1)$-entry). Moreover, by Corollary 3.1.5 for any $1 \leq b \leq \lfloor \frac{1}{2}(t-m+1) \rfloor - 1$ the columns b and $b+1$ are maximally reducible, so the last entries in these columns are $T_{t-b,b} = T_{t-b-1,b+1} = n$. Hence $T_{t-b,b} - T_{t-b-1,b+1} = 0$, and the column diagonal property applies for $a = t-b-1$ too. □

The following three lemmas have very similar proofs.

Lemma 3.1.8 (generalization of [2, Corollary 3.10]). *Let T be a SSYT of shape λ. Suppose columns $b_0, b_0+1, ..., b_0+k$ of T are all strictly reducible and satisfy $\lambda'_{b_0} \geq k$ and $\lambda'_{b_0+i} \geq k-i+1$ for all $1 \leq i \leq k$. Then*

$$\text{for all } 0 \leq i \leq k \text{ and } 0 \leq a \leq k-i \text{ we have } T_{a,b_0+i} = a. \tag{3.5}$$

Proof. The proof is by induction on a. We can rewrite the indices in (3.5) as $0 \leq a \leq k$, $0 \leq i \leq k-a$. The base case when $a = 0$ holds by our convention $T_{0,b} = 0$.

Suppose $T_{a_0,b_0+i} = a_0$ for some $0 \leq a_0 \leq k$ and all $0 \leq i \leq k - a_0$. We want to show that $T_{a_0+1,b_0+i} = a_0+1$ for all $0 \leq i \leq k-a_0-1$. Pick some $0 \leq i_0 \leq k-a_0-1$, by our assumptions on the length of the columns of T and the induction hypothesis, the entries $T_{a_0+1,b_0+i_0}, T_{a_0+1,b_0+i_0+1}, T_{a_0,b_0+i_0}, T_{a_0,b_0+i_0+1}$ are entries of T and $T_{a_0,b_0+i_0} = T_{a_0,b_0+i_0+1} = a_0$. Since the columns b_0+i_0, b_0+i_0+1 of T are strictly reducible, we have the column diagonal property (part (2) of Lemma 3.1.6), therefore $T_{a_0+1,b_0+i_0} - T_{a_0,b_0+i_0+1} \leq 1$. On the other hand since T is a SSYT we have $T_{a_0+1,b_0+i_0} - T_{a_0,b_0+i_0} \geq 1$, so we must have $T_{a_0+1,b_0+i_0} = T_{a_0,b_0+i_0}+1 = T_{a_0,b_0+i_0+1}+1 = a_0+1$. □

Lemma 3.1.9. *Let T be a SSYT of shape λ and largest part at most n. Suppose columns $b_0, b_0+1, ..., b_0+k$ of T are all maximally reducible and their lengths $\lambda'_{b_0}, \lambda'_{b_0+1}, ..., \lambda'_{b_0+k}$ are all equal to l for some $l > k$. Then*

$$\text{for all } 0 \leq i \leq k \text{ and } l-i \leq a \leq l \text{ we have } T_{a,b_0+i} = n-l+a. \tag{3.6}$$

Proof. The proof is by induction on a going from l down to $l-k$. We can rewrite the indices in (3.6) as $l-k \leq a \leq l$, $l-a \leq i \leq k$. The base case when $a=l$ holds by maximal reducibility, since $T_{l,b_0} = T_{l,b_0+1} = \cdots = T_{l,b_0+k} = n$.

Suppose $T_{a_0,b_0+i} = n-l+a_0$ for some $l-k \leq a_0 \leq l$ and all $l-a_0 \leq i \leq k$. We want to show that $T_{a_0-1,b_0+i} = n-l+a_0-1$ for all $l-a_0+1 \leq i \leq k$. Pick some $l-a_0+1 \leq i_0 \leq k$, by our assumptions on the length of the columns of T and the induction hypothesis, the entries $T_{a_0-1,b_0+i_0}, T_{a_0-1,b_0+i_0-1}, T_{a_0,b_0+i_0}, T_{a_0,b_0+i_0-1}$ are entries of T and $T_{a_0,b_0+i_0} = T_{a_0,b_0+i_0-1} = n-l+a_0$. Since the columns b_0+i_0-1, b_0+i_0 of T are maximally reducible we have the column diagonal property (part (2) of Lemma 3.1.6), therefore $T_{a_0,b_0+i_0-1} - T_{a_0-1,b_0+i_0} \leq 1$. On the other hand since T is a SSYT we have $T_{a_0,b_0+i_0} - T_{a_0-1,b_0+i_0} \geq 1$, so we must have $T_{a_0-1,b_0+i_0} = T_{a_0,b_0+i_0} - 1 = T_{a_0,b_0+i_0-1} - 1 = n-l+a_0-1$. □

Lemma 3.1.10. *Let T be a SSYT of shape λ. Suppose rows $a_0, a_0+1, ..., a_0+k$ of T are all strictly reducible, and their lengths satisfy $\lambda_{a_0} \geq k$ and $\lambda_{a_0+i} \geq k-i+1$ for all $1 \leq i \leq k$. Then*

$$\text{for all } 0 \leq i \leq k \text{ and } 0 \leq b \leq k-i \text{ we have } T_{a_0+i,b} = a_0+i. \tag{3.7}$$

Proof. The proof here is by induction on b. It is almost identical to the proof of Lemma 3.1.8, except we are now using the row diagonal property. We can rewrite the indices in (3.7) as $0 \leq b \leq k$, $0 \leq i \leq k-b$. The base case when $b=0$ holds by our convention $T_{a,0} = a$.

Suppose $T_{a_0+i,b_0} = a_0+i$ for some $0 \leq b_0 \leq k$ and all $0 \leq i \leq k-b_0$. We want to show that $T_{a_0+i,b_0+1} = a_0+i$ for all $0 \leq i \leq k-b_0-1$. Pick some $0 \leq i_0 \leq k-b_0-1$, by our assumptions on the length of the rows of T and the induction hypothesis, the entries $T_{a_0+i_0,b_0+1}, T_{a_0+i_0,b_0}, T_{a_0+i_0+1,b_0}, T_{a_0+i_0+1,b_0+1}$ are entries of T and $T_{a_0+i_0,b_0} = a_0+i_0$, $T_{a_0+i_0+1,b_0} = a_0+i_0+1$. Since the rows a_0+i_0, a_0+i_0+1 of T are strictly reducible we have the row diagonal property (part (1) of Lemma 3.1.6), therefore $T_{a_0+i_0+1,b_0} - T_{a_0+i_0,b_0+1} \geq 1$. On the other hand since T is a SSYT we have $T_{a_0+i_0,b_0+1} \geq T_{a_0+i_0,b_0}$, so we must have $T_{a_0+i_0,b_0+1} = T_{a_0+i_0,b_0} = T_{a_0+i_0+1,b_0} - 1 =$

$a_0 + i_0$. □

3.2 Staircase shape

In this section we complete the analysis of L_t^n from Section 3.1 to determine $\sigma(L_t^n)$ and $K(L_t^n)$. In fact, the analysis of L_{p-1}^{p-1} in [2] extends perfectly to L_t^n including all the proofs, so we merely quote the main results of [2] in their more general setting for L_t^n. If the statement of a result requires certain definitions that we prefer to omit, we only quote the final conclusion of the result that does not use those definitions. We use the definition of K_t^n as in Section 3.1.

Proposition 3.2.1 (generalization of [2, Proposition 3.11]). *Suppose T is a tableau of shape δ_t filed with integer entries. Then $T \in K_t^n$ if and only if the following conditions are satisfied. Note that if $t \leq m$, only condition (3) is relevant.*

1. *For any $1 \leq b \leq \lfloor \frac{1}{2}(t-m+1) \rfloor$,*
   (a) *for any $1 \leq a \leq \lfloor \frac{1}{2}(t-m+1) \rfloor - b$, we have $T_{a,b} = a$;*
   (b) *among the $\lfloor \frac{1}{2}(t+m) \rfloor$ remaining values of a, viz. $\lfloor \frac{1}{2}(t-m+1) \rfloor - b + 1 \leq a \leq t-b$, we have that $m+b-1$ of them satisfy $T_{a,b} - T_{a-1,b} = 2$, and the remaining $\lfloor \frac{1}{2}(t+m) \rfloor - (m+b-1)$ of them satisfy $T_{a,b} - T_{a-1,b} = 1$.*
2. *For any $1 \leq b \leq \lfloor \frac{1}{2}(t-m+1) \rfloor - 1$ and $\lfloor \frac{1}{2}(t-m+1) \rfloor - b + 1 \leq a \leq t-b$ we have the diagonal property $T_{a,b} - T_{a-1,b+1} \leq 1$.*
3. *For any $\lfloor \frac{1}{2}(t-m+1) \rfloor + 1 \leq b \leq t-1$ and any $1 \leq a \leq t-b$, we have $T_{a,b} \leq n$, $T_{a,b} - T_{a-1,b} \geq 2$ and $T_{a,b} - T_{a,b-1} \geq 1$.*

It is evident from Proposition 3.2.1 that $T \in K_t^n$ can be split into two parts, left and right, each displaying a different behavior. In the case $t = n = p-1$ described in [2], these parts are two halves of δ_t. In the more general case $n \geq t \geq 1$ the right part gets larger as m grows, and in fact if $m \geq t$, there is no left part and all of T is in the right part. In the rest of this section we quote results that are relevant when there is a left part, and we note what happens in case there is no left part.

Let $(\delta_t^n)^{\mathcal{L}}$ be the shape that is the left $\lfloor \frac{1}{2}(t-m+1) \rfloor$ columns of δ_t. Note that the shape of the right side of δ_t is $\delta_{t-\lfloor \frac{1}{2}(t-m+1) \rfloor} = \delta_{\lfloor \frac{1}{2}(t+m) \rfloor}$.

Definition 3.2.2 (generalization of [2, Definition 4.1]). *Let $(K_t^n)^{\mathcal{L}}$ be the set of all the tableaux of shape $(\delta_t^n)^{\mathcal{L}}$ with integer entries satisfying conditions (1) and (2) of Proposition 3.2.1. For $c = 1$ or 2, let $(K_t^n)^{\mathcal{L},c}$ be the subset of $(K_t^n)^{\mathcal{L}}$ consisting of all the tableaux whose $(1, \lfloor \frac{1}{2}(t-m+1) \rfloor)$-entry is c. (Note that the $(1, \lfloor \frac{1}{2}(t-m+1) \rfloor)$-entry is the last entry in the first row of any tableau in $(K_t^n)^{\mathcal{L}}$, which has to be either 1 or 2 by condition (1) of Proposition 3.2.1.)*

For $c = 1$ or 2, let $(K_t^n)^{\mathcal{R},c}$ be the set of all the tableaux of shape $\delta_{\lfloor \frac{1}{2}(t+m) \rfloor}$ satisfying the following conditions:

1. $T_{1,1} \geq c+1$.
2. $T_{a,b} - T_{a,b-1} \geq 1$ *for any* $2 \leq b \leq \lfloor \frac{1}{2}(t+m) \rfloor - 1$, $1 \leq a \leq \lfloor \frac{1}{2}(t+m) \rfloor - b$.
3. $T_{a,b} - T_{a-1,b} \geq 2$ *for any* $2 \leq a \leq \lfloor \frac{1}{2}(t+m) \rfloor - 1$, $1 \leq b \leq \lfloor \frac{1}{2}(t+m) \rfloor - a$.
4. $T_{\lfloor \frac{1}{2}(t+m) \rfloor - b, b} \leq n$ *for any* $1 \leq b \leq \lfloor \frac{1}{2}(t+m) \rfloor - 1$.

We consider all the sets above as posets with the coordinate-wise partial ordering.

Lemma 3.2.3 (generalization of [2, Lemma 4.2]).

$$K_t^n \cong \left((K_t^n)^{\mathcal{L},1} \times (K_t^n)^{\mathcal{R},1}\right) \cup \left((K_t^n)^{\mathcal{L},2} \times (K_t^n)^{\mathcal{R},2}\right)$$

Proposition 3.2.4 (generalization of [2, Proposition 4.5]).

$$(K_t^n)^{\mathcal{L}} \cong L_{\lfloor \frac{1}{2}(t-m)+1 \rfloor}^{\lfloor \frac{1}{2}(t+m) \rfloor}$$

Lemma 3.2.5 (generalization of [2, Lemma 4.9]). *The last entry in the first row of the unique minimal element of $(K_t^n)^{\mathcal{L}}$ is 1 if $t+m$ is even and 2 if $t+m$ is odd.*

Lemma 3.2.6 (generalization of [2, Lemma 4.14]).

1. *If $t+m$ is even, say $t+m = 2l$, then $(K_t^n)^{\mathcal{R},1} \cong L_l^l$.*

2. If $t+m$ is odd, say $t+m=2l+1$, then $(K_t^n)^{\mathcal{R},2} \cong L_l^l$.

Remark If $t \leq m$ and the tableaux in K_t^n have no left side, we have $\sigma(L_t^n) = \binom{t}{2}$ and $K(L_t^n) = K_t^n \cong L_t^m$ with the same proof as part (1) of Lemma 3.2.6.

Corollary 3.2.7 (generalization of [2, Corollary 4.15]). *For any $n \geq t \geq 1$, the poset K_t^n is nonempty (hence $K(L_t^n) = K_t^n$).*

Corollary 3.2.8 (generalization of [2, Corollary 4.16]).

$$\sigma(L_t^n) = \sum_{b=1}^{t-1} \min(m+b-1, t-b) = \begin{cases} l(l-1) - \binom{m}{2} & \text{if } t+m=2l; \\ l^2 - \binom{m}{2} & \text{if } t+m=2l+1. \end{cases}$$

Corollary 3.2.8 in the case $t = n = p-1$ (so $m = 1$) as presented in [2, Corollary 4.16] is in fact another proof of a formula previously proven by Elkies.

The next two results describe $K(L_t^n)$. In their paper, Liu and Stanley use these results to compute a rank-generating function for $K(L_t^n)$ for the case $t = n = p-1$ and thus compute the size of $K(L_{p-1}^{p-1})$. This turns out to be harder in the general case.

Theorem 3.2.9 (generalization of [2, Theorem 5.6]). *Suppose $t+m$ is odd, and denote $t+m = 2l+1$. Then*

$$K(L_t^n) \cong L_{l-m+1}^l \times L_l^l.$$

Remark In their paper, Liu and Stanley analyze the case $t = n$ (so $m = 1$). In that case Theorem 3.2.9 reduces to

$$K(L_n^n) \cong L_l^l \times L_l^l$$

where $n = 2l$. Liu and Stanley compute ([2, Theorem 2.12]) the rank-generating function of L_l^l:

$$F(L_l^l, q) = (1+q)^{l-1}(1+q^2)^{l-2} \cdots (1+q^{l-1}),$$

and conclude ([2, Theorem 5.6]) that the rank-generating function of $K(L_n^n)$ (where $n = 2l$) is given by

$$F(K(L_n^n), q) = \left((1+q)^{l-1}(1+q^2)^{l-2}\cdots(1+q^{l-1})\right)^2$$

where $F(K(L_2^2), q) = 1$.

Theorem 3.2.10 (generalization of [2, Proposition 5.10]). *Suppose $t+m$ is even and denote $t+m = 2k$. Denote by U_t^n the poset of join-irreducibles of $K(L_t^n)$. Then U_t^n can be divided into two disjoint sets $(U_t^n)^l$ and $(U_t^n)^r$, each of which is divided into two disjoint sets $(U_t^n)^l = (U_t^n)^{l,1} \dot\cup (U_t^n)^{l,2}$ and $(U_t^n)^r = (U_t^n)^{r,1} \dot\cup (U_t^n)^{r,2}$ such that they satisfy the following conditions:*

1. *$(U_t^n)^l \cong$ the poset of join-irreducibles of L_{k-m+1}^k.*
2. *$(U_t^n)^r \cong$ the poset of join-irreducibles of L_k^k.*
3. *$(U_t^n)^{l,1} \cong$ the poset of join-irreducibles of L_{k-m}^{k-1}.*
4. *$(U_t^n)^{r,1} \cong$ the poset of join-irreducibles of L_{k-1}^{k-1}.*
5. *No element in $(U_t^n)^{l,1}$ is comparable to any element in $(U_t^n)^r$.*
6. *No element in $(U_t^n)^{r,1}$ is comparable to any element in $(U_t^n)^l$.*
7. *Each element of $(U_t^n)^{r,2}$ is smaller than any element in $(U_t^n)^{l,2}$*

Remark In the case $t = n = 2k-1, m = 1$ Liu and Stanley show ([2, Theorem 5.11]) that the rank-generating function of $K(L_n^n)$ is given by

$$\begin{aligned} F(K(L_n^n), q) &= \left((1+q)^{k-2}(1+q^2)^{k-3}\cdots(1+q^{k-2})\right)^2 \\ &\quad \cdot \left((1+q)(1+q^2)\cdots(1+q^{k-1}) \times \left(1+q^{\binom{k}{2}}\right) - q^{\binom{k}{2}}\right) \end{aligned}$$

where $F(K(L_3^3), q) = 1 + q + q^2$.

3.3 Rectangular shape

Let $\lambda_{r,c}$ be the rectangular partition with r rows and c columns,

$$\lambda_{r,c} = \underbrace{(c, c, ..., c)}_{r}.$$

Let $L^n_{r,c}$ be the coordinate-wise partial ordering on SSYT of shape $\lambda_{r,c}$ with largest part at most n, and denote $m := n - r$. The analysis of $L^n_{r,c}$ reveals three different behaviors depending on the value of m.

3.3.1 Large m

Theorem 3.3.1. *Suppose $m \geq r + c - 1$ (equivalently, $n \geq 2r + c - 1$). Then $\sigma(L^n_{r,c}) = rc$ and $K(L^n_{r,c}) \cong L^{n-(r+c-1)}_{r,c}$.*

Proof. Clearly $\sigma(L^n_{r,c}) \leq rc$, the total number of entries of the tableaux in $L^n_{r,c}$. Since $n-(r+c-1) \geq r$, the poset $L^{n-(r+c-1)}_{r,c}$ contains the (minimal) tableau $\mathcal{T}^0$ defined by $\mathcal{T}^0_{a,b} = a$, so it is nonempty. Denote by $K^n_{r,c}$ the subposet of $L^n_{r,c}$ consisting of all the tableaux with rc reducible entries. Then it suffices to show that $K^n_{r,c} \cong L^{n-(r+c-1)}_{r,c}$, in which case $K(L^n_{r,c}) = K^n_{r,c} \cong L^{n-(r+c-1)}_{r,c}$ and $\sigma(L^n_{r,c}) = rc$.

Consider the transformation ϕ defined on any tableau T filled with integer entries as follows: $\phi(T)$ has the same shape as T and entries $\phi(T)_{a,b} = T_{a,b} + a + b - 1$. With our convention $T_{0,b} = 0$ and $T_{a,0} = a$ it is easy to see that for any shape λ, ϕ transforms SSYT of shape λ into tableaux of the same shape with $\phi(T)_{a,b} - \phi(T)_{a-1,b} \geq 2$ and $\phi(T)_{a,b} - \phi(T)_{a,b-1} \geq 1$ for all a, b such that $T_{a,b}$ is an entry of T. In other words, ϕ transforms SSYT of shape λ into SSYT of the same shape in which all entries are reducible. Moreover, it is easy to see that ϕ is invertible and that it preserves the component-wise partial order on SSYT of the same shape, so it is a poset isomorphism.

In our case, we apply ϕ to SSYT of rectangular shape $\lambda_{r,c}$. If T is a SSYT of shape $\lambda_{r,c}$, all the entries of T are bounded by the entry $T_{r,c}$, which is increased by $r + c - 1$ under ϕ. Hence $T \in L^{n-(r+c-1)}_{r,c}$ if and only if $\phi(T) \in K^n_{r,c}$, and we have established the isomorphism $K^n_{r,c} \cong L^{n-(r+c-1)}_{r,c}$. □

3.3.2 Northwestern Corners of Regions

In order to analyze the remaining possible values of m we take a different approach. In the next few lemmas we translate the problem of finding $\sigma(L^n_{r,c})$ and describing $K(L^n_{r,c})$ into a problem of finding certain regions in the Young diagram $D_{\lambda_{r,c}}$ with a maximum number of northwestern corners.

Definition 3.3.2. *Let λ be a partition, and let D_λ be the Young diagram of shape λ. Let R be a nonempty set of entries of D_λ. We call R a* southeast-closed (or SE-closed) region of λ *if*

$$(a,b) \in R \text{ implies } (a,b+1) \in R \text{ and } (a+1,b) \in R$$

(whenever these entries are in D_λ).

Let $R_1, R_2, ..., R_k$ be nonempty sets of entries of D_λ. We call $R_1, R_2, ..., R_k$ a southeast-closed (or SE-closed) sequence of regions of the shape λ *if:*

1. *$R_1 \supseteq R_2 \supseteq \cdots \supseteq R_k$.*
2. *For any $1 \leq i \leq k$, R_i is a SE-closed region of λ.*

We call k the length *of the sequence, and we denote by $\mathcal{R}_\lambda$ the set of all SE-closed sequences of regions for the shape λ, including the sequence of length 0 that has no regions.*

Definition 3.3.3. *Let T be a SSYT of shape λ. Define a sequence of regions $R_1 \supseteq R_2 \supseteq \cdots$ as follows: R_i is the set of entries (a,b) for which $T_{a,b} \geq a+i$ (note that by the definition of SSYT, we always have $T_{a,b} \geq a$). Since T contains a finite number of integer entries, the sequence $R_1 \supseteq R_2 \supseteq \cdots$ has only a finite number of nonempty regions, so we omit the empty regions and think of the sequence as a finite sequence $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_k$ (note that if T has $T_{a,b} = a$ for all its entries, we get an empty sequence). We denote the sequence of regions $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_k$ by $S(T)$.*

Lemma 3.3.4. *For any SSYT T of shape λ, $S(T)$ is a SE-closed sequence of regions for λ.*

Proof. Part (1) of Definition 3.3.2 is clear from the definition of $S(T)$. Part (2) follows immediately from the definition of SSYT: The rows of T are weakly increasing so $T_{a,b} \geq a+i$ implies $T_{a,b+1} \geq T_{a,b} \geq a+i$, and the columns of T are strictly increasing so $T_{a,b} \geq a+i$ implies $T_{a+1,b} \geq T_{a,b}+1 \geq (a+1)+i$. □

Definition 3.3.5. *Let λ be a partition and let $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_k$ be a SE-closed sequence of regions of λ. By convention we let $R_0 = D_\lambda$ and $R_{k+1} = \emptyset$ even though they are not part of the sequence. Define a tableau T of shape λ by:*

$$T_{a,b} = a+i \text{ if } (a,b) \in R_i \setminus R_{i+1}.$$

We denote the tableau T by $S'(R_1, R_2, ..., R_k)$.

Lemma 3.3.6. *Let λ be a partition and let $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_k$ be a SE-closed sequence of regions of λ. Then the tableau $S'(R_1, R_2, ..., R_k)$ is a SSYT of shape λ.*

Proof. Clearly the entries of $T = S'(R_1, R_2, ..., R_k)$ are positive integers, so all we need to prove is that the rows are weakly increasing and the columns are strictly increasing. Suppose (a,b) and $(a,b+1)$ are both entries of T. Let $0 \leq i \leq k$ be such that $(a,b) \in R_i \setminus R_{i+1}$, then by property (2) of Definition 3.3.2, $(a,b+1) \in R_i$ so $T_{a,b+1} \geq a+i = T_{a,b}$. Hence the rows of T are weakly increasing. Similarly, it is easy to show that T is strictly increasing in columns. □

Lemma 3.3.7. *Let λ be a partition. Then the maps S and S' are inverses of one another, so they give a bijection between SSYT of shape λ and $\mathcal{R}_\lambda$. Under this bijection, the reducible entries of a SSYT T are precisely the northwestern corners of the regions of $S(T)$.*

Moreover, for the shape $\lambda_{r,c}$, S and S' give a bijection between the SSYT of $L^n_{r,c}$ (SSYT of shape λ and largest part at most n) and SE-closed sequences of regions of $\lambda_{r,c}$ of length at most m (where $m = n - r$).

Remark By a northwestern corner of a region R we mean an entry $(a,b) \in R$ such that $(a-1,b), (a,b-1) \notin R$.

Proof. Lemma 3.3.4 and Lemma 3.3.6 show that S and S' are indeed maps between SSYT of shape λ and SE-closed sequences of regions of λ, and it is easy to see from their definition that they are inverses of one another.

Let T be a SSYT. Recall that the entry $T_{a,b}$ is reducible if $T_{a,b} - T_{a,b-1} \geq 1$ and $T_{a,b} - T_{a-1,b} \geq 2$. If (a, b) is a northwestern corner of some region R_i in $S(T)$, so that $(a, b) \in R_i$ but $(a, b-1), (a-1, b) \notin R_i$, this implies that $T_{a,b} \geq a + i$, $T_{a,b-1} < a + i$ and $T_{a-1,b} < (a-1) + i$. Hence $T_{a,b}$ is reducible. If (a, b) is not a northwestern corner of any region, let i be such that $(a, b) \in R_i \setminus R_{i+1}$, so $T_{a,b} = a + i$. Since (a, b) is not a northwestern corner, either $(a, b-1)$ or $(a-1, b)$ are in R_i. If $(a, b-1) \in R_i$ then $T_{a,b-1} \geq a + i$, and if $(a-1, b) \in R_i$ then $T_{a-1,b} \geq (a-1) + i$. In either case, $T_{a,b}$ is not reducible.

For $\lambda_{r,c}$ we have two useful properties. First, by the monotonicity of SSYT the entries of any $T \in L^n_{r,c}$ are bounded by the entry $T_{r,c}$. Second, by property (2) of Definition 3.3.2 any region in $S(T)$ contains the (r, c) entry, hence if $S(T)$ is of length k we must have $T_{r,c} = r + k$. Therefore if T is a SSYT of shape $\lambda_{r,c}$, T has largest part at most n if and only if $T_{r,c} \leq n = r + m$, and that happens if and only if $S(T)$ has at most m regions. □

Lemma 3.3.7 implies that the problem of finding $\sigma(L^n_{r,c})$ and $K(L^n_{r,c})$ is equivalent to the problem of finding SE-closed sequences of regions of the shape $\lambda_{r,c}$ with a maximal number of distinct northwestern corners. The latter problem has a clear symmetry around the main diagonal of $\lambda_{r,c}$, so from now on we may assume that $r \geq c$. Note that this symmetry also holds for $m \geq r + c - 1$ (in that case it follows from our proof of Theorem 3.3.1).

Remark For any SE-closed region R of the shape λ, the boundary of R traces the boundary of λ going counterclockwise from the southwestern corner of λ to the northeastern corner. So R can be described by the part of its boundary that goes clockwise from the southwestern corner of λ to the northeastern corner, which can be thought of as a lattice path with steps up and right. Therefore the problem of finding $\sigma(L^n_{r,c})$ and $K(L^n_{r,c})$ is also equivalent to the problem of finding m noncrossing lattice paths going from the southwestern corner of $D_{\lambda_{r,c}}$ to its northeastern corner with steps up and right and with a maximal number of distinct northwestern corners.

3.3.3 Small m

Theorem 3.3.8. *Suppose $m \leq r - c + 1$. Then $\sigma(L^n_{r,c}) = mc$ and $K(L^n_{r,c}) \cong L^{r-c+1}_{(r-c+1)-m,c}$ (or $K(L^n_{r,c})$ has exactly one element if $m = r - c + 1$).*

The next few lemmas prove theorem 3.3.8 in a very similar way to the proof of [2, Proposition 4.5]. SE-closed regions do not play a role in this proof (they will reappear in the analysis of the remaining values of m). We mentioned them earlier in order to reduce the number of cases we need to analyze, thanks to the symmetry around the main diagonal of $D_{\lambda_{r,c}}$.

Lemma 3.3.9. *Suppose $m \leq r - c + 1$ and let $T \in L^n_{r,c}$. Then for any $1 \leq b \leq c$,*

$$\#\text{reducible entries in the } b\text{th column of } T \leq m.$$

Therefore,

$$\#\text{ reducible entries in } T \leq mc.$$

Proof. This follows easily from part (2) of Lemma 3.1.1. The length of each column of T is equal to r, and $\min(n - r, r) = \min(m, r) = m$. □

Let $K^n_{r,c}$ be the subposet of $L^n_{r,c}$ consisting of the tableaux that have mc reducible entries. The following corollary easily follows from Lemma 3.3.9 and part (2) of Corollary 3.1.2.

Corollary 3.3.10. *Suppose $m \leq r - c + 1$ and $T \in K^n_{r,c}$. Then all of the columns of T are maximally reducible.*

Lemma 3.3.11 (based on [2, Proposition 3.11]). *Suppose $m \leq r - c + 1$ and T is a tableau of shape $\lambda_{r,c}$ filled with integer entries. Then $T \in K^n_{r,c}$ if and only if T satisfies the following conditions.*

1. *For any* $1 \le b \le c$,

   (a) *for* $1 \le a \le c - b$ *we have* $T_{a,b} = a$, *so the entries* $T_{a,b}$ *for* $1 \le a \le c - b$ *are irreducible;*

   (b) *for* $r - b + 1 \le a \le r$ *we have* $T_{a,b} = n - r + a = m + a$ *so the entries* $T_{a,b}$ *for* $r - b + 2 \le a \le r$ *are irreducible;*

   (c) *out of the remaining* $r - c + 1$ *entries* $T_{a,b}$ *(for* $c - b + 1 \le a \le r - b + 1$*),* m *of them are satisfy* $T_{a,b} - T_{a-1,b} = 2$ *and* $r - c + 1 - m$ *of them satisfy* $T_{a,b} - T_{a-1,b} = 1$.

2. *For any* $1 \le b \le c - 1$ *and any* $c - b + 1 \le a \le r - b + 1$ *we have the column diagonal property* $T_{a,b} - T_{a-1,b+1} \le 1$.

Proof. Suppose $T \in K^n_{r,c}$. By Corollary 3.3.10, all the columns of T are maximally reducible (and in particular strictly reducible). Therefore property (1)(a) follows from Lemma 3.1.8, property (1)(b) follows from Lemma 3.1.9 and property (1)(c) follows from Lemma 3.3.9 and maximal reducibility. Property (2) follows from part (2) of Lemma 3.1.6.

Now suppose T is a tableau of shape $\lambda_{r,c}$ filled with integer entries satisfying properties (1) and (2). It is easy to verify that property (1) implies that the entries of T are positive integers, the columns of T are weakly increasing and $T_{r,b} = n$ for all $1 \le b \le c$. In order to prove the lemma, it suffices to show that for any $1 \le a \le r$ and any $1 \le b \le c$,

$$\begin{gathered} T_{a,b} - T_{a-1,b} = 1 \text{ implies } T_{a,b} - T_{a,b-1} \ge 0 \\ \text{and} \\ T_{a,b} - T_{a-1,b} = 2 \text{ implies } T_{a,b} - T_{a,b-1} \ge 1. \end{gathered} \tag{3.8}$$

We know (3.8) holds for $1 \le b \le c$ and $1 \le a \le c - b$ by property (1)(a) and for $1 \le b \le c$ and $r - b + 2 \le a \le r$ by property (1)(b). Note that (3.8) holds whenever $T_{a,b-1} - T_{a-1,b} \le 1$, since then $T_{a,b} - T_{a,b-1} \ge (T_{a,b} - T_{a-1,b}) - 1$.

If $b = 1$, we know $T_{a,b-1} = a$ and $T_{a-1,b} = \sum_{j=1}^{a-1} T_{j,b} - T_{j-1,b} \geq a - 1$, so we have $T_{a,b-1} - T_{a-1,b} \leq 1$. If $b > 1$ and $a = c - b + 1$, then by property (1)(a) $T_{a,b-1} = a$ and $T_{a-1,b} = a - 1$ therefore again $T_{a,b-1} - T_{a-1,b} \leq 1$. If $2 \leq b \leq c$ and $c - b + 2 \leq a \leq r - b + 1$, we have $T_{a,b-1} - T_{a-1,b} \leq 1$ by property (2). □

Corollary 3.3.12. *Suppose $m = r - c + 1$, then $K^n_{r,c}$ has exactly one element.*

Proof. If $m = r - c + 1$, the conditions of Corollary 3.3.11 define exactly one tableau and it is easy to check that this tableau is indeed a SSYT with mc reducible entries. See Figure 3-1 for an example of the only tableau in $L^9_{6,4}$ with 12 reducible entries. □

1	1	1	2
2	2	3	4
3	4	5	6
5	6	7	7
7	8	8	8
9	9	9	9

Figure 3-1: The only element of $K^9_{6,4}$

Corollary 3.3.12 shows that if $m = r-c+1$, $K(L^n_{r,c}) = K^n_{r,c}$ has exactly one element and $\sigma(L^n_{r,c}) = mc$, thus it proves Theorem 3.3.8 in that case. To prove Theorem 3.3.8 for the case $m < r - c + 1$, we show that $K^n_{r,c} \cong L^{r-c+1}_{(r-c+1)-m,c}$. Since $L^{r-c+1}_{(r-c+1)-m,c}$ is clearly not empty (e.g. it contains the minimal tableau $\mathcal{T}^0$ defined by $\mathcal{T}^0_{a,b} = a$), the theorem follows.

Definition 3.3.13 (based on [2, Definition 4.3]). *Let $A^n_{r,c}$ be the set of tableaux of shape $\lambda_{r,c}$ with integer entries that satisfy condition (1) of Lemma 3.3.11.*

Let $B^n_{r,c}$ be the set of tableaux of shape $\lambda_{r-c+1,c}$ with entries 1 or 2 where each column has m copies of 2 and $r - c + 1 - m$ copies of 1.

Let $C^n_{r,c}$ be the set of tableaux of shape $\lambda_{r-c+1-m,c}$ with integer entries in $\{1, 2, ..., r-c+1\}$ where the entries in each column are strictly increasing.

Define $\theta_1 : A^n_{r,c} \to B^n_{r,c}$ in the following way. For any $T \in A^n_{r,c}$, do the following three operations on T.

1. *For any* $1 \le a \le r$ *and* $1 \le b \le c$*, replace the number in the* (a,b)*-entry of* T *with* $T_{a,b} - T_{a-1,b}$.

2. *Remove all the entries* $T_{a,b}$ *with* $a + b \le c$ *or* $a + b \ge r + 2$ *(after this each column has length r-c+1).*

3. *Shift all the entries up to make a rectangular shape* $\lambda_{r-c+1,c}$.

Define $\theta_2 : B^n_{r,c} \to C^n_{r,c}$ *in the following way. For any* $T' \in B^n_{r,c}$ *we create* $\theta_2(T')$ *of shape* $\lambda_{r-c+1-m,c}$ *with entries*

$$\theta_2(T')_{a,b} := \text{the row index of the } a\text{th } 1 \text{ in column } b \text{ of } T'.$$

Define $\theta = \theta_2 \circ \theta_1 : A^n_{r,c} \to C^n_{r,c}$.

Example The following is an example of the maps θ_1 and θ_2 for $r = 6, c = 4, n = 8$ (so $m = 2$).

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3&3\\\hline 3&3&5&5\\\hline 5&5&6&6\\\hline 6&7&7&7\\\hline 8&8&8&8\\\hline\end{array} \in A^8_{6,4} \quad \xmapsto{\theta_1} \quad \begin{array}{|c|c|c|c|}\hline 2&1&2&2\\\hline 1&2&2&1\\\hline 2&2&1&2\\\hline\end{array} \in B^8_{6,4} \quad \xmapsto{\theta_2} \quad \begin{array}{|c|c|c|c|}\hline 2&1&3&2\\\hline\end{array} \in C^8_{6,4}$$

It is clear that θ_1 is a bijection from $A^n_{r,c}$ to $B^n_{r,c}$ and θ_2 is a bijection from $B^n_{r,c}$ to $C^n_{r,c}$. Hence θ is a bijection from $A^n_{r,c}$ to $C^n_{r,c}$. The following proposition completes the proof of Theorem 3.3.8.

Proposition 3.3.14 (based on [2, Proposition 4.5]). *Suppose* $m < r-c+1$*. Consider both* $A^n_{r,c}$ *and* $C^n_{r,c}$ *as posets with the coordinate-wise partial ordering. Then* θ *is a poset isomorphism from* $A^n_{r,c}$ *to* $C^n_{r,c}$.

Furthermore (noting that $K^n_{r,c}$ *is a subposet of* $A^n_{r,c}$*) the map* θ *induces a poset isomorphism from* $K^n_{r,c}$ *to* $L^{r-c+1}_{(r-c+1)-m,c}$*. Hence* $K^n_{r,c} \cong L^{r-c+1}_{(r-c+1)-m,c}$.

In order to prove [2, Proposition 4.5], Liu and Stanley use the following definitions. For any column O with entries in $\{1,2\}$ they define

$$\#Ones(O,i) := \text{the number of 1's in the first } i \text{ entries of } O,$$

and

$$RI(O,a) := \text{the row index of the } a\text{th 1 in } O.$$

With these definitions, the definition of θ_2 can be rewritten as

$$\theta_2(T')_{a,b} := RI(\text{column } b \text{ of } T', a).$$

Liu and Stanley then prove the following lemma.

Lemma 3.3.15 ([2, Lemma 4.6]). *Suppose O and O' are two columns of l entries in $\{1,2\}$. Then the following two conditions are equivalent.*

1. *For any $1 \leq i \leq l$, $\#Ones(O,i) \leq \#Ones(O',i)$.*

2. *$\#Ones(O,l) \leq \#Ones(O',l)$ and $RI(O,a) \geq RI(O',a)$ for any $1 \leq a \leq \#Ones(O,l)$.*

Lemma 3.3.16 (based on [2, Lemma 4.7]). *Suppose $T^{(1)}, T^{(2)} \in A^n_{r,c}$. then the following conditions are equivalent.*

1. *$T^{(1)} \leq T^{(2)}$.*

2. *For any $1 \leq j \leq c$ and $1 \leq i \leq r-c+1$,*

$$\#Ones(\text{column } j \text{ of } \theta_1(T^{(1)}),i) \geq \#Ones(\text{column } j \text{ of } \theta_1(T^{(2)}),i).$$

3. *$\theta(T^{(1)}) \leq \theta(T^{(2)})$.*

Proof. The proof is exactly the same as the proof of [2, Lemma 4.7]. The equivalence between (2) and (3) follows directly from Lemma 3.3.15. For any $T \in A^n_{r,c}$, any

$1 \leq j \leq c$ and any $1 \leq i \leq r - c + 1$, we have

$$\begin{aligned} T_{(c-j)+i,j} - T_{c-j,j} &= \sum_{a=c-j+1}^{c-j+i} (T_{a,j} - T_{a-1,j}) \\ &= \sum_{k=1}^{i} \theta_1(T)_{k,j} = 2i - \#\text{Ones}(\text{column } j \text{ of } \theta_1(T), i). \end{aligned}$$

By property (1)(a) of Lemma 3.3.11, $T_{c-j,j} = c - j$. Therefore

$$T_{c-j+i,j} = 2i + c - j - \#\text{Ones}(\text{column } j \text{ of } \theta_1(T), i), \tag{3.9}$$

and the equivalence between (1) and (2) follows. □

Lemma 3.3.17 (based on [2, Lemma 4.8]). *Suppose $T \in A^n_{r,c}$. Then the following conditions are equivalent.*

1. *T satisfies condition (2) of Lemma 3.3.11.*

2. *For any $1 \leq j \leq c - 1$ and $1 \leq i \leq r - c + 1$,*

$$\#\textit{Ones}(\textit{column } j \textit{ of } \theta_1(T), i) \geq \#\textit{Ones}(\textit{column } j + 1 \textit{ of } \theta_1(T), i).$$

3. *The entries are weakly increasing in each row of $\theta(T)$.*

Proof. This proof is similar to the proof of Lemma 3.3.16. The equivalence between (2) and (3) follows from Lemma 3.3.15, and the equivalence between (1) and (2) follows from equation (3.9). □

The first conclusion of Proposition 3.3.14 follows from Lemma 3.3.16, and the second conclusion of Proposition 3.3.14 follows from Lemma 3.3.17.

3.3.4 Medium m

Theorem 3.3.18. *Suppose $r - c + 1 < m < r + c - 1$.*

1. *If m and $r-c+1$ have the same parity, write $m-(r-c+1)=2l$. Then $\sigma(L^n_{r,c})=mc-l(l+1)$ and $K(L^n_{r,c})$ contains exactly one element.*

2. *Otherwise, write $m-(r-c+1)=2l-1$. Then $\sigma(L^n_{r,c})=mc-l^2$ and $K(L^n_{r,c})$ is a chain of two elements.*

We prove Theorem 3.3.18 by finding sequences of SE-closed regions of $\lambda_{r,c}$ with a maximum number of distinct northwestern corners (recall that these are the reducible entries of the corresponding tableaux). We will use the terms northwestern corners and reducible entries interchangeably. We start by proving an upper bound for $\sigma(L^n_{r,c})$.

Definition 3.3.19. *Let λ be a partition, and let $E=\{(a_1,b_1),(a_2,b_2),...,(a_k,b_k)\}$ be a set of entries of D_λ. We say E is* a set of corners *if $b_1<b_2<\cdots<b_k$ and $a_1>a_2>\cdots>a_k$.*

Remark We always think of a set of corners as ordered from left to right in D_λ. For example, we can say that if some entry (a,b) is part of a set of corners E then the next entry (a',b') has $a'>a$. Likewise, if E is a set of corners in $D_{\lambda_{r,c}}$ we can say that if $a=1$ or $b=c$, (a,b) is the last entry of E.

Lemma 3.3.20. *Let λ be a partition. There is a bijection between the SE-closed regions of λ and the sets of corners of λ. Under this bijection, the sets of corners are the northwestern corners of their corresponding SE-closed regions.*

Proof. Let R be a SE-closed region of λ. The northeastern corners of R are the entries $(a,b)\in R$ with $(a-1,b),(a,b-1)\notin R$. since R is SE-closed, if (a,b) is a northwestern corner of R any entry $(a',b')\neq(a,b)$ of D_λ with $a'\geq a$ and $b'\geq b$ is in R and is not a northwestern corner of R. It follows that the northwestern corners of R are a set of corners of λ.

For any set of corners $E=\{(a_1,b_1),(a_2,b_2),...,(a_k,b_k)\}$ of λ, define $\varphi(E)$ to be the following set of entries of D_λ.

$$\varphi(E)=\bigcup_{i=1}^{k}\{(a',b'):a'\geq a_i\text{ and }b'\geq b_i\}.$$

It is easy to verify that $\varphi(E)$ is a SE-closed region of λ whose northwestern corners are precisely the entries in E. It is also easy to verify that taking the northwestern corners of a SE-closed region of λ is the inverse operation of φ, and the lemma follows. $\square$

Definition 3.3.21. *Let T be a tableau of shape $\lambda_{r,c}$. We split the entries of T into c L-shaped strips as follows. For $1 \leq s \leq c$, the sth L-strip contains the entries (i, s) for $1 \leq i \leq r - s$ and the entries $(c - s + 1, j)$ for $s \leq j \leq c$ (see Figure 3-2 for an example of splitting a Young diagram into L-strips). The total number of entries in the sth L-strip is therefore $r + c - 2s + 1$.*

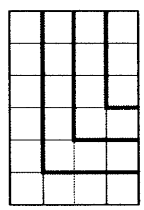


Figure 3-2: Splitting the boxes of the Young diagram of shape $\lambda_{6,4}$ into 4 L-strips

Lemma 3.3.22. *Suppose $T \in L^n_{r,c}$. Then for $1 \leq s \leq c$,*

$$\#\textit{reducible entries in the sth L-strip of } T \leq \min(m, r + c - 2s + 1).$$

Therefore,

$$\begin{aligned} \sigma(L^n_{r,c}) &\leq \sum_{s=1}^{c} \min(m, r + c - 2s + 1) \\ &= \begin{cases} mc - l(l+1) & \textit{if } m - (r - c + 1) = 2l \\ mc - l^2 & \textit{if } m - (r - c + 1) = 2l - 1 \end{cases} \end{aligned}$$

Proof. By Lemma 3.3.7, there are at most m regions in $S(T)$ and the reducible entries of T are precisely the set of distinct northwestern corners of these regions. By Lemma 3.3.20, the set of northwestern corners of any region $R_i \in S(T)$ is a set of corners of $\lambda_{r,c}$. Hence, R_i can have at most one northwestern corner in each L-strip. Since there are at most m regions in $S(T)$, there are at most m distinct reducible entries in each

L-strip. Clearly the number of reducible entries in the sth L-strip of T is at most the number of entries in the strip $r+c-2s+1$, so the first conclusion of the lemma follows. The second conclusion simply follows from a computation of the sum. $\square$

Let $K^n_{r,c}$ be the subposet of $L^n_{r,c}$ consisting of the SSYT that have $mc-l(l+1)$ reducible entries if $m-(r-c+1)=2l$, or $mc-l^2$ reducible entries if $m-(r-c+1)=2l-1$. In order to prove Theorem 3.3.18 it suffices to show that $K^n_{r,c}$ has a single element in the former case and is a chain of two elements in the latter case. We denote $k=c-l-1$ if $m-(r-c+1)=2l$ and $k=c-l$ if $m-(r-c+1)=2l-1$.

Lemma 3.3.23. *Suppose $T \in K^n_{r,c}$, Then $S(T)$ has m regions. Moreover,*

1. *for $1 \leq s \leq k$, each one of the m regions of $S(T)$ has a distinct northwestern corner in the sth L-strip of T;*
2. *for $k+1 \leq s \leq c$, the sth L-strip of T has less than m entries and all of them are northwestern corners of regions in $S(T)$.*

Proof. This is an easy consequence of the proof of Lemma 3.3.22 and the observation that $m < r+c-2s+1$ implies $s \leq k$. $\square$

Lemma 3.3.24. *Suppose $m-(r-c+1)=2l$ and $T \in K^n_{r,c}$. Then for $1 \leq s \leq k+1$, the m reducible entries in the sth L-strip of T are the entries (a,b) with $a \geq k-s+2 = c-l-s+1$ and $b \leq s+l$ (note that there are indeed m such entries).*

Remark Note that when $s=k+1$ Lemma 3.3.24 holds by Lemma 3.3.23. We have included it here only as a base case for an induction.

Proof. The proof is by induction on s going from $s=k+1$ down to $s=1$. The base case is the $(k+1)$-st L-strip of T. It has $r+c-2(k+1)+1=m$ entries, and by Lemma 3.3.23 all of them are reducible. These are indeed all the entries (a,b) of the sth L-strip with $a \geq k-s+2=1$ and $b \leq s+l=c$.

Suppose the lemma holds for the sth L-strip of T, we prove it for the $(s-1)$-st strip. Recall that the reducible entries of T come from m sets of corners, and that each of the m reducible entries in the sth L-strip belongs to a different set of corners, and likewise for the m reducible entries of the $(s-1)$-st strip. Hence each one of the reducible entries in the $(s-1)$-st strip must be in the same set of corners as one of the reducible entries in the sth strip. In particular, each reducible entry (a,b) in the $(s-1)$-st strip must have some entry (a',b') in the sth strip with $a' < a$ and $b' > b$. But by the induction hypothesis all the reducible entries (a',b') in the sth strip satisfy $a' \geq c-l-s+1$ and $b' \leq s+l$, therefore we must have $a \geq c-l-(s-1)+1$ and $b \leq (s-1)+l$ and the conclusion follows. □

Suppose $m-(r-c+1)=2l$ and $T \in K^n_{r,c}$. Lemma 3.3.24 together with Lemma 3.3.23 determine the locations of the reducible entries of T. These reducible entries are the union of the entries of m sets of corners. In order to prove Theorem 3.3.18 (for the case $m-(r-c+1)=2l$) we need to show that there exists a unique way to define m sets of corners $E_1, E_2, ..., E_m$ such that their union (in fact it will be a disjoint union) is the set of reducible entries of T and $R_1, R_2, ..., R_m$ (where $R_i := \varphi(E_i)$) is a SE-closed sequence of regions for $\lambda_{r,c}$. It would then follow that $R_1, R_2, ..., R_m$ define a unique tableau $T_0 \in K^n_{r,c}$, which implies $K(L^n_{r,c}) = K^n_{r,c}$ has a single element and $\sigma(L^n_{r,c}) = mc - l(l+1)$.

By Lemmas 3.3.23 and 3.3.24 the first L-strip of T contains m reducible entries, the entries (a,b) with $a \geq c-l$ and $b \leq l+1$, and each one of these belongs to a different set of corners. Going from $(c-l,1)$ down to $(r,1)$ and then right to $(r,l+1)$, denote these entries by $(a_1,b_1),(a_2,b_2),...,(a_m,b_m)$ and let (a_i,b_i) belong to E_i.

Lemma 3.3.25. *In the above setting we must have for any $1 \leq i \leq m$,*

$$E_i = \{(a_i,b_i),(a_i-1,b_i+1),(a_i-2,b_i+2),...,(1,b_i+a_i-1)\} \tag{3.10}$$

Remark We note a few properties of equation (3.10).

1. Each of the entries in equation (3.10) is in a different L-strip of T.

2. The sets E_i defined in equation (3.10) are all disjoint

3. The sets E_i defined in equation (3.10) are clearly sets of corners. Moreover, if we let $R_i := \varphi(E_i)$, it is easy to verify that $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_m$ and therefore the tableau $T_0 := S'(R_1, ..., R_m)$ is the unique element of $K^n_{r,c}$.

Proof. The proof goes by induction on $1 \leq s \leq c$, and in each step we assign the reducible entries in the sth L-strip of T to the sets $E_1, ..., E_m$. We have already assigned the reducible entries of the first L-strip of T. Suppose that we have assigned the reducible entries in the $(s-1)$-st strip for some $2 \leq s \leq k+1$ consistently with equation (3.10). By Lemma 3.3.24, the reducible entries in the $(s-1)$-st strip are the m entries (a', b') with $a' \geq c - l - (s-1) + 1$ and $b' \leq (s-1) + l$ and the reducible entries in the sth strip are the m entries (a, b) with $a \geq c - l - s + 1$ and $b \leq s + l$. We need to assign each of the latter entries to a set of corners that already contains one of the former entries. Using the property of sets of corners, it is easy to verify (e.g. by considering the entries from top to bottom to the right) that we must assign the entry (a, b) to the set E_i that contains the entry $(a+1, b-1)$. This is consistent with equation (3.10).

Now suppose we have assigned the reducible entries in the $(s-1)$-st strip for some $k+2 \leq s \leq c$ consistently with equation (3.10). By Lemma 3.3.23, all the entries in the strips $k+1, ..., s-1$ are reducible, including one entry with $a = 1$ and one entry with $b = c$ in each strip. These entries must be the last entries in the sets of corners they were assigned to, therefore out of the m sets of corners, $2(s-k-1)$ cannot be assigned entries from the sth strip. We are left with $m - 2(s-k-1) = r + c - 2s + 1$ sets that can still be assigned reducible entries. Note that the number of (reducible) entries in the $(s-1)$-st strip is $r + c - 2s + 3$ and two of these entries are the last entries in their respective sets of corners, so each of the $r + c - 2s + 1$ entries of the $(s-1)$-st strip with $a > 1$ and $b < c$ has been assigned to one of the $r + c - 2s + 1$ "open" sets. The number of (reducible) entries in the sth strip is also $r + c - 2s + 1$, so each of them must be assigned to an "open" set that contains one of the entries of the $(s-1)$-st strip with $a > 1$ and $b < c$. It is now easy to verify similarly to the above case that the entry (a, b) of the sth strip must be assigned to the set E_i that contains the entry $(a+1, b-1)$ of the $(s-1)$-st strip. This is again consistent with equation (3.10). ☐

We are now left with proving Theorem 3.3.18 for the case $m-(r-c+1)=2l-1$. Recall that in this case, $k=c-l$. The kth L-strip in a tableau $T \in K^n_{r,c}$ then has $r+c-2k+1=m+1$ entries and by Lemma 3.3.23, m of them are reducible.

Lemma 3.3.26. *Suppose $m-(r-c+1)=2l-1$ and $T \in K^n_{r,c}$. Then one of the following two cases holds.*

1. *For $1 \leq s \leq k$, the reducible entries in the sth L-strip of T are the entries (a,b) with $a \geq k-s+1$ and $b \leq s+l-1$.*

2. *For $1 \leq s \leq k$, the reducible entries in the sth L-strip of T are the entries (a,b) with $a \geq k-s+2$ and $b \leq s+l$.*

Proof. The $(k+1)$-st L-strip of T has $r+c-2(k+1)+1=m-1$ entries, and by Lemma 3.3.23 they are all reducible. Since they are all in the same L-strip, each one of them must belong to a different set from of the m sets of corners of T. Similarly, the kth L-strip of T has m reducible entries and each one of them belongs to a different set of corners of T, one reducible entry for each set. Hence $m-1$ of the reducible entries of the kth strip belong to a set of corners that also contains one of the entries of the $(k+1)$-st strip. Each entry (a,b) of these $m-1$ reducible entries of the kth strip therefore has some entry (a',b') of the $(k-1)$-st strip with $a > a'$ and $b < b'$, and since the entries in the $(k-1)$-st strip have $a' \geq 1$ and $b' \leq c$, we see that out of the m reducible entries in the kth strip, $m-1$ of them are the entries (a,b) with $a \geq 2$ and $b \leq c-1$. The remaining reducible entry can be either $(1,k)$ or $(r-k+1,c)$.

It is now left to prove the following.

1. If the reducible entries in the kth L-strip of T are the entries (a,b) with $a \geq 1$ and $b \leq c-1$, then case (1) of Lemma 3.3.26 holds.

2. If the reducible entries in the kth L-strip of T are the entries (a,b) with $a \geq 2$ and $b \leq c$, then case (2) of Lemma 3.3.26 holds.

The proof is by induction on the sth L-strip of T going down from $s = k$ to $s = 1$, similar to the proof of Lemma 3.3.24, with either one of the base cases $s = k$ described above. $\square$

Suppose $m - (r - c + 1) = 2l - 1$ and $T \in K^n_{r,c}$, then either T satisfies condition (1) or condition (2) of Lemma 3.3.26. In either case, denote the reducible entries of the first L-strip of T by $(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)$ going from top to bottom to the right. As in the case $m - (r - c + 1) = 2l$, we want to assign the reducible entries of T (determined by Lemma 3.3.23 and Lemma 3.3.26) to m sets of corners $E_1, ..., E_m$. Let (a_i, b_i) belong to E_i. The following lemma shows that there is a unique way to build $E_1, ..., E_m$.

Lemma 3.3.27. *In the above setting we must have for any $1 \leq i \leq m$,*

$$E_i = \{(a_i, b_i), (a_i - 1, b_i + 1), (a_i - 2, b_i + 2), ..., (1, b_i + a_i - 1)\} \tag{3.11}$$

Proof. The proof is the same as the proof of Lemma 3.3.25, except here there is one set of corners ending in the kth L-strip of T, 3 sets of corners ending in the $(k+1)$-st L-strip of T, etc. $\square$

The sets E_i defined in Lemma 3.3.27 are clearly sets of corners. Letting $R_i := \varphi(E_i)$ it is easy to verify that $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_m$, so these regions are a sequence of SE-closed regions for $\lambda_{r,c}$. If $(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)$ satisfy condition (1) of Lemma 3.3.26 denote $T_1 := S'(R_1, ..., R_m)$, and if they satisfy condition (2) of Lemma 3.3.26 denote $T_2 := S'(R_1, ..., R_m)$. It follows that T_1 and T_2 are the only elements of $K^n_{r,c}$, and it is easy to verify that $T_1 > T_2$. Therefore $K(L^n_{r,c}) = K^n_{r,c}$ is a chain of two elements and $\sigma(L^n_{r,c}) = mc - l^2$, and this completes the proof of Theorem 3.3.18 for the case $m - (r - c + 1) = 2l - 1$.

3.4 Double staircase

Let μ_s denote the partition $(2s, 2s - 2, ..., 2)$. We call μ_s the double staircase with s steps. Let L^n_s be the coordinate-wise partial ordering on the set of SSYT of shape μ_s

with largest part at most n. Denote $m := n - s$. In this section we compute $\sigma(L_s^n)$ and describe $K(L_s^n)$ for the various possible values of m. Note that unlike the previous sections, in this section we will use an upper bound on the number of reducible entries in each **row** (rather than column) and all of its consequences (for the shape μ_s, the upper bound on $\sigma(L_s^n)$ given by the columns is larger than the upper bound given by the rows).

Lemma 3.4.1. *Let $T \in L_s^n$. Then for any $1 \le a \le s$,*

$$\#\textit{reducible entries in the ath row of } T \le \min(n-a, 2(s+1-a)).$$

Therefore,

$$\#\textit{ reducible entries in } T \le \sum_{a=1}^{s} \min(n-a, 2(s+1-a)).$$

Proof. This is an easy consequence of part (1) of Lemma 3.1.1, since the length of row a of T is $2(s+1-a)$. □

We now give a few lemmas that hold for $1 \le m \le s$. We then move on to analyze the cases $m = 1$ and $1 < m \le s$ separately, since they display different behaviors.

Lemma 3.4.2. *Suppose $1 \le m \le s$ and $T \in L_s^n$. Then*

$$\sigma(L_s^n) \le \frac{1}{2}(s-m+1)(s+3m-2) + m(m-1)$$

(in particular, if $m = 1$ then $\sigma(L_s^n) \le \binom{s+1}{2}$).

Proof. Suppose $1 \le m \le s$ and $T \in L_s^n$. Applying Lemma 3.4.1, we see that the number of reducible entries in the ath row of T is at most

$$\min(n-a, 2(s-a+1)) = \begin{cases} n-a = m+s-a & \text{for } 1 \le a \le s-m+1 \\ 2(s-a+1) & \text{for } s-m+2 \le a \le s, \end{cases} \tag{3.12}$$

so the total number of reducible entries in T is at most

$$\sum_{a=1}^{s-m+1}(m+s-a)+\sum_{a=s-m+2}^{s}2(s-a+1)=\frac{1}{2}(s-m+1)(s+3m-2)+m(m-1).$$

□

Let K_s^n be the subposet of L_s^n consisting of the tableaux with the maximum possible number of reducible entries as described in Lemma 3.4.2.

Corollary 3.4.3. *Suppose $1 \leq m \leq s$ and $T \in K_s^n$. Then T satisfies the following conditions.*

1. *The first $s-m+1$ rows of T are maximally reducible.*

2. *For any $1 \leq a \leq s-m+1$,*

   (a) *for $1 \leq b \leq s-m+1-a$ we have $T_{a,b}=a$. Thus the first $s-m+1-a$ entries out of the $2(s+1-a)$ entries of the ath row of T are irreducible.*

   (b) *Out of the remaining $s+m-a+1$ entries in the ath row, i.e., the entries (a,b) for $s-m-a+2 \leq b \leq 2(s+1-a)$, $s+m-a$ entries are reducible (and satisfy $T_{a,b}-T_{a,b-1}=1$) and one entry is irreducible (and satisfies $T_{a,b}-T_{a,b-1}=0$)*

3. *For any $s-m+2 \leq a \leq s$, all the entries in the ath row of T are reducible.*

Proof. Property (1) follows from the proof of Lemma 3.4.2 and part (1) of Corollary 3.1.2. Property (2)(a) follows from Lemma 3.1.10. Property (2)(b) follows from the proof of Lemma 3.4.2 and property (1). Property (3) follows from the proof of Lemma 3.4.2. □

Definition 3.4.4. *Suppose $1 \leq m \leq s$ and let $T \in K_s^n$. We define the* location function $l:\{1,2,...,s-m+1\} \to \mathbb{N}$ *as follows. For each row $1 \leq a \leq s-m+1$ of T, let $l(a)$ be the single value $s-a-m+2 \leq b \leq 2(s-a+1)$ such that $T_{a,b}-T_{a,b-1}=0$. In other words, $l(a)$ is the location of the single irreducible entry which is not part of* *the left-aligned triangle of irreducible entries described in property (2)(a) of Corollary 3.4.3.*

Lemma 3.4.5. *Suppose $1 \le m \le s$. Suppose $T \in K_s^n$ has location function l and $l(a) < 2(s-a+1)$ for some $2 \le a \le s-m+1$. Then $l(a-1) \le l(a)+1$.*

Proof. By the maximal reducibility of the first $s-m+1$ rows of T, for any $1 \le i \le s-m+1$ and $1 \le j \le 2(s-i+1)$, we have

$$\begin{aligned} T_{i,j} &= T_{i,0} + \sum_{b=1}^{j} T_{i,b} - T_{i,b-1} \\ &= T_{i,0} + 1 \times \#\text{reducible entries in the first } j \text{ entries of row } i \text{ of } T + \\ &\quad\ 0 \times \#\text{irreducible entries in the first } j \text{ entries of row } i \text{ of } T \\ &= i + \#\text{reducible entries in the first } j \text{ entries of row } i \text{ of } T. \qquad (3.13) \end{aligned}$$

By our assumptions, $T_{a,l(a)+1}$ is a reducible entry of T. By Corollary 3.4.3 and Definition 3.4.4, out of the first $l(a)+1$ entries of the ath row of T, $s-m-a+2$ are irreducible and the remaining $l(a)-s+m+a-1$ are reducible. Applying equation (3.13) we see that $T_{a,l(a)+1} = l(a)-s+m+2a-1$.

Since $T_{a,l(a)+1}$ is reducible, we have

$$T_{a-1,l(a)+1} \le T_{a,l(a)+1} - 2 = l(a) - s + m + 2a - 3.$$

Applying equation (3.13) to row $a-1$, we see that

$$\begin{aligned} & l(a) - s + m + 2a - 3 \ge T_{a-1,l(a)+1} \\ = \ & a - 1 + \#\text{reducible entries in the first } (l(a)+1) \text{ entries of row } a-1 \text{ of } T. \end{aligned}$$

It follows that in the first $l(a)+1$ entries of row $a-1$ of T there are at most $l(a)-s+m+a-2$ reducible entries, and at least $s-m-(a-1)+2$ irreducible entries. But by Corollary 3.4.3 this is the total number of irreducible entries in row $a-1$ including the one denoted by $l(a-1)$, so we must have $l(a-1) \le l(a)+1$. □

3.4.1 The case $m = 1$

We now start analyzing the case $m = 1$. We will break down the proof of Theorem 3.4.6 into several lemmas.

Theorem 3.4.6. *Suppose $m = 1$, then $\sigma(L_s^n) = \binom{s+1}{2}$ and $K(L_s^n) \cong J(A_s)$ where A_s is the poset of pairs $\{(x, y) \in \mathbb{P}^2 | x + y \leq s + 1\}$ with the coordinate-wise partial ordering, and $J(P)$ is the poset of order ideals of the poset P.*

Corollary 3.4.7. *Suppose $m = 1$, $T \in K_s^n$ has location function l, and suppose $l(a) = 2(s - a + 1)$ for some row a of T (so the single irreducible entry is the last entry of row a). Then $l(a + 1) = 2(s - a)$ (the single irreducible entry of row $a + 1$ is the last entry of the row).*

Hence there exists some threshold $1 \leq t \leq s + 1$ such that $l(a) = 2(s - a + 1)$ ($l(a)$ is the last entry of its row) for all $a \geq t$ and $l(a) < 2(s - a + 1)$ ($l(a)$ is not the last entry of its row) for all $a < t$.

Proof. Since row $a + 1$ has length $2(s - a)$ and $l(a + 1) \leq 2(s - a)$, it is impossible to have $l(a) = 2(s - a + 1) \leq l(a + 1) + 1$. Therefore Lemma 3.4.5 does not hold, which implies $l(a + 1) = 2(s - a)$. □

Lemma 3.4.8. *Let T be a tableau of shape μ_s filled with integer entries. Then $T \in K_s^{s+1}$ if and only if there exists a location function $l : \{1, ..., s\} \to \mathbb{N}$ such that:*

1. *for all $1 \leq a \leq s$, $s - a + 1 \leq l(a) \leq 2(s - a + 1)$;*
2. *if $l(a) < 2(s - a + 1)$ then $l(a - 1) \leq l(a) + 1$;*
3. *for any $1 \leq a \leq s$ and $b \in \{1, 2, ..., s - a\} \cup \{l(a)\}$, $T_{a,b} - T_{a,b-1} = 0$ (with the convention $T_{a,0} = a$). For all other values of b such that $T_{a,b}$ is an entry of T, $T_{a,b} - T_{a,b-1} = 1$.*

Proof. If $T \in K_s^n$ has location function l, Corollary 3.4.3 and Lemma 3.4.5 prove properties (1)-(3). Now suppose T and l satisfy properties (1)-(3). We need to show that T is a SSYT with largest part at most $s + 1$ and that T has $\binom{s+1}{2}$ reducible entries. It is clear from property (3) that the entries of T are positive integers and the rows of T are weakly increasing. Moreover, the last entry $T_{a,2(s+1-a)}$ of each row a is

$$\sum_{b=1}^{2(s+1-a)} T_{a,b} - T_{a,b-1} = T_{a,0} + (s-a+1) \cdot 0 + (s-a+1) \cdot 1 = s+1$$

so T has largest part $s+1$.

By property (3), we can split the entries of any row a into four segments and write the entries of each segment as follows:

- Segment I: $1 \leq b \leq s-a$ we have $T_{a,b} = a$.
- Segment II: $s-a+1 \leq b < l(a)$ we have $T_{a,b} = a + (b - (s-a)) = 2a + b - s$.
- Segment III: $b = l(a)$ we have $T_{a,l(a)} = T_{a,l(a)-1} = 2a + l(a) - s - 1$.
- Segment IV: $b > l(a)$ we have $T_{a,b} = T_{a,l(a)} + b - l(a) = 2a + b - s - 1$.

Note that segment II may be empty (if $l(a) = s-a+1$) or segment IV may be empty (if $l(a) = 2(s-a+1)$ is the last entry of row a). In order to show that T is strictly increasing in columns and has $\binom{s+1}{2}$ reducible entries, it suffices to show that for $2 \leq a \leq s$ we have $T_{a,b} - T_{a-1,b} \geq 1$ when $T_{a,b}$ is in segment I or III, and $T_{a,b} - T_{a-1,b} \geq 2$ when $T_{a,b}$ is in segment II or IV (this is clearly true for $a = 1$ by our convention $T_{0,b} = 0$).

Now compare the entries of two consecutive rows a and $a-1$: when $T_{a,b}$ is in segment I, $T_{a-1,b}$ is also in segment I and $T_{a,b} - T_{a-1,b} = 1$. When $T_{a,b}$ is in segment II, $T_{a,b} - T_{a-1,b} \geq 2$ regardless of the segment of $T_{a-1,b}$. When $T_{a,b}$ is in segment III, $T_{a,b} - T_{a-1,b} \geq 1$ regardless of the segment of $T_{a-1,b}$. When $T_{a,b}$ is in segment IV (which implies that $l(a) < 2(s-a+1)$), by property (2) $T_{a-1,b}$ must be in segment III or IV and therefore $T_{a,b} - T_{a-1,b} = 2$. □

It follows from the construction described in property (3) of Lemma 3.4.8 that there is a bijection between K_s^n and location functions satisfying properties (1) and (2). Clearly, such location functions exist (for example, the function $l(a) = s-a+1$), therefore K_s^n is nonempty and we have $K(L_s^n) = K_s^n$) and $\sigma(L_s^n) = \binom{s+1}{s}$. We now turn to analyze the structure of K_s^n. The following corollary shows that if the set of location functions is given the coordinate-wise order, the bijection between K_s^n and location functions is in fact a poset isomorphism.

Corollary 3.4.9. *Let $T, T' \in K_s^n$, and denote their location functions by l, l' respectively. Then $T \leq T'$ in K_s^n if and only if $l(a) \leq l'(a)$ for any $1 \leq a \leq s$ and the equality holds only if $l = l'$.*

Proof. This is an easy consequence of the proof of Lemma 3.4.8. From the description of the four segments of each row, it is evident that for any $1 \leq a \leq s$, row a of T is coordinate-wise less than row a of T' if and only if $l(a) < l'(a)$, and the rows are equal if $l(a) = l'(a)$. □

Proposition 3.4.10. *Suppose $m = 1$. The join-irreducibles of K_s^n are isomorphic to the set A_s, the poset of pairs $\{(x, y) \in \mathbb{P}^2 | x + y \leq s + 1\}$ with the coordinate-wise partial order. Thus $K_s^n \cong J(A_s)$.*

Proof. The join irreducibles of K_s^n are the tableaux that cover exactly one element of K_s^n. In the rest of this proof we think of K_s^n in terms of location functions (functions l that satisfy properties (1) and (2) of Lemma 3.4.8).

Consider the following scenario: suppose l is a location function and there exists $1 \leq a_0 \leq s$ satisfying the following conditions, that we will refer to as *the coverage conditions.*

1. $l(a_0) > s - a_0 + 1$,

2. $a_0 = 1$ or $l(a_0 - 1) \leq l(a_0)$.

Then one can define a function l_{a_0} by

$$l_{a_0}(a) = \begin{cases} l(a) - 1 & \text{if } a = a_0 \\ l(a) & \text{otherwise.} \end{cases}$$

It is easy to verify that l_{a_0} is a location function, and by Corollary 3.4.9 l covers l_{a_0}. We claim that the functions $\{l_{a_0} : a_0 \text{ satisfies the coverage conditions for } l\}$ are precisely the location functions covered by l. Suppose l covers some location function l', then by Corollary 3.4.9 $l'(a) \leq l(a)$ for all $1 \leq a \leq s$. Let i_0 be the minimal row such that $l'(i_0) < l(i_0)$. Since $l(i_0) > l'(i_0) \geq s - i_0 + 1$ and either $i_0 = 1$ or $l(i_0 - 1) = l'(i_0 - 1) \leq l'(i_0) + 1 \leq l(i_0)$, i_0 satisfies the coverage conditions for l, so l covers l_{i_0}. If $l(i_0) - l'(i_0) > 1$ we see that $l > l_{i_0} > l'$, so l cannot cover l'. If $l(i_0) - l'(i_0) = 1$ and l and l' differ in another row $i_1 > i_0$, then l_{i_0} and l' also differ in that row and we see again that $l > l_{i_0} > l'$. Hence we must have $l' = l_{i_0}$.

It follows that K_s^n has a unique minimal element, the location function l_0 with $l_0(a) = s - a + 1$ for all $1 \leq a \leq s$. Moreover, for any other location function l there exists a minimal $1 \leq a_0 \leq s$ such that $l(a) = s - a + 1$ for all $a < a_0$ and $l(a_0) > s - a_0 + 1$. This a_0 satisfies the coverage conditions for l, so l covers l_{a_0}. We want to identify the location functions l for which this a_0 is the only row that satisfies the coverage conditions.

Suppose l is a location function in K_s^n, and a_0 is as defined above. By property (2) of Lemma 3.4.8 $l(a_0) > s - a_0 + 1$ implies $l(a) > s - a + 1$ for all $a > a_0$. Therefore the following conditions are equivalent:

1. $1 \leq a_0 \leq s$ is the only row that satisfies the coverage conditions for l.

2. $l(a - 1) > l(a)$ for all $a > a_0$.

By Corollary 3.4.7 there exists some $1 \leq t \leq s+1$ such that $l(a) = 2(s - a + 1)$ for all $a \geq t$ and $l(a) < 2(s - a + 1)$ for all $a < t$. Note that since row s of the tableaux in K_s^n has only two entries, either $l(s) = 1$, in which case we must have $l = l_0$ the unique minimal element of K_s^n, or $l(s) = 2$ in which case $t \leq s$. In the former case, l does not have a row a_0 so it is excluded from our discussion (the minimal elements of k_s^n are not considered join-irreducibles of K_s^n). Therefore we must have $1 \leq t \leq s$. Clearly by the definition of a_0, we must have $t \geq a_0$.

The discussion so far shows the following: let l be a location function, $l \neq l_0$, and let a_0 be the minimal row that satisfies the coverage conditions for l. Then l covers only one element in K_s^n if and only if $a_0 \le t \le s$ and

$$l(a) = \begin{cases} s-a+1 & \text{for } a < a_0 \\ l(a+1)+1 & \text{for } a_0 \le a < t \\ 2(s-a+1) & \text{for } a \ge t. \end{cases} \tag{3.14}$$

It follows that the two values $1 \le a_0 \le s$ and $a_0 \le t \le s$ determine the entire function l. On the other hand, it is easy to verify that a function l defined as in (3.14) is indeed a location function with a_0 being its only row that satisfies the coverage conditions. Therefore, the join-irreducibles of K_s^n are in bijection with the set $D_s := \{(a_0, t) : 1 \le t \le s, 1 \le a_0 \le t\}$. Moreover, if we think of D_s as a poset with the partial ordering

$$(a_0, t) \le (a_0', t') \text{ if and only if } a_0 \ge a_0' \text{ and } t \ge t',$$

it follows from Corollary 3.4.9 and equation (3.14) that the bijection between the join-irreducibles of K_s^n and the set D_s is in fact a poset isomorphism. Letting $x = s+1-t$ and $y = t+1-a_0$, it is now an easy exercise to verify that the poset D_s is isomorphic to the poset A_s. □

Let $M(s)$ be the set of all subsets of $[s]$, with the ordering $A \le B$ if the elements of A are $a_1 > a_2 > \cdots > a_j$ and the elements of B are $b_1 > b_2 > \cdots > b_k$, where $j \le k$ and $a_i \le b_i$ for $1 \le i \le j$. $M(s)$ is a well known poset, defined, e.g., in [4, page 68] and in [8, page 177]. As mentioned in [8, page 177], $M(s)$ is a distirbutive lattice and $M(s) \cong J(A_s)$. We therefore have the following corollary.

Corollary 3.4.11. *We have* $K(L_s^n) \cong M(s)$.

3.4.2 Medium m

Theorem 3.4.12. *For* $1 < m \le s$ *we have*

$$\sigma(L_s^n) = \frac{1}{2}(s-m+1)(s+3m-2) + m(m-1)$$

and there is exactly one element in $K(L^n_s)$.

Proof. Suppose $T \in K^n_s$. By Corollary 3.4.3, for $s - m + 2 \leq a \leq s$ all the entries in the ath row of T are reducible. This (and the fast that the entries of T are at most $n = s + m$) implies in particular:

- $T_{s,2} \leq s + m$ (the last entry of the sth row) ;
- $T_{s,1} \leq s + m - 1$;
- $T_{s-j,1} \leq (s + m - 1) - 2j$ for $1 \leq j \leq m - 1$.

Thus, $T_{s-m+1,1} \leq s-m+1$. But the columns of T are strictly increasing, so $T_{s-m+1,1} \geq s - m + 1$. Hence all of the above inequalities must be equalities, so we must have $T_{s-m+1,1} = s - m + 1$ and this entry is irreducible. Note that row $s - m + 1$ satisfies property (2) of Corollary 3.4.3, so we have $l(s-m+1) = 1$ which implies (by Lemma 3.4.5) $l(a) = (s - m + 1) - a + 1$ for all $1 \leq a \leq s - m + 1$. This determines all the entries in the first $s - m + 1$ rows of T.

Now for $0 \leq j \leq m - 2$, row $a = s - j$ of T satisfies:

- $T_{s-j,1} = (s + m - 1) - 2j$.
- All the entries in row a are reducible, so for any $2 \leq b \leq 2j + 2$ we have $T_{a,b} - T_{a,b-1} \geq 1$.
- There are $2(j + 1)$ entries in row a, and $T_{a,2j+2} \leq s + m$.

These three properties show that we must have equalities $T_{a,b} - T_{a,b-1} = 1$ for all $2 \leq b \leq 2j + 2$, and this determines all the entries of row a.

All in all, the above discussion describes a single tableau that can be in K^n_s, and it is easy to verify that this tableau is indeed in K^n_s. See Figure 3-3 for an example of the only element of L^9_6 with 32 reducible entries. Thus we have established that $K(L^n_s) = K^n_s$ has a single element and $\sigma(L^n_s) = \frac{1}{2}(s-m+1)(s+3m-2)+m(m-1)$ □

1	1	1	1	2	3	4	5	6	7	8	9
2	2	2	3	4	5	6	7	8	9		
3	3	4	5	6	7	8	9				
4	5	6	7	8	9						
6	7	8	9								
8	9										

Figure 3-3: The only element of K_6^9

3.4.3 Large m

Theorem 3.4.13. *For $m > s$ we have $\sigma(L_s^n) = s^2 + s$ and $K(L_s^n) \cong P_s^{m-s}$, where P_s^n is the coordinate-wise partial ordering on reverse (nonstrict) plane partitions of shape $(2s, 2s-2, ..., 2)$ with largest part at most n.*

Proof. Let $T \in L_s^n$. $m > s$ implies $m + s - a \geq 2(s - a + 1)$ for any $1 \leq a \leq s$, so by Lemma 3.4.1 the number of reducible entries in the ath row of T is at most $2(s - a + 1)$, and

$$\sigma(L_s^n) \leq \sum_{a=1}^{s} 2(s - a + 1) = s^2 + s.$$

Let K_s^n be the subposet of L_s^n consisting of the tableaux with $s^2 + s$ reducible entries. Let $T \in K_s^n$, so all the entries in all the rows of T are reducible, i.e., for any $1 \leq a \leq s$ and any $1 \leq b \leq 2(s - a + 1)$,

$$T_{a,b} - T_{a,b-1} \geq 1 \text{ and } T_{a,b} - T_{a-1,b} \geq 2. \tag{3.15}$$

For $T \in K_s^n$, let T' be the tableau of shape μ_s and entries $T'_{a,b} = T_{a,b} - (2a+b-2)$. By (3.15), the rows and columns of T' are weakly increasing and $T'_{1,1} = T_{1,1} - 1 \geq 1$ so all the entries of T' are positive integers. The largest part of each row a of T (T') is in the last entry $T_{a,2(s-a+1)}$ ($T'_{a,2(s-a+1)}$) and $T_{a,2(s-a+1)} \leq s + m$ if and only if $T'_{a,2(s-a+1)} \leq s + m - 2s = m - s$, so T has largest part at most $s + m$ if and only if T' has largets part at most $m - s$. It is easy to see that the map $T \mapsto T'$ is invertible, so it is a bijection between K_s^n and P_s^{m-s}. It is also clear that the map $T \mapsto T'$ is a poset isomorphism between these two sets, so we get $K(L_s^n) = K_s^n \cong P_s^{m-s}$ and $\sigma(L_s^n) = s^2 + s$. ☐

3.5 Double Staircase with a Short Step

For $1 \leq k \leq s$, let $\nu_{s,k}$ be the partition

$$(2s-1, 2s-3, ..., 2s-2k+1, 2s-2k, 2s-2k-2, ..., 4, 2)$$

with $s-1$ "double steps" and one shorter step (see Figure 3-4 for an example of the Young diagram of shape $\nu_{6,3}$ in which the third step is shorter). Let $L^n_{s,k}$ be the coordinate-wise partial ordering on SSYT of shape $\nu_{s,k}$ with largest part at most n. For the shape $\nu_{s,k}$ we only know how to analyze the case $n = s+1$.

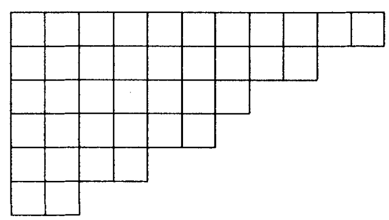


Figure 3-4: The Young diagram of shape $\nu_{6,3}$

Theorem 3.5.1. *We have* $\sigma(L^{s+1}_{s,k}) = \binom{s+1}{2}$ *and* $K(L^{s+1}_{s,k})$ *has only one element.*

Lemma 3.5.2. *Let* $T \in L^{s+1}_{s,k}$*. Then for any* $1 \leq a \leq s$,

$$\#\textit{reducible entries in the ath row of } T \leq s+1-a.$$

Therefore,

$$\#\textit{ reducible entries in } T \leq \sum_{a=1}^{s} s+1-a = \binom{s+1}{2}.$$

Proof. This is again an easy consequence of Lemma 3.1.1, since for all $1 \leq a \leq s$ the length of the ath row of T is either $2(s-a+1)$ (for $k+1 \leq a \leq s$) or $2(s-a+1)-1$ (for $1 \leq a \leq k$) and in any case it is not less than $s+1-a$. □

Let $K^{s+1}_{s,k}$ be the subposet of $L^{s+1}_{s,k}$ consisting of the tableaux with $\binom{s+1}{2}$ reducible entries.

If we take the element of $K(L^{s+1}_{s})$ (as defined in Section 3.4) with $l(a) = 2(s-a+1)$ and remove the last entries of rows $1, 2, ..., k$, the result is clearly a tableau T_0 in $L^{s+1}_{s,k}$.

We only removed irreducible entries, so by Theorem 3.4.6, the number of reducible entries of T_0 is $\binom{s+1}{2}$ hence $T_0 \in K^{s+1}_{s,k}$. We claim this is the only element of $K^{s+1}_{s,k}$.

Corollary 3.5.3. *Suppose $T \in K^{s+1}_{s,k}$. Then T satisfies the following conditions.*

1. *All the rows of T are maximally reducible.*

2. *For any $1 \leq a \leq s$,*

   (a) *for $1 \leq b \leq s-a$ we have $T_{a,b} = a$. Thus the first $s-a$ entries of the ath row of T are irreducible.*

   (b) *Out of the remaining entries in the ath row, $s-a+1$ entries (a,b) are reducible and satisfy $T_{a,b} - T_{a,b-1} = 1$. For $1 \leq a \leq k$, these are all the remaining entries in the ath row of T. For $k+1 \leq a \leq s$, there is one irreducible entry (a,b) that satisfies $T_{a,b} - T_{a,b-1} = 0$ for some $s-a+1 \leq b \leq 2(s-a+1)$.*

Proof. The proof is very similar to the proof of Corollary 3.4.3. We simply use the lengths of the rows of $\nu_{s,k}$ instead of the lengths of the rows of μ_s. □

Lemma 3.5.4. *Suppose $k < s$ and $T \in K^{s+1}_{s,k}$. Then for $k+1 \leq a \leq s$, the last entry $T_{a,2(s-a+1)}$ of the ath row of T is irreducible.*

Proof. We prove by induction on $1 \leq i \leq s-k$ that $T_{k+i,2(s-(k+i)+1)}$, the last entry of the $(k+i)$-th row of T, is irreducible. The base case is for $i=1$. By Corollary 3.5.3, the last two entries of the kth row of T are $T_{k,2s-2k} = s$ and $T_{k,2s-2k+1} = s+1$. But $T_{k+1,2s-2k} \leq s+1$ so we must have $T_{k+1,2s-2k} = s+1$, $T_{k+1,2s-2k} - T_{k,2s-2k} = 1$ and $T_{k+1,2s-2k}$ is irreducible.

Now suppose the last entry of row $k+i$ is irreducible, so $T_{k+i,2(s-(k+i)+1)} = T_{k+i,2(s-(k+i)+1)-1}$. By maximal reducibility in row $k+i$, the last two entries of the row equal $s+1$. By Corollary 3.5.3, the one before last entry of row $k+i$ is reducible, so $T_{k+i,2(s-(k+i)+1)-2} = s$. But just like the base case, this implies that the last entry of row $k+i+1$, $T_{k+i+1,2(s-(k+i+1)+1)}$, is irreducible. □

Corollary 3.5.3 and Lemma 3.5.4 determine all the entries of $T \in K_{s,k}^{s+1}$, and show that T_0 is indeed the only element of $K_{s,k}^{s+1}$. Therefore we have $\sigma(L_{s,k}^{s+1}) = \binom{s+1}{2}$ and $K(L_{s,k}^{s+1}) = K_{s,k}^{s+1}$ has only one element.

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