Parabolic Springer Resolution

by

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Abstract

Let $G$ be a reductive group over a field $k = \bar{k}$. Let $P$ be a parabolic subgroup. We construct a functor $\text{Groupoid}(V_c^0) \to \text{Cat}$, where $V_c^0$ is a connected space, which induces an action of $\pi_1(V_c^0)$ on $D^b(\text{Coh}(T^*G/P))$ generalizing a classical result. It is also a part of a study of natural equivalences between $D^b\text{Coh}(T^*G/P) \simeq D^b\text{Coh}(T^*G/Q)$ for $P, Q$ associated parabolic subgroups.

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Chapter 1

Introduction

We explain the motivation of the work.

**Group actions and symplectic resolutions.**

Let $Y$ be a symplectic singularity over an algebraically closed field $k$. Let $X$ be a symplectic resolution of $Y$. Consider the bounded derived category of coherent sheaves on $X$, $D^b\text{Coh}(X)$. In some cases such as the Springer resolution, an action of a group on this category was constructed.

Let $G$ be a reductive group over an algebraically closed field $k = \bar{k}$. Let $g := \text{Lie}(G)$, $B$ a Borel subgroup, $\mathcal{N}$ the nilpotent cone in $g$. The Springer resolution $\pi : T^*G/B \to \mathcal{N}$ is a symplectic resolution. Its derived category $D^b(\text{Coh}(T^*G/B))$ carries a known action of the affine braid group $\text{Braff}$. ([25][24][23][22][5]. See also [2] for a different realization of the action). One would like to find analogous actions for other symplectic resolutions. Let $P$ be a parabolic subgroup of $G$. Let $X := T^*G/P$. We suggest a construction of a local system of categories on a topological space $V_C^0$, with values different
realizations of the category

\[ C := D^b(\text{Coh}(T^*G/P)) \]  

(1.1)

This gives rise to an action of \( \pi_1(V^0)_C \) on the category \( C \). In the case of \( P = B \), this recovers the known action of \( \text{Br}_{\text{aff}} \) on the category. More precisely, \( \pi_1(V^0_C) = \text{Br}_{\text{aff}, \text{pure}} \) - the pure affine braid group. However in this case there is additional symmetry in the construction, that allows the action to extend to an action of \( \text{Br}_{\text{aff}} \).

A central role in the construction of the base \( V^0_C \) is played by the topic of quantizations in positive characteristic. The base of the construction has to do with the universal parameter space of quantizations of the symplectic resolution. The quantizations of a symplectic resolution \( X \), which correspond to points in \( V^0_C \), give rise to various t structures on the category \( D^b(\text{Coh}(X)) \). In some cases the construction enables a description of the variation of these t-structures along the parameter space \( V^0_C \).

**D equivalences**

Let \( F \) be an algebraically closed field. Let \( X^{(1)}, X^{(2)} \) be two smooth projective varieties over \( F \).

**Definition 1.** \( X^{(1)}, X^{(2)} \) are K equivalent if there is a smooth projective variety, \( Z \), and a birational correspondence

\[ X^{(1)} \leftarrow Z \rightarrow X^{(2)} \]  

(1.2)

such that the pullbacks of the canonical divisors to \( Z \), are linear equivalent.
A conjecture by Kawamata [16] suggests that 'K equivalence implies D equivalence'. That is a K equivalence of the varieties, implies an equivalence of the bounded derived categories of coherent sheaves as triangulated categories. One caveat of this conjecture is that it is not expected that there will be a construction of a canonical equivalence of the bounded derived categories. When the conjecture holds, one can further study the family of natural equivalences between the categories.

One case where the conjecture holds is the following. Let $Y$ be a symplectic singularity over $F$, let $X^{(i)}$ be its different symplectic resolutions. These varieties are K equivalent, (the canonical divisor of a symplectic resolution is trivial). Kaledin proved [13] [15], that the categories $D^b(\text{Coh}(X^{(i)}))$ are equivalent. His construction of the equivalence was non canonical. The key to his construction of an equivalence functor is a choice of a tilting generator of $D^b(\text{Coh}(X))$, which is a non canonically defined object.

A natural question in this case is to study the family of natural equivalences between

$$D^b(\text{Coh}(X^{(i)})), D^b(\text{Coh}(X^{(j)}))$$

In the case $X^{(i)} = T^*G/P^{(i)}$ for $P^{(i)}$ associated parabolic subgroups, there is an idea to attach the categories $D^b\text{Coh}(T^*G/P^{(i)})$ to certain points $pt_{P^{(i)}}$ of the base space as part of the construction above, to see the natural equivalences as the assignment of the local system to homotopy classes of paths between these points.
Chapter 2

Background

This section sets notations that will be used and background material.

Let $k$ be an algebraically closed field of characteristic $p >> 0$. Let $G$ be a reductive group over $k$. Let $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{B}$ be the variety of Borel subalgebras in $\mathfrak{g}$. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Let $T^*B \to \mathcal{N}$ be the Springer resolution. Let $U := U(\mathfrak{g})$ be the enveloping algebra.

Let $T$ be a maximal torus. Let $\Lambda$ be the root lattice of $G$. Let $\lambda \in \Lambda$. Denote by $O(\lambda)$ the corresponding $G$ equivariant line bundle on $G/B$. Let $W$ be the Weyl group.

Let $\lambda \in \Lambda$. Recall the dot action of $W$, $w \cdot \lambda := p\nu((\lambda + \rho)/p) - \rho$, $w \in W$. $\lambda \in \Lambda$ is called $p$-regular, if the stabilizer in $W$ of $\lambda + p\Lambda \in \Lambda/p\Lambda$ is trivial.

Let $P$ be a parabolic subgroup of $G$ such that $T \subset P \subset G$. Let $L$ be a Levi subgroup.

Let

$$\Lambda_L = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle = 0 \quad (\alpha \in R(L,T)) \}$$

(2.1)
Let $T^*G/P \to \mathcal{N}_L$ denote the parabolic Springer map.

Let $X$ be an algebraic variety. Let $O_X$ be the structure sheaf. Given a coherent sheaf $\mathcal{A}$ of associative $O_X$ algebras, let $\text{Coh}(X, \mathcal{A})$ be the category of sheaves of coherent $\mathcal{A}$ modules.

**$\mathcal{D}$ modules and Coherent sheaves in characteristic $p$**

(Reference [23] [22])

Let $k$ be a perfect field of characteristic $p >> 0$. Let $X$ be a smooth variety over $k$.

Let $\mathcal{D}_X$ be the sheaf of crystalline differential operators on $X$. In this section we will recall properties of $\mathcal{D}_X$ and $\mathcal{D}_X$ modules. In particular we will recall its relations with the structure sheaf of the cotangent space $O_{T^*X}$ and recall the relations between $\mathcal{D}_X$ modules and coherent sheaves on $T^*X$. Two main ideas are

1. $\mathcal{D}_X$ is a quantization of the sheaf $O_{T^*X}$.
2. $\mathcal{D}_X$ has a structure of a quasi coherent sheaf of algebras on $T^*X^{(1)}$ because the center of $\mathcal{D}_X$ is canonically isomorphic to the sheaf of commutative rings $O_{T^*X^{(1)}}$.

The superscript $(1)$ stands for the Frobenius twist of a variety $Y$ defined over a perfect field $k$. As an abstract scheme, $Y \simeq Y^{(1)}$. Moreover, when $Y$ is defined over $\mathbb{F}_p$ then $Y \simeq Y^{(1)}$ also as a $k$ scheme. Hence we omit the superscript from the notation, identifying $T^*X^{(1)}$ with $T^*X$. 

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In what follows we focus on properties and observations that we later use. For more a reference is [23][22].

Let $\mathcal{D}_X$ be the sheaf of crystalline differential operators on $X$. It assigns to an affine open subset $U \subset X$, an algebra $\mathcal{D}_X(U)$. By definition $\mathcal{D}_X(U)$ is generated by $\mathcal{O}_X(U)$ and $\text{Vect}_X(U)$ (the $\mathcal{O}_X(U)$-module of vector fields on $U$). The relations are the following:

1. For $\zeta_1, \zeta_2 \in \text{Vect}(U)$, $\zeta_1 \zeta_2 - \zeta_2 \zeta_1 = [\zeta_1, \zeta_2] \in \text{Vect}(U)$.
2. For $\zeta \in \text{Vect}(U), f \in \mathcal{O}_X(U)$, $\zeta f - f \zeta = \zeta(f)$.
3. $\mathcal{O}_X(U)$ is a subalgebra of $\mathcal{D}_X(U)$.

**Remark 1.** For $k$ an algebraically closed field of characteristic zero, this is the definition by generators and relations of the ordinary sheaf of differential operators.

We fix some notations. *Twisted differential operators*: Let $\mathcal{L} \in \text{Pic}(X)$ be a line bundle. We denote by $\mathcal{D}_{\mathcal{L},X}$ the sheaf of $\mathcal{L}$ twisted differential operators.

Let $G$ be as above. Let $P$ be a parabolic subgroup. In the special case $X := G/P$ Let $\mathcal{D}_{\lambda} := \mathcal{D}_{\lambda,X} := \mathcal{D}_{\mathcal{O}(\lambda),X}$ for $\lambda \in \Lambda_L$.

Let $\mathcal{D}_{\mathcal{L},X} \text{-mod}$ be the category of $\mathcal{L}$ twisted $\mathcal{D}$ modules on $X$. In the special case $X := G/P$, we also denote $\mathcal{D}_{\lambda,X} \text{-mod}$ for $\lambda \in \Lambda_L$, or $\mathcal{D}_{\lambda} \text{-mod}$ when there is no ambiguity for what $P$ is.
Claim 1. **It is still true, as in characteristic zero, that** $D_X$ **is a quantization of** $O_{T^*X}$. **That is, there is the natural increasing filtration on this sheaf** $D_X$ **by order of the differential operator, and**

$$gr(D_X) \cong O_{T^*X} \quad (2.2)$$

Claim 2. **In contrast to the characteristic zero situation, an element in** $D(U)$ **is not uniquely identified by its action on** $O_X(U)$. **The morphism**

$$D_X \rightarrow \text{End}(O_X) \quad (2.3)$$

**isn't injective. The basic example to keep in mind is where** $X = \text{Spec}(k[x])$ **is the affine line, and the differential operator is** $\partial^p$. **It has a trivial action on the structure sheaf, even though it is not zero as an element of** $D_X$.

Theorem 1. **The center of** $D_X$ **is big. It is canonically isomorphic to the sheaf of rings** $O_{(T^*X)^{(1)}}$.

This implies that $D_X$ has the structure of a quasi coherent sheaf of algebras on $T^*X^{(1)}$.

Theorem 2. **Given** $L \in \text{Pic}(X)$, **$D_L \times X$ is an Azumaya algebra on** $T^*X^{(1)}$.

Given an Azumaya algebra $\mathcal{A}$ over a variety $Y$, one can consider the category of coherent $\mathcal{A}$ modules $\text{Coh}(Y, \mathcal{A})$. [22]

An Azumaya algebra $\mathcal{A}$ is called **equivalent** to an Azumaya algebra $\mathcal{B}$ when they are Morita equivalent. One denotes $\mathcal{A} \sim \mathcal{B}$. In this case $\text{Coh}(Y, \mathcal{A}) \cong \text{Coh}(Y, \mathcal{B})$. [22]
An Azumaya algebra $A$ is called split over $Y$ if $A \sim O_Y$. In this case $\text{Coh}(Y, A) \simeq \text{Coh}(Y)$.

Specializing to the case $Y := T^*X$,

**Claim 3.** Given a line bundle $\mathcal{L}$ on $X$, there is a canonical Morita equivalence between $\mathcal{D}_{\mathcal{L}, X}$ and $\mathcal{D}_X$. This implies a canonical equivalence between the categories

$$\mathcal{D}_X\text{-mod} \simeq \mathcal{D}_{\mathcal{L}, X}\text{-mod}$$

(2.4)

**Theorem 3.** $A := \mathcal{D}_X$ doesn't split over $T^*X$, but it does splits on a formal neighborhood of the zero section of $T^*X$.

The splitting implies the equivalence:

$$\mathcal{D}_{\mathcal{L}, X}\text{-mod}_0 \simeq \text{Coh}_0(T^*X)$$

(2.5)

The subscript $0$ means we restrict to objects which are supported on the formal neighborhood of the zero section in $T^*X$. (We think of $\mathcal{D}_{\mathcal{L}, X}$ modules as sheaves on $T^*X^{(1)}$ with an additional structure).

In the special case that $X = T^*B$ this can be generalized.

**Theorem 4.** Consider $X = T^*B$

Consider the Springer map $\pi : T^*G/B \to N$. $\mathcal{D}_{G/B}^\wedge$ splits on the formal neighborhood of every fiber of the Springer resolution $\pi$. Hence obtain an equivalence

$$\mathcal{D}_{G/B}^\wedge\text{-mod}_e \simeq \text{Coh}_e(T^*(G/B)),$$

(2.6)

where the subscript stands for a support condition on a formal neighborhood of $\pi^{-1}(e)$ for $e \in N$. 

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We won’t use this generality in this work though.

**Definition 2.** Let $F_{D_{L},\text{coh}_{T^{*}X}}$ denote the equivalence functor

$$F_{D_{L},\text{coh}_{T^{*}X}} : \mathcal{D}_{L,X} \rightarrow \text{Coh}_{0}(T^{*}X)$$  \hspace{1cm} (2.7)

In the case $X = G/P$ Let

$$\mathcal{D}_{\lambda,P} := \mathcal{D}_{G/P}^{O(\lambda)}$$  \hspace{1cm} (2.8)

Let

$$\mathcal{D}_{\lambda} := \mathcal{D}_{G/P}^{O(\lambda)}$$  \hspace{1cm} (2.9)

when $P$ is unambiguous.

Let

$$F_{D_{\lambda},\text{coh}} := F_{D_{O(\lambda)},\text{coh}_{T^{*}G/P}}$$  \hspace{1cm} (2.10)

Let

$$F_{D,\text{coh}} := F_{D,\text{coh}_{T^{*}G/P}}$$  \hspace{1cm} (2.11)

when there is no twist.
Localization theorem in characteristic $p$

We set some notations first. Let $T \subset B \subset G$ be a maximal torus and a Borel subgroup in $G$. Let $t, b, g$ be the corresponding Lie algebras. Let $\Phi$ be the root system of $(G, T)$ and $\Phi^+$ be the positive roots. Let $W$ be the Weyl group of $\Phi$. $W$ acts on $\Lambda$ by the dot $\cdot$ action. Let $\mathfrak{h}$ be the universal Cartan subalgebra of $g$. There is a canonical isomorphism $\mathfrak{h} \simeq b/[b, b] \simeq t$

Let $\zeta$ be the center of the enveloping algebra $U(g)$. The center consists of two parts. The first is the Harish Chandra center, which is the $G$ invariants in the enveloping algebra.

$$\zeta_{HC} := U(g)^G$$ (2.12)

There is an isomorphism

$$U(g)^G \simeq S(\mathfrak{h})^{(W, \cdot)}$$ (2.13)

where $S(\mathfrak{h})$ is the symmetric algebra of $\mathfrak{h}$ and we take the $W$ invariants for the dot action.

Let $\lambda \in \Lambda$. $\lambda$ defines a maximal ideal of $\zeta_{HC}$ via the natural map

$$\Lambda/p\Lambda \rightarrow \mathfrak{h}^*/W, \lambda \mapsto d\lambda$$ (2.14)

Let

$$U_\lambda := U(g)_\lambda := U \otimes_{\zeta_{HC}} k$$ (2.15)

Let $U_\lambda$-mod := $U(g)_\lambda$-mod be the category of finitely generated $U(g)_\lambda$ modules.

The second part of the center of $U(g)$ is the Frobenius center $\zeta_{Fr}$, also
called the $p$-center. The subalgebra $\zeta_{Fr}$ is generated by the elements of the form $\lambda^{p} - \lambda^{[p]} x \in \mathfrak{g}$, where $x \mapsto x^{[p]}$ is the restricted power map, which is characterized by the formula:

$$ad(x^{[p]}) = ad(x)^{p}. \quad (2.16)$$

Maximal ideals of $\zeta_{Fr}$ are in bijection with points of $\mathfrak{g}^{*} \simeq \mathfrak{g}$.

$$\zeta_{Fr} \simeq S(\mathfrak{g}^{(1)}) \quad (2.17)$$

The right hand side means functions on $\mathfrak{g}^{*(1)}$, (the Frobenius twist of $\mathfrak{g}^{*}$).

The algebra $\zeta$ is built from these two parts. When $p >> 0$, $\zeta_{HC} \otimes_{\zeta_{Fr}, \zeta_{HC}} \zeta_{Fr} \simeq \zeta$. In other words, to give a central character of $U(\mathfrak{g})$, we need to give a pair of elements $(\lambda, e) \in \mathfrak{h}^{*} \times \mathfrak{g}^{*(1)}$ which are compatible.

Let us restrict to the case $\lambda \in \Lambda$, and $e$ is nilpotent. Given a compatible pair $\lambda \in \Lambda$ and $e \in \mathfrak{g}^{*}$ nilpotent we denote by $U_{\lambda}^{\text{mod}, e}$ the category of finitely generated $U_{\lambda}$ modules with generalized $p$ character $\lambda$. That is, $U_{\lambda}$ modules which are killed by a power of the maximal ideal of $e$ in $\zeta_{Fr}$.

**Derived localization in characteristic $p$**

For the classical BB localization theorem in characteristic zero see [4]. The following theorem, proved in [22],[21], is an analog in positive characteristic.

Let $\lambda \in \Lambda$.

**Theorem 5.** There is a natural isomorphism $\Gamma(D_{G/B, \lambda}) \simeq U(\mathfrak{g})_{\lambda}$.

**Theorem 6.** [22]. Let $\lambda \in \Lambda$ be $p$-regular, then the derived global sections
functor
\[ \Gamma^B : D^b(\mathcal{D}_{\lambda,G/B} \text{-mod}) \to D^b(U_\lambda \text{-mod}) \quad (2.18) \]
is an equivalence of categories.

Let \( \text{Loc}^B_\lambda : D^b(U_\lambda \text{-mod}) \to D^b(\mathcal{D}_{\lambda,G/B} \text{-mod}) \) denote the derived localization functor.

**Theorem 7.** Let \( \lambda, e \) be a compatible pair. The derived global sections functor induces an equivalence of the full subcategories

\[ \Gamma^B_{\lambda,e} : D^b(\mathcal{D}_{\lambda,G/B} \text{-mod}_e) \to D^b(U_\lambda \text{-mod}_e) \quad (2.19) \]

Note that combining the last theorem with the equivalence in equation (2.5), the following equivalence is obtained:

\[ D^b(\text{Coh}_0(T^*G/B)) \simeq D^b(\mathcal{D}_{\lambda,G/B} \text{-mod}_0) \simeq D^b(U_\lambda \text{-mod}_0). \quad (2.20) \]

Similarly, there is a parabolic version. We use the definitions from [21], section 1.

Let \( P \) be a parabolic subgroup of \( G \) with a levi \( L \), let \( X := G/P \), let \( \lambda \in \text{Pic}(G/P) \).

Define the algebra

\[ A_{\lambda,G/P} := \Gamma(G/P,\mathcal{D}_{\lambda,G/P}) \quad (2.21) \]

\[ A_{\lambda,G/P} \text{-mod} \]

Let \( A_{\lambda,G/P} \text{-mod} \) be the category of finitely generated modules over \( A_{\lambda,G/P} \).

Let \( \chi \in \mathfrak{g}^{(1)} \) be compatible with \( \lambda \). Let \( A_{\lambda,G/P} \text{-mod}_\chi \subset A_{\lambda,G/P} \text{-mod} \) be the
subcategory of finitely generated $A_{\lambda, G/P}$ modules with generalized Frobenious character $\chi$.

We claim the following

**Claim 4.** Let $P, Q$ be associated parabolic subgroups. There is a canonical equivalence $\text{Pic}(G/P) \simeq \text{Pic}(G/Q)$. Let $\lambda \in \text{Pic}(G/P) \simeq \text{Pic}(G/Q)$. Then the algebras $A_{\lambda, G/P}$ and $A_{\lambda, G/Q}$ are canonically equivalent. (See a proof below).

This justifies omitting the choice of $P$ in an association class from the notation for $A_{\lambda}$.

Let $A_{\lambda} := A_{\lambda, G/P}$

Let $A_{\lambda}-\text{mod}$ be the category of finitely generated $A_{\lambda}$ modules.

**Theorem 8.** Let $\lambda \in \Lambda_L$ be $p$-regular, then the derived global sections functor

$$\Gamma^P_\lambda : D^b(D_{\lambda, G/P}-\text{mod}) \to D^b(A_{\lambda}-\text{mod})$$

is an equivalence of categories, and

$$D^b(\text{Coh}_0(T^*G/P)) \simeq D^b(D_{\lambda, G/P}-\text{mod}_0) \simeq D^b(A_{\lambda}-\text{mod}_0)$$

We define notations:

- Let $\Gamma^P_\lambda$ be the derived global sections functor.
- Let $\text{Loc}^P_\lambda$ denote the derived localization functor.
- Let $F^P_{D_{\lambda, \text{Coh}}}$ be an equivalence functor $D^b(D_{\lambda}-\text{mod}_0) \to D^b(\text{Coh}_0(T^*G/P))$
- Let $F^P_{A_{\lambda, \text{Coh}}}$ be an equivalence functor $D^b(A_{\lambda}-\text{mod}_0) \to D^b(\text{Coh}_0(T^*G/P))$
Cone Order

Let $V_{\mathbb{R}}$ be a real vector space. Let $\Sigma$ be a real hyperplane arrangement. Let $V_{\mathbb{R}}^0$ be the complement of $\Sigma$ in $V_{\mathbb{R}}$. We call the connected components, alcoves. Let $C \subset V_{\mathbb{R}}$ be a cone. We will use the notion defined in [1] of two alcoves $A, A'$ having the relation $A'$ is above $A$ with respect to $C$, denoted $A' >_C A$.

Let $V_C$ be the complexification of $V_{\mathbb{R}}$. Let $V_C^0 := V_C - \cup_{H \in \Sigma} H_C$.

**Remark 2.** Given two alcoves $A, A'$ in $V_{\mathbb{R}}^0$ whose closure share a codimension one wall, $H$. Then, $A' >_C A$ if and only if the positive half loop from $A$ to $A'$ which goes around $H$, is a path which maps the interval $(0,1)$ to $V_{\mathbb{R}} + i\mathbb{C}$.

In particular, let $V_{\mathbb{R}} := h^*_{\mathbb{R}}$. Let $\Sigma$ be the affine coroot hyperplanes. Let $C_P^+$ be positive cone for the parabolic $P$. We get a partial order on the alcoves, $A <_{C_P^+} A'$. We denote it also by $A <_P A'$. Moreover, let $\lambda, \mu \in V_{\mathbb{R}}^0$ be points in the interior of the alcoves $A, A'$ respectively, then we also denote $\lambda <_P \mu$.

As described in the abstract, we answer the $D$ equivalence question by a construction of a local system of categories over a topological space $V_C^0$.

The moment map

The reference for the section is [20].

Let $G$ be a reductive over algebraically closed field $k$. Let $\mathfrak{g} := \text{Lie}(G)$. Let $Z/k$ be a variety with an action of the group $G$. The varieties $T^*Z$ and $\mathfrak{g}^*$ are Poisson varieties. The $G$ action on $Z$, gives rise to a $G$ equivariant morphism
of Poisson varieties $T^*Z \to \mathfrak{g}^*$. We compose with the natural isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ from the killing form, and refer to $\mu : T^*Z \to \mathfrak{g}$ as the moment map.

In this work, we let $P$ be a parabolic subgroup of $G$, let $Z := G/P$ and consider the moment map

$$\mu : T^*G/P \to \mathfrak{g}$$

(2.24)

In the case that $P = B$ is a Borel, the image of this map is the nilpotent cone $\mathcal{N}$, and the map is a symplectic resolution, the Springer resolution. More generally, the image of this map is the closure of a nilpotent orbit in $\mathfrak{g}$. The map $T^*G/P \to \text{Image}(\mu)$ is generically a finite cover.

**Symplectic resolutions**

The section doesn’t contain new results. The material can be found concentrated in [29], [20] and [8], and we include it for completeness. In order to be able to generalize the construction in this work to other symplectic resolutions $X^{(t)} \to Y$, it is necessary to review the geometry of symplectic resolutions, their quantizations and deformations, derived localization and Namikawa’s action on the Picard group of $X$. That’s what we briefly recall in the following subsections. Finally at the end we mention the interesting development from [28] on the notion of Symplectic duality. In this section we work over an algebraically closed field of characteristic zero.

A symplectic singularity $Y$, is a normal variety, with an algebraic symplectic two form, $\Omega$, on the smooth locus $Y^{sm}$ of $Y$, such that for some smooth projective resolution of $Y$, $\pi : X \to Y$, the pullback of $\Omega$ from the smooth locus extends to a two form on all of $X$, possibly degenerate.
Beauville [3] showed that if $\Omega$ extends to one smooth resolution, then it extends to any other resolution.

A symplectic resolution $X$ of a symplectic singularity $Y$, is a resolution as above, where the two form on $X$ is a symplectic form.

When the symplectic singularity $Y$ is affine, $Y$ must be the affinization of $X$. $Y := X^{aff} X^{aff} := Spec \Gamma(X, O_X)$. Hence in such case it is enough to specify $X$, without specifying $Y$.

Observation: A symplectic manifold, is in particular a Poisson variety.

Observation: The canonical bundle of a symplectic resolution is trivial $K_X \cong O_X$.

Standard examples of symplectic resolutions include:

1. The Springer resolution $\mu : T^*G/B \to N$, or more generally the restriction of this resolution to a Slodowy slice.

**Slodowy slice:**[26] Let $e \in N$ be a nilpotent element.

The Slodowy slice, $\Sigma_e \subset N$, is a subvariety of $N$, such that the restriction of the Springer resolution $\mu$ to $\mu^{-1}(\Sigma_e)$ is another symplectic resolution. In other words, the preimage of $\Sigma_e$ under the Springer resolution is a smooth manifold, and the restriction of the symplectic two form on $T^*G/B$ to $\mu^{-1}(\Sigma_e)$ is non degenerate.

To define $\Sigma_e$, using Jacobson-Morozov theorem there is an $sl_2$ triple $e, h, f$ that includes $e$. Define $\Sigma_e$ to be the intersection of $e$ plus the centralizer of $f$ in $g$, with the nilpotent cone. One place where the Slodowy slice and the symplectic resolution $\mu^{-1}(\Sigma_e)$ plays a role is in the work [23], [22], where this is used to generalize the known braid group action on the category $D^b Coh_{\mu^{-1}(0)}(T^*G/B)$ (coherent sheaves with support on
a formal neighborhood of the zero section $G/B \subset T^*G/B$), to an action on the bounded derived category of coherent sheaves with support on a formal neighborhood of $\mu^{-1}(e)$.

2. Quiver varieties. See [18]. These varieties have important applications in representation theory: for example, in Nakajima’s geometric construction of the representations and crystals of Kac-Moody algebras.

3. Symplectic resolutions of quotient singularities. [8] A basic example is a symplectic resolution of Kleinian singularities. Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup. Consider the quotient $\mathbb{C}^2/\Gamma$. This is a Poisson variety, where the Poisson structure is induced from the canonical symplectic structure on $\mathbb{C}^2$. The canonical minimal resolution of $\mathbb{C}^2/\Gamma$ is a symplectic resolution. ([7], [12],[11])

More generally, we can talk about symplectic quotients. Consider any finite dimensional symplectic vector space, $(W, \omega)$ and a symplectic subgroup $G \subset Sp(W, \omega)$. Consider the Poisson variety $W/G$. where the Poisson structure is induced from the symplectic form on $W$. Symplectic resolutions of such varieties (when exists), have been studied.

Remark 3 (Slodowy slice and Kleinian singularities). McKay correspondence gives a correspondence between finite subgroups as above $\Gamma \subset SL_2(\mathbb{C})$, and types $A,D,E$ Dynkin graphs. Recall - Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup. Let $I:=\text{Irreps}(\Gamma)$ parametrize the irreducible representations $L_i$, $i \in I$ of $\Gamma$. the corresponding Dynkin diagram is defined by letting $I$ parametrize the vertices, and the number of edges from $i$ to $j$ be $e_{ij} := \dim \text{Hom}_\Gamma(L_i, L_j \otimes \mathbb{C}^2)$, where $\mathbb{C}^2$ is the standard $SL_2$ representation, restricted to $\Gamma$. (Observe that $e_{ij} = e_{ji}$)
Given $\Gamma$, take $g$ to be of the corresponding type. Let $e \in g$ be a subregular nilpotent. Then there is an equivalence of Poisson varieties between the quotient singularity $\mathbb{C}^2/\Gamma$ and the Slodowy slice for $e$: $\mathbb{C}^2/\Gamma \simeq \Sigma_e$.

Another quotient singularity that has been studied is the following. Let $g$ be a finite dimensional semi simple lie algebra. Assume $g$ is of type $A$, $B$, or $C$. Let $\mathfrak{h}$ be a Cartan subalgebra. Take the vector space $W$, to be the cotangent bundle of the Cartan. The Weyl group $W \subset GL(\mathfrak{h})$, is embedded in the symplectic group $Sp(W)$. In this setup, the quotient $W/W$ has a symplectic resolution. (In fact the condition that $g$ was of type $A$, $B$, or $C$ is also necessary). More generally one could take any coxeter group rather than $W$.

Moreover, specializing the above setup to type $A$, a symplectic resolution of the quotient space is well known. This is the Hilbert Chow morphism. $\pi : Hilb^n(\mathbb{C}^2) \to T^*\mathbb{C}^n/S_n$. 

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Conical symplectic resolutions

The reference for the material is [29] and [28].

**Definition 3.** A conical symplectic resolution of weight $n \in \mathbb{Z}$, is a symplectic smooth complex variety $(X, w)$, where $w$ is the symplectic form on $X$, with the following extra data:

An action of $\mathbb{C}^*$ on $X$, such that

1. The induced action on the symplectic form satisfies that for some $n \geq 1$
   which is called the weight.

   \[ s^*w = s^nw \]  (2.25)

2. The action of $\mathbb{C}^*$ on the coordinate ring of $X$ is via nonnegative weights.

3. $\mathbb{C}[X]^\mathbb{C}^*$ consists only of the constant functions.

4. The canonical map from $X$ to its affinization is a projective resolution of singularities.

Examples include the following cases:

1. A Nakajima quiver variety $X$ is a conical symplectic resolution of weight two.

   Nakajima quiver varieties are a family of examples that will overlap with some of the other examples that we present below.
2. Let $H$ denote a simple unimodular rational hyperplane arrangement. Let $X$ be the hypertoric variety associated with this arrangement and $X_0$ be the hypertoric variety associated with the centralization of the arrangement. Then $X_0$ is the affinization of $X$, and $X \to X_0$ is a conical symplectic resolution with a natural $\mathbb{C}^*$ action of weight two.

The cases of Nakajima quiver varieties and hypertoric varieties, were carefully examined in the work of [29]

3. A case of a special relevance to this work is as follows. Let $G$ be a reductive algebraic group, Let $P$ be a parabolic subgroup. then

$$X = T^* G / P$$

(2.26)

is a conical symplectic resolution, where $\mathbb{C}^*$ acts by inverse scaling of the fibers of the cotangent bundle. The weight of the action is one.

4. Let $\Gamma \subset SL(2, \mathbb{C})$ be a finite subgroup. A crepant resolution of $\mathbb{C}^2 \Gamma$, with a $\mathbb{C}^*$ action induced from the inverse diagonal action on $\mathbb{C}^2$ is an example of a conical symplectic resolution of weight two.

Note that this is a special case of quiver variety where the underlying graph of the quiver is the extended Dynkin diagram of $\Gamma$.

5. Resolutions of symplectic quotients of vector spaces. Most of the examples above are in fact special cases of this construction.
Cohomological properties of conical symplectic resolution

Let $X$ be a conical symplectic resolution.

Claim 5.

$$H^{2p+1}(X, \mathbb{C}) = 0$$ (2.27)

for $p \in \mathbb{Z}^+$. That is, odd cohomology groups vanish.

For even cohomology groups, we have the following result

$$H^{2p}(X, \mathbb{C}) = H^{pp}(X, \mathbb{C})$$ (2.28)

for all $p \in \mathbb{Z}^+$.

The base space $V^0_C$ has to do with deformations and quantizations of symplectic resolutions.

Deformations of symplectic resolutions

The reference for the material is [29].

Proposition 1 (Namikawa). [19]

A conical symplectic resolution $X$ of weight $n$, has a flat universal Poisson deformation.

$$\pi : M \to H^2(X, \mathbb{C}).$$ (2.29)

Moreover, consider the induced $\mathbb{C}^*$ action on $H^2(X, \mathbb{C})$. There is a $\mathbb{C}^*$ action on $M$ which extends the $\mathbb{C}^*$ action on $X \subset M$, the fiber over $0 \in H^2(X, \mathbb{C})$ and $\pi$ is equivariant with respect to these two actions.
A corollary of the proposition is that the cohomology of any two fibers of $\pi$ are canonically isomorphic. That’s because the inclusions of all fibers of $\pi$ are homotopy equivalent.

Quantizations of symplectic resolutions

Examples for filtered quantizations of a structure sheaf of a Poisson variety, include: $U(g)$ as the filtered quantization for $\text{Sym}(g)$, symplectic reflection algebras as quantizations of quotient singularities $V/\Gamma$ for $(V, w)$ symplectic finite dimensional vector space and $\Gamma \subset \text{Sp}(V)$ finite subgroup, and the sheaf of differential operators $\mathcal{D}(X)$ for a cotangent space $T^*X$.

Quantizations of symplectic resolutions have been studied in [27][15][14]. Given a conical symplectic resolution $X \to Y$, to every point in the vector space $H^2(X, \mathbb{R})$ there corresponds a quantization of the structure sheaf $\mathcal{O}_X$.

In the special case $X = T^*G/P$ these are the familiar quantizations by twisted $\mathcal{D}$ modules on $G/P$. Indeed for every conical symplectic resolution there is an isomorphism $\text{Pic}(X) \otimes \mathbb{R} \simeq H^2(X; \mathbb{R})$. In the special case $X = T^*G/P$, $\text{Pic}(T^*G/P) \simeq \text{Pic}(G/P)$. Given $\mathcal{L} \in \text{Pic}(G/P)$, the quantization that corresponds to $\mathcal{L}$ comes from the sheaf of $\mathcal{L}$ twisted differential operators on $G/P$.

For different symplectic resolutions $X^{(i)}, X^{(j)}$ of the same symplectic singularity, the birational isomorphism between $X^{(i)}, X^{(j)}$ induces an isomorphism on the Picard groups $\text{Pic} := \text{Pic}(X^{(i)}) \simeq \text{Pic}(X^{(j)})$. Hence it makes sense to compare the quantizations $\mathcal{O}_{X^{(i)}}^\mathcal{L}$ and $\mathcal{O}_{X^{(j)}}^\mathcal{L}$ parametrized by the same $\mathcal{L} \in \text{Pic}$. We will make use of this fact later.
Derived localization for a conical symplectic resolution

We mentioned derived localization for $X = T^*G/P$ in characteristic $p$. More generally: Fix a conical symplectic resolution $X \to Y$. Let

$$V_R := H^2(X, \mathbb{R}) \simeq \text{Pic}(X) \otimes \mathbb{R}$$

(2.30)

One can ask when does derived localization hold. That is, for which $\mathcal{L} \in H^2(X, \mathbb{R})$ the global sections functor from sheaves of modules over $\mathcal{O}_X$, to modules over the global sections

$$\Gamma : D^b(\mathcal{O}_X \text{-mod}) \to D^b(\Gamma(\mathcal{O}_X^\mathcal{L}) \text{-mod})$$

(2.31)

an equivalence.

It's a conjecture that for any symplectic resolution, derived localization holds away from a discrete set of hyperplanes $H_i$. Let me call these hyperplanes the 'walls' in $V_R$.

Let

$$V^0_R \subset V_R$$

(2.32)

be the complement of these walls.

As mentioned above, the conjecture was proved for the case

$$X = T^*G/P$$

(2.33)

$P$ a parabolic subgroup.

In the paper [29], a precise version of the conjecture is formulated in the
more general case of conical symplectic resolutions. Given a conical symplectic resolution $X$, and a quantization of it $\mathcal{O}_X$, they define an appropriate category of modules over the quantization. This is denoted by $\mathcal{D}$-mod. Indeed in the case when the symplectic resolution is a cotangent bundle $T^*Z$, this category is finitely generated twisted $\mathcal{D}$ modules, where the twist is determined by the quantization. They give a related result on the area where derived localization holds.

Furthermore, they describe results on the area where localization holds even at the level of abelian categories, and not just at the derived categories level.

Another interesting observation is a comparison between the category of modules over quantizations in the analytic verses the algebraic setup.

In the literature one can find results on quantizations either in analytic or algebraic setup. In order to be able to use results that were obtained in either setup, it is important to have a Serre's GAGA type theorem. Indeed, [29] give a precise definition for the category of modules over the analytic quantization $\mathcal{D}^{an}$-mod and prove that the analytification functor from the algebraic category to the analytic category.

$$\mathcal{D} \text{-mod} \rightarrow \mathcal{D}^{an} \text{-mod} \quad (2.34)$$

is an equivalence of categories.

**Namikawa's generalized Weyl group**

The references are [29] and [19].

Let $\phi : X \rightarrow X_0$, be a conical symplectic resolution, where $X_0$ is the
affinization of $X$. Namikawa constructs a group, $W$, called the generalized Weyl group of $X$, which is a Coxeter group. He then constructs an action of $W$ on $H^2(X, \mathbb{R})$.

The quotient $H^2(X, \mathbb{C})/W$ is a vector space. Namikawa proves that it is isomorphic to the Poisson cohomology group $HP^2(X_0, \mathbb{C})$.

The definition of $W$: (In the form presented in [29])

Let $X_s$ be the smooth part of the singular locus of $X_0$. Consider $\{\Sigma_j\} \subset X_s$, the codimension two connected components of $X_s$. For each $j$ we will define a group $W_j$ and let $W := \prod W_j$.

$W_j$ is defined as follows: the preimage under $\phi$ of an arbitrary point in $\Sigma_j$ is a union of $\mathbb{P}^1$'s that form the shape of a finite type ADE Dynkin diagram. $\pi_1(\Sigma_j)$ acts on the Dynkin diagram by diagram automorphism. $W_j$ is the centralizer of $\pi_1(\Sigma_j)$ in the Coxeter group associated to the Dynkin diagram.

**Proposition 2.** [19] Namikawa explains that this action allows the construction of a universal Poisson deformation of $X_0$ as follows: Start with the universal Poisson deformation of $X \pi : M \to H^2(X, \mathbb{C})$ that was mentioned above. This map factors through the affinization $M_0$ of $M$, $M_0 := \text{Spec} \mathbb{C}[M]$, hence get a map $\text{Spec} \mathbb{C}[M] \to H^2(X, \mathbb{C})$. This map is equivariant with respect to Namikawa's action of $W$ on $H^2(X, \mathbb{C})$ and with respect to a symplectic action of $W$ on $M_0$. The induced map between the quotient spaces is the universal Poisson deformation of $X_0$.

In the case that $X = T^*(G/B)$. $X_0 \simeq \mathcal{N}$ and $X_s$ consists of a single component, which is the subregular orbit.
\[ \mathbb{W} = W_1 \]  

is the Weyl group of \( G \).

**Geometry of conical symplectic resolutions**

The reference is [29].

For a conical symplectic resolution \( X \), there is the canonical isomorphism

\[ \text{Pic}(X) \otimes \mathbb{R} \cong H^2(X, \mathbb{R}) \]  

Let \( X^{(i)} \) be a set of conical symplectic resolutions of a fixed cone \( X_0 \). For a given pair of indexes, \( i, j \), consider the natural birational map

\[ f_{ij} : X^{(i)} \dashrightarrow X^{(j)}. \]  

It can be shown that \( f_{ij} \) induces an isomorphism on an open subvarieties \( U_{ij} \subset X^{(i)}, U_{ji} \subset X^{(j)} \), whose complement have codimension at least 2.

This claim can be strengthened. Consider the collection \( X^{(i)} \) of symplectic resolutions of \( X_0 \). For each index \( i \), there exists a fixed open subvarieties \( U_i \) of \( X^{(i)} \) whose complement is of codimension a least 2, such that for all other \( j \), \( f_{ij} \) is an isomorphism from \( U_i \) to \( U_j \).

**Corollary 1.** There is a canonical isomorphism between Picard groups of different conical symplectic resolutions of a fixed cone \( X_0 \).

Denote \( V_{\mathbb{R}} := \text{Pic}(X^{(i)}) \otimes \mathbb{R} \). It is independent of the choice of the conical resolution \( X^{(i)} \). The action of \( \mathbb{W} \) on \( V_{\mathbb{R}} \) is well defined.
Given a cone $X_0$, a natural question is how many conical symplectic resolutions exists. Namikawa proved that the number is finite. Moreover, he gives a result comparing the ample cones of the different symplectic resolutions and relating it to the action of the Namikawa’s Weyl group $W$ on the second cohomology group.

First observe that conical symplectic resolutions of a fixed $X_0$ which are not isomorphic, has a different ample cone.

**Proposition 3.** [19] Given a cone $X_0$ and $X$ a symplectic resolution of $X_0$. Consider $F : M \rightarrow H^2(X, \mathbb{C})$, the universal Poisson deformation of $X$. Consider its affinization $M_0$. Remember that $F$ factors through $M_0$.

There is a finite real hyperplane arrangement $\mathcal{H}$ in $V_\mathbb{R}$, s.t the union of the complexification of the hyperplanes in $V_\mathbb{R} \otimes \mathbb{C}$ is the locus over which the map $M \rightarrow M_0$ is not an isomorphism. Denote this locus $H^{(\mathbb{C})}$.

$\mathcal{H}$ is invariant under the action of $W$. Moreover, Let $V_C^0$ be the complement of $\mathcal{H}$. Then for the ample cones of the different symplectic resolutions of $X_0$ are chambers in $V_C^0$.

Moreover there is a bijection between pairs $X(i), w$ where $w \in W$ and the chambers in $V_C^0$. Where the pair $X(i), w$ represents the chamber which is the transition of the ample cone of $X(i)$ in $V_\mathbb{R}$ by the action of $w$.

Symplectic resolutions and representations of Lie algebras

Reference [28], [29].

A major study on symplectic resolutions has been recently done. As mentioned above, the universal enveloping algebra is ring of global sections of a quantization of a conical symplectic resolution.
Representations of lie algebras have a rich theory which was studied for many years. One can ask to what extent this can be generalized when using different conical symplectic resolutions. The idea is to think about enveloping algebras and the rich theory associated with them as one example of a much larger theory. [29] prove that some properties of the theory can indeed be generalized when studying quantizations of symplectic resolutions of affine cones. In particular to every conical symplectic resolution a category $\mathcal{O}$ is attached, generalizing the BGG category $\mathcal{O}$, and is shown to share some of the properties of the BGG category $\mathcal{O}$, such as the highest weight structure and a theory of Harish Chandra bimodules and a generalization of Beilinson Bernstein localization theorem which we already mentioned above.

Examples of conical symplectic resolutions of affine cones for which this theorem was particularly studied are in particular quiver varieties and hypertoric varieties.

**Symplectic duality**

Reference [28]

An interesting result in [28] is the proof that the generalized category $\mathcal{O}$ which is attached to a conical symplectic resolution is Koszul in many cases, and its Koszul dual happens to be category $\mathcal{O}$ of a different symplectic resolution. That symplectic resolution is called the *symplectic dual*.

There are important consequences of the duality. One interesting corollary is an identification of two geometric realization of weight spaces of simple representations of simply laced simple algebraic groups. (Construction of Nakajima and construction of Ginzburg/ Mirkovic Vilonen)
Relation to physics

In physics there is a phenomena called mirror duality. It’s a duality between certain gauge theories. There is a moduli space attached to a gauge theory. For two mirror dual gauge theories the duality exchanges certain components in their moduli spaces. The Higgs branch and the Coulomb branch.

It seems that the examples for the two sort of dualities seem to coincide.

In recent years there has been progress to define the notion of the Higgs Branch and the Coulomb branch in a precise mathematical manner.

D equivalence of symplectic resolutions

Reference is [8] and [16]

Recall definition 1 of $K$-equivalent varieties.

Two smooth projective varieties $X, X'$ over an algebraically closed field, which are birationally equivalent, are called $K$-equivalent, if there is a birational correspondence $X \leftarrow Z \rightarrow X'$, where $Z$ is some smooth projective variety, such that the pullbacks to $Z$ of the canonical divisors are linear equivalent.

Kawamata conjectured that K equivalence implies an equivalence of the bounded derived categories of coherent sheaves on these varieties.

One case in which this holds is when $X$ and $X'$ are two symplectic resolutions that resolve the same variety. K equivalence holds, since the canonical bundles of symplectic resolutions are trivial. it is a proof of Kaledin that indeed in this case the bounded derived categories are equivalent.

Theorem 9. Given $X, X'$ symplectic resolutions of a fixed symplectic singularity $Y$, there is an equivalence $D^b(X) \simeq D^b(X')$, Where $D^b(-) := D^b(Coh(-))$
The proof involves a construction of such an equivalence. A main ingredient in the construction is a choice of a tilting generator for $D^b(X)$.

**Definition 4.** A tilting generator for a smooth variety $X/k$, is a locally free sheaf $\mathcal{F}$ on $X$, such that

1. $\text{Ext}^i(\mathcal{F}, \mathcal{F}) = 0, \quad (2.38)$

for all $i > 0$,

2. The functor

$$R \text{Hom}(\mathcal{F}, -) : D^b \text{Coh}(X) \to D^b(A_{\mathcal{F}} \text{-mod}) \quad (2.39)$$

is an equivalence of categories. where,

$$A_{\mathcal{F}} := \text{End}(\mathcal{F})^{op} \quad (2.40)$$

**Theorem 10 ([9]).** A conical symplectic resolution has a tilting generator.

Using the existence of the tilting generator, the proof of the $D$ equivalence begins by let $\mathcal{E}$ be a choice of a tilting generator for $X$. As we saw before $X$ and $X'$ agree on an open subset whose complement is of codimension at least two. $\mathcal{E}$ can be extended to a vector bundle $\mathcal{E}'$ on $X'$, such that $H^i(X', \text{End}(\mathcal{E}')) = 0$ for $i > 0$. Yet $\mathcal{E}'$ is not necessarily the tilting generator of $X$. $\mathcal{E}, \mathcal{E}'$ agree on an open set whose complement is of codimension at least two, and the algebras $R := \text{End}(\mathcal{E}) \cong \text{End}(\mathcal{E}') =: R'$ are isomorphic. Hence we can consider the functor $\text{Hom}(\mathcal{E}', -) : D^b(X') \to D^b(R'^{op} \text{-mod}) \cong D^b(X)$. That's the functor that is proved by Kaledin to be an equivalence. [9], [13].

The caveat is that the equivalence constructed is highly non canonical,
since a tilting generator is not. The local system constructed below can lead to a better understanding of a family of natural D equivalences between these categories.

The Springer action

Reference [30].

For simplicity assume we work over $k = \mathbb{C}$ in this section. Recall that $G$ is an algebraic group defined over $k$, $g := \text{Lie}(G)$, $B$ is the flag variety, $W$ is the Weyl group. Let $\pi : T^*G/B \to \mathcal{N}$ be the Springer resolution. $\mathcal{N}$ be the nilpotent cone. Let $e \in \mathcal{N}$, let $\mathcal{B}_e \subset T^*G/B$ be the Springer fiber over $e$.

One of our motivations is generalizing a known action of the affine braid group, $B_{aff,g}$, on the category $D^b(\text{Coh}_e(T^*\mathcal{B}))$ for $e \in \mathcal{N}$. The subscript $e$ stands for restricting the support to be on the formal neighborhood of $\pi^{-1}(e)$. This action is in some sense a categorification of the Springer action of $W$ on the (co)homology of a Springer fiber $\mathcal{B}_e$, (By looking at the induced action on the Grothendieck group of the category $D^b(\text{Coh}_e(T^*\mathcal{B}))$).

Moreover, there is also an analogy in the techniques by which both actions were constructed. Especially if we consider the construction of the action on the category via kernels. We recall some classical facts.

Properties of the Springer fiber

Let $e \in \mathcal{N}$, $\mathcal{B}_e := \{B \in \mathcal{B} \mid e \in \text{Lie}(n_B)\}$.

For example, Let $V/k$ be an $n$ dimensional vector space. Let $G = SL_n = SL(V)$. Identifying Borel subgroups with complete flags in $V$ we get a description:
\[ B_e = \left\{(0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V) \mid e.V_i \subset V_{i-1}\right\} \quad (2.41) \]

Properties of \( B_e \): \( \text{dim} B_e = \frac{1}{2}(\text{dim} B - \text{dim} O_e) \) where \( O_e \) is the adjoint orbit of \( e \) in \( \mathcal{N} \).

In addition \( B_e \) is always connected and often singular.

Extreme cases for Springer fibers are \( e = 0 \), for which \( B_e = B \), and regular nilpotent \( e \), for which \( B_e = \text{pt} \) (indeed the regular locus of \( \mathcal{N} \), is exactly where the birational morphism \( T^*(G/B) \to \mathcal{N} \) is an isomorphism). Explicitly for \( SL_n \), a regular nilpotent element, \( e \), is (up to conjugation) a single Jordan block in some basis \( v_1 \ldots v_n \in V \), and the single Borel subgroup in the fiber of the Springer map over the element \( e \), is the upper triangular matrices in this basis. The corresponding flag is that which is determined by this basis \( <v_1> \subset <v_1, v_2> \subset \ldots \).

Explicit examples:

1. Let \( G = SL_3 \), let \( e \) be of Jordan type \((2,1)\). Then \( B_e \cong \mathbb{P}^1 \cup_{pt} \mathbb{P}^1 \). (two projective lines, glued at a point). Lets write \( e \) in a basis \( v_1, v_2, v_3 \in V \) as a matrix which has all zeros except for upper right corner. Then explicitly in terms of flags, the first \( \mathbb{P}^1 \) is flags of the form \( 0 \subset <v_1> \subset V_2 \subset V \). Here the free part is in \( V_2 \), which is any 2 dimensional subvector space between the fixed \( <v_1> \) and \( V \). The second \( \mathbb{P}^1 \) is flags of the form \( 0 \subset V_1 \subset <v_1, v_2> \subset V \). Here the free component is the choice of \( V_1 \) which is a one dimensional vector space between 0 and \( <v_1, v_2> \). These sets of flags indeed have exactly a single point in common, and each is isomorphic to \( \mathbb{P}^1 \) hence \( \text{dim} B_e = 1 \)
2. Let $G = Sl_4$, let $e$ be of Jordan type $(2,2)$. Then there are two irreducible components of $B_e$: $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}(O \oplus O(-2))$ ($\mathbb{P}$ stands for projectivization). They are glued along a $\mathbb{P}^1$ that is embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ diagonally and as the zero section of the other component. $\dim B_e = 2$

The use of Grothendieck resolution $\mathfrak{g} \to \mathfrak{g}$ in the construction of the Springer action

The Springer action was constructed using the Grothendieck resolution $\mathfrak{g} \to \mathfrak{g}$. Similarly, in [24] the construction of the action action of $Br_{aff}$ on $D^b(\text{Coh}_e(T^*G/B))$, which uses correspondences and kernels, also used the Grothendieck resolution. There is an underlying reason for that.

Grothendieck resolution is closely related to $T^*G/B \to \mathcal{N}$. At the level of $k$ points $\mathfrak{g} := \{(x,B) \mid x \in \mathfrak{g}, B \in \mathcal{B}, x \in \text{Lie}(B)\}$. Hence at the level of $k$ points, the preimage of $\mathcal{N} \subset \mathfrak{g}$ under the Grothendieck resolution is $T^*G/B(k)$. (That's not true at the level of schemes. The preimage of $\mathcal{N} \subset \mathfrak{g}$ is a non reduced scheme, whose associated reduced scheme is $T^*(G/B)$). ([6])

$\mathfrak{g} \to \mathfrak{g}$ is useful because it is small whereas the Springer map $T^*G/B \to \mathcal{N}$ is only semi-small. That's the reason that some constructions work more easily when starting with $\mathfrak{g} \to \mathfrak{g}$.

The action of $W$ on $H^*(B_e)$

Let $W$ be the Weyl group. For every nilpotent $e \in \mathcal{N}$ there is an action of $W$ on the cohomology of the Springer fiber $H^*(B_e)$. In general, there is no action of $W$ on the space $B_e$ itself.

Considering the extreme cases.

- The case of regular $e$: is not of interest since, $B_e$ is a point.
The case $e = 0$: For $e = 0$ one gets an interesting action on $H^*(B)$. In this case, the action coincides with an older known action that is described in algebraic terms: The cohomology ring of $G/B$ is well understood. Let $\mathfrak{h}$ be the Cartan subalgebra. There is an isomorphism $H^*(B) \to \text{sym}(\mathfrak{h}^*)/\text{sym}(\mathfrak{h}^*_>)^W$. The natural action of $W$ on $\text{sym}(\mathfrak{h}^*)/\text{sym}(\mathfrak{h}^*_>)^W$, coming from the action of $W$ on the Cartan corresponds to the Springer action on the left hand side.

Now, Consider the general case: Reference [17]. Let $e \in \mathcal{N}$.

Let $\pi_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ be the Grothendieck resolution. One works in the setup of constructible and perverse sheaves. Let $\mathbb{Q}$ be the constant sheaf. One first constructs an action of $W$ on $R\pi_{\mathfrak{g}}^*\mathbb{Q}$: Since $\pi_\mathfrak{g}$ is small, $R\pi_{\mathfrak{g}}^*\mathbb{Q}[\text{dim}\mathfrak{g}]$ is a perverse sheaf. Moreover it is of the form $IC(\mathfrak{g}^{rs}, \mathcal{L})$, $\mathcal{L} := ((\pi_{\mathfrak{g}}^*\mathbb{Q})|_{g^{rs}}.$ (Pushforward of constant sheaf restricted to $\mathfrak{g}^{rs}$). Here $\mathfrak{g}^{rs}$ stands for regular semi simple elements, those over which $\mathfrak{g}$ is an etale cover of rank $|W|$. Since $\mathfrak{g}^{rs} \to \mathfrak{g}^{rs}$ is an etale cover of rank $|W|$, $W$ acts on $\mathcal{L}$. By functoriality of the IC extension $W$ acts on $R\pi_{\mathfrak{g}}^*\mathbb{Q}$. Then the restriction to the stalk $e \in \mathcal{N}$ gives the action on $H^*(B_e)$. 

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Chapter 3

Construction

Let $G$ be a reductive algebraic group over $k = \bar{k}$ with $\text{char}(k) > 0$. Let $P_0 \subset G$ be a fixed parabolic subgroup. Let $L$ be a Levi subgroup. Let $X^{(i)}$ be the varieties $T^*G/P^{(i)}$ for different parabolic subgroups $P^{(i)}$ which are associated to $P_0$. $X^{(i)}$ are symplectic resolutions that resolve the same variety. The categories $D^b(\text{Coh}_0(X^{(i)}))$ are all equivalent. We construct a local system of categories on a topological space $V_C^0$, with values the categories $D^b(\text{Coh}_0 X^{(i)})$ and $D^b(A_\lambda \cdot \text{mod}_0)$, where $\lambda \in \Lambda_L$ is $p$-regular. These categories are equivalent, however we choose to assign different realizations of the categories to different points, so that it allows us to describe the functors attached to paths between points in a natural way. In particular, for every $X^{(i)}$, there is a special point $pt_{X^{(i)}} \in V_C^0$ (a contractible subset of points), to which the local system assigns the category $D^b(\text{Coh}(X^{(i)}))$.

This suggests a picture, where natural D equivalences between the derived categories $D^b(\text{Coh } X^{(i)}), D^b(\text{Coh } X^{(j)})$, are parametrized by homotopy classes of paths between the associated points $pt_{X^{(i)}}, pt_{X^{(j)}}$ in the topological space $V_C^0$. For abbreviation, we denote the point $pt_{T^*G/P^{(i)}}$ by $pt_{P^{(i)}}$. 

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In the first chapter we associated a connected topological space $V_C^0$ to a complement $V_R^0$ of a real discrete hyperplane arrangement, $\Sigma$. Here let $V_R^0 := h^{reg}_{R,L} \subset \Lambda_L \otimes \mathbb{R}$ be the $p$-regular weights.

In particular for $P=B$ a Borel subgroup, the hyperplane arrangement in this case is the union of affine coroot hyperplanes.

$$\Sigma = \cup_{\alpha^\vee,n} H_{\alpha^\vee,n}$$

(3.1)

$$H_{\alpha^\vee,n} = \{ \lambda \in h^* \mid \langle \lambda, \alpha^\vee \rangle = np \}$$

(3.2)

Where $n \in \mathbb{Z}$ and $\alpha^\vee$ is a coroot.

The fundamental group of the connected space $V_C^0$ is $\pi_1(V_C^0) = \text{The pure affine braid group}$.

We call the connected components in $V_R^0$, the alcoves. Let $\lambda \in V_R^0$. We let $A_\lambda$ be the alcove that contains $\lambda$.

**Remark 4.** In the case $P=B$, the induced action on $D^b(\text{Coh}_0 T^*G/B)$ generalizes the well known weak action of the affine Braid group on this category.

Note that by a local system we mean a weak local system. Attaching categories to points in $V_C^0$, attaching functors up to isomorphism to homotopy classes of paths between points, such that concatenation of paths corresponds to composition of the functors. We do not discuss higher compatibilities.
Consider the general case of different symplectic resolutions $X^{(i)}$ which resolve a fixed variety $Y$. Let $X$ be one of the resolutions. In order to generalize the construction of the local system to the general case it is a natural suggestion to let $V_\mathbb{R}$ be $H^2(X, \mathbb{R})$.

Recall that for a symplectic resolution there is a canonical isomorphism $H^2(X, \mathbb{R}) \simeq \text{Pic}(X) \otimes \mathbb{R}$. And these vector spaces in fact depend only on $Y$.

There is a quantization of $X$ corresponding to each point in $H^2(X, \mathbb{R})$. Hence we can ask whether derived localization holds for the quantization associated to the point $\lambda \in V_\mathbb{R}$.

It is suggested that we should take $V_\mathbb{R}^0$ to be the area where derived localization theorem does not hold. It is a conjecture, which is known for the case $X = T^*G/P$, that this turns out to be a complement of a discrete hyperplane arrangement.

Given symplectic resolutions $X^{(i)}$ of a fixed variety $Y$, Namikawa constructed in [19] a generalized Weyl group $\mathbb{W}$. A coxeter group which acts on the Pic. It may be possible to generalize the construction we explain below to other symplectic singularities using this action.

Using Namaikawa’s action the space $V_\mathbb{R}^0$ can be described in terms of ample cones of different symplectic resolutions and their translations under $\mathbb{W}$.

Recall that for all symplectic resolutions $X^{(i)}$ of $Y$, the Picard groups are canonically identified. Denote this group Pic. One can take the ample cones $C(X^{(i)})_+$ of different symplectic resolutions of $Y$.

Consider the $\mathbb{W}$ action on Pic. Consider all the ample cones of different symplectic resolutions of $Y$, and their transitions via the action of $\mathbb{W}$. Let $V_\mathbb{R}^0$ be the complement of the boundaries of these cones in $\text{Pic} \otimes \mathbb{R}$. 

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It's a conjecture that there is an isomorphism between \( V_0^0 \simeq V_0^0 \).

Going back to the special case, we make the construction in stages. We construct a local system on \( V_0^0 \) with values \( D^b(A_\lambda \text{-mod}) \); \( \lambda \in V_0^0 \). We observe that the construction remains valid when restricting to the categories with an additional support condition \( D^b(A_\lambda \text{-mod}_0) \simeq D^b \text{Coh}_0(\mathbb{T}^n) \) and add \( \text{pt}_\mu(\lambda) \). We lift to characteristic zero and generalize to the category without support conditions.

Let \( \mathcal{G} \) be the groupoid of \( V_0^0 \). Such a groupoid has a nice presentation by generators and relations.

**Generators and Relations**

Given a discrete hyperplane arrangement in a real vector space \( \mathbb{R}^n \), let \( W_\mathbb{R}^0 \) be the complement. Let \( W_\mathbb{C}^0 \) be associated to \( W_\mathbb{R}^0 \) as in chapter 1. We describe a presentation for the groupoid of that space following Salvetti's work. The idea of his work was that by constructing a CW complex embedded in \( W_\mathbb{C}^0 \), whose embedding is a homotopy equivalence, it makes it possible to describe the generators and relations for the groupoid in a combinatorial way. Generators are the one cells, relations are given by the attaching maps of the two cells, and the zero cells in the CW complex correspond to the real alcoves in \( W_\mathbb{R}^0 \). More precisely,

**Generators:** The generators of the groupoid are as follows. For every pair of alcoves \( \mathcal{A}, \mathcal{A}' \) that share a codimension one face, \( \mathcal{H} \), we take as a generator the positive half loop going between alcoves \( \mathcal{A}, \mathcal{A}' \) along \( \mathcal{H} \). Let \( l_{\mathcal{A},\mathcal{A}'} \) be the generator for the path from \( \mathcal{A} \) to \( \mathcal{A}' \).
Relations: To express the relations we define a notion,

Given a path in $V_1^\circ$ consisting of generators, the length of the path is the number of generators involved.

First Set of Relations:
One way to describe the relations is: for each two alcoves $\mathcal{A}, \mathcal{A}'$: all minimal length orbits between $\mathcal{A}$ and $\mathcal{A}'$ are homotopic.

A Smaller set of relations:
It is sufficient to take a smaller set of relations.
Let $\mathcal{A}$ be an alcove, and let $F$ be a codimension two boundary of $\mathcal{A}$. Consider $\mathcal{A}_F^-$, the opposite alcove with respect to $F$. There are exactly two positive minimal paths between $\mathcal{A}$ and $\mathcal{A}_F^-$. The only relations that we need to impose on the generators is that for any $\mathcal{A}, F$ these two paths are equivalent.

We construct a functor from the groupoid $G(V_1^\circ)$ to Cat, attaching a category to each alcove, and a functor between the categories of the alcoves for each path generator of the groupoid. To prove it is indeed a functor from the groupoid, we check the relations that guarantee that given two homotopy equivalent paths, the composition of the corresponding functors gives isomorphic functors.

Attaching categories and functors
Recall that $P^{(i)}$ are associated parabolic subgroups of $G$. We construct a local system with values

$$D^b(A_\lambda \text{-mod}, A_\lambda = \Gamma(D_{\lambda, G/P^{(i)}}) \quad (3.3)$$
We claim,

**Claim 6.** For associated parabolic subgroups $P$ and $Q$, and $\lambda \in \Lambda_L$

\[
\Gamma(D_{\lambda,G/P}) \simeq \Gamma(D_{\lambda,G/Q})
\]

(3.4)

This justifies denoting the algebra above by $A_\lambda$ without referring to the choice of a parabolic in the notation.

Indeed, consider the natural morphism

\[
U(-) = F(DA,G_1s) \to F(DAG/Q)
\]

(3.5)

It is surjective for $\text{char}(k) >> 0$ ([22]). Let $\ker Q$ be its kernel. Similarly define $\ker P$. It is enough to check that $\ker P = \ker Q$ as subgroups of $U_\lambda$. It is enough to check this in a similar setup over the complex numbers.

Let $j : U_Q \hookrightarrow G/Q$ be an affine open subset of $G/Q$. Let $M'_Q$ be a $D_\lambda$ module on $U_Q$. Let $M_Q := j_!M'_Q$, Let $M_Q := \Gamma(G/Q,M_Q)$. It is a $U(g)_\lambda$ module. There is an equality $\ker Q = \text{Ann}_{U(g)_\lambda} M_Q$ since in the natural composition $\Gamma(D_{\lambda,G/Q}) \hookrightarrow \Gamma(U_Q,D_{\lambda,G/Q}) \hookrightarrow \text{End}(M_Q)$ both maps are injective. Hence it's enough to find $D_\lambda$ modules on the partial flag varieties that come from $D_\lambda$ modules on affine open subsets, and compare their global sections annihilator.

Let $[Q] := 1Q \in G/Q, j : U_Q := P.[Q] \hookrightarrow G/Q, M_Q := j_*O_{U_Q}, [P] := 1P \in G/P, M_P := \delta_{[P]}$.

Going back to the construction, to each real alcove $A \subset V^0_\mathbb{R}$ attach the category $D^b(A_\lambda\text{-mod})$ using some $\lambda \in A \cap \Lambda_L$. The construction will be independent of the choice of $\lambda$ in an alcove up to a canonical isomorphism.
Recall the notion of order of alcoves that share a codimension one wall, with respect to a parabolic subgroup.

Given adjacent alcoves $A, A'$ in $V_R^0$ that share a codimension one face, there is a parabolic subgroup $P$, with the property that $A <_P A'$. Equivalently,

$$l_{A,A'} \subset V_R + iC_F^+.$$

(3.6)

Let $\lambda_A, \lambda_A'$ be weights in these alcoves. Define the functor associated to $l_{A,A'}$ to be

$$F_{A,A'}^P : D^b(\Gamma(D_{\lambda_A, G/P}) \text{-mod}) \to D^b(\Gamma(D_{\lambda_A', G/P}) \text{-mod})$$

(3.7)

$$F_{A,A'}^P := \Gamma^P_{\lambda_A}(- \otimes O(\lambda_{A'} - \lambda_A))Loc^P_{\lambda_A}.$$

(3.8)

Claim 7. The functor is independent of a choice of a parabolic subgroup $P$, for which $A <_P A'$

For now we assume this, the proof of this independence will be given later. This is the reason that we are allowed to change the notation of this functor to $F_{A,A'}$, omitting $P$ from the notation.

Theorem 11. This is a local system.

To prove this defines a functor from $\mathcal{G}$ to Cat we need to check the relations of $\mathcal{G}$ hold.

Consider any hyperplane arrangement $V_R^0$. Let $F$ be a codimension 2 face. Let $\mathcal{A}$ be an alcove which contains $F$ in its boundary. Let $A_{\mathcal{A}}$ denote the
opposite alcove with respect to $F$. Let $C$ be a cone in $V_R^0$, defined by the hyperplanes in the boundary of the alcove $\mathcal{A}$, that intersect the face $F$. Then the two minimal paths in $V_R^0$ from $\mathcal{A}_F$ to $\mathcal{A}$ are going up with respect to the cone $C$.

Specializing to our setup, given an alcove $\mathcal{A} \subset V_R^0$, and a codimension 2 face $F$ in its boundary. There is a parabolic $P$ in the fixed association class, such that the two minimal orbits in our generators from $\mathcal{A}_F$ to $\mathcal{A}$ are orbits that always go up with respect to the partial order on alcoves given by $C^+_P$.

Finally, observe that $F_{\mathcal{A},\mathcal{A}'}$ satisfies the following relation: Any two paths between two alcoves, that go through increasing alcoves according to a fixed parabolic $P$, have isomorphic corresponding functors.

This follows from definition since $\Gamma^{P,\lambda_{\mathcal{A}'}}(\text{Loc}^{P,\lambda_{\mathcal{A}'}} \simeq \text{Id}$, and $(- \otimes O(\lambda' - \lambda)) \otimes O(\lambda'' - \lambda') \simeq (- \otimes O(\lambda'' - \lambda))$.

Next, we would like to check that $F_{\mathcal{A},\mathcal{A}'}$ is well defined. Recall, let $\mathcal{A}, \mathcal{A}'$ be adjacent alcoves in $V_R^0$ with shared codimension one face. Let $P$ and $Q$ be two associated parabolic subgroups of $G$, which both impose increasing relation on the two alcoves, $\mathcal{A} <_P \mathcal{A}'$ and $\mathcal{A} <_Q \mathcal{A}'$. The claim is that there is an isomorphism of functors

$$F_{\mathcal{A},\mathcal{A}'}^P \simeq F_{\mathcal{A},\mathcal{A}'}^Q$$

where

$$F_{\mathcal{A},\mathcal{A}'}^P := \Gamma^P_{\lambda_A}(- \otimes O(\lambda_{\mathcal{A}' - \lambda_A}))\text{Loc}_{\lambda_{\mathcal{A}'}}^P.$$ (3.9)

To prove this, we start with a special case where the association class of
The parabolic subgroups that we work with is that of Borel subgroups. That is $P, Q$ are in particular Borel subgroups.

In this case, the functor $F_{\lambda,\mu}^B$ attached to weights $\lambda, \mu \in \Lambda$ has interpretation in terms of a translation functor between representation categories [23]. Hence it follows that it is independent of the choice of the Borel subgroup $B$ as long as $B$ satisfies $\lambda <_B \mu$.

Explicitly,

**Lemma 1.** The functor $F_{\lambda,\mu}^B : \Gamma(D_{\lambda,G/B})\text{-mod} \to \Gamma(D_{\mu,G/B})\text{-mod}$ is isomorphic to the translation functor $T_{\lambda,\mu} : u(g)_\lambda\text{-mod} \to u(g)_\mu\text{-mod}$

i.e.

$$D^b(\Gamma(D_{\lambda,G/B})\text{-mod}) \xrightarrow{F_{\lambda,\mu}^B} D^b(\Gamma(D_{\mu,G/B})\text{-mod})$$

$$\cong$$

$$D^b(u(g)_\lambda\text{-mod}) \xrightarrow{T_{\lambda,\mu}} D^b(u(g)_\mu\text{-mod})$$

Using a commutative diagram we relate the case of a general Parabolic subgroup to that of a Borel subgroup.

Let P be a parabolic such that $A <_P A'$. Let $\lambda \in A, \mu \in A'$ be two regular integral weights in $\Lambda_L$. Let $B$ be contained in $P$ and satisfies $\lambda <_B \mu$. Let $\pi : G/B \to G/P$ be the projection.

The pullback functors

$$\pi^* : D^b(D_{\lambda,G/P}\text{-mod}) \to D^b(D_{\lambda,G/B}\text{-mod}) \quad (3.11)$$

$$\pi^* : D^b(D_{\mu,G/P}\text{-mod}) \to D^b(D_{\mu,G/B}\text{-mod}) \quad (3.12)$$
fit into the following commutative diagram:

\[
\begin{array}{c}
D^b(\Gamma(D_{\lambda,G/B} \text{-mod})) \xrightarrow{F_{\lambda,\mu}^B} D^b(\Gamma(D_{\mu,G/B} \text{-mod})) \\
\pi^* & \pi^* \\
D^b(\Gamma(D_{\lambda,G/P} \text{-mod})) \xrightarrow{F_{\lambda,\mu}^P} D^b(\Gamma(D_{\mu,G/P} \text{-mod}))
\end{array}
\]

Corollary 2. Combining the above two diagrams, we get the following commutative diagram.

\[
(3.13)
\]

\[
\begin{array}{c}
D^b(U(g)_{\lambda} \text{-mod}) \xrightarrow{T_{\lambda,\mu}} D^b(U(g)_{\mu} \text{-mod}) \\
\pi^* & \pi^* \\
D^b(A_{\lambda} \text{-mod}) \xrightarrow{F_{\lambda,\mu}^P} D^b(A_{\mu} \text{-mod})
\end{array}
\]

where the vertical maps \(\pi^*\) come from the surjection of algebras \(u(g)_{\lambda} \to A_{\lambda}\). It is the derived tensor product with \(U_{\lambda} \otimes_{A_{\lambda}}\) on the left vertical map, (similarly \(U_{\mu} \otimes_{A_{\mu}}\) on the right vertical map)

Observe that since \(U_{\lambda} \to A_{\lambda}\) is surjective, the functor \(A_{\lambda} \text{-mod} \to u(g)_{\lambda} \text{-mod}\) is fully faithful and conservative in the abelian level.

Even though the functor \(\pi^*\) is fully faithful at the abelian level, this doesn't a priory imply that at the derived categories level - the upper horizontal functor of (3.13) \(T_{\lambda,\mu}\), determines the lower horizontal functor \(F_{\lambda,\mu}^P\). Yet it is the case.

The two horizontal functors of the commutative diagram, \(T_{\lambda,\mu}\) and \(F_{\lambda,\mu}^P\), are given by derived tensoring with bimodules. The bimodules are the values of the functors on the algebras \(F_{\lambda,\mu}^B(U_{\lambda})\), \(F_{\lambda,\mu}^P(A_{\lambda})\) respectively. By the definition
of $F^B_{\lambda,\mu}, F^P_{\lambda,\mu}$, these are $R\Gamma(\lambda D_\mu)$, where $D_\lambda$ is the sheaf of differential operators either on $G/B$ or $G/P$, and

$$\lambda D_\mu := D_\lambda \otimes O(\mu - \lambda)$$

(3.14)

We denote the bimodules by $B_U, B_A$ respectively. We claim that these bimodules live in the heart of the categories of modules $D^b(U_\lambda \otimes U_\mu^{op} \text{-mod})$ $D^b(A_\lambda \otimes A_\mu^{op} \text{-mod})$ respectively. It is enough to prove for $P$.

It is enough to prove that there is a filtration on $\mathcal{F} := \lambda D_\mu$, whose associated graded $\text{gr}\mathcal{F}$ has vanishing higher cohomology $H^i(\text{gr}\mathcal{F}) = 0$ for $i > 0$. Since then it follows that $H^1(\mathcal{F}) = 0$ hence $R\Gamma(\mathcal{F})$ is in the heart. Indeed, $\lambda D_\mu$ has a filtration with associated graded $O_{T^*(G/P)}(\mu - \lambda)$, and for $\mu - \lambda > 0$ $H^1(O_{T^*G/P}(\mu - \lambda)) = 0$.

We denoted the bimodules by $B_U \in U_\lambda \otimes U_\mu^{op} \text{-mod}$ and $B_A \in A_\lambda \otimes A_\mu^{op} \text{-mod}$ respectively. Let $B'_A \in A_\lambda \otimes U_\mu^{op} \text{-mod}$ be the module obtained from $B_A$ by making the $A_\mu$ right action to $U_\mu$ right action, through the map $U_\mu \to A_\mu$. Let $B'_U := A_\lambda \otimes U_\mu \otimes B_U$. $B'_U \in U_\lambda \otimes U_\mu^{op} \text{-mod}$. By the commutativity of diagram (3.13), it follows that $B'_A \simeq B'_U$. Hence $B'_A$ is determined from $B_U$. Since $A_\mu \text{-mod} \to U_\mu \text{-mod}$ is fully faithful and conservative, (similarly for $A_\lambda \otimes A_\mu^{op} \text{-mod} \to A_\lambda \otimes U_\mu^{op} \text{-mod}$) it follows that $B_A$ is determined from $B'_A$. Hence $B_A$ is determined from $B_U$. Hence $F_{A,A'}$ is well defined. This concludes the first stage of the construction. This concludes the first stage of the construction. Observe that $F_{A,A'}$ respects restriction to the full subcategories $D^b(A_\lambda \text{-mod}_0) \subset D^b(A_\lambda \text{-mod})$ hence the local system can be restricted to have values these full subcategories.

We add generators in order to see in the picture the points $pt_P$ for a
parabolic $P$ in the association class. Let $pt_P$ be a point in $V_\mathbb{R} + iC_\mu^+$. Let $D^b(\text{Coh}_0(T^*G/P))$ be the attached category. Add a generator $l_{A_\lambda, \text{Coh}_P}$ to a natural path from the alcove $A_\lambda$ to the point $pt_P$. Let the corresponding functor be $F_{A_\lambda, \text{Coh}_P} : D^b(A_\lambda\text{-mod}_0) \to D^b(\text{Coh}_0(T^*G/P))$, $F_{A_\lambda, \text{Coh}_P} := F_{D, \text{Coh}}(\otimes O(0 - \lambda))\text{Loc}_\lambda^P$. $F_{D, \text{Coh}}$ was defined in section 2.

A generator $l_{A_\lambda, A_\mu}$ is contained in $V_\mathbb{R} + iC_\mu^+$ if and only if $\lambda <_P \mu$.

If this is satisfied for $P$ then the generator breaks to a composition of $l_{A_\lambda, pt_P}$ followed by $l_{pt_P, A_\mu}$.
Extension

Let $P_0$ be a fixed parabolic in the association class. Let $\mathcal{P} := G/P_0$ the associated partial flag variety. Consider the local system using the Salvetti generators $l_{A,A'}$.

Using the canonical equivalence $F_{A,A',\text{Coh}_P} : D^b(A_{-}\text{mod}_0) \to D^b(\text{Coh}_0(T^*\mathcal{P}))$ we can consider the above as a local system with constant value the category $D^b\text{Coh}_0(T^*\mathcal{P})$, where $G$ is over an algebraically closed field $k$ of $\text{char}(k) >> 0$.

Let $F_{\lambda\mu,\text{Coh}_P}$ denote the functors attached to $l_{A,A_{-}\mu}$ in this setup. If $\lambda <_{P_0} \mu$, then $F_{\lambda\mu,\text{Coh}_P} \simeq \text{Id}$

The above local system was constructed when $G$ lives over a field $k$ of characteristic $p$, and the value was the category of bounded derived category of coherent sheaves with a support condition. We would like to extend. We would like to construct a local system on $V^{_C}_C$ with value the category $D^b(\text{Coh}(T^*\mathcal{P}))$, (without the support condition), such that the restriction of the functors to the subcategory $D^b(\text{Coh}_0(T^*\mathcal{P}))$ recovers the above local system. We would also like to extend to case that $G$ lives over a characteristic zero field/ring.

We discuss the kernels by which these functors $F_{\lambda\mu,\text{Coh}_P}$ are defined, and some properties of these kernels with respect to a base change.

Let $R$ be a $\mathbb{Z}[1/h]$ algebra, where $h$ is the Coxeter number. Let $G_R$ be a split connected, simply connected, semi simple algebraic group over $R$. Consider its corresponding Lie algebra $\mathfrak{g}_R$. Consider a maximal torus and a Borel subgroup that contains it and their Lie algebras respectively. $T_R, B_{0,R}, t_R, b_{0,R}$. Let $B_R := (G/B_0)_R$. Consider the corresponding root system $\phi$, together with positive roots $\phi^+$ for $b_{0,R}$ and the simple roots $\Sigma$. Consider the Weyl group $W$.  

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Consider the set $I$ of Coxeter generators $s_\alpha$ of $W$, associated to $\Sigma$. Consider the affine Weyl group $W_{\text{aff}} := W \ltimes \Lambda$, which is a Coxeter group. To each Coxeter group there is an associated braid group. Consider the associated braid group $\text{Br}_{\text{aff}}$.

Notation: We denote a base change to a ring $R'$ by the subscript $R'$. 

$F_{\lambda,\mu,\text{Coh}_{B_0}}$ and the classical action of the affine braid group on $D^b(\text{Coh}(T^*B))$

There exists an action of $\text{Br}_{\text{aff}}$ on $D^b(\text{Coh}(T^*B)_R)$, constructed in [24][25]. By base changing to an algebraically closed field $k$, of characteristic $p > h$, and then restricting to a formal neighborhood of the zero section $B \subset T^*B$, this induces an action on $D^b(\text{Coh}_0(T^*(B)_k)$. Observe that $\pi_1(V_C^0) \simeq \text{Br}_{\text{aff, pure}} \neq \text{Br}_{\text{aff}}$. However in the case $B$, there is an additional symmetry on the parameter space $V_C^0$. $W_{\text{aff}}$ acts on the alcoves in $V_C^0\mathbb{R}$, simply transitively, and the affine braid group is the fundamental group for the quotient space $\pi_1(V_C^0/W_{\text{aff}}) \simeq \text{Br}_{\text{aff}}$.

**Definition 5.** Consider the projection $\text{Br}_{\text{aff}} \to W_{\text{aff}}$. There exists a natural section of the projection $\text{Br}_{\text{aff}} \to W_{\text{aff}}$, [BMR2,2.1]. Denote this section $w \mapsto T_w$. It is not a group homomorphism, however, there is the length function $l : W_{\text{aff}} \to \mathbb{N}$, and the section satisfies the rule $T_{w_1 w_2} = T_{w_1} T_{w_2}$ when $l(w_1) + l(w_2) = l_{w_1 w_2}$.

Let $F_w$ denote the action functor of $T_w$ on $D^b(\text{Coh}(T^*B))_{\mathbb{Z}[1/h]}$. More generally, let $F_{w,R}$ denote the action functor on a base change to $R$, $D^b(\text{Coh}(T^*G/B)_R)$. In particular Let $F_{w,k}$ denote the action of $T_w$ on $D^b(\text{Coh}_0(T^*G/B_0)_k)$. Let $\lambda, \mu$ belong to two adjacent alcoves in $V_C^0\mathbb{R}$, that share a codimension one wall. There exists a unique $w \in W_{\text{aff}}$ such that $\mu = w \lambda$. If $\lambda >_{B_0} \mu$, then $F_{w,k} \simeq F_{\lambda,\mu,\text{Coh}_{B_0}}$. 

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The functors $F_{w,R}$ have kernels $F_w \in D^b(Coh(T^*(B)_R \times_R T^*(B)_R))$, which are in fact in the heart of the natural $t$ structure, and are simple to describe. We recall their description, [23].

Recall,

**A presentation of $\text{Br}_{\text{aff}}$**

There exist several natural presentations of $\text{Br}_{\text{aff}}$. We consider the following one:

**Generators:** Let $s \in I$ be the simple reflections. To each simple reflection we associate a generator of $\text{Br}_{\text{aff}}$ called $T_s$. ($T_s$ will indeed be the natural lift of $s \in W_{\text{aff}}$ under the projection $\text{Br}_{\text{aff}} \to W_{\text{aff}}$ mentioned above). Another form of generators is $\theta_x \ x \in \Lambda$.

**Relations:** Let $s, t \in I$ be simple reflections. Let $n_{\alpha, \beta}$ denote the order, $\text{Ord}(s, t \in W)$ of $st \in W$. It is enough to impose the following relations:

Relations between the $T_s, s \in I$ is the braid relations: $T_s T_i \ldots = T_i T_s \ldots n_{s,t}$ compositions on each side.

Relations between the $\theta$, are saying that $\theta : \Lambda \to \text{Br}_{\text{aff}}$ is a group homomorphism, $\theta_x \theta_y = \theta_{x+y}$.

Relations between $T_s$ and $\theta_x$: They commute if $\alpha(x) = x$, and otherwise, $\theta_x = T_s \theta_{x-\alpha(x)} T_s$, $s := s_\alpha$ the simple reflection for a simple root $\alpha$, $s(x) = x - \alpha$.

**The kernels of the functors $F_{\lambda,\mu,\text{Coh}_{B_0}}$**

The kernel of $F_{\lambda,\mu,\text{Coh}_{B_0}}$ has a nice description. As proved in [24][25], there is a nice description of the kernel of the action $F_{w,R}$. This description is compatible with base change.

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The description is most natural in the Koszul dual situation [10] where $T^*G/B_R \to N_R$ is replaced by the Grothendieck resolution $\tilde{g}_R \to g_R$. (The fact that the Grothendieck resolution is small, whereas the Springer resolution is only semi small, makes the Grothendieck resolution better to work with).

The idea of the kernels construction is as follows:

Consider the case $w \in W \subset W_{aff}$. Let $F_{w, R}^{\tilde{g}}$ be the notation for the functor which is the action of $T_w$ on the category $D^b(\text{Coh}(\tilde{g}_R))$.

Let $\pi : \tilde{g} \to g$. Let $\tilde{g}_{reg} \subset \tilde{g}$ the regular elements. Let $\tilde{g}_{reg} := \pi^{-1}(g_{reg})$.

The kernels that describe $F_{w, R}^{\tilde{g}}$ are constructed by the following idea: Recall the construction of the action of the Weyl group $W$ on the cohomology of a Springer fiber $B_e e \in N$, preformed using the minimal Goresky MacPherson extension of a perverse sheaf [17]. The idea is to use the fact that $\tilde{g}_{reg}$ is a ramified Galois covering with Galois group $W$. Thus $W$ acts on $\tilde{g}_{reg}$ by deck transformations. Then, this action extends to an action on the cohomology of the Springer fiber.

The idea of the construction of the Braid group action on $D^b(\text{Coh}(\tilde{g}))$ is similar. Starting the action of $W$ on $\tilde{g}_{reg}$ and extending it to an action on the category of derived coherent sheaves on $\tilde{g}$.

**Remark 5.** When restricting $\tilde{g} \to g$ to the regular semi simple locus this is a cover, hence $W$ acts on $\tilde{g}_{reg-ss}$ by Deck transformations. Then, this action can be extended to the regular locus.

The formula of the functors of $F_{w, R}^{\tilde{g}}$ in terms of kernels is as follows:

Let $(B_0)_R$ be the fixed Borel, let $B_R$ be the flag variety. Let $w \in W \subset W_{aff}$. Let $T_w \in \text{Br}_{aff}$. The action of $T_w$ on $D^b(\text{Coh}(\tilde{g}_R))$ is defined to be a functor $F_{w, R}^{\tilde{g}}$, whose kernel is as follows [24].
Let $\Gamma_{w,R}$ be the graph of the action of $w \in W$ on $\tilde{g}_{\text{reg},R}$. Let $Z_{w,R}$ be the closure of the graph. Another description is by considering the natural morphisms $\tilde{g}_R \times_R \tilde{g}_R \to B_R \times R B_R$. $Z_{w,R}$ is the inverse image of the $G_R$ orbit, $(B_{0,R}, w^{-1} B_{0,R})$.

**Definition 6.** Let the kernel of $F_{w,R}^\hat{g}$ be $O_{Z_{w,R}} \in D^b((T^*B \times T^*B)_R)$.

In the dual version where the category is $D^b\text{Coh}(T^*B)_R$, the kernel is given by the structure sheaf of a closed subscheme of $T^*G/B_{0,R} \times_R T^*G/B_{0,R}$. The subscheme is defined to be the scheme theoretic intersection $Z'_{w,R} := Z_{w,R} \cap (T^*G/B_{0,R} \times_R T^*G/B_{0,R})$. The functors in this case are denoted $F_{w,R}$.

**Claim 8.** [24] These functors $F_{w,R}^\hat{g}$, $F_{w,R}$ define an action of the Braid group on the categories $D^b(\text{Coh}(\hat{g}_R)), D^b(\text{Coh}(T^*B)_R)$ respectively. In other words, the functors satisfy the braid relations.

Above we recalled the action of $T_w$ for $w \in W$. To extend it to an action of the affine braid group, the extra action of $\Lambda \subset \text{Br}_{\text{aff}}$, is given by twist by the line bundles corresponding to the lattice elements. Let $\theta_x x \in \Lambda$ act by twist by the line bundle corresponding to $x$. The kernel is the direct image of $O((T^*G/B)_R)(\lambda)$ under the diagonal embedding, $\Delta \hookrightarrow (T^*G/B)_R \times (T^*G/B)_R$. It is clear that the relation $\theta_x \theta_y \simeq \theta_{x+y}$ for $x, y \in \Lambda$ holds. The relation between $T_s, \theta_x$ holds as well.

Observe the nice behavior of the functors with respect to base change. Let $R \to R'$ be a ring morphism. By definition, the kernel of the functor $F_w^\hat{g}_{R'}$ is a base change of the kernel of $F_w^\hat{g}_R$.

A similar observation is true for the kernels of the actions of $T_w$ on $D^b(\text{Coh}(T^*G/B_R))$. 
The actions of $\text{Br}_{\text{aff}}$ on $D^b(\text{Coh}(\mathfrak{g})_R)$, $D^b(\text{Coh}(T^*G/B)_R)$ are weak geometric actions. See next subsection for the definition of that notion.

Aside: Convolution of kernels and weak geometric action

Let $X$ be a smooth scheme over a commutative ring $R$. Consider the two natural projections $\pi_1, \pi_2 : X \times_R X \to X$. Consider a sheaf $\mathcal{F} \in D^b \text{Coh}(X \times_R X)$. We say that $\mathcal{F}$ is good, if it is in the image of the derived pushforward from derived coherent sheaves on a closed subscheme of $X \times_R X$, for which the two projections to $X$ are proper. If $\mathcal{F}$ is a good sheaf, then it is a kernel of a well defined functor $F_\mathcal{F} : D^b(\text{Coh}(X)) \to D^b(\text{Coh}(X))$

$$F_\mathcal{F}(\mathcal{G}) := \pi_{2*}(\mathcal{F} \otimes \pi_1^* \mathcal{G})$$  \hspace{1cm} (3.15)

(Where all operations on the Right hand side are derived)

The formula for the composition of two such functors in terms of the kernels is well known:

Let $\mathcal{F}_1, \mathcal{F}_2$ be two kernels that define functors $D^b \text{Coh}(X) \to D^b \text{Coh}(X)$. The composition of $\mathcal{F}_1, \mathcal{F}_2$ is another functor given by a kernel. The kernel is the convolution of the two kernels. Let $\pi_{12}, \pi_{13}, \pi_{23} : X \times_R X \times_R X \to X \times_R X$ be the three possible projections. Then

$$\mathcal{F}_1 \ast \mathcal{F}_2 := \pi_{1*}(\pi_{12}^* \mathcal{F}_1 \otimes \pi_{23}^* \mathcal{F}_2)$$  \hspace{1cm} (3.16)

(Again all operations are derived)

In other words - $D^b(X \times X)$ has a monoidal structure given by the convolution product, and this monoidal category acts on the category $D^b(\text{Coh}(X))$,
letting $\mathcal{F}$ act by $F_\mathcal{F}$.

Moreover, observe that this assertion is compatible with a base change to another base ring $R'$. That is:

Let $R \to R'$ be a ring morphism, let $X_{R'}$ be the base change. Consider the map $B : X_{R'} \times_{R'} X_{R'} \to X \times_R X$. The derived pullback of the kernel $B^* : D^b \text{Coh}(X \times_R X) \to D^b \text{Coh}(X_{R'} \times_{R'} X_{R'})$ satisfies $B^* \mathcal{F}_1 \ast B^* \mathcal{F}_2 \simeq B^*(\mathcal{F}_1 \ast \mathcal{F}_2)$. In other words, the pullback functor $B^* : D^b \text{Coh}(X_R \times_R X_R) \to D^b \text{Coh}(X_{R'} \times_{R'} X_{R'})$ is a monoidal functor. The action of $D^b(X_{R',R} \times_{R',R} X_{R',R})$ on $D^b(X_{R',R})$ is compatible with this pullback map.

Let $Y$ be a variety over $R$, Let $X := T^*Y$, then the formalism of convolution is also compatible with restriction to a formal neighborhood of the zero section $Y \subset T^*Y$.

These observations lead to the definition of a weak geometric action of a group on a category. As follows. [23]

**Definition 7.** A weak homomorphism from an abstract group $H$ to a monoidal category, is a morphism from $H$ to the group of isomorphism classes of invertible objects.

**Definition 8.** A weak geometric action of an abstract group $H$ on the category $D^b \text{Coh}(X_R)$ is a weak morphism from $H$ to the monoidal category $(D^b \text{Coh}(X_R \times_R X_R), \ast)$.

This notion behaves well under base change. Let $R \to R'$ be a commutative ring morphism. A weak geometric action on $D^b \text{Coh}(X_R)$ induces a weak
geometric action on $D^b(X_{R'})$ by using a base change of the kernels. (That's what we saw in the case of the Br$_{aff}$ action on $D^b(Coh(T^*B))$).

We will use the following obvious claim,

Claim 9 (Relations form). Let $X$ be a smooth scheme over $R$. Let $H$ be a group. Let $H$ be generated by a set $S$ of elements $h \in S \subset H$. The relations can be described in the form

$$h_{i_1}^\pm \ldots h_{i_k}^\pm = 1, h_{i_j}^\pm \in S \quad (3.17)$$

To define a weak action of $H$ on the category $D^b(Coh(X))$, let $\mathcal{F}_h : D^b(Coh(X)) \rightarrow D^b(Coh(X))$ be an equivalence functor assigned to $h \in S$, which is given by a kernel $\mathcal{F}_h \in D^b(Coh(X \times X))$. Let $(\mathcal{F}_h)^{-1} := \mathcal{F}_h^{-1}$ defined to be the kernel of the inverse equivalence. Then these functors define a weak action of $H$ on $D^b(Coh(X))$ if there are isomorphisms of sheaves (in the derived category) corresponding to the relations (3.17):

$$\mathcal{F}_{h_{i_1}^\pm} \ast \ldots \ast \mathcal{F}_{h_{i_k}^\pm} \simeq O_\Delta \quad (3.18)$$

Where $O_\Delta$ is the structure sheaf of the diagonal.

Next, let $B_0 \subset P_0$. Remember that $k$ is the field. We need to understand the pullback functor of $D$ modules $D^b(D_{\lambda}(G/P_0)_{k}$-mod) $\rightarrow D^b(D_{\lambda}(G/B_0)_{k}$-mod), at the level of the corresponding bounded derived category of coherent sheaves, as in [23].
A special pullback functor $\pi^*: D^b(\text{Coh}_0(T^*G/P_0)_k) \to D^b(\text{Coh}_0(T^*G/B_0)_k)$

We are working over a field $k$ of characteristic $p >> 0$. Let $\pi : (G/B_0)_k \to (G/P_0)_k$ be the projection. The ordinary pullback functor at the level of derived category of $\mathcal{D}$ modules, $D^b(\mathcal{D}_\lambda(T^*G/P_0)_k \text{-mod}_0) \to D^b(\mathcal{D}_\lambda(T^*G/B_0)_k \text{-mod}_0)$ does not correspond to the ordinary pullback functor at the level of derived category of coherent sheaves. Rather the corresponding pullback of coherent sheaves is as follows:

We describe a functor $\text{Coh}(T^*G/P_0)_R \to \text{Coh}(T^*G/B_0)_R$ ($R$ as before). In the case $R=k$, and after restricting to $\text{Coh}_0(T^*G/P_0)_k$ this is compatible with the pullback functor of $\mathcal{D}$ modules.

**Remark 6.** To simplify notations, I remove mentioning the base ring $R$ from the notation.

Let $B_0 \subset G$ be a Borel subgroup. Let $P_0 \supset B_0$ be a parabolic which contains it. Consider the varieties $\mathcal{B} := G/B_0, \mathcal{P} := G/P_0$. Consider the projection $\pi : \mathcal{B} \to \mathcal{P}$, a smooth surjective map. Consider the morphism of vector bundles $d\pi : T^*\mathcal{P} \times_{\mathcal{P}} \mathcal{B} \to T^*\mathcal{B}$. $d\pi$ is a closed embedding.

For example, let $G=GL_n$. Let $P_0$ be a minimal parabolic, in standard form, its Levi is $(n-2)$ boxes of size one and one box of size two by two. Consider the projection $\mathcal{B} \to \mathcal{P}$. We can describe the varieties and this projection in terms of moduli space of flags.

Identifying $\mathcal{B}$ with the moduli space of full flags of length $n$ and $\mathcal{P}$ with the moduli space of partial flags, $0 \subset V_1 \subset V_2 \subset \ldots V_{i-1} \subset V_{i+1} \subset \ldots V_n$ with
dimV_i = i, the morphism B \to \mathcal{P} is forgetting V_i. The fiber is \mathbb{P}^1 (It corresponds to choices of a line in V_{i+1}/V_{i-1}). The closed embedding \dd \pi is a divisor.

Another example is to let \mathcal{P}_0 = G. \text{Then } \mathcal{P} = pt \Rightarrow T^*Y = pt, \dd \pi : B \to T^*B is the zero section.

Let \( f : T^*(G/P_0) \times_{G/P_0} G/B_0 \to T^*(G/P_0) \) be the projection map.

**Definition 9.** Using the correspondence \((d \pi, f)\) define the special pullback functor \([23]\)

\[
\pi^* := d \pi_* f^* : D^b\text{Coh}(T^*(G/P_0)) \to D^b\text{Coh}(T^*(G/B_0))
\]

(3.19)

Since \( f \) is flat and \( d \pi \) is a closed embedding, it follows that,

**Claim 10.** This functor is exact.

Consider the case where \( R = k \) (algebraically closed field of char \( p > h \)).

Restricting the domain to \( D^b\text{Coh}_0(T^*(G/P_0)) \), consider the functor \( d \pi_* f^* : D^b\text{Coh}_0(T^*(G/P_0)) \to D^b\text{Coh}_0(T^*(G/B_0)) \). Under the equivalence of Coherent with D modules, the functor \( d \pi_* f^* : D^b\text{Coh}_0(T^*(G/P_0)) \to D^b\text{Coh}_0(T^*(G/B_0)) \) corresponds to the pullback of D modules \( D^b(D\lambda(G/P_0) - \text{mod}) \to D^b(D\lambda(G/B_0) - \text{mod}) \).

Under the equivalence with the categories of representations, it corresponds to the functor \( D^b(A^0_\lambda - \text{mod}) \to D^b(U^0_\lambda - \text{mod}) \) that is induced by the surjective morphism of algebras \( U^0_\lambda \to A^0_\lambda \).

**Remark 7.** As mentioned above the affine braid group action on \( D^b(\text{Coh}(T^*G/B_0)) \) has another version when considering the category \( D^b(\text{Coh}(\bar{\mathcal{G}})) \), where \( \bar{\mathcal{G}} \) is the
Grothendieck resolution. We can construct a local system for the Parabolic version, $D^b(Coh(\tilde{g}_P))$. (At the level of $k$ points $g^P := \{(x, p) \mid p \in G/P, x \in p\}$.

In this version the situation is simpler. The functor between coherent sheaves that corresponds to the pullback functor of $D$ modules is the ordinary pullback functor of coherent sheaves.

**Remark 8.** We don't discuss the case $X = \tilde{g}_P$ in length since it is not a symplectic resolution, just a Poisson variety. (Studying the family of quantization of a Poisson variety, may allow to generalize the picture that we suggest of a local system on a subset of the universal parameter space of quantizations to such case.)

Using this functor we extend the picture to coherent sheaves with no support condition as follows: Let $R = k$ in this part.

We will define a local system on $V_C^0$ with the value $D^b(Coh(T^*G/P_0))$, such that restriction of the functors to the category $D^b Coh_0(T^*G/P_0)$ recovers the previous local system.

Remember, the generators of the groupoid of $V_C^0$ are $l_{\lambda, \mu}$, where $\lambda, \mu$ are two weights in adjacent real alcoves that share a codimension one wall.

In the local system for $D^b Coh_0(T^*G/P_0)$ we attached the functor $F_{\lambda, \mu, CohP_0}$ to $l_{\lambda, \mu}$.

Let $\tilde{F}_{\lambda, \mu, CohP_0}$ denote the functor we will attach to the generator $l_{\lambda, \mu}$ in the local system for $D^b Coh(T^*G/P_0)$

When $P_0$ is a Borel $B_0$, we know a definition of $\tilde{F}_{\lambda, \mu, CohB_0}$ by its kernel, whose restriction to the subcategory with the support condition is $F_{\lambda, \mu, CohB_0}$. The kernels live in the heart. In the abelian category $Coh(T^*G/B_0 \times T^*G/B_0)$.
Let $\pi^*$ be the special pullback functor defined above.

The following diagram commutes:

$$
\begin{array}{c}
D^b(\text{Coh}_0(T^*G/B_0)) \\
\pi^* \\
\downarrow \\
D^b(\text{Coh}_0(T^*G/P_0)) \\
\pi^* \\
\end{array}
\xrightarrow{F_{\lambda, \mu, \text{Coh}_{P_0}}} 
\begin{array}{c}
D^b(\text{Coh}_0(T^*G/B_0)) \\
\pi^* \\
\downarrow \\
D^b(\text{Coh}_0(T^*G/P_0)) \\
\pi^* \\
\end{array}
$$

Moreover, as before, the functors $F_{\lambda, \mu, \text{Coh}_{P_0}}$ are uniquely characterized as functors that make the diagram commutes.

There exists and unique functors $\tilde{F}_{\lambda, \mu, \text{Coh}_{P_0}}$ that complete the following diagram to a commutative diagram.

$$
\begin{array}{c}
D^b(\text{Coh}(T^*G/B_0)) \\
\pi^* \\
\downarrow \\
D^b(\text{Coh}(T^*G/P_0)) \\
\pi^* \\
\end{array}
\xrightarrow{F_{\lambda, \mu, \text{Coh}_{P_0}}} 
\begin{array}{c}
D^b(\text{Coh}(T^*G/B_0)) \\
\pi^* \\
\downarrow \\
D^b(\text{Coh}(T^*G/P_0)) \\
\pi^* \\
\end{array}
$$

**Claim 11.** We claim that these functors $\tilde{F}_{\lambda, \mu, \text{Coh}_{P_0}}$ define a local system on $V^0_C$ with values the category $D^b \text{Coh}(T^*G/P_0)$.

We need to check that the relations of the groupoid hold. These relations are described in terms of equations on the kernels of the form, equation (3.18).

Let $X$ be a smooth projective variety over $k$ that admits a projective map to an affine variety. A relation of the form of equation (3.18) for sheaves in $D^b(\text{Coh}(T^*X \times T^*X))$, holds if and only if it holds after passing to a formal neighborhood of the zero section $X \subset T^*X$. ([23])
It follows that it is enough to prove equation (3.18) for the restriction of the kernels to the formal neighborhood of the zero section. The restricted kernels are the kernels of the functors $F_{\lambda,\mu,Coh_{P_0}}$, which indeed form a local system and hence their kernels satisfy equation (3.18).

Now, let $G$ be over the ring $R := \mathbb{Z}[1/h]$. Let $B_0 \subset P_0$ be the fixed borel subgroup and fixed parabolic subgroup. We will define a local system on $V^0_C$ with value the category $D^b \text{Coh}(T^*G/P_0)$, such that base changing the functors to the category $D^b(\text{Coh}(T^*G/P_0))_k$, recovers the previous local system.

Remember the generators of the groupoid are $I_{\Lambda, \Lambda}$, as before. Let $F_{\lambda,\mu,Coh_{P_0}}$ denote the functors we will attach to $I_{\Lambda, \Lambda}$.

Consider the case $P_0 = B_0$ a Borel. In this case we already explained the existence of the functors $\overline{F}_{\lambda,\mu,\text{Coh}_{B_0}}$, defined by their kernel, that form a local system, and whose base extension to $k$, recovers the the local system on $V^0_C$ with value $D^b \text{Coh}(T^*G/B_0)_k$.

Moreover, there exists and unique functors $\overline{F}_{\lambda,\mu,\text{Coh}_{P_0}} : D^b(\text{Coh}(T^*G/P_0)) \rightarrow D^b(\text{Coh}(T^*G/P_0))$ that make the following diagram commute.

\[
\begin{array}{ccc}
D^b(\text{Coh}(T^*G/B_0)) & \xrightarrow{\overline{F}_{\lambda,\mu,\text{Coh}_{B_0}}} & D^b(\text{Coh}(T^*G/B_0)) \\
\pi^* \downarrow & & \pi^* \\
D^b(\text{Coh}(T^*G/P_0)) & \xrightarrow{\overline{F}_{\lambda,\mu,\text{Coh}_{P_0}}} & D^b(\text{Coh}(T^*G/P_0))
\end{array}
\]

**Claim 12.** We claim that attaching these functors to the generators $I_{\Lambda, \Lambda}$ defines a local system on $V^0_C$.

Observe that the base change of $\overline{F}_{\lambda,\mu,\text{Coh}_{P_0}}$ to $k$, is $\overline{F}_{\lambda,\mu,\text{Coh}_{P_0}}$. 

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To prove the functors \( \overline{F}_{\lambda,\mu,Coh_{P_0}} \) satisfy the relations of a local system, we need to prove that the equations of the form, equation (3.18), on the convolution of the kernels of \( \overline{F}_{\lambda,\mu,Coh_{P_0}} \) hold. The claim follows from the following fact,

Let \( R \) be a finite localization of \( \mathbb{Z} \). Let \( X := T^*G/B \) where \( X \) is considered as a variety over \( R \). Let \( \Delta_X \subset X \times_R X \) denote the diagonal and let \( \Delta_X \otimes_{q} \) denote the base extension of this diagonal to \( \overline{F}_q \). \( q \) a prime number.

Let \( \mathcal{F} \in D^b \text{Coh}^G(X \times_R X) \). If for any prime \( q \), which is not invertible in \( R \), the base extension of \( \mathcal{F} \) to \( \overline{F}_q \) (in the derived sense), satisfies the relation \( \mathcal{F} \otimes_R \overline{F}_q \simeq O_{\Delta_X \otimes_{F_q}} \). Then \( \mathcal{F} \simeq O_{\Delta_X \otimes_{F_q}} \).

To see this assertion, use the following property. Let \( A \) be a finitely generated flat \( R \) algebra. Let \( A_q := A \otimes_R \overline{F}_q \). The assertion above follows from a claim on the behavior of \( A \) modules, and their derived extension to \( \overline{F}_q \), where \( q \) is a prime which is not invertible in \( R \).

Given a finitely generated flat \( R \) algebra \( A \), and \( M \) an object in the bounded derived category of \( A \) modules. If the derived extension of \( M \) to \( \overline{F}_q \) is concentrated in degree 0 for every prime \( q \) not invertible in \( R \), then \( M \) is flat over \( R \) and concentrated in degree 0. If the derived extension is zero for every \( q \) as above, then \( M \) is zero.

Using these observations, we reduce to proving that the relations (3.18) hold after a base extension to \( \overline{F}_q \) for various primes \( q \), not invertible in \( R \). In this case the relations become relations for the kernels of \( \overline{F}_{\lambda,\mu,Coh_{P_0},F_q} \) which we already know to hold.
Remark 9 (An application of the local system). Let $X$ be a symplectic resolution over a field $k$. Recall that $V_\mathbb{R}$ is a parameter space for quantization of $X$. These quantizations give rise to natural $t$ structures on the category of coherent sheaves $D^b(\text{Coh}(X))$. In particular, in the area where localization theorem holds $V_\mathbb{R}^0$, there is a $t$ structure attached to each alcove. It is possible to describe the changes in the $t$ structures when crossing a wall, using the local system above, and a variant of Lusztig $a$-function for the Weyl group.
Bibliography


