Fields of Rationality of Cuspidal Automorphic Representations

by

John Binder

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Abstract

This thesis examines questions related to the growth of fields of rationality of cuspidal automorphic representations in families. Specifically, if $\mathcal{F}$ is a family of cuspidal automorphic representations with fixed central character, prescribed behavior at the Archimedean places, and such that the finite component π^∞ has a Γ-fixed vector, we expect the proportion of $\pi \in \mathcal{F}$ with bounded field of rationality to be close to zero if Γ is small enough. This question was first asked, and proved partially, by Serre for families of classical cusp forms of increasing level. In this thesis, we will answer Serre's question affirmatively by converting the question to a question about fields of rationality in families of cuspidal automorphic $\mathrm{GL}_2(\mathbb{A})$ representations. We will consider the analogous question for certain sequences of open compact subgroups Γ in $U_{E/F}(n)$. A key intermediate result is an equidistribution theorem for the local components of families of cuspidal automorphic representations.

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This doctoral thesis has been examined by a Committee of the Department of Mathematics as follows:

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Chapter 1

Introduction

1.1 History and motivation

Let $S_k(\Gamma_0(N))$ denote the space of classical cusp forms of weight k and level N. For each prime $p \nmid N$, the *Hecke operator* T_p is semisimple on $S_k(\Gamma_0(N))$. Since the T_p commute with one another, they are simultaneously diagonalizable. By Atkin-Lehner theory ([AL70]), there is in fact a canonical eigenbasis for the space, which we denote by $B_k(\Gamma_0(N))$.

If $f \in B_k(\Gamma_0(N))$ then f has a Fourier expansion

$$f(q) = q + a_2 q^2 + a_3 q^3 + \dots.$$

We define the *field of rationality* $\mathbb{Q}(f)$ of f to be the field $\mathbb{Q}(a_2, a_3, \dots)$ generated by all its Fourier coefficients. If $\sigma \in \mathrm{Aut}(\mathbb{C})$, then the form

$${}^{\sigma}f(q) = q + \sigma(a_2)q^2 + \sigma(a_3)q^3 + \dots$$

is also a member of the canonical basis of Hecke eigenforms. Since this space is finite-dimensional, we discern that $\mathbb{Q}(f)$ is a finite extension of $\mathbb{Q}$. Let $B_k^{\leq A}(\Gamma_0(N))$ be the set of those $f \in B_k(\Gamma_0(N))$ such that $[\mathbb{Q}(f) : \mathbb{Q}] \leq A$.

In [Ser97], Serre examined the growth of $\mathbb{Q}(f)$ in level-families of Hecke eigenforms.

Specifically, he proved the following Theorem:

Theorem 1.1.0.1. *Fix $A \geq 1$ and a rational prime p_0. Let $N_\lambda \to \infty$ be a sequence of levels such that $(p_0,\, N_\lambda) = 1$ for all N_λ.*

Then

$$\lim_{\lambda\to\infty} \frac{|B_k^{\leq A}(\Gamma_0(N_\lambda))|}{|B_k(\Gamma_0(N_\lambda))|} = 0.$$

On page 87 of that paper, Serre posited it was possible to allow $\{N_\lambda\}$ to be an arbitrary sequence of levels, removing the necessity of the auxiliary prime. Our primary motivation was to answer this question.

Serre used a trace formula argument to examine the asymptotic distribution of the Hecke eigenvalues a_{p_0} in families of high level (this is the part of the argument that required the existence of the auxiliary prime). The limiting distribution is the *Plancherel measure* on $[-2p_0^{(k-1)/2},\, 2p_0^{(k-1)/2}]$; this measure is absolutely continuous with respect to the Lebesgue measure on the interval, so points have measure 0. Moreover, if a_{p_0} is the Hecke eigenvalue of a cusp form of weight k, then it is of the form $\alpha + \overline{\alpha}$, where α is a Weil-p-integer of weight $k-1$. If $[\mathbb{Q}(a_{p_0}) : \mathbb{Q}] \leq A$ then $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 2A$. However, there are finitely many Weil-p_0-integers of given weight and given degree, so they occur with asymptotic density zero. This completes the proof.

The Plancherel measure arises naturally from the representation theory of GL_2. There is a classical correspondence $f \mapsto \pi_f$ between Hecke eigenforms and cuspidal automorphic $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ representations. If f has level coprime to p_0, then π_{f,p_0} is an unramified (tempered) principal series representation and therefore is defined by its (unitary) Satake parameters $\{\alpha_0,\, \beta_0\}$. Since f is a Γ_0-level form, then π_f has trivial central character, and so $\beta_0 = \overline{\alpha}_0$. If a_{p_0} is the p_0-Fourier coefficient of f, then $a_{p_0} = p_0^{(k-1)/2}(\alpha_0 + \beta_0)$. There is a natural Plancherel measure on the subspace of the unitary spectrum of $\mathrm{GL}_2(\mathbb{Q}_p)$ with trivial central character, and the Plancherel measure on $[-2p_0^{(k-1)/2}, 2p_0^{(k-1)/2}]$ arises via the restriction of this Plancherel measure to the unramified spectrum.

Waldspurger [Wal85] has defined an action of $\mathrm{Aut}(\mathbb{C})$ on the space of admissible

$\mathrm{GL}_2(\mathbb{Q}_p)$ representations. Given π_p, let $\mathrm{Stab}(\pi_p) = \{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^\sigma\pi_p \cong \pi_p\}$; the field of rationality $\mathbb{Q}(\pi_p)$ of π_p is the fixed field of $\mathrm{Stab}(\pi_p)$. Under the identification $f \mapsto \pi_f$, $\mathbb{Q}(f)$ is equal to the compositum of the fields $\mathbb{Q}(\pi_{f,p})$ for all finite primes p.

To relax the dependence on the auxiliary prime p_0, we need to work with cuspidal automorphic representations that are ramified at p_0. This requires two steps: first, we need to prove a Plancherel equidistribution result for cuspidal automorphic representations. The precise definition of Plancherel equidistribution is given in 3.3.0.9. The method of proof is by now standard (following, for instance, Shin and Templier ([Shi12] and [ST12]) and Finis, Lapid, and Mueller ([FL13], [FLM14], [FL15])) and depends on a density theorem of Sauvageot ([Sau97]), and an appropriate version of the trace formula. This will tell us that, if we take an appropriate family $\{\pi\}$ of automorphic $\mathrm{GL}_2(\mathbb{A})$ representations, then their p-components $\{\pi_p\}$ are equidistributed according the Plancherel measure on an appropriate subspace of the spectrum of $\mathrm{GL}_2(\mathbb{Q}_p)$. We must then show that the subset of representations π_p with small field of rationality has small Plancherel measure. Via a finiteness result of Shin and Templier (Corollary 5.7 of [ST14]), the set of representations we need to avoid will be finite, and so in practice we need concern ourselves only with discrete series representations, which can be examined explicitly.

1.2 Families of automorphic representations

Now let G be a connected reductive group over a number field F, and let Z denote the center of G. Fix the following data:

- An irreducible, algebraic, finite-dimensional representation ξ of $G(F_\infty)$;
- an automorphic character $\chi : Z(F)\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ such that χ_∞ is equal to the central character of ξ; and
- a compact open subgroup Γ of $G(\mathbb{A}^\infty)$, with χ trivial on $\Gamma \cap Z(\mathbb{A}^\infty)$.

We define the *automorphic family* $\mathcal{F}_{\mathrm{disc}}(\xi, \chi, \Gamma)$ as the multiset of discrete automorphic representations π such that $\chi_\pi = \chi$, and where $\pi = \pi^\infty \otimes \pi_\infty$ occurs with multiplicty

$$a_{\mathcal{F}}(\pi) = m_{\text{disc}}(\pi) \cdot \dim(\pi^{\infty})^{\Gamma} \cdot (-1)^{q(G)} \chi_{\text{EP}}(\pi_\infty \otimes \xi^\vee).$$

Define $\mathcal{F}_{\text{cusp}}$ similarly, but with m_{cusp} replacing m_{disc}.

We will give a precise definition of the χ_{EP} term later. For now, we make the following assumptions. First, let K_∞ be a maximal compact subgroup of $G(F_\infty)$; we assume K_∞ and $(G/Z)(F_\infty)$ have the same rank. Moreover, we assume ξ has *regular highest weight.* Under these hypotheses, there is a unique discrete-series L-packet Π_ξ of $G(F_\infty)$ representations such that $\chi_{\text{EP}}(\pi_\infty \otimes \xi^\vee) = (-1)^{q(G)}$ if $\pi_\infty \in \Pi_\xi$, and zero otherwise. Moreover, there is a test function ϕ_ξ on $G(F_\infty)$ such that $\operatorname{tr} \pi_\infty(\phi_\xi) = \chi_{\text{EP}}(\pi_\infty \otimes \xi^\vee)$.

As such, if we define

$$|\mathcal{F}| = \sum_{\pi} a_{\mathcal{F}}(\pi),$$

then $|\mathcal{F}|$ arises as the trace of a test function $\phi_\Gamma \phi_\xi$ on $L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$; this is what we mean when we say the families 'arise naturally in the trace formula'.

It is our goal to examine the growth of fields of rationality in families. Fix $A \in \mathbb{Z}_{\geq 1}$ and define the family $\mathcal{F}^{\leq A}(\xi, \chi, \Gamma)$ as the subfamily with multiplicities

$$a_{\mathcal{F}^{\leq A}}(\pi) = \begin{cases} a_{\mathcal{F}}(\pi) & [\mathbb{Q}(\pi) : \mathbb{Q}] \leq A \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we will examine the following question: let $\{\Gamma_\lambda\}$ be a sequence of subgroups whose index in a fixed open compact subgroup approaches infinity. Under what circumstances does

$$\lim_{\lambda \to \infty} \frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} = 0? \tag{1.2}$$

We will investigate this question in the following situations: let F be a totally real field and E/F a totally imaginary quadratic extension. Let $G = \mathrm{GL}_2$ over F or $U(n) = U_{E/F}(n)$, the quasi-split unitary group in n^2 variables with respect to E/F.

Let Γ_λ denote *either*:

- The principal congruence subgroup $\Gamma(\mathfrak{n}_\lambda)$ corresponding to an ideal $\mathfrak{n}_\lambda$, or
- the conductor level subgroup $K_n(\mathfrak{n}_\lambda)$, where $\mathfrak{n}_\lambda$ is divisible only by primes where G splits.

In either of these situations, if $N(\mathfrak{n}_\lambda) \to \infty$ as $\lambda \to \infty$, then 1.2 holds.

We give a brief remark on our choice of algebraic group. As noted above, after we have applied Plancherel equidistribution, we need to examine the fields of rationality of discrete series representations of $G(F_\mathfrak{p})$. When $G(F_\mathfrak{p}) \cong \mathrm{GL}_n(F_\mathfrak{p})$, the discrete series representations are parameterized in terms of the supercuspidal representations of $\mathrm{GL}_m(F_\mathfrak{p})$ for smaller m. Moreover, if $\mathfrak{p} \mid p > n$, the supercuspidal representations are parameterized by abelian characters of the multiplicative groups of extensions of $F_\mathfrak{p}$. Moreover, $\mathrm{Aut}(\mathbb{C})$ respects these parameterizations, so it is straightforward to determine a lower bound on the size of the field of rationality. Moreover, the representation-theoretic properties of discrete series $\mathrm{GL}_n(F_\mathfrak{p})$ representations are much better understood than those of other groups. This explains our decision to use twists of GL_n.

We use $G = \mathrm{GL}_2$ and $U(n)$ because for these groups, $G(F_\infty)$ has discrete-series representation. Let ξ be an irreducible finite dimensional algebraic representation such that the set of ξ-cohomological $G(F_\infty)$ representations is a discrete-series L-packet Π_ξ. Therefore, if π is a discrete automorphic representation with $\pi_\infty \in \Pi_\xi$, then $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is finite. This ensures our question is not vacuous.

1.3 Outline of the paper

In chapter 2, we will discuss the representation-theoretic basics we will need later in our paper. In particular, we will will describe the tempered spectrum of $\mathrm{GL}_n(L)$ as a countable union of compact real orbiforlds, and we will discuss the tensor decomposition of automorphic representations.

In chapters 3, 4, and 5, we discuss Plancherel equidistribution. In chapter 3, we will define the fixed-central-character Plancherel measure on the set $\Pi(G(L), \chi)$ of unitary $G(L)$ representations with central character χ. We give a definition of Plancherel equidistribution, and reduce the proof of Plancherel equidistribution to an asymptotic vanishing of trace formula terms using the trace formula and Sauvageot's density theorem. In chapter 4, we prove this asympotic vanishing for sequences of subgroups $\Gamma_\lambda \leq G(\mathbb{A}^{S,\infty})$ with $\mathrm{lev}(\Gamma_\lambda) \to \infty$. When $G = \mathrm{GL}_2$ and $\Gamma_\lambda = \Gamma_0(\mathfrak{n}_\lambda)$, we give an explicit upper bound on the trace formula terms using the reduced Bruhat-Tits building for GL_2. In more generality, we appeal to the bounds of Section 5 of [FL13]. In chapter 5, we prove a refined limit multiplicity result for cuspidal automorphic representations π such that the $\mathbb{A}_{\mathrm{spl}}$ part of π has conductor exactly $\mathfrak{n}$. This will depend on the construction and analysis of an explicit test function e^{new} whose trace is zero on all generic representations whose conductor is not $\mathfrak{n}$.

In chapter 6, we define the field of rationality of a smooth representation and determine an explicit lower bound for the degree of the field corresponding to supercuspidal, discrete-series, and tempered representations of $\mathrm{GL}_n(L)$ which have positive depth. In chapter 7, we will explicitly define our families of automorphic representations, and state and prove our main theorem, contingent upon some results from $\mathfrak{p}$-adic representation theory. Finally, in chapter 8 we prove these representation theoretic results.

Throughout, we fix the following notation:

- F will be used to denote a totally real field and E will denote a totally imaginary quadratic extension of F. $U(n) = U_{E/F}(n)$ will always denote the quasi-split unitary group. On the other hand, L and L' will be used to denote $\mathfrak{p}$-adic fields.

- $\mathbb{A}$ will denote the adèle ring of F. We let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{A}^\infty$ denote the restricted direct product of the $F_\mathfrak{p}$ over the finite primes $\mathfrak{p}$ of F; then $\mathbb{A} = \mathbb{A}^\infty \times F^\infty$. If $G = U(n)$ or GL_2, we let $\mathcal{V}_{\mathrm{spl}}$ denote the set of finite places of F at which G splits and $\mathcal{V}_{\mathrm{nsp}}$ denote the places where it does not split. We let $\mathbb{A}_{\mathrm{spl}}$, $\mathbb{A}_{\mathrm{nsp}}$ denote the split and nonsplit components of the finite adèles, so that

$\mathbb{A} = \mathbb{A}_{\mathbf{spl}} \times \mathbb{A}_{\mathbf{nsp}} \times F_\infty$.

- K, and Γ will denote open compact subgroups of $G(\mathbb{A}^\infty)$. Usually these subgroups will be contained in a fixed maximal open compact subgroup $\mathbf{K} = \prod_{\mathfrak{p}} \mathbf{K}_{\mathfrak{p}}$.

- In almost all circumstances, we will use lower-case fraktur to denote ideals in either F or a local field. In general, $\mathfrak{n}$ will denote an ideal of F. In either the global or local case, $\mathfrak{o}$ will denote the ring of integers and $\mathfrak{p}$ will denote a prime ideal. The only exception to this rule will be $\mathfrak{g}$ and $\mathfrak{h}$, which denote the Lie algebras of groups G, H respectively.

Chapter 2

Basics on Representation Theory and Unitary Groups

2.1 Admissible representations of $\mathfrak{p}$-adic groups

Throughout this section, L is a $\mathfrak{p}$-adic field and G/L is a connected reductive algebraic group. When the context is clear, we will identify G with its group $G(L)$ of L points.

Definition 2.1.0.1. *Let π be a G representation on a complex vector space V. We say π is* smooth *if every $v \in V$ is fixed by $\pi(K)$ for some open subgroup $K \leq G$.*

We say π is admissible *if it is smooth and, for every open subgroup K, the space π^K of K-fixed vectors is finite dimensional.*

Henceforth we will assume π is irreducible and admissible. In this situation, the result of Schur's lemma holds, so $\mathrm{End}_G(V) = \mathbb{C}$. As such, the center Z of G acts via a character χ_π; we call χ_π the *central character* of π.

We say π is *unitary* if V admits a non-degenerate hermitian form $\langle \cdot, \cdot \rangle$ such that $\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$ and $g \in G$. We write $\Pi(G)$ for the set of isomorphism classes of irreducible, admissible, unitary G representations. If π is unitary, its central character χ_π takes values in $S^1 \leq \mathbb{C}^\times$. For a unitary character χ, we let $\Pi(G, \chi)$ denote the subset of $\Pi(G)$ consisting of those representations π with $\chi_\pi = \chi$.

Let $v \in V$, $v^* \in V^*$. We define the *matrix coefficient* $m_{v,v^*} : G \to \mathbb{C}$ via

$$m_{v^*,v}(g) = \langle v^*, \, \pi(g)v \rangle \, .$$

The support of a matrix coefficient is invariant under multiplication by Z.

Definition 2.1.0.2. *Let π be an irreducible unitary G representation.*

- *We say π is* supercuspidal *if its matrix coefficients are compactly supported modulo Z.*
- *We say π is a* discrete series representation *if its matrix coefficients are in $L^2(G/Z)$.*
- *We say π is* tempered *if its matrix coefficients are in $L^{2+\epsilon}(G/Z)$ for every $\epsilon > 0$.*

Throughout this thesis, when we use the terminology *supercuspidal representation* we will specifically mean a *unitary* supercuspidal representation, unless otherwise noted. It is theorem of Tate that a supercuspidal representation is unitary if and only if its central character is unitary.

We denote the set of tempered G representations as $\Pi^t(G)$; define $\Pi^t(G, \chi)$ as the subset consisting of those representations with central character χ.

Let P be a parabolic subgroup of G with Levi factorization $P = MN$, so that N is the unipotent radical of P. Let $\delta_P : M \to \mathbb{C}^\times$ denote the modulus character of the action of M on N by conjugation (so that $\mu_N(m^{-1}Am) = \delta_P(m)\mu_N(A)$ for a Haar measure μ_N on N).

Definition 2.1.0.3. *We define the* normalized induction functor *from the Grothendieck group of admissible M representations to the Grothendieck group of admissible G representations as follows: given an admissible representation π_M of M, we consider the representation $\pi_M \otimes \delta_P$ and extend this to a representation π_P of P via the surjection $P \twoheadrightarrow P/N \cong M$. Let $I_M^G \pi_M = \mathrm{Ind}_P^G \pi_P$.*

The isomorphism class of $I_M^G \pi$ depends on the choice of parabolic P with Levi component M, but its image in the Grothendieck group does not. Moreover, I_M^G takes *irreducible admissible unitary representations to finite length, admissible unitary representations (see 3.13-3.15 of [BZ76])*

Throughout, we will use I_M^G to denote the *normalized* induction functor. Moreover, to avoid the dependence upon the parabolic $P = MN$, we will make a consistent choice of parabolic. In particular, when $G = \mathrm{GL}_n$, let P_0 denote the minimal parabolic consisting of upper-triangular matrices and let M_0 denote the minimal Levi subgroup consisting of diagonal matrices. We say P is a standard parabolic if $P \geq P_0$ and M is a standard Levi if $M \geq M_0$. Given a standard Levi subgroup M, there is a unique standard parabolic P whose Levi component is M.

It is worth noting the following property of supercuspidal representations, which follows from [BZ76] and [BZ77]:

Proposition 2.1.0.4. *Let π be a unitary G representation. The following are equivalent:*

(a) π is supercuspidal;

(b) π does not occur as a subquotient of the induced representation $I_M^G \pi_M$ for any $M \neq G$; and

(c) π does not occur as a subrepresentation of $I_M^G \pi_M$ for any $M \neq G$.

If π is any representation, there is a Levi subgroup M and a (not necessarily unitary) supercuspidal M representation π_M such that π is isomorphic to a subquotient of $I_M^G \pi_M$. Moreover, the pair (M, π_M) is unique in the following sense: if $(M', \pi_{M'})$ is another such pair, there is a $g \in G$ with $M' = gMg^{-1}$, and such that g conjugates π into π'.

Therefore, the supercuspidal representations form the building blocks of the representation theory of $\mathfrak{p}$-adic groups. We will only use this fact indirectly, but it is worth noting nonetheless.

2.2 The tempered spectrum of $\mathrm{GL}_n(L)$

In this section, we'll briefly describe the tempered spectrum of $G = \mathrm{GL}_n(L)$; let Z denote the center of G. The classifications contained in this chapter are also stated in [Kud91], an extremely readable introduction to the topic and to the non-Archimedean case of the local Langlands correspondence.

As before, let P_0 denote the subgroup of consisting of the upper-triangular matrices; this is a minimal parabolic subgroup. We say P is an *standard parabolic* subgroup if $P \geq P_0$. In this case, the Levi component M of P is a standard Levi subgroup.

Let $m = nd$ and let ρ be a unitary supercuspidal representation of $\mathrm{GL}_m(L)$. Let M be the standard Levi subgroup of $\mathrm{GL}_n(L)$ isomorphic to $\mathrm{GL}_m(L)^d$. Let ρ_M denote the (external) tensor product

$$\left(\rho \otimes |\det|^{\frac{1-d}{2}}\right) \otimes \left(\rho \otimes |\det|^{\frac{3-d}{2}}\right) \otimes \ldots \otimes \left(\rho \otimes |\det|^{\frac{d-1}{2}}\right).$$

Then

Lemma 2.2.0.5. *(i) $I_M^G \rho_M$ has a unique irreducible quotient module, which we call* $\mathrm{Sp}(\rho,\, d)$.

(ii) $\mathrm{Sp}(\rho,\, d)$ *is a discrete series G representation.*

(iii) All discrete series representations of G arise as $\mathrm{Sp}(\rho,\, d)$, *for some $d \mid n$ and supercuspidal representation ρ of* $\mathrm{GL}_{n/d}(L)$.

(iv) $\mathrm{Sp}(\rho,\, d) \cong \mathrm{Sp}(\rho',\, d')$ *if and only if $\rho \cong \rho'$ and $d = d'$.*

Proof. (i) is Proposition 2.10 of [Zel80]. (ii) follows from [BZ77]. (iii) is Proposition 11 of [Rod82]. (iv) follows because ρ is the unique unitary supercuspidal representation with an unramified twist occurring in the supercuspidal support of $\mathrm{Sp}(\rho,\, d)$. ∎

It is worth noting that $I = I_M^G \rho_M$ is equal to the *unnormalized* induction $\mathrm{Ind}_P^G \rho^{\otimes d} \otimes \mathbf{1}_N$. This will be useful when computing the field of rationality of $\mathrm{Sp}(\rho,\, d)$ later.

Lemma 2.2.0.6. *Let ω_i be a discrete series $\mathrm{GL}_{n_i}(L)$ representation for $i = 1, \ldots, r$, with $n_1 + \ldots + n_r = n$. Let M denote the subgroup of block diagonal matrices isomorphic to $\prod_i \mathrm{GL}_{n_i}(L)$. Let $M' = \prod_j \mathrm{GL}_{n'_j}(L)$.*

(i) $I^G_{M'}(\omega_1 \otimes \ldots \otimes \omega_r)$ is irreducible and tempered,

(ii) all tempered $\mathrm{GL}_n(L)$ representations are of this form, and

(iii) $I^G_M(\omega_1 \otimes \ldots \otimes \omega_r) \cong I^G_{M'}(\omega'_1 \otimes \ldots \otimes \omega'_{r'})$ if and only if $r = r'$ and there is a permutation s of $\{1, \ldots, r\}$ such that $\omega_i \cong \omega'_{s(i)}$.

Proof. (i) and (ii) are proven in [Jac77]. (iii) follows by examining the supercuspidal support of the two representations. □

In view of these facts, the tempered spectrum of $\mathrm{GL}_n(L)$ acquires a topological structure as a countable union of countable many compact (real) orbifolds; we invite the reader to see the first section of [AP05]. Let M be a Levi subgroup of $\mathrm{GL}_n(L)$ and let $X_u(M)$ denote the group of unramified (unitary) characters of M. Then $X_u(M)$ acts naturally on the set of discrete series M representations by twisting. Let $\mathcal{O}_M$ denote an orbit under this action. If we pick a basepoint $\omega_0 \in \mathcal{O}_M$, we get a surjective map $X_u(M) \to \mathcal{O}_M$ given by $\chi_M \mapsto \omega_0 \otimes \chi_M$; we give $\mathcal{O}_M$ the quotient topology under this map. The stabilizer of ω_0 is finite, so $\mathcal{O}_M$ is isomorphic to a principal homogeneous space under a real torus.

Let Ω denote the set of pairs $(M, \mathcal{O}_M)$, where M is a standard Levi subgroup and $\mathcal{O}_M$ is an $X_u(M)$ orbit of discrete series M representations. Given $(M, \mathcal{O}_M) \in \Omega$, we have a map $I = I^G_M : \mathcal{O}_M \to \Pi^t(\mathrm{GL}_n(L))$. Then the map

$$\coprod_{(M, \mathcal{O}_M) \in \Omega} \mathcal{O}_M \to \Pi^t(\mathrm{GL}_n(L))$$

is surjective; we give the set $\Pi(\mathrm{GL}_n(L))$ the quotient topology. An *orbit* $\mathcal{O}$ in $\Pi(\mathrm{GL}_n(L))$ is a connected component under this topology. Equivalently, $\mathcal{O}$ is the image of some orbit $\mathcal{O}_M$ under I^M_G. As a topological space, an orbit $\mathcal{O}$ has the structure of a quotient of a real torus by a finite group. It is worth noting that $\mathcal{O}$ does not always have the structure of a smooth manifold. For instance, if $M = M_0$ is the diagonal torus and $\mathcal{O}_M$ is the orbit of unramified (one-dimensional) characters of M and is isomorphic to $(S^1)^n$. Then $\mathcal{O} = I_M^G(\mathcal{O}_M)$ is the principal series orbit and is isomorphic to $(S^1)^n/\Sigma_n$, which is not smooth along the subspaces $\{x_i = x_j\}$.

We briefly remark on the fixed-central-character case: fix a (unitary) central character $\chi : L^\times \to \mathbb{C}^\times$. Let $X_u(M)_0$ denote the kernel of the restriction map $X_u(M) \to X_u(Z(G))$. If $\chi_M \in X_u(M)_0$ and ω is an irreducible admissible M representation, then $I_M^G\omega$ and $I_M^G(\chi_M \otimes \omega)$ have the same central character, so we have an action of $X_u(M)_0$ on the set of discrete-series representations ω such that $I_M^G\omega$ has central character χ. Let $\mathcal{O}_{M,\chi}$ denote an orbit under this action, and let $\mathcal{O}_\chi \subset \Pi(\mathrm{GL}_n(L), \chi)$ denote the image of an orbit $\mathcal{O}_{M,\chi}$.

Fix a central character $\chi : L^\times \to \mathbb{C}^\times$ and a (non-fixed central character) orbit $\mathcal{O} \subset \Pi(\mathrm{GL}_n(L))$. Then the restriction $\chi_\pi|_{\mathfrak{o}_L^\times}$ is independent of $\pi \in \mathcal{O}$. If $\chi_\pi|_{\mathfrak{o}_L^\times} = \chi|_{\mathfrak{o}_L^\times}$, then $\mathcal{O} \cap \Pi(\mathrm{GL}_n(L), \chi)$ is a sub-orbifold of codimension 1 in $\mathcal{O}$, and is precisely equal to an orbit $\mathcal{O}_\chi \subset \Pi(\mathrm{GL}_n(L), \chi)$. Otherwise, the intersection is empty.

In particular, if π is a discrete series $\mathrm{GL}_n(L)$ representation, then the orbit of π in $\Pi(\mathrm{GL}_n(L), \chi)$ is a finite set; this motivates the terminology 'discrete series.'

In the case where the residue characteristic of L is $p > n$, the supercuspidal representations of $\mathrm{GL}_n(L)$ have been classified by Howe ([How77]) and Moy ([Moy86]). They parameterize the supercuspidal representations in terms of equivalence classes of *admissible pairs* (L', η) where L'/L is a field extension of degree n and $\eta : L'^\times \to \mathbb{C}^\times$ is an 'admissible character.' Two pairs (L_1', η_1) and (L_2', η_2) are equivalent if there is an isomorphism $\tau : L_1' \to L_2'$ over L with $\eta_1 = \eta_2 \circ \tau$. We invite the reader to see Section 6.1 for details.

A mental picture of the tempered representation is important since it is the support of the *Plancherel measure*, as defined in the next chapter. The unitary spectrum does not admit such a nice topological characterization. Nonetheless, it is prudent to give a characterization of the unitary spectrum.

Definition 2.2.0.7. *Let τ be a discrete series $\mathrm{GL}_m(L)$ representation and fix $r \in \mathbb{Z}_{\geq 1}$.*

Define $u(\tau, r)$ as follows: let $M \cong \mathrm{GL}_m(L)^r \leq G = \mathrm{GL}_{mr}(L)$ be a standard Levi *subgroup and let*

$$\tau_M = \left(\tau \otimes |\cdot|^{\frac{r-1}{2}}\right) \otimes \ldots \otimes \left(\tau \otimes |\cdot|^{\frac{1-r}{2}}\right).$$

Then $I_M^G \tau_M$ has a unique irreducible quotient, which we denote by $u(\tau, r)$.

Moreover, given $\alpha \in (0, 1/2)$, the representation

$$I_{\mathrm{GL}_{mr} \times \mathrm{GL}_{mr}}^{\mathrm{GL}_{2mr}} \left(u(\tau, r) \otimes |\cdot|^{\alpha}\right) \otimes \left(u(\tau, r) \otimes |\cdot|^{-\alpha}\right)$$

is irreducible and unitary. We denote it by $u(\tau, r; \alpha)$.

Proposition 2.2.0.8. *[Tad86, Theorem D]. Let $\mathcal{B}$ denote the set of all representations of the form $u(\tau, r)$ and $u(\tau, r; \alpha)$. For any $\pi_1, \ldots, \pi_m \in \mathcal{B}$, the induced representation $I_M^G(\pi_1 \otimes \ldots \otimes \pi_m)$ is unitary. Moreover, all unitary representations arise in this way.*

2.2.1 Generic representations and conductors

Throughout this section, let $G = \mathrm{GL}_n(L)$ and let $U \leq G$ be the subgroup of strictly upper-triangular matrices. If ψ is a nontrivial additive character of L; then we extend ψ to a character of U as follows: for $u = (u_{ij})$, set

$$\psi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right).$$

Let (ψ, V) be a G representation. A *Whittaker functional* is a smooth linear functional $\lambda : V \to \mathbb{C}$ such that $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for $v \in V$, $u \in U$.

A representation (π, V) is *generic* if it admits a nonzero Whittaker functional. If π is irreducible and generic, then it admits exactly one Whittaker functional (up to scalar multiplication). By Frobenius reprocity, π has a Whittaker functional if and only if it embeds into $I = \mathrm{Ind}_U^G(\psi)$. The realization of π as a subspace of I is a *Whittaker model* for π.

Proposition 2.2.1.1. *A tempered $\mathrm{GL}_n(L)$ representation is generic.*

Proof. This is basically Theorem 4.9 of [BZ77]; they prove that a tempered representation π has a Kirillov model, which implies the existence of a Whittaker model. $\square$

Definition 2.2.1.2. *[JPS81] Let L be a $\mathfrak{p}$-adic group with ring of integers $\mathfrak{o}$. Let $\mathbf{K}_n = \mathbf{K} = \mathrm{GL}_n(\mathfrak{o})$. The subgroup $K_n(\mathfrak{p}^r) = K(\mathfrak{p}^r)$ is the subgroup consisting of those matrices*

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

where $X \in \mathbf{K}_{n-1}$, Y is an $(n-1)\times 1$ vector of elements of $\mathfrak{o}$, Z is a $1 \times (n-1)$ vector of elements in $\mathfrak{p}^r$, and $W \in 1 + \mathfrak{p}^r$.

Let π be a generic $\mathrm{GL}_n(L)$ representation. For $c \in \mathbb{Z}_{\geq 1}$, the following are equivalent:

(a) The ϵ factor $\epsilon(s, \pi, \phi)$ is equal to Cq^{-cs} for some $C \in \mathbb{C}$; and

(b) $c \geq 0$ is minimal so that π has a nonzero $K(\mathfrak{p}^c)$-fixed vector.

In this case, we say c is the conductor *of π.*

If π is a generic representation of conductor c, then π has a *unique* nonzero $K_n(\mathfrak{p}^c)$-fixed vector (up to scalar multiplication) [JPS81]. Moreover, the dimension of the space of $\pi^{K_n(\mathfrak{p}^r)}$ has been computed by Reeder:

Theorem 2.2.1.3 ([Ree91], Theorem 1). *Let π be a generic irreducible admissible $\mathrm{GL}_n(L)$ representation of conductor c. Then*

$$\dim \pi^{K_n(\mathfrak{p}^r)} = \binom{r - c + n - 1}{n-1}$$

2.3 The global situation: discrete and cuspidal automorphic representations

Throughout this section, F denotes a number field and $\mathbb{A} = \mathbb{A}_F$ its adèle ring. Let G/F be a connected reductive group. Then $G(\mathbb{A})$ acts on the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$ via right translation. We say π is an *automorphic representation* if it occurs as a subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}))$.

Let Z denote the center of G and let $\chi : Z(F)\backslash Z(\mathbb{A})$ be an automorphic (unitary) character. Let $L^2(G(F)\backslash G(\mathbb{A}), \chi)$ denote the subspace of functions $f \in L^2$ such that $f(gz) = \chi(z)f(g)$. If π is an automorphic representation with central character χ, we say π is a *discrete* automorphic representation if it occurs as a direct summand of $L^2(G(F)\backslash G(\mathbb{A}), \chi)$. We define L^2_{disc} as the subspace of L^2 spanned by the discrete automorphic representations, and for an automorphic representation π, set $m_{\text{disc}}(\pi)$ as the multiplicity of π in L^2_{disc}.

Finally, we define L^2_{cusp} as the space of those $f \in L^2$ such that

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)\, dg = 0$$

for almost all $g \in G(\mathbb{A})$ and for any subgroup N which is the unipotent radical of a proper parabolic subgroup P. A representation π is a *cuspidal* automorphic representation if it occurs as a subrepresentation of L^2_{cusp}; then $m_{\text{cusp}}(\pi)$ is the multiplicity of π in L^2_{cusp}.

It is a theorem of Borel and Jacquet (see page 197 of [BJ79]) that $L^2_{\text{cusp}} \subseteq L^2_{\text{disc}}$; the complementary subspace is called the *residual* spectrum.

We will need a decomposition theorem of Flath. Let π denote an automorphic $G(\mathbb{A})$ representation and let π_0 denote the subspace consisting of smooth functions (with respect to the $\mathfrak{p}$-adic topology at the finite places of F and the standard topology at the Archimedean places). Then π_0 is a dense subrepresentation of π. For each finite place $\mathfrak{p}$ of F, fix a special maximal compact subgroup $K_\mathfrak{p}$ of $G(F_\mathfrak{p})$, such that $K_\mathfrak{p}$ is hyperspecial almost everywhere.

Let π_v be a smooth irreducible $G(F_v)$ representation for each place v. Assume there is a finite set S_{ram} of places, containing the infinite places, such that π_v has a K_v-fixed vector w_v for all $v \notin S_{\text{ram}}$. We define the restricted tensor product

$$\bigotimes_v{}' \pi_v$$

as the subspace of $\bigotimes_v V_{\pi,v}$ spanned by

$$\left(\bigotimes_{v\in S} V_{\pi,v}\right)\otimes\left(\bigotimes_{v\notin S} w_v\right)$$

for all finite sets $S \supseteq S_{\mathrm{ram}}$.

Theorem 2.3.0.4 ([Fla79], Theorem 4). *Let π be an automorphic $G(\mathbb{A})$ representation with smooth part π_0. Then there are irreducible, admissible, unitary representations π_v for each place v of F such that*

$$\pi_0 \cong \bigotimes_v{}' \pi_v.$$

In this situation, we say that π decomposes as a tensor product of the π_v. Moreover, the central character χ_π decomposes as a product of local characters χ_v, then π_v has central character χ_v.

2.4 The groups $U_{E/F}(n)$; maximal special subgroups

In this section, we give a quick primer on the quasi-split unitary group $U_{E/F}(n)$. Let F be a totally real number field and E/F a totally imaginary quadratic extension; then E is a CM field and the nontrivial element in $\mathrm{Gal}(E/F)$ acts as complex conjugation for every embedding $E \hookrightarrow \mathbb{C}$: we denote this automorphism $x \mapsto \overline{x}$.

Let $\Phi = \Phi_n$ denote the matrix with entries

$$\Phi_{ij} = \begin{cases} (-1)^{i-1} & i+j=n+1 \\ 0 & \text{otherwise} \end{cases}$$

and let $U_{E/F}(n, R)$ be the set of $g \in \mathrm{GL}_n(E \otimes R)$ with $g\Phi_n\overline{g}^t = \Phi$. This defines an algebraic group over F. The algebraic subgroup of upper-triangular matrices is a Borel subgroup, so that $U(n)$ is quasi-split over F. Moreover, $U_{E/F}(n)$ becomes isomorphic GL_n after base-changing to E. Therefore, if v is any place of F such that

E splits at v, then $U_{E/F}(n, F_v) \cong \mathrm{GL}_n(F_v)$. In this case, we say $U(n)$ splits at v.

Let $\mathfrak{p}$ be a finite prime of F. If $U(n)$ splits at $\mathfrak{p}$ then $\mathbf{K}_\mathfrak{p} = \mathrm{GL}_n(\mathfrak{o}_{F,\mathfrak{p}})$ is a hyperspecial maximal compact subgroup of $\mathrm{GL}_n(F_\mathfrak{p})$. Otherwise, $U(n, F_\mathfrak{p})$ has a maximal hyperspecial $K_\mathfrak{p}$ subgroup whenever E/F is unramified. If E/F is ramified, then $U(n, F_\mathfrak{p})$ will only have a *special* maximal compact subgroup $\mathbf{K}_\mathfrak{p}$. In either case, there is a group scheme $\mathscr{G}$ over $\mathfrak{o}_{F,\mathfrak{p}}$ whose fiber over $F_\mathfrak{p}$ is isomorphic to U_n, and $\mathbf{K}_\mathfrak{p} = \mathscr{G}(\mathfrak{o}_{F,\mathfrak{p}})$. If E/F is unramified, (so that $\mathbf{K}_\mathfrak{p}$ is chosen to be hyperspecial), then the special fiber of $\mathscr{G}$ is a connected reductive group.

We invite the reader to see [Tit79] for the definition of special and hyperspecial subgroups. In particular, special (resp. hyperspecial) subgroups are the stabilizers of special (resp. hyperspecial) *points* in the Bruhat-Tits building; these are defined in 1.9 (resp 1.10). In the non-split case, we will not give an explicit description of the maximal special and hyperspecial subgroups of $U(n)$. Rather, we invite the reader to see Section 3 of [GHY01].

Definition 2.4.0.5. *Let $\mathfrak{p}$ be a prime of F, let $\mathbf{K}_\mathfrak{p}$ be a maximal special subgroup of $U_n(F_\mathfrak{p})$, and let $\mathscr{G}$ be a group scheme over $\mathfrak{o}_{F,\mathfrak{p}}$ whose generic fiber is isomorphic to $U(n)$, with $\mathscr{G}(\mathfrak{o}_{F,\mathfrak{p}}) = \mathbf{K}_\mathfrak{p}$. For $r > 0$, we define the* full level subgroups $\Gamma(\mathfrak{p}^r) \leq \mathbf{K}_\mathfrak{p}$ *as the kernel of the canonical map $\mathscr{G}(\mathfrak{o}_{F,\mathfrak{p}}) \to \mathscr{G}(\mathfrak{o}_{F,\mathfrak{p}}/\mathfrak{p}^r)$.*

If $U(n)$ splits at $\mathfrak{p}$, we will assume $\mathbf{K}_\mathfrak{p} = \mathrm{GL}_n(\mathfrak{o}_{F,\mathfrak{p}})$. Then $\Gamma(\mathfrak{p}^r)$ is subgroup $1 + \mathfrak{p}^r M_n(\mathfrak{o}_{F,\mathfrak{p}})$.

If $\mathfrak{n} = \prod_\mathfrak{p} \mathfrak{p}^{r_\mathfrak{p}}$ we set

$$\Gamma(\mathfrak{n}) = \left(\prod_{\mathfrak{p}|\mathfrak{n}} \mathfrak{p}^{r_\mathfrak{p}}\right) \times \left(\prod_{\mathfrak{p}\nmid\mathfrak{n}} K_\mathfrak{p}\right);$$

this is an open compact subgroup of $G(\mathbb{A}^\infty)$.

Note that this definition is equivalent to the definition given in [ST12] (see page 65 of that paper).

Chapter 3

Plancherel Equidsitribution

3.1 Hecke algebras and Plancherel measure

Throughout, let L denote a $\mathfrak{p}$-adic field. Let G/L be a connected reductive group with center Z and let $\chi : Z(L) \to \mathbb{C}^\times$ be a unitary character. Recall that $\Pi(G(L))$ is the set of irreducible, admissible, unitary $G(L)$ representations and $\Pi(G(L), \chi)$ is the subset consisting of those representations π with $\chi_\pi = \chi$. Moreover, Π^t denotes the subset of Π consisting tempered representations.

Definition 3.1.0.1. *We define the* Hecke algebra $\mathcal{H}(G(L))$ *as the convolution algebra of locally constant, compact supported functions* $G(L) \to \mathbb{C}$.

If $\phi \in \mathcal{H}(G(L))$ *and* π *is an irreducible, admissible* $G(L)$ *representation, then the map*

$$\pi(\phi) : v \mapsto \int_{G(L)} \phi(g)\, \pi(g) \cdot v\, dg$$

is well-defined and of trace class. We define $\widehat{\phi}(\pi) = \operatorname{tr} \pi(\phi)$. *The map* $\phi \mapsto \widehat{\phi}$ *is a linear map from* $\mathcal{H}(G(L))$ *to the set of bounded, continuous functions on* $\Pi(G(L))$ *that are supported on a finite number of Bernstein components.*

We define the fixed central character Hecke algebra $\mathcal{H}(G(L), \chi)$ *as the convolution algebra of locally constant functions* $\phi : G(L) \to \mathbb{C}$ *such that*

- ϕ *is compactly supported modulo* $Z(L)$, *and*

- *for $g \in G(L)$, $z \in Z(L)$, we have $\phi(gz) = \chi^{-1}(z)\phi(g)$*

If $\phi_\chi \in \mathcal{H}(G(L), \chi)$ and π is an irreducible, admissible $G(L)$ representation with central character χ, the map

$$\pi(\phi_\chi) : v \mapsto \int_{G(L)/Z(L)} \phi(g)\, \pi(g) \cdot v\, dg$$

is well-defined and of trace class: we define $\widehat{\phi}_\chi(\pi) = \operatorname{tr} \pi(\phi_\chi)$. As above, this gives a linear map from $\mathcal{H}(G(L), \chi)$ to the space of functions on $\Pi(G(L), \chi)$.

There is an averaging map $\mathcal{H}(G(L)) \to \mathcal{H}(G(L), \chi)$ *given by $\phi \mapsto \phi_\chi$, where*

$$\phi_\chi(g) = \int_{Z(L)} \phi(gz)\chi(z)\, dz.$$

We have stated the above definition for $G(L)$ but will often apply the notation more generally. Specifically, if F is a number field and G/F a connected reductive algebraic group, we may refer to the Hecke algebras $\mathcal{H}(G(\mathbb{A}_F^\infty))$ and $\mathcal{H}(G(\mathbb{A}_F^\infty), \chi)$ for a central character χ. If S is a finite set of finite places of F we may moreover replace $\mathbb{A}_F$ by $F_S = \prod_{\mathfrak{p} \in S} F_\mathfrak{p}$ or $\mathbb{A}^{\infty, S}$.

The following lemma is a simple application of Fubini's theorem, but will come up often in the following chapters:

Lemma 3.1.0.2. *Assume Haar measures on $Z(L)$, $G(L)$, $G(L)/Z(L)$ are chosen compatibly. Fix $\phi \in \mathcal{H}(G(L))$ and let $\phi_\chi \in \mathcal{H}(G(L), \chi)$ be its image under the averaging map. If $\pi \in \Pi(G(L), \chi)$, then $\widehat{\phi}(\pi) = \widehat{\phi}_\chi(\pi)$.*

If $\Gamma \leq G(L)$ is an open compact subgroup, let $e_\Gamma = \operatorname{vol}(\Gamma)^{-1}\mathbf{1}_\Gamma$. This is an idempotent in $\mathcal{H}(G(L))$ (that is, $e_\Gamma \star e_\Gamma = e_\Gamma$). Moreover, if π is an irreducible admissible $G(L)$-representation, then

$$\widehat{e_\Gamma}(\pi) = \operatorname{tr} \pi(e_\Gamma) = \dim \pi^\Gamma,$$

where π^Γ denotes the space of Γ-fixed vectors in the space of π. Moreover, let $e_{\Gamma,\chi}$ denote the image of e_Γ under the averaging map $\mathcal{H}(G(L)) \to \mathcal{H}(G(L), \chi)$. We note that $e_{\Gamma,\chi} = 0$ unless χ is trivial on $\Gamma \cap Z$, and in this case, $e_{\Gamma,\chi}(1) = \mathrm{vol}(\Gamma Z/Z)^{-1}$. Moreover, it follows immediately that if π has central character χ, then

$$\widehat{e}_{\Gamma,\chi}(\pi) = \dim \pi^{\Gamma}.$$

Proposition 3.1.0.3. *There is a unique measure $\widehat{\mu}^{\mathrm{pl}}$ on $\Pi(G(L))$, called the* Plancherel measure, *such that, for any $\phi \in \mathcal{H}(G(L))$ the following equality holds:*

$$\phi(1) = \widehat{\mu}^{\mathrm{pl}}(\phi) := \int_{\Pi(G(L))} \widehat{\phi}(\pi)\, d\widehat{\mu}^{\mathrm{pl}}(\pi).$$

Moreover, $\widehat{\mu}^{\mathrm{pl}}$ is supported on the tempered spectrum $\Pi^t(G(L))$.

For $\mathfrak{p}$-adic groups, the Plancherel measure was described in [Wal03]. In the case of $G = \mathrm{GL}_n$, a completely explicit description of the Plancherel measure is given in [AP05]. We will need a fixed-central-character version of the Plancherel measure:

Proposition 3.1.0.4. *There is a unique measure $\widehat{\mu}^{\mathrm{pl}}_{\chi}$ on $\Pi(G(L), \chi)$ such that, for any $\phi_{\chi} \in \mathcal{H}(G(L), \chi)$ the following equality holds:*

$$\phi_{\chi}(1) = \widehat{\mu}^{\mathrm{pl}}_{\chi}(\phi_{\chi}) := \int_{\Pi(G(L),\chi)} \widehat{\phi}_{\chi}(\pi)\, d\widehat{\mu}^{\mathrm{pl}}_{\chi}(\pi).$$

We call $\widehat{\mu}^{\mathrm{pl}}_{\chi}$ the fixed central character Plancherel measure*; it is supported on the tempered spectrum $\Pi^t(G(L), \chi)$. For any π which is not a discrete-series representation, we have $\widehat{\mu}^{\mathrm{pl}}_{\chi}(\pi) = 0$. If π is a discrete series representation, then $\widehat{\mu}^{\mathrm{pl}}_{\chi}(\pi) = \deg(\pi)$, the formal degree of π.*

To our knowledge, the construction of the fixed central character Plancherel measure has not been written down explicitly. However, the construction follows from abelian Fourier analysis and the non-fixed central character Plancherel measure as in [Bin15].

3.2 Euler-Poincaré functions at the Archimedean places

Let $G/\mathbb{R}$ be a reductive group. Throughout this chapter, we will assume that G has a maximal torus which is anisotropic modulo the center. Let A_G denote the maximal split torus in the center of $G \times_{\mathbb{Q}} \mathbb{R}$ and let $A_{G,\infty}$ denote the connected component of $A_G(\mathbb{R})$ (with respect to the real topology). Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and let $K'_\infty = K_\infty A_{G,\infty}$. Fix an irreducible finite dimensional algebraic $G(\mathbb{R})$ representation ξ and let ω_ξ denote the central character of ξ on $A_{G,\infty}$. Let π be an irreducible admissible representation whose central character on $A_{G,\infty}$ is ω_ξ. Let $\mathfrak{g} = \operatorname{Lie} G(\mathbb{R})$. The *Euler-Poincaré characteristic* of π (with respect to ξ) is defined as

$$\chi_{\mathrm{EP}}(\pi \otimes \xi^\vee) = \sum_{i \geq 0} (-1)^i \dim H^i(\mathfrak{g},\, K'_\infty,\, \pi \otimes \xi^\vee)$$

(here the cohomology term is Harish-Chandra's $(\mathfrak{g},\, K)$ cohomology).

We say π is ξ-cohomological if there is an $i \geq 0$ such that

$$H^i(\mathfrak{g},\, K'_\infty,\, \pi \otimes \xi^\vee) \neq 0.$$

More generally, if π is an automorphic $G(\mathbb{A})$ representation such that π_∞ is ξ-cohomological, we also call π ξ-cohomological. It is clear that $\chi_{\mathrm{EP}}(\pi_\infty \otimes \xi^\vee) = 0$ if π is not ξ-cohomological.

Definition 3.2.0.5. *Let ξ be an irreducible finite dimensional algebraic representation of $G(\mathbb{R})$ and let T be a compact torus of G of maximal dimension. We recall that $\xi\ |_{T(\mathbb{R})}$ decomposes as a direct sum of abelian characters $\{\lambda\}$. A choice of positive roots of T determines an ordering of the roots $\{\lambda\}$, and with respect to this ordering ξ has a unique positive weight λ_ξ. We say ξ has* regular highest weight *if for every coroot $\alpha^\vee$, we have $\langle \lambda,\, \alpha^\vee \rangle \neq 0$.*

Proposition 3.2.0.6. *Let the highest weight of ξ be regular and let π be an automorphic, ξ-cohomological representation. Let $q(G) = \frac{1}{2} \dim_{\mathbb{R}} G(\mathbb{R})/K'_\infty$.*

(a) If π is ξ-cohomological, then π_∞ is a discrete series representation, and $\chi_{\mathrm{EP}}(\pi_\infty \otimes$

$\xi^\vee) = (-1)^{q(G)}$. Moreover, all ξ-cohomological representations are in the same discrete series L-packet.

(b) π occurs in the discrete spectrum of $G(\mathbb{A})$ if and only if it occurs in the cuspidal spectrum, and $m_{\text{disc}}(\pi) = m_{\text{cusp}}(\pi)$.

(c) For any place v of F, π_v is tempered.

(d) The field of rationality $\mathbb{Q}(\pi)$ is a finite extension of $\mathbb{Q}$.

Proof. (a) is the second bullet point of page 44 of [ST12]. (b) is Theorem 4.3 of [Wal84]. (c) is a statement of Corollary 4.16 of [ST12]. Finally, (d) follows from Proposition 2.15 of [ST14], since a ξ-cohomological discrete automorphic representation is cuspidal, in view of (b). □

Let G' be the compact inner form of G. There is a unique Haar measure on $G(\mathbb{R})/Z(\mathbb{R})$ such that the induced measure on $G'(\mathbb{R})/Z'(\mathbb{R})$ has total measure 1; we call this measure the *Euler-Poincaré measure.*

In [CD90], Clozel and Delorme construct a bi-K_∞-invariant function $\phi_\xi \in C^\infty(G(\mathbb{R}))$ which satisfies

$$\phi_\xi(gz) = \omega_\xi^{-1}(z)\phi_\xi(g) \quad g \in G(\mathbb{R}),\, z \in Z(\mathbb{R})$$

and such that, for any π with $\chi_\pi = \chi_\xi$, we have

$$\operatorname{tr}(\phi_\xi) = \chi_{\mathrm{EP}}(\pi \otimes \xi^\vee) \tag{3.7}$$

where the trace is taken with respect to the Euler-Poincaré measure, on $G(\mathbb{R})$.

Throughout the paper, we'll need the following two facts:

- ϕ_ξ is *cuspidal*; that is, its orbital integrals vanish off of elliptic elements in $G(\mathbb{R})$ (see, for instance, page 267 of [Art89]).

- $\phi_\xi(1) = \dim \xi$; this is implicit in [Art89] and follows basically because $\dim \xi$ is the Plancherel measure of the L-packet of discrete-series representations which are ξ-cohomological.

A priori, when the highest weight of ξ is not regular, we have $\phi_\xi(gz) = \chi_\xi^{-1}(z)\phi_\xi(g)$ only for $z \in A_{G,\infty}$. However, for $G = \mathrm{GL}_2$ we can check that this holds whenever $z \in Z(\mathbb{R})$. In the first case, the Clozel-Delorme functions have been explicitly computed in [KL06, Section 14], and we can check explicitly that

$$\phi_\xi(gz) = \mathrm{sgn}(z)^{\dim \xi + 1}\phi_\xi(g)$$

as desired.

As such, we can consider ϕ_ξ as a function in the Hecke algebra $\mathcal{H}(G(\mathbb{R}), \chi_\xi^{-1})$ and can pick a measure μ_{EP} on $G(\mathbb{R})/Z(\mathbb{R})$ such that

$$\chi_{\mathrm{EP}}(\pi \otimes \xi^\vee) = \mathrm{tr}_{G/Z}\, \phi_\xi(\pi)$$

for π with central character χ_ξ.

3.3 Counting measures and Plancherel equidistribution

For this section, we place ourselves in the global setting. To this end, we fix a totally real number field F and a connected reductive group G/F with center Z. We continue to assume that the base change of G to F_∞ has a maximal torus which is compact modulo the center. Let $\mathbb{A} = \mathbb{A}_F$ denote the adèle ring of F. We fix moreover the following data:

- A finite set S of finite places,
- an irreducible, finite-dimensional, algebraic representation ξ of $G(F_\infty)$,
- an automorphic character $\chi : Z(F)\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ with $\chi|_{Z(F_\infty)} = \chi_\xi$, the central character of ξ, and
- an open compact subgroup $\Gamma \leq G(\mathbb{A}^S)$ such that χ is trivial on $\Gamma \cap Z(\mathbb{A}^S)$.

Let $\Pi_{\mathrm{disc}}(G, \chi)$ (resp. $\Pi_{\mathrm{cusp}}(G, \chi)$) denote the set of discrete (resp. cuspidal) automorphic $G(\mathbb{A})$ representations with central character χ.

Definition 3.3.0.8. *Fix the above data, and define the* counting measure $\widehat{\mu}_\Gamma$ *on* $\Pi(G(F_S), \chi_S)$ *(with respect to ξ) as follows: for a subset* $A \subseteq \Pi(G(F_S), \chi_S)$, *set*

$$\widehat{\mu}_\Gamma^{\mathrm{disc}}(A) = \frac{(-1)^{q(G)} \cdot \mathrm{vol}(\Gamma Z/Z)}{\dim(\xi) \cdot \mathrm{vol}(G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}))} \cdot \sum_{\pi \in \Pi_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \cdot \chi_{\mathrm{EP}}(\pi \otimes \xi^\vee) \cdot \dim(\pi^{S,\infty})^\Gamma \cdot \mathbf{1}_A(\pi_S).$$

Define μ^{cusp} *similarly, but with* m_{cusp} *replacing* m_{disc}.

If the highest weight of ξ is regular then $\mu_{\mathrm{cusp}} = \mu_{\mathrm{disc}}$ by Proposition 3.2.0.6.

Definition 3.3.0.9. *Let F, G, S, ξ, χ be as above, and let $\{\Gamma_\lambda\}_{\lambda \geq 0}$ be a sequence of open compact subgroups of $G(\mathbb{A}^S)$. We say that $\{\Gamma_\lambda\}$ satisfies* Plancherel equidistribution *with respect to ξ if the following hold:*

- *Whenever A is a bounded subset of $\Pi(G(F_S), \chi_S)$ that does not intersect the tempered spectrum $\Pi^t(G(F_S), \chi_S)$, we have*

$$\lim_{\lambda \to \infty} \widehat{\mu}_{\Gamma_\lambda}(A) = 0.$$

- *Whenever A is a Jordan-measurable subset of $\Pi_t(G(F_S), \chi_S)$, we have*

$$\lim_{\lambda \to \infty} \widehat{\mu}_{\Gamma_\lambda}(A) = \widehat{\mu}^{\mathrm{pl}}_{\chi_S}(A).$$

It is worth noting some discrepancies between our notation (which follows that of [Shi12] and [ST12]), and that of [FL13], [FLM14], [FL15]. In the latter three papers, Finis, Lapid, and Mueller consider the *limit multiplicity problem*, which differs from our definition in one important respect: instead of fixing an algebraic representation at ∞, they allow S to contain the infinite places and as such consider a more general set of representations at ∞. This makes their work more general, but forces them to work with more difficult versions of the trace formula, and the asymptotic vanishing of these trace formula terms is still unproven and depends on some analytic prerequisites. In contrast, in the formulation of Shin and Templier, we may use the trace formula on test functions whose infinite components are Euler-Poincare functions (see 3.7), which considerably simplifies the formula. Moreover, we are ultimately interested in fields of rationality and an important source of discrete automorphic representations with finite fields of rationality are those that are cohomological with respect to certain algebraic representations. It seems reasonable that our results could be replicated in a more general setting using more difficult versions of the trace formula after the analytic difficulties are resolved, but we have not examined this problem.

Remark 3.3.0.10. *The definitions of the Plancherel measure and our counting measures both depend on a choice of Haar measure on the group $G(\mathbb{A})$; in the statement of Plancherel equidistribution we assume that we have made the same choice of Haar measure on each side. Throughout the remainder of the paper, we will make the following choice of Haar measure. At the infinite places, we use the Euler-Poincaré measure. In the case where $G = U(n)$ or GL_n, we'll pick Haar measures on $G(F_\mathfrak{p})$ so that $\mathbf{K}_\mathfrak{p} Z/Z$ has measure 1, where $\mathbf{K}_\mathfrak{p}$ is as defined in Subsection 2.4.*

3.4 The trace formula and density theorem

In this section, we discuss two important steps that go into the proof of Plancherel equidistribution: a density theorem of Sauvageot and Arthur's trace formula.

Let $\widehat{f}_S$ be an arbitrary function on $\Pi(G(F_S), \chi_S)$. Then

$$\begin{aligned}\widehat{\mu}_\Gamma(\widehat{f}_S) &= \int_{\Pi(G(F_S),\chi_S)} \widehat{f}_S(\pi)\, d\mu_\Gamma(\pi) \\ &= \frac{(-1)^{q(G)} \cdot \mathrm{vol}(\Gamma Z/Z)}{\dim(\xi) \cdot \mathrm{vol}(G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}))} \\ &\quad \cdot \sum_{\pi \in \Pi_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \cdot \chi_{\mathrm{EP}}(\pi \otimes \xi^\vee) \cdot \dim(\pi^{S,\infty})^\Gamma \cdot \widehat{f}_S(\pi_S)\end{aligned}$$

and similarly $\widehat{\mu}^{\mathrm{pl}}(\widehat{f}_S)$ is the integral of $\widehat{f}_S$ with respect to the Plancherel measure on

$\Pi(G(F_S), \chi_S)$.

The following theorem is due to Sauvageot (see Proposition 7.1 of [Sau97]) in the non-fixed central character case. The extension to the fixed-central-character case follows using only abelian Fourier analysis, as in Lemma 11.12.7 of [Bin15].

Theorem 3.4.0.11. *[Sau97]. Let $\widehat{f}_S : \Pi(G(F_S), \chi_S) \to \mathbb{C}$ be bounded, have bounded support, and be discontinuous on a set of Plancherel measure zero. Fix $\epsilon > 0$. Then there are functions ϕ_S, ψ_S in the Hecke algebra $\mathcal{H}(G(F_S), \chi_S)$ such that*

- $|\widehat{f}_S(\pi) - \widehat{\phi}_S(\pi)| \leq \widehat{\psi}_S(\pi)$ *for all* $\pi \in \Pi(G(F_S), \chi_S)$, *and*
- $\widehat{\mu}^{\mathrm{pl}}_{\chi_S}(\psi_S) < \epsilon$.

Corollary 3.4.0.12. *Fix ξ and χ. If, for every function $\phi_S \in \mathcal{H}(G(F_S), \chi_S)$ we have*

$$\lim_{\lambda \to \infty} \widehat{\mu}_{\Gamma_\lambda}(\phi_S) = \widehat{\mu}^{\mathrm{pl}}(\phi_S)$$

then Plancherel equidistribution holds for the sequence $\{\Gamma_\lambda\}$.

Proof. This argument is by now standard: see Proposition 1.3 of cite [Sau97], Corollary 9.2 of [ST12], Section 2 of [FLM14], or Theorem 9.0.3 of [Bin15]. □

To prove Plancherel equidistribution, we will also need the (fixed-central-character) trace formula. Following [Art89], we will give a user-friendly version which applies to test functions of the form $\phi = \phi^\infty \cdot \phi_\xi$, where ϕ_ξ is an Euler-Poincare function. We'll need a definition:

Definition 3.4.0.13. *Let G/F be a reductive group. At each finite place, let $\mathbf{K}_\mathfrak{p}$ denote a special maximal compact subgroup (that is hyperspecial at all places where G is unramified; see 3.3.0.10), and let $\mathbf{K}^\infty = \prod_\mathfrak{p} \mathbf{K}_\mathfrak{p}$. Let P be a parabolic subgroup with Levi decomposition $P = MN$ and let $\gamma \in M$. If $\phi^\infty : G(\mathbb{A}^\infty) \to \mathbb{C}$ is locally constant and compactly-supported modulo the center, define the* constant term

$$\phi^\infty_M(\gamma) = \int_{\mathbf{K}^\infty} \int_{N(\mathbb{A}^\infty)} \phi^\infty(k^{-1}\gamma n k)\, dn\, dk.$$

Moreover, if $\phi_M^\infty : M(\mathbb{A}^\infty) \to \mathbb{C}^\times$ *is locally constant and compactly supported modulo the center, and* $\gamma \in M(\mathbb{A}^\infty)$, *let* M_γ *denote the identity component of the centralizer of* γ *in* M. *Define the* orbital integral

$$O_\gamma(\phi_M^\infty) = \int_{M_\gamma(\mathbb{A}^\infty)\backslash M(\mathbb{A}^\infty)} \phi_M^\infty(m^{-1}\gamma m)\, dm.$$

Definition 3.4.0.14. *Let* $\phi = \phi^\infty \phi_\xi$, *and let* $\chi : Z(\mathbb{A}) \to \mathbb{C}^\times$ *be such that* $\chi_\infty = \chi_\xi$. *The* geometric expansion *of the trace formula is*

$$I_{\text{geom}}(\phi^\infty, \phi_\xi, \chi) = \sum_{M \geq M_0} \sum_{\gamma \in M(F)/\sim} C(M, \xi, \gamma) \cdot O_\gamma(\phi_M^\infty).$$

Here the outer sum runs over the set of cuspidal Levi subgroups containing a fixed minimal Levi subgroup M_0. *The inner sum runs over representatives of equivalence classes of semisimple elements of* $M(F)$, *where* $\gamma \sim \gamma'$ *if* γ *is conjugate to* $z\gamma'$ *for some* $z \in Z(F)$.

We have $C(G, \xi, 1_G) = \dim(\xi)\operatorname{vol}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$. *The exact values of the other constants* $C(M, \xi, \gamma)$ *are unnecessary for our purposes: we invite the reader to see the explanation after (4.3) of [Shi12].*

The spectral expansion $I_{\text{spec}}(\phi^\infty, \phi_\xi, \chi)$ *is*

$$\begin{aligned} &(-1)^{q(G)} \sum_{\chi_\pi = \chi} m_{\text{disc}}(\pi) \cdot \operatorname{tr} \phi_\xi(\pi_\infty) \cdot \widehat{\phi}^\infty(\pi^\infty) \\ &= (-1)^{q(G)} \sum_{\chi_\pi = \chi} m_{\text{disc}}(\pi) \cdot \chi_{\text{EP}}(\pi_\infty \otimes \xi^\vee) \cdot \widehat{\phi}^\infty(\pi^\infty) \end{aligned}$$

here the sum runs over the set of discrete automorphic representations π *with central character* χ.

If the highest weight of ξ is regular, then $\operatorname{tr} \phi_\xi(\pi_\infty)$ is zero unless π_∞ is a discrete-series representation that is ξ-cohomological; in this case, $\operatorname{tr} \phi_\xi(\pi_\infty) = (-1)^{q(G)}$. Moreover, in this situation, all ξ-cohomological discrete automorphic representations are cuspidal, so we may replace the m_{disc} with m_{cusp} in the definition of I_{spec}.

Theorem 3.4.0.15 ([Art89], Theorem 6.1). *Assume $\phi = \phi^\infty\phi_\xi \in \mathcal{H}(G(\mathbb{A}), \chi)$. Then*

$$I_{\text{geom}}(\phi^\infty\phi_\xi, \chi) = I_{\text{spec}}(\phi^\infty\phi_\xi, \chi).$$

Proof. The non-fixed central character version of this is precisely Theorem 6.1 of [Art89]. The fixed-central-character version can be derived using abelian Fourier analysis. □

Proposition 3.4.0.16. *Fix G, F, S as above. Let $\mathbf{K} = \mathbf{K}^{S,\infty}$ be a fixed maximal compact subgroup of $G(\mathbb{A}^{S,\infty})$; assume it is maximal special everywhere and hyperspecial whenever $G_\mathfrak{p}$ is unramified. Let $\{\Gamma_\lambda\}$ be a sequence of open compact subgroups of $\mathbf{K}$. Assume Γ_λ decomposes as a product of local open compact subgroups $\Gamma_{\lambda,\mathfrak{p}}$ and that χ is trivial on $\Gamma_\lambda \cap Z(\mathbb{A})$.*

Assume for any pair (M, γ) with $\gamma \in M(F)$, $M \neq G$, or $\gamma \neq 1$, we have

$$O_\gamma(\mathbf{1}^M_{\Gamma_\lambda \cdot Z}) \to 0 \text{ as } \lambda \to \infty.$$

Then Plancherel equidistribution holds for $\mu^{\text{disc}}_{\Gamma_\lambda}$ with respect to ξ. If the highest weight of ξ is regular, then it also holds for $\mu^{\text{cusp}}_{\Gamma_\lambda}$.

Proof. At each finite place $\mathfrak{p} \notin S$, let $e_{\Gamma,\mathfrak{p}}$ denote the idempotent corresponding to $\Gamma_\mathfrak{p}$ in the Hecke algebra $\mathcal{H}(G(F_\mathfrak{p}))$, and let $e_{\Gamma,\mathfrak{p},\chi}$ be the image of $e_{\Gamma,\mathfrak{p}}$ under the averaging map $\mathcal{H}(G(F_\mathfrak{p})) \to \mathcal{H}(G(F_\mathfrak{p}), \chi)$. Then $e_{\Gamma,\mathfrak{p},\chi}(1) = \text{vol}(\Gamma Z/Z)^{-1}$ and $\widehat{e}_{\Gamma,\mathfrak{p},\chi}(\pi_\mathfrak{p}) \dim \pi_\mathfrak{p}^\Gamma$. Let $\phi_\lambda^{S,\infty} = \prod_{\mathfrak{p}\notin S} e_{\Gamma_\lambda,\mathfrak{p},\chi}$.

Let $\phi_S \in \mathcal{H}(G(F_S), \chi_S)$ be an arbitrary test function, and let ϕ_λ denote the test function $\phi_S\phi_\lambda^{S,\infty}\phi_\xi$. Then

$$\begin{aligned}\mu_{\Gamma,\chi}(\widehat{\phi}_S) &= \frac{\text{vol}(\Gamma Z/Z)}{\dim(\xi)\cdot \text{vol}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A}))} I_{\text{spec}}(\phi_S\phi_\lambda^{S,\infty}\phi_\xi, \chi) \\ &= \frac{\text{vol}(\Gamma Z/Z)}{\dim(\xi)\cdot \text{vol}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A}))} I_{\text{geom}}(\phi_S\phi_\lambda^{S,\infty}\phi_\xi, \chi)\end{aligned}$$

We first note

$$|\operatorname{vol}(\Gamma_\lambda Z/Z)\phi_\lambda^{S,\infty}(g)| = \begin{cases} 1 & g \in \Gamma_\lambda Z \\ 0 & \text{otherwise} \end{cases}$$

so in particular

$$\operatorname{vol}(\Gamma Z/Z)|O_\gamma(\phi_\lambda^{S,\infty,M})| \le O_\gamma(\mathbf{1}^M_{\Gamma_\lambda Z}).$$

Moreover, each of these functions is bounded above by $O_\gamma(\mathbf{1}^M_{\mathbf{K}Z})$, and because there are only finitely many similarity classes $[\gamma] \in M(F)$ with γ_∞ elliptic, $\gamma_S \in \operatorname{supp}(\phi_S^M)$, $\gamma^{S,\infty} \in \mathbf{K}^S$, we have

$$\begin{aligned}
&\lim_{\lambda\to\infty} \operatorname{vol}(\Gamma_\lambda Z/Z)\cdot I_{\text{geom}}(\phi_S\phi_\lambda^{S,\infty}\phi_\xi,\, \chi) \\
&\qquad = \operatorname{vol}(\Gamma_\lambda Z/Z) \sum_{M\ge M_0} \sum_{\gamma\in M(F)/\sim} C(M,\, \xi,\, \gamma)\left(\lim_{\lambda\to\infty} O_\gamma(\phi_S^M)\cdot O_\gamma(\phi_\infty^M)\cdot O_\gamma(\phi_\lambda^{S,\infty,M})\right) \\
&\qquad = C(G,\, \xi,\, 1)\phi_S(1)\phi_\infty(1)
\end{aligned}$$

Finally, since

$$\widehat{\mu}_{\Gamma,\chi}(\phi_S) = \frac{\operatorname{vol}(\Gamma_\lambda Z/Z)\cdot I_{\text{spec}}(\phi_S\phi_\infty\phi^{S,\infty})}{C(G,\, \xi,\, 1)\phi_\infty(1)} = \frac{\operatorname{vol}(\Gamma_\lambda Z/Z)\cdot I_{\text{geom}}(\phi_S\phi_\infty\phi^{S,\infty})}{C(G,\, \xi,\, 1)\phi_\infty(1)}$$

we have that $\lim_{\lambda\to\infty}\widehat{\mu}_{\Gamma_\lambda,\chi}(\phi_S) = \phi_S(1) = \widehat{\mu}^{\text{pl}}_\chi(\phi_S)$ completing the proof. □

We will use the following theorem of Shin and Templier (see Theorem 9.16 of [ST12]).

Theorem 3.4.0.17. *Fix ξ, S, and χ as above, and let $\{\mathfrak{n}_\lambda\}$ be a sequence of ideals, coprime to S, such that $N(\mathfrak{n}_\lambda) \to \infty$. Let $\Gamma(\mathfrak{n}_\lambda)$ be the full level subgroups defined in 2.4.0.5. Then the sequence $\{\Gamma(\mathfrak{n}_\lambda)\}$ satisfies Plancherel equidistribution.*

Proof. Since Shin and Templier do not fix the central character, we briefly run through the steps of the proof (all references in this proof are to [ST12]). First assume γ is not of the form zu, where $z \in Z$ and $u \in U$. Then we check, by passing to G^{ad} and using the result of Lemma 8.4, that if $\mathfrak{n}$ is small enough, then γ is not conjugate to any element of $Z \cdot \Gamma(\mathfrak{n})$. This tells us that the contribution from any $\gamma \neq 1$ eventually vanishes, for any Levi subgroup M. When $\gamma \in Z(F)$, we can assume $\gamma = 1$. Then we can bound these terms exactly as at the end of the proof of 9.16. □

In the next chapter, we will use 3.4.0.16 to prove Plancherel equidistribution in a number of situations. First, in the case $G = \mathrm{GL}_2$ we will give explicit bounds on trace formula terms in the case where $\Gamma = \Gamma_1(\mathfrak{n})$. We will also discuss some results of Finis-Lapid bounding trace formula terms and apply these bounds to the situation of $U(n)$ and GL_n.

3.5 Plancherel equidistribution for more general ξ

In this section, we briefly comment on the proof of Plancherel equidistribution in the situation where $G = \mathrm{GL}_2$ and ξ is not necessarily algebraic or, alternatively when its highest weight is not regular. The first situation holds when the central character of ξ is non-algebraic. The second situation occurs when ξ is one-dimensional; for instance, if f is a holomorphic cusp form of level 2 and $\pi_{\infty,f}$ is the real component of the associated cuspidal representation. Therefore, the case when $G = \mathrm{GL}_2$ and ξ does not have regular highest weight is of particular interest to the question of fields of rationality of cusp forms.

The first situation is particularly simple. In this case if π is an automorphic representation that is ξ-cohomological, then (a), (b), and (c) of 3.2.0.6 still hold; this is because ξ is a twist of an algebraic representation ξ' by an abelian character χ_∞. As such, the rest of the argument goes through exactly as follows.

In this second situation, we have shown that a vanishing of trace formula terms is enough to imply that

$$\lim_{\lambda} \widehat{\mu}^{\mathrm{disc}}_{\Gamma_\lambda,\chi}(A) = \widehat{\mu}^{\mathrm{pl}}_{\chi}(A)$$

when A is a Jordan measurable subset of the tempered spectrum, and that

$$\lim_{\lambda} \widehat{\mu}^{\mathrm{disc}}_{\Gamma_\lambda,\chi}(A) = 0$$

when A is disjoint from the tempered spectrum.

We wish to show the same is true that the same is true when $\widehat{\mu}^{\text{disc}}$ is replaced by $\widehat{\mu}^{\text{cusp}}$. For the first inequality, we note that the residual spectrum of GL_2 consists precisely of one-dimensional characters $\chi_0 \circ \det$, which are nowhere tempered. Therefore, if A is supported on the tempered spectrum we have

$$\widehat{\mu}^{\text{disc}}(A) = \widehat{\mu}^{\text{cusp}}(A).$$

If A is disjoint from the tempered spectrum, it's enough to show that the contribution from the residual spectrum is asymptotically zero. We note that the central character of $\chi_0 \circ \det$ is χ_0^2.

Lemma 3.5.0.18. *Given an ideal $\mathfrak{n}$ of F, let $P(\mathfrak{n})$ denote the number of primes dividing $\mathfrak{n}$. Fix a central character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. There is a constant C, depending only on F and χ, such that the number of characters χ_0 of conductor $\mathfrak{n}$ with $\chi_0^2 = \chi$ is bounded above by $C \cdot 2^{P(\mathfrak{n})}$.*

Proof. Let $\widehat{\mathfrak{o}}_F = \prod_{\mathfrak{p}} \mathfrak{o}_{F,\mathfrak{p}}$ and let $C_1 = |(F^\times \cdot \widehat{\mathfrak{o}}_F^\times) \backslash \mathbb{A}_F^\times|$. If we fix a character χ' on $\widehat{\mathfrak{o}}_F^\times$, there are at most C_1 automorphic characters of $\mathbb{A}_F^\times$ extending χ'.

As such, it's enough to count the number of characters χ_0 on $\widehat{\mathfrak{o}}_F^\times$ such that $\chi_0^2 = \chi|_{\widehat{\mathfrak{o}}_F^\times}$. We first note that, at any prime $\mathfrak{p}$, the set

$$\{\chi_{0,\mathfrak{p}} : \mathfrak{o}_{F,\mathfrak{p}}^\times \to \mathbb{C}^\times \mid \chi_{0,\mathfrak{p}}^2 = \chi_{\mathfrak{p}}|_{\mathfrak{o}_{F,\mathfrak{p}}^\times}\}$$

is finite. Let its cardinality be $C_{2,\mathfrak{p}}$. Let C_2 be the product of the $C_{2,\mathfrak{p}}$ at all the finite places $\mathfrak{p}$ such that either $\mathfrak{p} \mid 2$ or $\chi_{\mathfrak{p}}$ is ramified.

Assume $\chi_{\mathfrak{p}}$ is unramified and $\mathfrak{p} \nmid 2$, and assume $\chi_{0,\mathfrak{p}}^2 = \chi_{\mathfrak{p}}$. Then $\chi_{0,\mathfrak{p}}$ must be trivial on $1 + \mathfrak{p}$ since this is a pro-p-group for some $p \neq 2$. Therefore $\chi_{0,\mathfrak{p}}^2$ factors through $\mathfrak{o}_{F,\mathfrak{p}}^\times/(1 + \mathfrak{p})$. Since this group is cyclic, there are only two possible square roots of $\chi_{\mathfrak{p}}$.

Therefore, the number of square roots χ_0 of χ with conductor $\mathfrak{n}$ is bounded above by $C_1 C_2 2^{P(\mathfrak{n})}$ as desired. $\square$

Finally, we can prove:

Proposition 3.5.0.19. *Let $G = \mathrm{GL}_2$ and let ξ be* any *irreducible finite-dimensional representation of $\mathrm{GL}_2(F_\infty)$. Fix a central character $\chi : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ with $\chi_\infty = \chi_\xi$, a finite set S of finite places. Let Γ_λ be either:*

- *A full level subgroup $\Gamma(\mathfrak{n}_\lambda)$; or*
- *a subgroup $K_2(\mathfrak{n}_\lambda)$.*

Then Plancherel equidistribution for the sequence of measures $\{\widehat{\mu}^{\mathrm{disc}}_{\Gamma_\lambda}\}$ implies it for $\{\widehat{\mu}^{\mathrm{cusp}}_{\Gamma_\lambda}\}$.

Proof. If the highest weight of ξ is regular, the proof is already finished. Otherwise, it's enough to show that, asymptotically, the contribution of the residual spectrum to the family

$$\mathcal{F}_{\mathrm{disc}}(\chi, \xi, \Gamma_\lambda)$$

is asymptotically zero. By Plancherel equidistribution, the $|\mathcal{F}_{\mathrm{disc}}(\chi, \xi, \Gamma_\lambda)|$ grows as $\mathrm{vol}(Z\Gamma_\lambda/Z)^{-1}$, which is polynomial in $N(\mathfrak{n}_\lambda)$. However, the residual spectrum contribution comes from the one-dimensional characters χ_0 with $\chi_0^2 = \chi$ and whose conductor divides $\mathfrak{n}_\lambda$. We have already shown that the number of such characters is bounded above by $C \cdot 2^{P(\mathfrak{n}_\lambda)} = o(N(\mathfrak{n}_\lambda))$. For each such character we have

$$\dim\left(\chi_0 \circ \det\right)^{\Gamma_\lambda} = 1.$$

Finally, $\chi_{0,\infty}$ takes one of two values, so the Euler characteristic of $(\chi_0 \circ \det) \otimes \xi^\vee$ is bounded. Therefore the multiplicity of $\chi_0 \circ \det$ in $\mathcal{F}_{\mathrm{disc}}$ is bounded above, completing the proof. $\square$

Chapter 4

Bounds on trace formula terms

In this chapter, will give bounds on the trace formula terms for certain sequences of subgroups in the case where $G = \mathrm{GL}_n$ or $G = U(n)$. We'll begin by using the (reduced) Bruhat-Tits building for GL_2 to give very explicit computations of trace formula terms for the subgroups $\Gamma_0(\mathfrak{p}^r) \leq \mathrm{GL}_2(F_\mathfrak{p})$; this will allow us to determine a Plancherel equidistribution result for classical cusp forms. We will also discuss results of Finis-Lapid that give somewhat-less-explicit bounds for open-compact subgroups of $U(n)$ in terms of the *level* of the subgroup.

We remark here that the bounds of Finis-Lapid are more general than ours and certainly good enough to imply Plancherel equidistribution for Γ_0 subgroups of GL_2. We hope that our explicit bounds will not be redundant for two reasons: first, they give an explicit bound on the trace formula terms in question. Second, the geometric methods used may be applicable, though perhaps with some difficulty, to a larger class of reductive groups.

4.1 Bounds on trace formula terms for $\Gamma_0(\mathfrak{p}^r) \leq \mathrm{GL}_2(F_\mathfrak{p})$

Let T denote the diagonal torus of $\mathrm{GL}_2(F_\mathfrak{p})$; this is a minimal Levi subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$, and there are only two Levi subgroups containing it: T and $\mathrm{GL}_2(F_\mathfrak{p})$. Therefore, there are precisely two types of trace formula terms:

- *Orbital integrals*
$$O_\gamma(f) = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\,dg$$
for semisimple $\gamma \in F$, and

- *constant terms*
$$Q_\gamma(f) = \int_{\mathbf{A}^\infty}\int_{\mathbf{K}_\mathfrak{p}^\infty} f\left(k^{-1}\gamma\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}k\right)\,dk\,da$$
for diagonal elements $\gamma \in T(F)$.

The goal of this section is to bound these terms when f is the characteristic function of an open subgroup $Z \cdot \Gamma_0(\mathfrak{n})$. We will begin by computing local orbital integrals and constant terms and then summarize the global consequences in Subsection 4.1.3.

Throughout, we will choose measures on $G = \mathrm{GL}_2(F_\mathfrak{p})$, T the diagonal torus, N the subgroup of upper-triangular unipotent matrices, and $\mathbf{K}_\mathfrak{p} = \mathrm{GL}_2(\mathfrak{o}_{F,\mathfrak{p}})$ so that maximal compact subgroups are given measure 1; this also ensures that $dg = dt\,dn\,dk$ under the Iwasawa decomposition $G = TN\mathbf{K}_\mathfrak{p}$.

The key tool will be an analysis of the Bruhat-Tits tree for SL_2. We recall a definition:

Definition 4.1.0.1. *Consider the p-adic field $F_\mathfrak{p}$. The* Bruhat-Tits tree X *of* $\mathrm{SL}_2(F_\mathfrak{p})$ *is a graph consisting of the following data:*

- *The set of vertices is the set of equivalence classes of rank-two lattices $\Lambda \subseteq F_\mathfrak{p}^2$, with $\Lambda \sim \Lambda'$ if they differ by a scalar multiple.*
- *Two equivalence classes $[\Lambda]$, $[\Lambda']$ are adjacent if and only if there are lattices $\Lambda \in [\Lambda]$, $\Lambda' \in [\Lambda']$ such that $\Lambda \supsetneq \Lambda' \supsetneq \varpi \cdot \Lambda$.*

We briefly recall some facts:

1. The degree of every vertex $v \in X$ is $q+1$. To see this, fix a lattice Λ. If $\Lambda' \subset \Lambda$ with index q, then $\varpi\Lambda \subset \Lambda' \subset \Lambda$, and so Λ' corresponds uniquely to a one-dimensional subspace in $\Lambda/\varpi\Lambda \cong \mathbb{F}_q^2$. On the other hand, if $\Lambda' \supset \Lambda$

with index q, then Λ' is equivalent to $\varpi\Lambda'$, which is a sublattice of Λ of index q. Moreover, if $\Lambda_1 \sim \Lambda$ then all index-q sublattices of Λ_1 are equivalent to an index-q sublattice of Λ.

2. X is a tree [Ser80, Theorem 1].

The action of $\mathrm{GL}_2(F_\mathfrak{p})$ on the set of lattices in $F_\mathfrak{p}^2$ descends to an action on X by graph automorphisms.

Let $\{e_1, e_2\}$ be the standard basis of $F_\mathfrak{p}^2$. X has a distinguished line A_0 whose vertices correspond to the lattices with bases $\{e_1, \varpi^i e_2\}$; this is known as the *standard apartment*. For given $g \in \mathrm{GL}_2(F_\mathfrak{p})$, $A = g \cdot A_0$ is called an *apartment*. Given a vertex w and an apartment A, let $d(w, A)$ be the distance from w to A. Because X is a tree, there is a unique vertex $w' \in A$ such that $d(w, A) = d(w, w')$; we define $b_A(w) = w'$.

By the Iwasawa decomposition, every vertex has an associated lattice Λ with basis $\{e_1, ae_1 + \varpi^s e_2\}$, where $s \in \mathbb{Z}$ and $a \in F_\mathfrak{p}$. We denote this vertex by $w_{a,s}$. Note that $w_{a,s} = w_{a',s'}$ if and only if $s = s'$ and $a - a' \in \mathfrak{o}_{F,\mathfrak{p}}$, and that $w_{a,s} \in A_0$ if and only if $a \in \mathfrak{o}_{F,\mathfrak{p}}$. It is elementary to check by induction that if $a \notin \mathfrak{o}_{F,\mathfrak{p}}$ then $d(w_{a,s}, A_0) = -v_P(a)$ and that $b_{A_0}(w_{a,s}) = w_{0,s-v(a)}$; this follows because $w_{a,s}$ is adjacent to $w_{\varpi\cdot a,s+1}$.

We say a set of vertices $\{w_0, \ldots, w_r\}$ is a *segment* (of length r) if $d(w_i, w_j) = i - j$ for all $0 \le i, j \le r$.

We have the following:

Lemma 4.1.0.2. *Let $\gamma \in \mathrm{GL}_2(F_\mathfrak{p})$. Then $\gamma \in Z \cdot \Gamma_0(\mathfrak{p}^r)$ if and only if γ fixes the length-r segment $S_r = \{w_{0,0}, w_{0,1}, \ldots, w_{0,r}\}$.*

Moreover, $g^{-1}\gamma g \in Z \cdot \Gamma_0(\mathfrak{p}^r)$ if and only if γ fixes $g \cdot S_r$.

Proof. The second statement follows from the first. To prove the first, a quick computation yields that γ fixes the lattice Λ_i if and only if $\gamma \in Z \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi^i \end{smallmatrix}\right)^{-1} \mathbf{K}_\mathfrak{p} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi^i \end{smallmatrix}\right)$. The intersection of such subgroups from $i = 0$ to $i = r$ is $Z \cdot \Gamma_0(\mathfrak{p}^r)$. □

4.1.1 Computation of constant terms

Let $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right)$; we wish to compute the constant term $Q_t(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)})$. We begin with a lemma:

Lemma 4.1.1.1. *Let $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in \mathbf{K}_\mathfrak{p}$ and let $w \in X$ be a vertex. Then t fixes w if and only if $d(w,\, A_0) \leq v_\mathfrak{p}(t_1 - t_2)$.*

Proof. Write $w = w_{a,s}$, and note that t fixes $w_{a,s}$ if and only if

$$\begin{pmatrix} t_1 & (t_1 - t_2)a \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & \varpi^s \end{pmatrix}^{-1} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & \varpi^s \end{pmatrix} \in \mathbf{K}_\mathfrak{p} \cdot Z$$

which occurs if and only if $(t_1 - t_2)a \in \mathfrak{o}_{F,\mathfrak{p}}$.

Since $d(w_{a,s},\, A_0) = -v_\mathfrak{p}(a)$, this completes the proof. ☐

Proposition 4.1.1.2. *Let $t_1 \neq t_2 \in \mathfrak{o}_{F,\mathfrak{p}}^\times$. Then*

$$Q_\gamma(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)}) \leq \begin{cases} 1 & r \leq v_\mathfrak{p}(t_1 - t_2) \\ 2q^{v_\mathfrak{p}(t_1 - t_2)} \operatorname{vol}(\Gamma_0(\mathfrak{p}^r)) & r > v_\mathfrak{p}(t_1 - t_2). \end{cases}$$

Proof. Fix a strictly upper-triangular matrix n and note that for any $k \in \mathbf{K}_\mathfrak{p}$ we can only have $k^{-1}tnk \in \mathbf{K}_\mathfrak{p}$ if $n \in \mathbf{K}_\mathfrak{p}$. Since $t_1 \neq t_2$, there is a g so that $g^{-1}tng = t$ and therefore the set X^{tn} of vectors fixed by tn is of the form $g \cdot X^t$. If $A = g \cdot A_0$, then $w \in X^{tn}$ if and only if $d(w,\, A) \leq v_\mathfrak{p}(t_1 - t_2)$.

Fix $n \in N \cap \mathbf{K}_\mathfrak{p}$; we have $k^{-1}tnk \in Z \cdot \Gamma_0(\mathfrak{p}^r)$ if and only if the segment $k \cdot S_r \subset X^{tn}$. We note that the initial vertex of $k \cdot S_r$ is $w_{0,0}$; we will show that there number of such segments contained in X^{tn} is at most $[\mathbf{K}_\mathfrak{p} : \Gamma_0(\mathfrak{p}^r)]$ if $r \leq v(t_1 - t_2)$, and is at most $2q^{v_\mathfrak{p}(t_1 - t_2)}$ otherwise. The first statement is clear by counting the total number of segments of length r with a given initial point.

For the second case, we note the following: since X is a tree, if $S = (w_0, \ldots, w_\ell)$ is a segment with $d(w_1,\, A) > d(w_0,\, A)$, then $d(w_{i+1},\, A) > d(w_i,\, A)$ for all i. As such, if $k \cdot S_r$ is a segment contained in X^{tn}, then for all $1 \leq i \leq r - v_\mathfrak{p}(t_1 - t_2)$, we have $d(w_i, A) \leq d(w_{i-1}, A)$. As such, we claim that there are at most $2q^{v_\mathfrak{p}(t_1-t_2)}$ segments of the form $k \cdot S_r = \{w_0', \ldots, w_r'\}$ contained in X^{tn}. Because $k \in \mathbf{K}_\mathfrak{p}$, we have $w_0' = w_0$. For each $1 \leq i \leq r - v_\mathfrak{p}(t_1 - t_2)$, if $w_{i-1} \notin A$, then w_i is the unique neighbor of w_{i-1} with $d(w_i, A) < d(w_{i-1}, A)$. If the $w_{i-1} \in A$ and $w_{i-2} \notin A$, then w_i must be one of the two neighbors of w_{i-1} in A. Finally, if $w_{i-1}, w_{i-2} \in A$, then w_i must be the other neighbor of w_{i-1} in A. Finally, if $i > r - v_\mathfrak{p}(t_1 - t_2)$, then w_i can be any of the q neighbors of w_{i-1} which are not equal to w_{i-2}. This completes the proof of the claim.

Therefore, for any $n \in N \cap \mathbf{K}_\mathfrak{p}$ we have

$$\int_{\mathbf{K}_\mathfrak{p}} \mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)}(k^{-1}tnk)\,dk \leq \begin{cases} 1 & r \leq v_\mathfrak{p}(t_1 - t_2) \\ 2q^{v_\mathfrak{p}(t_1-t_2)}\,\mathrm{vol}(\Gamma_0(\mathfrak{p}^r)) & r > v_\mathfrak{p}(t_1 - t_2). \end{cases}$$

and so integrating over $n \in N \cap \mathbf{K}_\mathfrak{p}$ completes the proof. □

We will also need to compute the constant term $Q_z(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)})$ for a central element z.

Proposition 4.1.1.3. *Let $z \in Z(F_\mathfrak{p})$. Then*

$$Q_z(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)}) = \begin{cases} \frac{2}{q+1}q^{-k} & r = 2k+1 \\ q^{-k} & r = 2k. \end{cases}$$

In particular, $Q_z(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)}) \leq q^{-r/2}$.

Proof. We can assume that $z = 1$ and once again find the fixed subspace X^n for $n = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathbf{K}_\mathfrak{p}$. Let $w_{a,s}$ be as in the beginning of the section. Since

$$\begin{pmatrix} 1 & a \\ 0 & \varpi^s \end{pmatrix}^{-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & \varpi^s \end{pmatrix} = \begin{pmatrix} 1 & b\varpi^s \\ 0 & 1 \end{pmatrix}$$

we see that $w_{a,s} \in X^n$ if and only if $s \geq -v(b)$. In particular, if $b_{A_0}(w) = w_{0,s}$, then $d(w, w_{0,s}) \leq s + v(b)$. Alternatively, X^n is the union of balls of radius $s + v(b)$ around

$w_{0,s} \in A_0$, for $s \geq -v(b)$.

For fixed n, the volume of the set

$$\{k \in \mathbf{K}_\mathfrak{p} : k^{-1}nk \in Z \cdot \Gamma_0(\mathfrak{p}^r)\}$$

is the product of $\mathrm{vol}(\Gamma_0(\mathfrak{p}^r))$ with the number of segments $\{w_0, \ldots, w_r\}$ whose basepoint is $w_0 = w_{0,0}$ and which are contained in X^n. Let $n = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. If $v_\mathfrak{p}(b) \geq r$ then all length-r segments with basepoint w_0 are contained in X^n, so the total volume is 1. If $r > v_\mathfrak{p}(b)$, then for any $i \leq \lceil \frac{r-v(b)}{2} \rceil$ we must have $w_i = w_{0,i}$; for each subsequent step there are q choices, so the total number of segments contained in X^n is $q^{\lfloor \frac{v(b)}{2} \rfloor}$.

As such, we compute

$$\int_N \int_{\mathbf{K}_\mathfrak{p}} \mathbf{1}_{Z \cdot \Gamma_0(\mathfrak{p}^r)}(k^{-1}nk)\, dk\, dn = q^{-r} + \frac{1}{q^{r-1}(q+1)} \sum_{j=0}^{r-1} (q-1) q^{-j-1} q^{\lfloor j/2 \rfloor}.$$

An elementary computation using induction shows that this is equal to the quantity stated. $\square$

4.1.2 Computation of orbital integrals

The goal of this section is to prove:

Proposition 4.1.2.1. *Let γ be a non central, semisimple element of* $\mathrm{GL}_2(F_\mathfrak{p})$. *Then*

$$O_\gamma(\mathbf{1}_{Z \cdot \Gamma_0(\mathfrak{p}^r)}) \leq 2 \cdot \mathrm{vol}(\Gamma_0(\mathfrak{p}^r)) \cdot O_\gamma(\mathbf{1}_{Z \cdot \mathbf{K}_\mathfrak{p}})^2.$$

We'll break this into two cases: the case where γ is elliptic, and the case where γ is non-elliptic.

Lemma 4.1.2.2. *If γ is elliptic and noncentral then the set X^γ is finite.*

Proof. We can compute the fixed set directly, assuming $\gamma \in \mathbf{K}_\mathfrak{p}$ by conjugating and multiplying by an element of the center. If γ is elliptic then it is conjugate to a matrix of the form

$$\begin{pmatrix} x & y \\ \alpha y & x \end{pmatrix}$$

where α is either a unit that is not a square, or α is a uniformizer.

If α is a unit, then the X^γ is the single point $\{w_{0,0}\}$. If α is a uniformizer, then X^γ consists of those vertices w with $d(w, S_1) \leq v(y)$, where S_1 is the length-one segment $\{w_{0,0}, w_{0,1}\}$. In either case, X^γ is finite. □

We will now prove Proposition 4.1.2.1.

Proof of Proposition 4.1.2.1. Assume first that γ is elliptic, and by conjugating assume $\gamma \in \Gamma_0(\mathfrak{p}^r)$. Then $O_\gamma(\mathbf{1}_{Z\cdot\mathbf{K}_\mathfrak{p}})$ is the cardinality of X^γ. As such, for a given length r, there are at most $O_\gamma(\mathbf{1}_{Z\cdot\mathbf{K}_\mathfrak{p}})^2$ segments of length r contained in X^γ since each segment is determined uniquely by its two endpoints. For a given segment S'_r, the volume of the set $\{g \in G_\gamma\backslash \mathrm{GL}_2(F_\mathfrak{p}) : g \cdot S_r = S'_r\}$ is $\mathrm{vol}(\Gamma_0(\mathfrak{p}^r))$. This finishes the proof when γ is elliptic.

If γ is diagonalizable, we can assume $\gamma = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in \mathbf{K}_\mathfrak{p}$. In this case, [van72, Lemma 9] tells us that

$$O_\gamma(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)}) = |D_T^G(\gamma)|_\mathfrak{p}^{-1/2} Q_\gamma(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)})$$

where $D_M^G(\gamma)$ is the determinant of $1 - \mathrm{Ad}(\gamma)$ acting on $\mathrm{Lie}(G)/\mathrm{Lie}(T)$. In our situation we have

$$|D_T^G(\gamma)| = \left|\left(1 - \frac{t_1}{t_2}\right)\left(1 - \frac{t_2}{t_1}\right)\right| = |t_1 - t_2|^2.$$

Along with Proposition 4.1.1.2, this proves that $O_\gamma(\mathbf{1}_{Z\cdot\mathbf{K}_\mathfrak{p}}) = |t_1 - t_2|_\mathfrak{p}^{-1} = q^{v(t_1-t_2)}$. Applying these results to $\mathbf{1}_{Z\cdot\Gamma(\mathfrak{p}^r)}$ gives

$$O_\gamma(\mathbf{1}_{Z\cdot\Gamma_0(\mathfrak{p}^r)}) \leq 2 \cdot O_\gamma(\mathbf{1}_{Z\cdot\mathbf{K}_\mathfrak{p}})^2 \cdot \mathrm{vol}(\Gamma_0(\mathfrak{p}^r))$$

completing the proof. □

4.1.3 Summary of global consequences

We summarize the global consequences for use in subsequent sections below:

Proposition 4.1.3.1. *Let $\gamma \in \mathrm{GL}_2(F)$ be semisimple and let $\mathfrak{n} \subseteq \mathfrak{o}_F$ be an ideal. Then*

1. *If $\gamma \in Z(F)$, then*
$$Q_\gamma(\mathbf{1}_{Z(\mathbb{A}^\infty)\Gamma_0(\mathfrak{n})}) \leq N(\mathfrak{n})^{-1/2}$$

2. *If $\gamma = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T(F) - Z(F)$, then*
$$Q_\gamma(\mathbf{1}_{Z(\mathbb{A}^\infty)\Gamma_0(\mathfrak{n})}) \leq |N_{F/\mathbb{Q}}(t_1 - t_2)|_{\mathbb{R}} \cdot 2^{P(\mathfrak{n})} \cdot N(\mathfrak{n})^{-1}$$
where $P(\mathfrak{n})$ is the number of primes dividing $\mathfrak{n}$.

3. *If $\gamma \in \mathrm{GL}_2(F) - Z(F)$ is semisimple, then*
$$O_\gamma(\mathbf{1}_{Z(\mathbb{A}^\infty)\Gamma_0(\mathfrak{n})}) \leq O_\gamma(\mathbf{K}^\infty)^2 \cdot 2^{P(\mathfrak{n})} \cdot N(\mathfrak{n})^{-1}.$$

Proof. This follows from Propositions 4.1.1.2, 4.1.1.3, and 4.1.2.1 upon decomposing the orbital integrals and constant terms as a product of local orbital integrals and constant terms. □

Because $2^{P(\mathfrak{n})} \cdot N(\mathfrak{n})^{-1}$ decreases as $o(N(\mathfrak{n})^{-1+\epsilon})$ for every $\epsilon > 0$, we have the following

Corollary 4.1.3.2. *For every semisimple, noncentral $\gamma \in \mathrm{GL}_2(F)$ and every $\epsilon > 0$, there is a $C_{\epsilon,\gamma} > 0$ such that*
$$Q_\gamma(\mathbf{1}_{Z(\mathbb{A}^\infty)\Gamma_0(\mathfrak{n})}),\ O_\gamma(\mathbf{1}_{Z(\mathbb{A}^\infty)\Gamma_0(\mathfrak{n})}) < C_{\epsilon,\gamma} N(\mathfrak{n})^{-1+\epsilon}$$
for all ideas $\mathfrak{n} \subseteq \mathfrak{o}_{F,\mathfrak{p}}$.

4.2 General bounds for trace formula terms on $U(n)$

In this section, we will apply quite general computations from Section 5 of [FL15] to trace formula terms of $U(n)$. We first note some reductions. First, because we are working with a fixed-central-character version of the trace formula, we can pass from $U(n)$ to $U(n)^{\mathrm{ad}} = U(n)/Z$. Then the adjoint representation on the Lie algebra $\mathfrak{g}$ gives a natural faithful embedding $U(n)^{\mathrm{ad}} \hookrightarrow \mathrm{GL}_{n^2}$.

In section 2.4, we made a choice of a special maximal compact subgroup $\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$ at each finite prime $\mathfrak{p}$. For each $\mathfrak{p}$ we may pick a $\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$-stable lattice $\Lambda_{\mathfrak{p}}$ in the Lie algebra $\mathfrak{g}_{\mathfrak{p}}$. In our situation, we choose $\Lambda_{\mathfrak{p}}$ as follows: let $\mathscr{G}_{\mathfrak{p}}$ denote the group scheme over $\mathfrak{o}_{F,\mathfrak{p}}$ whose generic fiber is $U(n)$ and such that $\mathscr{G}_{\mathfrak{p}}(\mathfrak{o}_{F,\mathfrak{p}}) = \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$. Then there is a natural embedding $\mathrm{Lie}(\mathscr{G})(\mathfrak{o}_{F,\mathfrak{p}}) \hookrightarrow \mathrm{Lie}(\mathscr{G})(F_{\mathfrak{p}}) = \mathfrak{g}_{\mathfrak{p}}$. We define $\Lambda_{\mathfrak{p}}$ to be the image of this embedding, and $\Lambda_{\mathfrak{p}}$ is clearly $\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$-stable.

Definition 4.2.0.3 (Compare to page 37 of [FL13]). *Let $\mathbf{K}^{\infty} \leq U(n, \mathbb{A}^{\infty})$ be as in Section 2.4 and let $\mathbf{K}^{\infty,\mathrm{ad}}$ denote its image in the adjoint group. Let $\Gamma(\mathfrak{n})$ be the full level subgroup defined in 2.4.0.5, with image $\Gamma(\mathfrak{n})^{\mathrm{ad}} \leq \mathbf{K}^{\infty,\mathrm{ad}}$. Given a subgroup $K \leq \mathbf{K}^{\infty,\mathrm{ad}}$, we define the* level *of K to be the largest ideal $\mathfrak{n}$ such that $\Gamma(\mathfrak{n})^{\mathrm{ad}} \leq K$.*

We need a second definition. Let $x_{\mathfrak{p}} \in U(n)^{\mathrm{ad}}(F_{\mathfrak{p}})$ and set

$$\lambda_{\mathfrak{p}}(x_{\mathfrak{p}}) = \max\{n \in \mathbb{Z} \cup \{\infty\} : (\mathrm{Ad}(x_{\mathfrak{p}}) - 1)\Lambda_{\mathfrak{p}} \subset \mathfrak{p}^n \Lambda_{\mathfrak{p}}\}.$$

If $x \in U(n)^{\mathrm{ad}}(F)$ is not the identity, then $\lambda_{\mathfrak{p}}(x)$ is always finite and $\lambda_{\mathfrak{p}}(x) = 0$ for all but finitely many $\mathfrak{p}$. We define

$$\lambda(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{\lambda_{\mathfrak{p}}(x)};$$

We invite the reader to compare to Definition 5.2 of [FL13]. Since $U(n)^{\mathrm{ad}}$ is simple, we avoid the extra complication of projecting to a nontrivial ideal in the Lie algebra. Moreover, it is clear from this definition that $\lambda_{\mathfrak{p}}(x_{\mathfrak{p}}) \geq r$ if and only if $x_{\mathfrak{p}} \in \Gamma(\mathfrak{p}^r)^{\mathrm{ad}}$ for any $r > 0$.

Lemma 4.2.0.4. *Let $P = MN$ be a parabolic subgroup of $U(n)^{\mathrm{ad}}$. Fix $\gamma \in M(F)$ that is semisimple and not equal to 1. Then:*

(i) For almost all primes $\mathfrak{p}$, the following holds: for every $m \in M(F_{\mathfrak{p}})$ and $n \in N(F_{\mathfrak{p}})$, we have $\lambda_{\mathfrak{p}}(m^{-1}\gamma mn) = 0$.

(ii) For every prime $\mathfrak{p}$, the quantity $\lambda_{\mathfrak{p}}(m^{-1}\gamma mn)$ is bounded independent of $m \in M(F_{\mathfrak{p}})$, $n \in N(F_{\mathfrak{p}})$.

(iii) Given $\gamma \in U(n, F)^{\mathrm{ad}}$, there is an ideal $\mathfrak{n}$ such that, for any $m \in M(\mathbb{A}^{\infty})$, $n \in N(\mathbb{A}^{\infty})$, we have $\lambda(m^{-1}\gamma mn) \mid \mathfrak{n}$.

Proof. We invite the reader to compare to the proof of Lemma 5.26 of [FL13].

We first note that if $P = MN$ is a Levi decomposition, $m' \in M(F_{\mathfrak{p}})$ and $n' \in N(F_{\mathfrak{p}})$ with $m'n' \in \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$, then $m', n' \in \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$. Moreover, if we look at the reduction modulo $\mathfrak{p}$ of $\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$, we see that $(M\cap\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}})(N\cap\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}})$ is a Levi decomposition of a parabolic subgroup. As such, if $m'n' \in \Gamma(\mathfrak{p})$ then we must have $m', n' \in \Gamma(\mathfrak{p})$. Therefore, we can only have $m^{-1}\gamma mn \in \Gamma(\mathfrak{p})$ if $m^{-1}\gamma m \in \Gamma(\mathfrak{p})$. Moreover, if $m^{-1}\gamma m \in \Gamma(\mathfrak{p})$, then the characteristic polynomial $f_{\gamma}(T)$ of γ is congruent to $(T-1)^{\dim U(n)^{\mathrm{ad}}}$ modulo $\mathfrak{p}$. Since $\gamma \neq 1$, these polynomials are not equal, so they can only be congruent modulo finitely many primes.

For the second statement, note that the map $(x_{\mathfrak{p}}, u_{\mathfrak{p}}) \mapsto \lambda_{\mathfrak{p}}(x_{\mathfrak{p}}u_{\mathfrak{p}})$ is continuous as a function from $M(F_{\mathfrak{p}}) \times N(F_{\mathfrak{p}}) \to \mathbb{Z} \cup \{\infty\}$ (where the neighborhoods of ∞ are the cofinite sets). As above, we note that $\lambda_{\mathfrak{p}}(x_{\mathfrak{p}}u_{\mathfrak{p}}) \geq 1$ only if $x_{\mathfrak{p}}, u_{\mathfrak{p}} \in \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$, so we restrict to the intersection with $\mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$ in each coordinate. Finally, if $\gamma \neq 1$ is a semisimple element of $M(F_{\mathfrak{p}})$, then its conjugacy class C_{γ} is closed in $M(F_{\mathfrak{p}})$, and therefore $C_{\gamma} \cap \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}$ is compact. As such, the image of $(C_{\gamma} \cap \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}}) \times (N(F_{\mathfrak{p}}) \cap \mathbf{K}_{\mathfrak{p}}^{\mathrm{ad}})$ is compact in $\mathbb{Z} \cup \{\infty\}$. Moreover, since $\gamma \neq 1$, the value ∞ is not attained, so the image of $\lambda_{\mathfrak{p}}$ must be bounded, completing the proof.

Statement (iii) follows from (i) and (ii). □

Proposition 4.2.0.5. *Let $\gamma \in U(n, F)^{\mathrm{ad}}$ be semisimple and noncentral. There are* *constants C, $\epsilon > 0$ (depending only on γ) such that, for any $K \leq \mathbf{K}^{\infty,\mathrm{ad}}$, we have*

$$O_\gamma(\mathbf{1}_K) \leq C\,\mathrm{lev}(K)^{\epsilon}.$$

Proof. We may assume that $\gamma \in \mathbf{K}^{\infty,\mathrm{ad}}$. Let $k_1, \ldots, k_r \in \mathbf{K}^\infty$ be conjugate to γ such that, if $i \neq j$ then k_i and k_j are not conjugate by an element of $\mathbf{K}^\infty$. We can pick a finite set of representatives since $\gamma \in U(n, F)$, and therefore $O_\gamma(\mathbf{1}_K) \leq \infty$.

We therefore have

$$O_\gamma(\mathbf{1}_K) \leq \sum_{i=1}^{r} \mathrm{vol}\{k \in \mathbf{K}^{\infty,\mathrm{ad}} : k^{-1}k_ik \in K\}.$$

By (iii) of 4.2.0.4 and Remark 5.4 of [FL13], each of the terms in the sum is bounded above by $C_i\,\mathrm{lev}(K)^{-\epsilon_i}$. This completes the proof. □

Proposition 4.2.0.6. *Let $P \neq U(n)^{\mathrm{ad}}$ be a parabolic subgroup with Levi decomposition $P = MN$ and let $\gamma \in M(F)$. There are constants C, ϵ, depending only on γ, such that for any $K \leq \mathbf{K}^{\infty,\mathrm{ad}}$, we have*

$$O_\gamma(\mathbf{1}_K^M) \leq C\,\mathrm{lev}(K)^{-\epsilon}.$$

Proof. If $\gamma = 1$ then

$$O_\gamma(\mathbf{1}_K^M) \int_{N(\mathbb{A}^\infty)} \int_{\mathbf{K}^\infty} \mathbf{1}_{K'}(k^{-1}uk)\,dk\,du$$

and the result follows immediately from [FL13, Corollary 5.28].

Now assume γ is noncentral. We first note that $\mathbf{1}_K^M$ is supported on $\mathbf{K}^\infty \cap M$, so we can assume $\gamma \in \mathbf{K}^\infty$. Since $\lambda_{\mathfrak{p}}(m^{-1}\gamma m u)$ is bounded independently of u and m by 4.2.0.4, then by Theorem 5.3 and the logic of Remark 5.4 of [FL13], we have that

$$\mathbf{1}_K^M(m^{-1}\gamma m) = \int_{\mathbf{K}^{\infty,\mathrm{ad}}} \int_{N(\mathbb{A}^\infty)\cap\mathbf{K}^{\infty,\mathrm{ad}}} \mathbf{1}_K(k^{-1}m^{-1}\gamma m u k)\,du\,dk \leq C\,\mathrm{lev}(K)^{-\epsilon}.$$

Therefore

$$O_\gamma(\mathbf{1}^M_{K'}) \leq C \cdot \mathrm{lev}(K)^{-\epsilon} \cdot O_\gamma(\mathbf{1}_{\mathbf{K}^\infty \cap M}),$$

where the orbital integral on the right-hand side is taken in M. This completes the proof. □

As a corollary, we have

Corollary 4.2.0.7. *Let $G = \mathrm{GL}_2$ or $U(n)$. Fix an automorphic character χ of $Z(\mathbb{A})$, a finite set S of finite places, and a sequence $\{\Gamma_\lambda\}$ of open compact subgroups of $G(\mathbb{A}^{S,\infty})$ such that $\mathrm{lev}(\Gamma_\lambda) \to \infty$ as $\lambda \to \infty$. Fix a finite dimensional irreducible representation ξ of $G(\mathbb{R})$. If $G \neq \mathrm{GL}_2$ assume moreover that ξ is algebraic and has regular highest weight. Then Plancherel equidistribution holds for $\{\Gamma_\lambda\}$.*

Proof. This follows directly from the previous proposition and 3.4.0.16. In the case where $G = \mathrm{GL}_2$, we apply 3.5.0.19. □

Chapter 5

Refined Plancherel equidistribution for varying conductor

Throughout this chapter, let $G = U(n)$ or GL_2 over a totally real number field F; throughout, L will denote a $\mathfrak{p}$-adic field. Let $\mathcal{V}_{\mathrm{spl}}$ denote the set of finite places at which G splits, and let $\mathcal{V}_{\mathrm{nsp}}$ denote the set of finite places at which G does not split. Let $\mathbb{A}_{\mathrm{spl}}$ denote the restricted direct product of the fields $F_{\mathfrak{p}}$, for $\mathfrak{p} \in \mathcal{V}_{\mathrm{spl}}$; define $\mathbb{A}_{\mathrm{nsp}}$ similarly; then $\mathbb{A}^{\infty} = \mathbb{A}_{\mathrm{spl}} \times \mathbb{A}_{\mathrm{nsp}}$. As usual, fix an irreducible algebraic representation ξ of $G(\mathbb{R})$ whose highest weight is regular, and let χ be a central character with $\chi_{\infty} = \chi_{\xi}$.

Since the highest weight of ξ is regular, if π is an automorphic, ξ-cuspidal $G(\mathbb{A})$ representation, then each local component $\pi_{\mathfrak{p}}$ is tempered. In particular, if $\mathfrak{p} \in \mathcal{V}_{\mathrm{spl}}$, then $\pi_{\mathfrak{p}}$ is generic.

Recall the definition of $K_n(\mathfrak{p}^r)$ and the conductor of a generic representation from Definition 2.2.1.2. If we define $K_n(\mathfrak{n}) \leq \mathrm{GL}_n(\mathbb{A}_{\mathrm{spl}})$ analogously to $K_n(\mathfrak{p}^r)$, then π has a $K_n(\mathfrak{p})$-fixed vector if and only if $c(\pi) \mid \mathfrak{n}$.

Fix a character $\chi = \chi_{\mathrm{spl}} : \mathbb{A}_{\mathrm{spl}}^{\times} \to \mathbb{C}^{\times}$. We will define a test function $e_{\mathfrak{n},\chi}^{\mathrm{new}}$ such that, for any generic representation π_{spl} of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{spl}})$, we have

$$\operatorname{tr}\pi_{\mathrm{spl}}\left(e^{\mathrm{new}}_{\mathfrak{n},\chi}\right)=\begin{cases}1 & c(\pi_{\mathrm{spl}})=\mathfrak{n}\\ 0 & c(\pi_{\mathrm{spl}})\neq\mathfrak{n}.\end{cases}$$

We will then bound the trace formula terms of e^{new} to determine a Plancherel equidistribution theorem for varying conductor.

We invite the reader to compare to the results of [Bin16].

5.1 Definition of e^{new}

Let S be a finite set of finite places, and write $G(\mathbb{A}^{S,\infty})=G(\mathbb{A}^S_{\mathrm{spl}})\times G(\mathbb{A}^S_{\mathrm{nsp}})$.

Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}_F$, coprime to S, and so that $\mathfrak{n}$ is only divisible by primes in $\mathcal{V}_{\mathrm{spl}}$. Throughout, χ_{spl} will denote a character $\left(\mathbb{A}^S_{\mathrm{spl}}\right)^\times\to\mathbb{C}^\times$ whose conductor $\mathfrak{f}$ divides $\mathfrak{n}$. In this section, we will construct explicit test functions $e^{\mathrm{new}}_{n,\mathfrak{n}}\in\mathcal{H}(\mathrm{GL}_n(\mathbb{A}^S_{\mathrm{spl}}))$ and $e^{\mathrm{new}}_{n,\mathfrak{n},\chi}\in\mathcal{H}(\mathrm{GL}_n(\mathbb{A}^S_{\mathrm{spl}}),\chi)$ such that, for any *generic* representation $\mathrm{GL}_n(\mathbb{A}^S_{\mathrm{spl}})$-representation $\pi^S_{\mathrm{spl}}=\bigotimes_{\mathfrak{p}\notin S}\pi_{\mathfrak{p}}$, we have

$$\widehat{e}^{\mathrm{new}}_{n,\mathfrak{n}}(\pi^S_{\mathrm{spl}})=\operatorname{tr}\pi^S_{\mathrm{spl}}(e^{\mathrm{new}}_{n,\mathfrak{n}})=\begin{cases}1 & c(\pi^S)=\mathfrak{n}\\ 0 & c(\pi^S)\neq\mathfrak{n}.\end{cases}$$

and similarly for $e^{\mathrm{new}}_{n,\mathfrak{n},\chi}$ when π^S_{spl} has central character χ_{spl}.

We'll construct e^{new} as a product of local test function $e^{\mathrm{new}}_{n,\mathfrak{p}^r}$ for $\mathfrak{p}^r\mid\mathfrak{n}$; we'll later construct $e^{\mathrm{new}}_{n,\mathfrak{n},\chi}$ as the image of $e^{\mathrm{new}}_{n,\mathfrak{n}}$ under the averaging map $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}^S_{\mathrm{spl}}))\to\mathcal{H}(\mathrm{GL}_n(\mathbb{A}^S),\chi)$. We'll need two inputs: a theorem of Reeder and a combinatorial identity. Recall the definition of $K_n(\mathfrak{p}^r)$ from 2.2.1.2.

Theorem 5.1.0.1 ([Ree91], Theorem 1). *Let $\pi_{\mathfrak{p}}$ be a generic irreducible admissible representation of* $\mathrm{GL}_n(F_{\mathfrak{p}})$ *of conductor $c=c(\pi_p)$. Then*

$$\dim\pi_p^{K_n(\mathfrak{p}^r)}=\binom{r-c+n-1}{n-1}$$

It's worth remarking that the genericity condition is necessary. For example, if π is the trivial representation then $\dim \pi^{K_n(\mathfrak{p}^r)} = 1$ for all r.

Proposition 5.1.0.2. *For any $n \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}$, the following identity holds:*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{k-i+n-1}{n-1} = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $k = 0$ then the only nonzero term of the right-hand side is the $i = 0$ term, which is 1. If $k < 0$ then all terms of the sum are zero.

If $k > 0$, consider the polynomial function $g_{k,n}(x) = (x-1)^n x^{k-1}$. This polynomial vanishes with order n at $x = 1$, so $g_{k,n}^{(n-1)}(1) = 0$. On the other hand, we may expand $g_{k,n}$ as

$$g_{k,n}(x) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}x^{k-1+n-i}$$

so that

$$g_{k,n}^{(n-1)}(x) = (n-1)!\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{k-1+n-i}{n-1}x^{k-1+n-i}$$

(note that if $k-1+n-i < 0$ then $\binom{k-1+n-i}{n-1} = 0$) and therefore

$$0 = g_{k,n}^{(n-1)}(1) = (n-1)!\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{k-1+n-i}{n-1}$$

completing the proof. $\square$

This motivates the following definition:

Definition 5.1.0.3. *Given a prime $\mathfrak{p}$ and a conductor r, let*

$$e_{n,\mathfrak{p}^r}^{\text{new}} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}e_{K_n(\mathfrak{p}^{r-i})} \in \mathcal{H}(\mathrm{GL}_n(F_\mathfrak{p}))$$

where $e_{K_n(\mathfrak{p}^{r-i})} \in \mathcal{H}(\mathrm{GL}_n(F_\mathfrak{p}))$ is the idempotent function corresponding to the open compact subgroup $K_n(\mathfrak{p}^{r-i})$ of $\mathrm{GL}_n(F_\mathfrak{p})$. (By abusing notation, if $r-i<0$, we set $e_{K_n(\mathfrak{p}^{r-i})} = 0$).

If $\mathfrak{n} = \prod_{\mathfrak{p}} \mathfrak{p}_{\mathfrak{p}}^{r}$, *define*

$$e_{n,\mathfrak{n}}^{\text{new}} = \left(\prod_{\mathfrak{p}|\mathfrak{n}} e_{n,\mathfrak{p}^{r_{\mathfrak{p}}}}^{\text{new}}\right) \times \left(\prod_{\mathfrak{p}\nmid\mathfrak{n}} 1_{K_{\mathfrak{p}}}\right).$$

Proposition 5.1.0.4. *(1) Let* $\pi_{\mathfrak{p}}$ *be a generic representation of* $\mathrm{GL}_n(F_{\mathfrak{p}})$*. Then*

$$\operatorname{tr} \pi_{\mathfrak{p}}(e_{\mathfrak{p}^r}^{\text{new}}) = \begin{cases} 1 & c(\pi_{\mathfrak{p}}) = r \\ 0 & \textit{otherwise.} \end{cases}$$

(2) Let π_{spl}^S *be a generic automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_{\text{spl}}^S)$ *and let* $\mathfrak{n}$ *be an ideal coprime to* S *and divisible only by primes in* $\mathcal{V}_{\text{spl}}$*. Then*

$$\operatorname{tr} \pi^S(e_{\mathfrak{n}}^{\text{new}}) = \begin{cases} 1 & c(\pi^S) = \mathfrak{n} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The first statement follows from Reeder's theorem 2.2.1.3, our combinatorial identity 5.1.0.2, and the fact that if $K \leq \mathrm{GL}_n(F_{\mathfrak{p}})$ is an open compact subgroup, then $\operatorname{tr} \pi_{\mathfrak{p}}(e_K) = \dim \pi_{\mathfrak{p}}^K$. The second statement follows directly from the first. □

If $\chi_{\text{spl}}^S : (\mathbb{A}_{\text{spl}}^S)^\times \to \mathbb{C}^\times$ is a character of conductor $\mathfrak{f}^S$ and $\mathfrak{f}^S \mid \mathfrak{n}$ we define $e_{n,\mathfrak{n},\chi}^{\text{new}}$ to be the image of $e_{n,\mathfrak{n}}^{\text{new}}$ under the averaging map $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_{\text{spl}}^S)) \to \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_{\text{spl}}^S), \chi)$. The following corollary follows immediately from Proposition 5.1.0.4 and Lemma 3.1.0.2.

Corollary 5.1.0.5. *Let* π^S *be a generic automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_{\text{spl}}^S)$ *with central character* χ^S *and let* $\mathfrak{n}$ *be an ideal coprime to* S*, divisible only by the primes in* $\mathcal{V}_{\text{spl}}$*, and divisible by* $\mathfrak{f}$*. Then*

$$\operatorname{tr} \pi^S(e_{\mathfrak{n},\chi}^{\text{new}}) = \begin{cases} 1 & c(\pi^S) = \mathfrak{n} \\ 0 & \textit{otherwise.} \end{cases}$$

We'll need a description of $e_{\mathfrak{n},\chi}^{\text{new}}$ as a product of local functions. Recall that $e_{\mathfrak{p}^r}^{\text{new}}$ is given by a linear combination of idempotent functions $e_{K(\mathfrak{p}^r)}$; let $e_{K(\mathfrak{p}^r),\chi}$ be their images in $\mathcal{H}(\mathrm{GL}_n(F_\mathfrak{p}), \chi)$ under the averaging map. Let $K'(\mathfrak{p}^r)$ be the set of matrices

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in \mathrm{GL}_n(F_\mathfrak{p})$$

with $X \in \mathrm{GL}_{n-1}(\mathfrak{o}_\mathfrak{p})$, Y is an $(n-1) \times 1$-vector of elements of $\mathfrak{o}$, Z is a $1 \times (n-1)$ vector of elements in $\mathfrak{p}^r$, and $W \in \mathfrak{o}_\mathfrak{p}^\times$.

The following lemma is an easy computation:

Lemma 5.1.0.6. *(i) $e_{K(\mathfrak{p}^r),\chi}$ is supported on $\mathbf{K}_\mathfrak{p} \cdot Z$*

(ii) $\mathbf{K}_\mathfrak{p} \cap \mathrm{supp}(e_{K(\mathfrak{p}^r),\chi}) = K'(\mathfrak{p}^r)$

(iii) For our choice of Haar measure,

$$|e_{K(\mathfrak{p}^r),\chi}(g)| = [\mathbf{K}_\mathfrak{p} : K'(\mathfrak{p}^r)] = \frac{q^n - 1}{q - 1} q^{(r-1)(n-1)}$$

for any $g \in K'(\mathfrak{p}^r)$.

5.2 Analysis of trace formula terms and Plancherel equidistribution

Let χ_{spl}^S be as in the previous section. Let $\mathfrak{n}$ be an ideal of F that is divisible only by primes in $\mathcal{V}_{\mathrm{spl}} - S$, and such that $\mathfrak{n}$ is divisible by the conductor $\mathfrak{f}_{\mathrm{spl}}^S$ of χ_{spl}.

Let $h_{n,\mathfrak{n},\chi}(g) = e_{n,\mathfrak{n},\chi}(g)/e_{n,\mathfrak{n},\chi}(1)$. We have the following lemma:

Lemma 5.2.0.7. *Given a prime $\mathfrak{p}$ of norm q, $n \geq 2$ and conductor r, we have*

$$|h_{n,\mathfrak{p}^r,\chi}| \leq 6 \sum_{i=0}^{n} q^{-(n-1)i} \mathbf{1}_{Z \cdot K_n'(\mathfrak{p}^{r-i})},$$

where we take the characteristic function to be zero if $r - i < 0$.

Proof. By the definition of h, it's enough to show that $e_{\mathfrak{p}^r,\chi}^{\mathrm{new}}(1) \geq \frac{1}{6}\frac{q^n-1}{q-1} q^{(n-1)(r-1)}$.

We compute

$$e^{\text{new}}(1) = \frac{q^n-1}{q-1}\left(q^{(r-1)(n-1)} - \binom{n}{1}q^{(r-2)(n-1)} + \binom{n}{2}q^{(r-3)(n-1)} - \dots\right)$$
$$\geq \frac{q^n-1}{q-1}q^{(r-1)(n-1)}\left(1 - \binom{n}{1}q^{-(n-1)} - \binom{n}{3}q^{-3(n-1)} - \dots\right)$$

Consider the function

$$g(n,q) = \sum_{i=1}^{\lceil n/2 \rceil}\binom{n}{2i-1}q^{-(2i-1)(n-1)} = \frac{1}{2}\left((1+q^{1-n})^n - (1-q^{1-n})^n\right).$$

By taking derivatives, we can see that g is decreasing in q and n in the region where $q, n \geq 2$. When $q=2$ and $n=3$, this quantity is $\frac{49}{64} < 5/6$, and when $q=3$ and $n=2$ the quantity is $2/3$. We'll examine the case $n=q=2$ separately.

In the case $r \geq 3$ then

$$e^{\text{new}}(1) = (q+1)q^{r-1}\left(1-2q^{-1}+q^{-2}\right) = \frac{1}{4}(q+1)q^{r-1}.$$

If $r=2$ then

$$e^{\text{new}}(1) = (q+1)q\left(1-2q^{-1}+\frac{1}{q(q+1)}\right) = \frac{1}{6}(q+1)q.$$

If $r=1$ then

$$e^{\text{new}}(1) = (q+1)\left(1-\frac{2}{q+1}\right) = \frac{1}{3}(q+1).$$

Finally, if $r=0$ then $e^{\text{new}}(1)=1$.

As such, for any $\mathfrak{p}$, n, r, χ we have

$$e^{\text{new}}_{\mathfrak{p}^r,\chi}(1) \geq \frac{1}{6}\frac{q^n-1}{q-1}q^{(n-1)(r-1)} = \frac{1}{6}e_{Z\cdot K'(\mathfrak{p}^r)}$$

completing the proof. □

With this in hand, we can prove the asymptotic vanishing of trace formula terms:

Proposition 5.2.0.8. *Let M denote an F-rational Levi subgroup of G, let $\gamma \in M(F)$,* *and consider the trace formula term $O_\gamma(h^M_{\mathfrak{n},\chi})$ (where we consider $h^M_{\mathfrak{n},\chi}$ as a function on $M(\mathbb{A}^S_{\mathrm{spl}})$). We have*

$$|O_\gamma(h^M_{\mathfrak{n},\chi})| \leq C(\gamma)(6(n+1))^{P(\mathfrak{n})}N(\mathfrak{n})^{-\epsilon}$$

where $P(\mathfrak{n})$ denotes the number of primes dividing $\mathfrak{n}$. In particular, $O_{\gamma^S}(h_{n,\mathfrak{n},\chi}) \to 0$ as $N(\mathfrak{n}) \to \infty$.

Proof. Fix $\gamma \in M(F)$ such that $O_\gamma(\mathbf{1}_{\mathbf{K}^S_{\mathrm{spl}}})$ is nontrivial. By conjugating and shifting by an element of the center, we can in fact assume $\gamma \in \mathbf{K}^S_{\mathrm{spl}}$. Now if $g\gamma g^{-1} \in Z \cdot \mathbf{K}^S_{\mathrm{spl}}$, we must have $g\gamma g^{-1} \in \mathbf{K}^S_{\mathrm{spl}}$ by taking determinants. As such, if $K \leq \mathbf{K}^S_{\mathrm{spl}}$ with $ZK \cap \mathbf{K}^S_{\mathrm{spl}} = K'$ then $O_{\gamma^S}(\mathbf{1}_{ZK}) = O_{\gamma^S}(\mathbf{1}_{K'})$.

By the result of 5.2.0.7, we have

$$|h_{\mathfrak{n},\chi}| \leq 6^{P(\mathfrak{n})} \sum_{\mathfrak{d}} \left(\frac{N(\mathfrak{d})}{N(\mathfrak{n})}\right)^{n-1} \mathbf{1}_{ZK_n(\mathfrak{d})}$$

for a collection of ideals $\mathfrak{d} \mid \mathfrak{n}$, where the number of terms is bounded above by $(n+1)^{P(\mathfrak{n})}$. Moreover, there are C and $\epsilon > 0$, depending only on γ, such that

$$O_\gamma(\mathbf{1}^M_{ZK_n(\mathfrak{d})}) \leq C\,\mathrm{lev}(K'_n(\mathfrak{d}))^{-\epsilon} = CN(\mathfrak{d})^{-\epsilon}.$$

As such, $O_\gamma(h^M_{\mathfrak{n},\chi})$ is bounded above by a sum of $(n+1)^{P(\mathfrak{n})}$ terms, each of which is bounded above by a $C \cdot (6(n+1))^{P(\mathfrak{n})}N(\mathfrak{n})^{-\epsilon}$, completing the proof. □

With this in hand, we can complete the proof of Plancherel equidistribution for increasing conductor:

Theorem 5.2.0.9. *Fix the following data:*

- *An algebraic group $G = \mathrm{GL}_2$ or $U(n)$ over a totally real field F,*
- *an irreducible, finite dimensional, algebraic representation ξ of $G(F_\infty)$,*
- *an automorphic character $\chi : Z(\mathbb{A}) \to \mathbb{C}^\times$ satisfying $\chi_\infty = \chi_\xi$,*

- *a finite set S of finite places,*
- *a compact open subgroup $\Gamma_{\text{nsp}} \leq \mathcal{V}_{\text{nsp}}$ such that χ is trivial on $\Gamma_{\text{nsp}} \cap Z(\mathbb{A}_{\text{nsp}})$,*
- *A sequence of ideals $\{\mathfrak{n}_\lambda\}$, divisible only by primes $\mathfrak{p} \in \mathcal{V}_{\text{spl}} - S$, and such that $\mathfrak{n}_\lambda$ is divisible by the conductor of χ_{spl}*

Define the counting measure $\widehat{\mu}_\lambda = \widehat{\mu}_{\mathfrak{n}_\lambda, \xi, \Gamma}$ *on* $\Pi(G(F_S), \chi_S)$ *as follows: for a set* $A \subset \Pi(G(F_S), \chi_S)$*, set*

$$\mu_\lambda(A) = \frac{\mathrm{vol}(Z\Gamma_{\text{nsp}}/Z)}{e^{\text{new}}_{\mathfrak{n},\chi}(1)\dim(\xi)\,\mathrm{vol}(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))} \cdot \sum_{\substack{\pi \\ \chi_\pi = \chi \\ c(\pi^S_{\text{spl}}) = \mathfrak{n} \\ \pi_S \in A}} m_{\text{disc}}(\pi) \cdot \dim(\pi^S_{\text{nsp}})^\Gamma \cdot \chi_{\text{EP}}(\pi_\infty \otimes \xi)$$

Then Plancherel equidistribution holds for the measures μ_λ, in the sense of 3.3.0.9.

Proof. By Sauvageot's density theorem, it's enough to show that when $\phi_S \in \mathcal{H}(G(F_S), \chi_S)$, we have $\mu_\lambda(\widehat{\phi}_S) \to \widehat{\mu}^{\text{pl}}(\widehat{\phi}_S) = \phi_S$. We note that $\mu_\lambda(\phi_S)$ is the spectral side of the trace formula applied to the test function

$$\phi = \phi_S \cdot \cdot h_{\Gamma_{\text{nsp}}} \cdot h_{\mathfrak{n},\chi} \cdot \phi_\xi,$$

where $h_{\Gamma_{\text{nsp}}} = \frac{e_{\Gamma_{\text{nsp}},\chi_{\text{nsp}}}}{e_{\Gamma_{\text{nsp}},\chi_{\text{nsp}}}(1)}$.

Then for any M, γ we have

$$O_\gamma(\phi^\infty_M) = O_\gamma(\phi^\infty_{S,M}) O_\gamma(h_{\Gamma_{\text{nsp}},M}) O_\gamma(h_{\mathfrak{n},\chi,M}).$$

As $\lambda \to \infty$, we have $O_\gamma(h_{\Gamma_{\lambda,\text{nsp}},M}) \to 0$ by 5.2.0.8. This completes the proof as in Proposition 3.4.0.16. $\square$

Remark 5.2.0.10. *We conclude with a couple of brief remarks. First, we remark on the necessity of the open compact subgroup Γ_{nsp}. If our character χ were trivial at all places $\mathfrak{p} \in \mathcal{V}_{\text{nsp}}$, then it would make sense to simply take Γ_{nsp} to be the maximal compact subgroup group $\mathbf{K}_{\text{nsp}}$; this will simply count representations which are* *unramified on $\mathcal{V}_{\text{nsp}}$. On the other hand, since $Z(F_{\mathfrak{p}})$ is compact at all places $\mathfrak{p}$ where G does not split, then if $\chi_{\mathfrak{p}}$ is nontrivial for such a $\mathfrak{p}$, then all representations with central character $\chi_{\mathfrak{p}}$. As such, allowing representations with some fixed ramification at the nonsplit places is necessary to achieve greater generality.*

In fact, if we want, we could achieve even greater generality by allowing a varying sequence of open-compact subgroups Γ_λ. Then we would achieve Plancherel equidistribution as long as

$$N(\mathfrak{n}_\lambda \cdot \operatorname{lev}(Z\Gamma_\lambda)) \to \infty \text{ as } \lambda \to \infty;$$

this is apparent from the proof.

Chapter 6

Basics on fields of rationality; local representations

In this section, we examine fields of rationality in the abstract. If G is a topological group, call a G representation (π, V) *smooth* if every $v \in V$ has an open stabilizer.

Definition 6.0.0.1. *(See section 1.1 of [Wal85]). Let G be a group and let (π, V) be a smooth, complex G representation. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. We define a representation ${}^{\sigma}\pi$ as follows: let V' be a vector space and let $t : V \to V'$ be a σ-linear isomorphism, so that*

$$t(cv) = \sigma(c)t(v)$$

for $c \in \mathbb{C}$, $v \in V$.

Then we define ${}^{\sigma}\pi : G \to \mathrm{Aut}(V')$ via ${}^{\sigma}\pi(g) = t \circ \pi(g) \circ t^{-1}$.

Definition 6.0.0.2. *Let π be a smooth G representation. We define*

$$\mathrm{Stab}_{\mathrm{Aut}(\mathbb{C})}(\pi) = \{\sigma \in \mathrm{Aut}(\mathbb{C}) : {}^{\sigma}\pi \cong \pi\}.$$

The field of rationality $\mathbb{Q}(\pi)$ *of π is the fixed field of* $\mathrm{Stab}_{\mathrm{Aut}(\mathbb{C})}(\pi)$.

It is worth going through a very simple example: let $\chi : G \to \mathbb{C}^{\times}$ be a one-dimensional character. We claim ${}^{\sigma}\chi = \sigma \circ \chi$ and $\mathbb{Q}(\chi)$ is the field generated by all values of χ. To prove the first statement, identify V with $\mathbb{C}$ and let $t : \mathbb{C} \to \mathbb{C}$ be the σ-linear map $t(a) = \sigma(a)$. Then

$$({}^\sigma\chi(g))(a) = \sigma(\chi(g) \cdot \sigma(a)) = \sigma(\chi(g)) \cdot a.$$

The second claim follows from the first, once we realize that ${}^\sigma\chi = \chi$ if and only if σ fixes $\chi(g)$ for all $g \in G$.

The following are elementary but necessary:

Lemma 6.0.0.3. *Let G be a group and $H \leq G$ a subgroup. All representations below are assumed to be smooth.*

(i) If π, π' are G representations, then

$$\mathrm{Hom}_G({}^\sigma\pi,\, {}^\sigma\pi') = \mathrm{Hom}_G(\pi,\, \pi').$$

In particular, if π is a subquotient of π', then ${}^\sigma\pi$ is a subquotient of ${}^\sigma\pi'$.

(ii) If $\rho = \mathrm{Res}_H^G \pi$, then ${}^\sigma\rho = \mathrm{Res}_H^G\, {}^\sigma\pi$.

(iii) If $\pi = \mathrm{Ind}_H^G \rho$, then ${}^\sigma\rho = \mathrm{Ind}_H^G\, {}^\sigma\pi$.

(iv) If G is compact and π is a finite-dimensional G representation with character c_π, then $c_{{}^\sigma\pi}(g) = \sigma(c_\pi(g))$. In particular, $\mathbb{Q}(\rho)$ is the field generated by the values of c_π.

Proof. Statements (i), (ii), and (iii) are all clear. For statement (iv), let $\pi : G \to \mathrm{Aut}(V)$ and let $t : V \to V'$ be a σ-linear map. Since G is finite, $\pi(g)$ is semisimple and therefore V has a basis of eigenvectors $\{e_i\}$ with eigenvalues λ_i; let $t(v_i) = f_i \in V'$. Then

$${}^\sigma\pi(g)(f_i) = t(\pi(g)e_i) = t(\lambda_i e_i) = \sigma(\lambda_i)t(e_i) = \sigma(\lambda_i)f_i.$$

Thus ${}^\sigma\pi(g)$ has eigenvalue $\sigma(\lambda_i)$, so $\mathrm{tr}^\sigma\, \pi(g) = \sigma(\mathrm{tr}\, \pi(g))$, completing the first statement. The second part follows because the isomorphism class of π is determined by its character. $\square$

6.1 Fields of rationality of supercuspidal representations in the tame case

Throughout this chapter, we will assume L has residue characteristic $p > n$; we call this the *tame situation* since all supercuspidal representations of $\mathrm{GL}_n(L)$ arise from Howe's construction of tame supercuspidal representations. Under this assumption, there is a nice characterization of supercuspidal representations of $\mathrm{GL}_n(L)$. We begin with several definitions.

Let L be a local field and L'/L a tamely ramified extension degree n. Let $G = \mathrm{GL}_n(L)$. Throughout, given an extension L'/L, we let $U_i' = 1 + \mathfrak{p}_{L'}^i \leq L'^{\times}$. We write $U' = U_1'$. Let $\mathbb{F}_L$, $\mathbb{F}_{L'}$ denote the residue fields of L, L' respectively.

Definition 6.1.0.4. *Let η be a character of $L'^{\times}$ and $A \leq L'^{\times}$ be a subgroup. We say η is* nondegenerate *on A if there is no proper subextension L'' of L' such that* $\ker N_{L'/L''} \cap A \subseteq \ker \eta \cap A$. *Alternatively, $\eta|_A$ does not factor through $N_{L'/L''}$ for any proper subextension L''/L.*

We say η is admissible *if the following two conditions hold:*

(a) η is nondegenerate on $L'^{\times}$, and

(b) If $\eta|_{U'} = \eta' \circ N_{L'/L''}$, where η' is nondegenerate on U'', then L'/L'' is unramified.

We note that if $\eta : L'^{\times} \to \mathbb{C}^{\times}$ is an admissible character that is trivial on U', then L'/L is unramified.

Definition 6.1.0.5. *An* admissible pair *is a pair (L', η), where L'/L is a field extension of degree n and η is an admissible character of $L'^{\times}$. We say two admissible pairs (L_1', η_1) and (L_2', η_2) are* equivalent *if there is an isomorphism $\tau : L_1' \to L_2'$ (over L) such that $\eta_1 = \eta_2 \circ \tau$*

Proposition 6.1.0.6. *In the tame situation, there is a bijection $\eta \mapsto \pi_\eta$ between admissible pairs (L', η) (up to equivalence) and supercuspidal* $\mathrm{GL}_n(L)$ *representations.*

Proof. The construction of tame supercuspidal representations from an admissible pair is given in [How77] (with an alternate description in [Moy86]). The fact that

$\pi_{\eta_1} \cong \pi_{\eta_2}$ if and only if (L'_1, η_1) and (L'_2, η_2) are equivalent is Theorem 2 of [How77]. Surjectivity is Corollary 3.4.9 of [Moy86]. $\square$

We will recall Howe's construction in order to prove the following:

Proposition 6.1.0.7. *Let (L', η) be an admissible pair and let $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then*

$$\pi_{{}^\sigma\eta} \cong {}^\sigma\pi_\eta.$$

Moreover, the central character of π_η is $\eta|_{L^\times}$.

The construction depends on a *Howe factorization* of the admissible character η: see the Corollary on page 450 of [How77] or Lemma 2.2.4 of [Moy86].

Proposition 6.1.0.8. *Let η be an admissible character of $L'^\times$. Then there are integers $j(1) > j(2) > \ldots > j(r)$ and a tower of fields $L = L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_r = L'$ such that*

- $\eta = \eta_1 \cdot \eta_2 \cdot \ldots \cdot \eta_r$,
- *the conductor of η_α is $j(\alpha) = (l(\alpha) - 1)e_\alpha + 1$, where e_α is the ramification degree of L'/L_α*
- $\eta_1 \cdot \ldots \cdot \eta_\alpha = \eta^{(\alpha)} \circ N_{L'/L_\alpha}$, *where $\eta^{(\alpha)} : L_\alpha^\times \to \mathbb{C}^\times$ is nondegenerate on $(U_{L_\alpha})_{l(\alpha)-1}$.*

Equivalently, there are characters η'_α such $\eta_\alpha = \eta'_\alpha \circ N_{L'/L_\alpha}$.

The following is clear but necessary:

Lemma 6.1.0.9. *If η has Howe factorization $\eta = \eta_1 \cdot \ldots \eta_r$, then ${}^\sigma\eta$ has a Howe factorization as the product of ${}^\sigma\eta_i$. The intermediate subfields are the same, and the conductors of η_α and ${}^\sigma\eta_\alpha$ are both equal to $j(\alpha)$.*

If $j(\alpha)$ is even, let $i(\alpha) = i'(\alpha) = j(\alpha)/2$. If $j(\alpha)$ is odd, let $i(\alpha) = (j(\alpha) - 1)/2$ and $i'(\alpha) = (j(\alpha) + 1)/2$.

Henceforth, assume we have fixed an admissible pair (L, α) and a Howe factorization on α. We identify $M_n(L)$ with $\mathrm{End}_L(L')$, and consider the lattice flag

$$\ldots \supset \mathfrak{p}_{L'}^{-1} \supset \mathfrak{o}_{L'} \supset \mathfrak{p}_{L'} \supset \ldots$$

in L'.

For an intermediate subfield L_α let

- $M_\alpha = \mathrm{End}_{L_\alpha}(L')$,
- $G_\alpha = M_\alpha^\times$,
- $R_\alpha = \{g \in M_\alpha : g\mathfrak{p}_{L'}^h \leq \mathfrak{p}_{L'}^h \text{ for all } h \in \mathbb{Z}\}$
- $K_\alpha = R_\alpha^\times$
- $\xi_\alpha = \{g \in M_\alpha : g\mathfrak{p}_{L'}^h \supseteq \mathfrak{p}_{L'}^{h+1} \text{ for all } h \in \mathbb{Z}\}$.
- $J_\alpha = 1 + \xi_\alpha^{i(\alpha+1)} \leq R_\alpha$.

We will construct an inducing subgroup K_η and a representation κ_η of K_η such that $\pi_\eta = \mathrm{Ind}_{K_\eta}^G \kappa_\eta$.

Fix an additive character $\psi : L^+ \to \mathbb{C}^\times$ and let $\psi_{L'} = \psi \circ \mathrm{tr}_{L'/L}$. Let η_α be a character on $L'^\times$ of conductor $j(\alpha)$ as in the Howe factorization, so that η'_α is trivial on $1 + \mathfrak{p}_{L'}^{j(\alpha)}$ but not on $1 + \mathfrak{p}_{L'}^{j-1}$. Then there is a c_α in L' such that $\eta_\alpha(1+x) = \psi_{L'}(c_\alpha x)$ for $x \in \mathfrak{p}_{L'}^{i'(\alpha)}$; we call c_α a *representative* of η_α with respect to ψ; it is clear that if c_α is a representative of η_α with respect to ψ then it is also a representative of ${}^\sigma\eta_\alpha$ with respect to ${}^\sigma\psi$. The construction will use a choice of additive character and representative c_α, but we will prove in Lemma 6.1.0.13 that the construction is independent of our choices.

We now describe the construction. We'll consider two different cases: where $j(r) > 1$ and where $j(r) = 1$.

In the first case let

$$K_\eta = L'^\times J_{r-1} J_{r-2} \dots J_1$$

and we'll construct κ_η as a K_η representation $\kappa_r \otimes \dots \otimes \kappa_2 \otimes \kappa_1$. We note here that $K_{^\sigma\eta} = K_\eta$, and we'll prove that $\kappa_{\alpha,{}^\sigma\eta} \cong {}^\sigma\kappa_{\alpha,\eta}$. Throughout, we'll assume we have taken the Howe factorization $\prod_\alpha {}^\sigma\eta_\alpha$ of ${}^\sigma\eta$.

Following [Moy86], the representation $\kappa_{1,\eta}$ is $\eta'_1 \circ \det$, so it is clear that ${}^\sigma\kappa_{1,\eta} = \kappa_{1,{}^\sigma\eta}$.

For $\alpha > 2$, when $j(\alpha) = 2i(\alpha)$, then $\kappa_{\eta,\alpha}$ is a one-dimensional character. Then there is a character η'_α of L_α such that $\eta_\alpha = \eta'_\alpha \circ N_{L'/L_\alpha}$. We define κ_α on $L'^\times J_{r-1}J_{r-2}\dots J_\alpha$ via

$$\kappa_\alpha = \eta'_\alpha \circ \det . \tag{6.10}$$

If c_α is a representative of η_α with respect to ψ, we extend κ_α to $J_{\alpha-1} \cdot \dots \cdot J_1$ via

$$\kappa_\alpha(1+x) = \psi(\mathrm{tr}(c_\alpha x)). \tag{6.11}$$

From this description, it is clear that

$${}^\sigma\kappa_{\alpha,\eta,\psi,c_\alpha} = \kappa_{\alpha,{}^\sigma\eta,{}^\sigma\psi,c_\alpha}. \tag{6.12}$$

Now assume $j(\alpha) = 2i(\alpha)+1$. Let

$$H_\alpha = L^\times(1+\mathfrak{p}_{L'})J_{r-1}\dots J_{\alpha-1}$$

and

$$H'_\alpha = L^\times(1+\mathfrak{p}_{L'})J_{r-1}\dots J_\alpha(1+\xi_{\alpha-1}^{i(\alpha)+1}),$$

then set

$$J = L'^\times H,\ J' = L'^\times H'.$$

As above, fix $\eta'_\alpha : L_\alpha^\times \to \mathbb{C}^\times$ so that $\eta_\alpha = \eta'_\alpha \circ N_{L'/L_\alpha}$. Define a character τ_v on $J' \cap G(E_\alpha)$ via 6.10, then extend it to $1+\xi_\alpha^{i(\alpha)+1}$ via 6.11. Then the representation

$$\mathrm{Ind}_{H'}^{H}\tau_\alpha$$

is a multiple of a single representation κ'_α. We then extend κ'_α to a representation κ_α of $J_{\alpha-1}\dots J_1$ via 6.11. It is clear once again that ${}^\sigma\kappa_{\alpha,\eta,\psi,c_\alpha} = \kappa_{\alpha,{}^\sigma\eta,{}^\sigma\psi,c_\alpha}$.

Lemma 6.1.0.13. *(a) Given ψ, the representation $\kappa_{\alpha,\eta,\psi} = \kappa_{\alpha,\eta,\psi,c_\alpha}$ is independent of the c_α chosen.*

(b) The representation $\kappa_{\alpha,\eta} = \kappa_{\alpha,\eta,\psi}$ is independent of the ψ chosen.

Proof. We begin with a sub-lemma:

Lemma 6.1.0.14. *Let e be the ramification degree of L'/L and assume $g \in \mathrm{End}_L(L')$ satisfies $g\mathfrak{p}_{L'}^h \subseteq \mathfrak{p}_{L'}^{h-e+1}$ for all h. Then $\mathrm{tr}(g) \in \mathfrak{o}_L$.*

Proof. Let f denote the degree of the extension of $\mathbb{F}_{L'}/\mathbb{F}_L$, so that $ef = n$. Let $w_1, \ldots, w_f \in \mathfrak{o}_{L'}$ be such that their reductions modulo $\mathfrak{p}_{L'}$ are linearly independent over $\mathbb{F}_L$. Then the set

$$\{\varpi_{L'}^k w_m : 0 \leq k \leq e-1,\, 1 \leq m \leq f\}$$

is a basis for L'/L.

Fix k_0, m_0 and assume

$$g\varpi_{L'}^{k_0} w_{m_0} = \sum_k \varpi_{L'}^k \sum_m a_{k,m} w_m, \qquad a_{k,m} \in L.$$

Because $\{w_m\}$ are linearly independent over modulo $\mathfrak{p}_{L'}$, then

$$v_{L'}\left(\sum_m a_{k,m} w_m\right) = e \cdot \min_m v_L(a_{k,m}).$$

In particular, for given k,

$$v_{L'}\left(\varpi_{L'}^k \sum_m a_{k,m} w_m\right) = k + e \cdot \min_m v_L(a_{k,m}).$$

Since the k are distinct modulo e, all of these terms have different valuations, so that

$$v_{L'}\left(\sum_k \varpi_{L'}^k \sum_m a_{k,m} w_m\right) = \min_{m,k} \left(k + e \cdot v_L(a_{k,m})\right).$$

In particular, if $g\mathfrak{p}_{L'}^k \leq \mathfrak{p}_{L'}^{k+1-e}$ then we must have $a_{k_0,m_0} \in \mathfrak{o}_L$. As such, with respect to our chosen basis, all diagonal entries of g must lie in $\mathfrak{o}_L$, completing the proof. $\square$

We now proceed to the proof of part (a) of the lemma. We can assume wlog that ψ is trivial on $\mathfrak{o}_L$ but not on $\varpi_L^{-1}\mathfrak{o}_L$. Then $\psi \circ \mathrm{tr}_{L'/L}$ is trivial on $\mathfrak{p}_{L'}^{1-e}$, where $e = e(L'/L)$.

Let η_α have conductor $j(\alpha)$ (recall that η_α is a character on $L'^\times$), and recall that $i'(\alpha) = \lceil j(\alpha)/2 \rceil$; then $i'(\alpha) = i(\alpha)$ if $j(\alpha)$ is even, and $i'(\alpha) = i(\alpha) + 1$ if $j(\alpha)$ is odd.

and assume c_1, $c_2 \in L'$ are chosen so that

$$\psi(\mathrm{tr}(c_1 x)) = \eta_\alpha(1+x) = \psi(\mathrm{tr}(c_2 x))$$

for $x \in \mathfrak{p}_{L'}^{i'(\alpha)}$. As such, for all such x we have $\psi(\mathrm{tr}((c_1 - c_2)x)) = 1$, whence $c_1 - c_2 \in \mathfrak{p}^{1-e-i'(\alpha)}$ by Lemma 6.1.0.14.

If j is even, we use c_α to define $\kappa_\alpha(1+x) = \phi(\mathrm{tr}\, c_\alpha x)$ for $1 + x \in J_\alpha - 1 \cdot \ldots \cdot J_1$. For such an x, we have $x\mathfrak{p}_{L'}^h \subset \mathfrak{p}_{L'}^{h+i(\alpha)}$ by definition of J. If c_1, c_2 are two such representatives, we have $(c_1 - c_2)\mathfrak{p}^h \subset \mathfrak{p}^{h+1-e-i'(\alpha)}$. Therefore,

$$(c_1 - c_2)x\mathfrak{p}^h \leq \mathfrak{p}^{h+1-e-i'(\alpha)+i(\alpha)} = \mathfrak{p}^{h+1-e}$$

since $i'(\alpha) = i(\alpha)$. Thus, $\mathrm{tr}((c_1 - c_2)x) \in \mathfrak{o}_L$ by Lemma 6.1.0.14, so $\psi(\mathrm{tr}(c_1 x)) = \psi(\mathrm{tr}(c_2 x))$.

If j is odd, the proof is similar. We use c_α in two steps of the construction. First, we use it to define a character for $x + 1 \in 1 + \xi_\alpha(i(\alpha)+1) = 1 + \xi_\alpha^{i'(\alpha)}$. Second we use it to define a character for $x + 1 \in J_1 \cdot \ldots \cdot J_{\alpha-2} \leq \mathrm{Aut}_L(L')$. In the first situation, x satisfies $x\mathfrak{p}_{L'}^h \leq \mathfrak{p}_{L'}^{h+i(\alpha)}$. In the second situation, that x satisfies $x\mathfrak{p}_{L'}^h \subseteq \mathfrak{p}_{L'}^{h+i(\alpha-1)}$. Moreover, since j is odd we have $i'(\alpha) \leq i(\alpha - 1)$. Therefore, as above we have

$$(c_1 - c_2)x\mathfrak{p}_{L'}^h \subseteq \mathfrak{p}_{L'}^{h+1-e}$$

and therefore $\psi(\mathrm{tr}(c_1 x)) = \psi(\mathrm{tr}(c_2 x))$ for the x in question, proving that the characters constructed are independent of choice of c.

Having proved (a), we prove (b). Let $\psi^{(1)}$, $\psi^{(2)}$ be two nontrivial additive characters. There is a c such that $\psi^{(1)}(x) = \psi^{(2)}(cx)$. We can pick representatives $c_\alpha^{(1)}$, $c_\alpha^{(2)}$

such that $\eta_\alpha(1+x) = \psi^{(i)}(c_\alpha^{(i)}x)$ for x close to 0. Because the construction is independent of the representative chosen we can take $c_\alpha^{(2)} = c_\alpha^{(1)}c_{1,2}$. Then it is clear that

$$\kappa_\alpha^{(1)}(1+x) = \psi^{(1)}(\mathrm{tr}(c_\alpha^{(1)}x)) = \psi^{(2)}(c_{1,2}\,\mathrm{tr}(c_\alpha^{(1)}x)) = \psi^{(2)}(\mathrm{tr}(c_\alpha^{(2)}x)) = \kappa_\alpha^{(2)}(1+x)$$

in any situation where κ_α is defined in reference to ψ. $\square$

The construction in the case where $j(r) = 1$ is similar; in this case, we must have L'/L_{r-1} unramified. Then we set

$$K_\eta = L'^\times K_{r-1}J_{r-2}\dots J_1$$

and we construct a representation $\kappa_\eta = \kappa_r \otimes \dots \otimes \kappa_2 \otimes \kappa_1$. As above, $K_\eta = K_{\sigma\eta}$ for $\sigma \in \mathrm{Aut}(\mathbb{C})$. The construction of the representations $\kappa_1, \dots, \kappa_{r-1}$ are entirely analogous to the constructions above, so the claim goes through by the same argument.

The only discrepancy is in the construction of κ_r. Since η_r has conductor 1, then L'/L_{r-1} is unramified. Therefore, $\eta_r|_{\mathfrak{o}_{L'}}$ descends to a character θ of $\mathbb{F}_{L'}^\times$, which is the group of points of a torus in $\mathrm{GL}_{[L':L_{r-1}]}(\mathbb{F}_{L_{r-1}})$ which is anisotropic modulo the center. Moreover, because η_r is nondegenerate over L_{r-1}, θ is in 'general position' (i.e. does not factor through the norm map $N_{L'/L''}$ for any $L'' \subsetneq L'$). Therefore, by the construction of Green [Gre55] or Deligne-Luzstig [DL76], θ determines a cuspidal representation τ_θ of $K_{r-1}/(1+\xi_{r-1})$, which we lift to a representation τ_θ of K_{r-1}. We extend this representation to a representation κ_r of $L'^\times K_{r-1}$ by having the scalar $\varpi_{L'}$ act as $\eta_r(\varpi_{L'})$.

In either case, we may now prove that the central character of π_η is $\eta|_{L^\times}$. For each $\alpha = 1, \dots, r$, we see that $\kappa_\alpha|_{L'^\times}$ is a multiple of the character $\eta'_\alpha \circ \det = \eta'_\alpha \circ N_{L/L_\alpha}$. The product of these characters is η by the Howe factorization.

To prove that ${}^\sigma\pi_\eta = \pi_{\sigma\eta}$, we need one last lemma.

Lemma 6.1.0.15. *Let $\mathbb{F}$ be a finite field and let $\mathbb{F}_n$ denote the extension of $\mathbb{F}$ of degree n. Let $\theta : \mathbb{F}_n^\times \to \mathbb{C}^\times$ be a character in 'general position' and let τ_θ denote the cuspidal* *representation associated to θ via the construction of Deligne-Lusztig. If $\sigma \in \mathrm{Aut}(\mathbb{C})$ then*

$$^{\sigma}\tau_{\theta} \cong \tau_{^{\sigma}\theta}.$$

Proof. We'll briefly recall the construction of Deligne and Lusztig. Let F denote the Frobenius morphism on G and let T be a minimally-split, F-stable torus in GL_n, so that T is defined over $\mathbb{F}$. Let B be a Borel containing T and let $U \leq B$ be the unipotent radical so that $B = TU$. Define

$$\begin{aligned}\tilde{X}_{T\subset B} &= \{g \in G | g^{-1}F(g) \in F(U)\}/(U \cap F(U)) \\ X_{T\subset B} &= \{g \in G | g^{-1}F(g) \in F(U)\}/(T(\mathbb{F}) \cdot (U \cap F(U)))\end{aligned}$$

so that $\tilde{X}_{T\subset B} \to X_{T\subset B}$ is a $G(\mathbb{F})$-equivariant torsor, equipped with a right $T(\mathbb{F})$ action. Let $R_{T\subset B}$ denote the virtual representation

$$\sum_{i\geq 0}(-1)^i H^i_c(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}_l})$$

and let $R^{\theta}_{T\subset B}$ be the virtual sub-representation where $T(\mathbb{F})$ acts via θ; this subspace is $G(\mathbb{F})$-invariant since the actions of $T(\mathbb{F})$ and $G(\mathbb{F})$ commute. If θ is in general position, then $\pm R^{\theta}_{T\subset B}$ is irreducible and is equal to τ_{θ}. We therefore need to show that if ρ occurs in $R^{\theta}_{T\subset B}$, then $^{\sigma}\rho$ occurs in $R^{^{\sigma}\theta}_{T\subset B}$. We note that ρ occurs in $R^{\theta}_{T\subset B}$ if and only if $\mathrm{Hom}_{G(\mathbb{F})\times T(\mathbb{F})}(\rho \otimes \theta,\, R_{T\subset B}) \neq 0$, or alternatively if $\langle \rho \otimes \theta,\, R_{T\subset B}\rangle \neq 0$ (here we identify representations with their characters).

Because $(g,\, t)$ acts via automorphism on $\tilde{X}_{T\subset B}$, the trace of the associated action on étale cohomology is rational, so $R_{T\subset B}(g,\, t)$ has rational trace. Therefore, the character of $R_{T\subset B}$ has rational values, so we have

$$\langle {}^{\sigma}\rho \otimes {}^{\sigma}\theta,\, R_{T\subset B}\rangle = \langle {}^{\sigma}\rho \otimes {}^{\sigma}\theta,\, {}^{\sigma}R_{T\subset B}\rangle = \sigma(\langle \rho \otimes \theta,\, R_{T\subset B}\rangle) = \langle \rho \otimes \theta,\, R_{T\subset B}\rangle \neq 0$$

so that $^{\sigma}\rho$ occurs in $R^{^{\sigma}\theta}_{T\subset B}$ as desired. ∎

Therefore, for each $\kappa_{\alpha} = \kappa_{\alpha,\eta}$, we have $^{\sigma}\kappa_{\alpha,\eta} \cong \kappa_{\alpha,{}^{\sigma}\eta}$. Since κ_{η} is the tensor product of these representations and $\pi_{\eta,\psi} = \text{c-}\operatorname{Ind}_{K_\eta}^G \kappa_{\eta,\psi}$, we have

$$^\sigma\pi_\eta \cong \pi_{^\sigma\eta}.$$

This completes the proof of 6.1.0.7.

Recall that a $\mathrm{GL}_n(L)$ representation π is said to be *depth-zero* if it has a nonzero $\Gamma(\mathfrak{p})$-fixed vector.

Lemma 6.1.0.16. *Let (L', η) be an admissible pair. The following are equivalent:*

(i) η is trivial on $1 + \mathfrak{p}_{L'}$

(ii) L'/L is unramified and η is trivial on $1 + \mathfrak{p}_{L'}$

(iii) π_η is depth-zero.

Proof. *(i)* $\implies$ *(ii)* by the definition of an admissible character. The implication *(ii)* $\implies$ *(iii)* follows from the construction above: in this case, η has a Howe factorization $\eta = \eta_0\eta_1$ with $j(1) = 1$ and one checks directly that κ_η has a $\Gamma(\mathfrak{p})$-fixed vector.

To see that *(iii)* $\implies$ *(i)*, it follows from Lemma 3.13 of [ST14] that if π has depth zero, it has conductor $c(\pi) \leq n$. By Corollary 3.4.6 of [Moy86], in this situation we have $c(\pi_\eta) = n + f(L'/L)(j(\eta) - 1)$, which is only less than n when $j(\eta) \leq 1$. But there are no unramified admissible characters, completing the proof. ☐

Proposition 6.1.0.17. *Let the residue characteristic of L be $p > n$.*

(a) If (L, η) is an admissible pair, then $\mathbb{Q}(\pi_\eta) \subseteq \mathbb{Q}(\eta)$, and $[\mathbb{Q}(\eta) : \mathbb{Q}(\pi_\eta)] \leq n$.

(b) If π is a supercuspidal representation of nonzero depth, then $[\mathbb{Q}(\pi) : \mathbb{Q}] \geq \frac{p-1}{n}$.

Proof. We first prove (a). If $\sigma \in \mathrm{Aut}(\mathbb{C})$ and σ stabilizes η, then σ stabilizes π_η by Proposition 6.1.0.7. This proves the first statement. For the second statement, let $\eta = \eta_1, \ldots, \eta_d$ denote the conjugates of η under $\mathrm{Aut}(L'/L)$, so that $d \leq n$. Then $^\sigma\pi_\eta \cong \pi_\eta$ if and only if $^\sigma\eta = \eta_i$ for some $i = 1, \ldots, d$. Therefore, the stabilizer of η in $\mathrm{Aut}(\mathbb{C})$ is index at most d in the stabilizer of π_η, completing the second statement.

We now prove (b). Let (L', η) be an admissible pair and assume π_η has positive depth. Then η is nontrivial on $1+\mathfrak{p}_{L'}$, a pro-p-group, so $\eta(x) = \zeta_p$ for some $x \in 1+\mathfrak{p}_{L'}$. Therefore, we by part (a) we have

$$[\mathbb{Q}(\pi_\eta) : \mathbb{Q}] \geq \frac{1}{n}[\mathbb{Q}(\eta) : \mathbb{Q}] \geq \frac{1}{n}[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = \frac{p-1}{n}$$

completing the proof. □

Remark 6.1.0.18. *In fact, we can prove something stronger. Let G_0 denote the set of elements of $\mathrm{GL}_n(L)$ with $|\det(g)| = 1$. Then $\pi_\eta|_{G_0}$ depends only on $\eta|_{\mathfrak{o}_L^\times}$. Therefore we can prove that π is a supercuspidal representation of positive depth, then in fact $[\mathbb{Q}(\pi|_{G_0}) : \mathbb{Q}] \geq \frac{p-1}{n}$. This will be useful later when we compute the fields of rationality of tempered representations, since it will allow us to ignore the effect of twisting by the modulus character in the definition of normalized induction.*

Remark 6.1.0.19. *For more general reductive groups, we have the following analog of the results above:*

Proposition 6.1.0.20. *Let G/F be a connected reductive group, and let $|W|$ denote the cardinality of the Weyl group of a maximal torus in $G \times \bar{F}$. Let $\mathfrak{p}$ be a prime such that $F_\mathfrak{p}$ has sufficiently high residue characteristic p (depending only on G). If π is a supercuspidal representation of $G(F_\mathfrak{p})$ of positive depth, then*

$$[\mathbb{Q}(\pi) : \mathbb{Q}] \geq \frac{p-1}{|W|}.$$

The proof uses analogs by Kim, Murnaghan, and Yu to the results of Howe and Moy for GL_n. In particular, in [Yu01], Yu constructs a tame supercuspidal representation associated to a cuspidal datum $\Phi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ (we refer the reader to Section 3 of [Yu01] for precise definitions). In [Kim07], Kim proves that when the residue characteristic is large enough, this construction is exhaustive.

We'll briefly recall Yu's construction: the starting point is a datum

$$\Phi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi}),$$

where

- $\vec{G} = (G^0, \ldots, G^d)$ *is a sequence of subgroups with* $G^i \leq G^{i+1}$, *and for each* i *there is a tamely ramified extension* L_i/L *such that* $G^i(L_i)$ *is a Levi subgroup of* $G(L_i)$. *We assume moreover that* $Z(G^0)$ *is anisotropic modulo* $Z(G)$,

- y *is a point in the building* $\mathcal{B}(G, L) \cap \mathcal{A}(G, L', T)$, *where* T *is a maximal torus in* G^0 *that splits over* L',

- $\vec{r}$ *is a sequence of real numbers* $0 \leq r_0 < r_1 < \ldots < r_{d-1} \leq r_d$. *If* $d > 0$ *we insist* $0 < r_0$,

- ρ *is an irreducible representation of* K^0, *the stabilizer of* y *in* $G^0(L)$. *We assume* ρ *is trivial on* $G^0(L)_{0+}$ *and that the compactly induced representation* $\mathrm{ind}_{K^0}^{G^0} \rho$ *is irreducible supercuspidal, and*

- $\vec{\phi} = (\phi_0, \ldots \phi_d)$ *is a sequence of quasi-characters* $\phi_i : G^i(L) \to \mathbb{C}^\times$. *We assume* ϕ_i *is trivial on* $G^i(L)_{r_i^+}$ *but not on* $G^i(L)_{r_i}$ *for* $0 \leq i \leq d-1$. *If* $r_{d-1} < r_d$ *we assume the same for* $i = d$; *otherwise we assume* $\phi_d = 1$.

The Yu constructs a supercuspidal representation π_Φ *from this datum.*

If $\sigma \in \mathrm{Aut}(\mathbb{C})$ *and* $\Phi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$, *let* ${}^\sigma\Phi = (\vec{G}, y, \vec{r}, {}^\sigma\rho, {}^\sigma\vec{\phi})$, *where* ${}^\sigma\vec{\phi} = ({}^\sigma\phi_0, \ldots, {}^\sigma\phi_d)$. *It needs to be checked that if* Φ *is a cuspidal datum, then* ${}^\sigma\pi_\Phi = \pi_{{}^\sigma\Phi}$.

Let Φ *be a cuspidal datum and let*

$$\rho' = \rho \otimes \prod_{i=0}^{d} (\phi^i)^{-1}|_{K^0}.$$

It is a result of Hakim and Murnaghan that $\pi_{\Phi_0} = \pi_{\Phi_1}$ *if and only if there is a* $g \in G(L)$ *with* $\mathrm{Ad}(g)K_0^0 = K_1^0$, *and such that* $\mathrm{Ad}(g)$ *takes* ρ_0' *to* ρ_1'.

Let $W(G)$ *be the Weyl group of a maximal torus in* G *(after base change to the algebraic closure): one can prove that* $[N_{G(L)}(K^0) : K^0] \leq |W(G)|$. *Now if* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *fixes* π_Φ, *then* ${}^\sigma\rho'$ *must be equal to* $\mathrm{Ad}(x)\rho'$ *for some* $x \in N_{G(L)}(K^0)/K^0$. *Therefore, we have* $\mathbb{Q} \leq \mathbb{Q}(\pi_\Phi) \leq \mathbb{Q}(\rho_\Phi')$, *and* $[\mathbb{Q}(\rho_\Phi') : \mathbb{Q}(\pi_\Phi)] \leq |W(G)|$. *This* *gives an analog of Proposition 6.1.0.17 (a). To get an analog to (b), we note that* $\mathrm{dep}(\pi_\Phi) = r_d$, *and if* $r_d \neq 0$ *then* ρ' *is trivial on* $G(L)_{r_d}$ *but not on* $G(L)_{r_d^+}$, *and the quotient is a* p*-group.*

6.2 Fields of rationality of discrete series and tempered representations

In this section, we will use our result on the field of rationality of supercuspidal representations to discern similar results on the fields of rationality of discrete series and tempered representations.

Lemma 6.2.0.21. *Let* $n = md$, *let* ρ *be a supercuspidal representation of* $\mathrm{GL}_m(L)$, *and let* $\pi = \mathrm{Sp}(\rho, d)$.

(i) ${}^\sigma\pi = \mathrm{Sp}({}^\sigma\rho, d)$

(ii) $\mathbb{Q}(\pi) = \mathbb{Q}(\rho)$

(iii) If $\mathrm{dep}(\pi) > 0$ *then* $[\mathbb{Q}(\pi) : \mathbb{Q}] \geq \frac{p-1}{n}$. *In fact,* $[\mathbb{Q}(\pi|_{G_0}) : \mathbb{Q}] \geq \frac{p-1}{n}$.

Proof. Let $M \leq \mathrm{GL}_n(L)$ be the block-diagonal subgroup isomorphic to $\mathrm{GL}_m(L)^d$. Let $\pi = \mathrm{Sp}(\rho, d)$, so that π occurs as the unique irreducible quotient representation of

$$I = I_M^G\left(\left(\rho \otimes |\cdot|^{\frac{1-d}{2}}\right) \otimes \ldots \otimes \left(\rho \otimes |\cdot|^{\frac{d-1}{2}}\right)\right).$$

In particular, if $P = MN$ is the standard parabolic with M as its Levi component, then I is isomorphic to the *unnormalized* induction $\mathrm{Ind}_P^G(\rho^{\otimes d}) \otimes \mathbf{1}_N$. Therefore, ${}^\sigma\pi$ occurs as the unique irreducible quotient of ${}^\sigma I$ which is the unnormalized induction $\mathrm{Ind}_P^G({}^\sigma\rho^{\otimes d}) \otimes \mathbf{1}_N$. This proves ${}^\sigma\pi = \mathrm{Sp}({}^\sigma\rho, d)$, completing (i).

(ii) follows from (i) because $\mathrm{Sp}(\rho, d) \cong \mathrm{Sp}(\rho', d')$ if and only $d = d'$ and $\rho \cong \rho'$. Finally, (iii) follows from (ii) once we note that $\mathrm{dep}(\mathrm{Sp}(\rho, d)) = \mathrm{dep}(\rho)$. The claim about the restriction of π to G_0 follows because $\pi|_{G_0}$ depends only on $\rho|_{G_0}$ and $[\mathbb{Q}(\rho|_{G_0}) : \mathbb{Q}] \geq \frac{p-1}{n}$ by Remark 6.1.0.18. $\square$

Proposition 6.2.0.22. *Let π be a tempered representation of positive depth. Then* $[\mathbb{Q}(\pi):\mathbb{Q}] \geq \frac{p-1}{n}$.

Proof. Since π is tempered, there is a Levi subgroup $M \cong \mathrm{GL}_{n_1}(L) \times \ldots \times \mathrm{GL}_{n_r}(L)$ of $\mathrm{GL}_n(L)$ and a discrete series representation $\omega_M = \omega_1 \otimes \ldots \otimes \omega_r$ of M such that $\pi = I_M^G \omega_M$ (recall I_M^G denotes *normalized* induction). Let ω_i^0 denote the restriction of ω_i to $\mathrm{GL}_{n_i}(L)_0$.

If π has positive depth, then ω_M has positive depth, so one of the ω_i has positive depth; assume WLOG that ω_1 has positive depth. If ${}^\sigma\pi \cong \pi$; then σ permutes the representations ω_i^0. In particular, σ permutes the set $\{\omega_i^0\}$. In particular, ${}^\sigma\omega_1^0$ is isomorphic to ω_i^0 for some i with $n_i = n_1$. There are at most n/n_1 blocks of M isomorphic to $\mathrm{GL}_{n_i}(L)$, so ${}^\sigma\omega_1^0$ must be one of at most n/n_1 representations. Therefore

$$[\mathbb{Q}(\pi):\mathbb{Q}] \geq \frac{n_1}{n}[\mathbb{Q}(\omega_1^0):\mathbb{Q}] \geq \frac{n_1}{n} \cdot \frac{p-1}{n_1} = \frac{p-1}{n}$$

by (iii) of the previous lemma. $\square$

Chapter 7

Statements and Proofs of the Main Theorems

7.1 Fields of rationality in families: statements of the main theorems

Throughout this section G will be used to denote GL_2 or $U(n)$ over a totally real field F.

Definition 7.1.0.1. *Fix a finite-dimensional, irreducible, algebraic representation ξ of $G(F_\infty)$, a character χ of $Z(F)\backslash Z(\mathbb{A})$ such that $\chi_\infty = \chi_\xi$, and a subgroup Γ of $G(\mathbb{A}^\infty)$.*

We define the automorphic family $\mathcal{F}_{\mathrm{disc}}(\xi, \chi, \Gamma)$ as the multiset consisting of discrete automorphic representations with $\chi_\pi = \chi$, where a representation π occurs with multiplicity

$$a_{\mathcal{F}}(\pi) = (-1)^{q(G)} \cdot m_{\mathrm{disc}}(\pi) \cdot \dim(\pi^\infty)^\Gamma \cdot \chi_{\mathrm{EP}}(\pi_\infty \otimes \xi^\vee).$$

We define the family $\mathcal{F}_{\mathrm{cusp}}(\xi, \chi, \Gamma)$ similarly, but with m_{cusp} replacing m_{disc}.

For a multiset $\mathcal{F}$, let $|\mathcal{F}| = \sum_\pi a_{\mathcal{F}}(\pi)$.

We remark that the multiplicities arise naturally in the fixed-central-character trace formula. Let $e_\Gamma \in \mathcal{H}(G(\mathbb{A})^\infty)$ denote the idempotent element in the Hecke algebra corresponding to the open compact subgroup Γ, and let $e_{\Gamma,\chi}$ denote its image in $\mathcal{H}(G(\mathbb{A}^\infty), \chi)$. Let ϕ_ξ denote the Euler-Poincaré function defined in 3.7. Then $|\mathcal{F}|$ is the value of the geometric side of the trace formula applied to the function $(-1)^{q(G)} e_{\Gamma,\chi}\phi_\xi$. Moreover, we will assume χ is trivial on $\Gamma \cap Z(\mathbb{A}^\infty)$; otherwise $\mathcal{F}$ is empty.

We will often examine the case where the highest weight of ξ is regular. In this case, the formula simplifies in the following ways (see Propositon 3.2.0.6). First, all ξ-cohomological $G(F_\infty)$ representations π_∞ are discrete series representations, all occur in the same L-packet, and for such a representation we have $\chi_{\mathrm{EP}}(\pi_\infty \otimes \xi^\vee) = (-1)^{q(G)}$. Moreover, for any such representation we have $m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi)$, so $\mathcal{F}_{\mathrm{cusp}} = \mathcal{F}_{\mathrm{disc}}$ as multisets. Finally, if $\pi_\mathfrak{p}$ occurs as a local component of a ξ-cohomological representation, then $\pi_\mathfrak{p}$ is tempered.

Given a multiset $\mathcal{F}$, let $\mathcal{F}^{\leq A}$ denote the subfamily of A consisting only of those representations π with $[\mathbb{Q}(\pi) : \mathbb{Q}] \leq A$.

For the rest of this thesis, we will examine sequences $\{\Gamma_\lambda\}$ such that

$$\lim_{\lambda\to\infty} \frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} = 0.$$

Recall that $\mathcal{V}_{\mathrm{spl}}$ is the set of finite places where G splits and $\mathcal{V}_{\mathrm{nsp}}$ is the finite places where it does not split. Let $\mathbb{A}_{\mathrm{spl}}$ and $\mathbb{A}_{\mathrm{nsp}}$ denote the split and nonsplit components of the finite adèles, and let χ_{spl}, χ_{nsp} denote the restriction of χ to $Z(\mathbb{A}_{\mathrm{spl}})$, $Z(\mathbb{A}_{\mathrm{nsp}})$ respectively. We will prove the following result:

Theorem 7.1.0.2. *Assume that either $G = \mathrm{GL}_2$ or the highest weight of ξ is regular. Let $\mathfrak{n}_\lambda$ be a sequence of ideals, divisible by the conductor $\mathfrak{f}$ of χ of F and let Γ_λ denote the full level subgroup $\Gamma(\mathfrak{n}_\lambda) \leq G(\mathbb{A}^\infty)$. If $N(\mathfrak{n}_\lambda) \to \infty$, then*

$$\lim_{\lambda\to\infty} \frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} = 0.$$

Alternatively, fix $\Gamma_{\mathrm{nsp}} \leq G(\mathbb{A}_{\mathrm{nsp}})$ such that χ_{nsp} is trivial on Γ_{nsp}, and let $\mathfrak{n}_\lambda$ be a *sequence of ideals of F that are only divisible by primes in $\mathcal{V}_{\mathrm{spl}}$, with $N(\mathfrak{n}_\lambda) \to \infty$ as $\lambda \to \infty$.. Assume each $\mathfrak{n}_\lambda$ is divisible by the conductor of χ_{spl}. Let $\Gamma_\lambda = \Gamma_{\mathrm{nsp}} K_0(\mathfrak{n}_\lambda)$. Then*

$$\lim_{\lambda\to\infty} \frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} = 0.$$

For the relation between this problem and Serre's question regarding fields of rationality of classical cusp forms, see [Bin15]. We give a brief explanation here. Consider the case where $F = \mathbb{Q}$ and $G = \mathrm{GL}_2$, so that $\mathcal{V}_{\mathrm{spl}}$ is the set of all finite places. Let V_0 denote the standard representation of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C}^2$ and let $V_1 = V_0 \otimes |\det|^{-1/2}$, so that the central character of V_1 is the sign character. Let $\xi_k = \mathrm{Sym}^{k-2} V_1$. If f is a cusp form of weight k, then π_f is ξ_k-cohomological. Given $\mathfrak{n} \in \mathbb{Z}$, there is a bijection

$$\{\text{Newforms of level } \mathfrak{n}, \text{ character } \chi, \text{ and weight } k\}$$

and

$$\{\text{Cuspidal automorphic representations } \pi \text{ with } \chi_\pi = \chi, \text{ conductor } \mathfrak{n}, \text{ and } \pi_\infty = \pi_k\}$$

that extends to a bijection between the standard eigenbasis of $S_k(\mathfrak{n}, \chi)$ and the *multiset* $\mathcal{F}_{\mathrm{cusp}}(\xi_k, \chi, K_2(\mathfrak{n}))$. By the strong multiplicity one theorem, this bijection preserves fields of rationality (see, for instance, Theorem 1.4(5) of [RT11]).

As such, a resolution to Serre's question on fields of rationality of cusp forms follows from the $F = \mathbb{Q}$, $G = \mathrm{GL}_2$ case of the above theorem. In fact, the result easily generalizes to larger totally real fields.

We may also wish to count representations of a given (fixed) conductor with multiplicity one. In particular, fix χ, ξ, and Γ_{nsp} as above, and let $\mathfrak{n}$ be an ideal of F that is only divisible by primes in $\mathcal{V}_{\mathrm{spl}}$. Let

$$\mathcal{F}_{\mathrm{new,disc}}(\xi, \chi, \Gamma_{\mathrm{nsp}}, \mathfrak{n})$$

denote the multiset consisting of discrete automorphic representations π such that the conductor of π_{spl} is precisely $\mathfrak{n}$, counted with multiplicity

$$m_{\mathrm{disc}}(\pi) \cdot \dim \pi_{\mathrm{nsp}}^{\Gamma_{\mathrm{nsp}}} \cdot \chi_{\mathrm{EP}}(\pi \otimes \xi^{\vee}).$$

We henceforth refer to $\mathcal{F}_{\mathrm{new}}$ as a *conductor family.* Define $\mathcal{F}_{\mathrm{new,cusp}}$ similarly.

Then $|\mathcal{F}|$ is the value of the spectral side of the trace formula applied to the test function

$$\phi = e_{\Gamma_{\mathrm{nsp}},\chi} e_{\mathfrak{n},\chi}^{\mathrm{new}} \phi_{\xi}$$

where e^{new} is the new-vector counting function from Chapter 5.

We have an analogous result for fields of rationality of automorphic representations in conductor families:

Theorem 7.1.0.3. *Let $G = U(n)$ for $n \geq 3$, and let χ, ξ, Γ_{nsp} as above; assume the highest weight of ξ is regular, so $\mathcal{F}_{\mathrm{new,disc}} = \mathcal{F}_{\mathrm{new,cusp}}$. Let $\{\mathfrak{n}_\lambda\}$ be a sequence of ideals, divisible only by split primes, with $N(\mathfrak{n}_\lambda) \to \infty$ as $\lambda \to \infty$. Then*

$$\lim_{\lambda \to 0} \frac{|\mathcal{F}_{\mathrm{new}}^{\leq A}(\xi, \chi, \Gamma_{\mathrm{nsp}}, \mathfrak{n})|}{|\mathcal{F}_{\mathrm{new}}(\xi, \chi, \Gamma_{\mathrm{nsp}}, \mathfrak{n})|} = 0.$$

In the next two sections, we will prove these results, contingent upon some necessary facts from the representation theory of $\mathrm{GL}_n(L)$, which we prove in the next chapter.

Remark 7.1.0.4. *It is worth remarking on our stipulation that $G = \mathrm{GL}_2$ or $U(n)$. First, our results will explicitly use the representation theory of $\mathrm{GL}_n(L)$. In particular, the discrete series representations of $\mathrm{GL}_n(L)$ are well-understood in terms of the supercuspidal representations, and in particular it is easy to compute lower bounds on the fields of rationality of discrete-series representations. Moreover, we will be able to show, using results of Murnghan that are specific to $\mathrm{GL}_n(L)$, that if ρ_1, ρ_2 are supercuspidal $\mathrm{GL}_m(L)$ representations and $\pi_i = \mathrm{Sp}(\rho_i, d)$, then π_1 and π_2 give the same contribution to the automorphic families above (see Proposition 8.0.0.7 for a precise formulation). This will ensure that if $F_{\mathfrak{p}}$ has large enough residue characteristic, the contribution of discrete series representations with small field of rationality* *is small. To our knowledge, analogs of Murnaghan's results have not been proved for other* $\mathfrak{p}$*-adic groups. In view of this, our methods of proof require that our group is a twist of* GL_n*.*

Moreover, it is necessary to use groups G *such that* $G(\mathbb{R})$ *has discrete series representations. This ensures that if* ξ *is an algebraic* $G(\mathbb{R})$ *representation that has regular highest weight and* π *is a* ξ*-cohomological automorphic representation, then* π_∞ *lives in a discrete-series* L*-packet and* $\mathbb{Q}(\pi)$ *is actually a number field. Moreover, in this situation, the trace formula applied to test functions whose infinite component is a Clozel-Delorme function* ϕ_ξ *is particularly simple; see 3.4.0.15, following Theorem 6.1 of [Art89].*

These two conditions together necessitate that we use $G = \mathrm{GL}_2$ *or* $U(n)$*. We hope that we will be able to extend these results to, for instance, other classical groups as their discrete series representations become better understood.*

7.2 A finiteness result for local components of cohomological representations

Let ξ be an irreducible, finite dimensional, algebraic representation of $G(F_\infty)$.

Definition 7.2.0.5. *Fix a finite place* $\mathfrak{p}$ *of* F*. We say a representation* $\pi_\mathfrak{p}$ *of* $G(F_\mathfrak{p})$ *is* potentially ξ-cohomological *if there is a* ξ*-cohomological automorphic representation* π *whose* $\mathfrak{p}$*-component is isomorphic to* $\pi_\mathfrak{p}$*.*

Given $A \geq 1$*, let* $\mathcal{Z}_\mathfrak{p}(A, \xi)$ *denote the set of potentially* ξ*-cohomological* $G(F_\mathfrak{p})$ *representations* $\pi_\mathfrak{p}$ *with* $[\mathbb{Q}(\pi_\mathfrak{p}) : \mathbb{Q}] \leq A$*. (We will drop the references to* $\mathfrak{p}$*,* A*,* ξ *when they are clear from context).*

Proposition 7.2.0.6. *Fix* $\mathfrak{p}$*,* A*,* ξ *as above and assume the highest weight of* ξ *is regular. The set* $\mathcal{Z}_\mathfrak{p}(A, \xi)$ *is finite.*

Proof. This is Corollary 5.7 of [ST14]. □

Remark 7.2.0.7. *When* $G = \mathrm{GL}_2$ *and the highest weight of* ξ *is not regular, the same result holds. This can be pieced together from other results of [ST14]. In this case, if*

π is ξ-cohomological, either π is a one-dimensional representation or π_∞ is a discrete series representation. In either case, we appeal to the Langlands correspondence and go over to the Galois side; the associated $W_{F_\mathfrak{p}}$ representation ρ will be pure of weight 1 (so that the eigenvalues of $\mathrm{Frob}_\mathfrak{p}$ will be Weil-q-integers of weight 1), so there are only finitely many possible eigenvalues of Frobenius by the argument at the end of Lemma 5.1 of [ST14]. Moreover, since the field of rationality of ρ is bounded, so is its depth, and as such $\rho|_{I_\mathfrak{p}}$ is one of finitely many representations, completing the proof.

Proposition 7.2.0.8. *Fix $A \in \mathbb{Z}_{\geq 1}$, $\epsilon > 0$, and a finite prime $\mathfrak{p}$ of F. There is an $r \in \mathbb{Z}_{\geq 1}$ (depending on A, ϵ, $\mathfrak{p}$) such that, for any ideal $\mathfrak{n}$ of F with $\mathfrak{p}^r \mid \mathfrak{n}$, we have*

$$\frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma(\mathfrak{n}))|}{|\mathcal{F}(\xi, \chi, \Gamma(\mathfrak{n}))|} < \epsilon \text{ and } \frac{|\mathcal{F}^{\leq A}(\xi, \chi, K_n(\mathfrak{n}))|}{|\mathcal{F}(\xi, \chi, K_n(\mathfrak{n}))|} < \epsilon.$$

Remark 7.2.0.9. *The reader should compare the first statement to the statement of Theorem 6.1 (ii) of [ST14]. We believe there to be a small mistake in the proof contained in that paper: namely, the authors forget to count representations π with appropriate multiplicity. Fortunately, the proof is correct in spirit and the only missing step is to note a bound on the growth of $\dim \pi_\mathfrak{p}^{\Gamma(\mathfrak{p}^r)}$; the proof below should correct this minor oversight.*

Proof. We'll prove the statement for full level subgroups $\Gamma(\mathfrak{n})$ first. By Corollary 7.2.0.6 there is a finite set $\mathcal{Z}$ (depending on ξ and A) of potentially ξ-cohomological representations $\pi_\mathfrak{p}$ with $[\mathbb{Q}(\pi_\mathfrak{p}) : \mathbb{Q}] \leq A$. By Harish-Chandra's local character expansion, for each $\pi_\mathfrak{p}$, there are constants $C_{\pi_\mathfrak{p}}$ and $d_{\pi_\mathfrak{p}}$ such that $\dim \pi_\mathfrak{p}^{\Gamma(\mathfrak{p}^r)} \sim C_{\pi_\mathfrak{p}} q^{d_{\pi_\mathfrak{p}}}$ (see Remark 8.4.0.22). Let $C_\xi = \mathrm{vol}(G(F)Z(\mathbb{A})\backslash G(\mathbb{A})) \dim(\xi)$.

Let r be large enough that

(a) each such $\pi_\mathfrak{p}$ has a $\Gamma(\mathfrak{p}^r)$-fixed vector,

(b) for $r' \geq r$ and $\pi_\mathfrak{p} \in \mathcal{Z}$, we have

$$(1-\epsilon)C_{\pi_\mathfrak{p}} q^{d_{\pi_\mathfrak{p}} r} < \dim \pi_\mathfrak{p}^{\Gamma(\mathfrak{p}^r)} < (1+\epsilon) C_{\pi_\mathfrak{p}} q^{d_{\pi_\mathfrak{p}} r},$$

and

(c) for any $\mathfrak{n}$ with $\mathfrak{p}^r \mid \mathfrak{n}$, we have

$$C_\xi(1-\epsilon) < \mathrm{vol}(Z\backslash\Gamma(\mathfrak{n})Z)|\mathcal{F}_\mathfrak{n}| < C_\xi(1+\epsilon).$$

(This holds for large enough r by Plancherel equidistribution).

Now let $\mathfrak{n} = \mathfrak{p}^{r'} \cdot \mathfrak{t}$, with $\mathfrak{t}$ coprime to $\mathfrak{p}$ and $r' \geq r$. Let $\mathfrak{n}' = \mathfrak{p}^r\mathfrak{t}$, and let $\mathcal{F}_{\mathfrak{n}'} = \mathcal{F}(\Gamma(\mathfrak{n}'), \widehat{f}_{S_0}, \xi)$. Assume $\pi \in \mathcal{F}_\mathfrak{n}$ with $\pi_\mathfrak{p} \in \mathcal{Z}$. Then $\pi_\mathfrak{p}$ has a $\Gamma(\mathfrak{p}^r)$-fixed vector (by (a)) and so π occurs in $\mathcal{F}_{\mathfrak{n}'}$. For such a π we have

$$a_\mathfrak{n}(\pi) < (1+3\epsilon) \cdot q^{d_{\pi_\mathfrak{p}}(r'-r)} \cdot a_{\mathfrak{n}'}(\pi) \qquad by (b).$$

(Here $a_\mathfrak{n}$, $a_{\mathfrak{n}'}$ denote the multiplicities in $\mathcal{F}_\mathfrak{n}$, $\mathcal{F}_{\mathfrak{n}'}$ respectively).

Moreover, by (c) we have

$$\sum_{\pi_\mathfrak{p} \in \mathcal{Z}} a_{\mathfrak{n}'}(\pi) < |\mathcal{F}_{\mathfrak{n}'}| \leq (1+\epsilon)\frac{C_\xi}{\mathrm{vol}(Z\backslash\Gamma(\mathfrak{n}')Z)}$$

so that

$$\sum_{\pi_\mathfrak{p} \in \mathcal{Z}} a_\mathfrak{n}(\pi) < (1+5\epsilon)\frac{C_\xi}{\mathrm{vol}(Z\backslash\Gamma(\mathfrak{n}')Z)} q^{d_{\pi_\mathfrak{p}}(r'-r)}.$$

We note here that $d_{\pi_\mathfrak{p}}$ is bounded above by some d, the maximal dimension of a nilpotent orbit in $\mathrm{Lie}(G)$, so that we have

$$\sum_{\pi_\mathfrak{p} \in \mathcal{Z}} a_\mathfrak{n}(\pi) < (1+5\epsilon)\frac{C_\xi}{\mathrm{vol}(Z\backslash\Gamma(\mathfrak{n}')Z)} q^{d(r'-r)}.$$

Moreover $|\mathcal{F}_\mathfrak{n}|$ is bounded above by $(1-\epsilon)\frac{C_\xi}{\mathrm{vol}(Z\backslash\Gamma(\mathfrak{n})Z)}$. Therefore, we have

$$\frac{\left(\sum_{\pi_\mathfrak{p} \in \mathcal{Z}} a_\mathfrak{n}(\pi)\right)}{|\mathcal{F}_\mathfrak{n}|} < (1+7\epsilon)\frac{\mathrm{vol}(Z\backslash\Gamma(\mathfrak{n})Z)}{\mathrm{vol}(Z\backslash\Gamma(\mathfrak{n}')Z)} q^{d(r'-r)} = (1+7\epsilon)q^{(d-\dim(G))(r'-r)}$$

which can be made arbitrarily small if r' is large enough, since $d < \dim(G)$.

The case of $\Gamma = K_n(\mathfrak{n})$ is even easier, since $\dim \pi^{K_n(\mathfrak{p}^r)}$ grows polynomially in r instead of exponentially by Reeder's theorem 2.2.1.3. $\square$

Remark 7.2.0.10. *In fact, we can check that if p is high enough (depending on ϵ and A), we can take $r = n + 1$. This follows because, for high enough p, all $\mathrm{GL}_n(L)$ representations $\pi_\mathfrak{p}$ with $[\mathbb{Q}(\pi_\mathfrak{p}) : \mathbb{Q}] \leq A$ have conductor at most n. The number of $K_n(\mathfrak{p}^r)$-fixed vectors in a given π increases polynomially in r, but by Plancherel equidistribution, the total size of the family $\mathcal{F}(\xi, \chi, K_n(\mathfrak{p}^r \cdot \mathfrak{d}))$ increases polynomially in $N(\mathfrak{p})^r$, at least if $N(\mathfrak{p})$ is high enough. For a full exposition of this proof, at least in the $n = 2$ case, we invite the reader to see Lemma 10.0.2 of [Bin15] (the lemma is proved in the language of cusp forms but is easily adapted to the situation of automorphic representations).*

7.3 Contingent completion of the proof

Throughout this section, we will assume the following:

Proposition 7.3.0.11. *Fix $\epsilon > 0$ and $A \in \mathbb{Z}_{\geq 1}$. There is a $P_0 \in \mathbb{Z}_{\geq 1}$ such that, for a $\mathfrak{p}$-adic field L with residue characteristic $p > P_0$ and any unramified character $\chi : L^\times \to \mathbb{C}^\times$, the following holds.*

Let $\Gamma \leq \mathrm{GL}_n(L)$ be either

- *the full level subgroup $\Gamma(\mathfrak{p}^r)$ when $n \geq 2$, or*
- *the subgroup $K_n(\mathfrak{p}^r)$ when $n \geq 3$.*

Then for any r

$$\sum_{\substack{\pi \text{ discrete series} \\ \chi_\pi = \chi \\ [\mathbb{Q}(\pi):\mathbb{Q}] \leq A}} \deg(\pi) \cdot \dim \pi^\Gamma \leq \epsilon \cdot \mathrm{vol}(\Gamma Z/Z)^{-1}. \tag{7.12}$$

Moreover, with L as above and $n \geq 3$, if $e^{\mathrm{new}}_{\mathfrak{p}^r,\chi}$ is the test function from Definition 5.1.0.3, then (again for any r)

$$\sum_{\substack{\pi \text{ discrete series} \\ \chi_\pi = \chi \\ [\mathbb{Q}(\pi):\mathbb{Q}] \leq A}} \deg(\pi) \cdot \widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(\pi) \leq \epsilon \cdot e^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(1). \tag{7.13}$$

Because the proof of the proposition involves delving more deeply into the representation theory of $\mathrm{GL}_n(L)$, we have opted to prove it in the following chapter. In this section, we will complete the proof of our main theorems, *contingent upon the proposition.* As usual, $\mathcal{F}$ is either $\mathcal{F}_{\mathrm{new}}(\xi, \chi, \Gamma_{\mathrm{nsp}}, \mathfrak{n})$ or $\mathcal{F}(\xi, \chi, \Gamma)$ where $\Gamma = \Gamma(\mathfrak{n})$ or $\Gamma = \Gamma_{\mathrm{nsp}} K_n(\mathfrak{n})$.

Proof. (Of the main theorem). We consider three cases. The first case is when $\Gamma = \Gamma(\mathfrak{n})$ or $\Gamma = \Gamma_{\mathrm{nsp}} K_n(\mathfrak{n})$ and $n \geq 3$, so that Proposition 7.3.0.11 holds. In the second case, we make an analogous argument for the conductor-level families $\mathcal{F}_{\mathrm{new}}(\xi, \chi, \Gamma_{\mathrm{nsp}}, \mathfrak{n})$. In the third case, we handle the case $n = 2$ and $\Gamma = \Gamma_{\mathrm{nsp}} K_2(\mathfrak{n})$.

Case 1: Fix $A \geq 1$ and $\epsilon > 0$. We can pick a prime $\mathfrak{p}$ such that

(a) G splits at $\mathfrak{p}$,

(b) χ is unramified at $\mathfrak{p}$, and

(c) The result of Proposition 7.3.0.11 holds for all r.

Given this $\mathfrak{p}$, let r_0 be large enough that the result of Proposition 7.2.0.8 holds. Recall that we have $\Gamma_\lambda = \Gamma(\mathfrak{n}_\lambda)$ or $\Gamma_{\mathrm{nsp}} K_n(\mathfrak{n}_\lambda)$ for a sequence of ideals $\{\mathfrak{n}_\lambda\}$ (in the second case, $\mathfrak{n}_\lambda$ is only divisible by split primes). Put the ideals $\{\mathfrak{n}_\lambda\}$ into subsequences $S_0, S_1, \ldots, S_{r_0 - 1}, S_{r_0}$, such that, for $i < r_0$, $\mathfrak{n}_\lambda$ goes into S_i if $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}_\lambda) = i$. Put $\mathfrak{n}_\lambda$ into S_{r_0} if $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}_\lambda) \geq r_0$. We will show that, for any subsequence S_i, either S_i is finite or

$$\lim_{\substack{\lambda \to \infty \\ \mathfrak{n}_\lambda \in S_i}} \frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} < \epsilon. \tag{7.14}$$

Since there are finitely many subsequences, this will complete the proof. For each $\mathfrak{n}_\lambda \in S_i$, the $\mathfrak{p}$-component $\Gamma_{\mathfrak{p}}$ of $\mathfrak{p}$ is equal to a fixed subgroup $\Gamma_{\mathfrak{p},i}$ (either $\Gamma(\mathfrak{p}^i)$ or $K_n(\mathfrak{p}^i)$).

If $i = r_0$, 7.14 follows immediately (in fact, without taking limits) by the result of 7.2.0.8. So let $i < r_0$. Let $e_{\Gamma_{\mathfrak{p},i},\chi_{\mathfrak{p}}}$ denote the image of $e_{\Gamma_{\mathfrak{p},i}}$ under the averaging map $\mathcal{H}(G(F_{\mathfrak{p}})) \to \mathcal{H}(G(F_{\mathfrak{p}}), \chi_{\mathfrak{p}})$. Let $\mathcal{Z}$ denote the set of representations in $\Pi(G(F_{\mathfrak{p}}), \chi_{\mathfrak{p}})$ that are potentially ξ-cohomological and that satisfy $[\mathbb{Q}(\pi_{\mathfrak{p}}) : \mathbb{Q}] \leq A$. By 7.2.0.6, this set is finite; let $\mathbf{1}_{\mathcal{Z}}$ denote its characteristic function. If $\mathfrak{n}'_\lambda = \mathfrak{n}_\lambda/\mathfrak{p}^i$, then

$$\frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} \leq \frac{\widehat{\mu}_{\mathfrak{n}'_\lambda,\xi,\chi}(\mathbf{1}_{\mathcal{Z}}\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}})}{\widehat{\mu}_{\mathfrak{n}'_\lambda,\xi,\chi}(\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}})}$$

where $\widehat{\mu}$ is the counting measure as defined 3.3.0.8. By the Plancherel equidistribution theorem 4.2.0.7,

$$\lim_{\lambda\to\infty} \frac{\widehat{\mu}_{\mathfrak{n}'_\lambda,\xi,\chi}(\mathbf{1}_{\mathcal{Z}}\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}})}{\widehat{\mu}_{\mathfrak{n}'_\lambda,\xi,\chi}(\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}})} = \frac{\widehat{\mu}^{\mathrm{pl}}(\mathbf{1}_{\mathcal{Z}}\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}})}{\widehat{\mu}^{\mathrm{pl}}(\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}})} = \mathrm{vol}(\Gamma_{\mathfrak{p},i}Z/Z)\cdot\widehat{\mu}^{\mathrm{pl}}(\mathbf{1}_{\mathcal{Z}}\widehat{e}_{\Gamma_{\mathfrak{p},i},\chi_\mathfrak{p}}).$$

Since the set of representations in $\mathcal{Z}$ that are not discrete series representation is finite, its Plancherel measure is zero. Moreover, because we have chosen $\mathfrak{p}$ so that the result of 7.3.0.11 holds, we have

$$\mathrm{vol}(\Gamma_{\mathfrak{p},i}Z/Z) \sum_{\substack{\pi \text{ discrete series} \\ [\mathbb{Q}(\pi):\mathbb{Q}]\leq A}} \deg(\pi)\cdot\dim \pi^{\Gamma_{\mathfrak{p},i}} < \epsilon.$$

Therefore, for high enough λ such that $\mathfrak{n}_\lambda \in S_i$ we have

$$\frac{|\mathcal{F}^{\leq A}(\xi, \chi, \Gamma_\lambda)|}{|\mathcal{F}(\xi, \chi, \Gamma_\lambda)|} < \epsilon$$

finishing the proof.

Case 2: The case of conductor-level families Pick a prime $\mathfrak{p}$ as in Case 1. In this situation, we analyze families of the form

$$\mathcal{F}(\xi, \chi, \Gamma_{\mathrm{nsp}}, \mathfrak{n}_\lambda)$$

for a sequence $\{\mathfrak{n}_\lambda\}$ of ideas with norm approaching ∞. As above, divide the $\mathfrak{n}_\lambda$ into subsequences $S_0, \ldots, S_n, S_{n+1}$, where $\mathfrak{n}_\lambda \in S_{n+1}$ if $\mathrm{ord}_\mathfrak{p}(\mathfrak{n}_\lambda) \geq n+1$ and $\mathfrak{n}_\lambda \in S_i$ if $\mathrm{ord}_\mathfrak{p}(\mathfrak{n}_\lambda) = i$. As above, we'll show that the limit is less than ϵ for each subsequence.

For $i = n+1$, we have already shown that every tempered representation of $\mathrm{GL}_n(L)$ with $[\mathbb{Q}(\pi):\mathbb{Q}] \leq A \leq \frac{p-1}{n}$ has conductor at most n, so for $\mathfrak{n}_\lambda$ in S_{i+1} we already have

$$|\mathcal{F}_{\text{new}}^{\leq A}(\xi,\, \chi,\, \Gamma_{\text{nsp}},\, \mathfrak{n}_\lambda)| = 0.$$

The cases $0 \leq i \leq n$ follow exactly as in Case 1, using the second inequality of Proposition 7.3.0.11.

Case 3: the K_n case, where $n = 2$. The full exposition of this argument is given in Section 10 of [Bin15]. As above, we break our sequence $\{\mathfrak{n}_\lambda\}$ into S_0, S_1, S_2, S_3 and handle each subcase separately. The cases $i = 0, 2, 3$ are entirely analogous to the cases $i < n-1$, $i = n$, $i > n$ cases above assuming we have picked a prime of large enough norm.

For the $i = 1$ case, we run into the following problem: for fixed $\mathfrak{p}$ with norm q, we have $\widehat{\mu}^{\text{pl}}(\widehat{e}_{K_2(\mathfrak{p}),\chi_{\mathfrak{p}}}) = q+1$, and we have two Steinberg representations of conductor 1, each of which has formal degree $\frac{q-1}{2}$. To remedy the situation, let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ where each $\mathfrak{p}_j$ satisfies (a), (b), and (c) as in Case 1, $\mathfrak{p}_j$ has norm q_j, and

$$\prod_{j=1}^{r} \frac{q_j - 1}{q_j + 1} < \epsilon.$$

Given a tuple $t = (i_1, \ldots, i_r)$ with $i_j = 0, 1, 2, 3$, let S_t denote the subsequence of $\mathfrak{n}_\lambda$ such that $\text{ord}_{\mathfrak{p}_j}(\mathfrak{n}_\lambda) = i_j$ (unless $i_j = 3$, in which case we include $\mathfrak{n}_\lambda$ if $\text{ord}_{\mathfrak{p}_j}(\mathfrak{n}_\lambda) \geq 3$). Since each $\mathfrak{p}_j$ is big enough, the same arguments as in Case 1 handle all the situations except $t_1 = (1, \ldots, 1)$.

For this last subsequence: Let $\mathcal{Z}_S$ denote the (finite) set of representations $\pi_S = \pi_1 \otimes \ldots \otimes \pi_r$ such that π_j is potentially ξ-cohomological and $[\mathbb{Q}(\pi_j) : \mathbb{Q}] \leq A$. Let $\widehat{e}_S = \prod_{j=1}^{r} \widehat{e}_{K_2(\mathfrak{p}_j),\chi_{\mathfrak{p}_j}}$, and let $\widehat{f}_S = \mathbf{1}_{\mathcal{Z}_S} \cdot \widehat{e}_S$. The Plancherel measure of $\widehat{e}_S$ is

$$\prod_{j=1}^{r} (q_j + 1).$$

Then the support of $\widehat{f}_S$ is finite, so in particular its Plancherel measure is equal to the Plancherel measure of the set of representations π_S such that each π_i is Steinberg.

Since the Plancherel measure of the two Steinbergs at $\mathfrak{p}_j$ is $(q_j - 1)$, then

$$\widehat{\mu}^{\mathrm{pl}}(\widehat{f}_S) = \prod_{j=1}^{r}(q_j - 1).$$

As such, as $\mathfrak{n}_\lambda \to \infty$ for $\mathfrak{n}_\lambda \in S_{t_1}$, we have

$$\frac{|\mathcal{F}^{\leq A}(\xi, \chi, K_2(\mathfrak{n}_\lambda)|}{|\mathcal{F}(\xi, \chi, K_2(\mathfrak{n}_\lambda))|} \to \frac{\widehat{\mu}^{\mathrm{pl}}(\widehat{f}_S)}{\widehat{\mu}^{\mathrm{pl}}(\widehat{e}_S)} < \epsilon.$$

This completes the proof of case 3. □

Remark 7.3.0.15. *It is enlightening to reinterpret the proof of the* GL_2 *case in the language of modular forms. Then we have the following statements:*

- *Fix a prime* $\mathfrak{p}$ *and let* $\mathfrak{n}_\lambda$ *be a sequence of ideals coprime to* $\mathfrak{p}$*. Then as* $\mathfrak{n}_\lambda \to \infty$*, the proportion of cusp forms* f *of weight* k*, character* χ*, and level* $\mathfrak{n}_\lambda$ *such that* $[\mathbb{Q}(a_\mathfrak{p}(f)) : \mathbb{Q}] \leq A$ *approaches* 0*. This is effectively the argument of Serre in [Ser97].*

- *If* $\mathfrak{p} \mid p$ *is a prime and* f *is a newform of level* $\mathfrak{n}$*, with* $\mathfrak{p}^3 \mid \mathfrak{n}$*, then* $[\mathbb{Q}(f) : \mathbb{Q}] \geq \frac{p-1}{2}$*; this follows immediately from 6.2.0.22. Then the analog of Remark 7.2.0.10 follows because if* $\mathfrak{p}$ *is large and* $\mathfrak{p}^3 \mid \mathfrak{n}$*, then most forms of level* $\mathfrak{n}$ *arise from newforms of level* $\mathfrak{n}'$ *with* $\mathfrak{p}^3 \mid \mathfrak{n}'$*.*

- *The argument at the end of Case 3 has a nice reinterpretation in terms of cusp forms. Assume* $\chi_\mathfrak{p}$ *is unramified and* f *is a newform of level* $\mathfrak{p}\mathfrak{n}'$*, where* $\mathfrak{n}'$ *is coprime to* $\mathfrak{p}$*. Then the representaton* $\pi_{f,\mathfrak{p}}$ *is one of two Steinberg representations with central character* $\chi_\mathfrak{p}$*. In particular, the field of rationality of* $\pi_{f,\mathfrak{p}}$ *is degree at most* 2 *over* $\mathbb{Q}(\chi_\mathfrak{p}) \leq \mathbb{Q}(\chi)$*. Therefore, the newforms at level* $\mathfrak{p}\mathfrak{n}'$ *do not have large field of rationality, at least at* $\mathfrak{p}$*.*

  However, if we take the entire family of cusp forms of level $\mathfrak{p}\mathfrak{n}'$*, then* $\frac{2}{q+1}$ *(by proportion) of these forms are newforms at some level not divisible by* $\mathfrak{p}$*. By the argument in the first bullet point, if* $\mathfrak{n}'$ *is large enough, most of these cusp forms have large field of rationality at* $\mathfrak{p}$*. Therefore, if we take* $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ *to be* *a large set of large primes, we can ensure that only a small proportion of the cusp forms of level* $\mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_r \mathfrak{n}'$ *arise from newforms of level some level divisible by* $\mathfrak{p}_i$ *for each* i.

Using these methods, we cannot prove an analogous result for fields of rationality for *newforms*. Indeed, we can see where the proof of the newform case (Theorem 7.1.0.3) breaks down in the case $G = \mathrm{GL}_2$: it is precisely when we have fixed a set S of primes and examine a representation π of conductor $\mathfrak{n}$, where $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) = 1$ for all $\mathfrak{p} \in S$. Then π_S is a tensor product of Steinberg representations. In particular, if we only consider π_S, we have an upper bound on the size of the field of rationality in this situation.

It seems that the following question should contain all the difficulty in finishing the 'conductor family' case of GL_2, but will nonetheless require a new piece of innovation:

Question 7.3.0.16. *Let* $\mathbf{1}$ *denote the trivial character and let* ξ *be a finite-dimensional irreducible algebraic* $\mathrm{GL}_2(\mathbb{R})$ *representation with trivial central character. Let* $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ *be an enumeration of the primes in* F *and let* $\mathfrak{n}_\lambda = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_\lambda$. *Is is true that*

$$\lim_{\lambda \to \infty} \frac{|\mathcal{F}_{\mathrm{new}}^{\leq A}(\xi, \mathbf{1}, \mathfrak{n}_\lambda)|}{|\mathcal{F}_{\mathrm{new}}(\xi, \mathbf{1}, \mathfrak{n}_\lambda)|} = 0?$$

Chapter 8

Properties of depth-zero discrete series representations

In this chapter, we will prove Proposition 7.3.0.11. Recall the statement:

Proposition 8.0.0.1. *Fix $\epsilon > 0$ and $A \in \mathbb{Z}_{\geq 1}$. There is a $P_0 \in \mathbb{Z}_{\geq 1}$ such that, for a $\mathfrak{p}$-adic field L with residue characteristic $p > P_0$ and any unramified character $\chi : L^\times \to \mathbb{C}^\times$, the following holds.*

Let $\Gamma \leq \mathrm{GL}_n(L)$ be either

- *the full level subgroup $\Gamma(\mathfrak{p}^r)$ when $n \geq 2$, or*
- *the subgroup $K_n(\mathfrak{p}^r)$ when $n \geq 3$.*

Then

$$\sum_{\substack{\pi \text{ discrete series} \\ \chi_\pi = \chi \\ [\mathbb{Q}(\pi):\mathbb{Q}] \leq A}} \deg(\pi) \cdot \dim \pi^\Gamma \leq \epsilon \cdot \mathrm{vol}(\Gamma Z/Z)^{-1}. \tag{8.2}$$

Moreover, with L as above, if $e^{\mathrm{new}}_{\mathfrak{p}^r,\chi}$ is the test function from Definition 5.1.0.3, then

$$\sum_{\substack{\pi \text{ discrete series} \\ \chi_\pi = \chi \\ [\mathbb{Q}(\pi):\mathbb{Q}] \leq A}} \deg(\pi) \cdot \widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(\pi) \leq \epsilon \cdot e^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(1). \tag{8.3}$$

We first note that it is enough to prove the following:

Proposition 8.0.0.4. *Let L, G, ϵ, A, Γ be as above and fix $d \mid n$. Then there is s a P_0 such that for every $p > P_0$ and every $r \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{\pi=\mathrm{Sp}(\rho,\, d) \\ \chi_\pi=\chi \\ [\mathbb{Q}(\pi):\mathbb{Q}]\leq A}} \deg(\pi) \cdot \dim \pi^\Gamma \leq \epsilon \cdot \mathrm{vol}(\Gamma Z/Z)^{-1}. \tag{8.5}$$

Moreover, if $n \geq 3$, we can find P_0 such that, for any r and L with residue characteristic $p > P_0$ we have

$$\sum_{\substack{\pi=\mathrm{Sp}(\rho,\, d) \\ \chi_\pi=\chi \\ [\mathbb{Q}(\pi):\mathbb{Q}]\leq A}} \deg(\pi) \cdot \widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(\pi) \leq \epsilon \cdot e^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(1). \tag{8.6}$$

Then 7.3.0.11 follows because there are finitely many $d \mid n$ and because if $p > nA$ and π is a discrete series representation with $[\mathbb{Q}(\pi) : \mathbb{Q}] \leq A$, then π has depth zero.

When $d = n$ (so that π is a standard Steinberg representation), we will show this statement directly. When $d < n$, we will not compute $\dim \pi^\Gamma$ directly. Rather, we will show that if p is large enough, the proportion of representations of the form $\pi = \mathrm{Sp}(\rho,\, d)$ satisfying $[\mathbb{Q}(\pi) : \mathbb{Q}] \leq A$ is at most ϵ. This is the result of the counting argument in the next section. In the following sections, we will show:

Proposition 8.0.0.7. *Fix $m > 1$, $n = md$, and assume $p > 2n$. Let ρ_1, ρ_2 be two depth-zero supercuspidal $\mathrm{GL}_m(L)$ representations and let $\pi_i = \mathrm{Sp}(\rho_i,\, d)$. Then:*

(a) $\deg(\pi_1) = \deg(\pi_2)$*;*

(b) $\dim \pi_1^\Gamma = \dim \pi_2^\Gamma$*; and*

(c) if $e^{\mathrm{new}}_{\mathfrak{p}^r,\chi} \in \mathcal{H}(\mathrm{GL}_n(L),\, \chi)$ is the 'new vector' test function constructed in Chapter 4, then $\widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(\pi_1) = \widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(\pi_2)$.

These two facts together will be enough to prove Proposition 8.0.0.4 in the case

$m > 1$, since

$$\begin{aligned}\mathrm{vol}(\Gamma Z/Z) &= \widehat{\mu}^{\mathrm{pl}}_{\chi}(\widehat{e}_{\Gamma,\chi}) \\ &\geq \sum_{\pi = \mathrm{Sp}(\rho,\, d)} \deg(\pi) \dim \pi^{\Gamma}\end{aligned}$$

where the second sum runs over the depth-zero supercuspidal $\mathrm{GL}_m(L)$ representations ρ (and similarly when we replace $\dim \pi^{\Gamma}$ with $\widehat{e}^{\mathrm{new}}(\pi)$ and $\mathrm{vol}(\Gamma Z/Z)$ with $\widehat{e}^{\mathrm{new}}(1)^{-1}$).

8.1 Counting discrete series representations with small field of rationality

In this section, we'll count the depth-zero discrete series representations of $\mathrm{GL}_n(L)$ with a fixed central character, and also prove a lower bound on the number of such representations satisfying $[\mathbb{Q}(\pi) : \mathbb{Q}] \leq A$ independent of L. Let $\mathbb{F}_L$ denote the residue field of L and let $|\mathbb{F}_L| = q$. Given $A \in \mathbb{Z}_{\geq 1}$, let $f(A)$ denote the number roots of unity ζ with $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leq A$; note that $f(A)$ is finite for any A.

Proposition 8.1.0.8. *Fix a central character $\chi_0 : L^{\times} \to \mathbb{C}^{\times}$ that is trivial on $1 + \mathfrak{p}_L$, and assume $m \geq 2$.*

(a) Let $\beta(L, m, \chi_0)$ denote the number of depth-zero supercuspidal $\mathrm{GL}_m(L)$ representations ρ with $\chi_\rho = \chi_0$. Then

$$\frac{1}{m}\left(q^{m-1} - 1\right) \leq \beta(L, m, \chi_0)$$

(b) The number of depth-zero supercuspidal representations ρ with $\chi_\rho = \chi_0$ and $[\mathbb{Q}(\rho) : \mathbb{Q}] \leq A$ is bounded above by $\frac{1}{m} f(mA)$.

Proof. Let L'/L be the unique unramified extension of degree m. Every depth-zero supercuspidal $\mathrm{GL}_m(L)$ representation is of the form π_η where $\eta : L'^{\times} \to \mathbb{C}^{\times}$ is an admissible character trivial on $1 + \mathfrak{p}_{L'}$. Moreover, if π_η has central character χ_0, then $\chi_0 = \eta \mid_{L^{\times}}$. If ϖ is a uniformizer of L, then $L'^{\times}$ is generated by $\mathfrak{o}_{L'}^{\times}$ and ϖ. Therefore, if we insist that $\eta|_{L^\times} = \chi_{\pi_\eta} = \chi_0$, then η is determined by its restriction η_0 to $\mathfrak{o}_{L'^\times}$. An admissible character η_0 on $\mathfrak{o}_{L'}^\times$ trivial on $1 + \mathfrak{p}_{L'}$ descends to a character θ on $\mathbb{F}_{q^m}^\times$ that does not factor through the norm map $\mathbb{F}_{q^m}^\times \to \mathbb{F}_{q^x}^\times$ for $x \neq m$; we say such a character is in *general position.* Moreover, we note that $\theta|_{\mathbb{F}_q}$ must be equal to a fixed character χ_0. Therefore, it's enough to count the number $\gamma(L, m, \chi_0)$ of characters θ in general position on $\mathbb{F}_{q^m}^\times \to \mathbb{C}^\times$ with $\theta|_{\mathbb{F}_q} = \chi_0$. If θ is in general position, then its orbit under $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ has cardinality m, so we have $\gamma(L, m, \chi_0) = m\beta(L, m, \chi_0)$.

We handle two separate cases: where $m > 2$ and $m = 2$. In the $m > 2$ case, we note that the total number of characters on $\mathbb{F}_{q^m}^\times$ is $q^m - 1$. Moreover, the restriction map $\widehat{\mathbb{F}}_{q^m}^\times \to \widehat{\mathbb{F}}_q^\times$ is surjective, so the total number of characters on $\mathbb{F}_{q^m}^\times$ with fixed restriction of $\mathbb{F}_q^\times$ is

$$\frac{q^m - 1}{q - 1} = q^{m-1} + q^{m-2} + \ldots + q + 1.$$

We note that the number that are *not* in general position is bounded above by

$$\sum_{\substack{x|m \\ x<m}} q^x.$$

If $m > 2$, $x \mid m$, and $x < m$, then $x < m - 1$, so at most $q^{m-2} + \ldots + q + 1$ characters that are not in general position, completing the proof in the $m > 2$ case.

in the $m = 2$ case, fix a central character $\chi_0 \in \widehat{\mathbb{F}}_q^\times$; we claim there are at most 2 characters θ that are not in general position and such that $\theta|_{\mathbb{F}_q^\times} = \chi_0$. If θ is not in general position, then it is of the form $\theta_0 \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ for some $\theta_0 : \mathbb{F}_q^\times \to \mathbb{C}^\times$. On $\mathbb{F}_q^\times$, $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ acts as $x \mapsto x^2$, so the induced map $\widehat{\mathbb{F}}_q^\times \to \widehat{\mathbb{F}}_q^\times$ is two-to-one. As such, given a character χ_0, there are at most two characters θ_0 on $\mathbb{F}_q^\times$ such that $\chi_0 = \left(\theta_0 \circ N_{\mathbb{F}_q^2/\mathbb{F}_q}\right)|_{\mathbb{F}_q^\times}$, completing the proof of (a).

We now prove (b). If $[\mathbb{Q}(\rho) : \mathbb{Q}] \leq A$ then $[\mathbb{Q}(\eta) : \mathbb{Q}] \leq mA$, so in particular $[\mathbb{Q}(\eta_0) : \mathbb{Q}] \leq mA$. The group $\mathbb{F}_{q^m}^\times$ is cyclic: let it have generator α. If $[\mathbb{Q}(\eta_0) : \mathbb{Q}] \leq mA$, then $\eta_0(\alpha)$ must be one of the $f(mA)$ roots of unity ζ with $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leq mA$. Moreover, if η_0 satisfies $[\mathbb{Q}(\eta_0) : \mathbb{Q}] \leq A$, then so does any of its Galois conjugates, so the number of Galois orbits of characters η_0 with $[\mathbb{Q}(\eta_0) : \mathbb{Q}] \leq A$ is at most

$\frac{1}{m}f(mA)$. ☐

Corollary 8.1.0.9. *Let $p > n$. Fix $m \geq 2$, given $\epsilon > 0$ and $A \in \mathbb{Z}_{\geq 1}$. There is a $Q_0 > 1$ such that, for any L with $|\mathbb{F}_L| = q \geq Q_0$ and any character $\chi_0 : L^\times \to \mathbb{C}^\times$, trivial on $1 + \mathfrak{p}_L$, the proportion of depth-zero supercuspidal representations ρ with $\chi_\rho = \chi_0$ that satisfy $[\mathbb{Q}(\rho) : \mathbb{Q}] \leq A$ is at most ϵ.*

Proof. This follows directly from the above proposition. ☐

Corollary 8.1.0.10. *Fix $m \geq 2$ with $n = md$, fix $\epsilon > 0$ and $A \in \mathbb{Z}_{\geq 1}$. Let Q_0 be as in the previous corollary, and let L be satisfy $|\mathbb{F}_L| \geq Q_0$. Let χ_0 an unramified character $\chi_0 : L^\times \to \mathbb{C}^\times$. The proportion of depth-zero discrete series representations $\pi = \mathrm{Sp}(\rho,\, d)$ satisfying $[\mathbb{Q}(\pi) : \mathbb{Q}] \leq A$ is at most ϵ.*

Proof. If $\pi = \mathrm{Sp}(\rho,\, d)$ then $\chi_\pi = \chi_\rho^d$. There are at most d^2 characters χ_1 with $\chi_1^d = \chi_0$, and since $p > n > d$, each such χ_1 is trivial on $1 + \mathfrak{p}_L$. If we fix such a character χ_1, then the proportion of ρ such that $\chi_\rho = \chi_d$ and $[\mathbb{Q}(\rho) : \mathbb{Q}] \leq A$ is at most ϵ. Since $\mathbb{Q}(\pi) = \mathbb{Q}(\rho)$ this completes the proof. ☐

We will also need the following lemma:

Lemma 8.1.0.11. *Fix an unramified central character χ_0 of $L^\times$, where L has residue characteristic $p > n$. The number of Steinberg representations $\mathrm{St}(\chi)$ of $\mathrm{GL}_n(L)$ with central character χ_0 is bounded above by n^2.*

Proof. The central character of $\mathrm{St}(\chi)$ is $\chi^n = \chi_0$, so $\chi(\varpi_L)$ is one of the n roots of $x^n = \chi_0(\varpi)$. Moreover $\chi^n|_{\mathfrak{o}_L^\times}$ is trivial, so χ must be trivial on $1 + \mathfrak{p}_L$: otherwise it attains the value ζ_p and $\zeta_p^n \neq 1$ since $p > n$. Therefore, $\chi|_{\mathfrak{o}_L^\times}$ factors through a cyclic group with generator α, and we must have $\chi(\alpha)^n = 1$, so $\chi|_{\mathfrak{o}_L^\times}$ is one of at most n characters. This completes the proof, since $L^\times = \mathfrak{o}_L^\times \times \varpi_L^{\mathbb{Z}}$. ☐

8.2 Proof of 8.0.0.4 in some cases

Let $n = md$. In this section, we will give a proof of 8.0.0.4 in the following cases:

- The case where $m \geq 2$ and $\Gamma = K_n(\mathfrak{p}^r)$, or in the 'conductor family' case. We'll prove Proposition 8.0.0.7 in this case, and along the way we will also prove (a) and (c) of Proposition 8.0.0.7 in the case $\Gamma = \Gamma(\mathfrak{p}^r)$.

- The case $m = 1$, so π is a (standard) Steinberg representation. We will prove this by directly computing $\dim \pi^\Gamma$ and $\deg(\pi)$, without appealing to 8.0.0.7

The only remaining part of the proof will be (b) of Proposition 8.0.0.7 in the case $\Gamma = \Gamma(\mathfrak{p}^r)$. We've postponed this proof until the next sections since it will take us deeper into the representation theory of $\mathrm{GL}_n(L)$.

We first prove (a) of Proposition 8.0.0.7.

Lemma 8.2.0.12. *Let ρ be a depth-zero supercuspidal $\mathrm{GL}_m(d)$ representation and let $\pi = \mathrm{Sp}(\rho, d)$. If St_m denote the Steinberg representation of $\mathrm{GL}_m(L)$, then*

$$\frac{\deg(\pi)}{\deg(\mathrm{St}_m)^d} = \frac{1}{d} \cdot m^d \cdot \frac{(q^m - 1)^d}{q^{md} - 1} \cdot \frac{|\,\mathrm{GL}_{dm}(\mathbb{F}_q)|}{|\,\mathrm{GL}_m(\mathbb{F}_q)|^d} \cdot q^{-m^2 \frac{d^2-d}{2}}.$$

Proof. First, if ρ is a depth-zero supercuspidal $\mathrm{GL}_m(F_\mathfrak{p})$ representation, we may compute the formal degree $\deg(\rho)$ using Theorem 2.2.8 of [CMS90]. In this case, ρ is associated to the admissible pair (L_m, η) where L_m/L is the unramified extension of degree m, and $\eta : L_m^\times \to \mathbb{C}^\times$ is trivial on $1 + \mathfrak{p}_{L_m}^\times$. Using the notation of [CMS90], we compute $\alpha(\theta) = m - 1$, $f = m$, $e = 1$, so

$$\deg(\rho) = m \cdot \deg(\mathrm{St}).$$

We now use this to compute $\deg \pi$; in view of Theorem 6.3 of [AP05] we have

$$\frac{\deg(\pi)}{\deg(\rho)^d} = \frac{m^{d-1}}{r^{d-1} d} \cdot q^{\frac{d^2-d}{2}(f(\rho^\vee \times \rho) + r - 2m^2)} \cdot \frac{(q^r - 1)^d}{q^{dr} - 1} \cdot \frac{|\,\mathrm{GL}_{dm}(\mathbb{F}_q)|}{|\,\mathrm{GL}_m(\mathbb{F}_q)|^d}.$$

Here r is the number of unramified characters $\chi : L^\times \to \mathbb{C}^\times$ such that $\rho \otimes (\chi \circ \det) \cong \rho$; (or the *torsion number* of $\rho 0$), and $f(\rho \times \rho^\vee)$ is the conductor of the pair $\rho \times \rho'$.

We first prove $r = m$. Let $\chi : L^\times \to \mathbb{C}^\times$ be an unramified character. We note that $\rho = \pi_\eta$ where η is a depth-zero character of $L_m^\times$ and L_m is the unramified extension of L of degree m. Moreover $(\chi \circ \det) \otimes \pi_\eta \cong \pi_{(\chi \circ N_{L_m/L}) \cdot \eta}$. If $\chi(\varpi_L)^m = 1$, then $\chi \circ N_{L_m/L} \equiv 1$, so the torsion number is at least m. To see it is exactly m, the central character of $\pi \otimes (\chi \circ \det)$ is $\chi_\pi \cdot \chi^m$. Thus, we must have $\chi_m(\varpi_L)^m = 1$, and there are m such unramified characters.

To prove that $f(\rho \times \rho^\vee) = m^2 - m$, we use the local Langlands correspondence. Let $\mathscr{L}(\rho)$ denote the associated Weil-Deligne representation; since ρ is supercuspidal, $\mathscr{L}(\rho)$ is irreducible, and since the monodromy is trivial and so we need only consider $\mathcal{L}(\rho)$ as a representation of the Weil group W_L. Since $f(\rho \times \rho^\vee) = f(\mathscr{L}(\rho) \otimes \mathscr{L}(\rho)^\vee)$, it's enough to compute the conductor of the Weil representation $f(V \otimes V^\vee)$ when V is an irreducible, depth-zero Weil representation.

Since the Langlands correspondence preserves depth, $\mathscr{L}(\rho)|_{I_L}$ is trivial on the ramification subgroup I_L^1. Since I_L/I_L^1 is abelian, then V decomposes as a direct sum of abelian characters $\theta_1, \ldots, \theta_m$. Because V is irreducible, the characters θ_i are pairwise distinct. Therefore $(V \otimes V^\vee)$ is trivial on I_L^1, and the subspace of I_L-fixed vectors has dimension m (corresponding to the spaces $\theta_i \otimes \overline{\theta}_i$). From the definition, we discern

$$f(V \otimes V^\vee) = \operatorname{codim}((V \otimes V^\vee)^{I_L})^{N=0} + \int_0^\infty (V \otimes V^\vee)^{I_L^j} \, dj = m^2 - m$$

as desired.

Now the proof is complete once we plug $f = m^2 - m$, $r = m$ into Aubert-Plymen's formula. $\square$

Lemma 8.2.0.13. *Let $\Gamma = K_n(\mathfrak{p}^r)$. Let ρ_i be a supercuspidal $\mathrm{GL}_n(L)$ representation and $\pi_i = \mathrm{Sp}(\rho_i, d)$. For any r, we have*

$$\dim \pi_1^{K_n(\mathfrak{p}^r)} = \dim \pi_2^{K_n(\mathfrak{p}^r)}.$$

and

$$\widehat{e}_{\mathfrak{p}^r}^{\text{new}}(\pi_1) = \widehat{e}_{\mathfrak{p}^r}^{\text{new}}(\pi_2).$$

Proof. It follows from Corollary 3.4.6 of [Moy86] that the conductor $c(\pi_i)$ of π is n.

Then

$$\dim \pi_i^{K_n(\mathfrak{p}^r)} = \binom{r-1}{n-1}$$

by Reeder's Theorem 2.2.1.3.

Moreover, $\widehat{e}_{\mathfrak{p}^r}^{\mathrm{new}}(\pi) = 1$ if $r = n$ and zero otherwise by the construction of e^{new}. $\square$

This proves Proposition 8.0.0.7 in the case $\Gamma = K_n(\mathfrak{p}^r)$ and also in the 'new vector' case when $m > 1$; therefore, we have completed the proof of 7.3.0.11 in the case $m > 1$, $\Gamma = K_n(\mathfrak{p}^r)$.

We will also prove Equation 8.6 of Proposition 8.0.0.4 in the case $m = 1$:

Lemma 8.2.0.14. *Fix $\epsilon > 0$. There is a $Q_0 \in \mathbb{Z}_{\geq 1}$ such that the following holds: Let $\Gamma = K_n(\mathfrak{p}^r)$ for $n \geq 3$ or $\Gamma = \Gamma(\mathfrak{p}^r)$ for $n \geq 2$, as subgroups of $\mathrm{GL}_n(L)$. Assume the residue field $\mathbb{F}_L$ of L has cardinality $q > Q_0$. Then*

$$\sum_{\substack{\pi = \mathrm{St}(\chi_0) \\ \chi_\pi = \chi}} \deg(\pi) \cdot \dim \pi^\Gamma \leq \epsilon \cdot \mathrm{vol}(\Gamma Z/Z)^{-1}.$$

and

$$\sum_{\substack{\pi = \mathrm{St}(\chi_0) \\ \chi_\pi = \chi}} \deg(\pi) \widehat{e}_{\mathfrak{p}^r, \chi}^{\mathrm{new}}(\pi) \leq \epsilon \cdot e_{\mathfrak{p}^r, \chi}^{\mathrm{new}}(1).$$

Proof. Let χ_0 be a character of $L^\times$ trivial on $1 + \mathfrak{p}_L$. By (2.2.2′) of [CMS90], we have

$$\deg(\mathrm{St}(\chi)) = \frac{1}{n} \prod_{k=1}^{n-1} (q^k - 1) \leq \frac{1}{n} q^{n(n-1)/2}.$$

Again using (3.4.6) of [Moy86] we have that

$$c(\mathrm{St}(\chi_0)) = \begin{cases} n-1 & \chi_0 \text{ unramified} \\ n & \text{otherwise} \end{cases}.$$

Therefore, using Reeder's formula, we have

$$\dim \mathrm{St}(\chi_0)^{K_n(\mathfrak{p}^r)} \leq \binom{r-1}{n-2}.$$

Since there are at most n^2 Steinberg representations with given central character,

$$\sum_{\chi_0^n=\chi} \deg(\mathrm{St}(\chi_0)) \dim \mathrm{St}(\chi_0)^{K_n(\mathfrak{p}^r)} \leq n\binom{n-1}{r-2} q^{n(n-1)/2}$$

whereas $\mathrm{vol}(K_n(\mathfrak{p}^r)Z/Z)^{-1} \geq cq^{r(n-1)}$, completing the proof in this case (since we assume $n \geq 3$).

This proves the first equation in the case $\Gamma = K_n(\mathfrak{p}^r)$. We now prove the second equality. We proved in Lemma 5.2.0.7 that

$$\widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r,\chi}(1) \geq \frac{1}{6}\mathrm{vol}(K_n(\mathfrak{p}^r)Z/Z)^{-1} \geq \frac{c}{6}q^{r(n-1)}.$$

Since $\widehat{e}^{\mathrm{new}}_{\mathfrak{p}^r}(\pi) = 1$ or 0 for a tempered representation π, this completes the proof of the second equality.

To prove the result in the case where Γ is the full level subgroup $\Gamma(\mathfrak{p}^r)$, we will need the following sublemma:

Lemma 8.2.0.15. *There is a $C > 0$ such that, for all L with $|\mathbb{F}_L| = q$ and all characters $\chi_0 : L^\times \to \mathbb{C}^\times$ of conductor at most 1, we have*

$$\dim \mathrm{St}(\chi_0)^{\Gamma(\mathfrak{p}^r)} \leq Cq^{rn(n-1)/2}.$$

Proof. Since χ has conductor 1 and $r \geq 0$, it is clear that $\mathrm{St}(\chi)$ and $\mathrm{St}(\mathbf{1})$ have the same dimension of $\Gamma(\mathfrak{p}^r)$-fixed vectors. Let I denote the (unnormalized) induction $\mathrm{Ind}_B^G \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of the Borel subgroup B. Then $\dim \mathrm{St}^{\Gamma(\mathfrak{p}^r)} \leq \dim I^\Gamma$ since St is a quotient of I (for admissible representations, $\dim \pi^\Gamma = \mathrm{tr}\,\pi(e_\Gamma)$, so the function $\pi \mapsto \dim \pi^\Gamma$ is additive in exact sequences).

Let V be the space of I. Using Mackey's Theorem, we have

$$\begin{aligned} V^\Gamma &\cong \bigoplus_{g \in B\backslash G/\Gamma} \mathbb{C}^{B\cap g\Gamma g^{-1}} \\ &= \bigoplus_{g \in B\backslash G/\Gamma} \mathbb{C} \end{aligned}$$

so it suffices to find the cardinality of $B\backslash G/\Gamma$. Since $BK = G$ and Γ is normal in K, we have

$$B\backslash G/\Gamma = (B \cap K)\backslash K/\Gamma = ((B \cap K)\Gamma)\backslash K.$$

The group $(B \cap K) \cdot \Gamma(\mathfrak{p}^r)$ consists of those matrices in K such that the elements below the diagonal are in $\mathfrak{p}^r$. As such, there is a fixed C such that $((B \cap K) \cdot \Gamma(\mathfrak{p}^r))\backslash K \leq Cq^{r(n-1)n/2}$, completing the proof. □

This completes the proof in the $\Gamma = \Gamma(\mathfrak{p}^r)$ case, since

$$\sum_{\chi_0} \deg(\mathrm{St}(\chi_0)) \dim \mathrm{St}(\chi_0)^{\Gamma(\mathfrak{p}^r)} \leq nCq^{(r+1)n(n-1)/2}$$

but $\mathrm{vol}(\Gamma(\mathfrak{p}^r)Z/Z)^{-1} \geq cq^{r(n^2-1)}$ for all r, completing the proof. □

At this point, we have proved Proposition 8.0.0.4 in all cases except for the case where $m \geq 2$ and Γ is the full level subgroup $\Gamma(\mathfrak{p}^r)$. In this case, we still need to show that if $\pi_i = \mathrm{Sp}(\rho_i, d)$ we have $\dim \pi_1^{\Gamma(\mathfrak{p}^r)} = \dim \pi_2^{\Gamma(\mathfrak{p}^r)}$. We will complete this final case over the next sections. In fact, we will prove something stronger. Let Θ_{π_i} denote the Harish-Chandra character of π_i. Then

$$\dim \pi_i^{\Gamma} = \mathrm{vol}(\Gamma)^{-1} \operatorname{tr} \pi_i(\mathbf{1}_\Gamma) = \mathrm{vol}(\Gamma)^{-1} \int_\Gamma \Theta_{\pi_i}(g)\, dg.$$

The proof therefore reduces to the following lemma, which we prove in the next sections:

Lemma 8.2.0.16. *Let $\pi_i = \mathrm{Sp}(\rho_i, d)$ for ρ_i a depth-zero supercuspidal representation of $\mathrm{GL}_m(L)$. Then we can choose characters Θ_i for π_i such that $\Theta_{\pi_1}(g) = \Theta_{\pi_2}(g)$ for $g \in \Gamma(\mathfrak{p})$.*

We note that, a priori, a character is only defined up to equality outside a set of measure zero.

It is possible that the fixed dimension $\dim \pi^{\Gamma(\mathfrak{p}^r)}$ may be computed directly as well by computing orbital integrals in the Lie algebra $\mathfrak{g}$ of $\mathrm{GL}_n(L)$. We have not considered this question here.

8.3 Hecke algebra isomorphisms

In this section we give a quick exposition of spherical Hecke algebras and Hecke algebra isomorphisms; the information in this section is taken from [How85]. Let L be a $\mathfrak{p}$-adic field and $G_n = \mathrm{GL}_n(L)$.

Definition 8.3.0.17. *Let $K \leq G = \mathrm{GL}_n(L)$ be a compact open subgroup and let (τ, W) be a finite-dimensional, irreducible K representation. The τ-*spherical Hecke algebra *$\mathcal{H}(G\,/\,/\,K, \tau)$ is the convolution algebra of functions $f : G \to \mathrm{End}(W)$ satisfying*

$$f(k_1 g k_2) = \tau(k_1) f(g) \tau(k_2) \quad k_1, k_2 \in K, \, g \in G.$$

There is a correspondence $r_{K,\tau}$ between

$$(\text{(nonzero) irreducible representations of } \mathcal{H}(G\,/\,/\,K, \tau))$$

and

$$(\text{irreducible representations } (\pi, V) \text{ of } G \text{ with } \mathrm{Hom}_K(W^*, V) \neq 0)$$

where W^* is the dual of W equipped with the contragredient representation τ^*.

Let $e_{K,\tau} \in \mathcal{H}(G\,/\,/\,K, \tau)$ be supported on K, with $e(k) = \tau(k)$ for $k \in K$. If the Haar measure of G is normalized so that K has measure 1, then $e_{K,\tau}$ is the identity of $\mathcal{H}(G\,/\,/\,K, \tau)$.

Let (π, V) be a G representation, and let ν_0 denote the standard action of $\mathrm{End}(W)$ on W. Then we get an action of $C_c(G; \mathrm{End}(W)) = C_c(G) \otimes \mathrm{End}(W)$ on $V \otimes W$ via $\pi \otimes \nu_0$. Under the identification $V \otimes W = \mathrm{Hom}(W^*, V)$, we see that the image of $e_{K,\tau}$ are those $T \in \mathrm{Hom}(W^*, V)$ with $\pi(k)T = T\tau^*(k)$. In particular, we get an action of

$$\mathcal{H}(G\,/\,/\,K, \tau) = e_{K,\tau} * C_c(G; \mathrm{End}(W)) * e_{K,\tau}$$

on $\mathrm{Hom}_K(W^*, V)$. Let $r_{K,\tau}(\pi)$ be the corresponding representation of $\mathcal{H}(G\,/\,/\,K, \tau)$ on $\mathrm{Hom}_K(\tau^*, \pi)$.

Proposition 8.3.0.18. *The map $r_{K,\tau}$ is a bijection.*

Now let $\pi = \mathrm{Sp}(\rho, d)$ be a depth-zero discrete series G representation, where ρ is a depth-zero supercuspidal representation of $\mathrm{GL}_m(L)$. Let $G_m = \mathrm{GL}_m(L)$ with center Z_m and let $\mathbf{K}_m = \mathrm{GL}_m(\mathfrak{o}_L)$. Then $\rho = \mathrm{Ind}_{\mathbf{K}_m Z_m}^{G_m} \tau_0$, where τ_0 is a depth-zero representation of $\mathbf{K}_m Z_m$.; in particular, τ_0 is trivial on $\Gamma(\mathfrak{p})$. Let $\tau = \tau_0|_{\mathbf{K}_m}$. Since τ is trivial on $\Gamma(\mathfrak{p})$, we can consider τ as a representation of $\mathrm{GL}_m(\mathbb{F}_L)$, and then we can consider $\tau^{\otimes d}$ as a representation of $\mathrm{GL}_m(\mathbb{F}_L)^d$. Consider $\mathrm{GL}_m(\mathbb{F}_L)^d = M(\mathbb{F}_L)$ as a Levi subgroup of $\mathrm{GL}_n(\mathbb{F}_L)$, and let $P(\mathbb{F}_L)$ be the standard be parabolic subgroup with Levi component $M(\mathbb{F}_L)$. Then we may inflate $\tau^{\otimes m}$ to a representation of $P(\mathbb{F}_L)$ via the surjection $P(\mathbb{F}_L) \to M(\mathbb{F}_L)$. Finally, let P be the inverse image of $P(\mathbb{F}_L)$ under the reduction map $\mathbf{K}_n \to \mathrm{GL}_n(\mathbb{F}_L)$; this is a parahoric subgroup of G_n. We consider $\tau^{\otimes d}$ as a P-representation. Then $\mathrm{Hom}_P(\tau, \pi) \neq 0$, so π corresponds to a representation of the spherical Hecke algebra $\mathcal{H}(G\,/\,/\,P, (\tau^*)^{\otimes d})$. In this situation, the pair $(P, \tau^{\otimes d})$ is a *refined minimal K-type* and we say π contains $(P, \tau^{\otimes d})$.

Proposition 8.3.0.19. *Let $G_n = \mathrm{GL}_n(L)$ with $n = md$, Let P, $\tau^{\otimes d}$ be as in the previous paragraph. Let L_m denote the unramified extension of L of degree M, and let $G_d^0 = \mathrm{GL}_d(L_m)$, considered as a subgroup of G_n, and let B_d^0 denote an Iwahori subgroup of G_d^0. Then there is an isomorphism*

$$\mathcal{H}(G_n\,/\,/\,P, \tau^{\otimes d}) \cong \mathcal{H}(G_d^0\,/\,/\,B_d^0, \mathbf{1}).$$

This yields a correspondence between irreducible G_n representations such that $\mathrm{Hom}_P(\tau^{\otimes D}, \pi) \neq 0$ and G_d^0 representations with a nonzero Iwahori-fixed vector. Under this correspondence, $\mathrm{Sp}(\rho, d)$ corresponds to the Steinberg representation of G_d^0.

Proof. The first statement is [How85, Theorem 1.2]. The second statement is follows because the isomorphism of Hecke algebras preserves Plancherel measure, and therefore takes discrete series representations to discrete series representations. ☐

8.4 The local character expansion

Let G be a reductive group over a $\mathfrak{p}$-adic field L, and let $\mathfrak{g} = \mathrm{Lie}(G(L))$. Let $\mathfrak{g}_{\mathrm{reg}}$, G_{reg} denote the set of regular semisimple elements in $\mathfrak{g}$, $G(L)$ respectively.

Fix once and for all an additive character ψ on L, and let $\langle \cdot, \cdot \rangle$ be the bilinear form $\langle X, Y \rangle = \psi(\mathrm{tr}(XY))$. Assume the Haar measure dX on $\mathfrak{g}$ is self-dual with respect to ψ. Given an $\mathrm{Ad}(G)$-orbit $\mathcal{O} \subset \mathfrak{g}$, let $\mu_{\mathcal{O}}(f)$ denote the integral of f over the orbit $\mathcal{O}$. If $\widehat{f}$ is the Fourier transform of f with respect to ψ, we define the distribution $\widehat{\mu}_{\mathcal{O}}$ via $\widehat{\mu}_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(\widehat{f})$.

The distribution $\widehat{\mu}_{\mathcal{O}}$ is representable by a locally integrable function on $\mathfrak{g}$, which by abuse of notation we also call $\widehat{\mu}_{\mathcal{O}}$. In particular,

$$\widehat{\mu}_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(\widehat{f}) = \int_{\mathfrak{g}} f(X)\widehat{\mu}_{\mathcal{O}}(X)\, dX.$$

Given a semisimple element $s \in \mathfrak{g}$, let $\Omega_G(s)$ denote the set of $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$ that contain s in their closure. In particular, $\Omega_G(0)$ denotes the set of nilpotent orbits in $\mathfrak{g}$.

We will henceforth restrict our analysis to the case of $G = \mathrm{GL}_n$ and assume moreover that L has residue characteristic $p > 2n$. In this situation the map $X \mapsto 1+X$ gives a bijection from $\mathfrak{g}_{0+} = \mathfrak{p}_L M_n(\mathfrak{o}_L) \leq \mathfrak{g}$ to $\Gamma(\mathfrak{p}) \leq G$. We'll assume we have picked Haar measures on $\mathfrak{g}$, G so that this bijection is measure-preserving.

Let π be an irreducible admissible $G(L)$-representation with character Θ_π, let $s \in \mathfrak{g}$ be a semisimple element. and let $\mathcal{V} \leq \mathfrak{g}_{0+}$ be a neighborhood of 0 in $\mathfrak{g}$. Following [Mur03], we say *the germ of Θ_π is s-asymptotic on $\mathcal{V}$* if there are constants $c_{\mathcal{O}}(\pi)$ for $\mathcal{O} \in \Omega_g(s)$ such that, for any $X \in \mathcal{V} \cap \mathfrak{g}_{\mathrm{reg}}$, we have

$$\Theta_\pi(1+X) = \sum_{\mathcal{O} \in \Omega_g(s)} c_{\mathcal{O}}(\pi)\widehat{\mu}_{\mathcal{O}}(X).$$

Remark 8.4.0.20. *In the classical formulation the exponential map was used in place of $X \mapsto 1+X$; however, in Murnaghan's situation the map $X \mapsto 1+X$ suffices, following Remark 5.3.3 of [KM03]. This has the advantage that it is defined on all of*

$\mathfrak{p} \cdot M_n(\mathfrak{o}_L)$, *whereas the exponential map might not be, depending on the ramification of* $L/\mathbb{Q}_p$. *In the larger context of our proof, we are picking the residue characteristic to be 'large enough' from the outset, but it is nice to know that this result holds whenever* $p > 2n$.

It is a classical theorem of Howe [How74] (in the case of GL_n) and Harish-Chandra [Har99] (for connected reductive groups) that for any representation π, the germ of Θ_π is 0-asymptotic on some neighborhood of 0. It is moreover a theorem of DeBacker [DeB02] and Waldspurger [Wal95] that if π has depth d and p is large enough then the local character expansion is valid on $\mathfrak{g}_{d^+}$ (here $\mathfrak{g}_{d^+}$ is the Moy-Prasad lattice defined in [MP94, Section 3]).

In our situation, however, an asymptotic expansion result around some $s \neq 0$ from [Mur03] is necessary.

Theorem 8.4.0.21. *[Mur03] Assume* $p > 2n$. *Let* π *be an irreducible admissible* G *representation. Then there is* $s_\pi \in \mathfrak{g}_{\mathrm{reg}}$ *such that the germ of* Θ_π *is s-asymptotic on* $\mathfrak{g}_{\mathrm{dep}(\pi)^+}$. *Let* H *be centralizer of* s *in* G. *Then there is a correspondence* $\Omega_H(0) \leftrightarrow \Omega_G(s)$ *given by* $\mathcal{O}_H \mapsto \mathcal{O}_G = \mathrm{Ad}(G) \cdot (s \cdot \mathcal{O}_H)$. *There is a representation* π_H *of* H *and a constant* $\lambda > 0$, *depending only on choices of Haar measure on* G *and* H *such that* $c_{\mathcal{O}_H}(\pi_H) = \lambda c_{\mathcal{O}_G}(\pi)$. *In the situation where* $\pi = \mathrm{Sp}(\rho, d)$, *then* s *may be chosen as follows:* s *is an element of* $\mathfrak{o}_{L_m}^\times$ *whose reduction modulo* $\mathfrak{p}_L$ *generates* $\mathbb{F}_{L_m}$ *over* $\mathbb{F}_L$. *Then* $H = C_G(s) = \mathrm{GL}_d(L_m) \leq \mathrm{GL}_n(L)$ *and* π_H *is the* H *representation corresponding to* π *under the Hecke algebra isomorphism* $\mathcal{H}(G \,/\!/\, P, \tau^{\otimes d}) \cong \mathcal{H}(H \,/\!/\, B, \mathbf{1})$.

Proof. All statements in the theorem are derived from [Mur03]. The first and second statements are the content of Theorem 14.5. The last statement is the content of Theorem 14.1, and the fact that s_π may be chosen as stated follows from the computation of $s_{\tau,h}$ for the depth-zero case in Lemma 10.9. ☐

Now let ρ_1, ρ_2 denote depth-zero supercuspidal representations of G_m and let $\pi_i = \mathrm{Sp}(\rho, d)$. Then we may choose $s_{\pi_1} = s_{\pi_2}$, and both $\Theta_{\pi_1} = \Theta_{\pi_2}$ have s-asymptotic expansions on $\mathfrak{g}_{0^+} = \mathfrak{p} \cdot M_n(\mathfrak{o}_L)$. Moreover, under the Hecke algebra isomorphisms

$$\mathcal{H}(G//P,\, \tau_1^{\otimes d}) \cong \mathcal{H}(G_d^0//B_d^0,\, \mathbf{1}) \cong \mathcal{H}(G//P,\, \tau_2^{\otimes d}),$$

both π_1 and π_2 correspond to the Steinberg representation on G_d^0. Therefore, the constants occurring in the s_π asymptotic expansions are the same. In particular, $\Theta_{\pi_1} = \Theta_{\pi_2}$ on $\Gamma(\mathfrak{p})$. This completes the proof of Lemma 8.2.0.16, and therefore the proof of Proposition 7.3.0.11.

Remark 8.4.0.22. *It is worth noting another application of Harish-Chandra's character expansion to the growth of the number of $\Gamma(\mathfrak{p}^r)$-fixed vectors. Assume $G = \mathrm{GL}_n$, so that $\Gamma(\mathfrak{p}^r)$ is the image of $\mathfrak{p}^r M_n(\mathfrak{o}_L)$ under the map $X \mapsto 1 + X$. Then we have*

$$\begin{aligned}
\dim \pi^{\Gamma}(\mathfrak{p}^r) &= \mathrm{vol}(\Gamma(\mathfrak{p}^r))^{-1} \int_{\Gamma(\mathfrak{p}^r)} \Theta_\pi(g)\, dg \\
&= \mathrm{vol}(\mathfrak{p}^r M_n(\mathfrak{o}_L))^{-1} \int_{\mathfrak{p}^r M_n(\mathfrak{o}_L)} \Theta_\pi(1+X)\, dX \\
&= \mathrm{vol}(\mathfrak{p}^r M_n(\mathfrak{o}_L))^{-1} \sum_{\mathcal{O} \in \Omega_G(0)} c_{\mathcal{O}}(\pi) \int_{\mathfrak{p}^r M_n(\mathfrak{o}_L)} \widehat{\mu}_{\mathcal{O}}(X)\, dX \\
&= \mathrm{vol}(\mathfrak{p}^r M_n(\mathfrak{o}_L)) \sum_{\mathcal{O} \in \Omega_G(0)} c_{\mathcal{O}}(\pi) \widehat{\mu}_{\mathcal{O}}(\widehat{\mathbf{1}}_{\mathfrak{p}^r M_n(\mathfrak{o}_L)}) \\
&= C \sum_{\mathcal{O} \in \Omega_G(0)} \mu_{\mathcal{O}}(\mathbf{1}_{\mathfrak{p}^{r_0 - r} M_n(\mathfrak{o}_L)})
\end{aligned}$$

where the constants C and r_0 depend on the Haar measure and additive character ψ. We note that $\mu_{\mathcal{O}}(\mathbf{1}_{\mathfrak{p}^{r_0 - r} M_n(\mathfrak{o}_L)})$ grows as $q^{r \cdot \dim \mathcal{O}}$. As such, for large r, $\dim \pi^{\Gamma(\mathfrak{p}^r)}$ is polynomial in q^r with degree at most the maximal dimension d of any nilpotent orbit.

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