# Yang-Mills Replacement

by

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## Abstract

We develop an analog of the harmonic replacement technique of Colding and Minicozzi in the gauge theory context. The idea behind harmonic replacement dates back to Schwarz and Perron, and the technique involves taking a function $v\colon \Sigma \to M$ defined on a surface $\Sigma$ and replacing its values on a small ball $B^2 \subset \Sigma$ with a harmonic function $u$ that has the same values as $v$ on the boundary $\partial B^2$. The resulting function on $\Sigma$ has lower energy, and repeating this process on balls covering $\Sigma$, one can obtain a global harmonic map in the limit. We develop the analogous procedure in the gauge theory context. We take a connection $B$ on a bundle over a four-manifold $X$, and replace it on a small ball $B^4 \subset X$ with a Yang-Mills connection $A$ that has the same restriction to the boundary $\partial B^4$ as $B$, and we obtain bounds on the difference $\|B - A\|^2_{L^2_1(B^4)}$ in terms of the drop in energy. Throughout, we work with connections of the lowest possible regularity $L^2_1(X)$, the natural choice for this context, and so our gauge transformations are in $L^2_2(X)$ and therefore almost but not quite continuous, leading to more delicate arguments than are available in higher regularity.

Thesis Supervisor: Tomasz Mrowka
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# Contents

# Chapter 1

# Introduction

The goal of this thesis is to adapt the harmonic replacement technique of Colding and Minicozzi [2] to the gauge theory context. In the classical harmonic replacement techniques of Schwarz [16] and Perron [12], given a real-valued function $v$ on a domain $\Omega$ and a ball $B^n \subset \Omega$, a function $u$ on $\Omega$ is constructed by replacing $v$ on $B^n$ with a harmonic function with the same values on the boundary of $B^n$. In other words, outside of $B^n$, $u$ is equal to $v$, and on $B^n$, $u$ is equal to the solution of the Dirichlet problem for the Laplacian with boundary value $v|_{\partial B^n}$. This procedure decreases energy, and, repeating this process for balls covering $\Omega$, one can obtain a harmonic function on all of $\Omega$. In [2], Colding and Minicozzi adapt this technique to the nonlinear context of maps $v\colon \Sigma \to M$ from a two-dimensional surface $\Sigma$ to a manifold $M$, where they replace $v$ on a small ball $B^2 \subset \Sigma$ with a harmonic map $u$. In this thesis, we do the analogous construction for connections on a principal $G$-bundle over a compact four-manifold $X$, where $G$ is compact. Given such a connection $B$ and a four-ball $B^4 \subset X$, we construct a connection $A$ by replacing $B$ with a Yang-Mills connection on $B^4$ whose restriction to the boundary $\partial B^4$ matches that of $B$. More precisely, we prove the following theorem, presented as Corollary 4.5.

**Theorem 1.1.** *Let $P \to X$ be a principal $G$-bundle over compact 4-manifold $X$ with compact gauge group $G$, and let $B^4 \subset X$ be a 4-ball. Let $\mathcal{C}$ be the space of $L^2_1(X)$ connections modulo $L^2_2(X)$ gauge transformations, and let $\mathcal{C}_{\varepsilon,B^4}$ be those gauge* *equivalence classes of connections $[B]$ with small energy on $B^4$, that is, $\|F_B\|_{L^2(B^4)} < \varepsilon$. Then for small enough $\varepsilon$ there is an energy-decreasing continuous map $\mathcal{C}_{\varepsilon,B^4} \to \mathcal{C}_{\varepsilon,B^4}$ sending $[B]$ to an equivalence class of connections $[A]$, where $A$ is Yang-Mills on $B^4$ and gauge equivalent to $B$ outside $B^4$.*

Note that we work with connections in the borderline $L^2_1(X)$ regularity, which is the natural choice in four dimensions, but leads to more delicate arguments than for smooth connections. In particular, in the borderline regularity, we do not have a Sobolev embedding $L^2_2(X) \not\hookrightarrow C^0(X)$, as a result of which the group of $L^2_2(X)$ gauge transformations is not a Hilbert Lie group. However, working with smooth connections would be insufficient for our purposes, because, after replacing a smooth connection with a Yang-Mills connection on a ball $B^4 \subset X$, the resulting connection is not smooth across the boundary $\partial B^4$.

We can also express the Yang-Mills replacement map above as a homotopy, at least on compact families of connections, presented as Corollary 4.6.

**Theorem 1.2.** *Let $P \to X$ be a principal $G$-bundle over compact 4-manifold $X$ with compact gauge group $G$, and let $\mathcal{C}$ be the space of $L^2_1(X)$ connections modulo $L^2_2(X)$ gauge transformations. Let $\mathcal{K}$ be a compact family in $\mathcal{C}$. Then around any point $x \in X$ there exists a ball $x \in B^4 \subset X$ and homotopy $h_t\colon \mathcal{K} \to \mathcal{C}$ such that $h_1$ is the identity, $h_0$ sends $\mathcal{K}$ to connections that are Yang-Mills on $B^4$, $h_t([B])$ has monotone nondecreasing energy, and restricting to the complement of $B^4$ the homotopy is constant $h_t([B]) = [B]$.*

One should think of $\mathcal{K}$ as representing a homology or homotopy class. Applying harmonic replacement to a compact family of maps is a key step in [2], where Colding and Minicozzi apply harmonic replacement to a one-parameter family of maps $v_t\colon \Sigma^2 \to M^3$ representing a sweep-out of $M^3$ in order to prove finite extinction time of Ricci flow on homotopy 3-spheres. In the gauge theory context, mapping compact families of connections to compact families of Yang-Mills connections is a way to relate the topology of the moduli space of anti-self-dual Yang-Mills connections to the much better understood space of all connections modulo gauge, as seen in work of Taubes

[19, 20, 21] and Donaldson [4]. In turn, the topology of the moduli space of anti-self-dual Yang-Mills connections gives rise to Donaldson invariants, which have myriad applications and have been used to show that a topological manifold has no smooth structures [3] or infinitely many smooth structures [9]. In addition, recent work of Feehan and Leness [7] connects Donaldson invariants to the newer Seiberg-Witten invariants [25].

Another potential source of applications of Yang-Mills replacement arises from its similarity to Yang-Mills gradient flow, in that both give energy-decreasing paths of connections. Yang-Mills gradient flow has been extensively studied recently [6, 18], and perhaps the local nature of each replacement step and the greater control afforded by the choice of the balls $B^4 \subset X$ will lead to the use of Yang-Mills replacement as an alternative to or in conjunction with Yang-Mills gradient flow.

In this thesis, we perform Yang-Mills replacement on a single ball $B^4 \subset X$, but we can repeat this process on balls covering the compact manifold $X$. Repeating this process indefinitely, one would like to pass to the limit to obtain a global Yang-Mills connection, but doing so is a delicate matter and is the natural direction in which to continue this work. The difficulty arises because, although energy is decreasing, it may concentrate around finitely many points. Because the Yang-Mills replacement theorems above require there to be small energy on the ball $B^4$, to continue the replacement process indefinitely, we would need to choose balls whose radii shrink to zero. This bubbling behavior is a common feature of all of the nonlinear contexts discussed and has been studied for sequences of maps on surfaces [15], for general sequences of connections [17], and for Yang-Mills gradient flow [6]. Based on Sedlacek's work [17], one expects a weak limit of the sequence of connections to exist, but potentially on a different bundle, and it is natural to ask if one can say more about the bubbling behavior for Yang-Mills replacement.

The key ideas of this thesis are in Chapters 3 and 4. In Chapter 3, we discuss the local question of finding a Yang-Mills connection on a ball $B^4$. More precisely, we prove the following theorem, presented more fully as Theorem 3.1.

**Theorem 1.3.** *Let $B^4$ be a smooth 4-ball with arbitrary metric, let $i\colon \partial B^4 \to B^4$ be* *the inclusion, and let $P \to B^4$ be a principal $G$-bundle with trivializing connection $d$. There exists an $\varepsilon > 0$ such that if $A_\partial = d + a_\partial$ is an $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ connection with $\|a_\partial\|_{L^2_{1/2}(\partial B^4)} < \varepsilon$, then $A_\partial$ extends to an $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ Yang-Mills connection $A$ with $i^*A = A_\partial$, and $A$ depends smoothly on $A_\partial$.*

Then, using the gauge fixing results of Chapter 5, in Theorems 3.5 and 3.6, we broaden the hypotheses of this theorem and prove a uniqueness result for the solutions. Along the way, we prove the following energy convexity result, presented in greater generality as Proposition 3.4.

**Proposition 1.4.** *Let $A = d + a$ and $B = d + b$ be $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections with bounds on energy, in Coulomb gauge $d^*a = d^*b = 0$, and with bounds on $\|a\|_{L^4(B^4)}$ and $\|b\|_{L^4(B^4)}$. If $A$ and $B$ have the same restrictions to the boundary $i^*A = i^*B$ and $A$ is Yang-Mills, then*

$$\|B - A\|^2_{L^2_1(B^4)} \leq C \left( \|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)} \right).$$

This problem has been discussed by Marini [11] and Rivière [14], but instead of a direct energy minimization method used in their work, we use the inverse function theorem, allowing us to conclude smooth dependence of the solution $A$ on the boundary value $A_\partial$. The inverse function theorem method motivates the definition of $L^{2,n}_{-1}(B^4; \bigwedge^* T^*B^4)$ in Chapter 2 as the dual of those forms in $L^2_1(B^4; \bigwedge^* T^*B^4)$ that are normal to the boundary $\partial B^4$. This space is a more appropriate codomain of the Yang-Mills operator $A \mapsto d^*_A F_A$ on $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections than the standard space $L^2_{-1}(B^4; \bigwedge^* T^*B^4)$, which is the dual of $L^2_1(B^4; \bigwedge^* T^*B^4)$ forms that vanish on the boundary. We finish Chapter 3 by constructing the local version of the energy-decreasing homotopy in Theorem 1.2.

As we pass to the global question in Chapter 4, the main issue to be addressed is that in Theorem 1.3, we are able to prescribe the tangential component $i^*A$ of $A$ on the boundary $\partial B^4$, but not the normal component. As a result, when we take a global $L^2_1(X)$ connection $B$ and construct a connection $A$ that is Yang-Mills on the ball $B^4$ and equal to $B$ outside the ball, the tangential components of $A$ and

$B$ match on the boundary $\partial B^4$, but the normal components might not, and so the resulting piecewise-defined global connection is not $L^2_1(X)$. This motivates the definition of $L^2_d(X;\bigwedge^* T^*X)$, a space between $L^4(X;\bigwedge^* T^*X)$ and $L^2_1(X;\bigwedge^* T^*X)$ defined in Chapter 2 as those forms $\alpha$ such that $\alpha \in L^4(X;\bigwedge^* T^*X)$ and $d\alpha \in L^2(X;\bigwedge^* T^*X)$. However, unlike forms in $L^2_1(X;\bigwedge^* T^*X)$, $d^*\alpha$ is not necessarily in $L^2(X;\bigwedge^* T^*X)$. We show that the resulting connection $A$ is an $L^2_d(X)$ connection. However, losing regularity after a Yang-Mills replacement step is unsatisfactory because it prevents us from repeating the Yang-Mills replacement process on an overlapping ball. Because $L^2_d(X)$ connections have well-defined $L^2(X)$ curvatures, one might ask if a gauge fixing argument could show that they are gauge equivalent to $L^2_1(X)$ connections. We answer this question in the affirmative, proving the following theorem, presented as Corollary 4.4.

**Theorem 1.5.** *The space of $L^2_d(X)$ connections modulo $L^4_1(X)$ gauge transformations is homeomorphic to the space of $L^2_1(X)$ connections modulo $L^2_2(X)$ gauge transformations.*

To prove this theorem locally, we show that Uhlenbeck's gauge fixing results extend to the $L^2_d(X)$ regularity, so, locally, an $L^2_d(X)$ connection is gauge equivalent to an $L^2_1(X)$ connection. However, patching together these local gauge transformations to obtain the global result is a delicate matter because $L^4_1(X)$ and $L^2_2(X)$ gauge transformations are at the borderline regularity and hence are not continuous and do not have a smooth exponential map, so naïve cutoff function methods fail.

Finally, in Chapter 5, we develop the gauge fixing machinery that powers the results in Chapters 3 and 4, building off of results by Uhlenbeck [23, 24] and Marini [11]. For all of these results, we start with an $L^2_1(B^4)$ connection $A$ on a ball $B^4$ with small energy $\|F_A\|_{L^2(B^4)} < \varepsilon$, and we find an $L^2_2(B^4;G)$ gauge transformation $g$ that sends $A$ to a connection $\tilde{A} = d + \tilde{a}$ satisfying the Coulomb condition $d^*\tilde{a} = 0$ and a bound on $\|\tilde{a}\|_{L^2_1(B^4)}$. However, there are two natural boundary conditions to impose on the connection, either the Neumann conditions $i^* {*\tilde{a}} = 0$ or the Dirichlet conditions $d^*_{\partial B^4} i^*\tilde{a}$, where $i^*$ is the restriction to the boundary. Uhlenbeck [23] provides a full treatment for the Neumann boundary conditions, but her treatment of the problem with Dirichlet boundary conditions in [24] and the later improvement by Marini [11] have additional regularity assumptions on $A$. We prove the result without these assumptions. Along the way, we also prove the Coulomb gauge fixing result where we impose Dirichlet boundary conditions on the gauge transformation instead of on the connection. Finally, we extend these results to $L^2_d(B^4)$ connections $A$, and we improve the weak $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ convergence of the Coulomb gauge representatives in [23] to strong $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ convergence.

# Chapter 2

# Preliminaries

## 2.1 Yang-Mills Connections

Following the standard references [10, 5], we introduce the notation we will use for principal $G$-bundles and connections. Let $G$ be a compact Lie group. We fix a unitary representation $G \hookrightarrow U_N \subset M_N$, where $M_N$ denotes the vector space of $N$ by $N$ complex matrices.

**Definition 2.1.** Let $P \to X$ be a principal $G$-bundle over a compact manifold $X$, and let $\operatorname{ad} P$ denote the associated bundle $P \times_G \mathfrak{g}$. Let $A$ be an $L^2_1(X)$ connection, and let $F_A \in L^2(X; \operatorname{ad} P \otimes \bigwedge^2 T^*X)$ be its curvature. The *energy* of $A$ is

$$\mathcal{E}(A) = \tfrac{1}{2} \int_X |F_A|^2 = \tfrac{1}{2} \|F_A\|^2_{L^2(X)}.$$

**Definition 2.2.** A *Yang-Mills connection* $A$ is a critical point of the functional $\mathcal{E}$. If $X$ has boundary, then we require $A$ to be a critical point with respect to variations $A_t$ such that $i^*A_t$ is gauge equivalent to $i^*A$ on the boundary, where $i\colon \partial X \hookrightarrow X$ is the inclusion.

Using variations that are fixed on the boundary, we see that a Yang-Mills connection $A$ satisfies the *Yang-Mills equations*

$$\langle F_A, d_A c \rangle_{L^2(X)} = 0$$

for all $c \in L^2_1(X; \operatorname{ad} P \otimes T^*X)$ with $i^*c = 0$ on $\partial X$. When we are working over a local trivialization $d$ of $P$ over $B^4 \subset X$, we will also make use of the *projected Yang-Mills equations*, where we only require that $\langle F_A, d_A c\rangle_{L^2(B^4)} = 0$ for $c \in L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ satisfying $d^*c = 0$ on $B^4$ in addition to $i^*c = 0$ on $\partial B^4$.

We now review gauge transformations and their action on connections.

**Definition 2.3.** A *gauge transformation* is an automorphism of $P$. A gauge transformation can be represented by a section of the associated bundle $\operatorname{Ad} P = P \times_G G \subset P \times_G M_N$ with the conjugation action of $G$ on $G \subset M_N$. By an $L^p_k$ gauge transformation we mean an $L^p_k$ section $g$ of the vector bundle $P \times_G M_N$ such that $g(x) \in \operatorname{Ad} P$ a.e. on $X$.

With respect to a local trivialization of $P$ over $B^4 \subset X$, a gauge transformation is a $G$-valued function, and we can write down how explicitly how it acts on a connection $A$ expressed in this trivialization as $d + a$, where $a$ is a $\mathfrak{g}$-valued one-form. We have

$$g(A) = d + gag^{-1} - (dg)g^{-1}.$$

Writing $g(A) = B = d + b$, we can rewrite the above equation as

$$dg = ga - bg,$$

where the terms in the equation are $M_N$-valued one-forms.

When $(k+1)p > \dim X$, it is well-known [8, 23] that the group of $L^p_{k+1}$ gauge transformations has smooth multiplication and inversion and acts smoothly on $L^p_k$ connections, using the multiplication map $L^p_{k+1} \times L^p_{k+1} \to L^p_{k+1}$ and the Sobolev embedding $L^p_{k+1} \hookrightarrow C^0$.

In the borderline case $(k+1)p = \dim X$, the matter is more delicate. Because gauge transformations are $G$-valued and $G$ is compact, they are uniformly bounded in $L^\infty$. As a result, multiplication of borderline $L^p_{k+1}$ gauge transformations is still well-defined, as is their action on $L^p_k$ connections. However, these maps are only smooth with the $L^p_{k+1} \cap L^\infty$ topology on gauge transformations. With just the $L^p_{k+1}$

topology, the situation is more subtle. With just the $L^p_{k+1}$ topology, multiplication of gauge transformations and the action of gauge transformations on connections are no longer smooth maps, but nonetheless they are continuous. We will prove this claim for $L^2_2$ gauge transformations on a 4-manifold, but the argument works for general borderline groups. The key idea that gives us just enough power to prove continuity is that $G$-valued functions act as isometries on $L^p$ spaces.

**Proposition 2.4.** *Let $P$ be a principal $G$-bundle over a compact 4-manifold $X$ with compact group $G \hookrightarrow U_N \subset M_N$. The group of $L^2_2(X)$ gauge transformations has continuous multiplication and inversion maps, and $L^2_2(X)$ gauge transformations act continuously on $L^2_1(X)$ connections.*

*Proof.* We work over a closed ball in a local trivialization $B^4 \subset X$. Consider a sequence of $L^2_2(B^4; G)$ gauge transformations $g_i$ and $h_i$ converging in $L^2_2(B^4; G)$ to $g$ and $h$, respectively. We aim to show that $g_i h_i$ converges to $gh$ in $L^2_2(B^4; G)$. By the Sobolev multiplication maps, we know that $g_i h_i$ converges to $gh$ in any weaker space, such as $L^2(B^4; G)$.

We compute that

$$\nabla^2(g_i h_i) = (\nabla^2 g_i)h_i + 2(\nabla g_i)(\nabla h_i) + g_i(\nabla^2 h_i).$$

The middle term is straightforward. We know that the sequences $\nabla g_i$ and $\nabla h_i$ converge in $L^2_1(B^4; M_N \otimes T^*B^4)$, and we have a Sobolev multiplication map $L^2_1 \times L^2_1 \to L^2$. The other two terms are more subtle. We know that $\nabla^2 g_i$ converges in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$ to $\nabla^2 g$, and to avoid repeating the argument later, we instead consider a general sequence $\phi_i$ converging in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$ to $\phi$, with $\phi_i = \nabla^2 g_i$ in this particular case.

We show that the $\phi_i h_i$ converge in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$ to $\phi h$ by showing that every subsequence of the $\phi_i h_i$ has a further subsequence that converges to $\phi h$. We begin by passing to a subsequence of the $\phi_i h_i$. The $h_i$, being $G$-valued a.e., are uniformly bounded in $L^\infty(B^4; M_N)$. As a result, the $\phi_i h_i$ are uniformly bounded in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$, and hence after passing to a further subsequence the $\phi_i h_i$ converge weakly in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$. This weak limit must be $\phi h$ because we know that $\phi_i h_i$ converges to $\phi h$ in any weaker norm, such as $L^1(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$. To upgrade this weak convergence to strong convergence, we note that multiplication by an element of $G$ is an isometry of $M_N$, and hence

$$\|\phi_i h_i\|_{L^2(B^4)} = \|\phi_i\|_{L^2(B^4)} \to \|\phi\|_{L^2(B^4)} = \|\phi h\|_{L^2(B^4)}$$

For $L^2$, and in general for $L^p$ spaces with $1 < p < \infty$ [13], weak convergence along with convergence of the sequence of norms to the norm of the limit implies strong convergence. We conclude then that this further subsequence of the $\phi_i h_i$ converges strongly in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$ to $\phi h$, and hence so does the original sequence. Similarly, the sequence $g_i(\nabla^2 h_i)$ converges to $g(\nabla^2 h)$ in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$, and so $\nabla^2(g_i h_i)$ converges to $\nabla^2(gh)$ in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$ as desired.

Inversion is continuous by a much simpler argument. Because we have chosen a representation $G \hookrightarrow U_N$, inversion is the same as the conjugate transpose, which is a linear, and hence smooth, map $L^2_2(B^4; M_N) \to L^2_2(B^4; M_N)$. Using this fact, we can show that the action of gauge transformations on connections is continuous by an analogous argument to the above.

We would like to show that the map $g(a)$ defined by

$$\begin{aligned} L^2_2(B^4; G) \times L^2_1(B^4; \mathfrak{g} \otimes T^*B^4) &\to L^2_1(B^4; \mathfrak{g} \otimes T^*B^4) \\ (g, a) &\mapsto gag^{-1} - (dg)g^{-1} \end{aligned}$$

is continuous. Again, we choose sequences $g_i$ and $a_i$ that converge in $L^2_2(B^4; G)$ and $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ to $g$ and $a$, respectively, and we aim to show that the $g_i(a_i)$ converge to $g(a)$ in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. From the Sobolev multiplication theorems, we have convergence of the $g_i(a_i)$ to $g(a)$ in any weaker space, such as $L^2(B^4; \mathfrak{g} \otimes T^*B^4)$. We compute

$$\nabla(g_i(a_i)) = (\nabla g_i)a_i g_i^{-1} + g_i(\nabla a_i)g_i^{-1} + g_i a_i(\nabla g_i^{-1}) - (\nabla dg_i)g_i^{-1} - dg_i(\nabla g_i^{-1}).$$

Using the Sobolev multiplication theorems, we know that $(\nabla g_i)a_i$, $\nabla a_i$, $a_i \nabla g_i^{-1}$, $\nabla dg_i$, and $dg_i(\nabla g_i^{-1})$ all converge in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$ to the expected limits. As a result, to prove convergence of the $\nabla(g_i(a_i))$ in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$, it suffices to prove the statement that if $\phi_i$ converges to $\phi$ in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$, then $g_i\phi_i$ and $\phi_i g_i^{-1}$ converge to $g\phi$ and $\phi g^{-1}$ in $L^2(B^4; M_N \otimes T^*B^4 \otimes T^*B^4)$. Since the $g_i^{-1}$ converge in $L^2_2(B^4; G)$ to $g^{-1}$, we proved this statement for $\phi_i g_i^{-1}$ above, and the statement for $g_i\phi_i$ is analogous. $\square$

## 2.2 The Hodge Decomposition Theorem with Boundary Conditions

In this section, we summarize the treatment in [22, Section 5.9]. Let $X$ be a smooth manifold with boundary $\partial X$, and let $i\colon \partial X \to X$ be the inclusion.

**Definition 2.5.** Let $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$ denote the $L_1^2(X)$ differential forms $\alpha$ that are normal to the boundary, that is, they satisfy the Dirichlet boundary condition $i^*\alpha = 0$. Likewise, let $L_1^{2,\mathrm{t}}(X; \bigwedge^* T^*X)$ denote the $L_1^2(X)$ differential forms $\alpha$ that are tangent to the boundary, that is, they satisfy the Neumann boundary condition $i^* {*\alpha} = 0$, where $*$ is Hodge star operator.

**Definition 2.6.** Let $\mathcal{H}^{\mathrm{n}}$ denote the harmonic forms in $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$. That is, $\mathcal{H}^{\mathrm{n}}$ contains those $L_1^2(X; \bigwedge^* T^*X)$ forms $\alpha$ such that $i^*\alpha = 0$, $d\alpha = 0$, and $d^*\alpha = 0$. Likewise, let $\mathcal{H}^{\mathrm{t}}$ denote those $L_1^2(X; \bigwedge^* T^*X)$ forms $\alpha$ such that $i^* {*\alpha} = 0$, $d\alpha = 0$, and $d^*\alpha = 0$.

**Proposition 2.7** ([22, 5.9.36,5.9.38]). *The forms in $\mathcal{H}^{\mathrm{n}}$ and $\mathcal{H}^{\mathrm{t}}$ are smooth.*

**Proposition 2.8** ([22, 5.9.9]). *The natural map from the Dirichlet harmonic forms into the cohomology of $X$ rel boundary is an isomorphism. That is, $\mathcal{H}^{\mathrm{n}} \cong H^*(X, \partial X)$. Likewise, $\mathcal{H}^{\mathrm{t}} \cong H^*(X)$.*

**Proposition 2.9** ([22, 5.9.8]). *There exists Green's functions*

$$G^{\mathrm{n}}\colon L^2(X;\textstyle\bigwedge^* T^*X) \to L^2_2(X;\bigwedge^* T^*X), \text{ and}$$
$$G^{\mathrm{t}}\colon L^2(X;\textstyle\bigwedge^* T^*X) \to L^2_2(X;\bigwedge^* T^*X)$$

*such that:*

1. *For all $L^2(X;\bigwedge^* T^*X)$ differential forms $\alpha$,*

$$\alpha = dd^*G^{\mathrm{n}}\alpha + d^*dG^{\mathrm{n}}\alpha + \pi^{\mathrm{n}}_{\mathcal{H}}\alpha,$$
$$\alpha = dd^*G^{\mathrm{t}}\alpha + d^*dG^{\mathrm{t}}\alpha + \pi^{\mathrm{t}}_{\mathcal{H}}\alpha,$$

   *where $\pi^{\mathrm{n}}_{\mathcal{H}}$ and $\pi^{\mathrm{t}}_{\mathcal{H}}$ denote the $L^2(X)$ projections to the finite-dimensional spaces $\mathcal{H}^{\mathrm{n}}$ and $\mathcal{H}^{\mathrm{t}}$, respectively.*

2. *The operators $dd^*G^{\mathrm{n}}$, $d^*dG^{\mathrm{n}}$, and $\pi^{\mathrm{n}}_{\mathcal{H}}$ are $L^2(X)$-projections whose ranges are $L^2(X)$-orthogonal to each other. Likewise, the operators $dd^*G^{\mathrm{t}}$, $d^*dG^{\mathrm{t}}$, and $\pi^{\mathrm{t}}_{\mathcal{H}}$ are $L^2(X)$-projections whose ranges are $L^2(X)$-orthogonal to each other.*

3. *The range of $G^{\mathrm{n}}$ satisfies the boundary conditions $i^*G^{\mathrm{n}}\alpha = 0$ and $i^*d^*G^{\mathrm{n}}\alpha = 0$.*

4. *The range of $G^{\mathrm{t}}$ satisfies the boundary conditions $i^*{*G^{\mathrm{t}}}\alpha = 0$ and $i^*d^*{*G^{\mathrm{t}}}\alpha = 0$.*

5. *For any $k \geq 0$, $G^{\mathrm{n}}, G^{\mathrm{t}}\colon L^2_k(X;\bigwedge^* T^*X) \to L^2_{k+2}(X;\bigwedge^* T^*X)$.*

**Corollary 2.10.** *Let $X$ be a smooth manifold with smooth boundary, and let $k \geq 0$. Let $L^{2,\mathrm{n}}_{k+1}(X;\bigwedge^* T^*X)$ denote those $L^2_{k+1}(X;\bigwedge^* T^*X)$ forms $\alpha$ such that $i^*\alpha = 0$ on $\partial X$. Then*

$$d + d^*\colon L^{2,\mathrm{n}}_{k+1}(X;\textstyle\bigwedge^* T^*X) \to L^2_k(X;\bigwedge^* T^*X)$$

*has kernel and cokernel $\mathcal{H}^{\mathrm{n}}$.*

*Likewise, let $L^{2,\mathrm{t}}_{k+1}(X;\bigwedge^* T^*X)$ denote those $L^2_{k+1}(X;\bigwedge^* T^*X)$ forms $\alpha$ such that $i^*{*\alpha} = 0$ on $\partial X$. Then*

$$d + d^*\colon L^{2,\mathrm{t}}_{k+1}(X;\textstyle\bigwedge^* T^*X) \to L^2_k(X;\bigwedge^* T^*X)$$

*has kernel and cokernel* $\mathcal{H}^{\mathrm{t}}$.

*Proof.* Assume that $\alpha \in L^{2,\mathrm{n}}_{k+1}(X; \bigwedge^* T^*X)$ is in the kernel of $d + d^*$. The condition $i^*\alpha = 0$ implies that

$$\langle d\alpha, d^*\alpha \rangle_{L^2_k(X)} = \langle \alpha, d^*d^*\alpha \rangle_{L^2_k(X)} + \int_{\partial X} \alpha \wedge *d^*\alpha = 0.$$

Hence, $(d + d^*)\alpha = 0$ implies $d\alpha = 0$ and $d^*\alpha = 0$, so $\alpha \in \mathcal{H}^{\mathrm{n}}$ by definition.

Next, we show that the range has trivial intersection with $\mathcal{H}^{\mathrm{n}}$. Let $(d + d^*)\alpha = \phi$ and $i^*\alpha = 0$, where $\phi \in \mathcal{H}^{\mathrm{n}}$. Then

$$\begin{aligned} \|\phi\|^2_{L^2_k(X)} &= \langle d\alpha, \phi \rangle_{L^2_k(X)} + \langle d^*\alpha, \phi \rangle_{L^2_k(X)} \\ &= \langle \alpha, d^*\phi \rangle_{L^2_k(X)} + \langle \alpha, d\phi \rangle_{L^2_k(X)} + \int_{\partial X} \alpha \wedge *\phi - \phi \wedge *\alpha = 0. \end{aligned}$$

Finally, we show that the range of $d+d^*$ contains all $\beta \in L^2_k(X; \bigwedge^* T^*X)$ where $\beta$ is $L^2$-orthogonal to $\mathcal{H}^{\mathrm{n}}$. Proposition 2.9 gives us $G^{\mathrm{n}} \colon L^2_k(X; \bigwedge^* T^*X) \to L^2_2(X; \bigwedge^* T^*X)$ such that $\Delta G^{\mathrm{n}}\beta = \beta$ if $\beta$ is orthogonal to $\mathcal{H}^{\mathrm{n}}$. Hence, our desired preimage is $\alpha = (d + d^*)G^{\mathrm{n}}\beta$. By Proposition 2.9, we have boundary conditions $i^*G^{\mathrm{n}}\beta = 0$ and $i^*d^*G^{\mathrm{n}}\beta = 0$. Hence,

$$i^*\alpha = i^*dG^{\mathrm{n}}\beta + i^*d^*G^{\mathrm{n}}\beta = di^*G^{\mathrm{n}}\beta = 0,$$

so $\alpha \in L^{2,\mathrm{n}}_{k+1}(X; \bigwedge^* T^*X)$, as desired.

The second claim is analogous, or, alternatively, it follows from the identity

$$d + d^* = (-1)^{n(p-1)+1} * (d + d^*) *,$$

where $(-1)^{n(p-1)+1}$ acts on $\bigwedge^* T^*X$ by $(-1)^{n(p-1)+1}$ on the degree $p$ component of the exterior algebra, along with the facts that the isometry $*$ sends $L^{2,\mathrm{n}}_{k+1}(X; \bigwedge^* T^*X)$ to $L^{2,\mathrm{t}}_{k+1}(X; \bigwedge^* T^*X)$ and vice versa, and $*$ sends $\mathcal{H}^{\mathrm{n}}$ to $\mathcal{H}^{\mathrm{t}}$ and vice versa. $\square$

## 2.3 The space $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$

The Yang-Mills operator $A \mapsto d_A^* F_A$ is a second-order operator, so if our connection $A$ is in $L_1^2(X; \operatorname{ad} P \otimes T^*X)$, then $d_A^* F_A \in L^2_{-1}(X; \operatorname{ad} P \otimes T^*X)$. However, the space $L^2_{-1}(X)$ ends up being insufficient for our purposes. By definition, $L^2_{-1}(X; \bigwedge^* T^*X)$ is the dual of $L_1^2(X; \bigwedge^* T^*X)_0$, that is, $L_1^2(X; \bigwedge^* T^*X)$ forms that vanish on the boundary $\partial X$. We need to instead define a larger space, $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$, as the dual of $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$, that is, $L_1^2(X; \bigwedge^* T^*X)$ forms whose *tangential* components vanish on the boundary $\partial X$. In the remainder of the section, we prove basic results about the space $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$ needed to show that $d_A^* F_A$ and $\pi_{d^*} d_A^* F_A$ remain well-defined when their target is $L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X)$ instead of $L^2_{-1}(X; \operatorname{ad} P \otimes T^*X)$.

**Definition 2.11.** Let $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$ denote the dual Hilbert space of $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$. (See Definition 2.5.)

**Proposition 2.12.** *The space $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$ is reflexive, and smooth functions are dense in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$.*

*Proof.* Since $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$ is a closed subspace of $L_1^2(X; \bigwedge^* T^*X)$, the reflexivity of $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$ follows from [1, 1.21] and the reflexivity of $L_1^2(X; \bigwedge^* T^*X)$ [1, 3.5]. Meanwhile, using the reflexivity of $L_1^{2,\mathrm{n}}(X; \bigwedge^* T^*X)$, an argument like in [1, 3.12] shows that $L^2(X; \bigwedge^* T^*X)$ is dense in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$, and so $C^\infty(X)$ is dense in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$ also. $\square$

**Lemma 2.13.** *The operators*

$$\begin{aligned} d&\colon L_1^2(X; \textstyle\bigwedge^* T^*X) \to L^2(X; \textstyle\bigwedge^* T^*X), \text{ and} \\ d^*&\colon L_1^2(X; \textstyle\bigwedge^* T^*X) \to L^2(X; \textstyle\bigwedge^* T^*X) \end{aligned}$$

*have closed ranges.*

*Proof.* The operators

$$dd^*G^{\mathrm{t}} \colon L^2(X;\textstyle\bigwedge^*T^*X) \to L^2(X;\bigwedge^*T^*X), \text{ and}$$
$$d^*dG^{\mathrm{n}} \colon L^2(X;\textstyle\bigwedge^*T^*X) \to L^2(X;\bigwedge^*T^*X)$$

are projections and hence have closed ranges, so we proceed by showing that range$(d) =$ range$(dd^*G^{\mathrm{t}})$ and range$(d^*) =$ range$(d^*dG^{\mathrm{n}})$.

To show that range$(d) \subseteq$ range$(dd^*G^{\mathrm{t}})$, consider $d\alpha$ for $\alpha \in L^2_1(X;\bigwedge^*T^*X)$. By Proposition 2.9, since $d\alpha \in L^2(X;\bigwedge^*T^*X)$, we have an orthogonal decomposition $d\alpha = dd^*G^{\mathrm{t}}d\alpha + d^*dG^{\mathrm{t}}d\alpha + \pi^{\mathrm{t}}_{\mathcal{H}}\alpha$. I claim that, in fact, $d\alpha = dd^*G^{\mathrm{t}}d\alpha$. Because the decomposition is orthogonal, we simply check that

$$\begin{aligned}
\big\langle d\alpha, d^*dG^{\mathrm{t}}d\alpha\big\rangle_{L^2(X;\bigwedge^*T^*X)} &= \big\langle dd\alpha, dG^{\mathrm{t}}d\alpha\big\rangle_{L^2(X;\bigwedge^*T^*X)} - \int_{\partial X} d\alpha \wedge *dG^{\mathrm{t}}d\alpha = 0,\\
\big\langle d\alpha, \pi^{\mathrm{t}}_{\mathcal{H}}d\alpha\big\rangle_{L^2(X;\bigwedge^*T^*X)} &= \big\langle \alpha, d^*\pi^{\mathrm{t}}_{\mathcal{H}}d\alpha\big\rangle_{L^2(X;\bigwedge^*T^*X)} + \int_{\partial X} \alpha \wedge *\pi^{\mathrm{t}}_{\mathcal{H}}d\alpha = 0.
\end{aligned}$$

Here, we used the boundary conditions $i^*{*}dG^{\mathrm{t}}d\alpha = \pm i^*d^*{*}G^{\mathrm{t}}d\alpha = 0$ and $i^*{*}\pi^{\mathrm{t}}_{\mathcal{H}}d\alpha = 0$, along with the fact that $i^*(d\alpha)$ is well-defined even though $d\alpha \in L^2(X;\bigwedge^*T^*X)$ because of the identity $i^*d\alpha = di^*\alpha$. Hence, range$(d) =$ range$(dd^*G^{\mathrm{t}})$.

Likewise, for $d^*$, we have

$$\begin{aligned}
\langle d^*\alpha, dd^*G^{\mathrm{n}}d^*\alpha\rangle_{L^2(X)} &= \langle d^*d^*\alpha, d^*G^{\mathrm{n}}d^*\alpha\rangle_{L^2(X)} + \int_{\partial X} d^*G^{\mathrm{n}}d^*\alpha \wedge *d^*\alpha = 0,\\
\langle d^*\alpha, \pi^{\mathrm{n}}_{\mathcal{H}}d^*\alpha\rangle_{L^2(X)} &= \langle \alpha, d\pi^{\mathrm{n}}_{\mathcal{H}}d^*\alpha\rangle_{L^2(X)} - \int_{\partial X} \pi^{\mathrm{n}}_{\mathcal{H}}d^*\alpha \wedge *\alpha = 0,
\end{aligned}$$

using $i^*d^*G^{\mathrm{n}} = 0$ and $i^*(\mathcal{H}^{\mathrm{n}}) = 0$, along with the fact that $i^*{*}d^*\alpha$ is well-defined in $L^2_{-1/2}(\partial X;\bigwedge^*T^*\partial X)$ by the equation $i^*{*}d^*\alpha = \pm i^*d * \alpha = \pm di^*{*}\alpha$. Hence $d^*\alpha = d^*dG^{\mathrm{n}}\alpha$, so range$(d^*) =$ range$(d^*dG^{\mathrm{n}})$. □

**Proposition 2.14.** *The operator* $d^* \colon C^\infty(X;\bigwedge^*T^*X) \to L^{2,\mathrm{n}}_{-1}(X;\bigwedge^*T^*X)$ *extends to a bounded operator* $d^* \colon L^2(X;\bigwedge^*T^*X) \to L^{2,\mathrm{n}}_{-1}(X;\bigwedge^*T^*X)$ *with closed range.*

*Proof.* Let $f \in C^\infty(X;\bigwedge^*T^*X)$, and let $\phi \in L^{2,\mathrm{n}}_1(X;\bigwedge^*T^*X)$. Because $i^*\phi = 0$, we have

$$\langle d^* f, \phi\rangle_{L^2(X)} = \langle f, d\phi\rangle_{L^2(X)} - \int_{\partial X} \phi \wedge *f = \langle f, d\phi\rangle_{L^2(X)}.$$

Hence,

$$\left|\langle d^* f, \phi\rangle_{L^2(X)}\right| \leq \|f\|_{L^2(X)} \|\phi\|_{L^2_1(X)},$$

so $\|d^* f\|_{L^{2,\mathrm{n}}_{-1}(X)} \leq \|f\|_{L^2(X)}$. Because $C^\infty$ is dense in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$, we conclude that $d^*$ extends to a bounded operator $d^*\colon L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$, defined by the equation

$$\langle d^* f, \phi\rangle_{L^2(X)} = \langle f, d\phi\rangle_{L^2(X)}$$

for $f \in L^2(X; \bigwedge^* T^* X)$ and $\phi \in L^{2,\mathrm{n}}_1(X; \bigwedge^* T^* X)$.

By the closed range theorem, $d^*\colon L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ having closed range is equivalent to its transpose $d\colon L^{2,\mathrm{n}}_1(X; \bigwedge^* T^* X) \to L^2(X; \bigwedge^* T^* X)$ having closed range. On a larger domain, we know that $d\colon L^2_1(X; \bigwedge^* T^* X) \to L^2(X; \bigwedge^* T^* X)$ has closed range by Lemma 2.13. Because $L^2_1(X; \bigwedge^* T^* X)$ is a Hilbert space, $\ker d$ has a closed complement $(\ker d)^\perp$, so $d\colon (\ker d)^\perp \to \mathrm{range}(d)$ is an isomorphism of Banach spaces. Therefore, the image under $d$ of the closed space $(\ker d)^\perp \cap \ker i^*$ is closed. Summing with $\ker d \cap \ker i^*$, we see that $d(\ker i^*) = d((\ker d)^\perp \cap \ker i^*)$, so $d$ also has closed range as an operator $L^{2,\mathrm{n}}_1(X; \bigwedge^* T^* X) \to L^2(X; \bigwedge^* T^* X)$. Therefore, by the closed range theorem, $d^*\colon L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ has closed range, as desired. □

**Proposition 2.15.** *The operator*

$$\pi_{d^*} = d^* d G^{\mathrm{n}}\colon L^2(X; \textstyle\bigwedge^* T^* X) \to L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$$

*extends to a bounded operator* $\pi_{d^*}\colon L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$*, and this operator is a projection to the range of* $d^*\colon L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$*.*

*Proof.* Let $y \in L^2(X; \bigwedge^* T^* X)$, and let $\phi \in L^{2,\mathrm{n}}_1(X; \bigwedge^* T^* X)$. Because $\pi_{d^*}$ is a projection operator to a factor in the Hodge decomposition, we have

$$\langle \pi_{d^*} y, \phi\rangle_{L^2(X)} = \langle y, \pi_{d^*} \phi\rangle_{L^2(X)}.$$

By Proposition 2.9, $\phi \in L^2_1(X; \bigwedge^* T^* X)$ implies $\pi_{d^*}\phi \in L^2_1(X; \bigwedge^* T^* X)$. Furthermore, I claim that $\phi \in L^{2,\mathrm{n}}_1(X; \bigwedge^* T^* X)$ implies $\pi_{d^*}\phi \in L^{2,\mathrm{n}}_1(X; \bigwedge^* T^* X)$. Indeed,

$$0 = i^*\phi = i^*(dd^*G^{\mathrm{n}}\phi + d^*dG^{\mathrm{n}}\phi + \pi_{\mathcal{H}^{\mathrm{n}}}\phi) = di^*d^*G^{\mathrm{n}}\phi + i^*\pi_{d^*}\phi = i^*\pi_{d^*}\phi$$

because $i^*(\mathcal{H}^{\mathrm{n}}) = 0$ by definition and $i^*d^*G^{\mathrm{n}} = 0$ by Proposition 2.9. Hence,

$$\left|\langle \pi_{d^*}y, \phi\rangle_{L^2(X)}\right| \leq \|y\|_{L^{2,\mathrm{n}}_{-1}(X)} \|\pi_{d^*}\phi\|_{L^2_1(X)} \leq C\,\|y\|_{L^{2,\mathrm{n}}_{-1}(X)} \|\phi\|_{L^2_1(X)},$$

for some constant $C$, so

$$\|\pi_{d^*}y\|_{L^{2,\mathrm{n}}_{-1}(X)} \leq C\,\|y\|_{L^{2,\mathrm{n}}_{-1}(X)}.$$

Since $L^2(X; \bigwedge^* T^* X)$ is dense in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$, we see that $\pi_{d^*}$ extends to a bounded operator $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$, defined by the equation

$$\langle \pi_{d^*}y, \phi\rangle_{L^2(X)} = \langle y, \pi_{d^*}\phi\rangle_{L^2(X)}.$$

To show that $\pi_{d^*}\colon L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ is a projection to the range of $d^*\colon L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$, first recall that in the proof of Lemma 2.13 we showed that the range of $d^*\colon L^2_1(X; \bigwedge^* T^* X) \to L^2(X; \bigwedge^* T^* X)$ is equal to the range of the projection $\pi_{d^*}\colon L^2(X; \bigwedge^* T^* X) \to L^2(X; \bigwedge^* T^* X)$. Hence, if $y \in d^*(L^2_1(X; \bigwedge^* T^* X))$, then $y = \pi_{d^*}y$. Since $L^2_1(X; \bigwedge^* T^* X)$ is dense in $L^2(X; \bigwedge^* T^* X)$ and $d^*\colon L^2(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ is bounded, we know that $d^*(L^2_1(X; \bigwedge^* T^* X))$ is dense in $d^*(L^2(X; \bigwedge^* T^* X)) \subset L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$. Because $\pi_{d^*}\colon L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ is continuous, the equation $y = \pi_{d^*}y$ remains true for all $y \in d^*(L^2(X; \bigwedge^* T^* X))$. Thus, $d^*(L^2(X; \bigwedge^* T^* X)) \subseteq \pi_{d^*}\left(L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)\right)$.

Conversely, for all $y \in L^2(X; \bigwedge^* T^* X)$, we know that $\pi_{d^*}y \in d^*(L^2_1(X; \bigwedge^* T^* X)) \subseteq d^*(L^2(X; \bigwedge^* T^* X))$. We know that $L^2(X; \bigwedge^* T^* X)$ is dense in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$, that $\pi_{d^*}\colon L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X) \to L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ is continuous, and that $d^*(L^2(X; \bigwedge^* T^* X))$ is closed in $L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^* X)$ by Lemma 2.14, so $\pi_{d^*}y \in d^*(L^2(X; \bigwedge^* T^* X))$ remains true for all $y \in L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)$. Hence, $\pi_{d^*}\left(L^{2,\mathrm{n}}_{-1}(X; \bigwedge^* T^*X)\right) = d^*(L^2(X; \bigwedge^* T^*X))$, and the above fact that $y = \pi_{d^*} y$ for all $y \in d^*(L^2(X; \bigwedge^* T^*X))$ now implies that $\pi_{d^*}$ is a projection, as desired. $\square$

**Corollary 2.16.** *The operator $A \mapsto \pi_{d^*} d_A^* F_A$ is well-defined and smooth as an operator $L^2_1(X; \operatorname{ad} P \otimes T^*X) \to L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X) \cap \operatorname{range}(d^*)$.*

*Proof.* We know that $A \mapsto F_A$ is smooth as an operator

$$L^2_1(X; \operatorname{ad} P \otimes T^*X) \to L^2(X; \operatorname{ad} P \otimes \textstyle\bigwedge^2 T^*X).$$

By Proposition 2.14, $d^* \colon L^2(X; \operatorname{ad} P \otimes \bigwedge^2 T^*X) \to L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X)$ is a bounded linear operator, and hence smooth. The multiplication map $L^2_1(X) \times L^2(X) \times L^{2,\mathrm{n}}_1(X) \hookrightarrow L^4(X) \times L^2(X) \times L^4(X) \to L^1(X) \to \mathbb{R}$ is bounded, and hence by duality so is the bilinear multiplication map $L^2_1(X) \times L^2(X) \to L^{2,\mathrm{n}}_{-1}(X)$, so $A \mapsto [a\wedge]^* F_A$ is smooth as a map $L^2_1(X; \operatorname{ad} P \otimes T^*X) \to L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X)$. Thus $A \mapsto d_A^* F_A = d^* F_A + [a\wedge]^* F_A$ is smooth as a map $L^2_1(X; \operatorname{ad} P \otimes T^*X) \to L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X)$. Finally, by Proposition 2.15, $\pi_{d^*} \colon L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X) \to L^{2,\mathrm{n}}_{-1}(X; \operatorname{ad} P \otimes T^*X)$ is a bounded linear operator, and hence smooth, and its range is $\operatorname{range}(d^*)$. $\square$

## 2.4 The space $L^2_d(X; \bigwedge^* T^*X)$

Given an $L^2_1(X; \operatorname{ad} P \otimes T^*X)$ connection $A$ on $X$ and a ball $B^4$ in $X$, we will replace $A$ with a $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ Yang-Mills connection $B$ on $B^4$ whose tangential components match $A$ on $\partial B^4$. The resulting piecewise-defined global connection $A'$ is in $L^4(X; \operatorname{ad} P \otimes T^*X)$, but because the normal component of $B$ does not match that of $A$ on $\partial B^4$, the new connection $A'$ is not in $L^2_1(X; \operatorname{ad} P \otimes T^*X)$. However, the fact that the tangential components match still gives us more regularity than $L^4(X; \operatorname{ad} P \otimes T^*X)$, In fact, $A'$ still has enough regularity to define curvature $F_{A'} \in L^2(X; \operatorname{ad} P \otimes \bigwedge^2 T^*X)$. This leads us to define a space inbetween $L^4(X; \operatorname{ad} P \otimes T^*X)$ and $L^2_1(X; \operatorname{ad} P \otimes T^*X)$ which we call $L^2_d(X; \operatorname{ad} P \otimes T^*X)$.

**Definition 2.17.** Let $X$ be a compact smooth manifold, and let $L^2_d(X; \bigwedge^* T^*X)$ be the completion of smooth forms $\alpha$ under the norm

$$\|\alpha\|_{L^4(X)} + \|d\alpha\|_{L^2(X)} .$$

**Proposition 2.18.** *Let $X$ be a compact smooth manifold that decomposes as the union $Y \cup Z$, where $Y \cap Z = S$ is a component of the boundary of $Y$ and $Z$, and let $i\colon S \hookrightarrow X$ be the inclusion. Let $\beta \in L^2_1(Y; \bigwedge^* T^*Y)$ and $\gamma \in L^2_1(Z; \bigwedge^* T^*Z)$ such that $i^*\beta = i^*\gamma$. Let $\alpha$ be the $L^4$ form on $X$ defined piecewise by $\beta$ and $\gamma$. Then $\alpha \in L^2_d(X; \bigwedge^* T^*X)$.*

*Proof.* The key idea is that we have a discontinuity in the normal component as we cross $S$, but when taking $d$ we never take the normal derivative of the normal component, so we never see the discontinuity.

The question is local, and the operator $d$ commutes with diffeomorphisms, so, taking a chart around a neighborhood of a point $x \in S$, we can work on $\mathbb{R} \times \mathbb{R}^{n-1}$, with $\beta$ and $\gamma$ compactly supported $L^2_1$ forms on $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$, respectively, such that the tangential components of $\beta$ and $\gamma$ match on the interface $\{0\} \times \mathbb{R}^{n-1}$. Moreover, we can assume without loss of generality that $\gamma = 0$. Indeed, we can extend $\gamma$ to an $L^2_1$ form $\hat{\gamma}$ on all of $\mathbb{R}^n$, and subtract it from $\alpha$, $\beta$, and $\gamma$. Since $L^2_d(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$ is contained in $L^2_1(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$, $\alpha$ is in $L^2_d(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$ if and only if $\alpha - \hat{\gamma}$ is.

Hence, we have reduced our problem to the situation of a compactly supported $\beta \in L^2_1(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$ such that the tangential component $i^*\beta$ is zero, where $i\colon \{0\} \times \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ is the inclusion, and we aim to show that if $\alpha$ is the extension of $\beta$ by zero to all of $\mathbb{R}^n$, then $\alpha \in L^2_d(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$. Our goal now is to construct smooth forms $\alpha_i$ on $\mathbb{R}^n$ that converge to $\alpha$ in $L^2_d(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$. That is, we need $\alpha_i \to \beta$ in $L^4(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$, $\alpha_i \to 0$ in $L^4(\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$, and $d\alpha_i$ to converge in $L^2(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$.

We can decompose $\beta$ into tangential and normal components $\beta = \beta^{\top} + \beta^{\perp}$. That is, if $x_0$ is the first coordinate of $\mathbb{R}^n$, we can decompose $\bigwedge^* T^*\mathbb{R}^n$ into the subbundle of normal forms generated by the standard basis elements that contain $dx_0$ and the subbundle of tangential forms generated by the standard basis elements that do not contain $dx_0$. By assumption, $\beta^\top$ is zero on $\{0\} \times \mathbb{R}^{n-1}$, and there is no condition on $\beta^\perp$. For $\beta^\top$, by [1, 7.54], the fact that $\beta^\top$ is zero on the boundary lets us choose a smooth sequence of tangential forms $\beta_i^\top$ converging to $\beta^\top$ in $L_1^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$ such that each $\beta_i^\top$ is compactly supported away from the boundary. Then we can extend $\beta_i^\top$ by zero to obtain a smooth form $\alpha_i^\top$ on all of $\mathbb{R}^n$.

For the normal component, extend $\beta^\perp$ arbitrarily to a compactly supported $L_1^2$ normal form $\hat{\beta}^\perp$ on all of $\mathbb{R}^n$. Then, construct a smooth approximating sequence of normal forms $\hat{\beta}_i^\perp$ converging to $\hat{\beta}^\perp$ in $L_1^2(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$. Finally, let $0 \leq \phi(x_0) \leq 1$ be a cutoff function supported on $x_0 \geq -1$ with $\phi(x_0) = 1$ for $x_0 \geq 0$. Let $\alpha_i^\perp = \phi(ix_0)\hat{\beta}_i^\perp$, and let $\alpha_i = \alpha_i^\top + \alpha_i^\perp$. It remains to show that the sequence $\alpha_i$ has the desired properties.

On $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, we have $\alpha_i = \alpha_i^\top + \alpha_i^\perp = \beta_i^\top + \hat{\beta}_i^\perp$, which converges to $\beta^\top + \beta^\perp = \beta$ in $L_1^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$, and hence also in $L^4(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$. Meanwhile, on $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$,

$$\begin{aligned} \|\alpha_i\|_{L^4(x_0 \leq 0)} &= \left\|\phi(ix_0)\hat{\beta}_i^\perp\right\|_{L^4(x_0\leq 0)} \\ &\leq \left\|\phi(ix_0)\hat{\beta}^\perp\right\|_{L^4(x_0\leq 0)} + \left\|\phi(ix_0)(\hat{\beta}_i^\perp - \hat{\beta}^\perp)\right\|_{L^4(x_0\leq 0)} \\ &\leq \left\|\hat{\beta}^\perp\right\|_{L^4(-1/i\leq x_0\leq 0)} + \left\|\hat{\beta}_i^\perp - \hat{\beta}^\perp\right\|_{L^4(x_0\leq 0)} \to 0. \end{aligned}$$

Finally, we compute

$$\begin{aligned} d\alpha_i = \sum_{k=0}^{n-1} dx_k \wedge \frac{\partial \alpha_i^\top}{\partial x_k} &+ \sum_{k=1}^{n-1} dx_k \wedge \frac{\partial}{\partial x_k}(\phi(ix_0)\hat{\beta}_i^\perp) \\ &= \sum_{k=0}^{n-1} dx_k \wedge \frac{\partial \alpha_i^\top}{\partial x_k} + \phi(ix_0) \sum_{k=1}^{n-1} dx_k \wedge \frac{\partial \hat{\beta}_i^\perp}{\partial x_k}. \end{aligned}$$

The key fact here is that because $\hat{\beta}_i^\perp$ already contains a $dx_0$ term, we never take the derivative in the $x_0$ direction of the cutoff function $\phi(ix_0)$, and so that term remains bounded. We can test $L^2(\mathbb{R}^n)$ convergence piecewise. First, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, we have $\frac{\partial \alpha_i^\top}{\partial x_k} = \frac{\partial \beta_i^\top}{\partial x_k}$, which converges in $L^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$ to $\frac{\partial \beta^\top}{\partial x_k}$ because the $\beta_i^\top$ converge to $\beta^\top$ in $L_1^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$. Likewise, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, we have $\phi(ix_0)\frac{\partial \hat{\beta}_i^\perp}{\partial x_k} = \frac{\partial \beta_i^\perp}{\partial x_k}$, which converges to $\frac{\partial \beta^\perp}{\partial x_k}$ in $L^2(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}; \bigwedge^* T^*\mathbb{R}^n)$. Meanwhile, on $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$, we show convergence to zero. Because the $\alpha_i^\top$ are supported away from $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$, the $\frac{\partial \alpha_i^\top}{\partial x_k}$ are simply zero on $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$. Finally,

$$\begin{aligned}
\left\| \phi(ix_0)\frac{\partial \hat{\beta}_i^\perp}{\partial x_k} \right\|_{L^2(x_0 \leq 0)} &\leq \left\| \phi(ix_0)\frac{\partial \hat{\beta}^\perp}{\partial x_k} \right\|_{L^2(x_0 \leq 0)} + \left\| \phi(ix_0)\left( \frac{\partial \hat{\beta}_i^\perp}{\partial x_k} - \frac{\partial \hat{\beta}^\perp}{\partial x_k} \right) \right\|_{L^2(x_0 \leq 0)} \\
&\leq \left\| \frac{\partial \hat{\beta}^\perp}{\partial x_k} \right\|_{L^2(-1/i \leq x_0 \leq 0)} + \left\| \frac{\partial \hat{\beta}_i^\perp}{\partial x_k} - \frac{\partial \hat{\beta}^\perp}{\partial x_k} \right\|_{L^2(x_0 \leq 0)} \to 0.
\end{aligned}$$

Hence, the $d\alpha_i$ converge in $L^2(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$ to the $L^2(\mathbb{R}^n; \bigwedge^* T^*\mathbb{R}^n)$ function defined by $d\beta$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $0$ on $\mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$, as desired. □

**Proposition 2.19.** *Let $X$ be a compact smooth manifold with a principal $G$-bundle $P$, and let $A$ be an $L_d^2$ connection on $P$. Then $F_A$ is in $L^2(X; \operatorname{ad} P \times \bigwedge^2 T^*X)$. Moreover, if $g$ is a $L_1^4(X; \operatorname{Ad} P)$ gauge transformation, then $g(A)$ is once again a $L_d^2$ connection.*

*Proof.* The question is local, so we work on a compact subset $K$ of a trivialization. Let $A = d + a$, so $F_A = da + \frac{1}{2}[a \wedge a]$. Since $a \in L_d^2(K; \mathfrak{g} \otimes T^*K)$, we know that $da \in LF2K$, and $a \in L^4(K; \mathfrak{g} \otimes T^*K)$, so $\frac{1}{2}[a \wedge a] \in L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$. Thus $F_A \in L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$, as desired. Likewise, for $g \in L_1^4(K; G)$, let $g(A) = B = d + b$. Then

$$b = gag^{-1} - (dg)g^{-1}.$$

We have $a \in L^4(K; \mathfrak{g} \otimes T^*K)$, $g \in L_1^4(K; G)$, and, since $g$ is a gauge transformation, $g \in L^\infty(K; G)$. We can compute that then $b \in L^4(K; \mathfrak{g} \otimes T^*K)$. It remains to show that $db \in L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$. We compute

$$db = F_B - \tfrac{1}{2}[b \wedge b] = gF_A g^{-1} - \tfrac{1}{2}[b \wedge b].$$

Since $F_A \in L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$ and $g \in L^\infty(K; G)$, we know that $gF_A g^{-1}$ is in $L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$. Likewise, we showed that $b \in L^4(K; \mathfrak{g} \otimes T^*K)$, so $\frac{1}{2}[b \wedge b] \in$

$L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$, and so $db \in L^2(K; \mathfrak{g} \otimes \bigwedge^2 T^*K)$, as desired. $\square$

# Chapter 3

# The Dirichlet problem

In this chapter, we solve the Yang-Mills equation on $B^4$ with prescribed small boundary data in $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ using the inverse function theorem. Using gauge fixing, we then extend this result to a more general class of boundary values in Theorem 3.5. In addition, the inverse function theorem gives us local uniqueness of the solution, which we strengthen in Theorem 3.6. Along the way, in Proposition 3.4, we prove strict convexity in Coulomb gauge of the energy functional near small-energy Yang-Mills connections. Earlier work on this problem includes a paper by Marini [11] thatsolves the Dirichlet problem with boundary data assumed to be smooth on general compact manifolds, as well as lecture notes of Rivière [14] which solve the Dirichlet problem on the ball in the critical $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ regularity using direct minimization methods of Sedlacek [17]. An advantage of the inverse function theorem method is that it gives smooth dependence of the Yang-Mills solution on the boundary data. Finally, in Section 3.1 we show energy monotonicity of the linear path between an a connection and the Yang-Mills replacement that matches it on the boundary.

**Theorem 3.1.** *Let $B^4$ be a smooth 4-ball with arbitrary metric, let $i\colon \partial B^4 \to B^4$ be the inclusion, and let $P \to B^4$ be a principal $G$-bundle with trivializing connection $d$. There exist an $\varepsilon > 0$ and $\delta > 0$ such that if $A_\partial = d + a_\partial$ is an $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ connection with $\|a_\partial\|_{L^2_{1/2}(\partial B^4)} < \varepsilon$, then $A_\partial$ extends to a* unique *$L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$*

*connection* $A = d + a$ *such that*

1. $\|a\|_{L^2_1(B^4)} < \delta$,

2. *A satisfies the Yang-Mills equation* $d_A^* F_A = 0$ *on* $B^4$,

3. $i^* A = A_\partial$, *and*

4. *A satisfies the Coulomb condition* $d^* a = 0$.

*Moreover, A depends smoothly on* $A_\partial$.

We first prove the theorem replacing the Yang-Mills equation $d_A^* F_A = 0$ with a weaker projected Yang-Mills equation $\pi_{d^*} d_A^* F_A = 0$ (see Section 2.3), and then prove that in this situation the weaker equation $\pi_{d^*} d_A^* F_A = 0$ actually implies the full equation $d_A^* F_A = 0$.

**Proposition 3.2.** *There exist an* $\varepsilon > 0$ *and* $\delta > 0$ *such that if* $A_\partial = d + a_\partial$ *is an* $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^* \partial B^4)$ *connection with* $\|a_\partial\|_{L^2_{1/2}(\partial B^4)} < \varepsilon$, *then* $A_\partial$ *extends to a* unique $L^2_1(B^4; \mathfrak{g} \otimes T^* B^4)$ *connection* $A = d + a$ *such that*

1. $\|a\|_{L^2_1(B^4)} < \delta$,

2. *A satisfies the projected Yang-Mills equation* $\pi_{d^*} d_A^* F_A = 0$ *on* $B^4$,

3. $i^* A = A_\partial$, *and*

4. *A satisfies the Coulomb condition* $d^* a = 0$.

*Moreover, A depends smoothly on* $A_\partial$.

*Proof.* We consider the projected Yang-Mills operator

$$\begin{aligned} pYM\colon\, & L^2_1(B^4; \mathfrak{g} \otimes T^* B^4) \cap \ker d^* \\ & \to L^{2,\mathrm{n}}_{-1}(B^4; \mathfrak{g} \otimes T^* B^4) \cap \operatorname{range}(d^*) \times L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^* \partial B^4) \end{aligned}$$

defined by

$$pYM(a) = (\pi_{d^*} d_A^* F_A, i^* a).$$

We have that $A \mapsto \pi_{d^*} d_A^* F_A$ is smooth by Corollary 2.16 and $i^* \colon L_1^2(B^4; \mathfrak{g} \otimes T^*B^4) \to L_{1/2}^2(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ is a bounded linear operator and hence smooth.

Proving the proposition amounts to showing that $pYM$ is an isomorphism on a neighborhood of $a = 0$, for our desired $L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ extension $a$ is the inverse image of $(0, a_\partial)$. We do so using the inverse function theorem. The linearization of $pYM$ at $a = 0$ is

$$\begin{aligned}(d^*d, i^*) \colon & L_1^2(B^4; \mathfrak{g} \otimes T^*B^4) \cap \ker d^* \\ & \to L_{-1}^{2,\mathrm{n}}(B^4; \mathfrak{g} \otimes T^*B^4) \cap \operatorname{range}(d^*) \times L_{1/2}^2(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)\end{aligned}$$

It remains to show that this operator is an isomorphism of Banach spaces.

Let $a \in L_1^2(B^4; \mathfrak{g} \otimes T^*B^4) \cap \ker d^*$, and assume that $d^*da = 0$ and $i^*a = 0$. Thus $a \in L_1^{2,\mathrm{n}}(B^4; \mathfrak{g} \otimes T^*B^4)$, and so Proposition 2.14 tells us that

$$\langle d^*da, a \rangle_{L^2(X)} = \langle da, da \rangle_{L^2(X)}.$$

Since $d^*da = 0$, we conclude that $da = 0$. Since $d^*a = 0$ and $i^*a = 0$, we conclude that $a \in \mathcal{H}^\mathrm{n}$. But $H^1(B^4, \partial B^4) = 0$, so $a = 0$. Hence $(d^*d, i^*)$ is injective.

We first prove surjectivity onto $L_{-1}^{2,\mathrm{n}}(B^4; \mathfrak{g} \otimes T^*B^4) \cap \operatorname{range}(d^*) \times 0$. Let $y = d^*f$ for $f \in L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$. Then

$$y = d^*f = d^*(dd^*G^\mathrm{n}f + d^*dG^\mathrm{n}f + \pi_{\mathcal{H}^\mathrm{n}}f) = d^*d(d^*G^\mathrm{n}f).$$

Clearly, $d^*G^\mathrm{n}f \in \ker d^*$. Moreover, $i^*(d^*G^\mathrm{n}f) = 0$ by Proposition 2.9. Hence, $d^*G^\mathrm{n}f$ is our desired preimage of $(y, 0)$ under the map $(d^*d, i^*)$.

Now, given $a_\partial \in L_{1/2}^2(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$, the inverse trace map [1, Theorem 7.53] gives us an $a_1 \in L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ such that $i^*a_1 = a_\partial$. Then $d^*da_1$ is in the space $L_{-1}^{2,\mathrm{n}}(B^4; \mathfrak{g} \otimes T^*B^4) \cap \operatorname{range}(d^*)$, so the previous paragraph gives us an $a_2 \in L_1^2(B^4; \mathfrak{g} \otimes T^*B^4) \cap \ker d^*$ such that $d^*da_2 = d^*da_1$ and $i^*a_2 = 0$. Hence $(d^*d, i^*)(a_1 - a_2) = (0, a_\partial)$, giving us surjectivity onto the other factor $0 \times L_{1/2}^2(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ also.

We conclude that

$$(d^*d, i^*)\colon L^2_1(B^4; \mathfrak{g}\otimes T^*B^4)\cap \ker d^* \\ \to L^{2,\mathrm{n}}_{-1}(B^4; \mathfrak{g}\otimes T^*B^4)\cap \mathrm{range}(d^*) \times L^2_{1/2}(\partial B^4; \mathfrak{g}\otimes T^*\partial B^4)$$

is an isomorphism of Banach spaces. Hence, by the inverse function theorem, the projected Yang-Mills operator

$$pYM\colon L^2_1(B^4; \mathfrak{g}\otimes T^*B^4)\cap \ker d^* \\ \to L^{2,\mathrm{n}}_{-1}(B^4; \mathfrak{g}\otimes T^*B^4)\cap \mathrm{range}(d^*) \times L^2_{1/2}(\partial B^4; \mathfrak{g}\otimes T^*\partial B^4)$$

is a diffeomorphism between a neighborhood of $a = 0$ in $L^2_1(B^4; \mathfrak{g}\otimes T^*B^4)\cap \ker d^*$ and a neighborhood of $(y, a_\partial) = (0, 0)$ in

$$L^{2,\mathrm{n}}_{-1}(B^4; \mathfrak{g}\otimes T^*B^4)\cap \mathrm{range}(d^*) \times L^2_{1/2}(\partial B^4; \mathfrak{g}\otimes T^*\partial B^4).$$

In particular, for $a_\partial$ small in $L^2_{1/2}(\partial B^4; \mathfrak{g}\otimes T^*\partial B^4)$, we can solve $pYM(a) = (0, a_\partial)$ for $a \in L^2_1(B^4; \mathfrak{g}\otimes T^*B^4)\cap \ker d^*$, giving us our desired small $a$ satisfying $\pi_{d^*}d^*_A F_A = 0$, $i^*a = a_\partial$, and $d^*a = 0$.

More precisely, choose $\delta$ and $\varepsilon$ such that the $\delta$-ball around $a = 0$ and the $\varepsilon$-ball around $(y, a_\partial) = (0, 0)$ are contained in the above neighborhoods between which $pYM$ is a diffeomorphism, and such that the $\varepsilon$-ball is contained in the image of the $\delta$-ball under $pYM$. Given $a_\partial$ with $\|a_\partial\|_{L^2_{1/2}(\partial B^4)} < \varepsilon$, let $a$ be the preimage under $pYM$ of $(0, a_\partial)$, so $a$ depends smoothly on $a_\partial$. In addition, $\|a\|_{L^2_1(B^4)} < \delta$, $\pi_{d^*}d^*_A F_A = 0$, $i^*a = a_\partial$, and $d^*a = 0$, as desired. Moreover, $a$ is uniquely determined by these conditions because $pYM$ is injective on the $\delta$-ball around $a = 0$. $\square$

To complete the proof of Theorem 3.1, it remains to prove the following.

**Proposition 3.3.** *There exists an $\varepsilon > 0$ such that for any $L^2_1(B^4; \mathfrak{g}\otimes T^*B^4)$ connection $A = d + a$, if*

1. $\|a\|_{L^2_1(B^4)} < \varepsilon$,

2. *A satisfies the projected Yang-Mills condition* $\pi_{d^*} d_A^* F_A = 0$, *and*

3. *A satisfies the Coulomb condition* $d^* a = 0$,

*then A satisfies the full Yang-Mills equation* $d_A^* F_A = 0$ *on* $B^4$.

In higher regularity, this proposition can be proved using bounds on $d_A^* F_A$, but at the critical regularity, we must proceed directly by showing that $A$ locally minimizes energy. We prove an inequality similar to one used by Colding and Minicozzi for harmonic maps [2, Theorem 3.1].

**Proposition 3.4.** *Let* $B^4$ *be a smooth 4-ball with arbitrary metric, let* $i\colon \partial B^4 \to B^4$ *be the inclusion, and let* $P \to B^4$ *be a principal G-bundle with trivializing connection* $d$. *There exist constants* $\varepsilon_4, \varepsilon_F$ *and* $C$ *with the following significance. Let* $A = d + a$ *and* $B = d + b$ *be* $L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ *connections such that*

1. $\|a\|_{L^4(B^4)} < \varepsilon_4$ *and* $\|b\|_{L^4(B^4)} < \varepsilon_4$,

2. $\|F_A\|_{L^2(B^4)} < \varepsilon_F$,

3. *A satisfies the projected Yang-Mills equation* $\pi_{d^*} d_A^* F_A = 0$,

4. *A and B match on the boundary, that is,* $i^* A = i^* B$, *and*

5. *we have a Coulomb condition* $d^* a = d^* b$.

*Then*

$$\|B - A\|_{L_1^2(B^4)}^2 \leq C \left( \|F_B\|_{L^2(B^4)}^2 - \|F_A\|_{L^2(B^4)}^2 \right).$$

*In particular,*

$$\|F_A\|_{L^2(B^4)} \leq \|F_B\|_{L^2(B^4)}.$$

*Proof.* The projected Yang-Mills equation $\pi_{d^*} d_A^* F_A = 0$ can be restated as the condition that $\langle F_A, d_A c \rangle_{L^2(B^4)} = 0$ for all $c \in L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ with $i^* c = 0$ and $d^* c = 0$. In particular, let $c = B - A = b - a$, so $i^* c = i^* B - i^* A = 0$ and

$d^*c = d^*b - d^*a = 0$. Hence, taking the square of the $L^2(B^4)$ norm of the curvature equation $F_B = F_A + d_A c + \frac{1}{2}[c \wedge c]$, we obtain

$$\begin{aligned}\|F_B\|^2_{L^2(B^4)} = \|F_A\|^2_{L^2(B^4)} + \|d_A c\|^2_{L^2(B^4)} + \left\|\tfrac{1}{2}[c \wedge c]\right\|^2_{L^2(B^4)} \\ + 2\left\langle F_A, \tfrac{1}{2}[c \wedge c]\right\rangle_{L^2(B^4)} + 2\left\langle d_A c, \tfrac{1}{2}[c \wedge c]\right\rangle_{L^2(B^4)}.\end{aligned}$$

We then have the inequality

$$\begin{aligned}\|d_A c\|^2_{L^2(B^4)} &\leq \|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)} - \left\|\tfrac{1}{2}[c \wedge c]\right\|^2_{L^2(B^4)} \\ &\quad + 2\|F_A\|_{L^2(B^4)}\left\|\tfrac{1}{2}[c \wedge c]\right\|_{L^2(B^4)} + 2\|d_A c\|_{L^2(B^4)}\left\|\tfrac{1}{2}[c \wedge c]\right\|_{L^2(B^4)} \\ &\leq \|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)} - \left\|\tfrac{1}{2}[c \wedge c]\right\|^2_{L^2(B^4)} \\ &\quad + 2\|F_A\|_{L^2(B^4)}\left\|\tfrac{1}{2}[c \wedge c]\right\|_{L^2(B^4)} + \tfrac{1}{2}\|d_A c\|^2_{L^2(B^4)} + 2\left\|\tfrac{1}{2}[c \wedge c]\right\|^2_{L^2(B^4)}.\end{aligned}$$

Rearranging,

$$\begin{aligned}\|d_A c\|^2_{L^2(B^4)} &\leq 2\left(\|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)}\right) \\ &\quad + \left(2\|F_A\|_{L^2(B^4)} + \tfrac{1}{2}\|[c \wedge c]\|_{L^2(B^4)}\right)\|[c \wedge c]\|_{L^2(B^4)} \\ &\leq 2\left(\|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)}\right) \\ &\quad + \left(2\|F_A\|_{L^2(B^4)} + \tfrac{1}{2}C_{\mathfrak{L}}\|c\|^2_{L^4(B^4)}\right)C_{\mathfrak{L}}C_S^2\|c\|^2_{L^2_1(B^4)},\end{aligned} \tag{3.1}$$

where $C_{\mathfrak{L}}$ is the operator norm of the Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and $C_S$ is the operator norm of the Sobolev embedding $L^2_1(B^4) \hookrightarrow L^4(B^4)$.

The next step is to bound $\|c\|_{L^2_1(B^4)}$ in terms of $\|d_A c\|_{L^2(B^4)}$. Since $H^1(B^4, \partial B^4) = 0$, by Corollary 2.10, $d + d^*\colon L^{2,\mathrm{n}}_1(B^4; \bigwedge^* T^* B^4) \to L^2(B^4; \bigwedge^* T^* B^4)$ is a Fredholm operator with no kernel on one-forms. Thus, we have the estimate $\|c\|_{L^2_1(B^4)} \leq C_G\|(d + d^*)c\|_{L^2(B^4)}$ for some constant $C_G$ independent of $c \in L^{2,\mathrm{n}}_1(B^4; \mathfrak{g} \otimes T^* B^4)$.

Recalling that $d^*c = 0$, we can compute

$$\begin{aligned}\|c\|_{L_1^2(B^4)} &\leq C_G \|dc\|_{L^2(B^4)} = C_G \|d_A c - [a \wedge c]\|_{L^2(B^4)} \\ &\leq C_G \left( \|d_A c\|_{L^2(B^4)} + C_{\mathfrak{L}} \|a\|_{L^4(B^4)} \|c\|_{L^4(B^4)} \right) \\ &\leq C_G \|d_A c\|_{L^2(B^4)} + C_G C_{\mathfrak{L}} C_S \|a\|_{L^4(B^4)} \|c\|_{L_1^2(B^4)}\end{aligned}$$

Requiring $\varepsilon_4 \leq \frac{1}{2}(C_G C_{\mathfrak{L}} C_S)^{-1}$, we obtain $\|c\|_{L_1^2(B^4)} \leq C_G \|d_A c\|_{L^2(B^4)} + \frac{1}{2} \|c\|_{L_1^2(B^4)}$. Rearranging, we obtain our desired bound

$$\|c\|_{L_1^2(B^4)} \leq 2C_G \|d_A c\|_{L^2(B^4)} .$$

Combining the above inequality with (3.1), we have

$$\begin{aligned}\|c\|^2_{L_1^2(B^4)} \leq 8C_G^2 &\left( \|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)} \right) \\ &+ \left( 8C_G^2 C_{\mathfrak{L}} C_S^2 \|F_B\|_{L^2(B^4)} + 2C_G^2 C_{\mathfrak{L}}^2 C_S^2 \|c\|^2_{L^4(B^4)} \right) \|c\|^2_{L_1^2(B^4)} . \quad (3.2)\end{aligned}$$

Requiring, for example, that $\varepsilon_F \leq \frac{1}{4}(8C_G^2 C_{\mathfrak{L}} C_S^2)^{-1}$ and $(2\varepsilon_4)^2 \leq \frac{1}{4}(2C_G^2 C_{\mathfrak{L}}^2 C_S^2)^{-1}$, and noting that $\|c\|_{L^4(B^4)} < 2\varepsilon_4$, the above inequality becomes

$$\|c\|^2_{L_1^2(B^4)} \leq 8C_G^2 \left( \|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)} \right) + \frac{1}{2} \|c\|^2_{L_1^2(B^4)} .$$

Rearranging, we obtain

$$\|c\|^2_{L_1^2(B^4)} \leq 16C_G^2 \left( \|F_A\|^2_{L^2(B^4)} - \|F_B\|^2_{L^2(B^4)} \right) ,$$

so our desired inequality is true with $C = 16C_G^2$. □

Under the assumptions of Propositon 3.3, Proposition 3.4 tells us that $A$ has smaller energy than any nearby connection $B$ that matches $A$ on the boundary and satisfies the Coulomb condition $d^*b = 0$. It remains to remove this last condition. To do so, we use gauge fixing results that we will prove in the Chapter 5.

*Proof of Proposition 3.3.* Let $B$ be any $L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ connection satisfying the bound $\|b\|_{L_1^2(B^4)} < \varepsilon$ and the boundary condition $i^*B = i^*A$. We will prove that $\|F_A\|_{L^2(B^4)} \le \|F_B\|_{L^2(B^4)}$.

Let $\varepsilon_U$ and $C_U$ be the constants from Proposition 5.4. Require $\varepsilon \le \varepsilon_U$, so we can apply Proposition 5.4 to $B$ to give us a gauge equivalent connection $\tilde{B}$ satisfying $d^*\tilde{b} = 0$, $i^*\tilde{B} = i^*B = i^*A$, and

$$\|\tilde{b}\|_{L_1^2(B^4)} \le C_U \left( \|F_B\|_{L^2(B^4)} + \|i^*b\|_{L_{1/2}^2(\partial B^4)} \right).$$

Let $\varepsilon_4$ and $\varepsilon_F$ be the corresponding constants from Proposition 3.4. We have the continuity of the trace map $L_1^2(B^4) \to L_{1/2}^2(\partial B^4)$ and Sobolev maps $L_{1/2}^2(\partial B^4) \to L^3(\partial B^4)$ and $L_1^2(B^4) \to L^4(B^4)$. Using these along with the continuiuty of $F_A$, we can choose $\varepsilon$ small enough so that $\|a\|_{L_1^2(B^4)} < \varepsilon$ implies $\|a\|_{L^4(B^4)} < \varepsilon_4$, $\|F_A\|_{L^2(B^4)} < \varepsilon_F$. Likewise, the inequality $\|\tilde{b}\|_{L_1^2(B^4)} \le C_U \left( \|F_B\|_{L^2(B^4)} + \|i^*b\|_{L_{1/2}^2(\partial B^4)} \right)$ and the continuity of $b \mapsto F_B$ and $b \mapsto i^*b$ let us choose $\varepsilon$ small enough so that $\|b\|_{L_1^2(B^4)} < \varepsilon$ implies $\|\tilde{b}\|_{L^4(B^4)} < \varepsilon_4$.

Since $i^*A = i^*\tilde{B}$, we can apply Proposition 3.4 to $A$ and $\tilde{B}$, since $d^*a = 0$ by assumption and $d^*\tilde{b} = 0$ by Theorem 5.4. We conclude that

$$\|F_A\|_{L^2(B^4)} \le \|F_{\tilde{B}}\|_{L^2(B^4)} = \|F_B\|_{L^2(B^4)},$$

as desired. Hence, $A$ locally minimzes energy among connections whose restrictions to $\partial B^4$ is $i^*A$.

However, in our definition of a Yang-Mills connection, we required that $A$ be a critical point of the energy functional with respect to variations whose restrictions to the boundary are gauge equivalent to $i^*A$, not necessarily equal to $i^*A$. Hence, we must show furthermore that $A$ has smaller energy that any connection $B$ in a neighborhood of $A$ such that $i^*A$ is gauge equivalent to $i^*B$. They key fact here is that, unlike the projected Yang-Mills condition, the condition that $A$ locally minimizes energy among connections whose restriction to the boundary is equal to $i^*A$ is a gauge-invariant condition.

Hence, we require $\varepsilon$ be small enough so that $\|a\|_{L_1^2(B^4)}, \|b\|_{L_1^2(B^4)} < \varepsilon$ implies that

$\|F_A\|_{L^2(B^4)}$ and $\|F_B\|_{L^2(B^4)}$ are small enough so that we can apply Theorem 5.10, giving us connections $\tilde{A}$ and $\tilde{B}$ gauge equivalent to $A$ and $B$, respectively, such that $d^*\tilde{a} = d^*\tilde{b} = 0$ and $d^*_{\partial B^4} i^*\tilde{a} = d^*_{\partial B^4} i^*\tilde{b} = 0$, along with bounds on $\|\tilde{a}\|_{L^2_1(B^4)}$ and $\|\tilde{b}\|_{L^2_1(B^4)}$. Choosing $\varepsilon$ small enough, we can bound $\|i^*\tilde{a}\|_{L^3(\partial B^4)}$ and $\|i^*\tilde{b}\|_{L^3(\partial B^4)}$ and apply Proposition 5.17 to find that the gauge transformation $g$ sending $i^*\tilde{A}$ to $i^*\tilde{B}$ is constant. We can apply this constant gauge transformation to $\tilde{A}$ on all of $B^4$ without affecting the Dirichlet Coulomb conditions $d^*\tilde{a} = 0$ and $d^*_{\partial B^4} i^*\tilde{a} = 0$, so we can assume without loss of generality that $g = 1$ and $i^*\tilde{A} = i^*\tilde{B}$. Since $A$ locally minimzes energy among connections whose restrictions match it on the boundary, so does $\tilde{A}$, so $\|F_A\|_{L^2(B^4)} = \|F_{\tilde{A}}\|_{L^2(B^4)} \le \|F_{\tilde{B}}\|_{L^2(B^4)} = \|F_B\|_{L^2(B^4)}$, as desired. □

Using gauge fixing, we can strengthen Theorem 3.1 to solve the Yang-Mills equation for a larger class of boundary values, namely restrictions to the boundary of small-energy connections on the ball. This result is exactly what we need for replacing a global connection on a small ball with a Yang-Mills connection.

**Theorem 3.5.** *There exists an $\varepsilon > 0$ with the following significance. Let $B$ be an $L^2_1(B^4)$ connection with $\|F_B\|_{L^2(B^4)} < \varepsilon$. Then we can construct a Yang-Mills connection $A$ that depends continuously on $B$ such that $i^*A = i^*B$.*

*Proof.* The idea is to apply a gauge fixing result to $B$ in order to make the boundary value small enough to apply Theorem 3.1. We will use Theorem 5.19, the gauge fixing result with Neumann boundary conditions, though Theorem 5.10 with Dirichlet boundary conditions would work equally well. Let $\varepsilon_U$ and $C$ be the constants from Theorem 5.19, and let $\varepsilon_\partial$ be the bound on the boundary value in Theorem 3.1. Require $\varepsilon \le \varepsilon_U$ and $C_T C\varepsilon \le \varepsilon_\partial$, where $C_T$ is the norm of the trace map $i^* \colon L^2_1(B^4) \to L^2_{1/2}(\partial B^4)$.

We can thus apply Theorem 5.19 to obtain a $L^2_2(B^4; G)$ gauge transformation $g$ sending $B$ to $\tilde{B} = d + \tilde{b}$ in Coulomb gauge with $\|\tilde{b}\|_{L^2_1(B^4)} \le C\,\|F_B\|_{L^2(B^4)}$, so

$$\|i^*\tilde{b}\|_{L^2_{1/2}(\partial B^4)} \le C_T\,\|\tilde{b}\|_{L^2_1(B^4)} \le C_T C\,\|F_B\|_{L^2(B^4)} < \varepsilon_\partial.$$

Thus we can apply Theorem 3.1 to $i^*\tilde{B}$ to obtain an $L^2_1(B^4)$ Yang-Mills connection $\tilde{A}$ such that $i^*\tilde{A} = i^*\tilde{B}$. We let $A = g^{-1}(\tilde{A})$, so $i^*A = i^*B$, as desired.

It remains to show that this construction is continuous, which is made more complex by the fact that $g$ is not uniquely determined by $B$ and hence might not depend contiuously on $B$. However, at least with appropriately chosen $\varepsilon$, $g$ is unique up to a constant gauge transformation $c$. We show that $A$ does not depend on the choice of $g$, so let $g' = cg$, let $\tilde{B}' = g'(B) = c(\tilde{B})$, and let $\tilde{A}'$ be the Yang-Mills connection given by applying Theorem 3.1 to $i^*\tilde{B}'$. I claim that $\tilde{A}' = c(\tilde{A})$. Indeed, $i^*(c(\tilde{A})) = i^*\tilde{B}'$, and the other conditions of Theorem 3.1 are preserved under constant gauge transformations and hence are true of $c(\tilde{A})$. Since the connection given by Theorem 3.1 is unique, we conclude that $\tilde{A}' = c(\tilde{A})$, and so $A' = (g')^{-1}(\tilde{A}') = g^{-1}(\tilde{A}) = A$, as desired.

We can use this uniqueness to show that this construction is continuous. Indeed, let $B_i \to B$ be a sequence of connections converging in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. Let $A_i$ and $A$ be the corresponding Yang-Mills connections constructed above. We will show that $A_i$ converges to $A$ by showing that any subsequence of the $A_i$ has a further subsequence that converges to $A$.

Hence, we begin by passing to a subsequence of the $B_i$, which, of course, still converges to $B$, and so from Proposition 5.22 we know that, after passing to a further subsequence, we can have the Coulomb gauge representatives $\tilde{B}_i$ converging to a Coulomb gauge representative $\tilde{B}$ of $B$. However, $\tilde{B}$ is only determined up to a constant gauge transformation and may depend on our initial choice of subsequence. In addition, Lemma 5.3 gives us that, after passing to a subsequence, the gauge transformations $g_i$ sending $B_i$ to $\tilde{B}_i$ converge in $L^2_2(B^4; G)$ to the gauge transformation $g$ sending $B$ to $\tilde{B}$. Theorem 3.1 gives us that the $\tilde{A}_i$ depend smoothly on the $i^*\tilde{B}_i$, which depend linearly on the $\tilde{B}_i$, so we know that the $\tilde{A}_i$ converge to $\tilde{A}$. Finally, because the $g_i$ converge strongly to $g$ in $L^2_2(B^4; G)$, we know that the $A_i = g_i^{-1}(\tilde{A}_i)$ converge strongly to $A$ in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. By our previous argument, even though $\tilde{B}$ might depend up to a constant gauge transformation on our initial choice of subsequence, $A$ is unique and thus is independent of the initial choice of subsequence of the $B_i$.

Thus, $A$ depends continuously on $B$, as desired. $\square$

We now use gauge fixing to prove a stronger uniqueness result for the Yang-Mills solution.

**Theorem 3.6.** *There exists an $\varepsilon > 0$ such that if $A$ and $B$ are $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ Yang-Mills connections with*

1. *energy bounds $\|F_A\|_{L^2(B^4)}, \|F_B\|_{L^2(B^4)} < \varepsilon$, and*
2. *gauge equivalent boundary values $i^*A$ and $i^*B$,*

*then $A$ is gauge equivalent to $B$.*

*Proof.* Choose $\varepsilon$ small enough so that we can apply Theorem 5.10, giving us connections $\tilde{A}$ and $\tilde{B}$ gauge equivalent to $A$ and $B$, respectively, satisfying the Dirichlet Coulomb conditions $d^*\tilde{a} = d^*\tilde{b} = 0$ and $d^*_{\partial B^4} i^*\tilde{a} = d^*_{\partial B^4} i^*\tilde{a} = 0$, as well as the bounds $\|\tilde{a}\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)}$ and $\|\tilde{b}\|_{L^2_1(B^4)} \leq C \|F_B\|_{L^2(B^4)}$. Because $i^*A$ and $i^*B$ are gauge equivalent, $i^*\tilde{A}$ and $i^*\tilde{B}$ are gauge equivalent. Require that $\varepsilon$ be small enough so that the bounds $\|\tilde{a}\|_{L^2_1(B^4)}, \|\tilde{b}\|_{L^2_1(B^4)} < C\varepsilon$ suffice to apply Proposition 5.17 to $i^*\tilde{A}$ and $i^*\tilde{B}$ using the Sobolev and trace maps $L^2_1(B^4) \xrightarrow{i^*} L^2_{1/2}(\partial B^4) \hookrightarrow L^3(\partial B^4)$. We conclude that a constant gauge transformation $c \in G$ sends $i^*\tilde{A}$ to $i^*\tilde{B}$. Now viewing $c$ as a gauge transformation on all of $B^4$, apply $c$ to $\tilde{A}$, and note that $c(\tilde{A})$ satisfies all of the properties above required of $\tilde{A}$. Hence, we may, without loss of generality replace $\tilde{A}$ by $c(\tilde{A})$, or, in other words, assume that $c = 1$, so $i^*\tilde{A} = i^*\tilde{B}$.

Next, we require $\varepsilon$ be small enough so that our bounds $\|F_{\tilde{A}}\|_{L^2(B^4)}, \|F_{\tilde{B}}\|_{L^2(B^4)} < \varepsilon$ and $\|\tilde{a}\|_{L^2_1(B^4)}, \|\tilde{b}\|_{L^2_1(B^4)} < C\varepsilon$ suffice to give us the bounds needed for Proposition 3.4 via the Sobolev embedding $L^2_1(B^4) \hookrightarrow L^4(B^4)$. The Yang-Mills condition is gauge-invariant, so $\tilde{A}$ is Yang-Mills, and, in particular, also satisfies the projected Yang-Mills equation. Hence, we can apply Proposition 3.4 to $\tilde{A}$ and $\tilde{B}$ to conclude that $\|F_{\tilde{A}}\|_{L^2(B^4)} \leq \|F_{\tilde{B}}\|_{L^2(B^4)}$. However, since $\tilde{B}$ is Yang-Mills, we can also apply Proposition 3.4 to $\tilde{B}$ and $\tilde{A}$ to conclude that $\|F_{\tilde{B}}\|_{L^2(B^4)} \leq \|F_{\tilde{A}}\|_{L^2(B^4)}$, concluding that $\|F_{\tilde{A}}\|_{L^2(B^4)} = \|F_{\tilde{B}}\|_{L^2(B^4)}$. The main inequality of Proposition 3.4 then gives us that $\tilde{A} = \tilde{B}$, so $A$ and $B$ are gauge equivalent, as desired. $\square$

## 3.1 Linear interpolation

We know that a small Yang-Mills $A$ connection on $B^4$ locally minimizes energy among connections $B$ with the same restriction to the boundary. We go further by showing that, in Coulomb gauge, the linear interpolation from $B$ to $A$ is an energy-decreasing path. As before, for small connections in Coulomb gauge it suffices to assume only that $A$ satisfies the projected Yang-Mills equation instead of the full Yang-Mills equaiton.

**Proposition 3.7.** *There exist constants $\varepsilon_4$ and $\varepsilon_F$ with the following significance. Let $A = d + a$ and $B = d + b$ be $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections such that*

1. $\|a\|_{L^4(B^4)}, \|b\|_{L^4(B^4)} < \varepsilon_4$,
2. $\|F_A\|_{L^2(B^4)} < \varepsilon_F$,
3. *$A$ satisfies the projected Yang-Mills equation $\pi_{d^*} d^*_A F_A = 0$.*
4. $i^*A = i^*B$, *and*
5. $d^*a = d^*b$.

*Let $B_t = tA + (1-t)A$ be the linear interpolation between $B_0 = A$ and $B_1 = B$. Then $\|F_{B_s}\|_{L^2(B^4)} \leq \|F_{B_t}\|_{L^2(B^4)}$ if $s \leq t$, with equality only if $A = B$.*

*Proof.* Since $B_t$ satisfies all of the conditions above required of $B_1 = B$, in order to prove the general statement it suffices to show that $\frac{d}{dt}\big|_{t=1} \|F_{B_t}\|^2_{L^2(B^4)} \geq 0$.

Let $c = B - A$. Using the equations

$$F_B = F_A + d_A c + \tfrac{1}{2}[c \wedge c], \text{ and}$$
$$F_A = F_B + d_B(-c) + \tfrac{1}{2}[(-c) \wedge (-c)] = F_B - d_B c + \tfrac{1}{2}[c \wedge c],$$

along with the Yang-Mills condition $\langle F_A, d_A c\rangle_{L^2(B^4)} = 0$, we compute

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\Big|_{t=1} \|F_{B_t}\|^2_{L^2(B^4)} &= \left\langle F_B, d_B \tfrac{d}{dt}\Big|_{t=1} B_t\right\rangle_{L^2(B^4)} \\
&= \langle F_B, d_B c\rangle_{L^2(B^4)} \\
&= \left\langle F_B, F_B - F_A + \tfrac{1}{2}[c\wedge c]\right\rangle_{L^2(B^4)} \\
&= \|F_B\|^2_{L^2(B^4)} + \left\langle F_B, \tfrac{1}{2}[c\wedge c]\right\rangle_{L^2(B^4)} \\
&\qquad - \left\langle F_A + d_A c + \tfrac{1}{2}[c\wedge c], F_A\right\rangle_{L^2(B^4)} \\
&= \|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)} + \left\langle F_B - F_A, \tfrac{1}{2}[c\wedge c]\right\rangle_{L^2(B^4)}.
\end{aligned}$$

We bound the last term:

$$\begin{aligned}
&\left|\left\langle F_B - F_A, \tfrac{1}{2}[c\wedge c]\right\rangle_{L^2(B^4)}\right| \\
&= \left|\left\langle dc + \tfrac{1}{2}[b\wedge b] - \tfrac{1}{2}[a\wedge a], \tfrac{1}{2}[c\wedge c]\right\rangle_{L^2(B^4)}\right| \\
&\le \left(\|c\|_{L^2_1(B^4)} + \tfrac{1}{2}C_{\mathfrak{L}}\|b\|^2_{L^4(B^4)} + \tfrac{1}{2}C_{\mathfrak{L}}\|a\|^2_{L^4(B^4)}\right)\tfrac{1}{2}C_{\mathfrak{L}}\|c\|^2_{L^4(B^4)} \\
&\le \left(\tfrac{1}{4}C_{\mathfrak{L}}^2C_S^2(\|a\|^2_{L^4(B^4)} + \|b\|^2_{L^4(B^4)}) + \tfrac{1}{2}C_{\mathfrak{L}}C_S\|c\|_{L^4(B^4)}\right)\|c\|^2_{L^2_1(B^4)}
\end{aligned}$$

Since $\|c\|_{L^4(B^4)} \le \|a\|_{L^4(B^4)} + \|b\|_{L^4(B^4)}$, we can choose $\varepsilon_4$ small enough so that $\|a\|_{L^4(B^4)}, \|b\|_{L^4(B^4)} < \varepsilon_4$ guarantees that

$$\left|\left\langle F_B - F_A, \tfrac{1}{2}[c\wedge c]\right\rangle_{L^2(B^4)}\right| \le \tfrac{1}{2}C^{-1}\|c\|^2_{L^2_1(B^4)},$$

where $C$ is the constant in Proposition 3.4. Choosing $\varepsilon_4$ and $\varepsilon_F$ small enough so that we can apply Proposition 3.4, we have $\|c\|^2_{L^2_1(B^4)} \le C\left(\|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)}\right)$, so

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\Big|_{t=1}\|F_{B_t}\|^2_{L^2(B^4)} &\ge \|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)} - \tfrac{1}{2}C^{-1}\|c\|^2_{L^2_1(B^4)} \\
&\ge \tfrac{1}{2}\left(\|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)}\right) \ge 0,
\end{aligned}$$

with equality only if $\|F_B\|^2_{L^2(B^4)} - \|F_A\|^2_{L^2(B^4)} = 0$, in which case $B = A$ by Proposition 3.4. □

Again, we can use gauge fixing to prove this energy monotonicity result for a wider class of connections.

**Theorem 3.8.** *There exists a constant $\varepsilon$ with the following significance. Let $B$ be an $L^2_1(B^4)$ connection with $\|F_B\|_{L^2(B^4)} < \varepsilon$, and let $A$ be the Yang-Mills connection with $i^*A = i^*B$ constructed in Theorem 3.5. Let $B_t = tA + (1-t)A$ be the linear interpolation between $B_0 = A$ and $B_1 = B$. Then $\|F_{B_s}\|_{L^2(B^4)} \leq \|F_{B_t}\|_{L^2(B^4)}$ if $s \leq t$, with equality only if $A = B$.*

*Proof.* From the construction in Theorem 3.5, there exists a gauge transformation $g$ sending $A$ and $B$ to $\tilde{A} = d + \tilde{a}$ and $\tilde{B} = d + \tilde{b}$, respectively, such that $d^*\tilde{b} = 0$, and we know that $d^*\tilde{a} = 0$ because we obtained it from Theorem 3.1. Finally, the construction in Theorem 3.5 gives us a bound $\|\tilde{b}\|_{L^2_1(B^4)} \leq C\,\|F_B\|_{L^2(B^4)}$, and with the Sobolev inequalities we know that a small enough $\varepsilon$ we can bound $\tilde{b}$ in $L^4(B^4; \mathfrak{g} \otimes T^*B^4)$. Since $\tilde{A}$ depends continuously on $\tilde{B}$, this lets us also bound $\tilde{a}$ in $L^4(B^4; \mathfrak{g} \otimes T^*B^4)$. Finally, $A$ is Yang-Mills, so $\tilde{A}$ is Yang-Mills and in particular satisfies the projected Yang-Mills equation. Hence we can apply Proposition 3.7 to $\tilde{A}$ and $\tilde{B}$, giving us that the linear interpolation between $\tilde{A}$ and $\tilde{B}$ has monotone energy. But both energy and affine combinations are preserved after applying a gauge transformation, so the linear interpolation between $A$ and $B$ also has monotone energy, as desired. □

# Chapter 4

# Yang-Mills replacement for global connections

In this chapter, we consider an arbitrary compact 4-manifold $X$ and use the solution to the Dirichlet problem for Yang-Mills connections on the ball in order to construct an energy-decreasing map on global $L_1^2(X)$ connections modulo $L_2^2(X)$ gauge transformations. Namely, given a connection $B$ on $X$ and a ball $B^4 \subset X$ on which $B$ has small energy, we will replace $B$ on $B^4$ with a Yang-Mills connection that has the same restriction to the boundary $\partial B^4$, thereby constructing a piecewise connection $A$ that is Yang-Mills on the ball and equal to $B$ outside the ball. However, only the tangential components of $A$ and $B$ match on the boundary $\partial B^4$, and the normal components may disagree. As a result, this new connection $A$ is no longer in $L_1^2(X)$, but it is still in the space $L_d^2(X)$ defined in Section 2.4. However, we will show that this piecewise-defined connection is nonetheless gauge equivalent to an $L_1^2(X)$ connection.

More generally, we prove that any $L_d^2(X)$ connection is gauge equivalent via a $L_1^4(X)$ gauge transformation to a $L_1^2(X)$ connection, so, the space of $L_d^2(X)$ connections up to $L_1^4(X)$ gauge transformations is actually the same as the space of $L_1^2(X)$ connections up to $L_2^2(X)$ gauge transformations. Theorem 5.20 tells us that, locally, every $L_d^2(B^4)$ connection is gauge equivalent via an $L_1^4(B^4)$ gauge transformation to an $L_1^2(B^4)$ connection. However, patching these local gauge transformations to a global transformation is a delicate matter because in the critical regularity $L_1^4(B^4)$

gauge transformations need not be continuous. A crucial lemma for dealing with this issue is due to Taubes.

**Lemma 4.1** ([19, Lemma A.1]). *Let $U$ be an open ball in a Riemannian 4-manifold. Let $\mathfrak{F}$ be the set of triples $(g, a, b) \in L^2_2(U; M_N) \times L^2_1(U; \mathfrak{g} \otimes T^*U) \times L^2_1(U; \mathfrak{g} \otimes T^*U)$ such that $g$ is a gauge transformation sending $A = d + a$ to $B = d + b$ and $d^*a = d^*b = 0$. Then the projection to the first factor $\mathfrak{F} \to L^2_2(U; M_N)$ sending $(g, a, b)$ to $g$ factors continuously through $C^0_{\mathrm{loc}}(U; M_N)$.*

The next lemma we need is due to Uhlenbeck, and states that if two bundles over a compact $X$ are described by transition functions $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ that are sufficiently close to each other in $C^0$, then the two bundles are isomorphic.

**Lemma 4.2** ([23, Proposition 3.2]). *Let $X$ be a compact manifold with principal $G$-bundles $P$ and $Q$ and a finite cover by local trivializations $\{U_\alpha\}$ with continuous transition maps $g_{\alpha,\beta}, h_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to G$, respectively. There exists an $\varepsilon$ depending on the cover but not on the transition maps such that if, for all $\alpha$ and $\beta$, $\phi_{\alpha,\beta}\psi_{\beta,\alpha}$ is a neighborhood of the identity on which $\exp^{-1}$ is defined and*

$$\left\| \exp^{-1}(\phi_{\alpha,\beta}\psi_{\beta,\alpha}) \right\|_{C^0(U_\alpha \cap U_\beta)} < \varepsilon,$$

*then there exists a cover $\{V_\alpha\}$ of $X$ with $V_\alpha \subset U_\alpha$ and an isomorphism between $P$ and $Q$ defined by $\rho_\alpha \colon V_\alpha \to G$ satisfying $\psi_{\alpha,\beta}\rho_\alpha = \rho_\beta\phi_{\alpha,\beta}$.*

With these tools in hand, we can extend Uhlenbeck's techniques in [23, Theorem 3.6] to the critical regularity.

**Theorem 4.3.** *Let $P$ be a principal $G$-bundle over a compact manifold $X$. Let $A$ be a $L^2_d(X)$ connection on $P$. Then there exists an $L^4_1(X)$ gauge transformation on $P$ sending $A$ to an $L^2_1(X)$ connection $B$. Moreover, the gauge equivalence class of $B$ modulo $L^2_2(X)$ gauge transformations depends continuously on $A$.*

*Proof.* Let $A_i$ be a sequence of smooth connections converging to $A$ in $L^2_d(X)$. Let $\{U_\alpha\}$ be a finite cover of $X$ by balls contained inside smooth trivializations of $P$ and small enough so that, for all $\alpha$, $\|F_A\|_{L^2(U_\alpha)} < \varepsilon$ for the $\varepsilon$ in Proposition 5.22. A priori, the $\varepsilon$ required depends on the metric of the ball $B^4$. However, as discussed in Uhlenbeck [23], we can note that the energy of connections is invariant under conformal changes of metric, and dilations in particular. Thus, we can rescale small exponential neighborhoods to balls of unit size with metric close to that of the standard unit ball, and choose an $\varepsilon$ uniformly for all of the balls. Take a tail of the sequence to guarantee that $\|F_{A_i}\|_{L^2(U_\alpha)} < \varepsilon$ also. For a fixed $U_\alpha$, pass to a subsequence of $A_i$ given by Proposition 5.22, giving us gauge transformations $g_{i,\alpha}$ and $g_\alpha$ on $U_\alpha$ sending $A_i$ to $\tilde{A}_{i,\alpha}$ and $A$ to $\tilde{A}_\alpha$, respectively, such that the $\tilde{A}_{i,\alpha}$ are in Coulomb gauge on $U_\alpha$ and converge to $\tilde{A}_\alpha$ strongly in $L^2_1(U_\alpha; \mathfrak{g} \otimes T^*U_\alpha)$, and the $g_{i,\alpha}$ converge to $g_\alpha$ in $L^4_1(U_\alpha; G)$. Repeat this construction for the other $U_\alpha$, taking further subsequences. By the smoothness and uniqueness claim for the gauge transformation doing the gauge fixing in [5, Theorem 2.3.7], we know that the $g_{i,\alpha}$ are smooth because the $A_i$ are smooth.

Let $\phi_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to G$ be the transition functions for $P$. Since the $g_{i,\alpha}$ are smooth, we define new transition functions by $\psi_{i,\alpha,\beta} = g_{i,\beta}\phi_{\alpha,\beta}g_{i,\alpha}^{-1}$ for a bundle $Q_i$ over $X$, so the $g_{i,\alpha}$ define a bundle isomorphism between $P$ and $Q_i$ sending $A_i$ to the connection defined by the $\tilde{A}_{i,\alpha}$ on $Q_i$.

We would like to pass to the limit bundle $Q$ defined by $\psi_{\alpha,\beta} = g_\beta\phi_{\alpha,\beta}g_\alpha^{-1}$, where the $g_\alpha$ define an $L^4_1$ bundle isomorphism between $P$ and $Q$ sending to $A$ to the $L^2_1$ connection on $Q$ defined by the $\tilde{A}_\alpha$. However, the issue is that the $g_\alpha$ are not necessarily continuous, so we do not yet know that the $\psi_{\alpha,\beta}$ define a continuous bundle $Q$, nor do we know that $P$ and $Q$ are isomorphic as continuous bundles.

However, we know that, on $U_\alpha \cap U_\beta$, $\psi_{\alpha,\beta}$ is a gauge transformation between $\tilde{A}_\alpha$ and $\tilde{A}_\beta$, that is,

$$d\psi_{\alpha,\beta} = \psi_{\alpha,\beta}\tilde{a}_\alpha - \tilde{a}_\beta\psi_{\alpha,\beta}.$$

Since $\tilde{a}_\alpha$ and $\tilde{a}_\beta$ are $L^2_1(U_\alpha \cap U_\beta; \mathfrak{g} \otimes T^*U_\alpha \cap U_\beta)$, by Lemma 5.1, for any compact $K \subset U_\alpha \cap U_\beta$, we have $\psi_{\alpha,\beta} \in L^2_2(K; G)$. By Lemma 4.1, for any open ball $V$ contained in $K$, $\psi_{\alpha,\beta} \in C^0_{\mathrm{loc}}(V; G)$. Consequently, by slightly shrinking the $U_\alpha$ to open balls $U'_\alpha \subset U_\alpha$ that still cover $X$, we can guarantee that the transition maps $\psi_{\alpha,\beta}$ are continuous on $U'_\alpha \cap U'_\beta$, so, indeed the bundle $Q$ is continuous.

Moreover, since the $\psi_{i,\alpha,\beta}$ are gauge transformations between $\tilde{A}_{i,\alpha}$ and the $\tilde{A}_{i,\beta}$. Since the $\tilde{A}_{i,\alpha}$ and the $\tilde{A}_{i,\beta}$ converge strongly in $L^2_1(U_\alpha \cap U_\beta; \mathfrak{g} \otimes T^*U_\alpha \cap U_\beta)$ to $\tilde{A}_\alpha$ and $\tilde{A}_\beta$, by Lemma 5.3, after passing to a subsequence, the $\psi_{i,\alpha,\beta}$ converge in $L^2_2(K;G)$. Again, the limit must be $\psi_{\alpha,\beta}$ because the $\psi_{i,\alpha,\beta}$ converge to $\psi_{\alpha,\beta}$ in a weaker norm. For example, in the formula $\psi_{i,\alpha,\beta} = g_{i,\beta}\phi_{\alpha,\beta}g_{i,\alpha}^{-1}$, we know that the $g_{i,\beta}$ and $g_{i,\alpha}$ converge strongly in $L^4(K;G)$ to $g_\beta$ and $g_\alpha$, so $\psi_{i,\alpha,\beta}$ converges strongly to $\psi_{\alpha,\beta}$ in $L^2(K;G)$. Consequently, by Lemma 4.1, the $\psi_{i,\alpha,\beta}$ converge to $\psi_{\alpha,\beta}$ in $C^0_{\mathrm{loc}}(V;G)$, so they converge in $C^0(K';G)$ for compact subsets $K'$ of $V$. Hence, we can choose the $U'_\alpha$ such that the $\psi_{i,\alpha,\beta}$ converge to $\psi_{\alpha,\beta}$ in $C^0(U'_\alpha \cap U'_\beta; G)$.

Hence, we can choose a sufficiently large $i$ so that $\psi_{i,\alpha,\beta}$ is sufficiently close in $C^0(U'_\alpha \cap U'_\beta; G)$ to $\psi_{\alpha,\beta}$ in order to satisfy the conditions of Lemma 4.2. Applying this lemma, we conclude that there is a continuous bundle isomorphism $\rho$ between $Q$ and $Q_i$, which in turn is smoothly isomorphic to $P$ via $g_{i,\alpha}^{-1}$. Moreover, the argument in [23, Corollary 3.3] applies also to $L^2_2 \cap C^0(U'_\alpha \cap U'_\beta; G)$, so $\rho$ is in fact an $L^2_2(X)$ bundle map that depends continuously on $\psi_{i,\alpha,\beta}$ and $\psi_{\alpha,\beta}$ in $L^2_2 \cap C^0(U'_\alpha \cap U'_\beta; G)$. The $g_\alpha$ define an $L^4_1(X)$ bundle map between $P$ and $Q$ sending $A$ to an $L^2_1(X)$ connection defined by the $\tilde{A}_\alpha$, so $g_{i,\alpha}^{-1} \circ \rho \circ g_\alpha$ is an $L^4_1(X)$ bundle isomorphism from $P$ to itself sending $A$ to an $L^2_1(X)$ connection $B$ on $P$, as desired.

To prove continuity, note that we can choose sufficiently large $i$ so that for $j \geq i$, we can construct an $L^2_2 \cap C^0(X)$ bundle isomorphism $\rho_j$ between $Q_j$ and $Q_i$ just like we constructed $\rho$ above. Because the construction of $\rho_j$ depends continuously on the transition maps and we have chosen a the cover $U'_\alpha$ so that the $\psi_{j,\alpha,\beta}$ converge to $\psi_{\alpha,\beta}$ in $L^2_2 \cap C^0(U'_\alpha \cap U'_\beta; G)$, we have that the $\rho_{j,\alpha}$ also converge in $L^2_2 \cap C^0(U'_\alpha \cap U'_\beta; G)$ to $\rho_\alpha$. The $\tilde{A}_{j,\alpha}$ converge in $L^2_1(U'_\alpha; \mathfrak{g} \otimes T^*U'_\alpha)$, so the $\rho_j(\tilde{A}_{j,\alpha})$ converge in $L^2_1(U'_\alpha; \mathfrak{g} \otimes T^*U'_\alpha)$ to $\rho(\tilde{A}_\alpha)$ as connections on the same bundle $Q_i$. Applying the smooth bundle map $g_{i,\alpha}^{-1}$, let $B_i$ be the connection on $P$ defined on trivializations by $(g_{i,\alpha}^{-1} \circ \rho_j)(\tilde{A}_{j,\alpha}) = (g_{i,\alpha}^{-1} \circ \rho_j \circ g_{j,\alpha})(A_j)$. We see that the $B_i$ converge as $L^2_1$ connections on the bundle $P$ to $(g_{i,\alpha}^{-1} \circ \rho)(\tilde{A}_\alpha) = (g_{i,\alpha}^{-1} \circ \rho \circ g_\alpha)(A) = B$. That is, we have constructed $L^2_1(X)$

connections $B_i$ and $B$ gauge equivalent to $A_i$ and $A$, respectively, such that the $B_i$ converge in $L^2_1(X)$ to $B$.

The first issue to complete the proof of continuity is that we passed to subsequences in the proof, and our choice of $B$ may depend on our choice of subsequence. However, the gauge equivalence class of $B$ modulo $L^2_2(X)$ gauge transformations does not depend on this choice. Indeed, if $A$ is gauge equivalent by $L^4_1(X)$ gauge transformations to $L^2_1(X)$ connections $B$ and $B'$, then $B$ and $B'$ are gauge equivalent via an $L^4_1(X)$ gauge transformation $g$. But by Lemma 5.1, on every trivialization, $g$ is in $L^2_2$, so it is in fact an $L^2_2(X)$ gauge transformation, and so $[B] = [B']$. Hence, for any subsequence of the $A_i$, our argument above shows that after passing to a further subsequence, the $[A_i] = [B_i]$ converge to $[B]$ in the space of $L^2_1(X)$ connections modulo $L^2_2(X)$ gauge transformations. Thus the original sequence also converges to $[B]$.

The second issue is that we assumed that the $A_i$ are smooth, but to show continuity we need a general sequence of $L^2_d(X)$ connections on $P$. Now let $A_i$ be an $L^2_d(X)$ sequence of connections converging in $L^2_d(X)$ to $A$, and let $[B_i]$ and $[B]$ be the gauge equivalence class of $L^2_1(X)$ connections constructed above. To show that the $[B_i]$ converge to $[B]$, we consider sequences of smooth connections $A_{i,j}$ that converge to $A_i$, so from the above argument we know that the $[A_{i,j}]$ converge to $[B_i]$. We then use a diagonalization argument, constructing $j(i)$ such that $A_{i,j(i)}$ converges to $A$ and such that $[A_{i,j(i)}]$ is within $1/i$ of $[B_i]$. Because the $A_{i,j(i)}$ are smooth, the argument above gives us that the $[A_{i,j(i)}]$ converge to $[B]$, and so the $[B_i]$ must also converge to $[B]$. $\square$

**Corollary 4.4.** *The space of $L^2_d(X)$ connections modulo $L^4_1(X)$ gauge transformations is homeomorphic to the space of $L^2_1(X)$ connections modulo $L^2_2(X)$ gauge transformations.*

Our Yang-Mills replacement results follow.

**Corollary 4.5.** *Let $P \to X$ be a principal $G$-bundle over compact 4-manifold $X$ with compact gauge group $G$, and let $B^4 \subset X$ be a 4-ball. Let $C$ be the space of $L^2_1(X)$ connections modulo $L^2_2(X)$ gauge transformations, and let $\mathcal{C}_{\varepsilon,B^4}$ be those gauge* *equivalence classes of connections* $[B]$ *with small energy on* $B^4$*, that is,* $\|F_B\|_{L^2(B^4)} < \varepsilon$*. Then for small enough* $\varepsilon$ *there is an energy-decreasing continuous map* $\mathcal{C}_{\varepsilon,B^4} \to \mathcal{C}_{\varepsilon,B^4}$ *sending* $[B]$ *to an equivalence class of connections* $[A]$*, where* $A$ *is Yang-Mills on* $B^4$ *and gauge equivalent to* $B$ *outside* $B^4$*.*

*Proof.* On $B^4$, we construct $\hat{A}$ continuously by Theorem 3.5, and outside $B^4$ we set it equal to $B$. By Proposition 2.18, the result is an $L^2_d(X)$ connection that depends continuously on $B$, and by Theorem 3.8, $\|F_{\hat{A}}\|_{L^2(B^4)} \leq \|F_B\|_{L^2(B^4)}$. By Theorem 4.3, $\hat{A}$ is gauge equivalent to an $L^2_1(X)$ connection $A$, so $[A]$ depends continuously on $B$. $\square$

**Corollary 4.6.** *Let* $P \to X$ *be a principal* $G$*-bundle over compact* 4*-manifold* $X$ *with compact gauge group* $G$*, and let* $\mathcal{C}$ *be the space of* $L^2_1(X)$ *connections modulo* $L^2_2(X)$ *gauge transformations. Let* $\mathcal{K}$ *be a compact family in* $\mathcal{C}$*. Then around any point* $x \in X$ *there exists a ball* $x \in B^4 \subset X$ *and homotopy* $h_t\colon \mathcal{K} \to \mathcal{C}$ *such that* $h_1$ *is the identity,* $h_0$ *sends* $\mathcal{K}$ *to connections that are Yang-Mills on* $B^4$*,* $h_t([B])$ *has monotone nondecreasing energy, and restricting to the complement of* $B^4$ *the homotopy is constant* $h_t([B]) = [B]$*.*

*Proof.* Since $\mathcal{K}$ is compact, we can choose a ball $B^4$ around $x$ small enough so that for all $[B] \in \mathcal{K}$, $\|F_B\|_{L^2(B^4)} < \varepsilon$. Then, for each $B$, inside $B^4$ we construct $A$ and $B_t$ as in Theorems 3.5 and 3.8. By Theorem 3.5, $A$ and $B_t$ depend continuously on $B$, and from the construction it is clear that if we choose a different representative $B' = g(B)$ of $[B]$, then the resulting $A'$ and $B'_t$ satisfy $A' = g(A)$ and so $B'_t = g(B_t)$, so $[B_t]$ is independent of our choice of $B \in [B]$. Outside $B^4$, we set $B_t$ equal to $B$. Again, by Proposition 2.18, the resulting $B_t$ is an $L^2_d(X)$ connection that depends continuously on $B$, and by Theorem 4.3, $B_t$ is gauge equivalent to an $L^2_1(X)$ connection $\tilde{B}_t$, so we can set $h_t([B]) = [\tilde{B}_t]$. $\square$

# Chapter 5

# Gauge fixing

In this chapter, we prove several gauge fixing results which are used to prove the main results. We begin with a few general lemmas.

**Lemma 5.1.** *Let $K$ be a compact 4-manifold with a trivial principal $G$-bundle $P$, where $G$ is compact. Let $A$ and $B$ be two $L^2_1(K; \mathfrak{g} \otimes T^*K)$ connections that are gauge equivalent via a gauge transformation $g$. Then $g \in L^2_2(K; G)$.*

*Proof.* We have the equation

$$dg = ga - bg.$$

Since $G$ is compact, we know that $g \in L^\infty(K; G)$. Using the Sobolev embedding theorems, we know that $a, b \in L^4(K; \mathfrak{g} \otimes T^*K)$. Thus, $dg = ga - bg \in L^4(K; M_N \otimes T^*K)$, so $g \in L^4_1(K; G)$. Next,

$$\nabla(dg) = (\nabla g)a - b(\nabla g) + g(\nabla a) - (\nabla b)g \in L^4 \cdot L^4 - L^4 \cdot L^4 + L^\infty \cdot L^2 - L^2 \cdot L^\infty \subset L^2.$$

Hence, $dg \in L^2_1(K; M_N \otimes T^*K)$, so $g \in L^2_2(K; G)$. □

**Lemma 5.2.** *Let $K$ be a compact 4-manifold with a trivial principal $G$-bundle $P$. Consider two sequences of $L^2_1(K; \mathfrak{g} \otimes T^*K)$ connections $A_i$ and $B_i$ converging weakly to $A$ and $B$ in $L^2_1(K; \mathfrak{g} \otimes T^*K)$, respectively, such that $A_i$ and $B_i$ are gauge equivalent via a gauge transformation $g_i$. Then a subsequence of the $g_i$ converges weakly in $L^2_2(K; G)$ to a gauge transformation $g$ sending $A$ to $B$.*

*If the connections are only $L^4(K;\mathfrak{g}\otimes T^*K)$ and converge weakly in $L^4(K;\mathfrak{g}\otimes T^*K)$, then a subsequence of the gauge transformations converges weakly in $L^4_1(K;G)$.*

*Proof.* The gauge transformations give us equations

$$b_i = g_i a_i g_i^{-1} - (dg_i)g_i^{-1},$$

which we can rewrite as

$$dg_i = g_i a_i - b_i g_i. \tag{5.1}$$

The gauge transformations are assumed to be unitary and hence uniformly bounded in $L^\infty(K;M_N)$. Since $a_i$ and $b_i$ in $L^4(K;\mathfrak{g}\otimes T^*K)$, we know that $g_i a_i - b_i g_i = dg_i$ is bounded in $L^4(K;\mathfrak{g}\otimes T^*K)$, and hence the sequence $g_i$ is bounded in $L^4_1(K;M_N)$. Passing to a subsequence, we can assume that $g_i$ has a weak limit $g$ in $L^4_1(K;M_N)$. In particular, $g_i$ converges strongly to $g$ in $L^4(K;M_N)$, so we know that $g$ is in $G$ a.e. because a subsequence of the $g_i$ converges pointwise a.e. to $g$.

It remains to show that $g$ sends $A$ to $B$. We would like to take the limit of the equation $dg_i = g_i a_i - b_i g_i$, but the issue is that the product of sequences that converge weakly need not converge, even weakly. However, because the $g_i$ converge weakly to $g$ in $L^4_1(K;M_N)$, we know that the $g_i$ converge strongly to $g$ in $L^4(K;M_N)$, and hence using the multiplication map $L^4(K)\times L^2_1(K)\to L^2(K)$, we know that the $g_i a_i$ converges weakly to $ga$ in $L^2(K;M_N)$, and, similarly, $b_i g_i$ converges weakly to $bg$ in $L^2(K;M_N)$. The weak convergence of $dg_i$ follows from the linearity of $d$. Therefore, we can take the weak limit of (5.1) in $L^2(K;M_N\otimes T^*K)$ to find that

$$dg = ga - bg,$$

so $g$ indeed sends $A$ to $B$.

Moreover, if the $a_i$ and $b_i$ are bounded in $L^2_1(K;\mathfrak{g}\otimes T^*K)$, because the $g_i$ are bounded in $L^4_1(K;M_N)$ and $L^\infty(K;M_N)$, we see that

$$\nabla(dg_i) = (\nabla g_i)a_i - b_i(\nabla g_i) + g_i(\nabla a_i) - (\nabla b_i)g_i$$

is bounded in $L^4 \cdot L^4 + L^4 \cdot L^4 + L^\infty \cdot L^2 + L^2 \cdot L^\infty \subset L^2$. Hence, the $g_i$ are bounded in $L^2_2(K; M_N)$, so, after passing to a subsequence, they converge weakly in $L^2_2(K; M_N)$, and the limit is $g$, because the weak $L^2_2(K; M_N)$ limit must agree with the weak $L^4_1(K; M_N)$ limit. □

In both the preceding and the following lemma, the reason we need to take a subsequence is that the limit connection might have a nontrivial, but compact, stabilizer, so we might even have constant sequences $A_i = A$ and $B_i = A$ such that the sequence of gauge transformations fails to converge. However, in a situation where there is no stabilizer and we have a unique gauge transformation between $A$ and $B$, we can eliminate the need for taking a subsequence, using the fact that if every subsequence of the $g_i$ has a further subsequence that converges to $g$, then the original sequence $g_i$ converges to $g$ also.

**Lemma 5.3.** *Let $K$ be a compact 4-manifold with a trivial principal $G$-bundle $P$. Consider two sequences of $L^2_1(K; \mathfrak{g} \otimes T^*K)$ connections $A_i$ and $B_i$ converging strongly to $A$ and $B$ in $L^2_1(K; \mathfrak{g} \otimes T^*K)$, respectively, such that $A_i$ and $B_i$ are gauge equivalent via a gauge transformation $g_i$. Then a subsequence of the $g_i$ converges strongly in $L^2_2(K; G)$ to a gauge transformation $g$ sending to $A$ and $B$.*

*If the connections are only $L^4(K; \mathfrak{g} \otimes T^*K)$ and converge strongly in $L^4(K; \mathfrak{g} \otimes T^*K)$, then a subsequence of the gauge transformations converges strongly in $L^4_1(K; G)$.*

*Proof.* By Lemma 5.2, after passing to a subsequence, the $g_i$ converge weakly in $L^4_1(K; G)$ to a gauge transformation $g$ sending to $A$ to $B$. The main difficulty in promoting weak convergence to strong convergence is that even though the $g_i$ are bounded in $L^\infty(K; G)$, in the borderline regularity we cannot get strong convergence in $L^\infty(K; G)$. As before, we use the equations

$$dg_i = g_i a_i - b_i g_i,$$
$$dg = ga - bg.$$

The key tool that lets us promote weak convergence of the terms in this equation to strong convergence is that, for $L^p$ spaces with $1 < p < \infty$, weak convergence along with convergence of the sequence of norms to the norm of the limit implies strong convergence [13]. In order to get the convergence of the sequence of norms, we use the fact that gauge transformations are isometries of $L^p(K; M_N)$.

We begin by showing that, after passing to a subsequence, the $g_i$ converge strongly to $g$ in $L^4_1(K;G)$. Since $L^4_1 \hookrightarrow L^4$ is compact, we know that the $g_i$ converge strongly to $g$ in $L^4(K;G)$, so it remains to show that the $dg_i$ converge strongly to $dg$ in $L^4(K; M_N \otimes T^*K)$. As before, the $g_i a_i$ are bounded in $L^4(K; M_N \otimes T^*K)$, so, after passing to a subsequence, the $g_i a_i$ converge weakly in $L^4(K; M_N \otimes T^*K)$. Moreover, their limit is $ga$ because the $g_i$ converge strongly in $L^4(K;G)$ to $g$ and the $a_i$ converge strongly in $L^4(K; \mathfrak{g} \otimes T^*K)$ to $a$, so the $g_i a_i$ converge strongly in $L^2(K; M_N \otimes T^*K)$ to $ga$, and the weak $L^4(K; M_N \otimes T^*K)$ limit must agree with the $L^2(K; M_N \otimes T^*K)$ limit.

We now promote the weak convergence of the $g_i a_i$ to strong convergence by showing that the sequence of norms converges to the norm of the limit. Indeed, because gauge transformations are isometries, $\|g_i a_i\|_{L^4(K)} = \|a_i\|_{L^4(K)}$, which converges to $\|a\|_{L^4(K)} = \|ga\|_{L^4(K)}$. Hence, we have weak convergence and convergence of the sequence of norms to the norm of the limit, implying strong $L^4(K; M_N \otimes T^*K)$ convergence of $g_i a_i$ to $ga$. Likewise, $b_i g_i$ converges strongly in $L^4(K; M_N \otimes T^*K)$ to $bg$, so $dg_i$ converges strongly in $L^4(K; M_N \otimes T^*K)$ to $dg$, as desired.

Next, we improve the strong $L^4_1(K;G)$ convergence of the $g_i$ to strong $L^2_2(K;G)$ convergence. Strong $L^4_1(K;G)$ convergence implies strong $L^2_1(K;G)$ convergence, so it remains to show that $\nabla dg_i$ converges to $\nabla dg$ in $L^2(K; M_N \otimes T^*K \otimes T^*K)$. We compute

$$\nabla dg_i = dg_i \otimes a_i + g_i \nabla a_i - \nabla b_i g_i - b_i \otimes dg_i.$$

Because the $dg_i$ converge to $dg$ in $L^4(K; M_N \otimes T^*K)$ and the $a_i$ converge to $a$ in $L^4(K; \mathfrak{g} \otimes T^*K)$, we know that the $dg_i \otimes a_i$ converge to $dg \otimes a$ in the space $L^2(K; M_N \otimes T^*K \otimes T^*K)$. Likewise, the $b_i \otimes dg_i$ converge to $b \otimes dg$ in the space $L^2(K; M_N \otimes T^*K \otimes T^*K)$. For the $g_i \nabla a_i$ term, we use a similar argument to the above. This sequence is bounded in $L^2(K; M_N \otimes T^*K \otimes T^*K)$, so after passing to a subsequence it converges weakly in $L^2(K; M_N \otimes T^*K \otimes T^*K)$. Moreover, the limit is $g\nabla a$ because the $g_i$ converge to $g$ in $L^2(K; G)$ and the $\nabla a_i$ converge to $\nabla a$ in $L^2(K; M_N \otimes T^*K \otimes T^*K)$, so the $g_i\nabla a_i$ converge to $g\nabla a$ in $L^1(K; M_N \otimes T^*K \otimes T^*K)$, and the weak limit in $L^2(K; M_N \otimes T^*K \otimes T^*K)$ must agree with the strong limit in $L^1(K; M_N \otimes T^*K \otimes T^*K)$. As for the sequence of norms, since gauge transformations are isometries, $\|g_i\nabla a_i\|_{L^2(K)} = \|\nabla a_i\|_{L^2(K)}$, which converges to $\|\nabla a\|_{L^2(K)} = \|g\nabla a\|_{L^2(K)}$. Hence, the $g_i\nabla a_i$ converge strongly in $L^2(K; M_N \otimes T^*K \otimes T^*K)$ to $g\nabla a$. Likewise, the $\nabla b_i g_i$ converge strongly in $L^2(K; M_N \otimes T^*K \otimes T^*K)$ to $\nabla bg$. Thus, the $\nabla dg_i$ converge to $\nabla dg$ in $L^2(K; M_N \otimes T^*K \otimes T^*K)$, so the $g_i$ converge to $g$ in $L^2_2(K; G)$, as desired. □

## 5.1 Coulomb gauge with fixed boundary

In this section, we prove a gauge fixing result where the gauge transformation is fixed to be the identity on the boundary $\partial B^4$, as a prelude to proving the gauge fixing result with Dirichlet boundary conditions on the connection in Section 5.2. This result is present in Uhlenbeck's paper [24] but with $L^\infty$ bounds on the connection. Here, our connection is in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ and may be unbounded in $L^\infty$. In our result in this section, we require bounds on the connection itself rather than its curvature as in [23] and Section 5.2. Indeed, because the boundary value of the connection $A = d + a$ is fixed under the gauge transformation, curvature bounds alone are insufficient, and we also need to control the $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ norm of the boundary value $i^*a$.

**Proposition 5.4.** *There exists constants $\varepsilon$ and $C$ such that if $A = d + a$ is any $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connection with $\|a\|_{L^2_1(B^4)} < \varepsilon$, then there exists an $L^2_2(B^4; M_N)$ gauge transformation $g$ sending $A$ to $\tilde{A} = d + \tilde{a}$ such that*

1. *$i^*g$ is the identity gauge transformation on $\partial B^4$,*

2. *$\tilde{A} = g(A)$ is in Coulomb gauge, that is, $d^*\tilde{a} = 0$, and*

3. *$\|\tilde{a}\|_{L^2_1(B^4)} \leq C \left( \|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{1/2}(\partial B^4)} \right)$.*

*Moreover, if $A$ is actually in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$, then $g$ is in $L^2_3(B^4; M_N)$.*

We would like to find $g$ using the implicit function theorem. However, doing so requires the gauge group to have a differentiable exponential map, which is only true in higher regularity. Hence, we proceed similarly to the proof of Uhlenbeck's gauge fixing theorem with Neumann boundary conditions in [23]: We show that the space of $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connections satisfying Proposition 5.4 is both open and closed in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$, and that the space of $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections satisfying Proposition 5.4 is closed in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$.

We first prove a priori bounds on connections in Coulomb gauge.

**Lemma 5.5.** *There exist constants $\varepsilon$ and $C$, such that if $A = d + a$ is an $L^2_1(B^4)$ connection with $\|a\|_{L^4(B^4)} < \varepsilon$ in Coulomb gauge $d^*a = 0$, then*

$$\|a\|_{L^2_1(B^4)} \leq C \left( \|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{1/2}(\partial B^4)} \right).$$

*Furthermore, if $A$ is an $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connection, then*

$$\|a\|_{L^2_2(B^4)} \leq C \left( \|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{3/2}(\partial B^4)} \right).$$

*Proof.* Because $H^1(B, \partial B^4) = 0$, Corollary 2.10 tells us that $d + d^* \colon L^{2,\mathrm{n}}_1(B^4; \bigwedge^* T^*B^4) \to L^2(B^4; \bigwedge^* T^*B^4)$ is injective on one-forms. Because the trace map $i^* \colon L^2_1(B^4) \to L^2_{1/2}(\partial B^4)$ is surjective, we conclude that

$$(d + d^*, i^*) \colon L^2_1(B^4; \textstyle\bigwedge^* T^*B^4) \to L^2(B^4; \textstyle\bigwedge^* T^*B^4) \times L^2_{1/2}(\partial B^4; \textstyle\bigwedge^* T^*\partial B^4)$$

is also a Fredholm operator that is injective on one-forms. Indeed, the kernel of $(d + d^*, i^*)$ on $L^2_1(B^4; \bigwedge^* T^*B^4)$ is the same as the kernel of $d + d^*$ on $L^{2,\mathrm{n}}_1(B^4; \bigwedge^* T^*B^4)$. Moreover, Corollary 2.10 tells us that $\mathrm{range}(d + d^*, i^*) \oplus (\mathcal{H}^{\mathrm{n}} \times \{0\})$ contains all of $L^2(B^4; \bigwedge^* T^*B^4) \times \{0\}$, and the surjectivity of $i^*$ tells us that, for any $\alpha_\partial \in L^2_{1/2}(\partial B^4; \bigwedge^* T^*\partial B^4)$, the image contains a $(\beta, \alpha_\partial)$ for some $\beta$. Hence, $\mathrm{range}(d + d^*, i^*) \oplus (\mathcal{H}^{\mathrm{n}} \times \{0\})$ contains all of $L^2(B^4; \bigwedge^* T^*B^4) \times L^2_{1/2}(\partial B^4; \bigwedge^* T^*\partial B^4)$, so $(d + d^*, i^*)$

is Fredholm. Likewise, because $d + d^*\colon L^{2,\mathrm{n}}_2(B^4; \bigwedge^* T^* B^4) \to L^2_1(B^4; \bigwedge^* T^* B^4)$ is an Fredholm and injective on one-forms and $i^*\colon L^2_2(B^4; \bigwedge^* T^* B^4) \to L^2_{3/2}(\partial B^4; \bigwedge^* T^* \partial B^4)$ is surjective, we know that

$$(d + d^*, i^*)\colon L^2_2(B^4; \textstyle\bigwedge^* T^* B^4) \to L^2_1(B^4; \bigwedge^* T^* B^4) \times L^2_{3/2}(\partial B^4; \bigwedge^* T^* \partial B^4)$$

is Fredholm and injective on one-forms. Hence, using $d^* a = 0$, we have bounds

$$\|a\|_{L^2_1(B^4)} \leq C_G \left( \|da\|_{L^2(B^4)} + \|i^* a\|_{L^2_{1/2}(\partial B^4)} \right), \tag{5.2}$$

$$\|a\|_{L^2_2(B^4)} \leq C_G \left( \|da\|_{L^2_1(B^4)} + \|i^* a\|_{L^2_{3/2}(\partial B^4)} \right). \tag{5.3}$$

It remains to bound $da$ in terms of $F_A$. Using $da = F_A - \frac{1}{2}[a \wedge a]$, we compute

$$\|da\|_{L^2(B^4)} \leq \|F_A\|_{L^2(B^4)} + \tfrac{1}{2} C_{\mathfrak{L}} C_S \|a\|_{L^4(B^4)} \|a\|_{L^2_1(B^4)},$$

where $C_{\mathfrak{L}}$ is the operator norm of the Lie bracket and $C_S$ is the operator norm of the Sobolev embedding $L^2_1(B^4) \hookrightarrow L^4(B^4)$. Hence, requiring $\varepsilon \leq (C_{\mathfrak{L}} C_S C_G)^{-1}$, we have that $\|a\|_{L^4(B^4)} < \varepsilon$ implies

$$\|da\|_{L^2(B^4)} \leq \|F_A\|_{L^2(B^4)} + \tfrac{1}{2} C_G^{-1} \|a\|_{L^2_1(B^4)}.$$

Combining with (5.2), we see that

$$\|a\|_{L^2_1(B^4)} \leq C_G \left( \|F_A\|_{L^2(B^4)} + \|i^* a\|_{L^2_{1/2}(\partial B^4)} \right) + \tfrac{1}{2} \|a\|_{L^2_1(B^4)}.$$

Hence,

$$\|a\|_{L^2_1(B^4)} \leq 2 C_G \left( \|F_A\|_{L^2(B^4)} + \|i^* a\|_{L^2_{1/2}(\partial B^4)} \right),$$

as desired.

We now proceed to the higher regularity. We have

$$\nabla da = \nabla F_A - \tfrac{1}{2} \nabla [a \wedge a] = \nabla_A F_A - [a \otimes F_A] - [\nabla a \wedge a].$$

Hence,

$$\|\nabla da\|_{L^2(B^4)} \leq \|\nabla_A F_A\|_{L^2(B^4)} + C_{\mathfrak{L}} \|a\|_{L^4(B^4)} \|F_A\|_{L^4(B^4)} + C_{\mathfrak{L}} \|\nabla a\|_{L^4(B^4)} \|a\|_{L^4(B^4)}.$$

We have $\|\nabla a\|_{L^4(B^4)} \leq C_S \|\nabla a\|_{L^2_1(B^4)} \leq C_S \|a\|_{L^2_2(B^4)}$. However, to obtain a similar bound for $F_A$, we first need to prove an inequality

$$\|[a \wedge a]\|_{L^2_1(B^4)} \leq C_s \|a\|_{L^2_2(B^4)} \|a\|_{L^4(B^4)}$$

for some constant $C_s$, which does not immediately follow from the Sobolev multiplication theorems because $L^2_2$ is the critical level of regularity in four dimensions. We compute

$$\begin{aligned}
\|[a \wedge a]\|^2_{L^2_1(B^4)} &= \|\nabla[a \wedge a]\|^2_{L^2(B^4)} + \|[a \wedge a]\|^2_{L^2(B^4)} \\
&= \|2[\nabla a \wedge a]\|^2_{L^2(B^4)} + \|[a \wedge a]\|^2_{L^2(B^4)} \\
&\leq 4C^2_{\mathfrak{L}} \|\nabla a\|^2_{L^4(B^4)} \|a\|^2_{L^4(B^4)} + C^2_{\mathfrak{L}} \|a\|^4_{L^4(B^4)} \\
&\leq 4C^2_{\mathfrak{L}} C^2_S \|a\|^2_{L^2_2(B^4)} \|a\|^2_{L^4(B^4)} + C^2_{\mathfrak{L}} C^2_S \|a\|^2_{L^2_1(B^4)} \|a\|^2_{L^4(B^4)} \\
&\leq 5C^2_{\mathfrak{L}} C^2_S \|a\|^2_{L^2_2(B^4)} \|a\|^2_{L^4(B^4)},
\end{aligned}$$

as desired. From here, we can bound $F_A$:

$$\begin{aligned}
\|F_A\|_{L^4(B^4)} &\leq C_S \|da\|_{L^2_1(B^4)} + \tfrac{1}{2} C_S \|[a \wedge a]\|_{L^2_1(B^4)} \\
&\leq C_S \|a\|_{L^2_2(B^4)} + \tfrac{1}{2} C_S C_s \|a\|_{L^2_2(B^4)} \|a\|_{L^4(B^4)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\nabla da\|_{L^2(B^4)} \leq{}& \|\nabla_A F_A\|_{L^2(B^4)} + C_{\mathfrak{L}} C_S \|a\|_{L^4(B^4)} \|a\|_{L^2_2(B^4)} \\
&+ \tfrac{1}{2} C_{\mathfrak{L}} C_S C_s \|a\|^2_{L^4(B^4)} \|a\|_{L^2_2(B^4)} + C_{\mathfrak{L}} C_S \|a\|_{L^2_2(B^4)} \|a\|_{L^4(B^4)}.
\end{aligned}$$

By requiring $\varepsilon < \frac{1}{12}(C_{\mathfrak{L}}C_S C_G)^{-1}$ and $\varepsilon^2 < \frac{1}{6}(C_{\mathfrak{L}}C_S C_s C_G)^{-1}$, we have

$$\|\nabla da\|_{L^2(B^4)} \leq \|\nabla_A F_A\|_{L^2(B^4)} + \tfrac{1}{4}C_G^{-1}\|a\|_{L^2_2(B^4)}.$$

Hence,

$$\begin{aligned}
\|da\|_{L^2_1(B^4)} &= \sqrt{\|\nabla da\|^2_{L^2(B^4)} + \|da\|^2_{L^2(B^4)}} \leq \|\nabla da\|_{L^2(B^4)} + \|da\|_{L^2(B^4)} \\
&\leq \|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} + \tfrac{1}{4}C_G^{-1}\|a\|_{L^2_2(B^4)} + \tfrac{1}{2}C_G^{-1}\|a\|_{L^2_1(B^4)} \\
&\leq \|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} + \tfrac{3}{4}C_G^{-1}\|a\|_{L^2_2(B^4)}.
\end{aligned}$$

Combining with 5.3, we have

$$\|a\|_{L^2_2(B^4)} \leq C_G\left(\|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{3/2}(\partial B^4)}\right) + \tfrac{3}{4}\|a\|_{L^2_2(B^4)}.$$

Thus,

$$\|a\|_{L^2_2(B^4)} \leq 4C_G\left(\|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{3/2}(\partial B^4)}\right).$$

□

Now, we prove that the space of connections satisfying Proposition 5.4 is open.

**Lemma 5.6.** *There exists an $\varepsilon$ with the following significance. Let $A = d + a$ be an $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connection with $\|a\|_{L^4(B^4)} < \varepsilon$ and $d^*a = 0$. Then there exists an open $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ neighborhood of $A$ such that any connection $B$ in this neighborhood has an $L^2_3(B^4; M_N)$ gauge transformation $g$ with $i^*g$ the identity that sends $B$ to a connection $\tilde{B}$ satisfying the Coulomb condition $d^*\tilde{b} = 0$. Moreover, $g$ depends smoothly on $B$.*

*Proof.* We search for a $g$ of the form $e^{-\gamma}$, where $\gamma$ is a $\mathfrak{g}$-valued $L^2_3(B^4)$ function. For $\alpha$ small in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$, we want a solution $\gamma$ to the equation

$$\begin{aligned}
d^*(e^{-\gamma}(a+\alpha)e^{\gamma} - (de^{-\gamma})e^{\gamma}) &= 0, \\
i^*\gamma &= 0.
\end{aligned} \tag{5.4}$$

Letting $L^{2,\mathrm{n}}_3(B^4;\mathfrak{g})$ denote $L^2_3(B^4;\mathfrak{g}\otimes T^*B^4)$ functions $\gamma$ satisfying the boundary condition $i^*\gamma = 0$, we consider the map

$$\begin{aligned} L^{2,\mathrm{n}}_3(B^4;\mathfrak{g}) \times L^2_2(B^4;\mathfrak{g}\otimes T^*B^4) &\to L^2_1(B^4;\mathfrak{g}) \\ (\gamma,\alpha) &\mapsto d^*(e^{-\gamma}(a+\alpha)e^{\gamma} - (de^{-\gamma})e^{\gamma}) \end{aligned}$$

Because we are above the critical regularity, the exponential map is smooth, as are the relevant multiplication maps and linear maps in the above formula. To apply the implicit function theorem, we must show that the derivative of this map with respect to the $\gamma$ variable at $(\gamma,\alpha) = (0,0)$ is an isomorphism. This derivative map is

$$\gamma' \mapsto d^*[a,\gamma'] + d^*d\gamma'.$$

Call this map $T\colon L^{2,\mathrm{n}}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g})$. By Propositions 2.8 and 2.9 and the fact that $H^0(B^4,\partial B^4) = 0$, we know that $\gamma' \mapsto d^*d\gamma'$ is an isomorphism as a map $L^{2,\mathrm{n}}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g})$.

Next, we show that the other term, $d^*[a,\gamma']$, is a compact operator as a map $L^{2,\mathrm{n}}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g})$, so $T$ is a compact perturbation of $d^*d$, and hence is a Fredholm operator of index zero. We compute

$$\begin{aligned} d^*[a,\gamma'] = -*d*[a,\gamma'] = -*d[*a,\gamma'] &= -*[d*a,\gamma'] + *[*a\wedge d\gamma'] \\ &= [d^*a,\gamma'] + *[*a\wedge d\gamma'] = *[*a\wedge d\gamma'], \end{aligned}$$

We can view this term as a composition

$$L^{2,\mathrm{n}}_3(B^4;\mathfrak{g}) \xrightarrow{d} L^2_2(B^4;\mathfrak{g}\otimes T^*B^4) \hookrightarrow L^2_{3/2}(B^4;\mathfrak{g}\otimes T^*B^4) \xrightarrow{*[*a\wedge\cdot]} L^2_1(B^4;\mathfrak{g}).$$

The Sobolev multiplication and embedding theorems tell us that that the above maps are continuous and the second inclusion is compact, so the composition is comapct.

Thus $T$ is Fredholm of index zero, so to show that $T$ is an isomorphism, it will suffice to prove that $T$ is injective. Working in one less degree of regularity, since

$d^*d\colon L^{2,\mathrm{n}}_2(B^4;\mathfrak{g}) \to L^2(B^4;\mathfrak{g})$ is an isomoprhism, we know that there is a constant $\varepsilon_\Delta$ such that $\|d^*d\gamma'\|_{L^2(B^4)} \geq \varepsilon_\Delta \|\gamma'\|_{L^2_2(B^4)}$. On the other hand, we know that

$$\|{*}[{*}a \wedge d\gamma']\|_{L^2(B^4)} \leq C_{\mathfrak{L}} C_S C_d \|a\|_{L^4(B^4)} \|\gamma'\|_{L^2_2(B^4)},$$

where $C_{\mathfrak{L}}$ is the operator norm of the Lie bracket bilinear form, $C_S$ is the operator norm of the Sobolev embedding $L^2_1(B^4) \hookrightarrow L^4(B^4)$, and $C_d$ is the operator norm of $d\colon L^2_2(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g}\otimes T^*B^4)$. Hence, by requiring that $\varepsilon < \varepsilon_\Delta(C_{\mathfrak{L}}C_SC_d)^{-1}$, we see that $\|a\|_{L^4(B^4)} < \varepsilon$ implies that $\|{*}[{*}a\wedge d\gamma']\|_{L^2(B^4)} < \varepsilon_\Delta \|\gamma'\|_{L^2_2(B^4)}$, so

$$\|T\gamma'\|_{L^2(B^4)} \geq \|d^*d\gamma'\|_{L^2(B^4)} - \|{*}[{*}a \wedge d\gamma']\|_{L^2(B^4)} > 0.$$

Thus, $T$ is injective, and hence an isomorphism. Thus, the implicit function theorem gives us a solution $g = e^{-\gamma}$ to (5.4) depending smoothly on $\alpha$ in a neighborhood of $\alpha = 0$. That is, we have a gauge transformation that is the identity on the boundary sending a connection in an $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$ neighborhood of $A$ into Coulomb gauge, as desired. ☐

**Corollary 5.7.** *The space of $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections $A = d+a$ with $\|a\|_{L^2_1(B^4)} < \varepsilon$ satisfying Proposition 5.4 is open in $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$.*

*Proof.* Let $\varepsilon_4$ be the smaller of the constants in Lemmas 5.5 and 5.6, and let $C$ be the constant from Lemma 5.5. Because $A \mapsto F_A$ is continuous as a map $L^2_1(B^4;\mathfrak{g}\otimes T^*B^4) \to L^2(B^4;\mathfrak{g}\otimes \bigwedge^2 T^*B^4)$, and $a \mapsto i^*a$ is continuous as a map $L^2_1(B^4;\mathfrak{g}\otimes T^*B^4) \to L^2_{1/2}(\partial B^4;\mathfrak{g}\otimes T^*\partial B^4)$, we can require that the $\varepsilon$ in Proposition 5.4 be small enough so that $\|a\|_{L^2_1(B^4)} < \varepsilon$ implies $\|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{1/2}(\partial B^4)} < \varepsilon_4(C_SC)^{-1}$, where $C_S$ is the norm of the Sobolev embedding $L^2_1(B^4) \hookrightarrow L^4(B^4)$.

Let $A$ be a connection with $\|a\|_{L^2_1(B^4)} < \varepsilon$ satisfying Proposition 5.4, so there is a gauge transformation $g$ sending $A$ to $\tilde{A}$ such that $i^*g$ is the identity and $d^*\tilde{a} = 0$. We will show that a neighborhood of $\tilde{A}$ satisfies Proposition 5.4, and then pull it back to a neighborhood of $A$. The final condition of Proposition 5.4 implies that

$$\|\tilde{a}\|_{L^4(B^4)} \leq C_S \|\tilde{a}\|_{L^2_1(B^4)} \leq C_S C \left( \|F_A\|_{L^2(B^4)} + \|i^* a\|_{L^2_{1/2}(\partial B^4)} \right) < \varepsilon_4.$$

Hence, we can apply Lemma 5.6 to $\tilde{A}$ and find an open $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$ neighborhood of $\tilde{A}$ such that for any connection $B$ in the neighborhood, there is a $L^2_3(B^4; M_N)$ gauge transformation that is the identity on the boundary and sends $B$ to a connection $\tilde{B}$ in Coulomb gauge. Moreover, the implicit funcion theorem tells us that $g$ depends continously on $B$. Hence, by shrinking the neighborhood of $\tilde{A}$, we can guarantee that $g$ is close to the identity in $L^2_3(B^4; M_N)$, and hence that $\tilde{B}$ is close to $B$ and hence to $\tilde{A}$ in $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$. In particular, since $L^4(B^4)$ is a weaker norm than $L^2_2(B^4)$, we can choose the neighborhood of $\tilde{A}$ small enough so that $\tilde{B}$ also satisfies $\|\tilde{b}\|_{L^4(B^4)} < \varepsilon_4$. Then, we can apply Lemma 5.5 to $\tilde{B}$, to find that

$$\|\tilde{b}\|_{L^2_1(B^4)} \leq C \left( \|F_{\tilde{B}}\|_{L^2(B^4)} + \|i^* \tilde{b}\|_{L^2_{1/2}(B^4)} \right) = C \left( \|F_B\|_{L^2(B^4)} + \|i^* b\|_{L^2_{1/2}(B^4)} \right),$$

because $\|F_B\|_{L^2(B^4)}$ is invariant under gauge transformations, and $i^* g = 1$ implies that $i^* \tilde{b} = i^* b$.

We conclude that Proposition 5.4 holds on an open $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$ neighborhood of $\tilde{A}$. Since the conclusion of Proposition 5.4 is invariant under $L^2_3(B^4; M_N)$ gauge transformations $g$ with $i^* g = 1$, we can pull back this open neighborhood of $\tilde{A}$ via $g^{-1}$ to a neighborhood of $A$ satisfying Proposition 5.4, as desired. □

Finally, we prove that the set of connections satisfying Proposition 5.4 is closed.

**Lemma 5.8.** *The space of $L^2_1(B^4; \mathfrak{g} \otimes T^* B^4)$ connections satisfying Proposition 5.4 is closed in $L^2_1(B^4; \mathfrak{g} \otimes T^* B^4)$. Likewise, the space of $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$ connections with satisfying Proposition 5.4 is closed in $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$.*

*Proof.* Let $A_i \to A$ be a sequence of connections converging in $L^2_1(B^4; \mathfrak{g} \otimes T^* B^4)$ with $\|a_i\|_{L^2_1(B^4)} < \varepsilon$ and $\|a\|_{L^2_1(B^4)} < \varepsilon$, such that there exist $L^2_2(B^4; M_N)$ gauge transformations $g_i$ sending $A_i$ to $\tilde{A}_i$ satisfying $i^* g_i = 1$, $d^* \tilde{a}_i = 0$, and $\|\tilde{a}_i\|_{L^2_1(B^4)} \leq$

$C\left(\|F_{A_i}\|_{L^2(B^4)} + \|i^*a_i\|_{L^2_{1/2}(\partial B^4)}\right)$. Our goal is to find a limit $g$ sending $A$ to $\tilde{A}$ that also satisfies these conditions.

Because the $A_i$ converge in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$, we know that the $F_{A_i}$ converge and are hence bounded in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$, and the $i^*a_i$ converge and are hence bounded in $L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$. Hence, the above inequality tells us that the $\tilde{a}_i$ are bounded in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$, so we can pass to a subsequence where the $\tilde{a}_i$ converge weakly in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. Let $\tilde{a}$ be the limit. By Lemma 5.2, after passing to a subsequence, the $g_i$ converge weakly in $L^2_2(B^4; M_N)$ to a gauge transformation $g$ sending $A$ to $\tilde{A}$.

Finally, since $i^*\colon L^4_1(B^4; M_N) \to L^4_{3/4}(\partial B^4; M_N)$ is continuous and linear, the $g_i$ converging weakly to $g$ in $L^4_1(B^4; M_N)$ implies that the $i^*g_i = 1$ converge weakly to $i^*g$, hence $i^*g = 1$. Likewise, because $d^*\colon L^2_1(B^4; \mathfrak{g} \otimes T^*B^4) \to L^2(B^4; \mathfrak{g} \otimes T^*B^4)$ is continuous and linear, the $\tilde{a}_i$ converging weakly to $\tilde{a}$ in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ imply that the $d^*\tilde{a}_i = 0$ converge weakly to $d^*\tilde{a}$, so $d^*\tilde{a} = 0$. Finally, for the inequality, we use the lower semicontinuity of norms under weak limits and the strong convergence of $F_{A_i}$ and $i^*a_i$ to compute

$$
\begin{aligned}
\|\tilde{a}\|_{L^2_1(B^4)} \leq \liminf \|\tilde{a}_i\|_{L^2_1(B^4)} &\leq C\left(\liminf \|F_{A_i}\|_{L^2(B^4)} + \liminf \|i^*a_i\|_{L^2_{1/2}(\partial B^4)}\right) \\
&= C\left(\|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{1/2}(\partial B^4)}\right).
\end{aligned}
$$

We now proceed to prove closedness in higher regularity. Let $A_i \to A$ be a sequence of connections converging in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ with $\|a_i\|_{L^2_1(B^4)} < \varepsilon$ and $\|a\|_{L^2_1(B^4)} < \varepsilon$, such that there exist $L^2_3(B^4; M_N)$ gauge transformations $g_i$ sending $A_i$ to $\tilde{A}_i$ satisfying $i^*g_i = 1$, $d^*\tilde{a}_i = 0$, and $\|\tilde{a}_i\|_{L^2_1(B^4)} \leq C\left(\|F_{A_i}\|_{L^2(B^4)} + \|i^*a_i\|_{L^2_{1/2}(\partial B^4)}\right)$.

As in the proof of Corollary 5.7, this inequality, along with a small enough $\varepsilon$, guarantess that $\|\tilde{a}_i\|_{L^4(B^4)}$ is small enough to apply Lemma 5.5, giving us

$$
\begin{aligned}
\|\tilde{a}_i\|_{L^2_2(B^4)} &\leq C\left(\|\nabla_{\tilde{A}_i} F_{\tilde{A}_i}\|_{L^2(B^4)} + \|F_{\tilde{A}_i}\|_{L^2(B^4)} + \|i^*\tilde{a}_i\|_{L^2_{3/2}(\partial B^4)}\right) \\
&= C\left(\|\nabla_{A_i} F_{A_i}\|_{L^2(B^4)} + \|F_{A_i}\|_{L^2(B^4)} + \|i^*a_i\|_{L^2_{3/2}(\partial B^4)}\right),
\end{aligned}
$$

because $\|F_A\|_{L^2(B^4)}$ and $\|\nabla_A F_A\|_{L^2(B^4)}$ are gauge invariant quantities, and $i^*a_i = i^*\tilde{a}_i$. Since $A_i$ converges to $A$ in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$, we conclude that the right-hand side of the inequality is bounded, and hence a subsequence of the $\tilde{a}_i$ converges weakly in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$. Let $\tilde{a}$ be its limit. The above argument for $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections gives us a gauge transformation $g \in L^2_2(B^4; M_N)$ sending $A$ to $\tilde{A}$ satisfying all of the conditions of Proposition 5.4, so it only remains to show that $g$ is actually in $L^2_3(B^4)$. We prove this claim in two steps from the equation $dg = ga - \tilde{a}g$. First, note that the multiplication $L^2_2(B^4) \times L^2_2(B^4) \to L^3_1(B^4)$ is continuous. Hence, since $g, a, \tilde{a} \in L^2_2(B^4)$, we know that $dg \in L^3_1(B^4; M_N \otimes T^*B^4)$, so $g \in L^3_2(B^4; M_N)$. Next, since the multiplication $L^3_2(B^4) \times L^2_2(B^4) \to L^2_2(B^4)$ is continuous, we have that $dg \in L^2_2(B^4; M_N \otimes T^*B^4)$, so $g \in L^2_3(B^4; M_N)$, as desired. $\square$

These lemmas complete the proof of Proposition 5.4.

*Proof of 5.4.* The space of $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connections $A$ with $\|a\|_{L^2_1(B^4)} < \varepsilon$ is connected, and by Corollary 5.7 and Lemma 5.8 the space of connections with $\|a\|_{L^2_1(B^4)} < \varepsilon$ satisfying Proposition 5.4 is both open and closed, and hence contains all connections with $\|a\|_{L^2_1(B^4)} < \varepsilon$. Meanwhile, any $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connection $A$ with $\|a\|_{L^2_1(B^4)} < \varepsilon$ is the limit in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ of sequence $A_i$ of $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connections with $\|a\|_{L^2_1(B^4)} < \varepsilon$. Because the $A_i$ satisfy Proposition 5.4, Lemma 5.8 tells us that so does $A$. $\square$

We also prove that the gauge transformation constructed by Proposition 5.4 is unique, at least with an appropriate choice of constants. We require $L^4(B^4; \mathfrak{g} \otimes T^*B^4)$ bounds on the Coulomb gauge representatives, but note that these follow from the condition $\|\tilde{a}\|_{L^2_1(B^4)} \leq C\left(\|F_A\|_{L^2(B^4)} + \|i^*a\|_{L^2_{1/2}(\partial B^4)}\right)$ and the bounds on $\|a\|_{L^2_1(B^4)}$ in Proposition 5.4. In addition, for use in the future, we will assume that $i^*g$ is a constant gauge transformation on $\partial B^4$ but not necessarily the identity.

**Proposition 5.9.** *There exists a constant $\varepsilon$ such that if $A = d + a$ and $B = d + b$ are two $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections gauge equivalent via a gauge transformation $g \in L^2_2(B^4; G)$ satisfying*

*1. bounds* $\|a\|_{L^4(B^4)}, \|b\|_{L^4(B^4)} < \varepsilon$*,*

*2. the boundary condition that* $i^*g$ *is equal to a constant* $c \in G$ *on* $\partial B^4$*, and*

*3. the Coulomb condition* $d^*a = d^*b = 0$*,*

*then* $g$ *is the constant gauge transformation* $c$ *on* $B^4$*.*

*Proof.* We have the gauge equivalence equation

$$dg = ga - bg.$$

Thus, using $d^*a = d^*b = 0$, we have

$$d^*dg = -{*d}(g{*a}) + {*d}({*bg}) = -{*}(dg \wedge {*a} + gd({*a})) + {*}(d({*b})g - {*b} \wedge dg)$$
$$= -{*}(dg \wedge {*a} + {*b} \wedge dg). \quad (5.5)$$

Hence,

$$\|d^*dg\|_{L^2(B^4)} \leq \left(\|a\|_{L^4(B^4)} + \|b\|_{L^4(B^4)}\right) \|dg\|_{L^4(B^4)}$$
$$\leq C_S \left(\|a\|_{L^4(B^4)} + \|b\|_{L^4(B^4)}\right) \|dg\|_{L^2_1(B^4)}.$$

On the other hand, since $H^1(B^4, \partial B^4) = 0$, Corollary 2.10 tells us that

$$d + d^* \colon L_1^{2,\mathrm{n}}(B^4; M_N \otimes \textstyle\bigwedge^* T^*B^4) \to L^2(B^4; M_N \otimes \textstyle\bigwedge^* T^*B^4)$$

is a Fredholm operator with no kernel on one-forms. The boundary condition on $g$ implies that $i^*dg = d_{\partial B^4} i^*g = d_{\partial B^4} c = 0$, so $dg$ is indeed in $L_1^{2,\mathrm{n}}(B^4; M_N \otimes T^*B^4)$. Thus, there is a constant $C_G$ independent of $g$ such that

$$\|dg\|_{L^2_1(B^4)} \leq C_G \|(d + d^*)dg\|_{L^2(B^4)} = C_G \|d^*dg\|_{L^2(B^4)}.$$

Combining these inequalities, we have

$$\|dg\|_{L^2_1(B^4)} \leq C_G C_S \left( \|a\|_{L^4(B^4)} + \|b\|_{L^4(B^4)} \right) \|dg\|_{L^2_1(B^4)} .$$

Thus, requiring $\varepsilon \leq \frac{1}{4}(C_G C_S)^{-1}$, the condition $\|a\|_{L^4(B^4)}, \|b\|_{L^4(B^4)} < \varepsilon$ implies that

$$\|dg\|_{L^2_1(B^4)} \leq \frac{1}{2} \|dg\|_{L^2_1(B^4)} ,$$

so $dg = 0$. Thus $g$ is constant on $B^4$. Since $i^* g = c$ on $\partial B^4$, we conclude that $g = c$ on all of $B^4$, as desired. □

## 5.2 Coulomb gauge with Coulomb gauge on the boundary

In this section, we prove a second gauge fixing result, where we show that if a connection has small energy, then it is gauge equivalent to a connection $\tilde{A} = d + \tilde{a}$ that satisfies the Coulomb condition $d^* \tilde{a} = 0$ on $B^4$ and whose restriction to the boundary $i^* \tilde{a}$ satisfies the the Coulomb condition $d^*_{\partial B^4}(i^* \tilde{a}) = 0$ on $\partial B^4$, where $d^*_{\partial B^4}$ denotes the adjoint of the differential $d_{\partial B^4}$ on $\partial B^4$ with respect to the metric on $\partial B^4$.

**Theorem 5.10.** *There exist constants $\varepsilon$ and $C$ such that if $A$ is any $L^2_1(B^4; \mathfrak{g} \otimes T^* B^4)$ connection with $\|F_A\|_{L^2(B^4)} < \varepsilon$, then there exists an $L^2_1(B^4; \mathfrak{g} \otimes T^* B^4)$ connection $\tilde{A}$ gauge equivalent to $A$ by an $L^2_2(B^4; G)$ gauge transformation such that*

1. *$\tilde{A}$ is in Dirichlet Coulomb gauge, that is, $d^* \tilde{a} = 0$ on $B^4$ and $d^*_{\partial B^4}(i^* \tilde{a}) = 0$ on $\partial B^4$, and*

2. *$\|\tilde{a}\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)}$.*

*Moreover, if $A$ is in $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$, then $g \in L^2_3(B^4; G)$.*

Gauge fixing with the Dirichlet Coulomb condition $d^* \tilde{a} = 0$ and $d^*_{\partial B^4} i^* \tilde{a} = 0$ is shown in Uhlenbeck's paper [24, Theorem 2.7], but again with $L^\infty$ bounds. Marini

[11] improves this result to $L^2_1$ connections, but with the additional assumption that, on the boundary, $\|i^*F_A\|_{L^2(\partial B^4)} < \varepsilon$. We remove this condition, so $\|i^*F_A\|_{L^2(\partial B^4)}$ need not even be finite. As in the previous section and [23], we first work in higher regularity and prove that the space of $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connections satisfying Theorem 5.10 is both open and closed in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$, and then prove the result for $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections by showing that the space of $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ connections satisfying 5.10 is closed in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. We begin by strenghtening the a priori bounds in Lemma 5.5 in this setting.

**Lemma 5.11.** *There exist constants $\varepsilon$ and $C$, such that if $A = d + a$ is an $L^2_1(B^4)$ connection with $\|a\|_{L^4(B^4)} < \varepsilon$ in Dirichlet Coulomb gauge $d^*a = 0$ and $d^*_{\partial B^4} i^*a = 0$, then*

$$\|a\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)}.$$

*Furthermore, if $A$ is an $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connection, then*

$$\|a\|_{L^2_2(B^4)} \leq C \left( \|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} \right).$$

*Proof.* Recall (5.2) and (5.3).

$$\|a\|_{L^2_1(B^4)} \leq C_G \left( \|da\|_{L^2(B^4)} + \|i^*a\|_{L^2_{1/2}(\partial B^4)} \right),$$
$$\|a\|_{L^2_2(B^4)} \leq C_G \left( \|da\|_{L^2_1(B^4)} + \|i^*a\|_{L^2_{3/2}(\partial B^4)} \right).$$

The key idea is to absorb the $i^*a$ terms by proving that $d^*_{\partial B^4} i^*a = 0$ implies that $\|i^*a\|_{L^2_{1/2}(\partial B^4)} \leq C_F \|da\|_{L^2(B^4)}$ and $\|i^*a\|_{L^2_{3/2}(\partial B^4)} \leq C_F \|da\|_{L^2_1(B^4)}$ for some constant $C_F$. Since $d_{\partial B^4} + d^*_{\partial B^4}$ is elliptic and $H^1(\partial B^4) = 0$, we know that $d_{\partial B^4} + d^*_{\partial B^4}$ is a Fredholm operator with no kernel on one forms. Thus, there is a constant $C_g$ such that

$$\begin{aligned} \|i^*a\|_{L^2_{3/2}(\partial B^4)} &\leq C_g \|(d_{\partial B^4} + d^*_{\partial B^4})(i^*a)\|_{L^2_{1/2}(\partial B^4)} = C_g \|d_{\partial B^4} i^*a\|_{L^2_{1/2}(\partial B^4)} \\ &= C_g \|i^*(da)\|_{L^2_{1/2}(\partial B^4)} \leq C_g C_T \|da\|_{L^2_1(B^4)}, \end{aligned}$$

where $C_T$ is the operator norm of the trace map $L^2_1(B^4) \to L^2_{1/2}(\partial B^4)$. We could do the same argument in lower regularity, except that the trace map $L^2(B^4) \to L^2_{-1/2}(\partial B^4)$ is unbounded. However, we can still get the inequality $\|d_{\partial B^4} i^* a\|_{L^2_{-1/2}(\partial B^4)} \le C_T \|da\|_{L^2(B^4)}$ using the Hodge decomposition, as we show in Lemma 5.12. For now, we continue with this assumption. Thus, by the same argument,

$$\begin{aligned} \|i^* a\|_{L^2_{1/2}(\partial B^4)} &\le C_g \|(d_{\partial B^4} + d^*_{\partial B^4})(i^* a)\|_{L^2_{-1/2}(\partial B^4)} \\ &= C_g \|d_{\partial B^4} i^* a\|_{L^2_{-1/2}(\partial B^4)} \le C_g C_T \|da\|_{L^2(B^4)}. \end{aligned}$$

Hence, we have

$$\|a\|_{L^2_1(B^4)} \le C_G(C_g C_T + 1)\|da\|_{L^2(B^4)}, \tag{5.6}$$

$$\|a\|_{L^2_2(B^4)} \le C_G(C_g C_T + 1)\|da\|_{L^2_1(B^4)}. \tag{5.7}$$

At this point, we can follow the argument of Lemma 5.5 with $C_G(C_g C_T + 1)$ in place of $C_G$. By choosing $\varepsilon$ small enough, we can have $\|a\|_{L^4(B^4)} < \varepsilon$ to imply

$$\|da\|_{L^2(B^4)} \le \|F_A\|_{L^2(B^4)} + \tfrac{1}{2} C_G^{-1}(C_g C_T + 1)^{-1} \|a\|_{L^2_1(B^4)}.$$

Combing with (5.6) and rearranging, we have

$$\|a\|_{L^2_1(B^4)} \le 2C_G(C_g C_T + 1)\|F_A\|_{L^2(B^4)},$$

as desired.

Likewise, in higher regularity, we can modify the argument in Lemma 5.5 to choose $\varepsilon$ small enough to guarantee

$$\|da\|_{L^2_1(B^4)} \le \|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)} + \tfrac{3}{4} C_G^{-1}(C_g C_T + 1)^{-1} \|a\|_{L^2_2(B^4)}.$$

Combining with (5.7) and rearranging, we have

$$\|a\|_{L^2_2(B^4)} \leq 4C_G(C_gC_T+1)\left(\|\nabla_A F_A\|_{L^2(B^4)} + \|F_A\|_{L^2(B^4)}\right),$$

as desired. $\square$

**Lemma 5.12.** *Let $X$ be a compact smooth manifold with boundary, and let $\alpha$ be a differential form in $L^2_1(X; \bigwedge^* T^*X)$. There is a constant $C_T$ independent of $\alpha$ such that*

$$\|d_{\partial X} i^*\alpha\|_{L^2_{-1/2}(\partial X)} \leq C_T \|d\alpha\|_{L^2(X)}.$$

*Proof.* It suffices to consider smooth $\alpha$ because smooth forms are dense in $L^2_1(X; \bigwedge^* T^*X)$ and the linear map $d_{\partial X} \circ i^* \colon L^2_1(X; \bigwedge^* T^*X) \to L^2_{-1/2}(\partial X; \bigwedge^* T^*\partial X)$ and the linear map $d \colon L^2_1(X; \bigwedge^* T^*X) \to L^2(X; \bigwedge^* T^*X)$ are continuous. By Proposition 2.9,

$$\alpha = dd^*G^{\mathrm{t}}\alpha + d^*dG^{\mathrm{t}}\alpha + \pi^{\mathrm{t}}_{\mathcal{H}}\alpha.$$

Let $\beta = d^*dG^{\mathrm{t}}\alpha$, which is smooth by Proposition 2.9 because $\alpha$ is smooth, so both $i^*\alpha$ and $i^*\beta$ are well-defined. By the above equation, we see that

$$d\alpha = d\beta,$$
$$d_{\partial X} i^*\alpha = i^*d\alpha = i^*d\beta = d_{\partial X} i^*\beta.$$

Hence, it suffices to prove our lemma for $\beta$.

It is clear that $d^*\beta = 0$. Moreover, by the boundary conditions on the range of $G^{\mathrm{t}}$ in Proposition 2.9, we see that

$$i^* {*}\beta = i^* {*} d^*dG^{\mathrm{t}}\alpha = \pm i^* d {*} dG^{\mathrm{t}}\alpha = \pm d_{\partial X} i^* {*} dG^{\mathrm{t}}\alpha = \pm d_{\partial X} i^* d^* {*} G^{\mathrm{t}}\alpha = 0.$$

Hence, we can apply Corollary 2.10, noting that Proposition 2.9 gives us that $\beta$ is orthogonal to $\mathcal{H}^{\mathrm{t}}$. In other words, we have a Fredholm operator

$$d + d^* \colon L^{2,\mathrm{t}}_1(X; \textstyle\bigwedge^* T^*X) \to L^2(X; \textstyle\bigwedge^* T^*X),$$

and $\beta$ is orthogonal to its kernel, so there is a constant $C_G$ independent of $\beta$ such that

$$\|\beta\|_{L^2_1(X)} \leq C_G \|(d + d^*)\beta\|_{L^2(X)} = C_G \|d\beta\|_{L^2(X)}.$$

At this point, proving the claim for $\beta$ is straightforward. Let $C$ be the operator norm of $d_{\partial X} \circ i^* \colon L^2_1(X; \bigwedge^* T^* X) \to L^2_{-1/2}(\partial X; \bigwedge^* T^* \partial X)$. We have

$$\|d_{\partial X} i^* \beta\|_{L^2_{-1/2}(\partial X)} \leq C \|\beta\|_{L^2_1(X)} \leq C C_G \|d\beta\|_{L^2(X)},$$

as desired, letting $C_T = CC_G$. Since $d\alpha = d\beta$ and $d_{\partial X} i^* \alpha = d_{\partial X} i^* \beta$, we also have

$$\|d_{\partial X} i^* \alpha\|_{L^2_{-1/2}(\partial X)} \leq C_T \|d\alpha\|_{L^2(X)}$$

for smooth $\alpha$, and the aforementioned density argument gives us the inequality for all $\alpha \in L^2_1(X; \bigwedge^* T^* X)$. □

As for the fixed boundary gauge fixing, our next step is to prove openness in higher regularity.

**Lemma 5.13.** *There exist constants $\varepsilon_4$ and $\varepsilon_3$ with the following significance. Let $A = d + a$ be an $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$ connection with $\|a\|_{L^4(B^4)} < \varepsilon_4$, $\|i^* a\|_{L^3(\partial B^4)} < \varepsilon_3$, $d^* a = 0$, and $d^*_{\partial B^4} i^* a = 0$. Then there exists an open $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$ neighborhood of $A$ such that any connection $B$ in this neighborhood has an $L^2_3(B^4; G)$ gauge transformation $g$ that sends $B$ to a connection $\tilde{B}$ satisfying the Dirichlet Coulomb conditions $d^* \tilde{b} = 0$ and $d^*_{\partial B^4} i^* \tilde{b} = 0$. Moreover, $g$ depends smoothly on $B$.*

*Proof.* We adapt the proof of Lemma 5.6. Again, we search for a $g$ of the form $e^{-\gamma}$ for $\gamma \in L^2_3(B^4; \mathfrak{g})$. For $\alpha$ small in $L^2_2(B^4; \mathfrak{g} \otimes T^* B^4)$, we want a solution $\gamma$ to the system

$$\begin{aligned} d^*(e^{-\gamma}(a + \alpha)e^{\gamma} - (de^{-\gamma})e^{\gamma}) &= 0, \\ d^*_{\partial B^4} i^*(e^{-\gamma}(a + \alpha)e^{\gamma} - (de^{-\gamma})e^{\gamma}) &= 0. \end{aligned} \tag{5.8}$$

This time, we need to deal with the fact that Dirichlet Coulomb representatives are unique only up to constant gauge transformations, so in order to obtain an isomorphism, we need to make sure that our spaces of infinitesimal gauge transformations do not contain nonzero constant gauge transformations. Let $L^{2,\perp}_{k-1/2}(\partial B^4;\mathfrak{g})$ denote those $L^2_{k-1/2}(\partial B^4;\mathfrak{g})$ functions that are $L^2(\partial B^4)$-orthogonal to $\mathcal{H}^0(\partial B^4)$, that is, orthogonal to the constant functions on the 3-sphere. Let $L^{2,\perp}_k(B^4;\mathfrak{g})$ denote the inverse image of the closed subspace $L^{2,\perp}_{k-1/2}(\partial B^4;\mathfrak{g})$ under the restriction map $i^*\colon L^2_k(B^4;\mathfrak{g}) \to L^2_{k-1/2}(\partial B^4;\mathfrak{g})$. We consider the map

$$\begin{gathered}
L^{2,\perp}_3(B^4;\mathfrak{g}) \times L^2_2(B^4;\mathfrak{g}\otimes T^*B^4) \to L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{1/2}(\partial B^4;\mathfrak{g}),\\
(\gamma,\alpha) \mapsto \left(d^*(e^{-\gamma}(a+\alpha)e^{\gamma} - (de^{-\gamma})e^{\gamma}), d^*_{\partial B^4} i^*(e^{-\gamma}(a+\alpha)e^{\gamma} - (de^{-\gamma})e^{\gamma})\right).
\end{gathered}$$

Again, this map is smooth because we are above the critical regularity. Moreover, the range of the second component is indeed in $L^{2,\perp}_{1/2}(\partial B^4;\mathfrak{g})$ because it is in the range of $d^*_{\partial B^4}$, and it is easy to verify that if $\phi$ is a constant map, then $\left\langle d^*_{\partial B^4}\beta,\phi\right\rangle_{L^2(B^4)} = \langle \beta, d_{\partial B^4}\phi\rangle_{L^2(B^4)} = 0$ for all $\beta$.

To apply the implicit function theorem, we show that the derivative of this map with respect to the $\gamma$ variable at $(\gamma,\alpha)=(0,0)$ is an isomorphism. Call this map $T$. This map is

$$\begin{aligned}
&T\colon L^{2,\perp}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{1/2}(\partial B^4;\mathfrak{g}),\\
&T\colon \gamma' \to (d^*[a,\gamma'] + d^*d\gamma', d^*_{\partial B^4} i^*[a,\gamma'] + d^*_{\partial B^4} i^* d\gamma').
\end{aligned}$$

As in the proof of Lemma 5.6, our goal is to show that $T$ is a Fredholm operator of index zero by decomposing $T$ as a sum $T = T_0 + K$ where

$$\begin{aligned}
&T_0\colon \gamma' \mapsto (d^*d\gamma', d^*_{\partial B^4} i^* d\gamma')),\\
&K\colon \gamma' \mapsto (d^*[a,\gamma'], d^*_{\partial B^4} i^*[a,\gamma']).
\end{aligned}$$

We then show that $T_0$ is an isomorphism and $K$ is compact. Note that $d^*_{\partial B^4} i^* d\gamma' = d^*_{\partial B^4} d_{\partial B^4} i^*\gamma'$. Hence, $T_0\colon L^{2,\perp}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{1/2}(\partial B^4;\mathfrak{g})$ is the composition of the maps

$$(\Delta, i^*)\colon L^{2,\perp}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{5/2}(\partial B^4;\mathfrak{g}),$$
$$(\mathrm{Id}, \Delta_{\partial B^4})\colon L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{5/2}(\partial B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{1/2}(\partial B^4;\mathfrak{g}).$$

The fact that $\Delta_{\partial B^4}\colon L^{2,\perp}_{k+3/2}(\partial B^4;\mathfrak{g}) \to L^{2,\perp}_{k-1/2}(\partial B^4;\mathfrak{g})$ is an isomorphism follows from the usual Hodge decomposition on closed manifolds, since, by definition, we restrict the domain and range to the orthogonal complement of the harmonic functions $\mathcal{H}^0(\partial B^4)$, that is, the constant functions.

As for $(\Delta, i^*)$, as before, we know from Propositions 2.8 and 2.9 and the fact that $\mathcal{H}^0(B^4, \partial B^4) = 0$ that $\Delta\colon L^{2,\mathrm{n}}_{k+2}(B^4;\mathfrak{g}) \to L^2_k(B^4;\mathfrak{g})$ is an isomorphism for $k \geq 0$, where $L^{2,\mathrm{n}}_{k+2}(B^4;\mathfrak{g}) = L^2_{k+2}(B^4;\mathfrak{g}) \cap \ker i^* = L^{2,\perp}_{k+2}(B^4;\mathfrak{g}) \cap \ker i^*$. The injectivity of $(\Delta, i^*)$ follows. For surjectivity, we use a standard argument using the surjectivity of $i^*$. Indeed, the inverse trace map [1, Theorem 7.53] gives us surjectivity of $i^*\colon L^2_{k+2}(B^4;\mathfrak{g}) \to L^2_{k+3/2}(\partial B^4;\mathfrak{g})$ for $k \geq -1$. Since $L^{2,\perp}_{k+2}(B^4;\mathfrak{g})$ is defined as in the inverse image of $L^{2,\perp}_{k+3/2}(\partial B^4;\mathfrak{g})$ under this map, we know that $i^*\colon L^{2,\perp}_{k+2}(B^4;\mathfrak{g}) \to L^{2,\perp}_{k+3/2}(\partial B^4;\mathfrak{g})$ is also surjective. Given $(\beta, \gamma_\partial) \in L^2_k(B^4;\mathfrak{g}) \times L^{2,\perp}_{k+3/2}(\partial B^4;\mathfrak{g})$, let $\gamma_1 \in L^{2,\perp}_{k+2}(B^4;\mathfrak{g})$ be such that $i^*\gamma_1 = \gamma_\partial$. Meanwhile, we use the surjectivity of $\Delta\colon L^{2,\mathrm{n}}_{k+2}(B^4;\mathfrak{g}) \to L^2_k(B^4;\mathfrak{g})$ to find a $\gamma_2 \in L^{2,\mathrm{n}}_{k+2}(B^4;\mathfrak{g})$ such that $\Delta\gamma_2 = \Delta\gamma_1 - \beta$. Then, $\gamma_2 - \gamma_1 \in L^{2,\perp}_{k+2}(B^4;\mathfrak{g})$, and we have $\Delta(\gamma_1 - \gamma_2) = \beta$ and $i^*(\gamma_1 - \gamma_2) = i^*\gamma_1 = \gamma_\partial$, as desired. Setting $k = 1$ gives us that $T_0\colon L^{2,\perp}_3(B^4;\mathfrak{g}) \to L^2_1(B^4;\mathfrak{g}) \times L^{2,\perp}_{1/2}(\partial B^4;\mathfrak{g})$, is an isomorphism.

Next, we show that $K$ is compact. In the proof of Lemma 5.6, we computed that if $d^*a = 0$, then

$$d^*[a, \gamma'] = *[*a \wedge d\gamma'].$$

The same argument shows that if $d^*_{\partial B^4} i^* a = 0$, then

$$d^*_{\partial B^4} i^*[a, \gamma'] = d^*_{\partial B^4}[i^*a, i^*\gamma'] = *_{\partial B^4}[*_{\partial B^4} i^*a \wedge d_{\partial B^4} i^*\gamma'],$$

where $*_{\partial B^4}$ denotes the Hodge star operator on the sphere $\partial B^4$. As before, we view

$\gamma' \mapsto d^*[a,\gamma']$ as the composition

$$L_3^{2,\perp}(B^4;\mathfrak{g}) \xrightarrow{d} L_2^2(B^4;\mathfrak{g}\otimes T^*B^4) \hookrightarrow L_{3/2}^2(B^4;\mathfrak{g}\otimes T^*B^4) \xrightarrow{*[*a\wedge\cdot]} L_1^2(B^4;\mathfrak{g}),$$

and the Sobolev multiplication and embedding theorems, along with the smoothness of $*$ and $a \in L_2^2(B^4;\mathfrak{g})$, tells us that the maps above are continuous, and the second one is compact, so the composition is compact. Likewise, $\gamma' \mapsto d^*_{\partial B^4} i^*[a,\gamma']$ is the composition of the maps

$$\begin{aligned} L_3^{2,\perp}(B^4;\mathfrak{g}) \xrightarrow{i^*} L_{5/2}^{2,\perp}(\partial B^4;\mathfrak{g}) &\xrightarrow{d_{\partial B^4}} L_{3/2}^2(\partial B^4;\mathfrak{g}\otimes T^*\partial B^4) \\ &\hookrightarrow L_1^2(\partial B^4;\mathfrak{g}\otimes T^*\partial B^4) \xrightarrow{*_{\partial B^4}[*_{\partial B^4} i^* a\wedge\cdot]} L_{1/2}^2(\partial B^4;\mathfrak{g}). \end{aligned}$$

Again, the inclusion is compact, so the composition is compact.

We conclude that $K$ is compact, so $T$ is indeed a Fredholm operator of index zero. Hence, to show that $T$ is an isomorphism, it suffices to show that $T$ is injective. We show that this is indeed the case, assuming $\|a\|_{L^4(B^4)} < \varepsilon_4$ and $\|i^*a\|_{L^3(\partial B^4)} < \varepsilon_3$ for $\varepsilon_4$ and $\varepsilon_3$ small enough. Now setting $k = 0$ in the above argument gives us that, in one degree lower regularity, $T_0\colon L_2^{2,\perp}(B^4;\mathfrak{g}) \to L^2(B^4;\mathfrak{g}) \times L_{-1/2}^{2,\perp}(\partial B^4;\mathfrak{g})$ is also an isomorphism, so there exists an $\varepsilon_\Delta$ such that

$$\|T_0\gamma'\|_{L^2(B^4)\times L^2_{-1/2}(\partial B^4)} \geq \varepsilon_\Delta \|\gamma'\|_{L_2^2(B^4)}.$$

Next we bound $\|K\gamma'\|_{L^2(B^4)\times L^2_{-1/2}(\partial B^4)}$ from above. Since $*$ is an isometry, we have

$$\begin{aligned} \|d^*[a,\gamma']\|_{L^2(B^4)} &= \|*[*a\wedge d\gamma']\|_{L^2(B^4)} \\ &\leq C_{\mathfrak{L}} \|a\|_{L^4(B^4)} \|d\gamma'\|_{L^4(B^4)} \leq C_{\mathfrak{L}} C_S C_d \|a\|_{L^4(B^4)} \|\gamma'\|_{L_2^2(B^4)}, \end{aligned}$$

where $C_{\mathfrak{L}}$ is the operator norm of the Lie bracket $[\cdot,\cdot]$, $C_S$ is the operator norm of the Sobolev embedding $L_1^2(B^4) \hookrightarrow L^4(B^4)$, and $C_d$ is the operator norm of $d\colon L_2^2(B^4;\mathfrak{g}) \to$

$L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. Likewise,

$$\begin{aligned}\|d^*_{\partial B^4} i^*[a,\gamma']\|_{L^2_{-1/2}(\partial B^4)} &\le C_s \left\| *_{\partial B^4}[*_{\partial B^4} i^*a \wedge d_{\partial B^4} i^*\gamma'] \right\|_{L^{3/2}(\partial B^4)} \\ &\le C_s C_{\mathfrak{L}} \|i^*a\|_{L^3(\partial B^4)} \|d_{\partial B^4} i^*\gamma'\|_{L^3(\partial B^4)} \le C_s^2 C_{\mathfrak{L}} C_d' C_T \|i^*a\|_{L^3(\partial B^4)} \|\gamma'\|_{L^2_2(B^4)},\end{aligned}$$

where $C_s$ denotes the operator norms of the embeddings $L^{3/2}(\partial B^4) \hookrightarrow L^2_{-1/2}(\partial B^4)$ and $L^2_{1/2}(\partial B^4) \hookrightarrow L^3(\partial B^4)$, $C_d'$ denotes the operator norm of $d_{\partial B^4}\colon L^2_{3/2}(\partial B^4; \mathfrak{g}) \to L^2_{1/2}(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$, and $C_T$ is the norm of the trace operator $i^*\colon L^2_2(B^4; \mathfrak{g}) \to L^2_{3/2}(\partial B^4; \mathfrak{g})$. Hence, by choosing $\varepsilon_4$ and $\varepsilon_3$ small enough, we can guarantee that $\|a\|_{L^4(B^4)} < \varepsilon_4$ and $\|i^*a\|_{L^3(\partial B^4)} < \varepsilon_3$ imlies that

$$\|K\gamma'\|_{L^2(B^4)\times L^2_{-1/2}(\partial B^4)} \le \tfrac{1}{2}\varepsilon_\Delta \|\gamma'\|_{L^2_2(B^4)}.$$

As a consequence, since $T = T_0 + K$, we know that

$$\|T\gamma'\|_{L^2(B^4)\times L^2_{-1/2}(\partial B^4)} \ge \tfrac{1}{2}\varepsilon_\Delta \|\gamma'\|_{L^2_2(B^4)},$$

so $T$ is injective. Since $T$ has Fredholm index zero, we know that $T$ is an isomorphism, so the implicit function theorem gives us a solution $g = e^{-\gamma}$ to the system (5.8) that depends smoothly on $\alpha$ in a neighborhood of $\alpha = 0$. That is, for any connection in a neighborhood of $A$, we have a gauge transformation sending it to a connection satisfying the Dirichlet Coulomb conditions, as desired. □

Note that if instead of using the multiplication map $L^3(\partial B^4) \times L^3(\partial B^4) \to L^2_{-1/2}(\partial B^4)$ above we had used the multiplication map $L^6_{-1/2}(\partial B^4) \times L^2_{1/2}(\partial B^4) \to L^2_{-1/2}(\partial B^4)$ we could weaken the condition that $\|i^*a\|_{L^3(\partial B^4)}$ be small to the condition that $\|i^*a\|_{L^6_{-1/2}(\partial B^4)}$ be small. Conversely, because of the continuity of the maps $L^2_1(B^4) \hookrightarrow L^4(B^4)$ and $L^2_1(B^4) \xrightarrow{i^*} L^2_{1/2}(\partial B^4) \hookrightarrow L^3(\partial B^4)$, we can replace the conditions $\|a\|_{L^4(B^4)} < \varepsilon_4$ and $\|i^*a\|_{L^3(\partial B^4)} < \varepsilon_3$ in the above lemma with $\|a\|_{L^2_1(B^4)} < \varepsilon_{21}$ for a suitable $\varepsilon_{21}$.

**Corollary 5.14.** *The space of $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connections $A$ with $\|F_A\|_{L^2(B^4)} < \varepsilon$*

*satisfying Theorem 5.10 is open in* $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$.

*Proof.* We let $\varepsilon_4$ be the smaller of the constants in Lemma 5.13 and Lemma 5.11, let $\varepsilon_3$ be the constant in Lemma 5.13, and let $C$ be the constant in 5.11. We can choose $\varepsilon$ small enough such that $\|\tilde{a}\|_{L^2_1(B^4)} < C\varepsilon$ implies $\|\tilde{a}\|_{L^4(B^4)} < \varepsilon_4$ and $\|i^*\tilde{a}\|_{L^3(\partial B^4)} < \varepsilon_3$.

Let $A$ be an $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ connection with $\|F_A\|_{L^2(B^4)} < \varepsilon$ satisfying Theorem 5.10. Then there exists an $L^2_2(B^4; G)$ gauge transformation $g$ sending $A$ to $\tilde{A}$ such that $d^*\tilde{a} = 0$, $d^*_{\partial B^4} i^*\tilde{a} = 0$, and $\|\tilde{a}\|_{L^2_1(B^4)} \leq C \|F_A\|_{L^2(B^4)} < C\varepsilon$. As discussed earlier, this implies that $\tilde{a}$ is small enough to apply Lemmas 5.11 and Lemma 5.13. Hence, we apply Lemma 5.13 to $\tilde{A}$ to find an open $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ neighborhood of $\tilde{A}$ such that for any connection $B$ in the neighborhood has a gauge transformation $g$ that sends $B$ to a connection $\tilde{B}$ satisfying the Dirichlet Coulomb conditions $d^*\tilde{b} = 0$ and $d^*_{\partial B^4} i^*\tilde{b} = 0$. Since $g$, and hence $\tilde{B}$, depends smoothly on $B$, by shrinking the neighborhood of $\tilde{A}$ we can guarantee that $\tilde{B}$ is close to $\tilde{A}$ in $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$. Hence, we can guarantee that $\tilde{B}$ also satisfies the bounds $\|\tilde{b}\|_{L^4(B^4)} < \varepsilon_4$, so we can apply Lemma 5.11 to $\tilde{b}$ to get

$$\|\tilde{b}\|_{L^2_1(B^4)} \leq \|F_{\tilde{B}}\|_{L^2(B^4)} = \|F_B\|_{L^2(B^4)}.$$

Hence, Theorem 5.10 holds for $B$ in this neighborhood of $\tilde{A}$. Since Theorem 5.10 is gauge invariant, we can pull back this neighborhood of $\tilde{A}$ via $g^{-1}$ to an open neighborhood of $A$ that satisfies Theorem 5.10, as desired. ☐

**Lemma 5.15.** *The space of* $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ *connections satisfying Theorem 5.10 is closed. Likewise, the space of* $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$ *connections satisfying Theorem 5.10 is closed in* $L^2_2(B^4; \mathfrak{g} \otimes T^*B^4)$.

*Proof.* Let $A_i \to A$ be a sequence of connections converging in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ with $\|F_A\|_{L^2(B^4)} < \varepsilon$, such that there exist $L^2_2(B^4; G)$ gauge transformations $g_i$ sending $A_i$ to $\tilde{A}_i = d + \tilde{A}_i$ such that $d^*\tilde{a}_i = 0$, $d^*_{\partial B^4} i^*\tilde{a}_i = 0$, and $\|\tilde{a}_i\|_{L^2_1(B^4)} \leq C \|F_{A_i}\|_{L^2(B^4)}$. Because the $A_i$ and hence the $F_{A_i}$ converge, we know that the $\tilde{a}_i$ are bounded in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. Hence, after passing to a subsequence, the $\tilde{a}_i$ converge weakly in

$L_1^2(B^4;\mathfrak{g}\otimes T^*B^4)$ to $\tilde{a}$. Applying Lemma 5.2, there exists a gauge transformation $g \in L_2^2(B^4;G)$ sending $A$ to $\tilde{A} = d + \tilde{a}$.

The operators $d^*$ and $d^*_{\partial B^4} i^*$ are continuous and linear, so the conditions $d^*\tilde{a}_i = 0$ and $d^*_{\partial B^4} i^*\tilde{a}_i$ are preserved in the weak $L_1^2(B^4;\mathfrak{g}\otimes T^*B^4)$ limit $d^*\tilde{a} = 0$ and $d^*_{\partial B^4}\tilde{a} = 0$. Finally, because norms are lower semicontinuous under weak limits, we have

$$\|\tilde{a}\|_{L_1^2(B^4)} \leq \liminf \|\tilde{a}_i\|_{L_1^2(B^4)} \leq C \lim \|F_{A_i}\|_{L^2(B^4)} = \|F_A\|_{L^2(B^4)}\,.$$

Hence, $A$ indeed satisfies Theorem 5.10.

Meanwhile, for the higher regularity claim, let $A_i \to A$ be a sequence of connections converging in $L_2^2(B^4;\mathfrak{g}\otimes T^*B^4)$ with $\|F_{A_i}\|_{L^2(B^4)} < \varepsilon$ and $\|F_A\|_{L^2(B^4)} < \varepsilon$, such that there exist $L_3^2(B^4;G)$ gauge transformations $g_i$ sending $A_i$ to $\tilde{A}_i = d + \tilde{a}_i$ with $d^*\tilde{a}_i = 0$, $d^*_{\partial B^4} i^*\tilde{a}_i = 0$, and $\|\tilde{a}_i\|_{L_1^2(B^4)} \leq C\,\|F_{A_i}\|_{L^2(B^4)} < C\varepsilon$. Again, let $\varepsilon_4$ be the constant from Lemma 5.11, and require $\varepsilon$ to be small enough so that $\|\tilde{a}_i\|_{L_1^2(B^4)} < C\varepsilon$ guarantees that $\|\tilde{a}_i\|_{L^4(B^4)} < \varepsilon_4$. Then, applying Lemma 5.11, we have

$$\|\tilde{a}_i\|_{L_2^2(B^4)} \leq C\left(\left\|\nabla_{\tilde{A}_i} F_{\tilde{A}_i}\right\|_{L^2(B^4)} + \left\|F_{\tilde{A}_i}\right\|_{L^2(B^4)}\right) = C\left(\|\nabla_{A_i} F_{A_i}\|_{L^2(B^4)} + \|F_{A_i}\|_{L^2(B^4)}\right).$$

The convergence of the $A_i$ in $L_2^2(B^4;\mathfrak{g}\otimes T^*B^4)$ guarantees the convergence of the right-hand side, so the $\tilde{a}_i$ are bounded in $L_2^2(B^4;\mathfrak{g}\otimes T^*B^4)$, and hence, after passing to a subsequence, they have a weak limit $\tilde{a}$. In particular, the $\tilde{A}_i$ converge to $\tilde{A}$ in $L_1^2(B^4;\mathfrak{g}\otimes T^*B^4)$, so we can apply the above argument to conclude that there is a $L_2^2(B^4;G)$ gauge transformation $g$ sending $A$ to $\tilde{A}$ and $\tilde{A}$ satisfies the conditions of Theorem 5.10. Finally, the same argument as in Lemma 5.8 shows that because $A$ and $\tilde{A}$ are in $L_2^2(B^4;\mathfrak{g}\otimes T^*B^4)$, the gauge transformation $g$ is in fact in $L_3^2(B^4;G)$. □

Unlike for Proposition 5.4, for Theorem 5.10 we need one more ingredient, which is to show that the space of $L_2^2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections $A$ satisfying $\|F_A\|_{L^2(B^4)} < \varepsilon$ is connected. We use an argument like in [23].

**Lemma 5.16.** *Let $\varepsilon$ be a constant, and let $k \geq 1$. Let $P$ be a trivialized principal $G$-bundle over $B^4$. The set of $L_k^2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections $A$ on $P$ such that*

$\|F_A\|_{L^2(B^4)} < \varepsilon$ *is connected.*

*Proof.* Let $A$ be an $L^2_k(B^4; \mathfrak{g} \otimes T^*B^4)$ connection with $\|F_A\|_{L^2(B^4)} < \varepsilon$. For $0 \le \lambda \le 1$, let $f_\lambda \colon B^4 \to B^4$ be the scaling map $f_\lambda(x) = \lambda x$. Using the fixed trivialization of the principal bundle $P$, we have a bundle map $\hat{f}_\lambda \colon P \to P$ over $f_\lambda \colon B^4 \to B^4$, which identifies $f_\lambda^* P$ with $P$. Let $A_\lambda = f_\lambda^* A$. Note that $A_1 = A$, and that $A_0 = d$, since $\hat{f}_0$ identifies every fiber with the fiber at $x = 0$ via the trivialization. Moreover, $\|F_A\|_{L^2(B^4)}$ is conformally invariant, so, viewing $f_\lambda$ as an conformal isomorphism $B^4 \to \lambda \cdot B^4$, we have

$$\left\|F_{f_\lambda^* A}\right\|_{L^2(B^4)} = \|F_A\|_{L^2(\lambda \cdot B^4)} \le \|F_A\|_{L^2(B^4)} < \varepsilon.$$

I claim that $A_\lambda$ is a continuous path of $L^2_k(B^4; \mathfrak{g} \otimes T^*B^4)$ connections. Let $A = d + a$ and $A_\lambda = d + a_\lambda$. Then $a_\lambda(x) = \lambda \cdot a(\lambda \cdot x)$. We have that

$$\begin{aligned}
\left\|\nabla^j a_\lambda\right\|^2_{L^2(B^4)} &= \int_{B^4} \left|\lambda^{j+1} (\nabla^j a)(\lambda \cdot x)\right|^2 \, dx \\
&= \lambda^{2(j+1)} \lambda^{-4} \int_{\lambda \cdot B^4} \left|(\nabla^j a)(\lambda \cdot x)\right|^2 \, d(\lambda \cdot x) \\
&= \lambda^{2(j-1)} \left\|\nabla^j a\right\|^2_{L^2(\lambda \cdot B^4)} \le \lambda^{2(j-1)} \left\|\nabla^j a\right\|^2_{L^2(B^4)}.
\end{aligned}$$

Hence, $a_\lambda$ is indeed in $L^2_k(B^4; \mathfrak{g} \otimes T^*B^4)$. We now prove continuity of this path of connections at $\lambda = 0$. When $j > 1$, it is clear that $\|\nabla^j a_\lambda\|^2_{L^2(B^4)} \to 0$ as $\lambda \to 0$. When $j = 1$, we note that $\|\nabla a_\lambda\|_{L^2(B^4)} = \|\nabla a\|^2_{L^2(\lambda \cdot B^4)}$, which also approaches zero as $\lambda \to 0$. Finally, note that, by assumption, $a \in L^2_1(B^4; \mathfrak{g} \otimes T^*B^4) \subset L^4(B^4; \mathfrak{g} \otimes T^*B^4)$, so

$$\begin{aligned}
\|a_\lambda\|_{L^2(B^4)} &= \lambda^{-1} \|a\|_{L^2(\lambda \cdot B^4)} \le \lambda^{-1} \|a\|_{L^4(\lambda \cdot B^4)} \|1\|_{L^4(\lambda \cdot B^4)} \\
&= \lambda^{-1} \|a\|_{L^4(\lambda \cdot B^4)} \left(\lambda^4 \operatorname{vol}(B^4)\right)^{1/4} = \operatorname{vol}\left(B^4\right)^{1/4} \|a\|_{L^4(\lambda \cdot B^4)}.
\end{aligned}$$

Hence, $\|a_\lambda\|_{L^2(B^4)} \to 0$ as $\lambda \to 0$, completing the proof that $\|a_\lambda\|_{L^2_k(B^4)} \to 0$ as $\lambda \to 0$.

We now use standard arguments to prove continuity at $0 < \lambda \le 1$. Let $h$ be such that $0 < \lambda + h \leq 1$. We compute that

$$\begin{aligned}\left\|\nabla^j(a_{\lambda+h} - a_\lambda)\right\|_{L^2(B^4)} &= \left\|(\lambda+h)^{j+1}(\nabla^j a)(f_{\lambda+h}x) - \lambda^{j+1}(\nabla^j a)(f_\lambda x)\right\|_{L^2(B^4)} \\ &\leq (\lambda+h)^{j+1}\left\|(\nabla^j a)(f_{\lambda+h}x) - (\nabla^j a)(f_\lambda x)\right\|_{L^2(B^4)} \\ &\quad + \left((\lambda+h)^{j+1} - \lambda^{j+1}\right)\left\|\nabla^j a(f_\lambda x)\right\|_{L^2(B^4)}.\end{aligned}$$

The second term approaches zero as $h \to 0$, so it suffices to prove that

$$\left\|(\nabla^j a)(f_{\lambda+h}x) - (\nabla^j a)(f_\lambda x)\right\|_{L^2(B^4)} \to 0$$

as $h \to 0$. Let $\delta > 0$. Approximate $\nabla^j a$ in $L^2$ by a continuous function $\alpha$ so that $\|\nabla^j a - \alpha\|_{L^2(B^4)} < \frac{\delta}{3}(\frac{\lambda}{2})^2$. Since $\alpha$ is continuous on a compact domain, it is uniformly continuous, so we can guarantee that $\|\alpha((\lambda+h)\cdot x) - \alpha(\lambda \cdot x)\|_{L^2(B^4)} < \frac{\delta}{3}$ as long as $h \cdot x$ is sufficiently small. Since $|x| \leq 1$, we can simply choose $h$ sufficiently small. Finally, note that

$$\begin{aligned}\left\|\nabla^j a(\lambda \cdot x) - \alpha(\lambda \cdot x)\right\|^2_{L^2(B^4)} &= \int_{B^4}\left|\nabla^j a(\lambda \cdot x) - \alpha(\lambda \cdot x)\right|^2 \lambda^{-4} d(\lambda \cdot x) \\ &= \lambda^{-4}\left\|\nabla^j a(x) - \alpha(x)\right\|^2_{L^2(\lambda \cdot B^4)}.\end{aligned}$$

Hence,

$$\left\|\nabla^j a(\lambda \cdot x) - \alpha(\lambda \cdot x)\right\|_{L^2(B^4)} < \lambda^{-2}\tfrac{\delta}{3}(\tfrac{\lambda}{2})^2 < \tfrac{\delta}{3}.$$

Likewise,

$$\left\|\nabla^j a((\lambda+h) \cdot x) - \alpha((\lambda+h) \cdot x)\right\|_{L^2(B^4)} < (\lambda+h)^{-2}\tfrac{\delta}{3}(\tfrac{\lambda}{2})^2 < \tfrac{\delta}{3}$$

provided that we choose $h > -\frac{\lambda}{2}$. Hence,

$$\left\|\nabla^j a((\lambda+h) \cdot x) - \nabla^j a(\lambda \cdot x)\right\|_{L^2(B^4)} < \delta$$

for $h$ sufficiently small, completing the proof of continuity at $\lambda$. $\square$

We can now put together these lemmas to prove Theorem 5.10.

*Proof of Theorem 5.10.* The space of $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections $A$ with $\|F_A\|_{L^2(B^4)} < \varepsilon$ is connected by Lemma 5.16, and by Corollary 5.14 and Lemma 5.15 the space of connections satisfying Theorem 5.10 is both open and closed in the space of $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections with $\|F_A\|_{L^2(B^4)} < \varepsilon$, and hence contains all $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections with $\|F_A\|_{L^2(B^4)} < \varepsilon$.

Meanwhile, because $A \mapsto F_A$ is continuous as a map

$$L^2_1(B^4;\mathfrak{g}\otimes T^*B^4) \to L^2(B^4;\mathfrak{g}\otimes {\textstyle\bigwedge}^2 T^*B^4),$$

any $L^2_1(B^4;\mathfrak{g}\otimes T^*B^4)$ connection with $\|F_A\|_{L^2(B^4)} < \varepsilon$ is the $L^2_1(B^4;\mathfrak{g}\otimes T^*B^4)$ limit of $L^2_2(B^4;\mathfrak{g}\otimes T^*B^4)$ connections $A_i$ with $\|F_A\|_{L^2(B^4)} < \varepsilon$. These connections $A_i$ satisfy Theorem 5.10, so by Lemma 5.15 so does $A$. $\square$

Again, we finish by proving that the gauge transformation constructed by Theorem 5.10 is unique up to a constant gauge transformation. To do so, we first prove uniqueness up to constants on the boundary.

**Proposition 5.17.** *There exists a constant $\varepsilon$ such that if $A = d+a$ and $B = d+b$ are two $L^2_{1/2}(\partial B^4;\mathfrak{g}\otimes T^*\partial B^4)$ connections gauge equivalent via a gauge transformation $g \in L^2_{3/2}(\partial B^4;G)$ satisfying*

1. *bounds $\|a\|_{L^3(\partial B^4)}, \|b\|_{L^3(\partial B^4)} < \varepsilon$, and*

2. *the Coulomb conditions $d^*_{\partial B^4}a = d^*_{\partial B^4}b = 0$,*

*then $g$ is constant on $\partial B^4$.*

*Proof.* Equation (5.5) is also valid on $\partial B^4$ given $d^*_{\partial B^4}a = d^*_{\partial B^4}b = 0$, so we have

$$d^*_{\partial B^4}d_{\partial B^4}g = -*_{\partial B^4}(d_{\partial B^4}g \wedge *_{\partial B^4}a + *_{\partial B^4}b \wedge d_{\partial B^4}g).$$

Then,

$$\begin{aligned}\|d^*_{\partial B^4} d_{\partial B^4} g\|_{L^2_{-1/2}(\partial B^4)} &\leq C_S \|d^*_{\partial B^4} d_{\partial B^4} g\|_{L^{3/2}(\partial B^4)} \\ &\leq C_S \left(\|a\|_{L^3(\partial B^4)} + \|b\|_{L^3(\partial B^4)}\right) \|d_{\partial B^4} g\|_{L^3(\partial B^4)} \\ &\leq C_S^2 \left(\|a\|_{L^3(\partial B^4)} + \|b\|_{L^3(\partial B^4)}\right) \|d_{\partial B^4} g\|_{L^2_{1/2}(\partial B^4)}.\end{aligned}$$

where $C_S$ is the operator norm of the Sobolev embeddings $L^{3/2}(\partial B^4) \hookrightarrow L^2_{-1/2}(\partial B^4)$ and $L^2_{1/2}(\partial B^4) \hookrightarrow L^3(\partial B^4)$.

Here, the standard theory of elliptic operators on closed manifolds tells us that the operator

$$d_{\partial B^4} + d^*_{\partial B^4} \colon L^2_{1/2}(\partial B^4; M_N \otimes \textstyle\bigwedge^* T^* \partial B^4) \to L^2_{-1/2}(\partial B^4; M_N \otimes \textstyle\bigwedge^* T^* \partial B^4)$$

is Fredholm. Moreover, it has no kernel on exact forms because

$$\langle (d_{\partial B^4} + d^*_{\partial B^4})(d_{\partial B^4} g), g\rangle_{L^2(\partial B^4)} = \langle d_{\partial B^4} g, d_{\partial B^4} g\rangle_{L^2(\partial B^4)}.$$

We conclude that there is a constant $C_G$ such that

$$\|d_{\partial B^4} g\|_{L^2_{1/2}(\partial B^4)} \leq C_G \|(d_{\partial B^4} + d^*_{\partial B^4})(d_{\partial B^4}) g\|_{L^2_{-1/2}(\partial B^4)} = C_G \|d^*_{\partial B^4} d_{\partial B^4} g\|_{L^2_{-1/2}(\partial B^4)}.$$

Putting these inequalities together, we have

$$\|d_{\partial B^4} g\|_{L^2_{1/2}(\partial B^4)} \leq C_G C_S^2 \left(\|a\|_{L^3(\partial B^4)} + \|b\|_{L^3(\partial B^4)}\right) \|d_{\partial B^4} g\|_{L^2_{1/2}(\partial B^4)}.$$

Hence, requiring that $\varepsilon \leq \frac{1}{4}(C_G C_S^2)^{-1}$ gives

$$\|d_{\partial B^4} g\|_{L^2_{1/2}(\partial B^4)} \leq \frac{1}{2} \|d_{\partial B^4} g\|_{L^2_{1/2}(\partial B^4)},$$

so $d_{\partial B^4} g = 0$ and $g$ is constant on $\partial B^4$, as desired. $\square$

We can now prove the uniqueness up to constants of the gauge transformation constructed in Theorem 5.10 on all of $B^4$. Again, we assume $L^4(B^4; \mathfrak{g} \otimes T^*B^4)$ and $L^3(\partial B^4; \mathfrak{g} \otimes T^*\partial B^4)$ bounds, but these are implied by the $L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ bound given to us by Theorem 5.10.

**Corollary 5.18.** *There exists constants $\varepsilon_4$ and $\varepsilon_3$ such that if $A = d + a$ and $B = d + b$ are two $L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ connections connections gauge equivalent via a gauge transformation $g \in L_2^2(B^4; G)$ satisfying*

1. *bounds $\|a\|_{L^4(B^4)}, \|b\|_{L^4(B^4)} < \varepsilon_4$ and $\|i^*a\|_{L^3(\partial B^4)}, \|i^*b\|_{L^3(\partial B^4)} < \varepsilon_3$, and*
2. *the Dirichlet Coulomb conditions $d^*a = d^*b = 0$ and $d^*_{\partial B^4} i^*a = d^*_{\partial B^4} i^*b = 0$,*

*then $g$ is constant on $B^4$.*

*Proof.* Choose $\varepsilon_3$ small enough to apply Proposition 5.17. Then $i^*g$ is a constant gauge transformation $c \in G$ on $\partial B^4$. Then choose $\varepsilon_4$ small enough to apply Proposition 5.9, giving us that $g$ is the constant gauge transformation $c$ on all of $B^4$. $\square$

## 5.3 Convergence of Coulomb gauge representatives

In this section, we extend Uhlenbeck's gauge fixing result with Neumann boundary conditions [23] to $L_d^2(B^4)$ connections. Moreover, we show that if a sequence of small-energy connections converges in $L_d^2(B^4; \mathfrak{g} \otimes T^*B^4)$, then a subsequence of the Coulomb gauge representatives converges strongly in $L_1^2(B^4)$. Analogous results can also be proved for the gauge fixing result with Dirichlet boundary conditions in Theorem 5.10. Either boundary condition will suffice for our purposes, so we only present the results for gauge fixing with Neumann boundary conditions, but the proofs for the Dirichlet case are analogous. We begin by presenting Uhlenbeck's gauge fixing theorem.

**Theorem 5.19** ([23, Theorem 2.1]). *There exist constants $\varepsilon$ and $C$ such that if $A$ is an $L_1^2(B^4)$ connection with $\|F_A\|_{L^2(B^4)} < \varepsilon$, then there exists an $L_2^2(B^4; M_N)$ gauge transformation sending $A$ to a connection $\tilde{A} = d + \tilde{a}$ satisfying*

1. *the Coulomb condition* $d^*\tilde{a} = 0$ *on* $B^4$,

2. *the boundary condition* $i^* {*}\tilde{a} = 0$ *on* $\partial B^4$,

3. *the bound* $\|\tilde{a}\|_{L^2_1(B^4)} \le C \|F_A\|_{L^2(B^4)}$.

We aim to extend this theorem to $L^2_d(B^4)$ connections.

**Theorem 5.20.** *There exist constants* $\varepsilon$ *and* $C$ *such that if* $A$ *is an* $L^2_d(B^4)$ *connection with* $\|F_A\|_{L^2(B^4)} < \varepsilon$, *then there exists an* $L^4_1(B^4; M_N)$ *gauge transformation sending* $A$ *to an* $L^2_1(B^4)$ *connection* $\tilde{A} = d + \tilde{a}$ *satisfying*

1. *the Coulomb condition* $d^*\tilde{a} = 0$ *on* $B^4$,

2. *the boundary condition* $i^* {*}\tilde{a} = 0$ *on* $\partial B^4$,

3. *the bound* $\|\tilde{a}\|_{L^2_1(B^4)} \le C \|F_A\|_{L^2(B^4)}$.

Theorem 5.19 tells us that Theorem 5.20 holds for $L^2_1(B^4)$ connections, so we use a closedness argument analogous to [23, Lemma 2.4] and Lemma 5.8 to extend the result to $L^2_d(B^4)$ connections.

**Proposition 5.21.** *Let* $A_i$ *be a sequence of* $L^2_d(B^4; \mathfrak{g} \otimes T^*B^4)$ *connections with the bound* $\|F_{A_i}\|_{L^2(B^4)} < \varepsilon$ *that satisfy Theorem 5.20 and converge in* $L^2_d(B^4; \mathfrak{g} \otimes T^*B^4)$ *to a connection* $A$. *Then* $A$ *also satisfies Theorem 5.20.*

*Proof.* Let $g_i$ and $\tilde{A}_i$ be the gauge transformations and connections given to us by Theorem 5.20. We know that the $F_{A_i}$ converge to $F_A$ in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$. Hence, the $\|F_{A_i}\|_{L^2(B^4)}$ are bounded, and, thus, by the assumption that $\|\tilde{a}_i\|_{L^2_1(B^4)} \le C \|F_{A_i}\|_{L^2(B^4)}$, so are the $\|\tilde{a}_i\|_{L^2_1(B^4)}$. Hence, passing to a subsequence, the $\tilde{a}_i$ have a weak limit, which we call $\tilde{a}$. The conditions $d^*\tilde{a}_i = 0$ and $i^*{*}\tilde{a}_i = 0$ are linear, and so are preserved under weak limits, giving us $d^*\tilde{a} = 0$ and $i^*{*}\tilde{a} = 0$. Meanwhile, the $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$ norm is lower semicontinuous under weak limits, and $\|F_{A_i}\|_{L^2(B^4)}$ converges to $\|F_A\|_{L^2(B^4)}$, so the inequality $\|\tilde{a}_i\|_{L^2_1(B^4)} \le C \|F_{A_i}\|_{L^2(B^4)}$ is preserved under weak limits, giving us $\|\tilde{a}\|_{L^2_1(B^4)} \le C \|F_A\|_{L^2(B^4)}$. Finally, since, in particular, both the $a_i$ and $\tilde{a}_i$ converge weakly in $L^4(B^4; \mathfrak{g} \otimes T^*B^4)$, we can apply Lemma 5.2

to show that there exists an $L_1^4(B^4; \mathfrak{g} \otimes T^*B^4)$ gauge transformation $g$ sending $A$ to $\tilde{A}$. $\square$

*Proof of Theorem 5.20.* Let $A$ be an $L_d^2(B^4)$ connection with $\|F_A\|_{L^2(B^4)} < \varepsilon$. Consider a sequence of smooth connections $A_i$ that converge to $A$ in $L_d^2(B^4)$ and also have $\|F_A\|_{L^2(B^4)} < \varepsilon$. The connections $A_i$ satisfy Theorem 5.19, and hence also Theorem 5.20. Then $A$ satisfies Theorem 5.20 by Proposition 5.21. $\square$

In addition, we show that the weak subsequence convergence of the Coulomb gauge representatives above can be strengthened to strong subsequence convergence. Note, however, that taking a subsequence is necessary because Coulomb gauge is invariant under constant gauge transformations. By applying constant gauge transformations to a fixed connection in Coulomb gauge, we can construct a sequence of gauge equivalent connections in Coulomb gauge that nonetheless does not converge. However, because the gauge group is compact, we still expect convergence of a subsequence. In higher regularity, we can resolve this issue by considering infinitesimal gauge transformations that are orthogonal to the constant gauge transformations, but in the critical regularity infinitemsimal gauge transformations are not so well-behaved, so a more delicate argument would be necessary. Subsequence convergence will suffice for our purposes, however.

**Proposition 5.22.** *There exist constants $\varepsilon$ and $C$ such that if $A_i$ is a sequence of $L_d^2(B^4)$ connections converging strongly in $L_d^2(B^4; \mathfrak{g} \otimes T^*B^4)$ to $A$ with $\|F_{A_i}\|_{L^2(B^4)} < \varepsilon$ and $\|F_A\|_{L^2(B^4)} < \varepsilon$, then there exist $L_1^4(B^4)$ gauge transformations $g_i$ and $g$ sending $A_i$ and $A$ to $L_1^2(B^4)$ connections $\tilde{A}_i$ and $\tilde{A}$ respectively, such that*

1. $d^*\tilde{a}_i = d^*\tilde{a} = 0$,

2. $i^* {*} \tilde{a}_i = i^* {*} \tilde{a} = 0$,

3. $\|\tilde{a}_i\|_{L_1^2(B^4)} \leq C \|F_{A_i}\|_{L^2(B^4)}$ *and* $\|\tilde{a}\|_{L_1^2(B^4)} \leq C \|F_A\|_{L^2(B^4)}$,

*and, after passing to a subsequence, the $g_i$ converge strongly to $g$ in $L_1^4(B^4; M_N)$ and the $\tilde{A}_i$ converge strongly in $L_1^2(B^4; \mathfrak{g} \otimes T^*B^4)$ to $\tilde{A}$.*

*Proof.* We construct the $g_i$ using Theorem 5.20. We construct $g$ as in the proof of Proposition 5.21, so, after passing to a subsequence we have $g_i$ weakly converging to $g$ in $L^4_1(B^4; M_N)$ and $\tilde{a}_i$ weakly converging to $\tilde{a}$ in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$. Hence, the three conditions of the proposition are satisfied, and it remains to show that the convergence of the $\tilde{a}_i$ to $\tilde{a}$ is strong, at least after passing to a subsequence.

First, note that $\left\|F_{\tilde{A}_i}\right\|_{L^2(B^4)} = \|F_{A_i}\|_{L^2(B^4)}$. Since the $F_{A_i}$ converge strongly in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$ to $F_A$, we know that the $F_{\tilde{A}_i}$ are bounded in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$, and hence converge weakly in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$ after passing to a subsequence. Moreover, $\left\|F_{\tilde{A}_i}\right\|_{L^2(B^4)}$ converges, and weak $L^2$ convergence and convergence of $L^2$ norms implies strong $L^2$ convergence. Thus, after passing to a subsequence, we have that the $F_{\tilde{A}_i}$ converge strongly in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$, but we do not yet know that the limit is $F_{\tilde{A}}$. Because the $g_i$ converge to $g$ weakly in $L^4_1(B^4; M_N)$, we know that the $g_i$ converge strongly to $g$ in $L^4(B^4; M_N)$. Using the multiplication map $L^4 \times L^2 \times L^4 \to L^1$, we have that $F_{\tilde{A}_i} = g_i F_{A_i} g_i^{-1}$ converges strongly to $g F_A g^{-1} = F_{\tilde{A}}$ in $L^1(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$. Since the $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$ limit of the $F_{\tilde{A}_i}$ must be the same as the $L^1(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$ limit, we know that the $F_{\tilde{A}_i}$ converge strongly to $F_{\tilde{A}}$ in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$.

The next step is to show that convergence of curvature and Coulomb gauge implies convergence of the connections. Since $H^1(B^4) = 0$, Corollary 2.10 tells us that $d + d^*\colon L^{2,\mathrm{t}}_1(B^4; \bigwedge^* T^*B^4) \to L^2(B^4; \bigwedge^* T^*B^4)$ is a Fredholm operator with no kernel on one-forms. Hence, for some constant $C_G$, we have the inequality

$$\|b\|_{L^2_1(B^4)} \leq C_G \left\|(d + d^*)b\right\|_{L^2(B^4)}$$

for all $b \in L^{2,\mathrm{t}}_1(B^4; \mathfrak{g} \otimes T^*B^4)$. In particular, noting that $d^*\tilde{a}_i = d^*\tilde{a} = 0$, we have

$$\begin{aligned}
\|\tilde{a}_i - \tilde{a}\|_{L^2_1(B^4)} &\leq C_G \left\|d(\tilde{a}_i - \tilde{a})\right\|_{L^2(B^4)} \\
&= \left\|F_{\tilde{A}_i} - F_{\tilde{A}} - \tfrac{1}{2}\left([\tilde{a}_i \wedge \tilde{a}_i] - [\tilde{a} \wedge \tilde{a}]\right)\right\|_{L^2(B^4)} \\
&\leq \left\|F_{\tilde{A}_i} - F_{\tilde{A}}\right\|_{L^2(B^4)} + \tfrac{1}{2}\left\|[(\tilde{a}_i + \tilde{a}) \wedge (\tilde{a}_i - \tilde{a})]\right\|_{L^2(B^4)} \\
&\leq \left\|F_{\tilde{A}_i} - F_{\tilde{A}}\right\|_{L^2(B^4)} + \tfrac{1}{2} C_{\mathfrak{g}} C_S \left\|\tilde{a}_i + \tilde{a}\right\|_{L^4(B^4)} \left\|\tilde{a}_i - \tilde{a}\right\|_{L^2_1(B^4)},
\end{aligned}$$

where $C_{\mathfrak{L}}$ is the operator norm of the Lie bracket $[\cdot,\cdot]$ and $C_S$ is the operator norm of the Sobolev embedding $L^2_1(B^4) \hookrightarrow L^4(B^4)$. Using the inequalities $\|\tilde{a}_i\|_{L^2_1(B^4)} \leq C\|F_{A_i}\|_{L^2(B^4)}$ and $\|\tilde{a}\|_{L^2_1(B^4)} \leq C\|F_A\|_{L^2(B^4)}$ and the Sobolev embeding $L^2_1(B^4) \hookrightarrow L^4(B^4)$, by shrinking $\varepsilon$ we can guarantee that $\left\|F_{\tilde{A}_i}\right\|_{L^2(B^4)} < \varepsilon$ and $\|F_{\tilde{A}}\|_{L^2(B^4)} < \varepsilon$ imply that $\|\tilde{a}_i\|_{L^4(B^4)} < \frac{1}{2}(C_{\mathfrak{L}}C_S)^{-1}$ and $\|\tilde{a}\|_{L^4(B^4)} < \frac{1}{2}(C_{\mathfrak{L}}C_S)^{-1}$. Hence, with a small enough choice of $\varepsilon$, the above inequality becomes

$$\|\tilde{a}_i - \tilde{a}\|_{L^2_1(B^4)} \leq \left\|F_{\tilde{A}_i} - F_{\tilde{A}}\right\|_{L^2(B^4)} + \tfrac{1}{2}\|\tilde{a}_i - \tilde{a}\|_{L^2_1(B^4)}.$$

Rearranging,

$$\|\tilde{a}_i - \tilde{a}\|_{L^2_1(B^4)} \leq 2\left\|F_{\tilde{A}_i} - F_{\tilde{A}}\right\|_{L^2(B^4)}.$$

Hence, because the $F_{\tilde{A}_i}$ converge strongly to $F_{\tilde{A}}$ in $L^2(B^4; \mathfrak{g} \otimes \bigwedge^2 T^*B^4)$, the $\tilde{a}_i$ converge storngly to $\tilde{a}$ in $L^2_1(B^4; \mathfrak{g} \otimes T^*B^4)$, as desired. Finally, by Lemma 5.3, after passing to a subsequence, the $g_i$ converge strongly to $g$ in $L^4_1(B^4; M_N)$. $\square$

# Bibliography

[1] Robert A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] Tobias H. Colding and William P. Minicozzi, II. Width and finite extinction time of Ricci flow. *Geom. Topol.*, 12(5):2537–2586, 2008.

[3] S. K. Donaldson. Self-dual connections and the topology of smooth 4-manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 8(1):81–83, 1983.

[4] S. K. Donaldson. The approximation of instantons. *Geom. Funct. Anal.*, 3(2):179–200, 1993.

[5] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1990. Oxford Science Publications.

[6] Paul M. N. Feehan. Global existence and convergence of smooth solutions to Yang-Mills gradient flow over compact four-manifolds, 2014. `http://arxiv.org/abs/1409.1525`.

[7] Paul M. N. Feehan and Thomas G. Leness. Superconformal simple type and Witten's conjecture, 2014. `http://arxiv.org/abs/1408.5085`.

[8] Daniel S. Freed and Karen K. Uhlenbeck. *Instantons and four-manifolds*, volume 1 of *Mathematical Sciences Research Institute Publications.* Springer-Verlag, New York, 1984.

[9] Robert Friedman and John W. Morgan. On the diffeomorphism types of certain algebraic surfaces. I. *J. Differential Geom.*, 27(2):297–369, 1988.

[10] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[11] Antonella Marini. Dirichlet and Neumann boundary value problems for Yang-Mills connections. *Comm. Pure Appl. Math.*, 45(8):1015–1050, 1992.

[12] Oskar Perron. Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$. *Math. Z.*, 18(1):42–54, 1923.

[13] Johann Radon. Theorie und anwendungen der absolut additiven mengenfunktionen. *Sitzungsber. Akad. Wiss. Wien.*, 122, 1913.

[14] Tristan Rivière. The variations of the Yang-Mills lagrangian. KIAS lecture notes. `https://people.math.ethz.ch/~riviere/papers/yang-mills-course-kias-06-14.pdf`, 2014.

[15] J. Sacks and K. Uhlenbeck. The existence of minimal immersions of 2-spheres. *Ann. of Math. (2)*, 113(1):1–24, 1981.

[16] H. A. Schwarz. Ueber einen Grenzübergang durch alternirendes verfahren. *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 15:272–286, 1870.

[17] Steven Sedlacek. A direct method for minimizing the Yang-Mills functional over 4-manifolds. *Comm. Math. Phys.*, 86(4):515–527, 1982.

[18] Michael Struwe. The Yang-Mills flow in four dimensions. *Calc. Var. Partial Differential Equations*, 2(2):123–150, 1994.

[19] Clifford Henry Taubes. Path-connected Yang-Mills moduli spaces. *J. Differential Geom.*, 19(2):337–392, 1984.

[20] Clifford Henry Taubes. A framework for Morse theory for the Yang-Mills functional. *Invent. Math.*, 94(2):327–402, 1988.

[21] Clifford Henry Taubes. The stable topology of self-dual moduli spaces. *J. Differential Geom.*, 29(1):163–230, 1989.

[22] Michael E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Basic theory.

[23] Karen K. Uhlenbeck. Connections with $L^p$ bounds on curvature. *Comm. Math. Phys.*, 83(1):31–42, 1982.

[24] Karen K. Uhlenbeck. Removable singularities in Yang-Mills fields. *Comm. Math. Phys.*, 83(1):11–29, 1982.

[25] Edward Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1(6):769–796, 1994.