Special gradient trajectories counted by simplex straightening
by
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Abstract

We prove three theorems based on lemmas of Gromov involving the simplicial norm on stratified spaces. First, the Gromov singular fiber theorem (with proof originally sketched by Gromov) relates the simplicial norm to the number of maximum-multiplicity critical points of a smooth map of manifolds that drops in dimension by 1. Second, the multitangent trajectory theorem (proved with Gabriel Katz) relates the simplicial norm to the number of maximum-multiplicity tangent trajectories of a nowhere-vanishing gradient-like vector field on a manifold with boundary. And third, the Morse broken trajectory theorem relates the simplicial volume to the number of maximally broken trajectories of the gradient flow of a Morse–Smale function. Corollary: a Morse function on a closed hyperbolic manifold must have a critical point of every Morse index.

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Chapter 1

Introduction

The main theorem of this paper is in the style of Morse inequalities. The Morse inequalities state that given a Morse function on a closed manifold, the number of critical points with a given Morse index is at least the corresponding Betti number of the manifold. One corollary of the main theorem is a statement as simple as the Morse inequalities, for hyperbolic manifolds.

**Theorem 1** (Morse index corollary). Let $M$ be a closed hyperbolic manifold of dimension $n \geq 2$. Then any Morse function $f : M \to \mathbb{R}$ must have a critical point of every Morse index $0, 1, \ldots, n$.

To my knowledge this corollary is new. The main theorem is stronger than the corollary. It says that a Morse function on a hyperbolic manifold must have an $n$-part broken trajectory, which is defined as follows and includes a critical point of every Morse index.

Let $M$ be a closed manifold of dimension $n$, and let $f : M \to \mathbb{R}$ be a Morse function. For any Riemannian metric $g$ on $M$, we consider the flow along the negative gradient vector field $-\nabla f$; for every critical point $p$ of $f$ with Morse index $\text{ind}(p)$, the flow determines the descending (or unstable) manifold $D(p)$ of dimension $\text{ind}(p)$ and the ascending (or stable) manifold $A(p)$ of dimension $n - \text{ind}(p)$. The function $f$ satisfies **Morse–Smale transversality** with respect to the metric $g$ if for every two critical points $p$ and $q$, the manifolds $D(p)$ and $A(q)$ have transverse intersection. In particular, if the indices of $p$ and $q$ differ by 1, then there is a discrete set of unparametrized flow lines between $p$ and $q$. An $n$-part broken trajectory is a maximum-length path through the resulting directed graph; that is, it is a sequence of critical points $p_n, p_{n-1}, \ldots, p_1, p_0$ where the index of point $p_i$ is $i$, and a sequence of unparametrized flow lines $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$ where each $\gamma_i$ runs from $p_i$ to $p_{i-1}$.

The main theorem actually makes a stronger statement than the existence of one $n$-part broken trajectory. It says that the number of $n$-part broken trajectories is at least proportional to the hyperbolic volume of the manifold.

**Theorem 2** (Morse broken trajectory theorem, specific version). Let $M$ be a closed, oriented hyperbolic manifold of dimension $n \geq 2$. Let $g$ be an arbitrary Riemannian metric on $M$, and let $f : (M, g) \to \mathbb{R}$ be a Morse function satisfying Morse–Smale
Then we have

\[ \#(n\text{-part broken trajectories of } -\nabla f) \geq \frac{\text{Vol}(M, \text{hyp})}{\text{Vol } \Delta^n}, \]

where Vol $\Delta^n$ denotes the suprernal volume of a straight simplex in $n$-dimensional hyperbolic space.

In this theorem, the hyperbolic volume plays the role of a topological invariant, analogous to the Betti numbers in the Morse inequalities. This volume does not depend on a choice of hyperbolic metric; in dimension 2, this is because of the Gauss-Bonnet formula, and in dimension at least 3, it is because of the Mostow rigidity theorem. The hyperbolic volume may seem at first not to be a topological quantity at all. However, in even dimension, the Gauss–Bonnet–Chern formula—which expresses the Euler characteristic in terms of the integral of the Pfaffian—implies that the hyperbolic volume is proportional to the Euler characteristic. So, the hyperbolic volume gives the same information as Euler characteristic in even dimension, but gives more information than the Euler characteristic in odd dimension, where Poincaré duality implies that the Euler characteristic is zero.

In even dimension, combining the Morse inequalities with the Gauss Bonnet–Chern formula implies the inequality

\[ \#(\text{critical points of } f) \geq \text{const}(n) \cdot \text{Vol}(M, \text{hyp}), \]

a theorem very similar to the theorem just stated. However, this inequality does not extend to odd dimension: this fact is implied by the following proposition, which we state for contrast. A similar example appears on page 30 of Gromov's paper [16].

The proof is not related to the rest of this paper, but we include it for completeness.

**Proposition 3.** There is a sequence $M_1, M_2, \ldots$ of closed, oriented hyperbolic manifolds of dimension 3, and a sequence of Morse functions $f_i: M_i \to \mathbb{R}$, such that the number of critical points of the Morse functions is uniformly bounded, but the hyperbolic volumes satisfy

\[ \lim_{i \to \infty} \text{Vol}(M_i, \text{hyp}) = \infty. \]

**Proof.** Let $\Sigma$ be an oriented surface of genus at least 2, and let $\varphi: \Sigma \to \Sigma$ be a pseudo-Anosov diffeomorphism, so that the mapping torus $M_1$ is a hyperbolic manifold [28]. Let $M_i$ be the mapping torus of the $i$th iterate $\varphi^i$ of $\varphi$. Then $M_i$ is an $i$–fold cover of $M_1$, and therefore has hyperbolic volume given by

\[ \text{Vol}(M_i, \text{hyp}) = i \cdot \text{Vol}(M_1, \text{hyp}). \]

To create a Morse function on $M_i$ with few critical points, we view $M_i$ as two copies of $\Sigma \times [0, 1]$ glued along the boundary. We find a Morse function $f$ on $\Sigma \times [0, 1]$ that has the boundary as a level set and is a standard projection near the boundary; that is, for some $\varepsilon > 0$, for all $(x, t) \in \Sigma \times [0, \varepsilon)$ we have $f(x, t) = t$ and for all $(x, t) \in \Sigma \times (1 - \varepsilon, 1]$ we have $f(x, t) = 1 - t$. To construct $f_i$, we take $f$ on one
copy of $\Sigma \times [0, 1]$, and $-f$ on the other copy, and glue them together. No matter how complicated the gluing map $\varphi^i$ is, a neighborhood of the glued set looks like two copies of $\Sigma \times (-\varepsilon, \varepsilon)$, and $f_i$ on that neighborhood looks like the projection to $(-\varepsilon, \varepsilon)$, so $f_i$ is smooth and has two critical points for each critical point of $f$. \[\square\]

Thus, in dimension 3 a Morse-Smale function on a manifold of large hyperbolic volume may have few critical points, but it must still have many 3-part broken trajectories.

In fact, Theorem 2 has a stronger formulation in terms of Gromov’s simplicial norm, which was introduced in [15] and is a function on singular homology classes with real coefficients. The simplicial norm is defined as follows. For every singular chain $c$ on a topological space $X$, the norm of $c$, denoted $\|c\|_\Delta$, is the sum of absolute values of the (real) coefficients. For every real homology class $h$, the simplicial norm of $h$, denoted $\|h\|_\Delta$, is the infimum of $\|c\|_\Delta$ over all cycles $c$ representing $h$. The simplicial norm is often called the simplicial volume because it generalizes hyperbolic volume: if $M$ is any closed, oriented hyperbolic manifold of dimension $n$, with fundamental class $[M]$, then the simplicial norm is related to the hyperbolic volume by the formula

$$\|\alpha_*[M]\|_\Delta = \frac{\text{Vol}(M, \text{hyp})}{\text{Vol} \Delta^n},$$

where $\text{Vol} \Delta^n$ denotes the supremal volume of a straight simplex in $n$-dimensional hyperbolic space (Proportionality Theorem, p. 11 of [15], or Theorem 6.2 of [27]).

The more general version of Theorem 2 applies not only to manifolds admitting a hyperbolic metric but to any closed manifolds with nonzero simplicial volume. Among these are manifolds with sectional curvature pinched between two negative constants; products of hyperbolic manifolds, which never admit a hyperbolic metric; and, generalizing the products, locally symmetric spaces of non-compact type, as proved by Lafont and Schmidt in [21].

**Theorem 4** (Morse broken trajectory theorem, general version). Let $M$ be a closed, oriented manifold of dimension $n$. Let $g$ be a Riemannian metric on $M$, and let $f: (M, g) \to \mathbb{R}$ be a Morse function satisfying Morse-Smale transversality. Let $Z$ be an aspherical topological space, and let $\alpha: M \to Z$ be a continuous map. Then the fundamental homology class $[M]$ has $\alpha$–image in $H_n(Z)$ with simplicial norm satisfying the bound

$$\|\alpha_*[M]\|_\Delta \leq \#(n\text{-part broken trajectories of } -\nabla f).$$

The Morse broken trajectory theorem (Theorems 2 and 4) is the third of three theorems that come from one proof strategy. The first of these theorems was introduced by Gromov in the paper [16]. It appears in the present paper as the Gromov singular fiber theorem (Theorem 5 and Conjecture 6). The second of the theorems is from a collaboration with Gabriel Katz, and appears as the multitangent trajectory theorem (Theorems 7 and 8). The remainder of this chapter is concerned with stating those theorems.
The Gromov singular fiber theorem (Theorem 5) is about a smooth map \( f : X^n \to Y^{n-1} \) between manifolds that differ in dimension by 1. Generically we should expect that the set \( \tilde{\Sigma} \subseteq X \) of critical points (that is, points where the derivative is not surjective) should have dimension \( n - 2 \). This is certainly true if \( f \) is **purely folded**, which means that for every point \( p \in X \) there are coordinates on \( X \) near \( p \) and on \( Y \) near \( f(p) \) such that \( f \) is given in those coordinates by

\[
 f(x_1, \ldots, x_n) = (h(x_1, x_2), x_3, \ldots, x_n),
\]

where \( h : \mathbb{R}^2 \to \mathbb{R} \) is a Morse function. In this case \( f \) maps the critical point set \( \tilde{\Sigma} \) into \( Y \) by an immersion, and the image \( f(\tilde{\Sigma}) \) has codimension 1 in \( Y \). If, in addition, \( f(\tilde{\Sigma}) \) has transverse self-intersections, then there is a 0-dimensional set of points in \( Y \) where \( f(\tilde{\Sigma}) \) has the maximum multiplicity, \( n-1 \). The theorem states that the number of such points is at least proportional to the simplicial volume. Gromov states the theorem as the “\(\Delta\)–Inequality for Generic Maps” in Section 3.3 of his paper [16], and sketches how to prove it. Here we prove it in detail for the special case where the manifold \( X \) is negatively curved and the function is purely folded with transverse self-intersection of critical values.

**Theorem 5** (Gromov singular fiber theorem, specific version, [16]). Let \( X \) be a closed, oriented manifold of dimension \( n \), admitting a metric of negative sectional curvature, and let \( Y \) be a manifold of dimension \( n - 1 \). Suppose that \( f : X \to Y \) is a smooth, purely folded map such that its restriction to the set \( \tilde{\Sigma} \) of critical points is an immersion into \( Y \) with transverse self-intersections. Then we have the inequality

\[
 \|X\|_\Delta \leq \text{const}(n) \cdot \#(\text{multiplicity } n - 1 \text{ self-intersections of } f(\tilde{\Sigma})).
\]

The following more general version of the theorem is closer to how Gromov stated it, and can probably be proved by adapting the same proof; I myself am not sure of all the details, though.

**Conjecture 6** (Gromov singular fiber theorem, general version, [16]). Let \( X \) be a closed, oriented manifold of dimension \( n \). Let \( L \) be an aspherical topological space, and suppose that \( \alpha : X \to L \) is a continuous map such that \( \pi_1(X) \) does not contain two subgroups \( H_1, H_2 \) for which every element of \( H_1 \) commutes with every element of \( H_2 \) and the subgroups \( \alpha_*H_1 \) and \( \alpha_*H_2 \) of \( \pi_1(L) \) are not amenable groups. Let \( Y \) be a manifold of dimension \( n - 1 \), and suppose that \( f : X \to Y \) is a stable smooth map. Then we have the inequality

\[
 \|\alpha_*[X]\|_\Delta \leq \text{const}(n) \cdot \#(\text{multiplicity } n - 1 \text{ critical values of } f).
\]

The Gromov singular fiber theorem introduced techniques that eventually led to the Morse broken trajectory theorem (Theorems 2 and 4). In between, though, came the multitangent trajectory theorem (Theorems 7 and 8, stated next) which is also based on the Gromov techniques and which led Katz to conjecture the Morse broken trajectory theorem.
The setting for the multitangent trajectory theorem is the following. We consider a smooth vector field $v$ on a space $X$, which is a compact smooth manifold with boundary, with $\dim X = n + 1$. For any such vector field, we may form the space of trajectories of the flow along $v$. In general the trajectory space may not be a nice space, but it is nicer if $v$ is a traversing vector field: a non-vanishing vector field such that every trajectory is either a singleton in $\partial X$ or a closed segment. Katz has explored this general setup in multiple papers beginning with [19], and in the paper [20] he introduces the class of traversally generic vector fields, which have certain nice properties. In Theorem 3.5 of that paper, he proves that the traversally generic vector fields form an open and dense subset of the traversing vector fields. Therefore, we study only the traversally generic vector fields; the definition and relevant properties appear in Section 5 which is devoted to the multitangent trajectory theorem.

Every traversally generic vector field $v$ has a well-defined multiplicity $m(a)$ with which $v$ meets $\partial X$ at a point $a$, and every trajectory $\gamma$ has a reduced multiplicity $m'(\gamma)$, which is the sum over all $a \in \gamma \cap \partial X$ of $(m(a) - 1)$. (The full definition of multiplicity appears in Chapter 5.) Every trajectory of a traversally generic vector field $v$ on a manifold $X^{n+1}$ has reduced multiplicity at most $n$, and so we denote by $\max\cdot\mult(v)$ the set of maximum-multiplicity trajectories; that is, those trajectories $\gamma$ with $m'(\gamma) = n$.

**Theorem 7** (Multitangent trajectory theorem, specific version). Let $M$ be a closed, oriented hyperbolic manifold of dimension $n + 1 \geq 2$, and let $X$ be the space obtained by removing from $M$ an open set $U$ satisfying the following properties:

- The boundary $\partial U = \partial X$ is a closed submanifold of $M$, possibly with multiple connected components; and
- The closure $\overline{U}$ is contained in a topological open ball of $M$, possibly very far from round.

Let $v$ be a traversally generic vector field on $X$. Then we have

$$\Vol M \leq \text{const}(n) \cdot \# \max\cdot\mult(v).$$

In particular, because $\Vol M$ is nonzero, there must be at least one maximum-multiplicity trajectory. This theorem generalizes Theorem 7.5 of [19], which addresses the case where $n + 1 = 3$ and $U$ is any finite disjoint union of balls, with constant $\Vol(\Delta^3)$, where $\Delta^3$ denotes the regular ideal simplex in hyperbolic 3-space.

Like the other two theorems, the multitangent trajectory theorem also has a more general version in terms of the simplicial norm. If $X$ is an oriented manifold with boundary, then let $D(X)$ denote the double of $X$, which is the oriented manifold obtained by gluing two copies of $X$ along their boundary $\partial X$.

**Theorem 8** (Multitangent trajectory theorem, general version). Let $X$ be a compact, oriented manifold with boundary, with $\dim X = n + 1$. Let $Z$ be a space with contractible universal cover, and let $\alpha : D(X) \to Z$ be a continuous map. Assume
that for each connected component of the boundary \( \partial X \), the corresponding subgroup of \( \pi_1(Z) \) is an amenable group. Then for every traversally generic vector field \( v \), we have

\[
\| \alpha_*[D(X)] \|_\Delta \leq \text{const}(n) \cdot \# \text{max-mult}(v).
\]

That is, the topological quantity \( \| \alpha_*[D(X)] \|_\Delta \) is an obstruction to the existence of a traversally generic vector field without maximum-multiplicity trajectories.

One special case of the multitangent trajectory theorem led to the Morse broken trajectory theorem. Let \( M \) be a closed, hyperbolic manifold of dimension \( n \), and let \( f: M \to \mathbb{R} \) be a Morse function. We construct \( X \) by removing from \( M \) a small neighborhood of each critical point of \( f \), and let \( v \) be the restriction of \(-\nabla f\) to \( X \). Then each maximum-multiplicity trajectory of \( v \) in \( X \) approximates an \( n \)-part broken trajectory of \(-\nabla f\) in \( M \): it must start on \( \partial X \) near an index \( n \) critical point of \( f \), must then be tangent to \( \partial X \) near one critical point of each intermediate index \( n-1, n-2, \ldots, 1 \), and then must end on \( \partial X \) near an index 0 critical point. This special case led Katz to conjecture the Morse broken trajectory theorem without the effective constant.

Chapter 2 provides an overview of the general strategy for all three theorems by outlining the proof of the Morse broken trajectory theorem. Chapter 3 gives the precise statements of the lemmas from Gromov’s paper [16] that all three theorems rely on. In Chapter 4 we prove the Gromov singular fiber theorem (Theorem 5), in Chapter 5 we prove the multitangent trajectory theorem (Theorems 7 and 8), and in Chapter 6 we prove the Morse broken trajectory theorem (Theorems 2 and 4). Chapter 7 gives some comment on future directions, and Appendix A includes the full proofs of the lemmas from Chapter 3.

Much of the text in this paper is taken from the paper [2], which is coauthored with Gabriel Katz and proves the multitangent trajectory theorem, and from the paper [1], which proves the Morse broken trajectory theorem. Some of the theorems proved here—including the amenable reduction lemma (Lemma 11), the localization lemma (Lemma 13), and the Gromov singular fiber theorem (Theorem 5)—are stated and proved by Gromov in the paper [16]. The proofs are given again here in order to provide more detail and to match the hypotheses to what is needed for the other theorems.
Chapter 2

Storybook justification of Morse broken trajectory theorem

This chapter consists of a simplified exposition of the proof of the Morse broken trajectory theorem (Theorem 2). It is based on Proposition 10 (rainbow simplex straightening lemma) which is a simplified version of the amenable reduction lemma (Lemma 11). The rest of the proof of the Morse broken trajectory theorem is given here in vague outline to emphasize the structure of the proof, which is similar to the proofs of the earlier theorems—the Gromov singular fiber theorem (Theorem 5) and the multitangent trajectory theorem (Theorems 7 and 8).

The rainbow simplex straightening lemma is a generalization of simplex straightening, which was introduced by Milnor and Thurston in the paper [24]. Here we review simplex straightening and then modify it to obtain the rainbow version.

Proposition 9 (Simplex straightening, [24]). Let $M$ be a closed, oriented, hyperbolic manifold of dimension $n$ and let $T$ be a triangulation of $M$. Then we have

$$\text{Vol}(M, \text{hyp}) \leq \text{Vol} \Delta^n \cdot \#(n\text{-dim simplices of } T).$$

Proof. The universal cover of $M$ is the $n$-dimensional hyperbolic space $H^n$. In $H^n$ every straight simplex—that is, every simplex equal to the convex hull of its vertices—has volume bounded above by a universal bound $\text{Vol} \Delta^n$. We straighten the simplices of $T$ to obtain a new triangulation $T'$: for each simplex of $T$, we lift to $H^n$, replace by the straight simplex on the same vertices, and project back to $M$. (Actually, $T'$ may not be a true triangulation, because some simplices may overlap. Still, $T'$ is homotopic to $T$ in a way that preserves the incidences between faces.) Then $T'$ covers $M$, and every simplex has volume at most $\text{Vol} \Delta^n$, so we have

$$\text{Vol}(M, \text{hyp}) \leq \text{Vol} \Delta^n \cdot \#(n\text{-dim simplices of } T'),$$

and $T'$ has one simplex for each simplex in $T$, giving the desired inequality. \hfill \Box

Rainbow simplex straightening is a generalization of simplex straightening and is an easier version of the amenable reduction lemma (Lemma 11) which is due to Gromov [16].

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**Proposition 10** (Rainbow simplex straightening). Let $M$ be a closed, oriented, hyperbolic manifold of dimension $n$, and let $T$ be a triangulation of $M$. Suppose that the vertices of $T$ are colored such that every loop in the 1-skeleton of $T$ that is non-contractible in $M$ must pass through vertices of more than one color. Then we have

$$\text{Vol}(M, \text{hyp}) \leq \text{Vol} \Delta^n \cdot \#(\text{essential simplices of } T),$$

where an $n$-dimensional simplex of $T$ is said to be essential if all $n+1$ of its vertices are different colors.

Note that we can recover the original simplex-straightening statement by coloring the vertices all different colors.

**Proof.** First we drag the vertices of $T$ so that the vertices of each color all go to the same point. Because of the hypothesis on the coloring, it is possible to do this in such a way that every 1-dimensional edge of $T$ with both endpoints the same color becomes a loop that is null-homotopic in $M$. After dragging the vertices in this way, we straighten the simplices. In the resulting (singular) triangulation, every simplex that is not an essential simplex has zero volume, because the two vertices of the same color lift to the same point of $H^n$ and so the lift of the simplex to $H^n$ has fewer than $n+1$ distinct vertices. Only the essential simplices contribute to the volume of $M$. \qed

The idea of the Morse broken trajectory theorem is to apply the rainbow simplex straightening lemma to a triangulation $T$ that has one essential simplex for each $n$-part broken trajectory of $-\nabla f$. It is based on the partition of $M$ into descending manifolds

$$M = \bigcup_{p \in \text{Crit } f} \mathcal{D}(p).$$

The triangulation $T$ is a subdivision of this partition, and the vertices are colored according to which descending manifold they are in. The hypothesis of the rainbow simplex straightening lemma is satisfied: every descending manifold is contractible, so every non-contractible loop must enter multiple cells.

**Example.** In order to see why it might be plausible to choose the triangulation so that it has one essential simplex per $n$-part broken trajectory, we try out the process on the example depicted in Figure 2-1. In this example the manifold $M$ is the 2-sphere, which is not hyperbolic, but we can still set up the input to the rainbow simplex straightening lemma. The sphere is embedded in $\mathbb{R}^3$; we use the induced Riemannian metric and let $f$ be the height function. The critical points are $p$, $a$, $b$, and $c$, and there are four 2-part broken trajectories: each starts by running either from $p$ to $b$ or from $a$ to $b$, and then there are two ways to get from $b$ to $c$.

We construct the triangulation by starting in dimension 0 and working our way up: first triangulate the point $\mathcal{D}(c) = \{c\}$, then extend to a fine triangulation of the circle $\mathcal{D}(b) \cup \mathcal{D}(c)$, and then extend to a fine triangulation of all of $M$. Then the
triangulation has four essential simplices, depicted in the right-hand side of Figure 2-1. From the vertex $c$ they extend in four different directions corresponding to the four 2-part broken trajectories.

The actual proof of the Morse broken trajectory theorem is more complicated than this glossed-over version. Why should we be able to subdivide the descending manifolds to get a triangulation? And why should the descending manifolds fit together in such a way that the essential simplices are in bijection with the $n$-part broken trajectories? Still, this general outline is the strategy for the Morse broken trajectory theorem (Theorems 2 and 4), as well as the Morse singular fiber theorem (Theorem 5) and the multitangent trajectory theorem (Theorems 7 and 8). In each setting, some function or vector field determines a partition of our space into strata each with simple topology. We construct a triangulation (or really a singular fundamental cycle) according to this stratification and apply the rainbow simplex straightening lemma to bound the volume of the space in terms of the local geometry near the 0-dimensional strata.
Chapter 3

Lemmas common to all three settings

The purpose of this chapter is to state Lemma 11 (amenable reduction lemma) and Lemma 13 (localization lemma), which come from Gromov's paper [16]. These two lemmas constitute the main strategy for proving all three main theorems: the Gromov singular fiber theorem (Theorem 5), the multitangent trajectory theorem (Theorems 7 and 8), and the Morse broken trajectory theorem (Theorems 2 and 4). We defer the proofs to Appendix A.

The amenable reduction lemma (Lemma 11) is a generalization of rainbow simplex straightening (Proposition 10). Instead of hyperbolic volume, we use simplicial norm; instead of a triangulation, we use a cycle in singular homology; and, instead of assuming that the loops in each color are all null-homotopic, we assume that they generate an amenable group.

In order to state the lemma, we view singular cycles as maps on simplicial complexes in the following way. Let $c$ be a singular cycle on a topological space $X$. We construct a space $\Sigma$ from $c$ as follows. Let $j$ be the dimension of $c$, and let $\Delta^j$ denote the abstract $j$-simplex. For each singular simplex $\sigma_i : \Delta^j \to X$ appearing in $c$, there are $j + 1$ face maps from $\Delta^{j-1}$ to $X$ obtained by restricting $\sigma_i$. We form $\Sigma$ by taking one copy of $\Delta^j$ for each $\sigma_i$ and identifying the faces that have the same face map. (A similar construction appears on pages 108–109 of Hatcher’s textbook [17].) Note that every face must be glued to at least one other face, because otherwise it would appear with nonzero coefficient in the linear combination $\partial c$ of face maps, contradicting the cycle hypothesis $\partial c = 0$. Then we can view $c$ as a triple $(\Sigma, c_\Sigma, \sigma)$, where $c_\Sigma$ is a simplicial cycle on $\Sigma$, and $\sigma : \Sigma \to X$ is a continuous map such that $c = \sigma \cdot c_\Sigma$. Note that the space $\Sigma$ is not necessarily an honest simplicial complex, linearly embeddable in Euclidean space, but is what Hatcher calls a $\Delta$-complex, which may have (for instance) edges in its 1-skeleton that are self-loops (i.e., both endpoints are the same vertex).

The statement of the amenable reduction lemma uses the ideas of partial coloring, induced 1-complex, and essential simplex, which are defined as follows. Let $c$ be a cycle, viewed as the triple $(\Sigma, c_\Sigma, \sigma)$. By a partial coloring of $c$ we mean a list $V_1, V_2, \ldots, V_\ell, \ldots$ of disjoint subsets of the set of vertices of $\Sigma$. The induced 1-complex $\Sigma_\ell$ of a given subset $V_\ell$ is the maximal 1-dimensional subcomplex of $\Sigma$ with vertex set $V_\ell$; it consists of the vertices in $V_\ell$ and the edges that have both
endpoints in $V_e$. According to the partial coloring we classify each simplex as either essential or non-essential; the non-essential simplices are the ones that can be made to disappear in a certain sense. A simplex $\Delta$ of $\Sigma$ is a non-essential simplex of $c$ if either of the following conditions holds:

- $\Delta$ has two distinct vertices in the same $V_e$ (the vertices are permitted to have the same image in $X$ as long as they are distinct in $\Sigma$); or

- $\Delta$ has two vertices that are the same point of $\Sigma$, and the edge between them is a null-homotopic loop in $X$.

An **essential simplex** of $c$ is any simplex $\Delta$ of $\Sigma$ that is not non-essential; that is, $\Delta$ is essential if in the 1-skeleton of $\Delta$ in $\Sigma$, every vertex is in a different $V_e$, and any edges that are self-loops map to non-contractible loops in $X$. In particular, any simplex $\Delta$ with vertices in $\dim(\Delta) + 1$ different sets $V_e$ is essential.

**Lemma 11** (Amenable reduction lemma, p. 25 of [16]). Let $c$ be a cycle on space $X$ with partial coloring $\{V_e\}$. Suppose that $Z$ is an aspherical topological space, and let $\alpha: X \to Z$ be a continuous map such that for every connected component of every induced 1-simplex $\Sigma'_i$ of $c$, the $\alpha$-image of its fundamental group in $\pi_1(Z)$ is an amenable group. Then the simplicial norm of the $\alpha$-image of the homology class $[c] \in H_1(X)$ represented by the cycle $c = \sum r_i \sigma_i$ (where $r_i \in \mathbb{R}$ are coefficients and $\sigma_i$ are simplices) satisfies the bound

$$\|\alpha_*[c]\|_\Delta \leq \sum_{\sigma_i \text{ essential}} |r_i|.$$

When we use the amenable reduction lemma, we use it indirectly by invoking the localization lemma (Lemma 13) instead. We start with a homology class and a stratification of the ambient space, and then the localization lemma uses the geometry of the stratification to construct a partially colored cycle with few essential simplices, as input to the amenable reduction lemma.

In Gromov’s paper [16], a **stratification** of a space is defined to be any partition with the following property: if a stratum $S$ intersects the closure $\overline{S'}$ of another stratum $S'$, then $S \subseteq S'$ and we write $S \preceq S'$. This definition of stratification is what we use for the Gromov singular fiber theorem (Theorem 5) and the multitangent trajectory theorem (Theorems 7 and 8). For the Morse broken trajectory theorem (Theorems 2 and 4) we want the theorems about stratifications to apply to CW-complexes, partitioned by the open cells; however, not all CW-complexes are stratifications by this definition. Therefore, we define a **generalized stratification** of a space to be any partition with the following property. If a stratum $S$ intersects the closure $\overline{S'}$ of another stratum $S'$, then we write $S' \to S$, pronounced as "$S'$ limits to $S". In order for the partition to be a generalized stratification, we require that the resulting directed graph on the set of strata have no directed cycles. In this case, the directed graph generates a partial order on strata: we write $S \preceq S'$ whenever there is a directed path from $S'$ to $S$. If neither $S \preceq S'$ nor $S' \preceq S$, then we say the two strata are **incomparable**. In this paper we usually assume that strata are connected; given an
arbitrary generalized stratification, the set of connected components of strata is also a generalized stratification.

The localization lemma uses a stratified version of the simplicial norm. The stratified simplicial norm, like the simplicial norm, is an infimum of sums of coefficients, but the infimum is taken over only those cycles $c$ that are consistent with the (generalized) stratification, in the following sense:

- The cellular condition requires that for each simplex of $c$, the image of the interior of each face (of any dimension) must be contained in one stratum.
- The order condition requires that the image of each simplex of $c$ must be contained in a totally ordered chain of strata; that is, the simplex does not intersect any two incomparable strata.
- The internality condition requires that for each simplex of $c$, if the boundary of a face (of any dimension) maps into a stratum $S$, then the whole face maps into $S$.

For technical reasons, we also require a fourth condition, which is stated in terms of the underlying space $\Sigma$ of $c$.

- The loop condition requires that for every edge in the 1-skeleton of $\Sigma$ that is a self-loop, its image in $X$ must be a single point.

We record here an easy sublemma that says that the cellular, order, internality, and loop properties are preserved by barycentric subdivision. Later we refer to this sublemma in the proofs of the localization lemma (Lemma 13) and the Morse broken trajectory theorem (Theorems 2 and 4).

**Sublemma 12.** Let $X$ be a space with a generalized stratification, and let $c = (\Sigma, c_\Sigma, \sigma)$ be a cycle satisfying the cellular property. Then the first barycentric subdivision of $c$ satisfies the cellular, order, internality, and loop properties.

**Proof.** The cellular property is immediate because every open face of the subdivision is contained in an open face of $c$. The order property is proven as follows. In any simplex of $c$, if $f$ is a face in the boundary of another face $f'$, and $S$ and $S'$ are the strata of $f$ and $f'$, then those strata are related by $S' \rightarrow S$. Then in the barycentric subdivision, every simplex is contained in a totally ordered chain of faces, so the corresponding strata are also totally ordered. The internality property holds because in any face of the subdivision, the stratum of the interior also appears as the stratum of one of the vertices (at least). The loop property holds because the barycentric subdivision of any $\Sigma$ cannot have any self-loop edges. \qed

In the paper [16], the stratified simplicial norm of a homology class is the infimal sum of coefficients of cycles that represent the homology class and satisfy the conditions above. However, for our purposes it is more convenient to count only the simplices for which every vertex is in a different stratum; these are the essential
simplices. For any cycle \( c = \sum r_i \sigma_i \) (where \( \sigma_i \) are singular simplices and \( r_i \) are real coefficients) and generalized stratification \( S \), let \( \| c \|_{\Delta, \text{ess}}^S \) be defined by

\[
\| c \|_{\Delta, \text{ess}}^S = \sum_{\text{essential } \sigma_i} |r_i|.
\]

The **essential stratified simplicial norm** of a homology class \( h \) with respect to a generalized stratification \( S \), denoted \( \| h \|_{\Delta, \text{ess}}^S \), is the infimum of \( \| c \|_{\Delta, \text{ess}}^S \) taken over all cycles \( c \) representing \( h \) that satisfy the cellular, order, internality, and loop conditions. When \( h \) is a relative homology class, the essential stratified simplicial norm is defined in the same way, as an infimum over relative cycles.

In the localization lemma, we would like the strata to sit in the ambient space somewhat like submanifolds in a manifold or like cells in a cell complex. We say that a generalized stratification on a space \( X \) has the **NDR property** if every stratum \( S \) is a neighborhood deformation retract, in the following sense: there is a neighborhood \( U_S \) of \( S \) in \( X \) and a homotopy \( H: U_S \times [0, 1] \to X \) such that the start map \( H_0: U_S \to X \) is the inclusion, the image \( H_1(U_S) \) of the end map is contained in \( S \), and the restriction \( H_t|_S: S \to X \) to \( S \) at every time \( t \) is the inclusion.

**Lemma 13** (Localization lemma). Let \( X \) be a compact metrizable space, and let \( S \) be a generalized stratification on \( X \) with the NDR property, such that \( S \) has finitely many strata and each stratum is connected. Let \( Z \) be an aspherical topological space, and suppose \( \alpha: X \to Z \) is a continuous map such that for every stratum \( S \), the group \( \alpha_* \pi_1(S) \subseteq \pi_1(Z) \) is amenable. Let \( h \in H_j(X) \) be a homology class, and suppose that \( A \) is a closed subset of \( X \) such that among the strata intersecting \( A \), there is no totally ordered chain of more than \( j \) strata. Let \( h_{\text{rel}} \) be the image of \( h \) in \( H_j(X, A) \). Then the simplicial norm of \( \alpha_* h \in H_j(Z) \) satisfies the bound

\[
\| \alpha_* h \|_{\Delta} \leq \| h_{\text{rel}} \|_{\Delta, \text{ess}}^S.
\]

This statement of the localization lemma is more general than the original version in Gromov's paper [16]. There, the closed set \( A \) is assumed to be the complement of a small neighborhood of the \((n - j)\)-skeleton of \( X \) determined by the stratification. Here we need the more general statement, because for the Morse broken trajectory theorem (Theorems 2 and 4) we need to take \( A \) to be the \((j - 1)\)-skeleton of \( X \).
Chapter 4

Gromov singular fiber theorem

The main goal of this chapter is to prove the following lemma, after which the proof of the Gromov singular fiber theorem (Theorem 5) is a quick application of the localization lemma (Lemma 13). Recall that a stratification of a topological space $X$ is a partition such that if a stratum $S$ intersects the closure $\overline{S'}$ of another stratum $S'$, then $S \subseteq \overline{S'}$. The strata are partially ordered by containment of their closures. We say that $S$ is an amenable stratum if the corresponding subgroup $i_*\pi_1(S)$ of $\pi_1(X)$ is an amenable group, where $i_*: S \hookrightarrow X$ denotes the inclusion.

Lemma 14 (Main lemma for Gromov singular fiber theorem). Let $X$ be a closed, oriented manifold of dimension $n$, admitting a metric of negative sectional curvature, and let $Y$ be a manifold of dimension $n-1$. Suppose that $f: X \to Y$ is a smooth, purely folded map such that its restriction to the set $\Sigma$ of critical points is an immersion into $Y$ with transverse self-intersections. There is a stratification $S(X)$ of $X$, and a subset $P$ of $X$, satisfying the following properties:

1. Every point of $P$ is its own stratum.

2. Every totally ordered chain of more than $n$ strata has a point of $P$ as one of the strata in the chain.

3. Every point of $P$ is in $\Sigma$ and maps under $f$ to a multiplicity $n-1$ point of $f(\Sigma)$.

4. There is a finite set of stratified model neighborhoods, depending only on the dimension $n$, such that for each point $p$ in $P$ there is a small neighborhood of $p$ in $X$ with a stratification-preserving diffeomorphism to one of the model neighborhoods.

5. Every stratum of $S(X)$ satisfies the NDR property.

6. Every stratum of $S(X)$ is amenable.

In order to construct the stratification $S(X)$, we first need to understand the fibers of $f$ over the various points of $Y$. The fibers change according to the stratification $S_{\text{mult}}(Y)$ of $Y$ by multiplicity, defined as follows. We say that the multiplicity of any
Figure 4-1: Each 0-dimensional stratum of $S_{\text{mult}}(Y)$ has a neighborhood where the set of critical values $f(\hat{\Sigma})$ looks like $n - 1$ intersecting hyperplanes in $Y$, shown right for the case of $\dim X = 3$ and $\dim Y = 2$. The fiber in $X$ over the 0-dimensional stratum looks locally like an interval except at $n - 1$ cross points, which are resolved in different ways in the fibers of the neighboring strata, shown left. (This picture is modeled on one in [6].)

Point $y \in Y$ is the number of critical points $x \in \widehat{\Sigma}$ such that $f(x) = y$. Then, the strata of $S_{\text{mult}}$ are obtained by dividing $Y$ according to multiplicity and then taking connected components.

Over each stratum of $S_{\text{mult}}(Y)$, the restriction of $f$ is a locally trivial bundle. The fiber over each multiplicity-0 stratum is a collection of circles, and crossing any multiplicity-1 stratum corresponds to a surgery on the fiber: either a circle shrinks to a point or grows from a point, or two pieces of curve pinch together to become a cross and then resolve into two other pieces of curve. Figure 4-1 depicts the strata near a high-multiplicity point of $Y$, and the relationship between the fibers of those strata.

In the statement of Lemma 14, the set $P$ should be thought of as the set of all 0-dimensional strata, so any reasonable stratification we might cook up should satisfy properties (1), (2), (4), and (5). The difficulty is in constructing a stratification that simultaneously has only amenable strata and not too many 0-dimensional strata.

Here are some simple strategies for how to stratify $X$, which do not quite satisfy the properties:

(a) Let $S_{\text{mult}}(X)$ be the stratification of $X$ obtained by pulling back $S_{\text{mult}}(Y)$ by $f$; that is, the strata of $S_{\text{mult}}(X)$ are the connected components of $f^{-1}(S)$ as $S$ ranges over strata of $S_{\text{mult}}(Y)$. Then let $S_{\text{mult},\hat{\Sigma}}(X)$ be the refinement of $S_{\text{mult}}(X)$ obtained by separating $\hat{\Sigma}$ from the regular points and taking connected components. The 0-dimensional strata of $S_{\text{mult},\hat{\Sigma}}(X)$ are exactly the points of $\hat{\Sigma}$ lying over the
multiplicity $n - 1$ points of $f(\hat{\Sigma})$, satisfying property (3) of Lemma 14. But, a stratum of $S_{\text{mult}, \xi}(X)$ might not be amenable if its image stratum in $S_{\text{mult}}(Y)$ has complicated topology.

(b) Suppose we find a triangulation of $Y$ that is a refinement of $S_{\text{mult}}(Y)$, and pull back the triangulation to get a stratification of $X$. In this stratification, we can take $P$ to be empty and still satisfy property (2) of Lemma 14. To check amenability: each face of the triangulation is contractible, so the topology of each stratum in $X$ comes entirely from the topology of the fiber. However, these fibers may or may not generate amenable subgroups of $\pi_1(X)$.

(c) We can refine the stratification in strategy (b) by separating $\hat{\Sigma}$ from the regular points. Then every stratum is either contractible or homotopy equivalent to a circle, and hence is amenable, since quotients of $\mathbb{Z}$ are amenable. But, there are far too many 0-dimensional strata; the extra ones come from any vertices of the triangulation that are in $f(\hat{\Sigma})$ but are not among the multiplicity $n - 1$ points.

The stratification $S(X)$ constructed for Lemma 14 is some combination of strategies (a) and (b). In order to say where we use each strategy, we introduce some more notation. Over each point of $Y$, the fiber in $X$ may have several connected components. We define $\tilde{Y}$ to be the space in which the points correspond to the connected components of fibers of $X$. Then the map $f: X \to Y$ factors through $\tilde{Y}$; over $\tilde{Y}$ each fiber in $X$ is connected, and over $Y$ each fiber in $\tilde{Y}$ has finitely many points.

We let $S_{\text{mult}}(\tilde{Y})$ denote the stratification of $\tilde{Y}$ by connected components of preimages of strata from $S_{\text{mult}}(Y)$. Because the restriction of $f$ to each stratum of $S_{\text{mult}}(X)$ is a locally trivial bundle over the corresponding stratum in $Y$, each stratum of $S_{\text{mult}}(\tilde{Y})$ maps to the corresponding stratum in $Y$ by a finite-sheeted covering-space map, and each stratum of $S_{\text{mult}}(X)$ maps to the corresponding stratum in $\tilde{Y}$ by a locally trivial bundle.

Over $\tilde{Y}$, each fiber $F$ in $X$ generates some subgroup of $\pi_1(X)$. That is, if $i: F \to X$ denotes the inclusion, then $i_*\pi_1(F)$ is a subgroup of $X$. We sort the fibers based on whether their corresponding subgroups $i_*\pi_1(F)$ are amenable groups: we let $\tilde{Y}_{\text{nam}}$ denote the set of all points in $\tilde{Y}$ such that the fiber $F$ generates a non-amenable subgroup $i_*\pi_1(F)$, and let $X_{\text{nam}}$ denote the preimage in $X$ of $\tilde{Y}_{\text{nam}}$.

**Lemma 15.** The space $\tilde{Y}_{\text{nam}}$ is a closed union of strata from $S_{\text{mult}}(\tilde{Y})$, so the space $X_{\text{nam}}$ is a closed union of strata from $S_{\text{mult}}(X)$.

**Proof.** The map from $X$ to $\tilde{Y}$ is a locally trivial bundle over each stratum of $S_{\text{mult}}(\tilde{Y})$. So, if $y, y' \in \tilde{Y}$ are nearby points of the same stratum, then their fibers $F$ and $F'$ are homotopic and satisfy $i_*\pi_1(F) = i_*\pi_1(F')$. Thus $\tilde{Y}_{\text{nam}}$ is a union of strata.

At each point of $X \setminus \hat{\Sigma}$, the fiber looks locally like an interval, and at each point of $\hat{\Sigma}$, it looks like either an isolated point or a cross. Moving away from $\hat{\Sigma}$ corresponds to expanding the isolated point into a circle or unpinching the cross to form two intervals.

We show that the complement of $\tilde{Y}_{\text{nam}}$ is open. Suppose that $y_0$ is a point of $\tilde{Y}$ and $y \in \tilde{Y}$ is in a small neighborhood of $y_0$. If the fiber $F_0$ over $y_0$ is an isolated point,
then the fiber $F$ over $y$ is either another isolated point or a small circle, so $i_*\pi_1(F)$ is a quotient of $\mathbb{Z}$ and so is amenable. If $F_0$ is not an isolated point, then $F$ is obtained from $F_0$ by unpinching some crosses and selecting a connected component, and so we have $i_*\pi_1(F) \subseteq i_*\pi_1(F_0)$. Subgroups of amenable groups are amenable, so if $i_*\pi_1(F_0)$ is amenable, then $i_*\pi_1(F)$ is also amenable. \qed

Lemmas 16 and 17, stated next, suggest that the stratification $S(X)$ should use strategy (a) on $X_{\text{nam}}$ and strategy (b) away from $X_{\text{nam}}$. In the remainder of this chapter, we abuse notation and let $i$ denote the inclusion of any subset into $X$. Additionally, if we have some other space $\tilde{S}$ equipped with a map into a subset $S$ of $X$, we also let $i$ denote the map from $\tilde{S}$ into $X$, so that $i_*\pi_1(\tilde{S})$ is a subgroup of $\pi_1(X)$. The maps denoted by $i$ all have target $X$ but have different domains.

**Lemma 16.** Let $B$ be a contractible subset of one stratum of $S_{\text{mult}}(\tilde{Y})$ that is not in $Y_{\text{nam}}$, and let $S$ be the preimage of $B$ in $X$. Then $S$ is amenable.

**Proof.** Because $S$ is a locally trivial bundle over $B$, and $B$ is contractible, the bundle must be trivial. Thus if $F$ is the fiber, we have $i_*\pi_1(S) = i_*\pi_1(F)$, which we have assumed to be amenable. \qed

**Lemma 17.** Suppose that $S \in S_{\text{mult},\mathcal{S}}(X)$ is contained in $X_{\text{nam}}$. Then $i_*\pi_1(S) = 0$, so $S$ is an amenable stratum.

To prove this lemma, we use Lemmas 18 and 19, which we reuse later to show that other strata of $S(X)$ are amenable. Lemma 18 uses some information about the fundamental group $\pi_1(X)$ that comes from the hypothesis that $X$ admits a metric of negative sectional curvature.

**Lemma 18.** Let $X$ be a closed Riemannian manifold of negative sectional curvature. Then for every nontrivial element $g \in \pi_1(X)$, the centralizer $C(g)$ is isomorphic to $\mathbb{Z}$.

**Proof.** The proof comes directly from the proof given in do Carmo’s textbook [7] of Preissman’s theorem, which says that for $X$ as above, every nontrivial abelian subgroup of $\pi_1(X)$ is isomorphic to $\mathbb{Z}$. First Cartan’s Theorem (Theorem 12.2.2 of [7]) states that every nontrivial free homotopy class of loops on $X$ contains a closed geodesic. This implies that every nontrivial element of $\pi_1(X)$, viewed as an isometry of the universal cover $\tilde{X}$, must take some geodesic to itself (Proposition 12.2.6 of [7]), and this invariant geodesic is unique (Lemma 12.3.3 of [7]). Any two commuting elements of $\pi_1(X)$ have the same invariant geodesic (Lemma 12.3.4 of [7]), and any nontrivial subgroup of $\pi_1(X)$ containing only elements with the same invariant geodesic must be isomorphic to $\mathbb{Z}$ (Lemma 12.3.5 of [7]). Applying these last two lemmas to the subgroup $C(g)$ containing all elements that commute with $g$, we obtain $C(g) \cong \mathbb{Z}$. \qed

To prove Lemma 17, we need Lemma 19 as well.

**Lemma 19.** Let $F$ be a 1-dimensional finite cell complex, and let $E \to B$ be a locally trivial bundle with fiber $F$, such that the local trivializations respect the cell structure. Then there is a finite-sheeted covering space $\hat{B} \to B$ such that the pullback bundle $\hat{E} \to \hat{B}$ is isomorphic to the trivial bundle $F \times \hat{B}$.
Proof. Let $G$ denote the group of self-homeomorphisms of $F$ that respect the cell structure. The locally trivial bundles with fiber $F$ correspond to principal $G$-bundles over the same base. We observe that $G$ has finitely many connected components, corresponding to the graph automorphisms of $F$. We can choose a representative for each connected component of $G$ as follows. Fix an identification of each 1-cell of $F$ with $(0, 1)$, and consider the elements of $G$ that map edges to edges by affine maps—either the identity or $t \mapsto 1 - t$. These representatives form a finite subgroup of $G$ which we call $\text{Aut}_F$, and $\text{Aut}_F$ is a deformation retract of $G$.

Thus the locally trivial bundles with fiber $F$ correspond to principal $\text{Aut}_F$ bundles over the same base. We take $\tilde{B}$ to be this principal $\text{Aut}_F$ bundle, which is finite-sheeted because $\text{Aut}_F$ is finite, and the pullback bundle is trivial.

When we prove that various strata are amenable, there are a few properties of amenable groups that we use over and over. The group $\mathbb{Z}$ is amenable. Subgroups, quotients, and direct sums of amenable groups are amenable. And, finite extensions of amenable groups are amenable; that is, if $H$ is a normal subgroup of $G$ with finite quotient, and $H$ is amenable, then $G$ is amenable.

Proof of Lemma 17. Let $E$ denote the stratum of $S_{\text{mult}}(X)$ containing $S$. Then $E$ is a locally trivial bundle over its image $B$ in $\hat{Y}$, with some connected fiber $F$, and we have assumed that $i_*\pi_1(F)$ is nonamenable. In particular, $F$ cannot be just a circle, and must intersect $\hat{\Sigma}$, so we view $F$ as a 1-dimensional cell complex in which the points in $\hat{\Sigma}$ are the vertices. By Lemma 19 there is a finite-sheeted cover $\tilde{E} \to \tilde{B}$ that is the trivial bundle $F \times B$.

Let $\hat{S}$ denote any connected component of the preimage of $S$ in $\hat{E}$, so that $\hat{S}$ is the product of $\tilde{B}$ with either a vertex or an edge of $F$. We can view $\pi_1(\hat{E})$ as $\pi_1(F) \times \pi_1(\hat{S})$. Because $\hat{S}$ is a finite-sheeted covering space of $S$, we know that $\pi_1(S)$ is a finite extension of $\pi_1(\hat{S})$, so $i_*\pi_1(S)$ is a finite extension of $i_*\pi_1(\hat{S})$. Thus we can find information about $i_*\pi_1(\hat{S})$ by first studying $i_*\pi_1(\hat{S})$.

We claim that $i_*\pi_1(\hat{S}) = 0$. Suppose to the contrary that $g \in i_*\pi_1(\hat{S})$ is a nontrivial element. Then because $\pi_1(\hat{S})$ and $\pi_1(F)$ commute in $\pi_1(\hat{E})$, we know that $i_*\pi_1(\hat{S})$ and $i_*\pi_1(F)$ commute in $\pi_1(X)$, so $i_*\pi_1(F)$ is contained in the centralizer $C(g)$, which by Lemma 18 is isomorphic to $\mathbb{Z}$. This contradicts the assumption that $i_*\pi_1(F)$ is nonamenable, so we conclude that $i_*\pi_1(\hat{S}) = 0$.

We know that $i_*\pi_1(S)$ is a finite extension of $i_*\pi_1(\hat{S})$, so it must be a finite group. But by Lemma 18 every nontrivial element of $\pi_1(X)$ has infinite order, so in fact $i_*\pi_1(S) = 0$.

The next lemma describes the structure of the stratification $S(X)$ that we build to satisfy Lemma 14. The stratification looks like strategy (a) on $X_{\text{nam}}$ and like strategy (b) away from $X_{\text{nam}}$. However, we cannot literally use strategy (a) on $X_{\text{nam}}$ and strategy (b) on $X \setminus X_{\text{nam}}$ and expect to get a stratification. For an arbitrary stratification, if we take a closed union of strata (such as $X_{\text{nam}}$) and substitute a finer stratification on that set, the result is a stratification; the corresponding statement for an open union of strata (such as $X \setminus X_{\text{nam}}$) is false. Thus, to build the stratification $S(X)$, we use strategy (a) on $X_{\text{nam}}$ and use strategy (b) on the complement of a small
neighborhood of \( X_{\text{nam}} \), and in between there is some interpolation between the two strategies.

**Lemma 20 (Construction of \( S(X) \)).** There is a stratification \( S(X) \) of \( X \) satisfying the following properties:

1. \( S(X) \) is a refinement of \( S_{\text{mult}}(X) \).

2. On \( X_{\text{nam}} \), the stratification \( S(X) \) is equal to \( S_{\text{mult}}(\tilde{X}) \).

3. On the complement of \( X_{\text{nam}} \), the stratification \( S(X) \) is obtained by pulling back a stratification \( S(\tilde{Y}) \) of \( \tilde{Y} \).

4. \( S(\tilde{Y}) \) is a refinement of \( S_{\text{mult}}(\tilde{Y}) \), and is equal to \( S_{\text{mult}}(\tilde{Y}) \) on \( \tilde{Y}_{\text{nam}} \).

5. The strata of \( S(\tilde{Y}) \) that neighbor \( \tilde{Y}_{\text{nam}} \)—that is, they are not in \( \tilde{Y}_{\text{nam}} \) but their closures intersect \( \tilde{Y}_{\text{nam}} \)—form a small neighborhood of \( \tilde{Y}_{\text{nam}} \), and each can be homotoped into a stratum of \( \tilde{Y}_{\text{nam}} \). The corresponding strata of \( S(X) \), which are the strata that neighbor \( X_{\text{nam}} \), are all amenable.

6. On the (closed) union of strata of \( S(\tilde{Y}) \) whose closures are disjoint from \( \tilde{Y}_{\text{nam}} \), the stratification \( S(\tilde{Y}) \) is a triangulation.

Figure 4-2 depicts what \( S(\tilde{Y}) \) might look like near \( \tilde{Y}_{\text{nam}} \). The tricky part of constructing \( S(X) \) is ensuring property (5), which says that the strata neighboring \( X_{\text{nam}} \) are all amenable. Roughly, the argument is as follows. For each stratum \( B \) neighboring \( \tilde{Y}_{\text{nam}} \), we consider the corresponding stratum \( B' \) in \( \tilde{Y}_{\text{nam}} \). Using the reasoning from Lemma 17, we know that the preimage of \( B' \) in \( X \) is a locally trivial bundle \( E' \) with nonamenable fiber and a base that corresponds to an amenable (indeed, trivial) subgroup of \( \pi_1(X) \). The stratum above \( B \) is a locally trivial bundle \( E \) with amenable fiber and a base that is, in some sense, the same as the base of \( E' \) up to finite covers and contractible sets. In this way, the amenability of \( E \) is inherited from the amenability of the strata that make up \( E' \) in \( \tilde{Y}_{\text{nam}} \).

The following lemma provides the technical framework needed to construct \( S(X) \). It constructs the stratification \( S_{\text{fat}}(\tilde{Y}) \), a "fattening" of the stratification of \( \tilde{Y}_{\text{nam}} \). Then, Lemmas 25 and 27 show that the resulting strata are amenable, completing the proof of Lemma 20.

**Lemma 21.** There is a stratification \( S_{\text{fat}}(\tilde{Y}) \) of \( \tilde{Y} \) satisfying the following properties:

- \( S_{\text{fat}}(\tilde{Y}) \) is a refinement of \( S_{\text{mult}}(\tilde{Y}) \), and on the subset \( \tilde{Y}_{\text{nam}} \), the stratification \( S_{\text{fat}}(\tilde{Y}) \) is the same as \( S_{\text{mult}}(\tilde{Y}) \).

- For every stratum \( B \) of \( S_{\text{fat}}(\tilde{Y}) \) such that \( B \) is not in \( \tilde{Y}_{\text{nam}} \) but its boundary \( \partial B \) intersects \( \tilde{Y}_{\text{nam}} \), there is a stratum \( B' \subseteq \partial B \) with \( \dim B' = \dim B - 1 \) and \( \overline{B'} \cap \tilde{Y}_{\text{nam}} = \partial B \cap \tilde{Y}_{\text{nam}} \), with a retraction \( r: B \cup B' \to B' \) that is a locally trivial bundle such that the fiber is a union of intervals intersecting only at their common endpoint in \( B' \).
Figure 4-2: Given $\tilde{Y}_{\text{nam}}$ (shown in bold), the stratification $\mathcal{S}(\tilde{Y})$ is constructed by extending the stratification $\mathcal{S}_{\text{mult}}(\tilde{Y})$ of $\tilde{Y}_{\text{nam}}$ to a small neighborhood of $\tilde{Y}_{\text{nam}}$ (shown in gray) and triangulating the rest of $\tilde{Y}$.

- There is a triangulation $\mathcal{S}_\Delta(\tilde{Y})$ that is a refinement of $\mathcal{S}_{\text{rat}}(\tilde{Y})$.

The stratification $\mathcal{S}(\tilde{Y})$ for Lemma 20 is the refinement of $\mathcal{S}_{\text{rat}}(\tilde{Y})$ obtained by substituting $\mathcal{S}_\Delta(\tilde{Y})$ on the (closed) union of strata of $\mathcal{S}_{\text{rat}}(\tilde{Y})$ whose closures are disjoint from $\tilde{Y}_{\text{nam}}$. Then $\mathcal{S}(X)$ is the refinement of the pullback of $\mathcal{S}(\tilde{Y})$ to $X$ obtained by substituting $\mathcal{S}_{\text{mult},\Sigma}(X)$ on $X_{\text{nam}}$.

The next few lemmas set up coordinates on a neighborhood of $f(\hat{S})$ in $Y$ so that we can define $\mathcal{S}_{\text{rat}}(\tilde{Y})$ explicitly. The following lemma is a version of the tubular neighborhood theorem.

**Lemma 22.** Let $M$ be a Riemannian manifold, and suppose that a point $p \in M$ is contained in $k$ embedded hypersurfaces $H_1, \ldots, H_k$ that have transverse intersection. In a small neighborhood $U$ of $p$ in $M$, let $\pi: U \rightarrow H_1 \cap \cdots \cap H_k$ be the function that selects the nearest point of $H_1 \cap \cdots \cap H_k$. Let $N$ denote the normal bundle of $H_1 \cap \cdots \cap H_k$. For $i = 1, \ldots, k$ let $n_i: H_1 \cap \cdots \cap H_k \rightarrow N$ denote the normal vector to $H_i$ for some choice of coorientation, and let $d_i: U \rightarrow \mathbb{R}$ denote the signed distance to $H_i$. Then in a sufficiently small neighborhood $U'$ of $p$, the map $\varphi_{H_1,\ldots,H_k}: U' \rightarrow N$ given by

$$\varphi_{H_1,\ldots,H_k}(q) = (\pi(q), d_1(q) \cdot n_1(\pi(q)) + \cdots + d_k(q) \cdot n_k(\pi(q)))$$

is a diffeomorphism from $U'$ onto its image in $N$.

**Proof.** It suffices to check that the derivative of $\varphi_{H_1,\ldots,H_k}$ at $p$ is an isomorphism. To do this, we construct another function in the reverse direction such that the composition sends $p$ to $p$ and has derivative at $p$ equal to the identity. This reverse function $\exp: N \rightarrow M$ is given by

$$(x, v) \mapsto \exp_x(v),$$

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so that the composition \( \exp \circ \varphi_{H_1,\ldots,H_k} : U' \to M \) is given by

\[
q \mapsto \exp_{\pi(q)}(d_1(q) \cdot n_1(\pi(q)) + \cdots + d_k(q) \cdot n_k(\pi(q))).
\]

This composition is the identity when restricted to \( H_1 \cap \cdots \cap H_k \) and when restricted to the geodesics in the directions \( n_1,\ldots,n_k \) through \( H_1 \cap \cdots \cap H_k \), so its derivative at \( p \) must be the identity. □

We apply this lemma in the case where \( p \in Y \) is a point of multiplicity \( k \), and \( H_1,\ldots,H_k \) are the images of small neighborhoods in \( \hat{\Sigma} \) of the preimages of \( p \) in \( \Sigma \). We call these embedded hypersurfaces the **local sheets** of \( f(\Sigma) \).

In the following lemma, we consider the normal bundle \( N \) of \( H_1 \cap \cdots \cap H_k \) to be a Riemannian manifold, with metric obtained by regarding it locally as \((H_1 \cap \cdots \cap H_k) \times \mathbb{R}^k\), where we choose the basis for \( \mathbb{R}^k \) to be the hypersurface normals \( n_1,\ldots,n_k \). In these coordinates the hypersurfaces \( H_1,\ldots,H_k \) correspond to the coordinate hyperplanes of \( \mathbb{R}^k \).

**Lemma 23.** There is a Riemannian metric on \( Y \) and a small neighborhood of \( f(\Sigma) \) such that the following is true. Suppose we apply Lemma 22, taking \( M \) to be \( Y \), taking the point \( p \) to be on \( f(\Sigma) \), and taking the hypersurfaces \( H_1,\ldots,H_k \) to be the local sheets of \( f(\Sigma) \) at \( p \). Then in this small neighborhood of \( f(\Sigma) \), the resulting maps \( \varphi_{H_1,\ldots,H_k} \) are isometries.

**Proof.** We start with an arbitrary Riemannian metric on \( Y \). To modify this metric, we start with each point \( p \in Y \) of multiplicity \( n-1 \). At the intersection of \( n-1 \) local sheets, the normal bundle is just \( \mathbb{R}^{n-1} \), and near \( p \) we can pull back the Riemannian metric on \( \mathbb{R}^{n-1} \) using the map \( \varphi_{H_1,\ldots,H_{n-1}} \) from Lemma 22. Then we can use a partition of unity to interpolate between that metric and the original metric on \( Y \).

Next we consider the points of multiplicity \( n-2, n-3 \), and so on. Suppose that we have modified the metric on \( Y \) such that the maps from Lemma 22 are isometries near all points of multiplicity greater than \( k \), and consider a multiplicity-\( k \) stratum \( S \) of \( S_{\text{mult}}(Y) \). Note that near the boundary of \( S \), the maps \( \varphi_{H_1,\ldots,H_k} \) are already isometries, because they are determined by the coordinates near the points of multiplicity \( k+1 \).

Thus we can pull back the metric of the normal bundle of \( S \) and interpolate between that and the metric on \( Y \) in a way that does not alter the metric near the strata we have already dealt with. In this way, going up one dimension at a time, we modify the metric on \( Y \) near all points of \( f(\Sigma) \). □

The stratum-wise retractions needed for Lemma 21 are constructed in coordinates using the following lemma.

**Lemma 24.** Consider the orthant \([0, \infty)^k\) of the Euclidean space \( \mathbb{R}^k \), and let \( \varepsilon > 0 \) be arbitrary. There is a smooth retraction \( r_{\mathbb{R}^k} \) from the \( \varepsilon \)-neighborhood of the boundary of the orthant to the boundary of the orthant, with the following properties:

- Every fiber of \( r_{\mathbb{R}^k} \) is a segment.
- The retraction \( r_{\mathbb{R}^k} \) commutes with any permutation of the coordinates.
• If \((x_1, \ldots, x_k) \in [0, \infty)^k\) is a point with last coordinate satisfying \(x_k \geq 2\varepsilon\), then the projection of \(r_k(x_1, \ldots, x_k)\) to the first \(k-1\) coordinates is \(r_{k-1}(x_1, \ldots, x_{k-1})\), and the \(k\)th coordinate is at least \(\varepsilon\). If in addition \(x_k \geq 3\varepsilon\), then the \(k\)th coordinate is equal to \(x_k\).

• If \((x_1, \ldots, x_k) \in [0, \infty)^k\) is a point such that the first \(\ell\) coordinates \(x_1, \ldots, x_\ell\) are all at most \(2\varepsilon\), then the projection of \(r_k(x_1, \ldots, x_k)\) to the first \(\ell\) coordinates is obtained by translating the point \((x_1, \ldots, x_\ell)\) in \([0, \infty)^\ell\) in the \((-1, \ldots, -1)\) direction until it hits the boundary.

Proof. We fix a smooth cutoff function \(t: [0, \infty) \to [0, 1]\) such that \(t([0, 2\varepsilon]) = \{0\}\), \(t([3\varepsilon, \infty)) = \{1\}\), and \(t\) is nondecreasing. For any point \((x_1, \ldots, x_k) \in [0, \infty)^k\), let \(\mu\) denote the minimum coordinate; we must define \(r_k\) on all such points with \(\mu < \varepsilon\). For such a point \((x_1, \ldots, x_k)\) with minimum coordinate \(\mu < \varepsilon\), we define \(r_k(x_1, \ldots, x_k)\) to be the point such that for \(i = 1, \ldots, k\) the \(i\)th coordinate is given by the expression

\[
t(x_i) \cdot x_i + (1 - t(x_i)) \cdot (x_i - \mu).
\]

In this formula, every coordinate \(x_i\) that is equal to \(\mu\) gets sent to zero, and every coordinate that is at least \(3\varepsilon\) stays the same. Furthermore, given \(r_k(x_1, \ldots, x_k)\) and \(\mu\) we can recover \((x_1, \ldots, x_k)\) because for fixed \(\mu\), each coordinate is monotonic in \(x_i\). Thus each fiber is a segment. The other properties follow immediately from the formula. \(\square\)

Proof of Lemma 21. We fix small numbers \(\varepsilon_1, \ldots, \varepsilon_{n-1} > 0\) such that the coordinates from Lemma 23 are isometries in a ball of radius \(3n\varepsilon_{n-1}\) around every point of \(f(\Sigma)\), and such that \(\varepsilon_{i+1} > 2\varepsilon_i\) for each \(i\). We also require \(\varepsilon_{n-1}\) to be small enough to satisfy a condition based on Lemma 26 that we specify later.

We define the stratification recursively, starting in dimension 0 and going up. On \(\tilde{Y}_{\text{nam}}\) and on any 0-dimensional strata of \(S_{\text{mult}}(\tilde{Y})\), the stratification \(S_{\text{fat}}(\tilde{Y})\) is the same as \(S_{\text{mult}}(\tilde{Y})\). Suppose that \(S\) is a \(k\)-dimensional stratum of \(S_{\text{mult}}(\tilde{Y})\) such that the stratification \(S_{\text{fat}}(\tilde{Y})\) has already been defined on the boundary \(\partial S\) but not on \(S\). Let \(S_{>\varepsilon_k}\) denote the points of \(S\) of distance greater than \(\varepsilon_k\) from the boundary. The connected components of \(S_{>\varepsilon_k}\) are strata of \(S_{\text{fat}}(\tilde{Y})\) and are not subdivided further.

On the remaining points of \(S\), we can define a retraction \(r\) onto \(\partial S\) by using the coordinates from Lemma 23 and patching the coordinate retractions from Lemma 24, each taken with parameter \(\varepsilon = \varepsilon_k\). Let \(S_{\leq \varepsilon_k}\) denote the points of distance less than \(\varepsilon_k\) from \(\partial S\), and let \(S_{= \varepsilon_k}\) denote those of distance \(\varepsilon_k\). The stratification \(S_{\text{fat}}(\tilde{Y})\) has already been defined on \(\partial S\), so suppose that \(S'\) is any such stratum of \(\partial S\). Then the connected components of \(r^{-1}(S') \cap S_{<\varepsilon_k}\) and the connected components of \(r^{-1}(S') \cap S_{=\varepsilon_k}\) are strata of \(S_{\text{fat}}(\tilde{Y})\). In this way, the stratification \(S_{\text{fat}}(\tilde{Y})\) is extended to all of \(S\).

Suppose that \(B\) is a stratum of \(S_{\text{fat}}(\tilde{Y})\) such that \(B\) is not in \(\tilde{Y}_{\text{nam}}\) but its boundary \(\partial B\) intersects \(\tilde{Y}_{\text{nam}}\). Let \(S\) be the stratum of \(S_{\text{mult}}(\tilde{Y})\) that contains \(B\). Then in order to touch \(\tilde{Y}_{\text{nam}}\), we know that \(B\) must be in \(S_{<\varepsilon_k}\). Then \(r(B)\) is some stratum \(B'\), and because \(\overline{B} = \overline{B} \cap \partial S\) we see that \(\overline{B'} \cap \tilde{Y}_{\text{nam}} = \partial B \cap \tilde{Y}_{\text{nam}}\). The geometry of \(r\) over \(B'\) comes directly from the construction of \(r\).
To construct $S_\Delta(\tilde{Y})$, we start by creating a stratification $S_{\text{fat}}(Y)$, obtained by starting with $S_{\text{mult}}(Y)$ and subdividing by the same process above, except without anything in the role of $Y_{\text{nam}}$. That is, for every stratum of $S_{\text{mult}}(Y)$ we push a neighborhood of the boundary to the boundary and pull back the stratification of the boundary. Using the assumption $\varepsilon_{i+1} > 2\varepsilon_i$ and the properties of the retraction from Lemma 24 we see that in coordinates every stratum can be nicely described as a polyhedron. This implies that $S_{\text{fat}}(Y)$ is a Whitney stratified space and therefore has a stratified triangulation (see [12] and [13]; the former includes a definition of stratified triangulation, and the latter includes a definition of Whitney stratified space). This triangulation of $Y$ pulls back to a triangulation $S_\Delta(\tilde{Y})$ of $\tilde{Y}$ that is a refinement of $S_{\text{fat}}(\tilde{Y})$.

In Lemmas 16 and 17 we have shown that strata of $S(X)$ are amenable if they do not neighbor $X_{\text{nam}}$ or are in $X_{\text{nam}}$. In Lemmas 25 and 27 we show that the strata neighboring $X_{\text{nam}}$ are amenable.

**Lemma 25.** Suppose that $B \in S_{\text{fat}}(\tilde{Y})$ is not contained in $X_{\text{nam}}$, but its closure intersects $X_{\text{nam}}$. Let $E$ be the preimage of $B$ in $X$, and suppose that $E$ intersects $\tilde{\Sigma}$. Then $E$ is amenable.

**Proof.** Let $F$ denote the fiber of $E$ over $B$. Because $E$ is not in $X_{\text{nam}}$ we know that $i_\ast \pi_1(F)$ is amenable. The proof of the lemma is by induction on the dimension of $B$. If $B$ is 0-dimensional, then $E = F$ and so there is nothing to prove.

Because $E$ intersects $\tilde{\Sigma}$, we can view $F$ as a 1-dimensional cell complex and apply Lemma 19, to get a trivial bundle $\hat{E} \to \hat{B}$ that is a finite-sheeted cover of the original bundle $E \to B$. Let $\hat{V} \subset \hat{E}$ be the product of $\hat{B}$ with a vertex in $F$, so that $\pi_1(\hat{E}) = \pi_1(F) \times \pi_1(\hat{V})$, and let $V$ be the image of $\hat{V}$ in $E$, so that $V \subseteq E \cap \tilde{\Sigma}$. We know that $i_\ast \pi_1(E)$ is a finite extension of $i_\ast \pi_1(\hat{E})$, and that $i_\ast \pi_1(\hat{E})$ is a quotient of $i_\ast \pi_1(F) \times i_\ast \pi_1(\hat{V})$, and that $i_\ast \pi_1(F)$ is amenable. Thus, to prove that $i_\ast \pi_1(E)$ is amenable, it suffices to show that $i_\ast \pi_1(\hat{V})$ is amenable.

To show that $i_\ast \pi_1(\hat{V})$ is amenable, it suffices to show that $i_\ast \pi_1(V)$ is amenable, which we do by showing that the inclusion $i: V \hookrightarrow X$ is homotopic to a map that lands in a stratum known to be amenable. The stratum $B \in S_{\text{fat}}(\tilde{Y})$ comes with a retraction $r$ onto a stratum $B' \subseteq \partial B$. We use $r$ to homotope $V$ into $\Sigma \cap f^{-1}B'$. (Here we abuse notation and use $f$ to denote the map $X \to \tilde{Y}$.) Specifically, we view $r$ as a homotopy on $B \cup B'$, and lift it to the covering space $V \cup (\tilde{V} \cap f^{-1}B')$ to get a homotopy that begins with $i: V \hookrightarrow X$ and ends with image in $\tilde{V} \cap f^{-1}B'$.

In the case that $B'$ is in $\tilde{Y}_{\text{nam}}$, then the image $\tilde{V} \cap f^{-1}B'$ is one of the strata in $X_{\text{nam}}$ lying over $B'$. By Lemma 17 this stratum is amenable, so $i_\ast \pi_1(V)$ is amenable. In the case that $B'$ is not in $\tilde{Y}_{\text{nam}}$, the inductive hypothesis guarantees that $f^{-1}B'$ must be amenable, so $i_\ast \pi_1(V)$ is amenable.

Because $i_\ast \pi_1(V)$ is amenable, then $i_\ast \pi_1(\hat{V})$ is amenable, so $i_\ast \pi_1(F) \times i_\ast \pi_1(\hat{V})$ is amenable, so $i_\ast \pi_1(F)$ is amenable, so $i_\ast \pi_1(E)$ is amenable, as desired.

The remaining strata that we want to show are amenable are those for which the closure intersects $X_{\text{nam}}$, but the stratum does not intersect $\tilde{\Sigma}$. In other words the
fiber is a circle. In this case we cannot directly use Lemma 19 to find a finite-sheeted cover that is a trivial bundle. Thus we do a little more work to assign vertices to the fibers. Roughly, as in Figure 4-1 the fibers above the higher-multiplicity points of \( Y \) are obtained from the fibers above lower-multiplicity points by pinching pairs of points together. These pairs of points become the vertices of the regular fibers near \( \Sigma \). The work is in making this choice in a continuous way across different fibers to get the right topological properties.

Using the purely folded property, we can sort the points in \( \tilde{\Sigma} \) according to the Morse index of the corresponding Morse function \( \mathbb{R}^2 \to \mathbb{R} \): either the index is 1, in which case the fiber looks locally like a cross, or it is not 1, in which case the fiber looks locally like an isolated point. Each connected component of \( \tilde{\Sigma} \) is either entirely index 1 or entirely index not 1. Let \( \tilde{\Sigma}_1 \) denote the set of index-1 critical points in \( X \).

Let \( U \) be a small tubular neighborhood of \( \tilde{\Sigma}_1 \). Then we can construct a manifold \( \tilde{Y}_U \) such that the restriction \( f|_U : U \to Y \) factors through \( \tilde{Y}_U \), the map from \( \tilde{\Sigma}_1 \) to \( \tilde{Y}_U \) is an embedding, and the map \( \tilde{Y}_U \to Y \) is an immersion. To do this, we cover \( U \) by many small balls \( B_i \), and let \( \tilde{Y}_U \) be the space covered by balls \( f(B_i) \) such that balls \( f(B_1) \) and \( f(B_2) \) are glued on their overlap whenever \( B_1 \) and \( B_2 \) intersect in \( U \). We can pull back the metric on \( Y \) to obtain a metric on \( \tilde{Y}_U \).

**Lemma 26.** The numbers \( d > 0 \) and \( \delta > 0 \) may be chosen small enough that the following is true. Let \( U \subseteq X \) denote the \( d \)-neighborhood of \( \tilde{\Sigma}_1 \), which is tubular if \( d \) is small enough, and let \( \tilde{Y}_U \) be constructed as above. If \( x, p \in U \) have images in \( \tilde{Y}_U \) that are within distance \( \delta \) of each other, then the distance between \( x \) and \( p \) is less than the injectivity radius of \( X \).

**Proof.** Let \( \rho \) denote the injectivity radius of \( X \). We choose \( d \) such that if two points of \( U \) map to the same point of \( \tilde{Y}_U \), then their distance in \( X \) is less than \( \frac{\rho}{4} \).

Then to choose \( \delta \), because \( f \) is purely folded we see that \( f|_U \) is an open mapping. Thus the map \( U \times U \to \tilde{Y}_U \times \tilde{Y}_U \) is also an open mapping. The open subset of \( U \times U \) of pairs of points within distance \( \frac{\rho}{4} \) has open image that contains the diagonal. Thus there exists \( \delta \) such that the subset of \( \tilde{Y}_U \times \tilde{Y}_U \) of pairs of points within distance \( \delta \) is contained in this open image.

That is, if \( x, p \in U \) have images in \( \tilde{Y}_U \) of distance less than \( \delta \), there are \( x', p' \in U \) in the same fibers as \( x \) and \( p \) respectively, such that \( \text{dist}(x', p') < \frac{\rho}{2} \). But then the choice of \( d \) guarantees that \( \text{dist}(x, x') < \frac{\rho}{4} \) and \( \text{dist}(p, p') < \frac{\rho}{4} \), so in total we have \( \text{dist}(x, p) < \rho \).

In Lemma 21 where we construct \( S_{\text{fat}}(\tilde{Y}) \), we should make sure to choose the parameter \( \varepsilon_{n-1} \) such that \( \sqrt{n} \cdot \varepsilon_{n-1} < \delta \), so that each retraction \( r : B \cup B' \to B' \) moves each point by less than distance \( \delta \). We also need to choose \( \varepsilon_{n-1} \) such that if \( U \) is the \( d \)-neighborhood of \( \tilde{\Sigma}_1 \) as above, then the image of \( U \) in \( \tilde{Y} \) contains an \( n\sqrt{n} \cdot \varepsilon_{n-1} \)-neighborhood of the image of \( \tilde{\Sigma}_1 \). This assumption ensures that if a stratum of \( S_{\text{fat}}(\tilde{Y}) \) has closure that intersects \( \tilde{Y}_{\text{ram}} \), then the stratum is contained in the image of \( U \). These two assumptions about the construction of \( S_{\text{fat}}(\tilde{Y}) \) are needed for the following lemma.
Lemma 27. Suppose that $B \in \mathcal{S}_{\text{fat}}(\hat{Y})$ is not contained in $X_{\text{nam}}$, but its closure intersects $X_{\text{nam}}$. Let $E$ be the preimage of $B$ in $X$, and suppose that $E$ does not intersect $\hat{\Sigma}$. Then $E$ is amenable.

Proof. Let $d$ be as in Lemma 26, and $U$ be the $d$-neighborhood of $\hat{\Sigma}_1$. The points in $U$ that are not in the same fiber as a point of $\hat{\Sigma}_1$ (that is, points not in $f^{-1}(\hat{\Sigma}_1)$) are partitioned into intervals which are the connected components of their fibers over $\hat{Y}_U$. Let $W$ denote the set in $U$ obtained by taking the midpoint of every such interval.

As in Lemma 25, let $F$ denote the connected fiber of $E$ over $B$. Because $E$ is a union of connected fibers, and its closure intersects $X_{\text{nam}}$, it must intersect $U$ and therefore $W$. Let $V$ be a connected component of $E \cap W$. Then $V$ is a finite-sheeted covering space of $B$, so if we consider $V$ to be the vertices of the fibers, then the bundle $E \to B$ has local trivializations that respect the cell structure on $F$ given by taking $F \cap V$ to be the vertices. Thus, as in Lemma 25 we can define $\hat{B}$ and $\hat{E}$, and as before in order to show that $i_*\pi_1(E)$ is amenable, it suffices to show that $i_*\pi_1(V)$ is amenable, which we do by homotoping the inclusion $i: V \to X$ into a stratum that we know to be amenable.

We have a retraction $r: B \cup B' \to B'$. Because $B'$ is contained in $f(\hat{\Sigma})$ and has closure intersecting $\hat{Y}_{\text{nam}}$, Lemmas 17 and 25 guarantee that $\hat{\Sigma} \cap f^{-1}B'$ is part of an amenable subset of $X$. Therefore it suffices to homotope $V$ into this set. Note that the set $\hat{\Sigma} \setminus \hat{\Sigma}_1$ of critical points of index not 1 is a closed submanifold of $X$ that embeds into $\hat{Y}$ and is disjoint from $\hat{Y}_{\text{nam}}$—the fiber is always a single point—so the part of $\hat{\Sigma}$ that maps into $B \cup B'$ is contained entirely in $\hat{\Sigma}_1$.

The homotopy taking $i: V \to X$ to a map with image in $\hat{\Sigma}_1 \cap f^{-1}B'$ relies on Lemma 26 about the injectivity radius. We construct a map $r_V: V \to X$ with image in $\hat{\Sigma}_1 \cap f^{-1}B'$ such that for all $x \in V$ the points $x$ and $r_V(x)$ have images in $\hat{Y}_U$ of distance less than $\delta$. Then by Lemma 26 we see that $x$ and $r_V(x)$ have distance less than the injectivity radius of $X$, so we can homotope $i$ to $r_V$ by moving in a straight line. Specifically, to construct $r_V(x)$, we start with the path in $\hat{Y}$ given by a fiber of $r$ that starts with the image of $x$ and ends in $B'$. Then the image of this path in $Y$ lifts to a path in $\hat{Y}_U$ starting with the image of $x$ and ending in the image of $\hat{\Sigma}_1$. We define $r_V(x)$ to be the point in $\hat{\Sigma}_1$ that maps to this second endpoint of the path in $\hat{Y}_U$. The path has length less than $\delta$, so Lemma 26 applies, and $V$ is homotoped into an amenable stratum, implying that $i_*\pi_1(V)$ is amenable.

Proof of construction of $S(X)$ (Lemma 20). To review: we construct $S(X)$ as follows. We start with $\mathcal{S}_{\text{fat}}(\hat{Y})$, constructed in Lemma 21. The strata of $\mathcal{S}_{\text{fat}}(\hat{Y})$ for which the closures do not intersect $\hat{Y}_{\text{nam}}$ form a closed subset of $\hat{Y}$, and on this closed subset we substitute the finer stratification $\mathcal{S}_\Delta(\hat{Y})$, constructed in Lemma 21. The result is the stratification $\mathcal{S}(\hat{Y})$ of $\hat{Y}$. We pull back the stratification $\mathcal{S}(\hat{Y})$ to $X$ by taking the preimage of each stratum in $\hat{Y}$ and then taking connected components. Then on the closed subset $X_{\text{nam}}$ we substitute the finer stratification $\mathcal{S}_{\text{mult}}(X)$, which is the refinement of $\mathcal{S}_{\text{mult}}(X)$ in which the critical points $\hat{\Sigma}$ are separated from the regular points. The result is the stratification $\mathcal{S}(X)$.

All of the desired properties except for property (5) are immediate from the construction. The homotopies of the strata neighboring $\hat{Y}_{\text{nam}}$ into the strata of $\hat{Y}_{\text{nam}}$
are obtained by iterating the retractions \( r \) from Lemma 21, which can be viewed as deformation retractions. And, the amenability of the strata neighboring \( X_{\text{nam}} \) is the content of Lemmas 25 and 27.

**Proof of main lemma for Gromov singular fiber theorem (Lemma 14).** Lemmas 16, 17, 25, and 27 show that every stratum of \( S(X) \) is amenable. Let \( P \) be the set of points of \( X_{\text{nam}} \cap \hat{\Sigma} \) that map to multiplicity \( n - 1 \) points. By construction each point of \( P \) is a stratum, and these are the only 0-dimensional strata in \( X_{\text{nam}} \).

Suppose that \( S_0, \ldots, S_n \) is a totally ordered chain of strata, where \( S_0 \) is the smallest and \( S_n \) is the largest. We want to show that \( S_0 \) is a point of \( P \). Suppose first that none of \( S_0, \ldots, S_n \) is contained in \( X_{\text{nam}} \). Then they are all pullbacks of strata in \( \hat{Y} \); this is impossible because their images in \( \hat{Y} \) would form a totally ordered chain with dimension increasing at every step, and \( \hat{Y} \) is only \((n - 1)\)-dimensional.

So, some stratum \( S_t \) is contained in \( X_{\text{nam}} \), which implies that all preceding strata are also in \( X_{\text{nam}} \), because \( X_{\text{nam}} \) is closed. Similarly, because \( \hat{\Sigma} \) is closed, there exist \( k, \ell \) with \( k \leq \ell \) and \( 0 \leq \ell \leq n \) such that strata \( S_0 \) through \( S_k \) are in \( X_{\text{nam}} \cap \hat{\Sigma} \), strata \( S_{k+1} \) through \( S_{\ell} \) are in \( X_{\text{nam}} \setminus \hat{\Sigma} \), and the remaining strata are not in \( X_{\text{nam}} \). Taking images in \( \hat{Y} \), we obtain a totally ordered chain in \( \hat{Y} \) for which the only possible repetition is between the images of \( S_k \) and \( S_{k+1} \). By keeping track of dimensions in \( \hat{Y} \) we see that this repetition must occur, so \( S_0 \) is a 0-dimensional stratum of \( X_{\text{nam}} \) and thus must be in \( P \).

To find the stratified neighborhood of a point \( p \in P \), we start by looking at the fiber \( F \) over its image \( y \) in \( \hat{Y} \). The fiber \( F \) is a 1-dimensional cell complex with at most \( n - 1 \) vertices and exactly 4 edges at each vertex; thus, there are finitely many possibilities for \( F \). The shape of \( F \) determines its neighborhood in \( X \), which comes from unpinching the vertices in all possible ways, so it also determines the neighborhood of \( y \) in \( \hat{Y} \). The stratification of \( \hat{Y} \) near \( y \) depends only on which strata of \( S_{\text{mult}}(\hat{Y}) \) at \( y \) are in \( \hat{Y}_{\text{nam}} \); there are finitely many possibilities for this set of local strata. This information then completely determines the stratification of \( X \) in a neighborhood of \( F \), and thus determines the stratification near \( p \).

It remains to show that every stratum \( S \) of \( S(X) \) has the NDR property. In the case that \( S \) is a submanifold of \( X \), we know that \( S \) is a deformation retract of a tubular neighborhood. This is the case if \( S \) is in \( X_{\text{nam}} \) or in \( \hat{\Sigma} \setminus \hat{\Sigma}_1 \), or if \( S \) is disjoint from \( \hat{\Sigma} \). In the remaining case, \( S \) is a union of connected fibers intersecting \( \hat{\Sigma}_1 \); we know that \( S \cap \hat{\Sigma}_1 \) is a submanifold of \( \hat{\Sigma}_1 \) and that \( S \setminus \hat{\Sigma}_1 \) is a submanifold of \( X \). In a tubular neighborhood of \( S \setminus \hat{\Sigma}_1 \) we can already retract by taking the closest point. We want to define a retraction on a neighborhood of \( S \cap \hat{\Sigma}_1 \) that agrees with the closest-point retraction on the overlap. We use the same strategy to define the retraction as when constructing \( S_{\text{rat}}(\hat{Y}) \): we use the coordinates from Lemma 22 and the coordinate retraction from Lemma 24.

Specifically, we first modify the metric on \( X \) so that the tubular neighborhood of \( S \cap \hat{\Sigma}_1 \) in \( \hat{\Sigma}_1 \) is isometric to the corresponding neighborhood of the normal bundle. Then we consider a small tubular neighborhood \( U \) of \( \hat{\Sigma}_1 \) in \( X \). Under the map to \( \hat{Y}_U \), locally in \( U \) the singular fibers form a pair of transverse embedded hypersurfaces \( H_1 \) and \( H_2 \) in \( U \). Thus we can modify the metric on \( X \) again so that the maps \( \varphi_{H_1,H_2} \)
from Lemma 22 are isometries near every point of $\hat{\Sigma}_1$.

Under this new metric on $X$, in a neighborhood of $S \cap \hat{\Sigma}_1$ we have coordinates $(p, v, d_1, d_2)$ where $p$ is a point in $S \cap \hat{\Sigma}_1$, the vector $v$ at $p$ is tangent to $\hat{\Sigma}_1$ but normal to $S \cap \hat{\Sigma}_1$, and $d_1$ and $d_2$ are the distances to the two local sheets $H_1$ and $H_2$. Using the coordinate retraction $r_{\mathbb{R}^2}$ from Lemma 24, we define a retraction given by

$$r(p, v, d_1, d_2) = (p, 0, r_{\mathbb{R}^2}(d_1, d_2))$$

to send a neighborhood of $S \cap \hat{\Sigma}_1$ to $S$. In the outer part of this neighborhood the new retraction agrees with taking the closest point of $S$, so we can combine it with the closest-point map to get a deformation retraction of a small neighborhood of $S$ onto $S$.

**Proof of Gromov singular fiber theorem (Theorem 5).** We apply the localization lemma (Lemma 13) to the stratification $S(X)$ on $X$. Because $X$ admits a metric of negative sectional curvature, it is aspherical, so we may take $\alpha$ to be the identity on $X$. We take $h$ to be the fundamental homology class $[X]$, and take the closed set $A$ to be the complement of a small neighborhood of the points in $P$. Then the essential stratified simplicial norm $\|[X]_{rel}\|^{S}_\Delta, ess$ is bounded by a constant times the number of points in $P$, where the constant is the maximum over all stratified model neighborhoods. There are at most $n - 1$ points in $P$ above each multiplicity $n - 1$ point in $Y$, so we obtain

$$\|[X]\|_\Delta \leq \text{const}(n) \cdot \#(\text{multiplicity } n - 1 \text{ self-intersections of } f(\hat{\Sigma})).$$
Chapter 5

Multitangent trajectory theorem

The main lemma for the multitangent trajectory theorem is Lemma 28, which describes the nice properties of a traversally generic vector field and which is a consequence of the work of Katz in the paper [20]. That paper introduces the definition of traversally generic (Definition 3.2) and proves that the traversally generic vector fields form an open dense subset of the traversing vector fields (Theorem 3.5). The machinery behind the proof of density comes from the theory of singularities of generic maps, in particular from Thom-Boardman theory (see Theorem 5.2 from Chapter VI of [11]). Below, before stating the main lemma (Lemma 28) we give the definitions of traversally generic vector fields and the reduced multiplicity of a trajectory.

The definition of traversally generic includes the notion of boundary generic (Definition 2.1 in [20]), which is defined as follows. Given a traversing vector field $v$ on $X$, we let $\partial_2 X$ denote the set of points where $v$ is tangent to $\partial X$. Alternatively, we view $v|_{\partial X}$ as a section of the normal bundle $TX/T\partial X$ of $\partial X$ in $X$, and let $\partial_2 X$ be the zero locus. If the section corresponding to $v$ is transverse to the zero section, then $\partial_2 X$ is a submanifold of $\partial X$ with codimension 1. Then we repeat the process using the following iterative construction. Let $\partial_0 X = X$ and $\partial_1 X = \partial X$. Once the submanifolds $\partial_j X$ have been defined for all $j \leq k$, we view $v|_{\partial_k X}$ as a section of the normal bundle of $\partial_k X$ in $\partial_{k-1} X$, and if it is transverse to the zero section, then the zero locus $\partial_{k+1} X$ is a submanifold of $\partial_k X$ with codimension 1. We say $v$ is boundary generic if for all $k$, when we view $v|_{\partial_k X}$ as a section of the normal bundle of $\partial_k X$ in $\partial_{k-1} X$, this section is transverse to the zero section.

If $v$ is boundary generic, then the multiplicity $m(a)$ of any point $a \in X$ is defined to be the greatest $j$ such that $a \in \partial_j X$. By definition, if $m(a) = j > 0$, this means that $v$ is tangent to $\partial_{j-1} X$ at $a$ but not tangent to $\partial_j X$ there. Because each $\partial_j X$ has dimension $n + 1 - j$, the greatest possible multiplicity is $m(a) = n + 1$.

Being traversally generic is a property of each trajectory $\gamma$ of $v$. Using the $v$–flow along $\gamma$, we may identify all fibers of the normal bundle of $\gamma$ in $X$; we denote the resulting quotient by $T_\gamma$, so that $T_\gamma$ is an $n$–dimensional vector space. For each point $a_i \in \gamma \cap \partial X$, the tangent space $T\partial_{m(a_i)} X$ is transverse to $\gamma$, so it can be viewed as a subspace $T_i \subseteq T_\gamma$. We say that a traversing vector field $v$ is traversally generic if $v$ is boundary generic and if for every trajectory $\gamma$ of $v$, the collection of subspaces
\{T_i(\gamma)\}_i is generic in \(T_*\); that is, the quotient map

\[ T_* \to \bigoplus_{a_i \in \gamma \cap \partial X} T_*/T_i \]

is surjective. Equivalently, for every subcollection of the subspaces, the sum of their codimensions is equal to the codimension of their intersection (and is in particular nonnegative). Recall that the \textbf{reduced multiplicity} of every trajectory \(\gamma\) is the sum over all \(a_i \in \gamma \cap \partial X\) of \(m(a_i) - 1\). Thus, because \(\dim T_* = n\) and \(\dim T_i = n - (m(a_i) - 1)\), the property of being traversally generic implies \(m'(\gamma) \leq n\).

Let \(\mathcal{T}(v)\) denote the space of trajectories of a vector field \(v\) on \(X\), and let \(\Gamma: X \to \mathcal{T}(v)\) denote the quotient map. The main lemma (Lemma 28) describes how every traversally generic vector field \(v\) gives rise to stratifications of \(\mathcal{T}(v)\) and of \(X\).

\textbf{Lemma 28} (Main lemma for multitangent trajectory theorem). \textit{Let} \(X\) \textit{be a compact manifold with boundary, with} \(\dim X = n + 1\). \textit{The traversally generic vector fields} \(v\) \textit{on} \(X\) \textit{satisfy the following properties:}

1. \textit{For} \(k = 0, \ldots, n\), \textit{define} \(Y_k \subseteq \mathcal{T}(v)\) \textit{by}

\[ Y_k := \{\gamma \in \mathcal{T}(v): m'(\gamma) = n - k\}. \]

Then \(Y_k\) is a \(k\)-dimensional manifold, the connected components of all \(Y_k\) constitute a stratification of \(\mathcal{T}(v)\), and the boundary of each stratum is a union of smaller-dimensional strata.

2. \textit{Let} \(S\) \textit{be any stratum of} \(\mathcal{T}(v)\). \textit{Then the restriction} \(\Gamma\mid: \Gamma^{-1}(S) \to S\) \textit{has the structure of a trivial bundle with fiber equal to either an interval or a point; and the restriction} \(\Gamma\mid: \partial X \cap \Gamma^{-1}(S) \to S\) \textit{has the structure of a finite covering space.}

3. \textit{For} \(k = 0, \ldots, n\), \textit{define} \(X^0_k \subseteq \partial X\) \textit{by}

\[ X^0_k := \Gamma^{-1}(Y_k) \cap \partial X, \]

\textit{and for} \(k = 1, \ldots, n + 1\), \textit{define} \(X^c_k \subseteq X \setminus \partial X\) \textit{by}

\[ X^c_k := \Gamma^{-1}(Y_{k-1}) \setminus \partial X. \]

Then \(X^0_k\) and \(X^c_k\) is a \(k\)-dimensional submanifold, the connected components of all \(X^0_k\) and \(X^c_k\) constitute a stratification of \(X\), and the boundary of each stratum is a union of smaller-dimensional strata.

4. \textit{There is a finite collection, depending only on the dimension} \(n + 1\) \textit{and not on} \(v\) \textit{or} \(X\), \textit{of stratified local models covering} \(X\). \textit{That is, each local model is an} \((n + 1)\)-\textit{dimensional stratified space with finitely many strata, and every point in} \(X\) \textit{has a neighborhood diffeomorphic to one of the local models in a way that preserves the stratification.}
The paper [20] proves (Theorem 3.1) an equivalent characterization of traversally generic vector fields, called versal vector fields (Definition 3.5); the main ingredient in the proof is the Malgrange preparation theorem (see, for instance, Theorem 2.1 from Chapter IV of [11]). When we use the description of versal vector fields, our main lemma (Lemma 28) becomes straightforward. In the discussion below, we define versal vector fields and explain why they satisfy the properties in the main lemma.

For a vector field to be versal means that in a neighborhood of each trajectory there are local coordinates of a certain form. In preparation for the definition of a versal vector field, we first define local coordinates near one point. For each \( m \) with \( 1 \leq m \leq n+1 \), we use variables \( u \in \mathbb{R} \), \( \bar{x} = (x_0, \ldots, x_{m-2}) \in \mathbb{R}^{m-1} \), and \( y' \in \mathbb{R}^{n-(m-1)} \), and define

\[
P_m(u, \bar{x}) = u^m + \sum_{\ell=0}^{m-2} x_\ell u^\ell = u^m + x_{m-2}u^{m-2} + \cdots + x_1u + x_0.
\]

We consider the vector field \( \frac{\partial}{\partial u} \) on the space

\[
X_+ = \{(u, \bar{x}, y') : P_m(u, \bar{x}) \geq 0 \}
\]

or on the space

\[
X_- = \{(u, \bar{x}, y') : P_m(u, \bar{x}) \leq 0 \}.
\]

The trajectories above each fixed \((\bar{x}, y')\) stretch between the roots of \( P_m(u, \bar{x}) \) as a function of \( u \). If \( m \) is odd, then \( X_+ \) has unbounded trajectories in the positive direction (that is, \( u \to +\infty \)), and \( X_- \) has unbounded trajectories in the negative direction (\( u \to -\infty \)). If \( m \) is even, then \( X_+ \) has unbounded trajectories in both directions, and \( X_- \) has only bounded trajectories. In particular, if \( m \) is even, then the trajectory in \( X_- \) through the point \((u, \bar{x}, y') = 0\) is only that one point. The vector fields in these local models are boundary generic, and the multiplicity of each point \((u, \bar{x}, y')\) in the sense defined earlier is equal to the multiplicity of vanishing of \( P_m(u, \bar{x}) \) as a function of \( u \).

A **versal vector field** is described by local coordinates in a neighborhood of each trajectory \( \gamma \), as follows. Suppose \( \gamma \) enters \( X \) at \( a_1 \in \partial X \), and then meets \( \partial X \) at \( a_2, \ldots, a_p \in \partial X \), in order, exiting at \( a_p \). For each \( i \) with \( 1 \leq i \leq p \), let \( x_i \) denote a variable in \( \mathbb{R}^{m(a_i)-1} \), and let \( y' \) denote a variable in \( \mathbb{R}^{n-m'(\gamma)} \). Then the coordinates are

\[
(u, x_1, \ldots, x_p, y') \in \mathbb{R}^{n+1},
\]

and \( X \) corresponds to the subset

\[
\{(u, x_1, \ldots, x_p, y') : P_{m(a_i)}(u - i, x_i) \geq 0 \ \forall i < p, \ P_{m(a_p)}(u - p, x_p) \leq 0 \},
\]

and \( v \) corresponds to the vector field \( \frac{\partial}{\partial u} \). The trajectory \( \gamma \) corresponds to the line \((x_1, \ldots, x_p, y') = 0\), and the points \( a_1, \ldots, a_p \) correspond to the points \( u = 1, \ldots, p \). Note that for \( X \) to be nonempty, we must have either \( p = 1 \) and \( m(a_p) \) is even, or \( p > 1 \) and both \( m(a_1) \) and \( m(a_p) \) are odd while all other \( m(a_i) \) are even.
Proof of main lemma for multitangent trajectory theorem (Lemma 28). Because every traversally generic vector field is versal (Theorem 3.1 of [20]), it suffices to check Lemma 28 for the versal vector fields. Part 4 is immediate: there is one local model for each sequence of multiplicities corresponding to reduced multiplicity at most \( n \). Parts 1 and 3 follow from examining the local models: near each trajectory \( \gamma \), the only trajectories with reduced multiplicity equal to \( m'(\gamma) \) are those with all coordinates \( x_i \) equal to 0 (with \( y \) and \( u \) varying). To prove Part 2, we see from the local models that \( \Gamma \) is a locally trivial bundle map over each stratum of \( T(v) \), with fiber equal to either an interval or a point. Then, because the interval is oriented, and every bundle of oriented intervals is trivial, the bundle must be trivial. \( \square \)

**Proof of multitangent trajectory theorem, general version (Theorem 8).** From Part 3 of the main lemma (Lemma 28), the vector field \( v \) gives rise to a stratification of \( X \); doubling this stratification produces a stratification of the closed manifold \( D(X) \) by submanifolds. From Part 4 of the main lemma, there are only finitely many strata, because the compact set \( X \) can be covered by finitely many neighborhoods each matching one of the local models.

In order to apply the localization lemma (Lemma 13), we need to check that for each stratum \( S \) of \( X \), the subgroup \( \alpha_\ast \pi_1(S) \) of \( \pi_1(Z) \) is an amenable group. We have assumed that this is true if \( S \subseteq \partial X \). (Every subgroup of an amenable group is amenable.) Otherwise, we apply Parts 2 and 3 of the main lemma: \( S \) is one connected component of \( \Gamma^{-1}(\sigma) \setminus \partial X \) for some stratum \( \sigma \) of \( T(v) \), and the entire preimage \( \Gamma^{-1}(\sigma) \) is a trivial bundle \( \sigma \times F \), for some fiber \( F \). Under the stratification of \( X \), the fiber \( F \) is an interval subdivided by finitely many points from \( \partial X \). There is a homotopy on the 1-dimensional part of \( F \) that pushes each open subinterval to the next point of \( \partial X \), which gives a homotopy on \( \Gamma^{-1}(\sigma) \setminus \partial X \) that starts with the inclusion into \( X \) and ends with a map into \( \partial X \). Applying this homotopy to loops in \( S \) we see that \( \pi_1(S) \) is contained in \( \pi_1(\partial X) \), so its \( \alpha \)-image is an amenable group.

Now we apply the localization lemma (Lemma 13) to the space \( D(X) \), with \( j = \dim X = n + 1 \). We take the closed set \( A \) to be the complement of a small neighborhood of the 0-dimensional strata, which are the intersections of the maximum-multiplicity trajectories with \( \partial X \). Let \( x_1, \ldots, x_r \) denote these 0-dimensional strata; then we have

\[
r \leq (n + 2) \cdot \max \text{-mult}(v),
\]

because each trajectory has at most \( n \) intermediate points of \( \partial X \), and \( n + 2 \) points in total. Applying Part 4 of the main lemma, around each point \( x_i \) we choose a neighborhood \( U_i \subseteq X \) matching one of the local models, small enough that the various \( U_i \) are disjoint, and let \( D(U_i) \subseteq D(X) \) denote the double of \( U_i \). We take \( U = \bigcup_{i=1}^r D(U_i) \), and take \( A = D(X) \setminus U \). If \( S \) denotes the stratification on \( D(X) \), then there exists some constant \( C_n \), depending only on \( n \), satisfying

\[
\|(D(U_i), \partial D(U_i))\|_{\Delta, \text{ess}}^S \leq C_n
\]
for all $i$. Thus, the conclusion of the localization lemma gives
\[ \|\alpha_\ast [D(X)]\|_\Delta \leq \|[D(X)]\|_{rel} \leq \sum_{i=1}^r C_n \leq C_n \cdot (n + 2) \cdot \text{max-mult}(v). \]

\[ \square \]

**Proof of multitanget trajectory theorem, specific version (Theorem 7).** We construct a degree-1 map $\alpha : D(X) \to M$ that sends all of the boundary $\partial X$ to a single point. Given such a map, we have $\alpha_\ast \pi_1(\partial X) = 0$ (and $0$ is an amenable group), so the general version (Theorem 8) gives
\[ \text{max-mult}(v) \geq \text{const}(n)\|\alpha_\ast [D(X)]\|_\Delta = \text{const}(n)\|[M]\|_\Delta \geq \text{const}(n) \cdot \text{Vol} M, \]
where the value of $\text{const}(n)$ is not fixed and may change between inequalities, but is always positive. To construct $\alpha$, let $B$ be an open ball containing $\overline{U}$, and let $B'$ be a slightly smaller ball with $\overline{U} \subset B' \subset \overline{B'} \subset B$. There is a degree-1 map $M \to M$ obtained by collapsing $\overline{B'}$ to a single point $\ast$, and stretching the cylinder $B \setminus B'$ to fill $B \setminus \ast$. We define $\alpha$ on $X$ as the restriction of this map on $M$, and define $\alpha$ on the second copy of $X$ as the constant map at $\ast$. Then $\alpha$ on all of $D(X)$ has degree 1, and we have $\alpha(\partial X) = \ast$. \[ \square \]
Chapter 6

Morse broken trajectory theorem

We begin by briefly summarizing the significance of the Morse–Smale condition; all of this information can be found in the book [3]. An example of a Morse function that is not Morse–Smale is obtained by taking the height function of a torus $T^2$ stood on end, as in Figure 6-1. The pair of index–1 critical points violates the transversality condition, because there is flow between them: the descending manifold of the upper point and the ascending manifold of the lower point both have dimension 1, and their intersection in the 2-dimensional surface has dimension 1, so it is not transverse. More generally, if $D(p)$ and $A(p)$ have a transverse and non-empty intersection, then the intersection has dimension $\text{ind}(p) - \text{ind}(q)$, which in particular must be strictly positive because the intersection contains 1-dimensional flow lines. Thus, in a Morse–Smale flow, for every flow line the starting critical point must have index strictly greater than the ending critical point.

If we tilt the torus slightly, then the height function becomes Morse–Smale. In fact, every gradient vector field may be approximated in $C^\infty$ by a Morse–Smale gradient vector field (this is called the Kupka–Smale theorem), and every $C^1$–small perturbation of a Morse–Smale gradient vector field is still Morse–Smale (this is a theorem of Palis). On the tilted torus, each of the index–1 critical points has two flow lines up to the index–2 critical point and two flow lines down to the index–0 critical point, as shown in Figure 6-1, giving eight 2-part broken trajectories in all. (Of course, the Morse broken trajectory theorem says nothing about this example because the torus has zero simplicial volume.) It is not obvious in general that there are only finitely many flow lines between critical points of index difference 1; this theorem is part of a larger family of compactness results, which say that the space of flow lines can be compactified by adding in the broken trajectories (with some appropriate notion of convergence). The usual reason to count flow lines between critical points of index difference 1 is because they define the differential in the Morse–Smale–Witten chain complex, which is a chain complex in which the set of $k$–chains is freely generated by the critical points of index $k$. The homology of the Morse–Smale–Witten complex is called Morse homology, and it is isomorphic to the singular homology of the manifold. In that setting the flow lines are counted with signs determined by orientations of the descending manifolds. In this paper we count the flow lines without any sign and do not use Morse homology.
Figure 6-1: The height function of a standard torus stood on end, shown on the left, is not Morse–Smale because there is flow between the two index–1 critical points. A slight perturbation of the torus, shown on the right, gives a Morse–Smale function with eight 2-part broken trajectories. (The picture is modeled on one in [3].)

Now we turn from the background to the specific theorems of Morse theory that are needed for the Morse broken trajectory theorem. Let \((M, g)\) be a closed Riemannian manifold, and let \(f: (M, g) \to \mathbb{R}\) be a Morse function satisfying Morse-Smale transversality. We would like to say that the descending manifolds \(D(p)\) are the open cells of a nice CW-complex structure on \(M\), in which the incidences between cells correspond somehow to breaking of trajectories and there is a triangulation that agrees with the CW-complex structure. This wish is mostly true, but in the proof, the necessary analysis requires a special condition at the critical points.

The pair \((f, g)\) is said to be Euclidean if at every critical point \(p\), there is a neighborhood with coordinates \((x_1, x_2) \in \mathbb{R}^{\text{ind}(p)} \times \mathbb{R}^{n-\text{ind}(p)}\) such that \(g\) is equal to the standard Euclidean metric and \(f\) is given by

\[
f(x_1, x_2) = f(p) - \frac{1}{2} |x_1|^2 + \frac{1}{2} |x_2|^2.
\]

Here, \(p\) has coordinates \((x_1, x_2) = (0, 0)\). We reduce the general case to the Euclidean case, and then work only in the Euclidean case; in [30], Wehrheim justifies this reduction, which was stated by Franks in [8] (or, see [26] for a longer discussion).

**Theorem 29** (Remark 3.6 in [30]). Let \(M\) be a closed manifold, and let \(\Psi_s\) be the flow along some Morse–Smale negative gradient vector field. Then there is a homeomorphism \(h: M \to M\) such that \(h\Psi_s h^{-1}\) is the flow along a Morse–Smale negative gradient vector field which in addition is Euclidean.

In particular, the number of \(n\)-part broken trajectories of \(h\Psi_s h^{-1}\) is equal to the number of \(n\)-part broken trajectories of \(\Psi_s\), so to prove the Morse broken trajectory theorem it suffices to consider the case where \((f, g)\) is Euclidean. The main lemma is the following.
Lemma 30 (Main lemma for Morse broken trajectory theorem). Let \((M, g)\) be a closed Riemannian manifold of dimension \(n\), and suppose \(f: (M, g) \to \mathbb{R}\) is Morse–Smale and Euclidean. Let \(S\) be the partition into descending manifolds \(D(p)\). Then \(M\) is a CW-complex for which the generalized stratification by open cells is equal to \(S\), and for each descending manifold \(D(p)\) of dimension \(n\), the corresponding relative homology class \([D(p)] \in H_n(M, M^{n-1})\) satisfies the bound

\[
||[D(p)]||_{S, \text{ess}}^S \leq \#(n\text{-part broken trajectories beginning at } p).
\]

We use Lizhen Qin’s exposition [25] as the reference for the Morse theory results; it presents self-contained proofs for a collection of Morse theory folk theorems. In order to make \(M\) into a CW-complex, the goal is to construct spaces \(\overline{D(p)}\) each homeomorphic to a closed ball with interior \(D(p)\), and maps \(\overline{D(p)} \to M\) for which the restriction to \(D(p)\) is the inclusion. In the following discussion we fix notation necessary for describing the structure of \(\overline{D(p)}\).

For any two critical points \(p\) and \(q\), there is a smooth manifold \(\mathcal{M}(p, q)\) which is the set of all unparametrized flow lines from \(p\) to \(q\). Let \(I = \{r_0, r_1, \ldots, r_{k+1}\}\) be a list of critical points, with

\[
\text{ind}(r_0) > \text{ind}(r_1) > \cdots > \text{ind}(r_{k+1}).
\]

The product manifolds \(\mathcal{M}_I\) and \(\mathcal{D}_I\) are defined by

\[
\mathcal{M}_I = \prod_{i=0}^k \mathcal{M}(r_i, r_{i+1}), \quad \mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_{k+1}).
\]

The length of \(I\) is defined to be \(|I| = k\), which is two less than the number of critical points in \(I\). Then the spaces \(\mathcal{M}(p, q)\) and \(\mathcal{D}(p)\) can be compactified as

\[
\overline{\mathcal{M}(p, q)} = \bigsqcup_I \mathcal{M}_I, \quad \overline{\mathcal{D}(p)} = \bigsqcup_I \mathcal{D}_I,
\]

where in \(\overline{\mathcal{M}(p, q)}\) each \(I\) is required to begin with \(p\) and end with \(q\), and in \(\overline{\mathcal{D}(p)}\) each \(I\) is required to begin with \(p\). Figure 6-2 depicts \(\overline{\mathcal{D}(p)}\) in an example where \(p\) is an index–2 critical point of a Morse–Smale function on \(S^2\).

The theorems that describe the topology of these compactifications show that they are smooth manifolds with corners; a **smooth manifold with corners** of dimension \(n\) is defined by an atlas of open subsets of \([0, \infty)^n\) with smooth diffeomorphisms as transition functions. The set of points that have exactly \(k\) coordinates equal to zero (in any chart) is a manifold of dimension \(n-k\), and we refer to it as the **codimension–\(k\) part**. A **smooth manifold with faces** is a smooth manifold with corners such that for every \(k\), every point of the codimension–\(k\) part belongs to the closures of \(k\) different connected components of the codimension–1 part.

The description of \(\overline{\mathcal{D}(p)}\) relies on the description of \(\overline{\mathcal{M}(p, q)}\). The following two theorems in [25] have proofs taken from the paper [5] of Burghelea and Haller, who...
Figure 6-2: In this example the vertical coordinate gives a Morse–Smale function on $S^2$, shown in the upper picture. The compactified disk $\overline{D(p)}$, shown in the lower picture, consists of the following parts: the open 2-cell $D(p)$, the open segments $\mathcal{M}(p, b) \times D(b)$ and $\mathcal{M}(p, c) \times D(c)$, and the pair of points $\mathcal{M}(p, b) \times \mathcal{M}(b, c) \times D(c)$. Under the evaluation map $e: \overline{D(p)} \to S^2$, the entire closed segment $\mathcal{M}(p, c) \times D(c)$, shown as the left-hand boundary of the disk, collapses to the point $c$. 
attribute the result to Latour [22].

**Theorem 31** (Theorem 3.3 of [25]). Let \((M, g)\) be a closed Riemannian manifold and suppose \(f : (M, g) \to \mathbb{R}\) is Morse–Smale and Euclidean. Then for every pair of critical points \((p, q)\), there is a smooth structure on \(\mathcal{M}(p, q)\) which satisfies the following properties:

- It is a compact smooth manifold with faces, and the codimension-\(k\) part is \(\bigcup_{|I|=k} \mathcal{M}_I\).
- The smooth structure is compatible with that of each \(\mathcal{M}_I\).
- For any critical point \(r\) with \(\text{ind}(p) > \text{ind}(r) > \text{ind}(q)\), the concatenation map \(\mathcal{M}(p, r) \times \mathcal{M}(r, q) \to \mathcal{M}(p, q)\) is a smooth embedding.

**Theorem 32** (Theorem 3.4 of [25]). Let \((M, g)\) be a closed Riemannian manifold and suppose \(f : (M, g) \to \mathbb{R}\) is Morse–Smale and Euclidean. Then for every critical point \(p\), there is a smooth structure on \(\overline{D(p)}\) which satisfies the following properties:

- It is a compact smooth manifold with faces, and the codimension-\(k\) part is \(\bigcup_{|I|=k-1} \mathcal{D}_I\).
- The smooth structure is compatible with that of each \(\mathcal{D}_I\).
- For any critical point \(r\) with \(\text{ind}(p) > \text{ind}(r)\), the concatenation map \(\overline{\mathcal{M}(p, r)} \times \overline{\mathcal{D}(r)} \to \overline{\mathcal{D}(p)}\) is a smooth embedding.
- The evaluation map \(e : \overline{\mathcal{D}(p)} \to M\) is smooth, where the restriction of \(e\) on \(\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_{k+1})\) is the coordinate projection onto \(\mathcal{D}(r_{k+1}) \subseteq M\).

To make \(M\) into a CW-complex, what remains is to show that \(\overline{\mathcal{D}(p)}\) with this topology is homeomorphic to a closed ball.

**Theorem 33** (Theorem 3.7 of [25]). Let \((M, g)\) be a closed Riemannian manifold and suppose \(f : (M, g) \to \mathbb{R}\) is Morse–Smale and Euclidean. Let \(D^k\) denote the closed ball of dimension \(k\), and let \(S^{k-1}\) denote its boundary. Then for every critical point \(p\) there is a homeomorphism between the pair \((\overline{D(p)}, \partial \overline{D(p)})\) and the pair \((D^{\text{ind}(p)}, S^{\text{ind}(p)-1})\).

In order to prove the main lemma of this chapter, Lemma 30, we want to find a triangulation of each \(n\)-dimensional cell \(\overline{D(p)}\) that satisfies the **cellular**, **order**, **internality**, and **loop** properties and does not have too many essential simplices. Let \(S_I\) denote the stratification of \(\overline{D(p)}\) consisting of the connected components of the codimension-\(k\) part for each \(k\). We will first find a finite triangulation of \(\overline{D(p)}\) with the **cellular** property with respect to \(S_I\); an iterated barycentric subdivision of this triangulation satisfies all four properties by Sublemma 12, and the next lemma counts the number of essential simplices with respect to \(S_I\). Figure 6-3 depicts an example triangulation of the disk \(\overline{D(p)}\) from Figure 6-2, labeled with its essential simplices.
Lemma 34. Let $X = [0, \infty)^n$, and let $S_1$ be the stratification of $X$ consisting of the connected components of the codimension $k$ part for each $k$. Let $c_1$ be a finite triangulation of the closure of a neighborhood of 0 in $X$, satisfying the cellular property with respect to $S_1$, and let $c_2$ denote the first barycentric subdivision of $c_1$. Then the map sending each simplex of $c_2$ to the list of strata of its vertices induces a bijection essential simplices of $c_2$ \leftrightarrow totally ordered chains of $n + 1$ strata.

Proof. The proof is by induction on $n$. The base case is $n = 0$, in which case $X$ is one point. For $n > 0$, we apply the inductive hypothesis to each $(n - 1)$-dimensional face of $X$ with the induced triangulation. The resulting essential $(n - 1)$-simplices of $c_2 \cap \partial X$ are in bijection with the totally ordered chains of $n$ strata in $\partial X$. There is only one $n$-dimensional stratum, which we append to the totally ordered chains of $n$ strata in $\partial X$ to get the totally ordered chains of $n + 1$ strata in $X$. An $n$-simplex of $X$ is essential if and only if it has one vertex in the interior of $X$ and the opposite face is an essential $(n - 1)$-simplex of $\partial X$; there is exactly one such $n$-simplex of $c_2$ for each $(n - 1)$-simplex of $c_2 \cap \partial X$. Thus the essential $n$-simplices are in bijection with the totally ordered chains of $n + 1$ strata in $X$.

We use the following easy lemma to show that when counting totally ordered chains of $n + 1$ strata, it is equivalent to count them globally on $D(p)$ or locally near the vertices.

Lemma 35. Let $D$ be an $n$-dimensional compact smooth manifold with faces. Let $v$ be a vertex of the codimension-$n$ part, and let $U$ be a neighborhood of $v$ corresponding to a ball around 0 in $[0, \infty)^n$. Then for every $k$, the connected components of the codimension-$k$ part of $U$ all come from different connected components of the codimension-$k$ part of $D$.

Proof. Every point $x$ in the codimension-$k$ part can be labeled by which $k$ connected components of the codimension-1 part of $D$ have closure containing $x$. This labeling is constant as $x$ varies within a connected component of the codimension-$k$ part of $D$. Coming together at $v$, there are $n$ different connected components of the codimension-1 part of $D$, so in $U$ every connected component of the codimension-$k$ part gets a different label. Thus none of them can be part of the same connected component of the codimension-$k$ part of $D$.

The final ingredient needed to prove the main lemma (Lemma 30) is the following lemma. It relates the stratification $S_1$, which comes from the structure of $\overline{D(p)}$ as a manifold with corners, to the generalized stratification of $M$ corresponding to its CW structure. Specifically, consider the evaluation map $e: \overline{D(p)} \to M$ from Theorem 32, and let $S_2$ be the generalized stratification of $\overline{D(p)}$ consisting of the space $e^{-1}(D(r)) = \mathcal{M}(p, r) \times D(r)$ for each critical point $r$. Note that every stratum of $S_2$ is a union of strata from $S_1$. Figure 6-3 illustrates the fact that some simplices are essential with respect to $S_1$ but are not essential with respect to $S_2$. 48
Lemma 36. Let $(M, g)$ be a closed Riemannian manifold and suppose $f : (M, g) \rightarrow \mathbb{R}$ is Morse–Smale and Euclidean. Let $p$ be any critical point. Let $S_1$ be the stratification of $\overline{D(p)}$ consisting of the connected components of the codimension-$k$ part for each $k$, and let $S_2$ be the stratification of $\overline{D(p)}$ consisting of the space $\mathcal{M}(p, r) \times D(r)$ for each critical point $r$. Then the number of totally ordered chains of $\text{ind}(p) + 1$ strata from $S_2$ such that those strata belong to $\text{ind}(p) + 1$ different strata from $S_2$ is the number of $\text{ind}(p)$-part broken trajectories starting at $p$.

Proof. Roughly, a totally ordered chain of $\text{ind}(p) + 1$ strata from $S_1$ corresponds to an increasing sequence of sets of critical points at which the flow lines from $p$ may break. For such a set of critical points, the corresponding stratum of $S_2$ is determined by the critical point of least index. Thus, in order to get $\text{ind}(p) + 1$ different strata from $S_2$, the critical points must be added in decreasing order of index. The number of ways to make such a chain of $S_1$-strata is the number of $\text{ind}(p)$-part broken trajectories.

The proof is by induction on $\text{ind}(p)$. The base case is $\text{ind}(p) = 0$, in which case $\overline{D(p)}$ is one point. For $\text{ind}(p) > 0$, every totally ordered chain of $\text{ind}(p) + 1$ strata from $S_1$ consists of the maximal stratum $D(p)$ along with some totally ordered chain of $\text{ind}(p)$ strata with maximum equal to some codimension-1 stratum. We know that the codimension-1 part of $\overline{D(p)}$ consists of various $\mathcal{M}(p, r) \times D(r)$, where $r$ is a critical point with $\text{ind}(r) < \text{ind}(p)$.

If $\text{ind}(r) = \text{ind}(p) - 1$, then $\mathcal{M}(p, r)$ is a finite set. We know that concatenation gives an embedding $\mathcal{M}(p, r) \times D(r) \rightarrow \overline{D(r)}$, by Theorem 32. The space $\mathcal{M}(p, r) \times D(r) = \mathcal{M}(p, r) \times D(r)$ is a finite disjoint union of copies of $D(r)$, so any totally ordered chain of strata with maximum equal to some stratum in $\mathcal{M}(p, r) \times D(r)$ is completely contained in one of the copies of $D(r)$. We apply the inductive hypothesis to each $D(r)$. Then, the $S_2$-stratum $D(p)$ is different from every $S_2$-stratum intersecting...
any $D(r)$, so the number of totally ordered chains of $\text{ind}(p) + 1$ strata from $S_1$, with maximum $D(p)$ and next stratum in $M(p, r) \times D(r)$, such that these strata belong to $\text{ind}(p) + 1$ different strata from $S_2$, is the number of $(\text{ind}(p) + 1)$-part broken trajectories starting with $p$ and then $r$.

If, on the other hand, $\text{ind}(r) < \text{ind}(p) - 1$, then as before we have an embedding $M(p, r) \times D(r) \to D(p)$. We claim that in $M(p, r) \times D(r)$ there are no totally ordered chains of $\text{ind}(p)$ strata from $S_1$ such that these strata belong to $\text{ind}(p)$ different strata from $S_2$. First we observe that if two strata $S$ and $S'$ of $S_1$ are related by $S \sim S'$, then the strata of $S_2$ containing $S$ and $S'$ satisfy the same relation. Thus for every totally ordered chain of strata from $S_1$, the corresponding strata from $S_2$ are also totally ordered. We also observe that for any two critical points $q_1$ and $q_2$ of the same index, the corresponding $S_2$-strata $M(p, q_1) \times D(q_1)$ and $M(p, q_2) \times D(q_2)$ are incomparable. This implies that the maximum possible length of a totally ordered chain of $S_2$-strata intersecting $M(p, r) \times D(r)$ is $\text{ind}(r) + 1$, which is less than $\text{ind}(p)$.

Thus, back in $D(p)$, for every totally ordered chain of $\text{ind}(p) + 1$ strata from $S_1$ such that those strata belong to $\text{ind}(p) + 1$ different strata from $S_2$, the next-to-maximum stratum is in $A_4(p, r) \times D(r)$ for some critical point $r$ with $\text{ind}(r) = \text{ind}(p) - 1$. Adding up the number of ways to do this for all such $r$, the total is the number of $\text{ind}(p)$-part broken trajectories starting at $p$.

**Proof of main lemma for Morse broken trajectory theorem (Lemma 30).** Theorems 32 and 33 imply that $M$ is a CW-complex with the descending manifolds as open cells. As a smooth manifold with corners, each $D(p)$ is a Whitney stratified space and therefore has a stratified triangulation (see [12] and [13]). The stratified triangulation satisfies the cellular property with respect to $S_1$, and there are finitely many simplices because $D(p)$ is compact. Let $c_{\text{rel}}(p)$ be an iterated barycentric subdivision of this original triangulation.

By Sublemma 12, the relative $(D(p), \partial D(p))$ cycle $c_{\text{rel}}(p)$ satisfies the cellular, order, internality, and loop properties with respect to $S_1$, so it also satisfies these four properties with respect to $S_2$, or equivalently, as a relative $(M, M^{n-1})$ cycle with respect to $S$. By Lemmas 34 and 35, the triangulation $c_{\text{rel}}(p)$ has one essential simplex with respect to $S_1$ for each totally ordered chain of $n + 1$ strata from $S_1$, so by Lemma 36, $c_{\text{rel}}(p)$ has one essential simplex with respect to $S_2$ (or equivalently $S$) for each $n$-part broken trajectory beginning at $p$. Thus we have

$$\| [D(p)] \|_{S, \text{ess}}^S \leq \| c_{\text{rel}}(p) \|_{S, \text{ess}}^S = \#(n\text{-part broken trajectories beginning at } p).$$

**Proof of Morse broken trajectory theorem, general version (Theorem 4).** By Theorem 29 we may assume that $(f, g)$ is Euclidean. Let $S$ denote the generalized stratification of $M$ corresponding to the CW structure from the descending manifolds. We apply the localization lemma (Lemma 13) to $S$. The generalized stratification of a CW-complex by open cells satisfies the NDR property: every skeleton $X^i$ is a neighborhood deformation retract in the next skeleton $X^{i+1}$, so if $S$ is a cell, we obtain the NDR
neighborhood $U_S$ by taking the preimage of $S$ under the composition of all such retractions that have $i$ at least the dimension of $S$. Every stratum is contractible, so its fundamental group is zero and hence amenable. We take $A = X^{j-1}$, in which there is no totally ordered chain of more than $j$ strata.

The relative class $[M]_{rel} \in H_n(M, M^{n-1})$ corresponding to $[M]$ is equal to the sum of $n$-cells

$$[M]_{rel} = \sum_{\text{ind}(p) = n} [D(p)],$$

and so by the main lemma (Lemma 30) its essential stratified simplicial norm satisfies the bound

$$\| [M]_{rel} \|^S_{\Delta, \text{ess}} \leq \sum_{\text{ind}(p) = n} \| [D(p)] \|^S_{\Delta, \text{ess}} \leq \#(n\text{-part broken trajectories}).$$

Applying the localization lemma (Lemma 13), we obtain the inequality

$$\| \alpha_* [M] \|_{\Delta} \leq \| [M]_{rel} \|^S_{\Delta, \text{ess}},$$

which proves the theorem. \qed

**Proof of Morse broken trajectory theorem, specific version (Theorem 2).** Suppose $M$ is hyperbolic. For $Z$ we take $M$, and for $\alpha: M \to Z$ we take the identity map on $M$. Combining the general version (Theorem 4) with the formula relating simplicial norm to hyperbolic volume, we obtain

$$\frac{\text{Vol}(M, \text{hyp})}{\text{Vol} \Delta^n} = \| [M] \|_{\Delta} \leq \#(n\text{-part broken trajectories}).$$

\qed

**Proof of Morse index corollary (Theorem 1).** Note that there is no need to worry about orientations, because any Morse function on a non-orientable manifold pulls back to a Morse function on the orientable double cover.

We prove a stronger statement of the Morse index corollary. Let $M$ be a closed manifold of dimension $n$. Let $Z$ be an aspherical topological space, and suppose that $\alpha: M \to Z$ is a continuous map such that some homology class $h \in H_j(M)$ in degree $j$ has $\alpha$-image with nonzero simplicial norm, that is, $\| \alpha_* h \|_{\Delta} > 0$. We claim that any Morse function $f: M \to \mathbb{R}$ must have a critical point of every index from 0 to $j$ and every index from $n - j$ to $n$.

It suffices to prove the statement about indices from 0 to $j$, because any statement about a Morse function $f$ can also be applied to $-f$. In order to apply the proof of the Morse broken trajectory theorem (Theorems 2 and 4) we need to modify $f$ to obtain a gradient vector field that is Morse–Smale and Euclidean. If we choose an arbitrary Riemannian metric $g$ on $M$, then $(f, g)$ may not be Morse–Smale. However, the Kupka–Smale theorem guarantees that some $C^\infty$-small perturbation of the vector field $\nabla f$ is a Morse–Smale gradient vector field. Because $f$ is Morse, its second derivative at each critical point is nondegenerate, so the inverse function theorem
guarantees that the perturbed $\nabla f$ has the same number of zeroes as $\nabla f$, with the same Morse indices. Then we apply Theorem 29 to obtain a gradient vector field that is both Morse-Smale and Euclidean, and has the same number of zeroes as $\nabla f$ with the same indices. By relabeling $f$ and $\nabla f$ to be this new function and its gradient, we may assume that $\nabla f$ is Morse-Smale and Euclidean.

The main lemma for the Morse broken trajectory theorem (Lemma 30) can be extended to arbitrary degree $j$, with the same proof. That is, for each descending manifold $D(p)$ of dimension $j$, the corresponding relative homology class $[D(p)] \in H_j(M, M^{j-1})$ satisfies the bound

$$\|\alpha, h\|_{\Delta, \text{ess}} \leq \#(j \text{-part broken trajectories beginning at } p).$$

Then we apply the localization lemma (Lemma 13) to the class $h$, taking the closed set $A$ to be the $(j-1)$-skeleton $M^{j-1}$. If $h$ is represented by the cellular cycle $\sum_{\text{ind}(p) = j} r_p \cdot [D(p)]$, then the localization lemma gives the inequality

$$\|\alpha, h\|_{\Delta} \leq \sum_{\text{ind}(p) = j} |r_p| \cdot \#(j \text{-part broken trajectories beginning at } p).$$

Because $\|\alpha, h\|_{\Delta} > 0$, there must be at least one $j$-part broken trajectory beginning at some critical point $p$ of index $j$, and such a trajectory includes a critical point of every index from 0 to $j$.

This stronger statement of the Morse index corollary applies to many different manifolds. First of all, if $M$ is a closed manifold of nonzero simplicial volume, then we may take $Z = K(\pi_1(M), 1)$ and take $\alpha: M \to Z$ to be the classifying map, in which case $\|\alpha, [M]\|_{\Delta} = \|[M]\|_{\Delta}$ (Corollary (B) of the Mapping Theorem, section 3.1 of [15]; or see Ivanov [18]). Thus, every Morse function on a closed manifold of nonzero simplicial volume has a critical point of every index.

Another nice application comes from taking $M$ to be a product $N \times X$, where $N$ has nonzero simplicial volume and $X$ is some other closed manifold. We may take $h$ to be the image in $H_{\dim N} (N \times X)$ of the fundamental class $[N]$ under a section of the projection $N \times X \to N$. If $\dim X \leq \dim N + 1$, then we see that any Morse function on $N \times X$ must have a critical point of every index. If $\dim X$ is larger, then sometimes we can reach the same conclusion if we know that $X$ has nonzero Betti numbers in suitable degrees. For instance, if $N$ has nonzero simplicial volume and $T$ is a torus of any dimension, then any Morse function on $N \times T$ must have a critical point of every index.

One question suggested by the Morse broken trajectory theorem (Theorems 2 and 4) is whether the inequality can be reversed. Is there some function of the simplicial volume that provides an upper bound for the minimum number of $n$-part broken trajectories of any Morse-Smale gradient vector field on a given manifold? There may be a nice answer in dimension at least 4, but Dylan Thurston proved for me that the answer in dimension 3 is no.
Proposition 37. For all $k \in \mathbb{N}$, there are only finitely many closed hyperbolic 3-manifolds that admit a Morse-Smale gradient vector field for which the number of 3-part broken trajectories is at most $k$.

Proof. In a 3-manifold $M$, the number of 3-part broken trajectories is exactly four times the number of isolated flow lines between index 2 and index 1; let us call these the 2-1 isolated flow lines. It suffices to show that there are finitely many closed hyperbolic 3-manifolds with $k$ 2-1 isolated flow lines. The idea is, the hyperbolic 3-manifold $M$ can be written as a complex of at most $k$ simplices glued along faces, and there are only finitely many such complexes.

Suppose that we are in the special case where the Morse function $f$ is self-indexing and there is only one index 0 critical point and only one index 3 critical point. Let $\Sigma$ denote the level surface halfway up, at value $\frac{3}{2}$. The descending manifolds of the index-2 critical points and the ascending manifolds of the index-1 critical points meet $\Sigma$ at loops, forming a Heegaard diagram for $M$. The loops have $k$ (transverse) intersections, corresponding to the $k$ 2-1 isolated flow lines.

We form a 2-dimensional stratified space $P$ by taking the union of $\Sigma$ with these disks: the portions above $\Sigma$ of the index-2 descending manifolds, and the portions below $\Sigma$ of the index-1 ascending manifolds. In the special case where the complement in $\Sigma$ of the loops is a disjoint union of cells, we see that $P$ is the dual complex to a triangulation of $M$ by $k$ simplices. Otherwise, we use the Matveev complexity developed in the paper [23]; there are only finitely many irreducible manifolds of a given complexity. Every hyperbolic 3-manifold $M$ is irreducible (i.e., every embedded sphere bounds a ball), and exhibiting $P$ shows that $M$ has complexity at most $k$.

In the case of an arbitrary Morse-Smale function $f$, we construct $P$ in a similar way, but we do not automatically have a level surface $\Sigma$ that separates the index-2 critical points from the index-1 critical points. Recall that $\mathcal{M}(p, q)$ denotes the space of flow lines from $p$ to $q$, and its compactification $\overline{\mathcal{M}(p, q)}$ is a compact smooth manifold with corners (Theorem 3.3 of [25], stated in the present paper as Theorem 31). Given a continuous function

$$\sigma_{p,q}: \overline{\mathcal{M}(p, q)} \to \mathbb{R}$$

with image contained in the interval from $f(q)$ to $f(p)$, it makes sense to view the graph of $\sigma_{p,q}$ as a subset of $M$ that intersects each flow line in $\mathcal{M}(p, q)$ exactly once. We can piece together several such graphs, if they are compatible, to get a surface like $\Sigma$, as follows.

We first construct the 0-dimensional part of $P$: for each pair of critical points $p$ and $q$ with $\text{ind}(p) = 2$ and $\text{ind}(q) = 1$, there are finitely many elements of $\mathcal{M}(p, q)$, so we choose an arbitrary function

$$\sigma_{p,q}: \overline{\mathcal{M}(p, q)} \to (f(q), f(p)).$$

Then we construct the 1-dimensional part of $P$, which means constructing $\sigma_{p,q}$ for either $\text{ind}(p) = 3$ and $\text{ind}(q) = 1$, or $\text{ind}(p) = 2$ and $\text{ind}(q) = 0$. For such $p$ and $q$, the value of $\sigma_{p,q}$ on the boundary of $\overline{\mathcal{M}(p, q)}$ is determined by the 0-dimensional step; we extend arbitrarily to the interior. Then we construct the 2-dimensional part
of $P$. For $\text{ind}(p) = 3$ and $\text{ind}(q) = 0$ we extend $\sigma_{p,q}$ from the boundary to all of $M(p,q)$. We also add in the portions of the index 2 descending manifolds and the index 1 ascending manifolds enclosed by the 1-dimensional part of $P$. The resulting space $P$ has $k$ vertices, and its complement is a disjoint union of 3-cells, one for each index 0 or index 3 critical point; it shows that the Matveev complexity of $M$ is at most $k$.

In dimension 3 there are infinite families of hyperbolic manifolds with a uniform bound on their volume. If we start with a complete hyperbolic manifold $N$ of dimension 3 with finite volume and one cusp, then there are infinitely many closed hyperbolic manifolds obtained by gluing a solid torus into the cusp (Theorem 5.8.2 of [27]), and their volumes are less than the volume of $N$ (Theorem 6.5.6 of [27]). Thus, there is no function of volume alone that is an upper bound on the minimum number of $n$-part broken trajectories.
Chapter 7

Future directions

Even though the inequality of the Morse broken trajectory theorem (Theorems 2 and 4) cannot be reversed in dimension 3, in higher dimensions it would be nice to see what kind of reverse inequality is possible. In each dimension at least 4, there are only finitely many hyperbolic manifolds with volume less than any given bound [29], and the same is true of manifolds with sectional curvature pinched between two negative constants [14]. Thus, for these special cases it is automatically true that the minimum number of $n$-part broken trajectories is bounded above by some function of the volume; the question is what the function might be. To my knowledge it is not known to what extent these finiteness statements extend to the more general setting of simplicial volume of manifolds; Fujiwara and Manning give some overview in their paper [10]. Thus it may or may not be true in dimensions at least 4 that the minimum number of $n$-part broken trajectories is bounded above by some function of the simplicial volume. Perhaps there is also an upper bound for the minimum number of $n$-part broken trajectories in terms of some topological quantity other than the simplicial volume.

Other questions arise from the Morse index corollary (Theorem 1):

- The manifolds that most obviously violate the conclusion of the Morse index corollary are spheres. Is the corollary true of aspherical manifolds? If not, what are the first counterexamples? This question was suggested to me by Alex Nabutovsky.

- Is the Morse index corollary true of manifolds with nonpositive sectional curvature? (We know that it is true for negative sectional curvature and for the product of a negative-sectional-curvature manifold and a torus.) This question was suggested to me by Vitali Kapovitch.

- One strengthening of the Morse index corollary would be the following: if $M$ is a closed hyperbolic manifold of dimension $n$, then for every $i = 0, \ldots, n$ there is a finite-sheeted covering space $\tilde{M}$ of $M$ that has nonzero homology in degree $i$. Is this stronger statement true? In dimension 3 this is a consequence of the virtual fibering theorem, which says that every closed hyperbolic 3-manifold has a finite-sheeted cover that fibers over the circle (see [9] for a survey). In
higher dimensions the statement is known in some cases (see the paper [4] of Bergeron, Haglund, and Wise). This question was suggested to me by Ian Agol, who pointed me to the latter reference, and by Vitali Kapovitch.
Appendix A

Proofs of amenable reduction and localization lemmas

The purpose of this chapter is to prove the amenable reduction lemma (Lemma 11) and the localization lemma (Lemma 13). First we prove the amenable reduction lemma, which for convenience is restated next.

**Lemma 11** (Amenable reduction lemma, p. 25 of [16]). Let $c$ be a cycle on space $X$ with partial coloring $\{V_i\}$. Suppose that $Z$ is an aspherical topological space, and let $\alpha : X \to Z$ be a continuous map such that for every connected component of every induced 1-complex $\Sigma_1$ of $c$, the $\alpha$-image of its fundamental group in $\pi_1(Z)$ is an amenable group. Then the simplicial norm of the $\alpha$-image of the homology class $[c] \in H_1(X)$ represented by the cycle $c = \sum r_i \sigma_i$ (where $r_i \in \mathbb{R}$ are coefficients and $\sigma_i$ are simplices) satisfies the bound

$$\|\alpha_*[c]\|_\Delta \leq \sum_{\sigma_i \text{ essential}} |r_i|.$$  

The amenable reduction lemma requires a couple of preliminary sublemmas that allow chains and simplicial norm to behave like straightened triangulations and hyperbolic volume; namely, any pair of simplices that differ by a homotopy and a transposition of two vertices should cancel each other out.

**Sublemma 38** ([15], p. 48). Let $Z$ be a space with contractible universal cover $\tilde{Z}$. Then there is a “straightening” operator

$$\text{straight}: C_*(Z) \to C_*(Z)$$

with the following properties:

- For each simplex $\sigma \in C_*(Z)$, the straightened version $\text{straight}(\sigma)$ is a simplex of the same dimension with the same sequence of vertices.

- If two simplices $\sigma_1, \sigma_2 \in C_*(Z)$ have the same sequence of vertices, and their lifts $\tilde{\sigma}_1, \tilde{\sigma}_2 \in C_*(\tilde{Z})$ to the universal cover also have the same sequence of vertices, then $\text{straight}(\sigma_1) = \text{straight}(\sigma_2)$.  

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straight commutes with the boundary map $\partial$; that is, straight is a chain-complex endomorphism.

- straight commutes with the standard action of the symmetric group $S_{j+1}$ on each $C_j(Z)$.

- straight is chain homotopic to the identity.

Proof. We construct the straightening operator and the chain homotopy simultaneously, one dimension at a time. The chain homotopy will be, for each simplex $\sigma: \Delta \to Z$, a map $H(\sigma): \Delta \times I \to Z$ such that the restriction of $H(\sigma)$ to $\Delta \times 0$ is $\sigma$, the restriction to $\Delta \times 1$ is straight($\sigma$), and for each face $F \in \partial \Delta$, the restriction of $H(\sigma)$ to $F \times I$ is $H(\sigma|_F)$.

For every 0 dimensional simplex $\sigma^0 \in C_0(Z)$, we have $\text{straight}(\sigma^0) = \sigma^0$ and a constant homotopy $H(\sigma^0)$. For a simplex $\sigma$ with $\dim \sigma = j > 0$, suppose the straightening and chain homotopy are already defined for every dimension less than $j$. In particular, $\text{straight}(\partial \sigma)$ depends only on the sequence of vertices of the lift $\tilde{\sigma}$ of $\sigma$ to the universal cover $\tilde{Z}$. We lift $\text{straight}(\partial \sigma)$ to $\tilde{Z}$; because $\tilde{Z}$ is contractible, there is some simplex filling in the lift of straight($\partial \sigma$), and we can choose straight($\sigma$) to be the corresponding simplex in $Z$. We make this choice only once per orbit of $\pi_1(Z) \times S_{j+1}$ on the set $(Z)^{j+1}$ of sequences of vertices in $Z$.

Having chosen straight($\sigma$), we lift $\sigma$, straight($\sigma$), and $H(\partial \sigma)$ to $\tilde{Z}$ to form a sphere of dimension $j$. Because $\tilde{Z}$ is contractible, we can fill in this sphere by a map $\tilde{H}(\sigma)$ on $\Delta \times I$ that has the prescribed boundary, and let $H(\sigma)$ be the corresponding map into $Z$.

We also use the anti-symmetrization operator,

$$\text{symm}: C_\ast(Z) \to C_\ast(Z)$$

given by

$$\text{symm}(\sigma^j) = \frac{1}{(j + 1)!} \sum_{q \in S_{j+1}} \text{sign}(q) \cdot q(\sigma^j)$$

for every simplex $\sigma^j \in C_j(Z)$. Gromov states that this operator is chain homotopic to the identity ([15], p. 29), and Fujiwara and Manning give the proof in [10].

Sublemma 39 ([10], Appendix B). Let $Z$ be any topological space. The anti-symmetrization operator $\text{symm}: C_\ast(Z) \to C_\ast(Z)$ is chain homotopic to the identity.

Thus, the composition of these two operators

$$\text{symm} \circ \text{straight}: C_\ast(Z) \to C_\ast(Z),$$

satisfies the following properties:

1. $\text{symm} \circ \text{straight}(\sigma)$ depends only on the list of vertices of the lift $\tilde{\sigma}$ to the universal cover.
2. For every \( q \in S_{j+1} \) and every \( \sigma^j \in C_j(Z) \), we have
\[
s\circ \text{straight} \circ q(\sigma^j) = \text{sign}(q) \cdot s\circ \text{straight}(\sigma^j).
\]

3. \( s\circ \text{straight} \) is a chain map, chain homotopic to the identity.

4. \( s\circ \text{straight} \) does not increase the norm; that is, for every chain \( c \in C_*(Z) \), we have
\[
\| s\circ \text{straight}(c) \|_\Delta \leq \|c\|_\Delta.
\]

Property 2 is our reason for introducing \( s \) at all: it allows homotopic simplices with opposite orientations to cancel in a sum or average.

Proof of the amenable reduction lemma (Lemma 11). Without loss of generality we may assume that every induced 1-complex \( \Sigma^1_\ell \) is connected; this is because if we subdivide \( V_\ell \) according to the connected components of \( \Sigma^1_\ell \), there is no change to which simplices are essential.

The proof relies on modifying the simplices of \( \alpha_a c \) by dragging the vertices around loops in \( \pi_1(Z) \). In order for this process to make sense, we first need to modify the cycle \( c \) so that all of its vertices are at the base point. Specifically, viewing \( c \) as the triple \( (\Sigma, c, \sigma) \), we choose one base point \( * \) in \( X \) and one base point \( *_\ell \) in each \( \Sigma^1_\ell \), and choose a path \( \gamma_\ell \) in \( X \) from \( \sigma(*_\ell) \) to \( * \). Then for each vertex \( v \) in \( \Sigma^1_\ell \) we choose a path in \( \Sigma^1_\ell \) from \( v \) to \( *_\ell \). We homotope \( \sigma \) in a small neighborhood of \( v \) in \( \Sigma \) so that \( v \) travels along the image in \( X \) of that path to \( \sigma(*_\ell) \), and then along \( \gamma_\ell \) to \( * \). We apply this sequence of homotopies to \( \sigma \), one for each vertex, to get some new map \( \sigma' \).

Let \( \Gamma_\ell \) denote the subgroup of \( \pi_1(Z) \) generated by the images under \( \alpha \circ \sigma \) of the edges in \( \Sigma^1_\ell \), all of which are now loops at the base point. We claim that \( \Gamma_\ell \) is an amenable group, as follows. For every edge in \( \Sigma^1_\ell \), the restriction of \( \sigma' \) to that edge is a loop in \( X \) which is obtained by taking the \( \sigma \)-image of a loop in \( \Sigma^1_\ell \) and conjugating by the path \( \gamma_\ell \). Thus, the subgroup \( \Gamma_\ell \) resulting from \( \sigma' \) is a subgroup of the conjugation of \( \alpha \circ \sigma \pi_1(\Sigma^1_\ell, *) \) by \( \alpha \circ \gamma_\ell \). We have assumed that \( \alpha \circ \sigma \pi_1(\Sigma^1_\ell) \) is an amenable group, and every subgroup of an amenable group is amenable, so \( \Gamma_\ell \) is amenable. By replacing \( \sigma \) by \( \sigma' \), we may now assume that all vertices of \( c \) are at the base point and that \( \Gamma_\ell = \alpha \circ \sigma \pi_1(\Sigma^1_\ell) \).

Next we imagine pushing the vertices in \( V_\ell \) along the loops in \( \Gamma_\ell \). Given a singular simplex \( \sigma_i \in C_*(X) \) and a path \( \gamma : [0,1] \to X \) beginning at one vertex of \( \sigma_i \), there is a homotopy \( \sigma_{i,t} \) pushing the vertex along \( \gamma \); the image of each \( \sigma_{i,t} \) is the union of the image of \( \sigma_i \) with the partial path \( \gamma|_{[0,t]} \). Given a singular cycle \( c \in C_*(X) \) and a path \( \gamma \) beginning at one vertex of \( c \), we may apply this process to every simplex of \( c \) containing that vertex, to obtain a homotopic (and thus homologous) cycle \( \gamma \ast c \). More precisely, if \( c \) is \( (\Sigma, c, \sigma) \) then we modify \( \sigma \) by a homotopy supported in a neighborhood of one vertex of \( \Sigma \). Likewise, we may take a path \( \gamma \) in \( Z \) rather than in \( X \), and obtain a cycle \( \gamma \ast \alpha_a c \), for which the straightened cycle \( \text{straight}(\gamma \ast \alpha_a c) \) depends on \( \gamma \) only up to homotopy. That is, if \( c \) is \( (\Sigma, c, \sigma) \) then \( \alpha_a c \) is \( (\Sigma, c, \alpha \circ \sigma) \) and we homotope \( \alpha \circ \sigma \).
Applying this process to every vertex of \( \Sigma \) in \( \bigcup L \) simultaneously, we obtain an action of the product group \( \times_\ell(\Gamma_\ell)^{V_\ell} \) on \( c \), given by

\[
g \mapsto \text{symm} \circ \text{straight}(g \ast \alpha_* c), \quad g \in \times_\ell(\Gamma_\ell)^{V_\ell}.
\]

That is, suppose that \( g \) is an element of the product group \( \times_\ell(\Gamma_\ell)^{V_\ell} \), and for each vertex \( v \) in the union \( \bigcup L \), let \( \gamma_v \) denote the corresponding coordinate of \( g \); we can think of \( \gamma_v \) as a loop in \( Z \). Then to find \( g \ast \alpha_* c \) we choose disjoint neighborhoods in \( \Sigma \) of all vertices \( v \in \bigcup L \) and for each \( v \) we homotope \( \alpha \circ \sigma \) in the chosen neighborhood of \( v \) to push \( \alpha \circ \sigma \) along \( \gamma_v \) in \( Z \). We will take the average of cycles \( \text{symm} \circ \text{straight}(g \ast \alpha_* c) \) as \( g \) ranges over a large finite subset of \( \times_\ell(\Gamma_\ell)^{V_\ell} \). To choose this subset, we use the definition of amenable group.

One characterization of (discrete) amenable groups is the Følner criterion: for every amenable group \( \Gamma \), every finite subset \( S \subset \Gamma \), and every \( \varepsilon > 0 \), there is a finite subset \( A \subset \Gamma \) satisfying the inequality

\[
\frac{|xA \Delta A|}{|A|} \leq \varepsilon \quad \forall x \in S,
\]

where \( \Delta \) denotes the symmetric difference. In our setting, we choose \( S \) to be the set of \( \alpha \)-images of edges in \( c \) with both endpoints in \( V_\ell \), and then apply the Følner criterion to find \( A \subset \Gamma_\ell \). We take the average of \( \text{symm} \circ \text{straight}(g \ast \alpha_* c) \) for \( g \in \times_\ell(A_\ell)^{V_\ell} \subset \times_\ell(\Gamma_\ell)^{V_\ell} \); the result is some cycle homologous to \( \alpha_* c \) which we show has small norm.

First we show that if a given simplex \( \sigma_i \) of \( c \) is not essential, then the average of \( \text{symm} \circ \text{straight}(g \ast \alpha_* \sigma_i) \) has norm at most \( \varepsilon \). If one edge of \( \sigma_i \) is a contractible loop in \( X \), then every \( \text{symm} \circ \text{straight}(g \ast \alpha_* \sigma_i) \) is equal to 0 (using properties 1 and 2 of \( \text{symm} \circ \text{straight} \)), so the average is 0. Thus, we address the case where \( \sigma_i \) has two distinct vertices \( v_1 \) and \( v_2 \) in some \( V_\ell \). When averaging over all \( g \in \times_\ell(A_\ell)^{V_\ell} \), we average separately over each slice where only the \( v_1 \) and \( v_2 \) components of \( g \) vary and all other components are fixed. It suffices to show that the average of \( \text{symm} \circ \text{straight}(g \ast \alpha_* \sigma_i) \) over each such slice \( A_\ell \times A_\ell \) has norm at most \( \varepsilon \).

Having fixed all components of \( g \) other than the \( v_1 \) and \( v_2 \) components, we let \( g(\gamma_1,\gamma_2) \) denote the element of \( \times_\ell(A_\ell)^{V_\ell} \) for which the \( v_1 \) and \( v_2 \) components are \( \gamma_1 \) and \( \gamma_2 \) in \( A_\ell \) and the other components have the specified fixed values. Let \( x \in \Gamma_\ell \) denote the \( \alpha \)-image of the edge in \( c \) between \( v_1 \) and \( v_2 \). Then the edge \( x \) in \( \alpha_* \sigma_i \) becomes an edge \( \gamma_1^{-1} x \gamma_2 \) in \( g(\gamma_1,\gamma_2) \ast \alpha_* \sigma_i \). Consider the involution

\[
(\gamma_1, \gamma_2) \mapsto (x \gamma_2, x^{-1} \gamma_1)
\]

on the square subset

\[
(\gamma_1, \gamma_2) \in (xA_\ell \cap A_\ell) \times (x^{-1} A_\ell \cap A_\ell) \subset A_\ell \times A_\ell.
\]

The path resulting from \( (x \gamma_2, x^{-1} \gamma_1) \) is the inverse of the path resulting from \( (\gamma_1, \gamma_2) \),
and thus (using property 2 of symm \circ straight) we have
\[
\sum_{(\gamma_1, \gamma_2) \in (x A_\ell \cap A_\ell) \times (x^{-1} A_\ell \cap A_\ell)} \text{symm} \circ \text{straight}(g(\gamma_1, \gamma_2) \ast \alpha_i \sigma_i) = 0.
\]

In other words, only those \((\gamma_1, \gamma_2)\) outside the square subset contribute to the average. By the Følner criterion we have
\[
|x A_\ell \cap A_\ell| \geq \left(1 - \frac{\varepsilon}{2}\right) |A_\ell|,
\]
and so
\[
|(x A_\ell \cap A_\ell) \times (x^{-1} A_\ell \cap A_\ell)| \geq \left(1 - \frac{\varepsilon}{2}\right)^2 |A_\ell| \times |A_\ell| > (1 - \varepsilon) |A_\ell \times A_\ell|.
\]

Thus the average over \(A_\ell \times A_\ell\) satisfies
\[
\frac{1}{|A_\ell \times A_\ell|} \left| \sum_{(\gamma_1, \gamma_2) \in A_\ell \times A_\ell} \text{symm} \circ \text{straight}(g(\gamma_1, \gamma_2) \ast \alpha_i \sigma_i) \right| \leq \frac{1}{|A_\ell \times A_\ell|} \sum_{(\gamma_1, \gamma_2) \notin (x A_\ell \cap A_\ell) \times (x^{-1} A_\ell \cap A_\ell)} 1 < \varepsilon.
\]

Taking the sum over all simplices \(\sigma_i\) of \(c\), we obtain the inequality
\[
\|\alpha_* [c]\|_\Delta \leq \sum_{\sigma_i \text{ essential}} |r_i| + \sum_{\sigma_i \text{ not essential}} \varepsilon |r_i|,
\]
and taking the limit as \(\varepsilon \to 0\) we obtain the inequality of the lemma statement. \(\square\)

Next we prove the localization lemma, restated here; this version is a slight generalization of the version stated and proved by Gromov in the paper [16].

**Lemma 13** (Localization lemma). Let \(X\) be a compact metrizable space, and let \(S\) be a generalized stratification on \(X\) with the NDR property, such that \(S\) has finitely many strata and each stratum is connected. Let \(Z\) be an aspherical topological space, and suppose \(\alpha : X \to Z\) is a continuous map such that for every stratum \(S\), the group \(\alpha_* \pi_1(S) \subseteq \pi_1(Z)\) is amenable. Let \(h \in H_j(X)\) be a homology class, and suppose that \(A\) is a closed subset of \(X\) such that among the strata intersecting \(A\), there is no totally ordered chain of more than \(j\) strata. Let \(h_{rel}\) be the image of \(h\) in \(H_j(X, A)\). Then the simplicial norm of \(\alpha_* h \in H_j(Z)\) satisfies the bound
\[
\|\alpha_* h\|_\Delta \leq \|h_{rel}\|^S_{\Delta, \text{ess}}.
\]

The idea of the proof is to take any relative cycle representing \(h_{rel} \in H_j(X, A)\) and extend it to a cycle representing \(h\), by adding a chain in \(A\) with no essential simplices with respect to a partial coloring that comes from the stratification. Then
we apply the amenable reduction lemma (Lemma 11) to show that the new non-essential simplices do not contribute to the simplicial norm of $\alpha_* h$, thus finishing the proof. The following sublemma states that any representative of $h_{rel}$ may be extended by a chain in $A$ to a representative of $h$.

**Sublemma 40.** Let $(X, A)$ be a pair of spaces, let $h \in H_j(X)$ be a homology class, and let $h_{rel} \in H_j(X, A)$ be the corresponding relative homology class. Let $c_{rel}$ be any relative cycle such that $[c_{rel}] = h_{rel}$. Then there exists a chain $c_A$ in $A$ such that $\partial c_A = -\partial c_{rel}$ (so $c_{rel} + c_A$ is a cycle in $X$) and $[c_{rel} + c_A] = h$.

**Proof.** The proof is by chasing the long exact sequence of homology for the pair $(X, A)$. By the exactness at $H_j(X, A)$, we see that the class of $Dc_{rel}$ must be zero in $H_{j-1}(A)$, so there is some chain $c_1$ in $A$ such that $\partial c_1 = \partial c_{rel}$. By the exactness at $H_j(X)$, because $h - [c_{rel} - c_1] \in H_j(X)$ maps to zero in $H_j(X, A)$ there is some cycle $c_2$ in $A$ such that $[c_2] = h - [c_{rel} - c_1]$. We set $c_A = -c_1 + c_2$. \qed

In order to construct a partial coloring on some chain $c_A$ in $A$ such that there are no essential simplices, we modify the strata to create a related partition of $A$, and then color the vertices of $c_A$ according to which set of the partition they are in. In the following sublemma we construct this partition of $A$.

**Sublemma 41.** Let $X$ be a compact metric space, with a generalized stratification satisfying the NDR property, such that there are finitely many strata. Let $A$ be any closed subset of $X$, and let $\varepsilon > 0$ be arbitrary. Then there exists a partition $\{P_S\}$ of $A$, containing one subset $P_S$ for each stratum $S$, and $\delta > 0$ such that the following properties hold:

- For every stratum $S$, the set $P_S$ is contained in the $\varepsilon$-neighborhood of $S$.
- If $S$ and $S'$ are incomparable strata, then the distances $\text{dist}(P_S, P_{S'})$, $\text{dist}(P_S, S')$, and $\text{dist}(S, P_{S'})$ are all greater than $\delta$.
- For every stratum $S$, the $\delta$-neighborhood of $P_S$ in $A$ is contained in an NDR neighborhood $U_S$ of $S$ in $X$.

**Proof.** We start by numbering the strata $S_1, \ldots, S_r$ such that if $S_j \preceq S_i$ then $j \leq i$. We construct $P_{S_i}$ in order, with a recursion assumption that if $P_{S_j}$ have been constructed for all $j < i$, then their union $\bigcup_{j<i} P_{S_j}$ is an $A$-open neighborhood of the (closed) union $A \cap \left( \bigcup_{j<i} S_j \right)$. To construct $P_{S_i}$, we let $K_i$ be the complement of $\bigcup_{j<i} P_{S_j}$ in $A \cap S_i$; the set $K_i$ is compact. If $S'$ is any stratum so that its closure $\overline{S'}$ is disjoint from $S_i$ (equivalently $S' \nvdash S_i$), then $\overline{S'}$ has positive distance from $K_i$. We select $\varepsilon_i < \varepsilon$ such that $3\varepsilon_i$ is less than this positive distance for all such strata $S'$ and such that the $3\varepsilon_i$-neighborhood $N_{3\varepsilon_i}(K_i)$ (taken in $A$ or $X$, it doesn't matter) is contained in an NDR neighborhood $U_{S_i}$ of $S_i$ in $X$. We let $P_{S_i}$ be the complement of $\bigcup_{j<i} P_{S_j}$ in $A \cap N_{3\varepsilon_i}(K_i)$. The recursion assumption is preserved: the new union $\bigcup_{j\leq i} P_{S_j}$ is an $A$-open set containing $A \cap \left( \bigcup_{j\leq i} S_j \right)$. 62
Having constructed all $P_{S_i}$, we set $\delta < \min \varepsilon_i$ and check that the construction satisfies the three conclusions of the lemma. The first conclusion is true because $P_{S_i}$ is in the $\varepsilon_i$-neighborhood of $S_i$. For the second conclusion, suppose that $S_i$ and $S_j$ are incomparable strata. Because $S_j \neq S_i$, all of $S_j$ lies further than $3\varepsilon_i$ away from $K_i$, whereas all of $P_{S_i}$ lies within $\varepsilon_i$ of $K_i$. Thus we have $\text{dist}(P_{S_i}, P_{S_j}) > 2\varepsilon_i - \varepsilon_j$. We may reverse the roles of $S_i$ and $S_j$ if necessary, to assume $\varepsilon_i \geq \varepsilon_j$, in which case $2\varepsilon_i - \varepsilon_j > \delta$. The third conclusion is true because $N_\delta(P_{S_i}) \subseteq N_{3\varepsilon_i}(K_i) \subseteq U_{S_i}$.

**Proof of the localization lemma (Lemma 13).** Let $c_{rel}$ be a relative $(X, A)$ cycle that represents $h_{rel}$ and satisfies the cellular, order, internality, and loop conditions. We apply Sublemma 40 to get a chain $c_A$ in $A$ such that $c_{rel} + c_A$ is a cycle representing $h$. We apply Sublemma 41 to get a partition $\{P_S\}$ of $A$ and a number $\delta > 0$. We construct, as input to the amenable reduction lemma (Lemma 11), a cycle $c = c_{rel} + c_1 + c_2 + c_3$ representing $h$ and a partial coloring on $c$ with one subset of vertices $V_\ell$ for each stratum $S_\ell$, as follows:

- To construct $c_5$, we start with $c_A$ and apply iterated barycentric subdivision until the diameter of each simplex in $X$ is less than $\delta$. For each vertex $v$ of $c_5$, if $v \in P_{S_\ell}$, then $v \in V_\ell$.

- $c_2$ is the cylinder $-\partial c_0 \times [0, 1]$ with a standard prism triangulation, mapped to $X$ by the projection $-\partial c_0 \times [0, 1] \rightarrow -\partial c_6$. The vertices of $-\partial c_0 \times 1$ are identified with the vertices of $c_5$, so their partial coloring is determined by their membership in $P_{S_\ell}$. The vertices of $-\partial c_0 \times 0$ have a different partial coloring: if $v \in V_\ell$, then $v \in V_\ell$.

- $c_1$ is a subdivision of the cylinder $\partial c_{rel} \times [0, 1]$, mapped to $X$ by the projection $\partial c_{rel} \times [0, 1] \rightarrow \partial c_{rel}$. The end $\partial c_{rel} \times 0$ is identified with $\partial c_{rel}$ and is not subdivided. The end $\partial c_{rel} \times 1$ is divided by barycentric subdivision so that it may be identified with the 0 end of $c_2$, which is equal to $-\partial c_5$. The middle of the cylinder is subdivided by concatenating the chain homotopies corresponding to barycentric subdivision, one for each iteration. For each vertex $v$ of $c_1$, if $v \in V_\ell$, then $v \in V_\ell$.

- For every vertex $v$ of $c_{rel}$, if $v \in S_\ell$, then $v \in V_\ell$.

First we verify that every simplex in $c_1$, $c_2$, and $c_5$ is not essential. Any loops in any of these chains are null-homotopic because of either the loop property on $c_{rel}$ or the barycentric subdivision as in Sublemma 12. To show that two of the $j + 1$ vertices of each simplex are labeled with the same subset $V_\ell$, we claim that for every two vertices of a simplex, the strata $S_\ell$ corresponding to their labels must be comparable strata. In $c_1$ this is true by the order property or Sublemma 12. In $c_5$ this is true because $\text{dist}(P_S, P_{S'}) > \delta$ whenever $S$ and $S'$ are incomparable strata. In $c_2$, to show that every two vertices in a simplex have labels corresponding to comparable strata, there are three cases: if both vertices are in $c_1$, we use the argument for $c_1$; if both vertices are in $c_5$, we use the argument for $c_5$; and if there is one vertex in each, we
use the fact that dist(S, P_{S'}) > \delta whenever S and S' are incomparable strata. Thus, for every simplex in c_1, c_2, and c_6, the j + 1 vertex labels correspond to a totally ordered chain of strata intersecting A, and every such chain has length at most j so two of the labels must be the same.

Let (\Sigma, \sigma) be the simplicial complex structure of c. Next we show that for every V_\ell, the induced 1-complex \Sigma^1 \ell \subseteq \Sigma is homotopic in X to something with image in S_\ell. Then the \alpha-image of the fundamental group of every connected component of \Sigma^1 \ell is a subgroup of the \alpha-image of the fundamental group of S_\ell, which we have assumed to be an amenable group; any subgroup of an amenable group is also amenable. We homotope \Sigma^1 \ell into S_\ell by showing that it is included in the NDR neighborhood U_{S_\ell}, which homotopes into S_\ell. Indeed, every edge in \Sigma^1 \ell is in S_\ell \cup N_\delta(P_{S_\ell}), where N_\delta denotes the \delta-neighborhood: if the edge is in c_{rel} or c_1, then both endpoints are in S_\ell, so the edge is in S_\ell by the internality condition or Sublemma 12; if the edge is in c_2 or c_6 but not in c_1, then at least one endpoint is in P_{S_\ell}, and the whole simplex has diameter less than \delta, so the edge is in N_\delta(P_{S_\ell}). So we have

\[ \Sigma^1 \ell \subseteq S_\ell \cup N_\delta(P_{S_\ell}) \subseteq U_{S_\ell}, \]

and thus the whole 1-complex may be homotoped into S_\ell.

Now every hypothesis of the amenable reduction lemma (Lemma 11) is satisfied, so we apply it to obtain

\[ \| \alpha_* h \|_\Delta \leq \| c_{rel} \|^{S}_{\Delta, ess}. \]

\[ \square \]
Bibliography


