SMALL DISTURBANCES IN COMPRESSOR ANNULI WITH SWIRL, THROUGHFLOW AND ENTROPY VARIATION

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INTRODUCTION

Control of the radiation of sound from the compressors and turbines of jet engines depends to a great extent on understanding of the propagation of acoustical modes in the ducting, the general design approach being to choose the numbers of blades in interacting rows so that no propagating mode will be strongly excited.[(1)] The conditions for non-propagation or "cutoff" are therefore critical to this procedure. Another application of modal analysis is found in the linear three dimensional flow theory of turbomachinery.[(2)] Here the complete isentropic flow field of the compressor rotor is represented as a superposition of normal modes.

To date, most such modal treatments have either neglected the effect of average flow velocity in the turbomachine duct, or considered the acoustical disturbances to propagate in a gas at rest in a coordinate system moving with the average flow velocity. This approach is correct if the resulting (moving) coordinate system is inertial, but in general is not correct for rotating coordinate systems. In the context of compressor analyses, it is valid for uniform axial flow, as applied by McCune,[(2)] for example, but incorrect for swirl, as applied by Morfey.[(3)]

Indeed, as we shall show, the classical technique of dividing small disturbances into the three classes[(4)] of vorticity, entropy, and sound fluctuations, which do not interact to first order, is not valid in rotating flows. Thus, a generalization of the concepts of sound and turbulence is needed. Such a generalization will not be achieved in the present work, but it is hoped that a few steps will be made in this direction.

The general equations for pressure disturbances in an inhomogeneous swirling gas have been given by Blokhintsev,[(5)] who also obtained the general equation for an isentropic gas. Apparently the only other analyses of pressure wave behavior in rotating fluids are those of Salant[(6)] and Sozou.[(7)] The former considered the effects of a solid body rotation on the symmetric normal modes, i.e., modes with no tangential nonuniformity. The latter treated the same type of disturbance in a Rankine vortex.

The main purpose of the present analysis is to provide a consistent modal acoustic treatment for compressor annuli with large swirl and throughflow, and with radial variations of entropy. The mean flow will be assumed uniform in the axial and tangential directions, so that the results are applicable only sufficiently far upstream and downstream of blading that the first order variations in these directions have died out.

As might be expected, the analysis is nevertheless somewhat complex. While a general treatment will be given, for arbitrary radial variations of entropy and tangential and axial velocity, analytical solutions for the radial eigenfunctions are available only for some special cases. These do include three important cases, namely, 1) isentropic flow with solid body rotation and constant axial velocity, 2) isentropic flow with free vortex rotation and constant axial velocity, and 3) flow with negligible mean velocity but with radial entropy variation. The first of these represents the conditions behind inlet guide vanes, though not with complete consistency, as will be noted below. The second represents quite accurately the conditions behind high-work fan rotors, except for the effects of entropy variation. The last case gives some insight into the effects of such variations.

GENERAL FORMULATION

We consider the flow of an inviscid perfect gas, through an annulus described in cylindrical coordinates r, θ, z by an inner radius r_i and an outer radius r_o. The steady (unperturbed) flow has axial velocity W(r), tangential velocity V(r) and entropy S(r). Denoting the total value of each of the variables by a prime, we expand about the mean values, so that the velocity components are u'=u, v'= V+v w'= W+w. Similarly the entropy, pressure, density, and temperature are s'= S+s, p'= P+p, ρ'= R+ρ, $\mathcal{T}$'= T+$\mathcal{T}$. To first order, the equations describing the gas are then,

$$L(u) - \frac{2V}{r}v = -\frac{1}{R}\frac{\partial p}{\partial r} + \frac{1}{R^2}\frac{dP}{dr}\rho \tag{1}$$

$$L(v) + \left(\frac{dV}{dr} + \frac{V}{r}\right) u = -\frac{1}{Rr}\frac{\partial p}{\partial \theta} \tag{2}$$

$$L(w) + \left(\frac{dW}{dr}\right) u = -\frac{1}{R}\frac{\partial p}{\partial z} \tag{3}$$

$$L(\rho) + R\frac{\partial u}{\partial r} + \left(\frac{dR}{dr} + \frac{R}{r}\right) u + \frac{R}{r}\frac{\partial v}{\partial \theta} + R\frac{\partial w}{\partial z} = o \tag{4}$$

$$L(s) + \left(\frac{dS}{dr}\right) u = o \tag{5}$$

$$s = c_p \frac{\mathcal{T}}{T} - \mathcal{R}\frac{p}{P} \tag{6}$$

$$\frac{p}{P} = \frac{\rho}{R} + \frac{\mathcal{T}}{T} \tag{7}$$

where L≡ ∂/∂t + (V/r)∂/∂θ + W∂/∂z is a convective derivative following the mean flow.

It is the second term of Eq.(1) and the second term of Eq.(2) which are neglected when the noninertial character of the rotating coordinate system is ignored. The last term of Eq.(1) is also usually neglected.

Within the above assumptions, a consistent mean flow must satisfy only the relations,

$$\frac{1}{R}\frac{dP}{dr} = \frac{V^2}{r} \tag{8}$$

$$S = C_p \log T - \mathcal{R} \log P \tag{9}$$

$$P = R\,\mathcal{R}\,T \tag{10}$$

If in addition the flow has constant stagnation temperature, then there is the added constraint $T_t = T + (W^2 + V^2)/2C_p$, which connects W(r) and V(r).

Some general observations can be made at this point concerning the behavior of fluctuations in a rotating flow. We note first that if V = 0, and therefore dP/dr = 0 and if dR/dr = 0 , dW/dr = 0 , and dS/dr = 0 , it can be shown that a general disturbance of the gas may be represented as a superposition of three types of elementary disturbances,[4] namely vorticity, entropy, and sound. The vorticity and entropy satisfy $L(\vec{\omega}) = 0$ and $L(s) = 0$ respectively, while the sound satisfies the wave equation in coordinates fixed in the moving gas, $a^2\nabla^2 p - L^2 p = 0$.

From Eqs.(1) to (5), we see that in a swirling gas with radial entropy gradient, none of these simple modes exist independently of the others. Eq.(5) states that it is the total entropy of the gas which is convected, rather than just the entropy perturbation, and this is intuitively reasonable. This conclusion implies, however, that the entropy perturbation is nonzero if $dS/dr \neq 0$ and there is a radial velocity perturbation. The entropy perturbation enters the radial momentum equations through ρ, so in general it will couple to both pressure and vorticity disturbances.

It does not seem fruitful to attempt reduction of Eqs.(1) - (5) to a set of equations relating the three perturbation modes, since in general the source terms are not expressible in terms of the separate modes. Some progress can however be made by means of a partial Fourier analysis of the general disturbance field. Suppose we represent each of the dependent variables by the form

$$\rho(r,\theta,z,t) = \int \rho(r,m,k,\omega)e^{i(kz + m\theta - \omega t)}dk\,dm\,d\omega \tag{11}$$

Subject to some questions of convergence of the integrals, this is a perfectly general representation of an arbitrary disturbance field. We may then, from an examination of the behavior of $\rho(r,m,k,\omega)$, $s(r,m,k,\omega)$ etc., infer the behavior of such an arbitrary field.

Transforming Eqs.(1) to (5) in this way we find,

$$i\lambda u - \frac{2V}{r} v = - \frac{a^2}{R}\frac{d\rho}{dr} + \left[(2-\gamma)\frac{V^2}{r} - \frac{a^2}{c_p}\frac{dS}{dr}\right]\frac{\rho}{R} - \frac{a^2}{c_p}\frac{ds}{dr} - \frac{\gamma}{c_p}\frac{V^2}{r} s$$

$$= - \frac{1}{R}\frac{dp}{dr} + \frac{1}{R}\frac{dP}{dr}\left(\frac{1}{\gamma}\frac{p}{P} - \frac{s}{c_p}\right) \tag{12}$$

$$i\lambda v + \left(\frac{dV}{dr} + \frac{V}{r}\right)u = -i\frac{ma^2}{r}\left(\frac{s}{c_p} + \frac{\rho}{R}\right) = - \frac{ima^2}{r\gamma}\frac{p}{P} \tag{13}$$

$$i\lambda w + \left(\frac{dW}{dr}\right)u = - ika^2\left(\frac{s}{c_p} + \frac{\rho}{R}\right) = \frac{ika^2}{\gamma}\frac{p}{P} \tag{14}$$

$$i\lambda\rho + R\frac{du}{dr} + \left(\frac{dR}{dr} + \frac{R}{r}\right)u + \left(\frac{imR}{r}\right)v + (ikR)w = 0 \tag{15}$$

$$i\lambda s + \left(\frac{dS}{dr}\right)u = 0 \tag{16}$$

where $\lambda = kW(r) + mV(r)/r - \omega$

(17)

A general procedure for reduction of these equations to a single second order differential equation in u can be given as follows. Since (13), (14) and (16) are algebraic in v, w, and s, they can be solved in terms of ρ and u. Eliminating s, v, w from (12) and (15) one finds that (15) involves u, du/dr, and ρ, but not $d\rho/dr$. Solving for ρ from (15) then permits its elimination from (12), resulting in a single second order equation for u. Unfortunately the coefficients are a bit complicated, so this result will not be written out. Rather, the special cases mentioned in the Introduction will be discussed.

Before proceeding to these detailed solutions, some observations are in order concerning the general behavior of a disturbance field in the rotating gas.

We note first that within the present formalism, the independent convection of vorticity and entropy in a uniform gas is recovered by putting $\lambda = o$, as this expresses the condition that the disturbance be convected. It is immediately clear, then, from Eqs.(12), (13) and (14) that u, v, and w can be prescribed arbitrarily, except for the condition $\mathrm{div}\,\vec{u} = o$ from Eq.(15). Similarly, from Eq.(16) s is arbitrary for convected disturbances, satisfying $\lambda = o$.

<u>"Shear" Flows:</u>

In the rotating nonuniform gas, on the other hand, the condition $\lambda = o$ does not always permit a velocity field without density or entropy fluctuations, because of the second terms in Eqs.(12), (13), (14) and (16). We may ask, however, whether there is a (different) condition on k, m, ω, which permits a velocity perturbation, independent of ρ and s, which is in this sense analogous to a solenoidal shear flow. One such set of conditions is simply that for solvability of the set of homogeneous linear equations in u, v, w, obtained by putting $\rho = s = o$, together with Eq.(15). The former is $\lambda D = o$, where

$$D = \lambda^2 - \frac{2V}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right) \tag{18}$$

The solution $\lambda = o$ corresponds to $u=v=o$, and an arbitrary w. It is valid for $dS/dr \neq o$. From the continuity equation (15), this disturbance must satisfy $k = o$, i.e., w must be uniform in z, and it then follows from $\lambda = o$ that $V = \Omega r$. It represents a disturbance of the axial velocity alone, variable in r and o, convected by a solid body mean flow. Such a disturbance might be produced, for example, by a set of non lifting blades rotating in a solid body flow of the same angular velocity as the blades.

The second solution ($D = o$) requires $dS/dr = o$ and Eqs.(12) and (14) imply that

$$\frac{v}{u} = \frac{i\lambda r}{2V}, \quad \frac{w}{u} = \frac{i}{\lambda}\frac{dW}{dr} \tag{19}$$

and the continuity equation (15) requires that

$$\frac{r}{u}\frac{du}{dr} + 1 + \frac{r}{R}\frac{dR}{dr} - \frac{m\lambda r}{2V} - \frac{kr}{\lambda}\frac{dW}{dr} = o$$

which integrates to

$$u(r) = \frac{c}{Rr} e^{\int\left(\frac{m\lambda}{2V} + \frac{k}{\lambda}\frac{dW}{dr}\right)dr} \tag{20}$$

This disturbance is therefore of the form,

$$u = \frac{c}{Rr} e^{i(kz + m\theta - \omega t) + \int\left(\frac{m\lambda}{2V} + \frac{k}{\lambda}\frac{dW}{dr}\right)dr} \tag{21}$$

which can be made specific when $V(r)$ and $W(r)$ are specified. The forms of v and w are then given by Eqs.(19). Tangential and radial velocity perturbations are allowed by this form, but they are not independent, and if $dW/dr = o$, $w = o$ except for the special case that $\lambda = o$ as well as $D = o$. Thus, for $dW/dr = o$ we have,

$$u = \frac{c}{Rr} e^{i(kz + m\theta - \omega t) + \frac{1}{2}\int m[k(W/V) + \frac{m}{r} - \frac{\omega}{V}]dr}$$

$$v = \frac{i}{2}\left[k\frac{W}{V}r + m - \frac{\omega r}{V}\right]u \qquad W = \text{ct.}$$

$$w = o \tag{22}$$

For a wheel flow, $V = \Omega r$,

$$u = \frac{c}{R} r^{\pm m\Omega - 1} e^{i(kz + m\theta - \omega t)}$$

$$v = \pm i \ \Omega u \qquad\qquad V = \Omega r$$

$$w = o \qquad\qquad W = ct. \qquad (22a)$$

where the condition $D = o$ has been used to write

$$\left(\frac{kW}{\Omega} + m - \frac{\omega}{\Omega}\right)^2 = 4\Omega^2$$

There seems to be no pure "shear" disturbance in a free vortex, since neither the condition $\lambda = kW + mV/r - \omega = o$ nor the condition $D = \lambda^2 = o$ can be satisfied for all r.

To illustrate the significance of these results, consider a viscous wake shed by a stationary radial vane set at zero incidence in a wheel flow. For steady flow $\omega = o$. We note that Eq.(22a) will not accommodate an axial velocity perturbation at all. Nor can a solution of the type $\lambda = o$ provide the axial velocity perturbation, since it is invariant in z. The conclusion which appears to follow is that the viscous wake of a blade in a rotating fluid in general cannot be in static equilibrium with the inviscid flow. It must have an associated pressure disturbance, and in this sense the analogy between the behavior of cascades and that of blades in swirling flows breaks down.

An additional conclusion which can be inferred is that a general "shear" disturbance convected through a rotating flow will produce a first order pressure disturbance. This conclusion will be quantified for $dS/dr = o$ below. At this point we simply note that this coupling implies a strong production of broad band noise in a turbulent rotating flow.

Entropy Spottiness:

Equation (16) states that the total entropy of the gas is convected, rather than just the entropy perturbation. Any radial velocity perturbation will, for $dS/dr \neq 0$, generate an entropy perturbation, s, as is obvious. The resulting s will interact through the radial momentum equation to produce velocity and pressure perturbations.

Furthermore, even if $dS/dr = 0$, so that the entropy perturbations are convected ($\lambda = o$), these convected disturbances will still interact in the radial momentum equation.

Thus we conclude that in the presence of swirl ($dP/dr \neq 0$) any entropy spottiness will produce velocity and sound fields. This is another first order source of broad band noise.

Pressure Waves:

There seem to be no very general results available for the behavior of pressure waves when $dS/dr \neq 0$ and $V \neq 0$. In the next section, pressure disturbances will be considered for $dS/dr = 0$, and in the following section for $dS/dr \neq 0$, $V = W = 0$.

ISENTROPIC FLOW

A uniform gas ingested by an ideal (lossless) turbomachine would satisfy the condition $dS/dr = 0$, and in addition $s = 0$, so that the special case of a completely isentropic gas is of quite general interest. It must be distinguished from the case of an isentropic mean flow, $dS/dr = 0$, which, as noted above, allows convected entropy fluctuations which in general couple to pressure fluctuations.

For the more restricted situation, Eqs. (12), (13), and (14) can be solved for u and v and w, with the results,

$$u = \frac{i}{D}\left\{\frac{\lambda a^2}{R}\frac{d\rho}{dr} + \left[\frac{2mVa^2}{Rr^2} - (2-\gamma)\frac{\lambda V^2}{Rr}\right]\rho\right\} \tag{23}$$

$$v = -\frac{1}{D}\left\{\left(\frac{dV}{dr} + \frac{V}{r}\right)\frac{a^2}{R}\frac{d\rho}{dr} - \left[(2-\gamma)\frac{V^2}{Rr}\left(\frac{dV}{dr} + \frac{V}{r}\right) - \frac{\lambda m a^2}{Rr}\right]\rho\right\} \tag{24}$$

$$w = \left(\frac{i}{\lambda}\frac{dW}{dr}\right) u - \left(\frac{k}{\lambda}\frac{a^2}{R}\right)\rho \tag{25}$$

where $$D = \lambda^2 - \frac{2V}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right) \neq 0$$

It is readily demonstrated that this velocity field is irrotational if V is irrotational , i.e., $V = \Gamma/r$, but it is otherwise rotational. Since the flow is for this case barotropic, the total fluid acceleration has a potential, but this does not mean that the velocity perturbations are irrotational. Indeed this is only the case when the mean flow is irrotational.

Eliminating u, v, w from Eq.(15), and expanding the coefficients to second order in the swirl Mach number gives the following equation for ρ,

$$r^2\frac{d^2\rho}{dr^2} + \left[1 + (2\gamma - 3)\frac{V^2}{a^2} - \frac{r}{D}\frac{dD}{dr}\right] r\frac{d\rho}{dr}$$

$$+ \left\{\frac{r^2 D}{a^2}\left[1 - \frac{a^2k^2}{\lambda^2}\right] - m^2 - \frac{2mV}{\lambda r}\left(1 + \frac{r}{D}\frac{dD}{dr} - \frac{r}{V}\frac{dV}{dr}\right) - \left(\frac{2kmV}{\lambda^2}\right)\frac{dW}{dr}\right\}\rho = 0 \tag{26}$$

as the general equation for the radial eigenfunction for the case of $dS/dr = 0$, $s = 0$.

Wheel Flow

The simplest case of general interest is that of a solid body rotation. We put

$$V = \Omega r \qquad , \ \lambda = m\Omega + kW - \omega$$

$$D = \lambda^2 - 4\Omega^2 \ , \ a^2 = a_c^2 + \frac{\gamma-1}{2}\Omega^2 r^2$$

It will also be assumed that W is constant, even though this implies a radial variation in stagnation temperature, which is not realistic for flows induced by inlet guide vanes, for example. The elimination of dW/dr makes it possible to see the effects of rotation in the simplest context.

With these conditions, Eq.(26) becomes,

$$r^2 \frac{d^2\rho}{dr^2} + [1 + (2\gamma - 3)\frac{\Omega^2 r^2}{a^2}]\, r \frac{d\rho}{dr} + \{(\frac{\lambda^2 - 4\Omega^2}{a_c^2})[(1 - \frac{k^2 a_c^2}{\lambda^2})r^2 - \frac{\gamma-1}{2}\frac{\Omega^2 r^4}{a_c^2} - m^2\}\rho = 0 \tag{27}$$

The form of this equation simplifies somewhat for the special case of $\gamma = 3/2$, when the r dependent term in the second coefficient is zero. Since $\gamma = 3/2$ is a "reasonable" value, it seems justified for the present purposes to drop this term. Furthermore the term in r^4 of the last coefficient is small compared to that in r^2 unless the tangential Mach number exceeds unity, so we drop it also. The solutions are then in terms of Bessel functions,

$$\rho = Z_m(\mu r) \tag{28a}$$

where

$$\mu^2 = \frac{\lambda^2 - 4\Omega^2}{a_c^2}(1 - \frac{k^2 a_c^2}{\lambda^2})$$

From Eq.(23), the radial velocity is

$$u = \frac{i}{D}\{\frac{\lambda a^2}{R}\frac{d\rho}{dr} + [\frac{2m\Omega a^2}{Rr} - \frac{\lambda\Omega^2}{2R}r]\,\rho\}$$

so that if $u(r_o) = u(r_i) = 0$, we have

$$\frac{r}{\rho}\frac{d\rho}{dr} = -\frac{2m\Omega}{\lambda} + \frac{\Omega^2 r^2}{2a_c^2}; \qquad r = r_o, \; r_i \tag{29}$$

Now for given frequency ω and azimuthal eigenvalue m, μ and k are determined by (29).

Supposing for the moment that the value of m is set by matching to a rotor (e. g. m equals the number of blades for the first harmonic), and that μ has been determined from Eq. (29), we may determine the conditions for propagation of the resulting mode. The condition is that k be real. Since $\lambda = m\Omega + kW - \omega$, this is equivalent to requiring that λ be real.

Writing Eq.(28a) in terms of λ, we find

$$(1 - M^2)\lambda^4 + (2\nu)\lambda^3 + [\nu^2 + \mu^2 W^2 - 4\Omega^2(1 - M^2)]\lambda^2 - (8\Omega^2\nu)\lambda - 4\Omega^2\nu^2 = 0 \tag{28b}$$

where $\nu = \omega - m\Omega$. The condition for existence of propagating modes is the condition for existence of real roots of this equation.

Consider first the limiting case of $\Omega \to 0$. The equation then reduces to

$$(1 - M^2)\lambda^4 + (2\nu)\lambda^3 + [\nu^2 + \mu^2 W^2]\lambda^2 = 0 \tag{28c}$$

which has a double root $\lambda = 0, 0$. For $M^2 < 1$, and $\nu = \omega - m\Omega = m(\Omega_R - \Omega) > 0$, where Ω_R is the angular velocity of the rotor with m blades, all coefficients of the quadratic are positive. According to Descartes Rule of Signs, there are then no positive real roots, and either two or no negative real roots. There are two negative real roots if the rotors angular velocity is "above cutoff" in the conventional sense, the condition for propagation being that $4\nu^2 - 4(1 - M^2)\,[\nu^2 + \mu^2 W^2] > o$

or $\nu^2 > (1 - M^2)\, a^2\, \mu^2$

For large m this is equivalent to the condition that the tip relative mach number exceed unity.

For $\Omega \neq o$, we note that because the last two terms of Eq.(28b) are negative, there will always be one change of sign for positive λ , and usually three changes of sign for negative λ . Thus, according to Descartes Rules of Signs, there will always be one positive real root, and either three or one negative real roots. It is readily seen that the two conditional negative roots correspond to the two found for $\Omega = o$. Thus, the effect of rotation of the flow is to add one positive and one negative real root. These roots are present for any nonzero Ω , so we conclude that there may always be propagating modes in the presence of a solid body rotation of the flow.

To determine the magnitude of these roots it is convenient to introduce the following notation

$$\varepsilon = \frac{\Omega}{\Omega_R}, \qquad y = \frac{\lambda}{m\Omega_R}$$

$$M_T = \frac{\Omega_R\, r_t}{a(r_t)}$$

where r_t is the tip radius, and Ω_R is the angular velocity of an exciting rotor. Then $\nu = m\Omega_R(1 - \varepsilon)$, and Eq.(28b) becomes

$$f(y) = (1-M^2)y^4 + 2(1-\varepsilon)y^3 + \left[(1-\varepsilon)^2 + \frac{\mu^2 W^2}{m^2 \Omega_R^{\,2}} - 4\frac{\varepsilon^2}{m^2}(1 - M^2)\right]y^2 - 8\frac{\varepsilon^2}{m^2}(1-\varepsilon)y$$
$$- 4\frac{\varepsilon^2}{m^2}(1-\varepsilon)^2 = o \qquad (28d)$$

In terms of y, k is given by $kr_t = 2m(1 - \varepsilon + y)$. For m large, the last two terms, which lead to the new roots, are small, with the result that these roots fall near $y = o$. The behavior of f(y) is shown schematically in Fig.(1).

Sketch a) shows the behavior without swirl, and just at cutoff, the full curve showing the usual quadratic form of f(y), the dashed curve,the full quartic. Adding swirl splits the double root at the origin to a positive and negative pair as shown in sketch b), which is drawn for a condition below cutoff but with swirl. Sketch c) shows the behavior above cutoff, and with swirl: The condition $\lambda = y = o$ characterizes a disturbance convected with the mean flow, so that as $\Omega \rightarrow o$, the swirl modes are purely convected. As the swirl is increased (increasing ε), one mode propagates upstream relative to the flow, and one downstream. Since the last two terms of Eq.(28d) are of order $1/m^2$, the propagation velocity relative to the flow will be small unless m is small, and it is unlikely that modes with m equal to 10 or larger will propagate upstream. Both modes should, however, be found in the flow field downstream of a rotor.

As an example consider a ten bladed rotor (m = 10) with swirl angular velocity one half the rotors angular velocity ($\varepsilon = \frac{1}{2}$), and axial velocity one half the tangential velocity of the rotor tip ($W/\Omega_R r_t = \frac{1}{2}$). Varying the axial mach number, M, leads to the roots shown (approximately) in Table I. The axial mach number at cutoff for the usual modes is 0.703. The modes due to swirl propagate with nearly constant axial wave numbers over the range of mach number from 0.5 to 0.8.

Values of the wave number, (times the tip radius), are shown at the right in Table I. The modes due to swirl have wave numbers slightly below and above the value of $kr_t = 10$ which corresponds to pure convection. As kr_t must be negative for a mode to propagate upstream, both swirl modes would appear only in the flow downstream of the rotor for this example.

A more detailed numerical analysis will be required to exhibit more fully the effects of swirl. It seems that the amplitude of the swirl modes must be connected to the amplitude of the swirl, so it should become small for

Table I

ROOTS OF EQ. (28d) AND CORRESPONDING AXIAL WAVE NUMBERS

M	y_1	y_2	y_3	y_4	$(kr_t)_1$	$(kr_t)_2$	$(kr_t)_3$	$(kr_t)_4$
.5	-.66 -.462	-.66 +.462	-.065	.075	-3.2 -9.22	-3.2 +9.22	8.7	11.5
.703	-.99	-.99	-.044	.103	-10	-10	9.1	12.1
.8	-2.13	-.65	-.064	.075	-32.6	-3	8.7	11.5

small swirl. They may nevertheless be dominant even for small swirl, when the usual modes are below cutoff.

Free Vortex Flow

Many high-work compressor and fan stages induce a free vortex flow, as this yields uniform work radially. If the flow has constant stagnation temperature and entropy initially, it also has constant stagnation temperature and entropy downstream of the rotor, and it is readily demonstrated that the first implies constant axial velocity. Thus, as the next case we take,

$$V = \frac{\Gamma}{r}, \quad W = ct.$$

$$\lambda = m\Gamma/r^2 + kW - \omega$$

$$D = \lambda^2, \quad a^2 = a_\infty^2 \left[1 - \frac{\gamma - 1}{2} \frac{\Gamma^2}{a_\infty^2 r^2}\right]$$

Eq.(26) may then be put in the form,

$$r^2 \frac{d^2\rho}{dr^2} + \left[1 + (2\gamma - 3)\frac{\Gamma^2}{r^2 a^2} + \frac{4m\Gamma}{(kW-\omega) r^2} - \frac{-4m^2\Gamma^2}{(kW-\omega)^2 r^4}\right] r dr$$

$$+ \left\{\left[\frac{(kW-\omega)^2}{a^2} - k^2\right] r^2 - \frac{4m\Gamma}{(kW-\omega) r^2} + \left[2(kW-\omega)\frac{m\Gamma}{a^2} - m^2\right] + \frac{8m^2\Gamma^2}{(kW-\omega)^2 r^4}\right.$$
$$\left. + \frac{m^2\Gamma^2}{a^2 r^2}\right\}\rho = 0$$

Substituting the expression for a^2 leads to additional terms in both coefficients, proportional to Γ^2. No analytical solution seems to be available for this equation, which is analogous to Eq.(27) in that it is correct to second order in the swirl mach number. Dropping terms of this order, i.e., deleting the last term of the first coefficient and the last two terms of the second coefficient, and replacing a^2 by $a_\infty^{\,2}$, we find the solution [9],

$$\rho = e^{\frac{m\Gamma}{(kW-\omega)r^2}} Z_\nu\,(\alpha r) \qquad (31)$$

where $\nu^2 = m^2 - 2(kW-\omega)\frac{m\Gamma}{a_\infty^{\,2}}$

$$\alpha^2 = \frac{(kW-\omega)^2}{a_\infty^{\,2}} - k^2$$

From Eq.(26), if we require $u(r_o) = u(r_i) = 0$,

$$\frac{r}{\rho}\frac{d\rho}{dr} = \frac{\Gamma}{r^2}\left[\frac{\Gamma}{2a_\infty^{\,2}} - \frac{2m}{\lambda r}\right], \quad r = r_o,\ r_i \qquad (32)$$

The effect of swirl enters in the form of the eigenfunction, and in its order ν, as well as in the boundary condition, so that a detailed numerical solution will be required to determine the total effect on conditions for propagation. It is interesting to note, however, that for a given mode, i.e., a given value of α, the tip speed for excitation of propagating modes is <u>independent</u> of the swirl mach number. Thus, solving for k in terms of α, we find

$$k = -\frac{M}{1-M^2}\left(\frac{\omega}{a_\infty}\right) \pm \sqrt{\frac{(\omega/a_\infty)^2}{(1-M^2)^2} - \frac{\alpha^2}{1-M^2}}$$

so that the condition for propagation is

$$\left(\frac{\omega}{a_\infty}\right)^2 > \alpha^2(1-M^2)$$

In terms of the rotor tip mach number,

$$M_T^{\,2} > \frac{(\alpha r_o)^2}{m^2}(1-M^2)$$

For the limit of $m \to \infty$, $(\alpha r_o/m)^2 \to 1$, and this becomes $M_T^2 + M^2 > 1$.

The interesting point here is that it is not the mach number relative to the flow which controls cutoff for this case, as in the wheel flow. Only the axial and rotor tangential mach numbers are relevant. This suggests, for example, that a rotor producing a vortex swirl may excite propagating modes downstream even though its exit relative mach number is less than unity. It also indicates that a rotor operating in a vortex swirl in the direction of rotation may have the same critical tip speed as one operating without inlet swirl.

As noted above, both of these conclusions are tentative, pending computation of the eigen values. Since the swirl modes stemmed from terms of order of the swirl mach number squared in the solid body rotation case, we should not expect to find these modes in the present solution which is valid only to first order in the swirl mach number. They may very well be present in the free vortex flow, but a numerical integration of the differential equation will be required to settle this question.

It seems worth noting that for this special case of free vortex flow an alternative formulation, in terms of the velocity potential, is possible. Thus, taking

$$u = \frac{\partial \phi}{\partial r}$$

$$v = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

$$w = \frac{\partial \phi}{\partial z}$$

Eq.(15) may be put in the form,

$$r^2 \frac{d^2\phi}{dr^2} + [1 + \frac{\Gamma^2}{r^2a^2}] r\frac{d}{dr} + \{[\frac{(kW-\omega)^2}{a^2} - k^2]r^2 + \frac{m^2\Gamma^2}{a^2r^2} + [\frac{2m\Gamma(kW-\omega)}{a^2} - m^2]\}\rho = 0$$

To first order in Γ/ra this has the solution, $\phi = Z_\nu(\alpha r)$ where ν and α are as given above. From (13) o (14), $\rho = (-iR\lambda / a^2)\phi$, so that we have

$$\rho = \lambda\phi = [1 + \frac{m\Gamma}{(kW-\omega)r^2}] Z_\nu(\alpha r)$$

which is equivalent to (31) to first order in Γ/ra.

The difficulty in carrying this formulation to $O(\Gamma/ar)^2$ is as great as for the ρ formulation.

Volumetric Coupling of Turbulence and Sound

As noted above, there is a first order coupling between pressure and velocity perturbations, unless the velocity satisfies special conditions such as Eq.(21). Some insight into this coupling can be gained by writing expressions for the vorticity, and for the pressure which are analogous to those expressing the convection of vorticity and the simple wave behavior of sound in a uniform fluid. Computing the components of vorticity from (1), (2), (3), we find that,

$$L(\omega_\theta) = \frac{2V}{r}\frac{\partial v}{\partial z} + \frac{d}{dr}(\frac{V}{r})\frac{\partial w}{\partial \theta} + \frac{dW}{dr}(-\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r}) + (\frac{d^2W}{dr^2}) u \qquad (33)$$

$$L(\omega_r) = + (\frac{dV}{dr} + \frac{V}{r})\frac{\partial u}{\partial z} - \frac{1}{r}(\frac{dW}{dr})\frac{\partial u}{\partial \theta} \qquad (34)$$

$$L(\omega_z) = -\frac{1}{r}\left(\frac{dV}{dr} + \frac{V}{r}\right)\frac{\partial v}{\partial \theta} - \left(\frac{dV}{dr} + \frac{V}{r}\right)\frac{\partial u}{\partial r} - \left(\frac{dW}{dr}\right)\frac{\partial v}{\partial z} - \left[\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr}\right]u \tag{35}$$

so that the vorticity field is coupled to the velocity field of the acoustical perturbation to the extent that it contributes to the right sides of these equations. For the special cases $V = \Omega r$, and $V = \Gamma/r$ taking $W = ct.$ for both, we have

$$L(\omega_\theta) = 2\Omega\frac{\partial v}{\partial z}$$

$$L(\omega_r) = +2\Omega\frac{\partial u}{\partial z} \qquad V = \Omega r$$

$$L(\omega_z) = -2\Omega\left[\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial r} + \frac{u}{r}\right]$$

$$L(\omega_\theta) = -\frac{2\Gamma}{r^2}\omega_r$$

$$V = \Gamma/r$$

$$L(\omega_r) = L(\omega_z) = 0$$

For the wheel flow, any z dependent tangential velocity will lead to a net production of ω_θ, a z dependent u will lead to ω_r and so forth. For the free vortex, a radial vorticity, such as would be shed by a radial blade, will lead to production of tangential vorticity.

By following the usual procedure of taking the divergence of the momentum equations, then eliminating the divergence of the velocity by the continuity equation, a wave-like equation for the pressure perturbation results, viz.:

$$\nabla^2 p - RL\left[\frac{1}{R}L(\rho)\right] - \frac{R}{r}\frac{d}{dr}\left(\frac{r}{R^2}\frac{dR}{dr}\right)p = RL\left[\frac{1}{R}\frac{dR}{dr}u\right]$$

$$+ \frac{2R}{r}\left[\frac{dV}{dr}\left(v - \frac{\partial u}{\partial \theta}\right) + V\frac{\partial v}{\partial r}\right] - \left(2R\frac{dW}{dr}\right)\frac{\partial u}{\partial z} \tag{36}$$

Specializing this as above, and in addition assuming the tangential Mach number of the swirl is small, so that R and a^2 are constant,

$$\nabla^2 p - \frac{1}{a^2} L^2(p) = 2R[\frac{1}{R}\frac{dV}{dr}(v - \frac{\partial u}{\partial \theta}) + \frac{V}{r}\frac{\partial v}{\partial r}] \tag{37}$$

$$= 2R\Omega\omega_z \;, \quad V = \Omega r \tag{37a}$$

$$= 2R\frac{\Gamma}{r^2}[\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}], \; V = \frac{\Gamma}{r} \tag{37b}$$

Equation (37a) is particularly interesting. It states that an axial vorticity perturbation in a wheel flow acts as a sound source. This means that turbulence convected through a set of inlet guide vanes producing a wheel flow should cause a first order volumetric production of noise downstream of the vanes. The result is a strong volumetric source of broad-band noise.

The second result, for vortex flow, does not seem to be expressible in terms of a vorticity perturbation alone. Nevertheless, the right side of (37b) will in general be nonzero for a turbulent field, so the above conclusion applies to this case as well.

FLOW WITH MEAN ENTROPY VARIATION

Returning to the general formulation, Eqs.(12) to (16), we consider the effects of radial variations of mean entropy, S. It appears that in the general case of $V \neq 0$, $W \neq 0$, the reduction to a single second order equation fails. This general case has not as yet been studied. Some information can be had for the special case of $V = W = 0$, however, as follows.

Define a new variable $\sigma = s/c_p + \rho/R$, then Eqs.(12), (13), (14) and (16) reduce to:

$$i\lambda u = -a^2 \frac{d\sigma}{dr}$$

$$\lambda v = -\frac{ma^2}{r}\sigma$$

$$\lambda w = -ka^2\sigma$$

$$\sigma - \frac{\rho}{R} = \frac{i}{\lambda}\frac{1}{c_p}\frac{dS}{dr}u$$

Substituting these into the continuity equation (15) results in

$$r^2\frac{d^2\sigma}{dr^2} + [1 - 2\delta]\, r\frac{d\sigma}{dr} + [(\frac{\omega^2}{a^2} - k^2)\, r^2 - m^2]\sigma = 0 \qquad (38)$$

where $\delta = -\frac{r}{2c_p}\frac{dS}{dr}$, which has the solution

$$\sigma = r^\delta Z_p(\mu r) \qquad (39)$$

where

$$p^2 = m^2 + \delta^2$$

$$\mu^2 = (\frac{\omega^2}{a^2} - k^2)$$

Applying the boundary condition $u(r_o) = u(r_i) = 0$, we find the condition,

$$\frac{Z_p'}{Z_p} = -\frac{\delta}{\mu r}\ ,\ r = r_o, r_i \qquad (40)$$

Evidently the effects of the entropy gradient are two. For given m, the order of the radial eigenfunction is increased, and in general this will increase μ^2. The effect in the boundary condition is to shift the phase, without much change of μ. Since the axial wave number is

$$k = \pm\sqrt{(\frac{\omega}{a})^2 - \mu^2}$$

we conclude that the effect of the entropy gradient is to increase the critical ω for propagation, the ratio of ω's being

$$\frac{\omega(\delta)}{\omega(o)} = \frac{\mu}{\mu(o)} \approx 1 + \frac{\delta^2}{2m^2} \qquad (41)$$

Clearly this effect will be significant only for small m, as σ is likely to be of order unity at most.

CONCLUSIONS

It is probably true that this analysis has raised as many questions as it has answered. Nevertheless some conclusions do emerge. Ordered, from the most general to the most specific, they are as follows:

1) The intuitive concept of independent vorticity, sound and entropy fluctuations is not applicable in a swirling flow. "Shear" and "entropy" disturbances in general have associated pressure fields. Vorticity perturbation fields are not simply convected, and interact with the swirl to produce noise.

2) A "turbulent" velocity field, convected through a swirling flow, will produce first order pressure fluctuations which should appear as broad band noise. The same is true of entropy spottiness.

3) Viscous wakes of turbomachine blades cannot exist in static equilibrium with the inviscid flow. They induce pressure fields, and in general are distorted by the swirl.

4) For an isentropic swirling gas, the radial eigenfunctions of pressure disturbances are modified by swirl. This is true for both solid body and vortex swirls.

5) For solid body swirl, "swirl modes" which propagate for any nonzero swirl are present in addition to the usual modes, which propagate for sufficiently large rotor tip relative mach number. The swirl modes have axial wave numbers nearly independent of the rotor tip (and axial flow) mach number, while the axial wave numbers for the usual modes vary considerably with these mach numbers. They will usually propagate slowly relative to the flow, so will be found only downstream of a rotor.

6) For free vortex swirl, the rotor tip relative mach number for cutoff is independent of the swirl mach number, for a given radial mode.

7) Radial entropy variations alter the radial eigenfunctions, and increase the cutoff frequencies. The effect is largest for small azimuthal eigenvalues.

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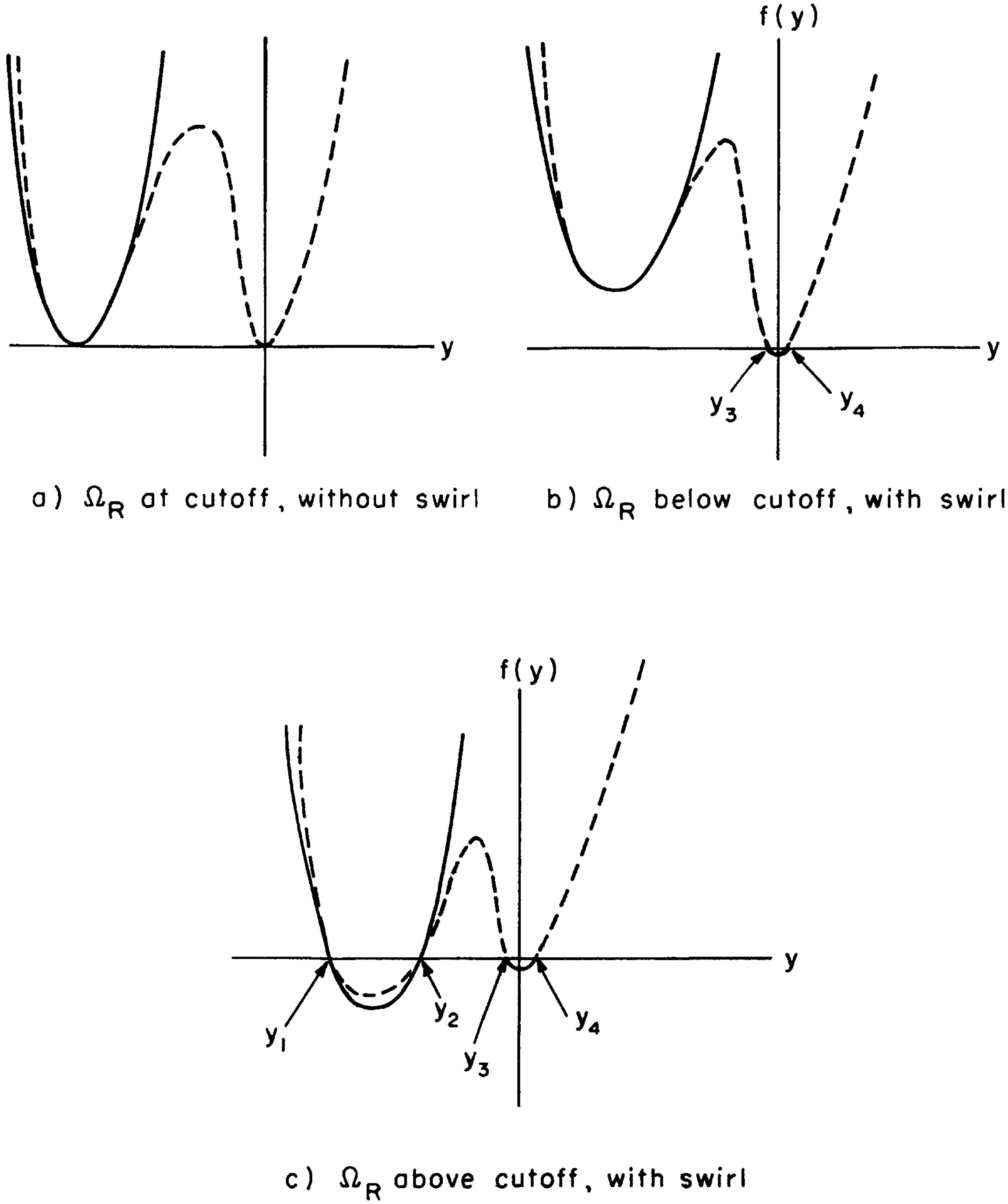

Figure 1: Behavior of the Characteristic Equation for Solid Body Rotation