THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER IN CURVED CHANNELS WITH ACCELERATION

YASUTOSHI SENOO

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Yasutoshi Senoo
Research Associate

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ABSTRACT

A theory is developed for two families of three-dimensional laminar boundary layers; namely, for the boundary layer on the parallel plane end walls of a curved channel with logarithmic spiral side walls, and for the boundary layer on the plane end wall of a concentric circular-arc channel having a particular family of accelerated or decelerated main flows. The second case shows the influence of acceleration and deceleration of a curved main flow. Numerical calculations show that acceleration makes the boundary layer thin and deceleration makes it thick, but the variation of thickness due to pressure gradient is very small compared with that in the two-dimensional case. The first case can be compared to the flow in a cascade. In this case, the variation of the width of the channel is directly related to the variation of the main flow velocity. According to the calculation, the boundary layer is thicker in an accelerated flow through a converging logarithmic spiral channel than in the decelerated flow through the same channel in the opposite direction. It is suspected that converging side walls make the end-wall boundary layer thick and that the effect of convergence is dominant over the effect of accelerated main flow.

Experimental data on the end wall of a turbine nozzle cascade were compared with theoretical predictions, with fair agreement across the nozzle and along the center line of the nozzle.
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1. Introduction

A few cases have been solved for three-dimensional laminar boundary layer. Sears and Cooke (Refs. 1 and 2) have solved the boundary layer on yawed cylinders of infinite length. Since the main flow and the boundary condition are uniform along the cylinder axis, the boundary layer is uniform along the axis too. This kind of flow was called "transferable across the main flow." Consequently, the velocity component in the plane perpendicular to the axis is independent of the axial velocity component, although the boundary layer is three-dimensional. A group of researchers (Refs. 3 and 4) has solved the boundary layers on flat plates with different kinds of curved main flow streamlines. However, theories are limited to transferable cases; i.e. the main flow and the boundary conditions are transferable along the leading edge of the flat plate. In practical cases, however, the cross flow is limited by the side walls of a channel. Therefore, even if the main flow is transferable, the boundary condition and consequently the boundary layer are not transferable. Additionally, it was assumed in that theory that the flow was bent by body force instead of surface force (pressure). That is, the main flow velocity increases or decreases in a constant-pressure field. Such a flow field is not realistic.

A few cases have been solved (Refs. 5, 6) in which the flow is not transferable across the main flow. In these cases, however, it was assumed that the turning angle or the curvature of the main flow streamline was very small and a linearized theory was used. In practical cases, e.g. the end-wall boundary layer of a turbine nozzle, neither the turning angle nor the curvature of the main flow is small. Therefore, a linearized theory is not a good approximation.

The main difficulty of the three-dimensional laminar boundary layer problem is that the equations are non-linear partial differential equations of three independent variables. The number of independent variables is reduced to two, if the flow is transferable across the main flow. This was the case solved in Refs. 1, 2, 3 and 4. The transferable flow across the main flow is not the only case to reduce the number of independent variables. If the flow is uniform along the main flow, one independent variable is excluded without sacrificing
the variation across the main flow (Ref. 7). More generally, if the flow is similar in a certain direction, the flow can be made uniform in this direction by suitable variation of the independent variables, and again one independent variable is excluded.

The equations can be further simplified by assuming the velocity profile as Pohlhausen did for the two-dimensional laminar boundary layer problem. This simplification sacrifices accuracy, but the non-linear partial differential equations become ordinary differential equations of the independent variable which is the coordinate across the main flow.

With these two simplifications, the main flow configuration is restricted and accurate velocity profiles are not obtained. However, several features of the three-dimensional boundary layer are easily obtained, which have not been known theoretically except in a few special cases.

2 Similarity Conditions for the Boundary Layer in a Polar Coordinate System

The analysis treated here is limited to steady, incompressible cases. Denoting a cylindrical coordinate system by $R$, $\theta$, and $z$, the three component velocities by $U_R$, $U_{\theta}$, and $U_z$, the static pressure by $p$, we obtain the equations of motion in the boundary layer near a flat plate $z = 0$ as follows:

$$
\begin{align*}
U_R \frac{\partial U_R}{\partial R} + \frac{U_{\theta}}{R} \frac{\partial U_R}{\partial \theta} + \frac{U_z}{R} \frac{\partial U_R}{\partial z} - \frac{U^2}{R} &= - \frac{\partial p}{\partial R} + \frac{1}{R} \frac{\partial (R U_{\theta})}{\partial \theta} \\
U_R \frac{\partial U_{\theta}}{\partial R} + \frac{U_{\theta}}{R} \frac{\partial U_{\theta}}{\partial \theta} + \frac{U_z}{R} \frac{\partial U_{\theta}}{\partial z} &= - \frac{1}{R} \frac{\partial p}{\partial \theta} + \frac{1}{R} \frac{\partial (R U_R)}{\partial z} \\
\frac{\partial U_z}{\partial R} + \frac{U_z}{R} \frac{\partial U_z}{\partial \theta} + \frac{1}{R} \frac{\partial U_z}{\partial z} &= 0
\end{align*}
$$

(1)

(2)

where it is assumed that $v_z/v_{\theta} \ll 1$, $z/R \ll 1$. The equation of continuity is

$$
\frac{\partial U_R}{\partial R} + \frac{U_z}{R} \frac{\partial U_R}{\partial \theta} + \frac{1}{R} \frac{\partial U_R}{\partial z} = 0
$$

(3)

Introducing new variables $y$ and $w$ which are defined by

$$
z = y/ \sqrt{Re}, \quad v_z = w/ \sqrt{Re} \quad \text{where} \quad Re = U_0 R_0 / \gamma ,
$$

(4)
we obtain

\[ v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{w}{r} \frac{\partial v_r}{\partial y} - \frac{v_\theta}{r} = - \frac{\partial p}{\partial r} + \frac{\partial^2 v_r}{\partial y^2} \]  

(5)

\[ v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \frac{\partial v_\theta}{\partial y} + \frac{w}{r} \frac{\partial v_\theta}{\partial y} = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial y^2} \]  

(6)

\[ \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial w}{\partial y} = 0 \]  

(7)

The boundary layer velocity components are related to the main flow velocity right outside the boundary layer \( V(r, \theta) \) by the following relations

\[ v_r = V(r, \theta) \phi_1(r, \theta, y), \]

\[ v_\theta = V(r, \theta) f_1(r, \theta, y), \]

\[ w = V(r, \theta) \psi_1(r, \theta, y). \]

The equations (5), (6), and (7) are

\[ \nabla^2 \phi_1 \frac{\partial \phi_1}{\partial r} + \nabla^2 \phi_1 \frac{\partial \phi_1}{\partial y} + \frac{v_\theta}{r} \frac{\partial \phi_1}{\partial \theta} + \frac{v_r}{r} \frac{\partial \phi_1}{\partial \theta} \]

\[ + \nabla^2 \psi_1 \frac{\partial \psi_1}{\partial r} - \frac{\partial^2 \phi_1}{\partial y^2} = - \frac{\partial p}{\partial r} + \nabla^2 \phi_1 \]

\[ \nabla^2 \phi_1 \frac{\partial f_1}{\partial r} + \nabla^2 \phi_1 \frac{\partial f_1}{\partial y} + \frac{v_\theta}{r} \frac{\partial f_1}{\partial \theta} + \frac{v_r}{r} \frac{\partial f_1}{\partial \theta} \]

\[ + \frac{v_\theta}{r} \frac{\partial \phi_1}{\partial \theta} + \nabla^2 \psi_1 \frac{\partial \psi_1}{\partial r} = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \nabla^2 f_1 \]

\[ \frac{\partial v_r}{\partial r} \phi_1 + \frac{\partial \phi_1}{\partial r} + \frac{v_r}{r} \frac{\partial \phi_1}{\partial \theta} - \frac{v_\theta}{r} \frac{\partial f_1}{\partial \theta} + \frac{f_1}{r} \frac{\partial v}{\partial \theta} + \nabla \frac{\partial \psi_1}{\partial y} = 0 \]  

(11)

New variables \( \rho, \eta, \alpha \) are introduced, which are defined by

\[ \rho = r/r_1, \quad \eta = y/\delta, \quad \alpha = \theta, \]  

(12)

where \( r_1 \) is a function of \( \theta \), \( \delta \) is a function of \( r \) and \( \theta \). (Figure 1)

The partial derivatives of a function \( \Phi(r, \theta, y) \) are
\[
\frac{\partial \Phi}{\partial r} = \frac{1}{r} \frac{\partial \Phi}{\partial \rho} + \eta \frac{\partial \Phi}{\partial \eta} \\
\frac{\partial \Phi}{\partial \theta} = \frac{\partial \Phi}{\partial \rho} \frac{\partial \rho}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \eta} + \alpha \frac{\partial \Phi}{\partial \alpha} \\
\frac{\partial \Phi}{\partial y} = \eta \frac{\partial \Phi}{\partial \eta} \\
\frac{\partial^2 \Phi}{\partial y^2} = \eta \frac{\partial \Phi}{\partial \eta} \\
\frac{\partial \Phi}{\partial \eta} = \frac{\partial \eta}{\partial y} \frac{\partial \Phi}{\partial \eta} \\
\frac{\partial \Phi}{\partial \alpha} = \frac{1}{\delta} \\
\frac{\partial \eta}{\partial y} = \frac{2}{\delta} \frac{y}{\delta} = \frac{1}{\delta}
\]

where
\[
\frac{\partial \eta}{\partial r} = -\eta \frac{\partial \delta}{\partial r} = \delta \frac{\partial \eta}{\partial r} \\
\frac{\partial \rho}{\partial \theta} = -\frac{\rho}{\delta} \frac{\partial \eta}{\partial \theta} = -\frac{\rho}{\delta} \frac{\partial \rho}{\partial \alpha} \\
\frac{\partial \eta}{\partial \theta} = -\eta \frac{\partial \delta}{\partial \rho} \frac{\partial \rho}{\partial \theta} + \frac{\partial \delta}{\partial \theta} = \eta \frac{\rho}{\delta} \frac{\partial \rho}{\partial \alpha} \frac{\partial \rho}{\partial \alpha} - \frac{\delta}{\delta} \frac{\partial \alpha}{\partial \alpha} \\
\frac{\partial \alpha}{\partial \theta} = 1 \\
\frac{\partial \eta}{\partial y} = \frac{2}{\delta} \frac{y}{\delta} = \frac{1}{\delta}
\]

With the new independent variables, the dependent variables are written as
\[
\phi_1(y, r, \theta) = \phi(\eta, \rho, \alpha) \\
f_1(y, r, \theta) = f(\eta, \rho, \alpha) \\
\psi_1(y, r, \theta) = \psi(\eta, \rho, \alpha).
\]

The Equations (9), (10) and (11) are
$$0 = \frac{ue \cdot le}{\phi e \cdot we} \Lambda + \left( \frac{pe}{ne} + \frac{de \cdot ge}{\phi e \cdot de} \right) \frac{du}{f} +$$

$$\left( \frac{pe}{fe} + \frac{le \cdot ge}{fe \cdot we} + \frac{de \cdot ge}{fe \cdot de} \right) \frac{du}{\Lambda} + \frac{du}{\phi \Lambda} +$$

$$\left( \frac{le \cdot le}{\phi e \cdot we} + \frac{de \cdot i}{fe} \right) \Lambda + \frac{de \cdot i}{\phi e} \phi$$

$$\frac{ue \cdot le}{fe \cdot we} \frac{du}{\phi e \cdot we} \Lambda + \frac{du}{fe} + \left( \frac{pe}{le} + \frac{de \cdot ge}{\phi e \cdot de} \right) \frac{du}{\phi \Lambda} +$$

$$\left( \frac{pe}{fe} + \frac{le \cdot ge}{fe \cdot we} + \frac{de \cdot ge}{fe \cdot de} \right) \frac{du}{f \Lambda} +$$

$$\frac{de \cdot i}{\phi e} f \phi \Lambda + \left( \frac{ue \cdot le}{fe \cdot we} + \frac{de \cdot i}{fe} \right) \phi \Lambda$$

$$\frac{ue \cdot le}{\phi e \cdot we} \frac{du}{f \Lambda} + \left( \frac{ue \cdot le}{we} + \frac{de \cdot i}{de} \right) - \quad =$$

$$\frac{du}{f \Lambda} - \frac{ue \cdot le}{\phi e \cdot we} \frac{du}{\phi \Lambda} + \left( \frac{pe}{le} + \frac{de \cdot ge}{\phi e \cdot de} \right) \frac{du}{f \phi \Lambda} +$$

$$\left( \frac{pe}{\phi e} + \frac{le \cdot ge}{\phi e \cdot we} + \frac{de \cdot ge}{\phi e \cdot de} \right) \frac{du}{f \Lambda} +$$

$$\frac{de \cdot i}{\phi e} \Lambda + \left( \frac{ue \cdot le}{\phi e \cdot we} + \frac{de \cdot i}{\phi e} \right) \phi \Lambda$$
If these three equations are independent of \( \alpha \), they can be greatly simplified. Such cases result from the following three conditions:

1. \( \psi = \psi (\eta, \rho) \), \( f = f (\eta, \rho) \)

2. The following terms are either zero or proportional to a function \( S(\alpha) \)

\[
\frac{\partial^2 \psi}{\partial \eta^2}, \quad \frac{\partial \psi}{\partial \eta}, \quad \frac{\psi}{\eta}, \quad \frac{\partial^2 \rho}{\partial \eta^2}, \quad \frac{\partial \rho}{\partial \eta}, \quad \frac{\partial^2 \psi}{\partial \eta \partial \theta}, \quad \frac{\partial \psi}{\partial \eta \partial \theta}, \quad \frac{\partial \rho}{\partial \theta},
\]

3. The following terms are either zero or proportional to a function \( T(\alpha) \)

\[
\frac{\partial^2 \psi}{\partial \eta^2}, \quad \frac{\partial \psi}{\partial \eta}, \quad \frac{\partial \psi}{\partial \theta}, \quad \frac{\partial \psi}{\partial \theta}, \quad \frac{\partial \psi}{\partial \eta}, \quad \frac{\partial \psi}{\partial \eta \partial \theta}
\]

The conditions (2) and (3) may be summarized as follows:

\[
\frac{\partial \rho}{\partial \eta}, \quad \frac{\partial^2 \rho}{\partial \eta^2}, \quad \frac{\partial \rho}{\partial \eta \partial \theta}, \quad \frac{\partial \rho}{\partial \theta}, \quad \frac{\partial \psi}{\partial \eta}, \quad \frac{\partial \psi}{\partial \eta \partial \theta}
\]

are independent of \( \alpha \).

Since

\[
\frac{\partial \eta}{\partial \rho} \xi = \frac{-\eta}{\delta} \frac{\partial \xi}{\partial \rho} ; \quad \frac{1}{\delta} \frac{\partial \xi}{\partial \rho}
\]

is to be a function of \( \rho \) alone,

\[
\log \delta = \delta'(\rho) + \delta'(\alpha), \quad \text{or} \quad \delta = \delta'(\rho) \delta'(\alpha)
\]

Since

\[
\frac{1}{\partial \rho} \frac{\partial \psi}{\partial \rho} \quad \text{is to be a function of \( \rho \) alone}, \quad \log V = V'(\rho) + V'(\alpha), \quad \text{or} \quad V = V'(\rho) V'(\alpha)
\]

Since

\[
\frac{\partial \rho}{\partial \theta} = -\frac{\rho}{\partial \alpha} \frac{\partial \eta}{\partial \alpha}
\]

is to be a function of \( \rho \) alone, \( \xi \frac{\partial \eta}{\partial \alpha} = K_1(\rho) \)

or

\[
\log r_1 = K_1(\rho) \alpha + K_2'(\rho), \quad \text{or} \quad r_1 = K_2(\rho) \exp K_1(\rho) \alpha
\]

Since \( r_1 \) was defined to be independent of \( \rho \), \( K_1(\rho) \) and \( K_2(\rho) \) are
constant. At $\alpha = 0, r_1 = 1$. Therefore, $r_1 = e^{k\alpha}$ where $k$ is an arbitrary constant.

Since $\frac{\partial \delta}{\partial \alpha} = -\frac{\eta}{\delta} \left( \frac{\partial \delta}{\partial \rho} \frac{\partial \rho}{\partial \alpha} + \frac{\partial}{\partial \alpha} \right)$ is to be independent of $\alpha$, $(1/\delta)(\partial \delta/\partial \alpha)$ must be a function of $\rho$ alone.

$log \delta = L_1(\rho) \alpha + L_2'(\rho)$, or $\delta = L_2(\rho) \exp L_1(\rho) \alpha$.

Since $(\partial \gamma/\partial \alpha)$ is a function of $\rho$ alone

$log V = M_1(\rho) \alpha + M_2'(\rho)$, or $V = M_2(\rho) \exp M_1(\rho) \alpha$.

In order to satisfy the condition $V = V_1(\rho) V_2(\alpha)$, $M_1(\rho)$ is a constant.

Consequently, $V = M(\rho) \exp^{m\alpha}$ where $m$ is an arbitrary constant.

Since $\frac{\partial}{\partial y} \left( \frac{\partial^2 \eta}{\partial y^2} \right) = \frac{\partial}{\partial x} \left( \frac{\partial \delta}{\partial \rho} \right) = \frac{1}{\rho(M(\rho) L_2'(\rho))} \exp (k \alpha - m\alpha - 2L_1(\rho) \alpha)$ is a function of $\rho$ alone, $k - m - 2L_1(\rho) = 0$ or $L_1 = \frac{k - m}{2}$.

If the pressure variation perpendicular to the end wall is small and negligible, the pressure in the boundary layer is equal to the main flow pressure. Therefore, $\frac{1}{V^2} \frac{\partial p}{\partial \rho}$ is a function of $\rho$ alone.

$\frac{\partial p}{\partial \rho} = M_1(\rho) \exp^{2m\alpha} \rho''(\rho) = \rho'(\rho) \exp^{2m\alpha}$

or

$\rho = \rho'(\rho) \exp^{2m\alpha} = \rho'(\alpha)$

Since $\frac{1}{V^2} \frac{\partial p}{\partial \alpha}$ is a function of $\rho$ alone

$\frac{\partial p}{\partial \alpha} = M_1(\rho) \exp^{2m\alpha} \rho''(\rho) = \rho'(\rho) \exp^{2m\alpha}$

$\rho = \frac{\rho'(\rho) \exp^{2m\alpha}}{2m} + \rho_4(\rho)$

These two conditions are satisfied by

$\rho = \rho'(\rho) \exp^{2m\alpha} + \rho_2$ where $\rho_2$ is a constant.

Since $\frac{\partial \gamma}{\partial y} \psi$ is not a function of $\alpha$ and $\frac{\partial}{\partial \alpha} \left( \frac{\partial \gamma}{\partial y} \right)$ is independent of $\alpha$, $\sqrt{\gamma} \frac{\partial \gamma}{\partial \alpha} \psi$ is not a function of $\alpha$.

That is, $\sqrt{\gamma} \frac{\partial \gamma}{\partial \alpha} \psi = \sqrt{M(\rho) \exp^{m\alpha} \rho} \exp^{k\alpha} \psi$

is not a function of $\alpha$,

or $\psi = \psi_2(\gamma, \rho) \exp (-\frac{m}{2} - \frac{k}{2}) \alpha$

If $\psi$ satisfies this relationship, $\frac{\partial \gamma}{\partial \alpha} \left( \frac{\partial \gamma}{\partial \alpha} \right) (\partial \gamma/\partial \gamma)$ is not a function.
of \( \alpha \).

As the summary, the following conditions are necessary in order that the differential equations are independent of \( \alpha \).

\[
\phi = \phi(\rho, \eta), \quad f = f(\rho, \eta) \quad (17)
\]

\[
\nu = M(\rho) \epsilon^{m\alpha} \quad (18)
\]

\[
\rho = \rho_i(\rho) \epsilon^{2m\alpha} + \rho_2 \quad (19)
\]

\[
\eta_i = \epsilon^{k\alpha} \quad (20)
\]

\[
\delta = L(\rho) \epsilon^{\frac{k-m\alpha}{2}} \quad (21)
\]

\[
\psi = \psi_2(\eta, \rho) \epsilon^{-\frac{m+k}{2} \alpha} \quad (22)
\]

Equations (18) and (19) are the conditions which are to be satisfied by the main flow, and (20) and (21) are the conditions to be satisfied by the transformation functions \( \eta_i \) and \( \delta \) defined in (12).

If there are boundary conditions at all, they must be similar along a constant \( \rho \) and constant \( \eta \) line. For example, if there is a side wall which affects the boundary layer behavior near the wall, the shape of the side wall must satisfy Equation (20) and the typical dimension of the boundary layer at the wall must satisfy Equation (21), otherwise a similar boundary layer with respect to \( \alpha \) does not exist.

If the main flow is irrotational, there is a relationship between the static pressure and the velocity, \( p + \frac{\rho \nu^2}{2} = \text{constant} \). This relation satisfies Equation (19).

3. **Main Flow**

In order that a similarity solution exist, the main flow must satisfy the conditions of Equations (18), (19) and (20). Since the
main flow must satisfy the equations of motion, the equations put additional restrictions on the main flow. The equations of motion and of continuity outside the boundary layer are:

\[
\begin{align*}
\nu_r \frac{\partial v_r}{\partial r} + \frac{\nu_0}{r} \frac{\partial v_r}{\partial \theta} - \frac{\nu_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} &= - \frac{\partial p}{\partial r} \\
\nu_r \frac{\partial v_\theta}{\partial r} + \frac{\nu_0}{r} \frac{\partial v_\theta}{\partial \theta} + \nu_\theta \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} &= - \frac{1}{r} \frac{\partial p}{\partial \theta} \\
\nu_r \frac{\partial v_z}{\partial r} + \frac{\nu_0}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial \theta} &= - \frac{\partial p}{\partial z} \\
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} &= 0
\end{align*}
\]

where viscous effect is neglected. Since \( v_z \) is very small compared to \( v_r \) and \( v_\theta \) near the boundary layer, \( v_z \) and the first partial derivatives of \( v_z \) with respect to \( \theta \) and \( r \) are negligible. Therefore, the equations are

\[
\begin{align*}
\nu_r \frac{\partial v_r}{\partial r} + \frac{\nu_0}{r} \frac{\partial v_r}{\partial \theta} - \frac{\nu_\theta^2}{r} &= - \frac{\partial p}{\partial r} \\
\nu_r \frac{\partial v_\theta}{\partial r} + \frac{\nu_0}{r} \frac{\partial v_\theta}{\partial \theta} + \nu_\theta \frac{\partial v_\theta}{\partial \theta} &= - \frac{1}{r} \frac{\partial p}{\partial \theta} \\
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} &= 0
\end{align*}
\]
With the transformation $v_r = V \phi$, $v_\theta = Vf_\theta$, $v_z = V \psi_\theta$ and $r = r_\theta \rho$, the equations become

$$
\frac{\nu_r^2 \phi_{r}}{r_1^2} + \frac{\nu_r^2 \phi_{\theta}}{r_1^2} + \frac{\nu_r^2 f_{\theta}}{r_1^2} (\frac{\partial \rho}{\partial \theta} \frac{\partial \phi_r}{\partial \rho} + \frac{\partial \phi_r}{\partial \theta}) + \frac{\nu_r^2 f_{\theta}^2}{r_1^2} \rho = - \frac{1}{r_1^2} \frac{\partial \rho}{\partial \theta}
$$

(23)

$$
\frac{\nu_r^2 \phi_{r}}{r_1^2} (\frac{\partial \rho}{\partial \theta} \frac{\partial \phi_r}{\partial \rho} + \frac{\partial \phi_r}{\partial \theta}) + \frac{\nu_r^2 f_{\theta}}{r_1^2} (\frac{\partial \rho}{\partial \theta} \frac{\partial \phi_r}{\partial \rho} + \frac{\partial \phi_r}{\partial \theta}) + \frac{\nu_r^2 f_{\theta}^2}{r_1^2} \rho = - \frac{1}{r_1^2} \frac{\partial \rho}{\partial \theta} (\frac{\partial \phi_r}{\partial \theta} + \frac{\partial \phi_r}{\partial \rho})
$$

(24)

$$
\frac{\phi_{r}}{r_1^2} \frac{\partial \phi_r}{\partial \rho} + \frac{\nu_r^2 \phi_{r}}{r_1^2} + \frac{\nu_r^2 f_{\theta}}{r_1^2} (\frac{\partial \rho}{\partial \theta} \frac{\partial \phi_r}{\partial \rho} + \frac{\partial \phi_r}{\partial \theta}) + \frac{\nu_r^2 f_{\theta}^2}{r_1^2} \rho = - \frac{1}{r_1^2} \frac{\partial \rho}{\partial \theta} (\frac{\partial \phi_r}{\partial \theta} + \frac{\partial \phi_r}{\partial \rho})
$$

(25)

The main flow is limited to a quasi-two-dimensional flow; that is, the velocity variation perpendicular to the end wall is small. Therefore, it is assumed that the partial derivatives of $\phi_0$ and $f_0$ with respect to $z$ are negligibly small, although the variation of $\psi_\theta$ with respect to $z$ is not negligible. If the flow is irrotational

$$
\frac{\partial \rho}{\partial \rho} = -V \frac{\partial \psi}{\partial \rho}, \quad \frac{\partial f}{\partial \alpha} = -V \frac{\partial \psi}{\partial \alpha}
$$

When the boundary layer is similar with respect to $\alpha$, the Equations (17), (18) and (20) must be satisfied. Substitution of the equations and the relation $\frac{\partial \rho}{\partial \theta} = \frac{\partial}{\partial \alpha}$ in Equations (23), (24) and (25) changes the equations to

$$
M \dot{\rho} \left( \phi_0^2 - f_\theta f_\theta k - 1 \right)
$$

$$
+ M \frac{f_\theta}{\rho} (\phi_0 m - f_\theta) + M (\phi_0 f_\theta - f_\theta f_\theta k) = 0
$$

(26)

$$
M \dot{\rho} \left( \phi_0^2 - k f_\theta^2 + k \right)
$$

$$
+ M \frac{f_\theta}{\rho} (\phi_0 f_\theta + m f_\theta^2 - m) + M (\phi_0 f_\theta - f_\theta f_\theta k) = 0
$$

(27)
where $M$, $\dot{\phi}_0$, and $\dot{f}_0$ mean the derivative with respect to $\rho$.

If all the main flow has a constant angular momentum,

$$\frac{d}{d\rho} (f_0 M \rho) = 0 \quad (29)$$

If $f_0$ is not a function of $\rho$, $M\rho$ is constant. Since $M$ can be defined to be unity at $\rho = 1$,

$$M = 1 \quad (30)$$

$$M = -1/\rho^2 = -M/\rho \quad (31)$$

$$\frac{df_0}{d\rho} = \frac{d\phi_0}{d\rho} = 0 \quad (32)$$

Since $\psi^2$ is very small, $\phi_0^2 + f_0^2 = \phi_0^2 + f_0^2 + \psi^2 = 1$.

Therefore, Equations (26) and (27) become

$$f_0 \phi_0 (k + m) = 0 \text{ and } \phi_0^2 (k + m) = 0 \quad (33)$$

That is, $\phi_0 = 0$ or $k + m = 0$.

Since (31) becomes

$$\mathcal{L}^{(m-k)\alpha} \frac{M}{\rho} f_0 (m+k) + M \mathcal{L}^{m\alpha} \frac{\partial \psi_0}{\partial \xi} = 0$$

if $m + k = 0$, $\partial \psi_0 / \partial \xi = 0$. That is, the main flow is two-dimensional. If $\phi_0 = 0$,

$$\frac{\partial \psi_0}{\partial \xi} = -\frac{(m+k)}{\rho} \mathcal{L}^{-k\alpha}$$

In a real flow field, a side wall of a curved channel is a main streamline, and the wall gives a boundary condition to the boundary layer on the end wall. Therefore, the flow is similar on a main streamline, or $r_1 = \mathcal{L}^{k\alpha}$ is a main streamline. Consequently, the first case of $m + k = 0$ is a logarithmic spiral flow; that is, a combination of free vortex flow and sink or source flow, which is well known as a two-
dimenSional potential flow. The second case is a circular arc flow, because $\theta_0 = 0$. Therefore, $r_1$ should be constant, or $k = 0$. If $m$ is positive the main flow is accelerated, and if $m$ is negative, the main flow is decelerated. Such flows are realized by a channel with concentric circular arc side walls, whose height varies along the length.

4 Momentum Equations

When the flow is similar with respect to $\alpha$, the velocity inside the boundary layer is a function of $\rho$ and $\eta$, and the equations of motion are a set of non-linear simultaneous differential equations. One way to solve these equations is to use momentum equations assuming the velocity profile in a similar way to the Karman-Pohlhausen method. However, in the present case, the momentum relation is expressed by a set of simultaneous equations and the flow should be represented by two parameters instead of one. The Equations (14), (15) and (16) are integrated with respect to $\eta$ from $\eta = 0$ to $\eta = 1$.

\[
\frac{V^2}{r} \int_\eta^1 \frac{\partial \phi'}{\partial \rho} d\eta - \frac{V^2}{\eta \delta} \int_\eta^1 \frac{\partial \phi'}{\partial \rho} d\eta + \frac{2V}{r} \int_\eta^1 \phi d\eta - \frac{V^2}{\eta \delta} \int_\eta^1 \frac{\partial (\phi \phi')}{\partial \rho} d\eta \\
+ \frac{V^2}{r \rho} \left( \frac{\partial \phi'}{\rho \eta \partial \rho} - \frac{\partial \phi'}{\delta \partial \alpha} \right) \int_\eta^1 \frac{\partial (\phi \phi')}{\partial \eta} d\eta + \frac{V^2}{r \rho} \int_\eta^1 \frac{\partial (\phi \phi')}{\partial \alpha} d\eta \\
- \frac{V^2}{r^2 \partial \alpha} \int_\eta^1 \phi' d\eta + \frac{1}{\eta \rho} \frac{\partial (\phi \phi')}{\partial \alpha} \int_\eta^1 \phi' d\eta + \frac{V^2}{\delta} \left[ \frac{\partial \phi'}{\partial \eta} \right]' \\
- \frac{V^2}{\rho \eta} \int_\eta^1 (f - \phi') d\eta = - \frac{1}{\eta \rho} \frac{\partial \rho}{\partial \rho} + \frac{V^2}{\delta} \left[ \frac{\partial \phi'}{\partial \eta} \right]' \quad (34)
\]

\[
\frac{V^2}{r} \int_\eta^1 \frac{\partial (\phi \phi')}{\partial \rho} d\eta - \frac{V^2}{r \delta} \int_\eta^1 \frac{\partial (\phi \phi')}{\partial \rho} d\eta + \frac{2V}{r} \int_\eta^1 \phi' d\eta \\
+ \frac{V^2}{r} \int_\eta^1 \frac{\partial \phi'}{\partial \rho} d\eta + \frac{1}{\eta \rho} \frac{\partial (\phi \phi')}{\partial \alpha} \int_\eta^1 \phi' d\eta + \frac{V^2}{r \rho} \int_\eta^1 \phi' d\eta \\
- \frac{1}{\eta^2 \partial \alpha} \int_\eta^1 \phi' d\eta + \frac{1}{\eta \rho} \frac{\partial \phi'}{\partial \alpha} \int_\eta^1 \phi' d\eta + \frac{2V}{r \rho} \int_\eta^1 \phi' d\eta + \frac{V^2}{\delta} \left[ \frac{\partial \phi'}{\partial \eta} \right]' \\
= \frac{1}{\eta \rho} \left( \frac{\partial \rho}{\partial \rho} - \frac{\partial \rho}{\partial \alpha} \right) + \frac{V^2}{\delta} \left[ \frac{\partial \phi'}{\partial \eta} \right]' \quad (35)
\]
If the main flow is irrotational and \( \psi, \phi, V, r_1 \) and \( \delta \) are expressed by Equations (17), (18), (20) and (21) and the boundary conditions are \( \psi = f = \psi = 0 \) at \( \eta = 0 \) and \( \partial f / \partial \eta = \partial \phi / \partial \eta = 0 \) at \( \eta = 1 \), the Equations (34), (35) and (36) become as follows,

\[
\begin{align*}
M^f_0 \frac{\partial f}{\partial \rho} d\eta - M^\phi_0 \frac{\partial \phi}{\partial \eta} d\eta &+ 2M^f_0 \int f^2 d\eta - M_k^f \int f \phi d\eta + 2mM^f_0 \int f \phi d\eta \\
\frac{M^\phi_0 (\psi \phi)_{\eta=1}}{\psi_0} - \frac{M^f_0 (f^2 - \phi^2)}{f_0} d\eta &+ M^\phi_0 \frac{\partial (f \phi)}{\partial \eta} d\eta - 2M^f_0 \int f \phi d\eta + 2mM^f_0 \int f \phi d\eta \\
\frac{M^f_0 \phi}{\phi_0} - M^\phi_0 \frac{\partial (f \phi)}{\partial \eta} d\eta &+ 2M^f_0 \int f \phi d\eta - M^\phi_0 \frac{\partial (f \phi)}{\partial \eta} d\eta - 2M^f_0 \int f \phi d\eta + 2mM^f_0 \int f \phi d\eta
\end{align*}
\]

where \( M \) and \( L \) mean the derivative with respect to \( \rho \).

There are three original differential equations and four unknowns \( \phi, f, \psi, p \). However, since \( p \) is related to the main flow velocity, there are virtually four equations to determine four unknowns. Since \( \delta \) is introduced in the equations, it looks like an additional unknown. However, \( \delta \eta = y \) is an independent variable and \( \delta \) always appears with \( \eta \) at the present stage. Therefore \( \delta \) is not yet an unknown variable.
Assumption of Velocity Distribution Inside the Boundary Layer

Experimental data are meager for the velocity distribution in a three-dimensional boundary layer. An example in the appendix shows that a set of polynomial expressions for the velocity distribution is not too bad for a wide range of pressure gradient.

If a fourth order polynomial of $\eta$ is chosen, the main flow component velocity is expressed by

$$\frac{u}{V} = (2\eta - 2\eta^3 + \eta^4) + A \frac{\eta}{6} (1 + \eta)^3$$  (40)

which satisfies the boundary conditions $u/V = 1$, $\frac{d(u/V)}{d\eta} = 0$ $\frac{d^2(u/V)}{d\eta^2} = 0$ at $\eta = 1$, and $u/V = 0$ at $\eta = 0$. Equations (5) and (6) show that

$$\frac{\partial^2 v_m}{\partial y^2} = \frac{\partial P}{\partial x_m} = -V \frac{\partial V}{\partial x_m}$$ at $y = 0$

where $v_m$ and $x_m$ are the velocity-component and length in the main flow direction. Therefore $A = 8^2 (\partial V/\partial x_m)$.

The cross-flow component velocity is artificially expressed in two parts:

$$\frac{v}{V} = B \frac{\eta}{6} (1 - \eta)^3 + C \frac{\eta}{6} (\eta - 3\eta^3 + 2\eta^4)$$  (41)

The first term has the second order of $\eta$ and the last term does not have the second order of $\eta$. That is, only the first term works against the pressure gradient, because Equations (5) and (6) show that

$$\frac{\partial^2 v_c}{\partial y^2} = \frac{\partial P}{\partial x_c} = -V \frac{\partial V}{\partial x_c}$$ at $y = 0$

where $v_c$ and $x_c$ are the velocity-component and length in the cross flow direction. Therefore $B = 8^2 \frac{\partial V}{\partial x_c}$.

If the main flow direction is $\frac{\pi}{2} - \beta$ from the radius (Figure 2), the tangential and radial component velocities are

---

* Since the equations which are to be used in the analysis are momentum equations; that is, the equations of motion are integrated with respect to $\eta$, the $\eta^2$ term is not very significant. Therefore, it is not necessary to divide the cross-flow component velocity into two terms as Equation (41).
\[ f = E \cos \beta + F(A \cos \beta - B \sin \beta) - CH \sin \beta \]
\[ \phi = E \sin \beta + F(A \sin \beta + B \cos \beta) + CH \cos \beta \]

where \( E = 2\eta - 2\eta^3 + \eta^4 \), \( F = \frac{1}{\delta} (1-\eta)^3 \), \( H = \frac{1}{\delta} (\eta^3 - 3\eta^3 + 2\eta^4) \) \hspace{1cm} (42)

From Equations (5) and (6)
\[
-\left( \frac{\partial^2 f}{\partial \eta^2} \right)_{\eta=0} = \left( A \cos \beta - B \sin \beta \right) = \delta^2 \frac{\partial V}{\partial \theta} = \frac{\delta^2}{\gamma \rho} \left( \frac{\partial \rho}{\partial \theta} \frac{\partial V}{\partial \rho} + \frac{\partial V}{\partial \alpha} \right) \\
\quad = \frac{\delta^2}{\gamma \rho} (k + m) \frac{A}{\rho} = \left( \frac{\delta^2}{\gamma \rho} \right) (k + m) \frac{A}{\rho} = \left( \frac{1}{\rho} \right)^2 (k + m)
\]

\[
\left( \frac{\partial^2 \phi}{\partial \eta^2} \right)_{\eta=0} = \left( A \sin \beta - B \cos \beta \right) = \delta^2 \frac{\partial V}{\partial \theta} = \frac{\delta^2}{\gamma \rho} (-1) \frac{A}{\rho} \\
\quad = -\frac{\delta^2}{\gamma \rho} \frac{A}{\rho} = -\left( \frac{1}{\rho} \right)^2
\]

Consequently,
\[ f = E \cos \beta + F (k + m) \left( \frac{1}{\rho} \right)^2 - CH \sin \beta \] \hspace{1cm} (43)
\[ \phi = E \sin \beta - F \left( \frac{1}{\rho} \right)^2 + CH \cos \beta \] \hspace{1cm} (44)

The substitution of Equations (43) and (44) into the momentum equations (37), (38) and (39) changes the unknowns to \( L, C \) and \( \Psi_1 \). Therefore, there are enough equations to determine these unknowns.

6 Circular Arc Main Flow with Acceleration or Deceleration

Equation (33) shows that a similar solution exists for a circular arc main flow, or \( \phi_0 = 0 \). Since the main flow is circular arc flow, \( r_1 \) is constant and \( k = 0 \). Equations (18), (19), (20) and (21) become
\[ r_1 = \epsilon^{k_1} = 1, \quad k = 0 \]
\[ V = M(\rho) \epsilon^{m_1} = \frac{1}{\rho} \epsilon^{m_1}, \quad M(\rho) = \frac{1}{\rho} \]
\[ \delta = L(\rho) \epsilon^{-m_1} \]
\[ \psi = \psi_1(\eta, \rho) \epsilon^{-m_1} \] \hspace{1cm} (45)
Substitution of Equation (45) into Equations (37), (38) and (39) gives

\[
\frac{L}{2 \rho} \int_{\eta}^{1} \phi^2 d\eta + \int_{\eta}^{1} \frac{\partial \phi^2}{\partial \rho} d\eta + \frac{3m}{2 \rho} \int_{\eta}^{1} f \phi d\eta
\]

\[
+ \frac{L}{2 \rho} \left(1 - f^2 - \phi^2 \right) d\eta + \frac{\rho}{L^2} \left( \frac{\partial \phi}{\partial \eta} \right)_{\eta=0} = 0
\]

(46)

\[
\frac{L}{2 \rho} \int_{\eta}^{1} f \phi d\eta + \int_{\eta}^{1} \frac{\partial (f \phi)}{\partial \rho} d\eta - \frac{m}{2 \rho} + \frac{3m}{2 \rho} \int_{\eta}^{1} f^2 d\eta
\]

\[
+ \left( \frac{\psi}{L} \right)_{\eta=1} + \frac{\rho}{L^2} \left( \frac{\partial f}{\partial \eta} \right)_{\eta=0} = 0
\]

(47)

\[
\left( \frac{\psi}{L} \right)_{\eta=1} = - \frac{L}{2 \rho} \int_{\eta}^{1} \phi d\eta - \int_{\eta}^{1} \frac{\partial \phi}{\partial \rho} d\eta - \frac{m}{2 \rho} - \frac{3m}{2 \rho} \int_{\eta}^{1} f d\eta
\]

(48)

Since the main flow is a circular-arc flow, \( k = 0 \) and \( \beta = 0 \), Equations (43) and (44) become

\[
f = E + m \left( \frac{L}{\rho} \right)^2 F
\]

\[
\phi = - \left( \frac{L}{\rho} \right)^2 F + CH
\]

(49)

Since Equations (46), (47) and (48) do not involve \( \left( \frac{\partial^2 f}{\partial \eta^2} \right)_{\eta=0} \)
and \( \left( \frac{\partial^2 \phi}{\partial \eta^2} \right)_{\eta=0} \), the significance of distinguishing \( H \) from \( F \) fades.

If a new unknown \( \chi \) is introduced and \( \phi \) is defined as

\[
f = E + m \Lambda F
\]

\[
\phi = \chi \Lambda F,
\]

where \( \Lambda = \left( \frac{L}{\rho} \right)^2 \)

Equations (46), (47) and (48) have three unknowns \( \psi, \chi \) and \( \Lambda \), and the equations become simple without increasing the degree of approximation compared to using (49).
Substitution of (50) and (48) into (46) and (47) makes

\[ \rho x \Lambda (2 \dot{x} \Lambda + 2.5 x \Lambda) \int^\prime \! F^2 \, d\eta + 1.5 m^2 x \Lambda^2 \int^\prime \! F^2 \, d\eta \\
+ 1.5 m x \Lambda \int^\prime \! EF \, d\eta + x \left( \frac{dE}{d\eta} \right)_{\eta=0} - m^2 \Lambda \int^\prime \! F^2 \, d\eta \\
- 2m \Lambda \int^\prime \! EF \, d\eta - \int^\prime \! E^2 \, d\eta + 1 = 0 \quad (51) \]

\[ \int^\prime \! x \Lambda^2 m \rho \int^\prime \! F^2 \, d\eta - 2.5 m x \rho \Lambda \int^\prime \! F^2 \, d\eta - m x \Lambda^2 \int^\prime \! F^2 \, d\eta \\
- \rho x \Lambda \left( \int^\prime \! EF \, d\eta - \int^\prime \! F \, d\eta \right) - 1.5 \rho x \Lambda \left( \int^\prime \! EF \, d\eta - \int^\prime \! F \, d\eta \right) \\
- x \Lambda \left( \int^\prime \! EF \, d\eta - \int^\prime \! F \, d\eta \right) - 1.5 m^3 \Lambda^2 \int^\prime \! F^2 \, d\eta \\
- m^2 \Lambda \left( 3 \int^\prime \! EF \, d\eta - 0.5 \int^\prime \! F \, d\eta \right) - 1.5 m \int^\prime \! E^2 \, d\eta - 0.5 m \int^\prime \! Ed\eta \\
- m^2 - \frac{1}{\Lambda} \left[ \left( \frac{dE}{d\eta} \right)_{\eta=0} - m \Lambda \left( \frac{dF}{d\eta} \right)_{\eta=0} \right] = 0 \quad (52) \]

The numerical values of the integral of \( E \) and \( F \) are

\[ \int^\prime \! E \, d\eta = 0.700000 \quad \int^\prime \! F \, d\eta = 0.008333 \]
\[ \int^\prime \! E^2 \, d\eta = 0.582540 \quad \int^\prime \! F^2 \, d\eta = 0.000110222 \]
\[ \int^\prime \! EF \, d\eta = 0.0046965 \quad \int^\prime \! F \, d\eta = 0.008333 \]
\[ \int^\prime \! \eta \, \frac{dE}{d\eta} \, d\eta = 0.300000 \quad \left( \frac{dF}{d\eta} \right)_{\eta=0} = 0.166667 \]
\[ \left( \frac{dE}{d\eta} \right)_{\eta=0} = 2 \]

Substitution of these values into (51) and (52) gives

\[ 2.5 \rho x^2 \Lambda \frac{dA}{d\rho} + 2 \rho x \Lambda^2 \frac{dx}{d\rho} + (1.5 x - 1) m^2 x^2 \]
\[ + 42.604 (1.5 x - 2) m \Lambda + 3778.2 + 15/2.4 x = 0 \quad (54) \]
These two equations are rearranged to
\[(−0.001819 + 0.0002755 \Lambda \rho \nabla \frac{d \Lambda}{d \rho}) \rho \nabla \frac{d \Lambda}{d \rho} + \rho \Lambda (−0.0002755 (m^3 + m \nabla^2 \Lambda) \Lambda^3 + (0.0090905 \nabla^2 \Lambda - 0.0153819 m^2 \nabla^2 \Lambda - 0.0180120 m^2) \Lambda^3 + (0.841738 \nabla^2 \Lambda + 1.508629m) \Lambda - 13.25317 \nabla^2 \Lambda - 20.6722 = 0)
\]
\[(0.001819 - 0.0002755 \Lambda \rho \nabla \frac{d \Lambda}{d \rho}) \rho \nabla \frac{d \Lambda}{d \rho} + (−0.0002204 m \nabla^2 \Lambda - 0.0001653 m^3 \nabla \Lambda - 0.0001102 m^3) \Lambda^3 + (0.007276 \nabla^2 \Lambda - 0.0182566 m^2 \nabla^2 \Lambda - 0.005752 m^2) \Lambda^3 + (0.553221 m \nabla^2 \Lambda + 0.727447 m) \Lambda - 9.502111 \nabla^2 \Lambda - 13.781472 = 0)
\]

As the boundary conditions, it is assumed that the boundary layer thickness is zero at \( \rho = 1 \). That is \( \Lambda = 0 \). It is assumed that \( d \nabla^2 \Lambda / d \rho \) is not infinity at \( \rho = 1 \). This assumption is justified by the results of computation. Under this assumption, Equation (56) gives the value of \( \nabla^2 \Lambda \) at \( \rho = 1 \). That is, \( \nabla^2 \Lambda = -1.55948 \). That is, the boundary conditions are \( \Lambda = 0, \nabla^2 \Lambda = -1.55948 \) at \( \rho = 1 \). (58)

The simultaneous Equations (56) and (57) are numerically integrated with the boundary conditions of (58). The results are shown in Figure 3. The abscissa is \( \rho \) and the ordinate is \( \rho \sqrt{\Lambda} = L = 8 \xi^\alpha \) and \( \nabla^2 \Lambda \). That is, the curves \( L \) in Figure 3 are dimensionless boundary layer thickness. Since \( V = (1/\rho) \xi^\alpha \), positive value of \( m \) means an accelerated main
flow. If \( m \) is positive, the two Equations (56) and (57) become identical at a certain value of \( \rho \). The critical value of \( \rho \) for \( m = 1.0 \) is 0.8836 and the critical value for \( m = 0.5 \) is 0.7143. The calculation cannot be continued beyond these values. The author suspects that the singular point has no particular significance. The equations used are the integrated equations instead of the equations of motion themselves. Therefore, it is suspected that there are enough conditions to specify the behavior of boundary layer near the singular point if the equations of motion are examined. However, because of integration, the equations are simplified to two momentum equations. Since the equations are non-linear differential equations, it is possible that the two equations become identical at particular values of the variables.

The velocity distributions in the main flow and cross flow directions, \( u/V \) and \( v/V \), are

\[
\frac{u}{V} = f = (2\eta - 2\eta^3 + \eta^4) + m \Lambda \eta(1 - \eta)^3/6
\]  
(59)

\[
\frac{v}{V} = \phi = \kappa \Lambda \eta(1 - \eta)^3/6
\]  
(60)

where

\[
\Lambda = (L/\rho)^2, \quad \eta = y/\delta
\]  
(61)

\( L \) and \( \kappa \) are shown in Figure 4.

The dimensionless boundary layer thickness \( \delta \) is

\[
\delta = L \varepsilon^{\frac{m\kappa}{2}} = L/\sqrt{\rho V} = L/\sqrt{V_{\rho=1}}
\]  
(62)

The real dimension of the boundary layer thickness \( R_{o}\delta_1 \) is

\[
R_{o}\delta_1 = R_{o}\delta/ \overline{Re} = R_{o}L/ \sqrt{V_{\rho=1}} \sqrt{U_{o}R_{o}/y}
\]

\[
= R_{o}L/ \sqrt{V_{o}R_{o}/y}
\]

**7. Logarithmic Spiral Main Flow**

Equation (33) shows that two families of flow are possible, one is \( \psi_0 = 0 \) or circular arc flow and the other is \( k + m = 0 \). If \( k + m = 0 \), Equations (13), (20), (21) and (22) become
If \( r_1 = e^{k\alpha} \) is a streamline, the flow is obviously a logarithmic spiral flow and
\[ k = \tan \beta. \] (64)

Equation (63) is substituted into Equations (37), (38) and (39) and a new unknown \( G \) is introduced.
\[ G = L/\rho \] (65)

Then, Equations (37), (38) and (39) become

\[
\begin{align*}
\int \rho \frac{\partial \phi'}{\partial \rho} d\eta - (1 + \rho \dot{G}) \left[ (\eta \phi^\prime)_{\eta=1} - \int_0^\eta \phi d\eta \right] \\
+ \int (1 - \phi - f') d\eta + \frac{l}{G} (\psi f)_{\eta=1} + \frac{l}{G^2} \left( \frac{\partial f}{\partial \eta} \right)_{\eta=0} \\
= k \rho \int \frac{\partial (f \phi)}{\partial \rho} d\eta - k \rho \frac{\dot{G}}{G} \left[ (\eta f \phi)^\prime - \int_0^\eta f \phi d\eta \right] \\
\int \rho \frac{\partial (f \phi)}{\partial \rho} d\eta - (1 + \rho \dot{G}) \left[ (\eta f \phi)^\prime - \int_0^\eta f \phi d\eta \right] \\
+ \frac{l}{G} (\psi f)_{\eta=1} + \frac{l}{G^2} \left( \frac{\partial f}{\partial \eta} \right)_{\eta=0} \\
= k \int \rho \frac{\partial f'}{\partial \rho} d\eta - k \rho \frac{\dot{G}}{G} \left[ (f' \eta)^\prime - \int_0^\eta f' d\eta \right] \\
\int \left( \frac{\partial}{\partial \rho} \right) \int_0^\eta \frac{\partial \phi}{\partial \rho} d\eta - (1 + \rho \dot{G}) \int_0^\eta \frac{\partial \phi}{\partial \eta} d\eta \\
= k \int (\rho \int_0^\eta \frac{\partial f}{\partial \rho} d\eta - \rho \frac{\dot{G}}{G} \int_0^\eta \frac{\partial f}{\partial \eta} d\eta)
\end{align*}
\]
Since \( m + k = 0 \), Equations (43) and (44) become

\[
f = E \cos \beta - CH \sin \beta
\]

(69)

\[
\phi = E \sin \beta - FG^2 + CH \cos \beta
\]

(70)

Substitution of (69) and (70) into (66), (67) and (68) gives a set of differential equations which are laborious to integrate.

As Equations (66), (67) and (68) do not involve \( (\partial f/\partial \eta)_{\eta=0} \) and \( (\partial \phi/\partial \eta)_{\eta=0} \), the significance of distinguishing \( H \) from \( F \) fades. A new unknown \( K \) is introduced and \( f \) and \( \phi \) are defined as follows.

\[
f = E \cos \beta
\]

(71)

\[
\phi = E \sin \beta + K F
\]

(72)

In Equations (40) and (41), consequently in Equations (69) and (70), the cross flow velocity component was adjusted with the \( H \) term, the magnitude of which was to be determined by the momentum equations, but the main flow velocity component was not adjusted with the \( H \) term. In Equations (71) and (72), the velocity component in the constant pressure direction was not adjusted with the \( H \) term but the radial component velocity was adjusted with the \( H \) term. Then, the \( H \) term was involved in the \( F \) term with the unknown coefficient \( K \) in Equation (72). Since there is no physical reason to prefer the set of Equations (69) and (70) to the set of Equations (71) and (72), the set of Equations (71) and (72) is chosen because it is simple and makes it easy to integrate Equations (66), (67) and (68).

The velocity profile assumptions (71) and (72) are substituted into (66), (67) and (68), \( k \) and \( \psi_2 \) in (66) and (67) are eliminated with the relations (64) and (68). Then (66) and (67) become as follows:

\[
\rho K \left( \sin \beta \int_0^1 EF \, d\eta - \sin \beta \int_0^1 F \, d\eta + 2K \int_0^1 F^2 \, d\eta \right)
\]

\[
+ \rho \frac{G}{G} \left( K \sin \beta \int_0^1 EF \, d\eta + K \sin \beta \int_0^1 F \, d\eta + K^2 \int_0^1 F^2 \, d\eta \right)
\]

\[
+ (1 - E^2 \, d\eta)(1 - \sin^2 \beta) + \sin^2 \beta \int_0^1 E \, d\eta + K \sin \beta \int_0^1 F \, d\eta
\]

\[
\frac{1}{\sigma^2} \left( \sin \beta E(0) + K F(0) \right) = 0
\]

(73)
\[ \rho K \left( \int_0^1 EF \, d\eta - \int_0^1 F \, d\eta \right) + \rho \frac{G}{G^2} K \left( \int_0^1 EF \, d\eta + \int_0^1 F \, d\eta \right) - \sin \beta \left( 1 - \int_0^1 E^2 \, d\eta \right) + K \left( \int_0^1 EF \, d\eta + \int_0^1 F \, d\eta \right) + \sin \beta \int_0^1 \eta \, E \, d\eta + \frac{2}{G^2} = 0 \]  

(74)

With the numerical values of the integral of \( E \) and \( F \) in Equation (53), Equations (73) and (74) become

\[ \rho K \left( -3.636833 \sin \beta + 0.220444 \right) \]

\[ + \rho \frac{G}{G^2} \left( -3.636833 \sin \beta + 0.110222 \right) \]

\[ + 417.460 - 117.460 \sin^2 \beta - 8.333 \, K \sin \beta \]

\[ + \frac{1}{G^2} \left( 2000 \sin \beta + 166.666 \, K \right) = 0 \]  

(75)

\[ \rho K + \rho \frac{G}{G^2} + K + 32.29762 \sin \beta - \frac{549.929}{G^2} = 0 \]  

(76)

These two equations are rearranged to

\[ 0.110222 \, K \rho = -417.459886 + 8.256373 \, K \sin \beta \]

\[ - 227.280940 \frac{K}{G^2} + 0.110222 \, K^2 \]  

(77)

\[ - 0.110222 \, \rho \, K^2 \frac{G}{G^2} = -417.459886 + 11.816246 \, K \sin \beta \]

\[ + 0.220444 \, K^2 - \frac{K}{G^2} \, 287.895214 \]  

(78)

For numerical integration, new variables are introduced

\[ Y = G^2, \mu = \frac{K}{G^2} \]  

(79)

As the working equations for numerical integration, Equations (77) and (78) become

\[ 0.055111 \, \rho \, \mu^2 \frac{Y \, dy}{d\rho} = 417.460 - 11.816246 \, \mu \, Y \sin \beta \]

\[ - 0.220444 \left( \mu \, Y \right)^2 + 287.895214 \, \mu \]  

(80)

\[ - 2 \left( 1 + \frac{Y}{\mu} \frac{d\mu}{dy} \right) \]

\[ = \frac{417.460 - 8.256373 \, \mu \, Y \sin \beta - 0.110222 \left( \mu \, Y \right)^2 + 227.280940 \, \mu}{417.460 - 11.816246 \, \mu \, Y \sin \beta - 0.220444 \left( \mu \, Y \right)^2 + 287.895214 \, \mu} \]
As the boundary condition, it is assumed that the boundary layer thickness is zero at ρ = 1; that is, Y = 0. If du/dY is not infinity at ρ = 1, Equation (81) gives the value μ = -1.55948 at ρ = 1; that is, Y = 0, μ = -1.55948 at ρ = 1.

The simultaneous Equations (80) and (81) are numerically solved with the boundary conditions (82). The results are shown in Figure 4. The abscissa is ρ and the ordinate is ρ \( \sqrt{Y} = ρ\beta = L = \delta \epsilon \) and μ. That is, the curves L in Figure 4 are dimensionless boundary layer thickness. Since tan β = k, a negative value of β means a converging channel with an accelerating main flow. β = 0 corresponds to a circular arc flow with no pressure gradient.

It should be noted that the boundary layer is thicker for a negative β, or an accelerated flow than for a positive β, or a decelerated flow. Since acceleration and convergence of the side walls are directly related, the author suspects that the converging effect of the side walls on the thickness is dominant over the effect of the accelerated main flow on the thickness.

The velocity distributions in the main flow and cross flow directions, \( u/V \) and \( v/V \), are

\[
\frac{u}{V} = (2\eta - 2\eta^3 + \eta^4) + K \sin \beta \ (1-\eta)^3/6
\]

\[
\frac{v}{V} = K \cos \beta \ (1-\eta)^3/6
\]

where \( K = \mu(L/\rho)^2 \), \( \eta = y/\delta \).

The values of L and μ are shown in Figure 4.

The dimensionless boundary layer thickness \( \delta \) is

\[
\delta = L \ \epsilon \ \delta = r_1L
\]

The real dimension of boundary layer thickness \( R_0\delta_1 \) is

\[
R_0\delta_1 = R_0\delta/ \sqrt{Re} = R_0r_1L/ \sqrt{U_0R_0} = R_0r_1L/ \sqrt{U_0VR_0} \]

8 Physical Interpretation of the Results

In two-dimensional flow, the velocity profile and the boundary layer thickness are very sensitive to the pressure gradient. However,
Figure 3 shows that in three-dimensional flow the pressure gradient affects the boundary layer behavior only slightly. This is due to the escaping effect of boundary layer fluid by the cross flow. Equation (59) shows that the cross flow velocity $v/V$ is proportional to $\Lambda$, or $\delta^2$. The amount of cross flow is proportional to $\delta^3$. Therefore, an adverse pressure gradient cannot make the three-dimensional boundary layer very thick, and a favorable pressure gradient cannot make the boundary layer very thin compared with the no pressure gradient case.

In two-dimensional flow, it was verified that no similar solution exists for adverse pressure gradient $U = U_0 \ell^a (a < 0)$ (Ref. 8). However, the three-dimensional theory gives a solution. As a matter of fact, the low momentum fluid in the boundary layer does not flow against the unfavorable pressure gradient in the main flow direction but flows in a sidewise direction in which the pressure gradient is favorable. Therefore, no separation occurs even for a steep unfavorable pressure gradient.

9 Comparison With Experiment

The end-wall boundary layer of a turbine nozzle cascade was measured and verified to be laminar in a companion paper (Ref. 9). For a comparison of the theory with the experimental results, the cascade was approximated by a pair of logarithmic spiral lines. In Figure 5, the thick full lines show the real channel shape and the thick broken lines show a pair of logarithmic spirals ($\log R/R_0 = \theta$) of spiral angle $\beta = 45^\circ$. The fine full lines across the channel are constant density (or pressure) lines observed by interferogram, and the fine dotted lines are constant pressure lines for the logarithmic spiral flow. Since the density does not vary in proportion to the pressure, the lines in the interferogram do not show an equal pressure difference between succeeding lines, but if the range of the variation of the pressure is small, the pressure difference between each pair of lines is nearly the same. Although the curvature of the logarithmic spiral is a little smaller than that of the real nozzle, the rate of convergence of the width of the channel is nearly the same. The constant pressure lines of the logarithmic spiral near station 4 - fine broken lines - show reasonable agreement with the constant density lines of the interferogram - fine full lines.
25.

9.1 Velocity Distribution in the Boundary Layer

The theoretically computed velocity distribution in the boundary layer was compared with the observed velocity distribution for the case with a laminar, upstream boundary layer. The experiment was done for the condition that the main flow velocity at station 4 (or 9) was 31.0 ft/s (the throat velocity was 49.4 ft/s), and the boundary layer flow was computed for this condition.

According to the theory, the velocity profile is similar along the center line of the nozzle and each velocity component is shown by a single curve, if

\[ \frac{u}{V}, \frac{v}{V} \text{ versus } \frac{R_0 - R_e}{R_0 - \rho} = \frac{u}{V_0} \frac{\sqrt{V_0 \rho r}}{\gamma} \]

is chosen as the coordinate system; where \( R_0 \) is the distance from the end wall, \( R_0 \rho r \) is the distance of the station from the center of the logarithmic spiral, \( U_0 V \) is the main flow velocity at the station, \( \gamma \) is the kinematic viscosity, \( U_0 u \) and \( U_0 v \) are the main flow and cross flow velocity components in the boundary layer. The theoretical velocity components in the main flow and cross flow directions are shown in Figure 6 by full lines, experimental values at station 3 by white circles, experimental values at station 4 by triangular points and experimental values at station 5 by black circles. The disagreement between the experimental values and the theoretical curves is the same order as the disagreement between the experimental values at different stations. The disagreement is partly due to the difference between the real nozzle shape and the assumed logarithmic spiral.

The theoretically predicted velocity distributions at stations 6, 7, 8, 9, 10 and 11 are shown in Figure 7A and the observed distributions are shown in Figure 7B, where real distance from the wall is used as the abscissa. Concerning the velocity component in the main flow direction, the theory predicts that the boundary layer displacement thickness increases from station 6 to station 9 where it is maximum and then it decreases slightly from station 9 to station 11 (see also Figure 9). In the experiment, the velocity distributions between station 7 and station 10 have a similar trend to the theoretical ones, although the
velocity profile is peculiar at station 6. Concerning the cross flow component, the theory predicts that the magnitude of velocity as well as the boundary layer thickness monotonically increases from station 6 to station 11, and the experiment supports the prediction although maximum cross flow velocity is about 30% larger than the theoretical value.

The boundary layer at station 11 was presumably obstructed by the inner wall of the passage which was not considered in the theory. The effect of the outer wall of the passage was taken into consideration in the theory by the assumption that the boundary layer thickness on the end wall was zero along the outer corner of the passage. The interferogram inserted in Figure 7 shows that the boundary layer on the pressure surface of the blade is very thick and turbulent and that station 6 is inside or very close to the layer. Consequently, the real effect of the outer blade is quite different from the assumed condition in the theory. It is not known how to take the corner effect into consideration. The corner effect may be partly responsible for the discrepancy between the experimental and the theoretical cross flows and for the peculiar velocity profile observed at station 6.

9.2 Displacement Thickness and Friction Force Along the Center Line of the Nozzle and Across the Nozzle

Boundary layer displacement thickness and wall friction force were measured from the observed velocity profile at four stations (2, 3, 4 and 5) along the center line of the nozzle and compared with the values predicted by the three-dimensional theory and by a two-dimensional theory. The two-dimensional theory was Walz's method utilizing the observed pressure distribution along the center line. In Figure 8A, the abscissa is the location along the center line and the ordinate is the friction force. The fine line is the friction force predicted by the two-dimensional laminar boundary layer theory, and the black circles are the experimental values. The dotted line, which connects the white circles, shows the value predicted by the three-dimensional boundary layer theory based on logarithmic spiral flow. Since a pair of logarithmic lines are similar to the shape of the nozzle only for a limited range, from station 3 to station 5, the
application of the theory is also limited to this range.

Concerning the main flow component of friction, the predictions of both the two-dimensional theory and the three-dimensional theory agree with the experimental value. However, it is suspected that the agreement of the prediction of the two-dimensional theory is fortuitous. According to the two-dimensional theory, the dimensionless thickness is radically influenced by the pressure gradient, while the three-dimensional theory shows that the dimensionless thickness is little influenced by the pressure gradient. The converging flow makes the boundary layer thicker than the two-dimensional flow with the same pressure distribution, and the cross flow due to turning of the main flow makes the boundary layer thin. Since the nozzle flow is a combination of converging flow and curved flow, these two effects may cancel each other giving a resultant friction force close to the simple two-dimensional theory prediction.

The cross flow component friction force is not predicted, of course, by a two-dimensional theory. The three-dimensional theory prediction for the cross flow component friction force agrees well with the observed value at station 4. The theoretically assumed pressure distribution of Figure 5 should create more boundary layer overturning than the real distribution at station 5. Accordingly, the disagreement of the theory with the experiment at station 5 is partly attributed to the discrepancy between the assumed and actual pressure distributions.

The displacement thickness of the boundary layer and the overturning angle at the wall are shown in Figure 8. The predicted boundary layer displacement thickness by the two-dimensional theory is again very close to that of the three-dimensional theory. The agreement of both theories with the observed values for the thickness is not as good as that for the friction force. The discrepancy is largest at station 4. This is mainly attributed to the very peculiar velocity profile at that station which is unlike that of neighboring stations. If the velocity distribution at station 4 was similar to that at a neighboring station, better agreement with both theories would result.
The overturning at the wall was computed as the arc-tangent of the ratio of the cross flow component friction force to the main flow component friction force; the values are shown in Figure 8 as black and white circles. Also, this angle was directly measured on the carbon-black-trace picture and plotted as the triangular points.

The displacement thickness, wall friction force and overturning angle across the nozzle were computed from Figure 7B and plotted on Figure 9. As explained before, since the end effect of the both blades is not properly accounted for in the theory, the agreement of the theory and the experiment is not perfect.

In summary, the three-dimensional laminar boundary layer theory predicts the boundary layer flow along the center line of the channel, as well as across the channel at station 4 with reasonable accuracy. Therefore, it is suspected that the theory is applicable for the curved part of the nozzle except near either blade surface.

10 Conclusion

Three-dimensional laminar boundary layer equations with three independent variables are reduced to a set of simultaneous ordinary differential equations, taking advantage of the similarity condition and of momentum equations. Although similarity conditions exist for several cases, only two families of them are realistic, i.e. the boundary layer on the parallel plane end walls of a curved channel with logarithmic spiral side walls, and the boundary layer on the plane end wall of a concentric circular-arc channel having a particular family of accelerated or decelerated main flows.

A few cases are solved for each of the families. The results show that acceleration makes the boundary layer thin and deceleration makes it thick, but the variation of thickness due to pressure gradient is small compared with that for two-dimensional cases. Particularly, the three-dimensional boundary layer does not separate with the unfavorable pressure gradient which would separate the two-dimensional boundary layer. The examples of logarithmic spiral channel show that the effect of convergence of the side walls is dominant over the effect of accelerated
main flow, which is related to the converging side walls.

A turbine nozzle cascade was represented by a pair of logarithmic spirals and the observed boundary layer behavior was compared with the prediction of the theory. The agreement was satisfactory.
APPENDIX

Velocity Profile of Three-Dimensional Laminar Boundary Layer

No experimental data are available for the velocity profile of a three-dimensional laminar boundary layer. The theoretical velocity profile was calculated by Cooke for a family of yawed cylinders. In the present analysis, the velocity profiles are assumed to be represented by polynomials. In order to check the validity of the assumption, the polynomial velocity profiles are compared with Cooke's velocity profiles for four main flow conditions.

In Cooke's analysis, the velocity components of the main flow $U$ and $V$ are assumed

$$U = U_0 x^m, \quad V = V_0$$  \hspace{1cm} (a-1)

and the velocity components in the boundary layer $u/U$ and $v/V$ are tabulated as functions of the distance from the wall $\zeta$ or $z \sqrt{\frac{m+1}{2} \frac{U}{y}}$. The angle $\sigma$ is defined as $\sigma = \tan^{-1} \frac{V}{U}$. The main-flow velocity-component $w_m$ is

$$\frac{w_m}{W} = \frac{u}{U} \cos \sigma + \frac{v}{V} \sin \sigma$$
$$= \frac{u}{U} - \sin \sigma \left( \frac{u}{U} - \frac{v}{V} \right)$$  \hspace{1cm} (a-2)

The cross-flow velocity component $w_c$ is

$$\frac{w_c}{W} = \frac{u}{W} \sin \sigma - \frac{v}{W} \cos \sigma = \sin \sigma \cos \sigma \left( \frac{u}{U} - \frac{v}{V} \right)$$  \hspace{1cm} (a-3)

Since $u/U$ and $v/V$ are independent of $\sigma$, the shape of the cross-flow component is independent of $\sigma$, but the shape of the main-flow component depends upon $\sigma$.

In the polynomial expression, it is assumed that

$$\frac{w_m}{W} = (2\eta - 2\eta^3 + \eta^4) + \left( \frac{\eta^2}{2} \frac{dW}{ds} \right) \frac{\eta}{6} (1 - \eta)^3$$  \hspace{1cm} (a-4)

$$\frac{w_c}{W} = \frac{K}{6} \eta (1 - \eta)^3$$  \hspace{1cm} (a-5)
The shape of the cross-flow component velocity profile does not vary, but the shape of the main-flow component velocity profile depends upon $\delta$ and $dW/ds$. The values of $\delta$ and $K$ are obtained as the solution of the momentum equations in the present analysis, but the analysis is not applicable to the cases solved by Cooke. Therefore, the values of $\delta$ and $K$ are arbitrarily chosen so that the best agreement is obtained between Cooke's velocity profiles and the polynomial profiles. For the comparison, the boundary layer thickness $\delta$ is expressed by the same abscissa

$$\zeta \text{ or } z \sqrt{\frac{m+1}{2}} \frac{U}{\nu x}.$$ The coefficient in $(a-4)$ is related to $\zeta$ as follows:

$$\frac{\delta^2}{\nu} \frac{dW}{ds} = \frac{\delta^2}{\nu} \cos \delta \frac{dU}{dx} = \frac{2m}{m+1} \cos \delta \zeta^2 \quad (a-6)$$

Comparison is made for $\delta = \pi/6$ and $\pi/4$, and for $m = 1$, stagnation point flow, and $m = -0.091$, separation flow. The results are shown in Fig. 10. Although agreement is not perfect, the polynomial expression is acceptable as approximate velocity profiles for all of these four cases.
**LIST OF SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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| $E$    | $2\eta - 2\eta^3 + \eta^4$  
| $F$    | $\eta(1 - \eta)^{3/6}$  
| $f$    | Ratio of tangential velocity component $v_0$ to main flow velocity $V$, expressed with $\rho, \alpha, \eta$ coordinate system, $f = v_0/V$  
| $f_0$  | $f$ in the main flow  
| $f_1$  | $f$ expressed with $r, \theta, \gamma$ coordinate system, $f_1(r, \theta, \gamma) = f(\rho, \alpha, \eta)$  
| $G$    | $L(\rho)/\rho$  
| $H$    | $(\eta - 3\eta^3 + 2\eta^4)/6$  
| $k$    | Arbitrary constant, $r_1 = e^{k\alpha}$  
| $K$    | Unknown function of $\rho$  
| $L$    | Function of $\rho$, $\delta = L(\rho) \epsilon^{\frac{k-m}{2}\alpha}$  
| $M$    | Function of $\rho$, $V = M(\rho) \epsilon^{ma}$  
| $m$    | Arbitrary constant, $V = M(\rho) \epsilon^{ma}$  
| $p$    | Dimensionless pressure  
| $R_0$  | Typical length at $\theta = 0$  
| $r$    | Dimensionless radius, $r = r_1\rho$  
| $r_1$  | Dimensionless typical radius at $\theta$, $r = r_1\rho$ (Figure 1)  
| $Re$   | Reynolds number, $Re = U_0R_0/\nu$  
| $S$    | Arbitrary function of $\alpha$  
| $T$    | Arbitrary function of $\alpha$  
| $U_0$  | Typical velocity  
| $V$    | Dimensionless velocity just outside the boundary layer  
| $v_r$  | Radial component of dimensionless velocity  
| $v_\theta$ | $\theta$ component of dimensionless velocity  
| $v_z$  | $z$ component of dimensionless velocity  
| $w$    | Transformed velocity in $z$ direction, $w = v_z \sqrt{Re}$  
| $Y$    | $Y = G^2 = \left\{ L(\rho)/\rho \right\}^2$  
| $Y$    | Transformed distance from the wall, $y = z \sqrt{Re}$  
| $z$    | Dimensionless distance from the wall  
| $\alpha$ | Angular location in $\rho, \alpha, \eta$ coordinate system  
| $\beta$ | Angle of logarithmic spiral, $\tan \beta = k$  
| $\delta$ | Boundary layer thickness, $\delta = L(\rho) \epsilon^{\frac{k-m}{2}\alpha}$ (Fig. 1)  
| $\delta^*$ | Displacement thickness of the main flow component of the boundary layer velocity |
\( \eta \) Ratio of distance from the end wall \( y \) to the boundary layer thickness \( \delta \), \( \eta = y/\delta \)

\( \theta \) Angular location in \( r, \theta, y \) coordinate system

\( \kappa \) Unknown function of \( \rho \)

\( \kappa = (L/\rho)^2 \) \hspace{1cm} (50)

\( \mu \) Kinematic viscosity

\( \nu \) Ratio of a radius \( r \) to the typical radius at the same angular location, \( \rho = r/r_1 \)

\( \rho_0 \) Density of fluid

\( \phi \) Ratio of radial velocity component \( v_r \) to main flow velocity \( V \) expressed with \( \rho, \alpha, \eta \) coordinate system, \( \phi = v_r/V \)

\( \phi_0 \) \( \phi \) in main flow

\( \phi_1 \) \( \phi \) expressed with \( r, \theta, y \) coordinate system

\( \psi \) Ratio of \( z \) component of velocity \( v_z \) to main flow velocity \( V \) expressed with \( \rho, \alpha, \eta \) coordinate system \( x = w/V \)

\( \psi_0 \) \( \psi \) in main flow

\( \psi_1 \) \( \psi \) expressed with \( r, \theta, y \) coordinate system

\( \psi_2 \) \( \psi = \psi_2(\eta, \rho) e^{-\frac{m+K\phi}{2}} \) \hspace{1cm} (22)
Bibliography


Velocity $v(y)$ at station $A(r, \theta)$ is similar to velocity $v_0(y)$ at station $A_0(r_0, 0)$. $r_1$ and $\theta$ are defined so that two corresponding points $B(r, \theta, y)$ and $B_0(r_0, 0, y_0)$ are identified by the same coordinates $(\rho_0, \eta_0)$; that is,

$B(r, \theta, y) = B(r_1, \theta, \delta \eta_0)$

$B_0(r, \theta, y) = B_0(\rho_0, 0, \delta \eta_0)$
FIG. 2 - LOGARITHMIC SPIRAL
FIG. 3 - COEFFICIENTS L AND −K FOR CIRCULAR-ARC MAIN FLOW
FIG. 4 - COEFFICIENTS $L$ AND $-\mu$ FOR LOGARITHMIC-SPIRAL MAIN FLOW
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FIG 6 BOUNDARY LAYER VELOCITY DISTRIBUTION ALONG CENTER-LINE OF NOZZLE
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FIG. 10 VELOCITY PROFILES OF THREE DIMENSIONAL LAMINAR BOUNDARY LAYERS