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A PARTIAL ANALOG OF THE INTEGRABILITY THEOREM FOR DISTRIBUTIONS ON p -ADIC SPACES AND APPLICATIONS

BY

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ABSTRACT

Let X be a smooth real algebraic variety. Let ξ be a distribution on it. One can define the singular support of ξ to be the singular support of the D_X -module generated by ξ (sometimes it is also called the characteristic variety). A powerful property of the singular support is that it is a coisotropic subvariety of T^*X . This is the integrability theorem (see [KKS, Mal, Gab]). This theorem turned out to be useful in representation theory of real reductive groups (see, e.g., [AG4, AS, Say]).

The aim of this paper is to give an analog of this theorem to the non-Archimedean case. The theory of D -modules is not available to us so we need a different definition of the singular support. We use the notion wave front set from [Hef] and define the singular support to be its Zariski closure. Then we prove that the singular support satisfies some property that we call weakly coisotropic, which is weaker than being coisotropic but is enough for some applications. We also prove some other properties of the singular support that were trivial in the Archimedean case (using the algebraic definition) but not obvious in the non-Archimedean case.

We provide two applications of those results:

- a non-Archimedean analog of the results of [Say] concerning Gel'fand property of nice symmetric pairs;
- a proof of multiplicity one theorems for GL_n which is uniform for all local fields. This theorem was proven for the non-Archimedean case in [AGRS] and for the Archimedean case in [AG4] and [SZ].

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1. Introduction

The theory of invariant distributions is widely used in representation theory of reductive algebraic groups over local fields. We can roughly divide this theory into two parts.

- Archimedean — distributions on smooth manifolds, Nash manifolds, real analytic manifolds, real algebraic manifolds, etc.
- Non-Archimedean — distributions on l -spaces, p -adic analytic manifolds, p -adic algebraic manifolds, etc.

In general, the non-Archimedean case of the theory of invariant distributions is easier than the Archimedean one, but there is one significant tool that is available only in the Archimedean case. This tool is the theory of differential operators. One of the powerful tools coming from the use of differential operators is the notion of singular support (sometimes it is also called the characteristic variety). The singular support of a distribution ξ on a real algebraic manifold X is a subvariety of T^*X . A deep and important property of the singular support is the fact that it is coisotropic. This fact is the integrability theorem (see [KKS, Mal, Gab]). This theorem turned out to be useful in the representation theory of real reductive groups (see, e.g., [AG4, AS, Say]).

The aim of this paper is to give an analog of this theorem to the non-Archimedean case. Though we didn't achieve a full analog of the integrability theorem, we managed to formulate and prove some partial analog of it. Namely we prove that the singular support satisfies some property that we call weakly coisotropic, which is weaker than being coisotropic but enough for some applications. We also prove some other properties of the singular support that were trivial in the Archimedean case but not obvious in the non-Archimedean case.

We provide two applications of those results.

- We give a non-Archimedean analog of the results of [Say] concerning the Gel'fand property of nice symmetric pairs.
- We give a proof of multiplicity one theorems for GL_n which is uniform for all local fields. This theorem was proven for the non-Archimedean case in [AGRS] and for the non-Archimedean case in [AG4] and [SZ].

The results of this paper are also applied in [Sun] where multiplicity one theorems for Fourier–Jacobi models are established.

1.1. THE SINGULAR SUPPORT AND THE WAVE FRONT SET. The theory of D -modules is not available to us so we need a different definition of singular support. We use the notion of wave front set from [Hef] and define the singular support to be its Zariski closure. Unlike the algebraic definition of the singular support, the definition of the wave front set is analytic and uses Fourier transform instead of differential operators; this is what makes it available for the non-Archimedean case.

Surprisingly, the fact that in the non-Archimedean case the singular support is weakly coisotropic follows quite easily from the basic properties of the wave front set developed in [Hef]. However, another important property of the the singular support that was trivial in the Archimedean case is not obvious in the non-Archimedean case. Namely in the presence of a group action one can exhibit some restriction on the singular support of invariant distribution. We also provide a non-Archimedean analog of this property.(X)

In general, our results are based on the work [Hef] where the theory of the wave front set is developed for the non-Archimedean case.

1.2. STRUCTURE OF THE PAPER. In Section 2 we give notations that will be used throughout the paper and give some preliminaries on distributions, including some results from [Hef] on the wave front set.

In Section 3 we introduce the notion of coisotropic variety and weakly coisotropic variety and discuss some of their properties.

In Section 4 we prove the main results on singular support and the wave front set. We sum up the properties of singular support in Subsection 4.2. In Subsection 4.3 we apply those properties to get some technical results that will be useful for proving the Gel'fand property.

In Section 5 we generalize the results of [Say] to arbitrary local fields of characteristic 0.

In Subsection 5.1 we give the necessary preliminaries for Section 5. In Subsubsection 5.1.1 we provide basic preliminaries on Gel'fand pairs. In Subsubsection 5.1.2 we review a technique from [AG2] for proving that a given pair is a Gel'fand pair. In Subsubsections 5.1.3–5.1.7 we review a technique from [AG2] and [AG3] for proving that a given symmetric pair is a Gel'fand pair.

In section 6 we indicate a proof of multiplicity one theorems for GL_n which is uniform for all local fields of characteristic 0. This theorem was proven for the

non-Archimedean case in [AGRS] and for the non-Archimedean case in [AG4] and [SZ].

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2. Notations and preliminaries

- Throughout the paper F is a local field of characteristic zero.
- All the algebraic varieties, analytic varieties and algebraic groups that we consider will be defined over F .
- Let X be an algebraic variety. By $X(F)$ we mean the set of F -points of X considered as a topological space, analytic manifold or Nash manifold.¹ We will treat vector spaces similarly, but when there is no possible confusion we will not distinguish between V and $V(F)$.
- By a reductive group we mean an algebraic reductive group.
- Let E be an extension of F . Let G be an algebraic group defined over F . We denote by $G_{E/F}$ the canonical algebraic group defined over F such that $G_{E/F}(F) = G(E)$.
- By Sp_{2n} we mean the symplectic group of $2n \times 2n$ matrices.
- The word manifold will always mean that the object is smooth (e.g., by algebraic manifold we mean smooth algebraic variety).
- For a group G acting on a set X and a point $x \in X$, we denote by Gx or by $G(x)$ the orbit of x and by G_x the stabilizer of x . We also denote by X^G the set of G invariant elements and for an element $g \in G$ denote by X^g the set of g invariant elements.
- An action of a Lie algebra \mathfrak{g} on a (smooth, algebraic, etc.) manifold M is a Lie algebra homomorphism from \mathfrak{g} to the Lie algebra of vector fields on M . Note that an action of a (Lie, algebraic, etc.) group on M defines an action of its Lie algebra on M .
- For a Lie algebra \mathfrak{g} acting on M , an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_xM$ the value at point x of the vector field corresponding to α . We denote by $\mathfrak{g}x \subset T_xM$ or by $\mathfrak{g}(x) \subset T_xM$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel. We denote

¹ See §§2.1.

$M^{\mathfrak{g}} := \{x \in M | \mathfrak{g}x = 0\}$ and $M^\alpha := \{x \in M | \alpha(x) = 0\}$, analogously to the group case.

- For manifolds $L \subset M$ we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M .
- Denote by $CN_L^M := (N_L^M)^*$ the conormal bundle.
- For a point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.
- Let M, N be (smooth, algebraic, etc.) manifolds. Let E be a bundle over N . Let $\phi : M \rightarrow N$ be a morphism. We denote by $\phi^*(E)$ the pullback of E .
- Let M, N be (smooth, algebraic, etc.) manifolds. Let $S \subset (T^*(N))$. Let $\phi : M \rightarrow N$ be a morphism. We denote $\phi^*(S) := d(\phi)^*(S \times_N M)$.
- Let M, N be topological spaces. Let E be over N . Let $\phi : M \rightarrow N$ be a morphism. We denote by $\phi^*(E)$ the pullback of E .
- Let V be a linear space. For a point $x = (v, \phi) \in V \times V^*$ we denote $\hat{x} = (\phi, -v) \in V^* \times V$; similarly for subset $X \subset V \times V^*$ we define \hat{X} . For a (smooth, algebraic, etc.) manifold and a subset $X \subset T^*(M \times V)$ we denote $\hat{X}_V \subset T^*(M \times V^*)$ in a similar way.
- Let B be a non-degenerate bilinear form on V . This gives an identification between V and V^* and therefore, by the previous notation, maps $F_B : V \times V \rightarrow V \times V$ and $F_B : T^*M \times V \times V \rightarrow T^*M \times V \times V$. If there is no ambiguity we will denote it by F_V .

2.1. DISTRIBUTIONS. In this paper we will refer to distributions on algebraic varieties over Archimedean and non-Archimedean fields. In the non-Archimedean case we mean the notion of distributions on l -spaces from [BZ], namely linear functionals on the space of locally constant compactly supported functions.

We will use the following notations.

NOTATION 2.1.1: *Let X be an l -space.*

- *Denote by $\mathcal{S}(X)$ the space of Schwartz functions on X (i.e., locally constant compactly supported functions). Denote $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ to be the dual space to $\mathcal{S}(X)$.*
- *For any locally constant sheaf E over X we denote by $\mathcal{S}(X, E)$ the space of compactly supported sections of E and by $\mathcal{S}^*(X, E)$ its dual space.*

- For any finite-dimensional complex vector space V we denote $\mathcal{S}(X, V) := \mathcal{S}(X, X \times V)$ and $\mathcal{S}^*(X, V) := \mathcal{S}^*(X, X \times V)$, where $X \times V$ is a constant sheaf.
- Let $Z \subset X$ be a closed subset. We denote

$$\mathcal{S}_X^*(Z) := \{\xi \in \mathcal{S}^*(X) \mid \text{Supp}(\xi) \subset Z\}.$$

For a locally closed subset $Y \subset X$ we denote $\mathcal{S}_X^*(Y) := \mathcal{S}_{X \setminus (\overline{Y} \setminus Y)}^*(Y)$. In the same way, for any locally constant sheaf E on X we define $\mathcal{S}_X^*(Y, E)$.

- Suppose that X is an analytic variety over a non-Archimedean field F . Then we define D_X to be the sheaf of locally constant measures on X (i.e., measures that locally are restriction of Haar measure on F^n). We denote $\mathcal{G}(X) := \mathcal{S}^*(X, D_X)$ and $\mathcal{G}(X, E) := \mathcal{S}^*(X, D_X \otimes E^*)$.
- For an analytic map $\phi : X \rightarrow Y$ of analytic manifolds over a non-Archimedean field we denote by $\phi^* : \mathcal{G}(Y) \rightarrow \mathcal{G}(X)$ the pullback; similarly, we denote $\phi^* : \mathcal{G}(Y, E) \rightarrow \mathcal{G}(X, \phi^*(E))$ for any locally constant sheaf E .

In the Archimedean case we will use the theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds, but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic*.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On \mathbb{R}^n it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

NOTATION 2.1.2: Let X be a Nash manifold.

Denote by $\mathcal{S}(X)$ the space of Schwartz functions on X . Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the dual space to $\mathcal{S}(X)$. We define D_X to be the bundle of densities on X for any Nash bundle E on X ; we define $\mathcal{S}^*(X, E), \mathcal{S}_X^*(Y), \mathcal{G}(X), \phi^*$, etc. analogously to the non-Archimedean case.

2.1.1. Invariant distributions.

PROPOSITION 2.1.3: Let an l -group G act on l -space X . Let $Z \subset X$ be a closed subset.

Let $Z = \bigcup_{i=0}^l Z_i$ be a G -invariant stratification of Z . Let χ be a character of G . Suppose that for any $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i)^{G,\chi} = 0$. Then $\mathcal{S}_X^*(Z)^{G,\chi} = 0$.

This proposition immediately follows from [BZ, Section 1.2].

PROPOSITION 2.1.4: Let a Nash group G act on a Nash manifold X . Let $Z \subset X$ be a closed subset.

Let $Z = \bigcup_{i=0}^l Z_i$ be a Nash G -invariant stratification of Z . Let χ be a character of G . Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i, \text{Sym}^k(CN_{Z_i}^X))^{G,\chi} = 0$. Then $\mathcal{S}_X^*(Z)^{G,\chi} = 0$.

This proposition immediately follows from [AGS, Corollary 7.2.6].

THEOREM 2.1.5 (Frobenius reciprocity): Let an l -group (respectively Nash group) G act transitively on an l -space (respectively Nash manifold) Z . Let $\varphi : X \rightarrow Z$ be a G -equivariant map. Let $z \in Z$. Let X_z be the fiber of z . Let χ be a character of G . Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z,\chi \cdot \Delta_G|_{G_z} \cdot \Delta_{G_z}^{-1}}$ where Δ denotes the modular character.

For a proof see [Ber, section 1.5] for the non-Archimedean case and [AG2, Theorem 2.3.8] for the non-Archimedean case.

2.1.2. *Fourier transform.* From now till the end of the paper we fix an additive character κ of F . If F is Archimedean we fix κ to be defined by $\kappa(x) := e^{2\pi i \text{Re}(x)}$.

NOTATION 2.1.6: Let V be a vector space over F . For any distribution $\xi \in \mathcal{S}^*(V)$ we define $\widehat{\xi} \in \mathcal{G}(V^*)$ to be its Fourier transform.

For a space X (an l -space or a Nash manifold depending on F), for any distribution $\xi \in \mathcal{S}^*(X \times V)$ we define $\widehat{\xi}_V \in \mathcal{G}(X \times V^*)$ to be its partial Fourier transform

Let B be a non-degenerate bilinear form on V . Then B identifies $\mathcal{G}(V^*)$ with $\mathcal{S}^*(V)$. We denote by $\mathcal{F}_B : \mathcal{S}^*(V) \rightarrow \mathcal{S}^*(V)$ and $\mathcal{F}_B : \mathcal{S}^*(M \times V) \rightarrow \mathcal{S}^*(M \times V)$ the corresponding Fourier transforms.

If there is no ambiguity, we will write \mathcal{F}_V , and sometimes just \mathcal{F} , instead of \mathcal{F}_B .

We will use the following trivial observation.

LEMMA 2.1.7: Let V be a finite-dimensional vector space over F . Let a group G act linearly on V . Let B be a G -invariant non-degenerate symmetric bilinear

form on V . Let $\xi \in \mathcal{S}^*(V)$ be a G -invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G -invariant.

NOTATION 2.1.8: Let V be a vector space over F . Consider the homothety action of F^\times on V by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of F^\times on $\mathcal{S}^*(V)$.

Also, for any $\lambda \in F^\times$ denote $|\lambda| := \frac{dx}{\rho(\lambda)dx}$, where dx denotes the Haar measure on F .

NOTATION 2.1.9: Let V be a vector space over F . Let B be a non-degenerate symmetric bilinear form on V . We denote

$$Z(B) := \{x \in V \mid B(x, x) = 0\}.$$

THEOREM 2.1.10 (Homogeneity Theorem): Let V be a vector space over F . Let B be a non-degenerate symmetric bilinear form on V .

(i) Suppose F is non-Archimedean. Let M be an l -space. Let

$$\xi \in \mathcal{S}_{V \times M}^*(Z(B) \times M)$$

such that $\mathcal{F}_B(\xi) \in L$.

Then there exists a unitary character u of F^\times such that $\rho(\lambda)\xi = \|\lambda\|^{\frac{\dim V}{2}}u(\lambda)\xi$ for any $\lambda \in F^\times$.

(ii) Suppose that F is Archimedean. Let M be a Nash manifold.

Let $L \subset \mathcal{S}_{V \times M}^*(Z(B) \times M)$ be a non-zero subspace such that $\forall \xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).

Then there exist a non-zero distribution $\xi \in L$ and a unitary character u of F^\times such that either $\rho(\lambda)\xi = \|\lambda\|^{\frac{\dim V}{2}}u(\lambda)\xi$ for any $\lambda \in F^\times$ or $\rho(\lambda)\xi = |\lambda|^{\frac{\dim V}{2}+1}u(\lambda)\xi$ for any $\lambda \in F^\times$.

For a proof see [AG2, section 5].

2.1.3. *The wave front set.* In this subsection F is a non-Archimedean field. We will use the notion of the wave front set of a distribution on analytic space from [Hef]. First we recall it for a distribution on an open subset of F^n .

Definition 2.1.11: Let $U \subset F^n$ be an open subset and $\xi \in \mathcal{S}^*(U)$ be a distribution. We say that ξ is **smooth** at $(x_0, v_0) \in T^*U$ if there are open neighborhoods A of x_0 and B of v_0 such that for any $\phi \in \mathcal{S}(A)$ there is an $N_\phi \in \mathbb{R}_{>0}$ for which, for any $\lambda \in F$ satisfying $|\lambda| > N_\phi$, we have $(\widehat{\phi\xi})|_{\lambda B} = 0$. The complement in

T^*U of the set of smooth pairs (x_0, v_0) of ξ is called the **wave front** set of ξ and denoted by $WF(\xi)$.

Remark 2.1.12: This notion appears in [Hef] with two differences.

- (1) The notion in [Hef] is more general and depends on some subgroup $\Lambda \subset F$, in our case $\Lambda = F$.
- (2) The notion in [Hef] defines the wave front set of ξ to be a subset in $T^*U - U \times 0$. In our notation this subset will be $WF(\xi) - U \times 0$.

The following lemmas are trivial

LEMMA 2.1.13: *Let $U \subset F^n$ be an open subset and $\xi \in \mathcal{S}^*(U)$ be a distribution. Then $WF(\xi)$ is closed, invariant with respect to the homothety $(x, v) \mapsto (x, \lambda v)$ and*

$$p_U(WF(\xi)) = WF(\xi) \cap (U \times 0) = \text{Supp}(\xi).$$

LEMMA 2.1.14: *Let $V \subset U \subset F^n$ be open subsets and $\xi \in \mathcal{S}^*(U)$. Then $WF(\xi|_V) = WF(\xi) \cap p_U^{-1}(V)$.*

LEMMA 2.1.15: *Let $U \subset F^n$ be an open subset, $\xi_1, \xi_2 \in \mathcal{S}^*(X)$ be distributions and f_1, f_2 be locally constant functions on X . Then $WF(f_1\xi_1 + f_2\xi_2) \subset WF(\xi_1) \cup WF(\xi_2)$.*

COROLLARY 2.1.16: *For any locally constant sheaf E on U we can define the wave front set of any element in $\mathcal{S}^*(U, E)$ and $\mathcal{G}(U, E)$.*

We will use the following theorem from [Hef]; see Theorem 2.8 there.

THEOREM 2.1.17: *Let $U \subset F^m$ and $V \subset F^n$ be open subsets, and suppose that $f : U \rightarrow V$ is an analytic submersion. Then for any $\xi \in \mathcal{G}(V)$ we have $WF(f^*(\xi)) \subset f^*(WF(\xi))$.*

COROLLARY 2.1.18: *Let $V, U \subset F^n$ be open subsets and $f : V \rightarrow U$ be an analytic isomorphism. Then for any $\xi \in \mathcal{G}(V)$ we have $WF(f^*(\xi)) = f^*(WF(\xi))$.*

COROLLARY 2.1.19: *Let X be an analytic manifold, E be a locally constant sheaf on X . We can define the wave front set of any element in $\mathcal{S}^*(X, E)$ and $\mathcal{G}(X, E)$. Moreover, Theorem 2.1.17 holds for submersions between analytic manifolds.*

3. Coisotropic varieties

Definition 3.0.1: Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M -**coisotropic** if one of the following equivalent conditions holds.

- (i) The ideal sheaf of regular functions that vanish on \overline{Z} is closed under a Poisson bracket.
- (ii) At every smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^\perp$. Here, $(T_z Z)^\perp$ denotes the orthogonal complement to $T_z Z$ in $T_z M$ with respect to ω .
- (iii) For a generic smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^\perp$.

If there is no ambiguity, we will call Z a coisotropic variety.

Note that every non-empty M -coisotropic variety is of dimension at least $\frac{1}{2} \dim M$.

NOTATION 3.0.2: For a smooth algebraic variety X we always consider the standard symplectic form on T^*X . Also, we denote by $p_X : T^*X \rightarrow X$ the standard projection.

Definition 3.0.3: Let (V, ω) be a symplectic vector space with a fixed Lagrangian subspace $L \subset V$. Let $p : V \rightarrow V/L$ be the standard projection. Let $Z \subset V$ be a linear subspace. We call it V -weakly coisotropic with respect to L if one of the following equivalent conditions holds.

- (i) $p(Z) \supset p(Z)^\perp$. Here, Z^\perp denotes the orthogonal complement with respect to ω .
- (ii) $Z^\perp \cap L \subset Z \cap L$. Here, $p(Z)^\perp$ denotes the orthogonal complement in L under the identification $L \cong (V/L)^*$.
- (iii) $p(Z)^\perp \subset Z \cap L$. Here, $p(Z)^\perp$ denotes the orthogonal complement in L under the identification $L \cong (V/L)^*$.

Definition 3.0.4: Let X be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it T^*X -**weakly coisotropic** if one of the following equivalent conditions holds.

- (i) For generic smooth point $x \in p_X(Z)$ and a generic smooth point $z \in p_X^{-1}(x) \cap Z$, the space $T_z(Z)$ is $T_z(T^*(X))$ -weakly coisotropic with respect to $\text{Ker}(dp_X)$.

- (ii) For any smooth point $x \in p_X(Z)$ and any smooth point $z \in p_X^{-1}(x) \cap Z$, the space $T_z(Z)$ is $T_z(T^*(X))$ -weakly coisotropic with respect to $\text{Ker}(dp_X)$.
- (iii) For any smooth point $x \in p_X(Z)$ and for any smooth point $y \in p_X^{-1}(x) \cap Z$, we have $CN_{p_X(Z),x}^X \subset T_y(p_X^{-1}(x) \cap Z)$.
- (iv) For any smooth point $x \in p_X(Z)$, the fiber $p_X^{-1}(x) \cap Z$ is locally invariant with respect to shifts by $CN_{p_X(Z),x}^X$, i.e., for any point $y \in p_X^{-1}(x)$ the intersection $(y + CN_{p_X(Z),x}^X) \cap (p_X^{-1}(x) \cap Z)$ is Zariski open in $y + CN_{p_X(Z),x}^X$.

If there is no ambiguity, we will call Z a weakly coisotropic variety.

Note that every non-empty T^*X -weakly coisotropic variety is of dimension at least $\dim X$.

The following lemma is straightforward.

LEMMA 3.0.5: *Any T^*X -coisotropic variety is T^*X -weakly coisotropic.*

PROPOSITION 3.0.6: *Let X be a smooth algebraic variety with a symplectic form on it. Let $R \subset T^*X$ be an algebraic subvariety. Then there exists a maximal T^*X -weakly coisotropic subvariety of R , i.e., a T^*X -weakly coisotropic subvariety $T \subset R$ that includes all T^*X -weakly coisotropic subvarieties of R .*

Proof. Let T' be the union of all smooth T^*X -weakly coisotropic subvarieties of R . Let T be the Zariski closure of T' in R . It is easy to see that T is the maximal T^*X -weakly coisotropic subvariety of R . ■

The following lemma is trivial.

LEMMA 3.0.7: *Let X be a smooth algebraic variety. Let a group G act on X ; this induces an action on T^*X . Let $S \subset T^*X$ be a G -invariant subvariety. Then the maximal T^*X -weakly coisotropic subvariety of S is also G -invariant.*

NOTATION 3.0.8: *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and let $R \subset T^*Y$ be any subvariety. We define the **restriction** $R|_Z \subset T^*Z$ of R to Z by $R|_Z := i^*(R)$, where $i : Z \rightarrow Y$ is the embedding.*

LEMMA 3.0.9: *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be a weakly coisotropic subvariety. Assume that any smooth point $z \in Z \cap p_Y(R)$ is also a smooth point of $p_Y(R)$ and we have $T_z(Z \cap p_Y(R)) = T_z(Z) \cap T_z(p_Y(R))$.*

Then $R|_Z$ is T^*Z -weakly coisotropic.

Proof. Let $x \in Z$, let $M := p_Y^{-1}(x) \cap R \subset p_Y^{-1}(x)$ and $L := CN_{p_Y(R),x}^Y \subset p_Y^{-1}(x)$. We know that M is locally invariant with respect to shifts in L . Let $M' := p_Z^{-1}(x) \cap R|_Z \subset p_Z^{-1}(x)$ and $L' := CN_{p_Z(R|_Z),x}^Y \subset p_Z^{-1}(x)$. We want to show that M' is locally invariant with respect to shifts in L' . Let $q : p_Y^{-1}(x) \rightarrow p_Z^{-1}(x)$ be the standard projection. Note that $M' = q(M)$ and $L' = q(L)$. Now clearly M' is locally invariant with respect to shifts in L' . ■

COROLLARY 3.0.10: *Let Y be a smooth algebraic variety. Let an algebraic group H act on Y . Let $q : Y \rightarrow B$ be an H -equivariant morphism. Let $O \subset B$ be an orbit. Consider the natural action of G on T^*Y and let $R \subset T^*Y$ be an H -invariant subvariety. Suppose that $p_Y(R) \subset q^{-1}(O)$. Let $x \in O$. Denote $Y_x := q^{-1}(x)$. Then:*

- if R is T^*Y -weakly coisotropic then $R|_{Y_x}$ is $T^*(Y_x)$ -weakly coisotropic.

COROLLARY 3.0.11: *In the notation of the previous corollary, if $R|_{Y_x}$ has no (non-empty) $T^*(Y_x)$ -weakly coisotropic subvarieties, then R has no (non-empty) $T^*(Y)$ -weakly coisotropic subvarieties.*

Remark 3.0.12: The results on weakly coisotropic varieties which we presented here have versions for coisotropic varieties; see [AG4, Section 5.1].

4. Properties of singular support and the wave front set

4.1. THE WAVE FRONT SET. In this subsection F is a non-Archimedean field.

THEOREM 4.1.1: *Let $Y \subset X$ be algebraic varieties, let $y \in Y(F)$ and suppose that X is smooth and Y is smooth at y . Let $\xi \in \mathcal{S}^*(X(F), E)$ and suppose that $\text{Supp}(\xi) \subset Y(F)$. Then $WF(\xi) \cap p_X^{-1}(y)(F)$ is invariant with respect to shifts by $CN_{Y,y}^X(F)$.*

This theorem immediately follows from the following one:

THEOREM 4.1.2: *Let $Y \subset X$ be analytic manifolds and let $y \in Y$. Let $\xi \in \mathcal{S}_X^*(Y)$ and suppose that $\text{Supp}(\xi) \subset Y$. Then $WF(\xi) \cap p_X^{-1}(y)$ is invariant with respect to shifts by $CN_{Y,y}^X$.*

In order to prove this theorem we will need the following standard lemma, which is a version of the implicit function theorem.

LEMMA 4.1.3: *Let $Y \subset X$ be analytic manifolds. Let $n := \dim(X)$ and $k := \dim(Y)$. Let $y \in Y$. Then there exist an open neighborhood $y \in U \subset X$ and an analytic isomorphism $\phi : U \rightarrow W$, where W is an open subset of F^n such that $\phi(Y \cap U) = W \cap F^k$, where $F^k \subset F^n$ is a coordinate subspace.*

Proof of Theorem 4.1.2.

CASE 1: $X = F^n, Y = F^k$. In this case the theorem follows from the fact that if a distribution on F^n is supported on F^k , then its Fourier transform is invariant with respect to shifts by the orthogonal complement to F^k .

CASE 2: $X = U \subset F^n, Y = F^k \cap U$, WHERE $U \subset F^n$ IS OPEN. This follows immediately from the previous case.

CASE 3: THE GENERAL CASE. This follows from the previous case using the lemma and Corollary 2.1.18. ■

THEOREM 4.1.4: *Let an algebraic group G act on a smooth algebraic variety X . Let \mathfrak{g} be the Lie algebra of G . Let $\xi \in \mathcal{S}^*(X)^G$. Then*

$$WF(\xi) \subset \{(x, v) \in T^*X(F) | v(\mathfrak{g}x) = 0\}.$$

We will prove a slightly more general theorem.

THEOREM 4.1.5: *Let an analytic group G act on an analytic manifold X . Let E be a G -equivariant locally constant sheaf on X . Let $\xi \in \mathcal{G}(X, E)^G$. Then $WF(\xi) \subset \{(x, v) \in T^*X(F) | v(\mathfrak{g}x) = 0\}$.*

In order to prove this theorem we will need the following easy lemma.

LEMMA 4.1.6: *Let X, Y be analytic manifolds. Let E be a locally constant sheaf on X . Let $\xi \in \mathcal{G}(X, E)$. Let $p : X \times Y \rightarrow X$ be the projection. Then $WF(p^*(\xi)) = p^*(WF(\xi))$.*

Proof of Theorem 4.1.5. Consider the action map $m : G \times X \rightarrow X$ and the projection $p : G \times X \rightarrow X$. Let $S := WF(\xi)$. We are given an isomorphism $p^*(E) \cong m^*(E)$ and we know that under this identification $p^*(\xi) = m^*(\xi)$. Therefore $WF(p^*(\xi)) = WF(m^*(\xi))$. By the lemma we have $WF(p^*(\xi)) = p^*(S)$. By Theorem 2.1.17 we have $WF(m^*(\xi)) \subset m^*(S)$. Thus we obtain $p^*(S) \subset m^*(S)$, which implies the requested inclusion. ■

4.2. SINGULAR SUPPORT.

Definition 4.2.1: Let X be a smooth algebraic variety; let $\xi \in \mathcal{S}^*(X(F))$. We will now define the singular support of ξ : it is an algebraic subvariety of T^*X and we will denote it by $SS(\xi)$.

In the case when F is non-Archimedean we define it to be the Zariski closure of $WF(\xi)$. In the case when F is Archimedean we define it to be the singular support of the D_X -module generated by ξ (as in [AG4]).

In [AG4, section 2.3] the following list of properties of the singular support for the Archimedean case was introduced:

Let X be a smooth algebraic variety.

- (1) Let $\xi \in \mathcal{S}^*(X(F))$. Then $\overline{\text{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(F)$, where $\overline{\text{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of $\text{Supp}(\xi)$.
- (2) Let an algebraic group G act on X . Let \mathfrak{g} denote the Lie algebra of G . Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g} \phi(\alpha(x)) = 0\}.$$

- (3) Let V be a linear space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V . Suppose that $\text{Supp}(\xi) \subset Z(F)$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X \times V}^{-1}(Z))$.
- (4) Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(F))$. Then $SS(\xi)$ is coisotropic.

Remark 4.2.2: Property (4) is a corollary of the integrability theorem (see [KKS, Mal, Gab]).

The result of the last subsection implies the following theorem

THEOREM 4.2.3: *The properties above are satisfied for the non-Archimedean case with the following modification: property (4) should be replaced by the following weaker one:*

(4') *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(F))$. Then $SS(\xi)$ is weakly coisotropic.*

We conjecture that property (4) holds for the non-Archimedean case without modification.

4.3. DISTRIBUTIONS ON NON DISTINGUISHED NILPOTENT ORBITS. In this subsection we deduce from the properties of singular support some technical results that are useful for proving the Gel'fand property.

NOTATION 4.3.1: Let V be an algebraic finite-dimensional representation over F of a reductive group G . We denote

$$Q(V) := (V/V^G)(F).$$

Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V(F)$. We also denote

$$\Gamma(V) = \{y \in V(F) \mid \overline{G(F)}y \ni 0\}.$$

Note that $\Gamma(V) \subset Q(V)$. We denote also $R(V) := Q(V) - \Gamma(V)$.

Definition 4.3.2: Let V be an algebraic finite-dimensional representation over F of a reductive group G . Suppose that there is a finite number of G orbits in $\Gamma(V)$. Let $x \in \Gamma(V)$. We will call it **G -distinguished**, if $CN_{G_{x,x}}^{Q(V)} \subset \Gamma(V^*)$. We will call a G orbit G -distinguished if all (or equivalently one of) its elements are G -distinguished.

If there is no ambiguity we will omit the “ G -”.

Example 4.3.3: For the case of a semi-simple group acting on its Lie algebra, the notion of G -distinguished element coincides with the standard notion of distinguished nilpotent element. In particular, in the case when $G = SL_n$ and $V = sl_n$ the set of G -distinguished elements is exactly the set of regular nilpotent elements.

PROPOSITION 4.3.4: Let V be an algebraic finite-dimensional representation over F of a reductive group G . Suppose that there is a finite number of G orbits on $\Gamma(V)$. Let $W := Q(V)$; let A be the set of non-distinguished elements in $\Gamma(V)$. Then there are no non-empty $W \times W^*$ -weakly coisotropic subvarieties of $A \times \Gamma(V^*)$.

The proof is clear.

COROLLARY 4.3.5: Let $\xi \in \mathcal{S}^*(W)$ and suppose that $\text{Supp}(\xi) \subset \Gamma(V)$ and $\text{supp}(\widehat{\xi}) \subset \Gamma(V^*)$. Then the set of distinguished elements in $\text{Supp}(\xi)$ is dense in $\text{Supp}(\xi)$

Remark 4.3.6: In the same way one can prove an analogous result for distributions on $W \times M(F)$ for any algebraic variety M .

5. Applications towards Gel'fand properties of symmetric pairs

In this section we will use the property of singular support to generate the results of [Say] for any local field of characteristic 0. Namely, we prove that a big class of a symmetric pairs are regular. The property of regularity of symmetric pair was introduced in [AG2] and was shown to be useful for proving the Gel'fand property. We will give more details on the regularity property and its connections with the Gel'fand property in Subsubsections 5.1.3–5.1.7.

5.1. PRELIMINARIES. In this subsection we give the necessary preliminaries for section 5.

5.1.1. *Gel'fand pairs.* In this subsection we recall a technique due to Gel'fand and Kazhdan (see [GK]) which allows one to deduce statements in representation theory from statements on invariant distributions. For a more detailed description see [AGS, section 2].

Definition 5.1.1: Let G be a reductive group. By an **admissible representation of G** we mean an admissible representation of $G(F)$ if F is non-Archimedean (see [BZ]) and an admissible smooth Fréchet representation of $G(F)$ if F is Archimedean.

We now introduce three notions of the Gel'fand pair.

Definition 5.1.2: Let $H \subset G$ be a pair of reductive groups.

- We say that (G, H) satisfy **GP1** if for any irreducible admissible representation (π, E) of G we have

$$\dim \text{Hom}_{H(F)}(E, \mathbb{C}) \leq 1.$$

- We say that (G, H) satisfy **GP2** if for any irreducible admissible representation (π, E) of G we have

$$\dim \text{Hom}_{H(F)}(E, \mathbb{C}) \cdot \dim \text{Hom}_{H(F)}(\tilde{E}, \mathbb{C}) \leq 1.$$

- We say that (G, H) satisfy **GP3** if for any irreducible unitary representation (π, \mathcal{H}) of $G(F)$ on a Hilbert space \mathcal{H} we have

$$\dim \text{Hom}_{H(F)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gel'fand and Kazhdan in certain p -adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p -adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gel'fand pair** in both the real and p -adic settings (see, e.g., [vD, BvD]).

We have the following straightforward proposition.

PROPOSITION 5.1.3: $GP1 \Rightarrow GP2 \Rightarrow GP3$.

We will use the following theorem from [AGS] which is a version of a classical theorem of Gel'fand and Kazhdan.

THEOREM 5.1.4: *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi- $H(F)$ -invariant distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

Remark 5.1.5: In many cases it turns out that GP2 is equivalent to GP1.

5.1.2. *Tame actions.* In this subsection we review some tools developed in [AG2] for solving problems of the following type. A reductive group G acts on a smooth affine variety X , and τ is an automorphism of X which normalizes the action of G . We want to check whether any $G(F)$ -invariant Schwartz distribution on $X(F)$ is also τ -invariant.

Definition 5.1.6: Let π be an action of a reductive group G on a smooth affine variety X . We say that an algebraic automorphism τ of X is **G -admissible** if:

- (i) $\pi(G(F))$ is of index ≤ 2 in the group of automorphisms of X generated by $\pi(G(F))$ and τ .
- (ii) For any closed $G(F)$ orbit $O \subset X(F)$, we have $\tau(O) = O$.

Definition 5.1.7: We call an action of a reductive group G on a smooth affine variety X **tame** if for any G -admissible $\tau : X \rightarrow X$, we have

$$\mathcal{S}^*(X(F))^{G(F)} \subset \mathcal{S}^*(X(F))^\tau.$$

Definition 5.1.8: We call an algebraic representation of a reductive group G on a finite-dimensional linear space V over F **linearly tame** if for any G -admissible linear map $\tau : V \rightarrow V$, we have $\mathcal{S}^*(V(F))^{G(F)} \subset \mathcal{S}^*(V(F))^\tau$.

We call a representation **weakly linearly tame** if for any G -admissible linear map $\tau : V \rightarrow V$, such that $\mathcal{S}^*(R(V))^{G(F)} \subset \mathcal{S}^*(R(V))^\tau$, we have $\mathcal{S}^*(Q(V))^{G(F)} \subset \mathcal{S}^*(Q(V))^\tau$.

THEOREM 5.1.9: *Let a reductive group G act on a smooth affine variety X . Suppose that for any G -semisimple $x \in X(F)$, the action of G_x on $N_{G_x, x}^X$ is weakly linearly tame. Then the action of G on X is tame.*

For a proof see [AG2, Theorem 6.0.5].

Definition 5.1.10: We call an algebraic representation of a reductive group G on a finite-dimensional linear space V over F **special** if for any $\xi \in \mathcal{S}_{Q(V)}^*(\Gamma(V))^{G(F)}$ such that, for any G -invariant decomposition $Q(V) = W_1 \oplus W_2$ and any two G -invariant symmetric non-degenerate bilinear forms B_i on W_i the Fourier transforms $\mathcal{F}_{B_i}(\xi)$ are also supported in $\Gamma(V)$, we have $\xi = 0$.

PROPOSITION 5.1.11: *Every special algebraic representation V of a reductive group G is weakly linearly tame.*

For a proof see [AG2, Proposition 6.0.7].

5.1.3. Symmetric pairs. In the coming 4 subsections we review some tools developed in [AG2] that enable us to prove that a symmetric pair is a Gel'fand pair.

Definition 5.1.12: A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. We call a symmetric pair **connected** if G/H is connected.

For a symmetric pair (G, H, θ) we define an **anti-involution** $\sigma : G \rightarrow G$ by $\sigma(g) := \theta(g^{-1})$, denote $\mathfrak{g} := \text{Lie}G$, $\mathfrak{h} := \text{Lie}H$, $\mathfrak{g}^\sigma := \{a \in \mathfrak{g} \mid \theta(a) = -a\}$. Note that H acts on \mathfrak{g}^σ by the adjoint action. Denote also $G^\sigma := \{g \in G \mid \sigma(g) = g\}$ and define a **symmetrization map** $s : G \rightarrow G^\sigma$ by $s(g) := g\sigma(g)$.

In the case when the involution is obvious we will omit it.

Remark 5.1.13: Let (G, H, θ) be a symmetric pair. Then \mathfrak{g} has a $\mathbb{Z}/2\mathbb{Z}$ grading given by θ .

Definition 5.1.14: Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be symmetric pairs. We define their **product** to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.

Definition 5.1.15: We call a symmetric pair (G, H, θ) **good** if for any closed $H(F) \times H(F)$ orbit $O \subset G(F)$, we have $\sigma(O) = O$.

PROPOSITION 5.1.16: *Every connected symmetric pair over \mathbb{C} is good.*

For a proof see, e.g., [AG2, Corollary 7.1.7].

Definition 5.1.17: We say that a symmetric pair (G, H, θ) is a **GK pair** if any $H(F) \times H(F)$ -invariant distribution on $G(F)$ is σ -invariant.

Remark 5.1.18: Theorem 5.1.4 implies that any GK pair satisfies GP2.

5.1.4. *Descendants of symmetric pairs.*

PROPOSITION 5.1.19: *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ such that HgH is closed. Let $x = s(g)$. Then x is a semisimple element of G .*

For a proof see, e.g., [AG2, Proposition 7.2.1].

Definition 5.1.20: In the notation of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a **descendant** of (G, H, θ) .

5.1.5. *Tame symmetric pairs.*

Definition 5.1.21:

- We call a symmetric pair (G, H, θ) **tame** if the action of $H \times H$ on G is tame.
- We call a symmetric pair (G, H, θ) **linearly tame** if the action of H on \mathfrak{g}^σ is linearly tame.
- We call a symmetric pair (G, H, θ) **weakly linearly tame** if the action of H on \mathfrak{g}^σ is weakly linearly tame.
- We call a symmetric pair (G, H, θ) **special** if the action of H on \mathfrak{g}^σ is special.

Remark 5.1.22: Evidently, any good tame symmetric pair is a GK pair.

THEOREM 5.1.23: *Let (G, H, θ) be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then (G, H, θ) is tame.*

For a proof see [AG2, Theorem 7.3.3].

5.1.6. *Regular symmetric pairs.*

Definition 5.1.24: Let (G, H, θ) be a symmetric pair. We call an element $g \in G(F)$ **admissible** if

- (i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
- (ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H -admissible.

Definition 5.1.25: We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(F)$ such that every $H(F)$ -invariant distribution on $R_{G,H}$ is also $Ad(g)$ -invariant, we have

- (*) every $H(F)$ -invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also $Ad(g)$ -invariant.

The following two propositions are evident.

PROPOSITION 5.1.26: *Let (G, H, θ) be symmetric pair. Suppose that any $g \in G(F)$ satisfying $\sigma(g)g \in Z(G(F))$ lies in $Z(G(F))H(F)$. Then (G, H, θ) is regular. In particular, if the normalizer of $H(F)$ lies inside $Z(G(F))H(F)$ then (G, H, θ) is regular.*

PROPOSITION 5.1.27: (i) *Any weakly linearly tame pair is regular.*
 (ii) *A product of regular pairs is regular (see [AG2, Proposition 7.4.4]).*

The importance of the notion of a regular pair is demonstrated by the following theorem.

THEOREM 5.1.28: *Let (G, H, θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

For a proof see [AG2, Theorem 7.4.5].

5.1.7. *Defects of symmetric pairs.* In this subsection we review some tools developed in [AG2] and [AG3] that enable us to prove that a symmetric pair is special.

Definition 5.1.29: We fix a standard basis e, h, f of $sl_2(F)$. We fix a grading on $sl_2(F)$ given by $h \in sl_2(F)_0$ and $e, f \in sl_2(F)_1$. A **graded representation of sl_2** is a representation of sl_2 on a graded vector space $V = V_0 \oplus V_1$ such that $sl_2(F)_i(V_j) \subset V_{i+j}$, where $i, j \in \mathbb{Z}/2\mathbb{Z}$.

The following lemma is standard.

LEMMA 5.1.30: (i) *Every graded representation of sl_2 which is irreducible as a graded representation is irreducible just as a representation.*

(ii) *Every irreducible representation V of sl_2 admits exactly two gradings. In one, the highest weight vector lies in V_0 , and in the other in V_1 .*

Definition 5.1.31: We denote by V_λ^w the irreducible graded representation of sl_2 with highest weight λ and highest weight vector of parity p , where $w = (-1)^p$.

The following lemma is straightforward.

LEMMA 5.1.32: $(V_\lambda^w)^* = V_\lambda^{w(-1)^\lambda}$.

Definition 5.1.33: Let π be a graded representation of sl_2 . We define the **defect** of π to be

$$def(\pi) = \text{Tr}(h|_{(\pi^e)_0}) - \dim(\pi_1).$$

The following lemma is straightforward.

LEMMA 5.1.34:

(1) $def(\pi \oplus \tau) = def(\pi) + def(\tau)$.

(2) $def(V_\lambda^w) = \frac{1}{2} \left(\lambda w + w \left(\frac{1 + (-1)^\lambda}{2} \right) - 1 \right) = \frac{1}{2} \begin{cases} \lambda w + w - 1, & \lambda \text{ is even,} \\ \lambda w - 1, & \lambda \text{ is odd} \end{cases}$

LEMMA 5.1.35: *Let \mathfrak{g} be a $(\mathbb{Z}/2\mathbb{Z})$ graded Lie algebra. Let $x \in \mathfrak{g}_1$ be a nilpotent element. Then there exists a graded homomorphism $\pi_x : sl_2 \rightarrow \mathfrak{g}$ such that $\pi_x(e) = x$.*

For a proof see, e.g., [AG2, Lemma 7.1.11].

LEMMA 5.1.36: *The morphism π_x is unique up to the exponentiated adjoint action of $(\mathfrak{g}_0)_x(\overline{F})$.*

For a proof see, e.g., [KR, Proposition 4].

Remark 5.1.37: In fact, the proof in [KR] also shows that π_x is unique up to the exponentiated adjoint action of $(\mathfrak{g}_0)_x(F)$.

Definition 5.1.38: Let \mathfrak{g} be a $(\mathbb{Z}/2\mathbb{Z})$ graded Lie algebra. Let $x \in \mathfrak{g}_1$. We define the **defect of x** by

$$def(x) = def(ad \circ \pi_x).$$

Lemma 5.1.36 implies that $def(x)$ is well defined.

LEMMA 5.1.39: *Let (G, H, θ) be a symmetric pair. Then there exists a G -invariant θ -invariant non-degenerate symmetric bilinear form B on \mathfrak{g} . In particular, $B|_{\mathfrak{h}}$ and $B|_{\mathfrak{g}^\sigma}$ are also non-degenerate and \mathfrak{h} is orthogonal to \mathfrak{g}^σ .*

For a proof see, e.g., [AG2, Lemma 7.1.9].

From now on we will fix such B and identify \mathfrak{g}^σ with $(\mathfrak{g}^\sigma)^*$.

LEMMA 5.1.40: *let (G, H, θ) be a symmetric pair. Assume that \mathfrak{g} is semi-simple. Then:*

- (i) *for any $x \in \mathfrak{g}^\sigma$ we have $CN_{Hx,x}^{\mathfrak{g}^\sigma} = (\mathfrak{g}^\sigma)^x$,*
- (ii) *$Q(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$.*

Proof. (i) is trivial.

(ii) Assume the contrary: there exist $0 \neq x \in \mathfrak{g}^\sigma$ such that $Hx = x$. Then $\dim(CN_{Hx,x}^{\mathfrak{g}^\sigma}) = \dim \mathfrak{g}^\sigma$, hence $CN_{Hx,x}^{\mathfrak{g}^\sigma} = \mathfrak{g}^\sigma$, which means $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^x$. Therefore x lies in the center of \mathfrak{g} which is impossible. ■

PROPOSITION 5.1.41: *Let (G, H, θ) be a symmetric pair. Let $\xi \in \mathcal{S}^*(Q(\mathfrak{g}^\sigma))$. Suppose that both ξ and $\mathcal{F}(\xi)$ are supported on $\Gamma(\mathfrak{g}^\sigma)$. Then the set of elements in $\text{Supp}(\xi)$ which have non-negative defect is dense in $\text{Supp}(\xi)$.*

The proof is the same as the proof of [AG2, Proposition 7.3.7].

5.2. ALL THE NICE SYMMETRIC PAIRS ARE REGULAR.

Definition 5.2.1: Let (G, H, θ) be a symmetric pair. Let $x \in \Gamma(\mathfrak{g}^\sigma)$ be a nilpotent element. We will call it distinguished if it is distinguished with respect to the action of H on \mathfrak{g}^σ .

LEMMA 5.2.2: *Our definition of distinguished element coincides with the one in [Sek]. Namely, an element $x \in \Gamma(\mathfrak{g}^\sigma)$ is distinguished iff $((\mathfrak{g}_s)^\sigma)^x$ does not contain semi-simple elements. Here \mathfrak{g}_s is the semi-simple part of \mathfrak{g} .*

This lemma follows immediately from Lemma 5.1.40.

Definition 5.2.3: We will call a symmetric pair (G, H, θ) a **pair of negative distinguished defect** if all the distinguished elements in $\Gamma(\mathfrak{g}^\sigma)$ have negative defect.

THEOREM 5.2.4: *Let (G, H, θ) be a symmetric pair of negative distinguished defect. Then it is special.*

Proof. Let $\xi \in \mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{H(F)}$ such that both ξ and $\mathcal{F}(\xi)$ are supported in $\Gamma(\mathfrak{g}^\sigma)$. Choose stratification

$$\Gamma(\mathfrak{g}^\sigma) = X_n \supset X_{n-1} \supset X_0 = 0 \supset X_{-1} = \emptyset$$

such that $X_i - X_{i-1}$ is an H -orbit which is open in X_i . We will prove by descending induction that ξ is supported on X_i . So we fix i and assume that ξ is supported on X_i ; our aim is to prove that ξ is supported on X_{i-1} . Suppose that $X_i - X_{i-1}$ is non-distinguished. Then by Corollary 4.3.5 we have $\text{Supp}(\xi) \subset X_{i-1}$. Now suppose that $X_i - X_{i-1}$ is distinguished. Then by Proposition 5.1.41 we have $\text{Supp}(\xi) \subset X_{i-1}$. ■

We will use the notion of nice symmetric pair from [LS]. We will use the following definition.

Definition 5.2.5: A symmetric pair (G, H, θ) is called nice iff the semi-simple part of the pair $(\mathfrak{g}, \mathfrak{h})$ decomposes, over the algebraic closure, to a product of pairs of the following types:

- $(g_1 \oplus g_1, g_1)$, where g_1 is a simple Lie algebra,
- (sl_m, so_m) ,
- $(sl_{2m}, sl_m \oplus sl_m \oplus \mathfrak{g}_a)$, where \mathfrak{g}_a is the one dimensional Lie algebra,
- $(sp_{2m}, sl_m \oplus \mathfrak{g}_a)$,
- $(so_{2m+k}, so_{m+k} \oplus so_m)$, for $k = 0, 1, 2$,
- (e_6, sp_8) ,
- $(e_6, sl_6 \oplus sl_2)$,
- (e_7, sl_8) ,
- (e_8, so_{16}) ,
- $(f_4, sp_6 \oplus sl_2)$,
- $(g_2, sl_2 \oplus sl_2)$.

This notion is motivated by [Sek], where the following theorem is proven (see Theorem 6.3).

THEOREM 5.2.6: *Let (G, H, θ) be a nice symmetric pair. Let $\pi : sl_2 \rightarrow \mathfrak{g}$ be a graded homomorphism such that $\pi(e)$ is distinguished. Consider \mathfrak{g} as a graded representation of sl_2 ; decompose it to irreducible representations by $\mathfrak{g} = \bigoplus V_{\lambda_i}^{\omega_i}$.*

Then

$$\sum_{i \text{ s.t. } \omega_i(-1)^{\lambda_i} = -1} (\lambda_i + 2) - \dim(\mathfrak{g}^\sigma) > 0.$$

COROLLARY 5.2.7: *Any nice symmetric pair is of negative distinguished defect. Thus by Theorem 5.2.4 it is special and hence weakly linearly tame and regular.*

This corollary follows immediately from the theorem using the following lemma and the fact that $\mathfrak{g} \cong \mathfrak{g}^*$ as a graded representation of sl_2 .

LEMMA 5.2.8: *Let V be a graded representation of sl_2 . Decompose it to irreducible representations by $V = \bigoplus V_{\lambda_i}^{\omega_i}$. Denote*

$$\delta(V) := \sum_{i \text{ s.t. } \omega_i(-1)^{\lambda_i} = -1} (\lambda_i + 2) - \dim(V_1).$$

Then

$$\delta(V) + \delta(V^*) + def(V) + def(V^*) = 0.$$

Proof. This lemma is a straightforward computation using Lemmas 5.1.34 and 5.1.32. ■

6. A uniform proof of multiplicity one theorems for GL_n

In this section we indicate a proof of multiplicity one theorems for GL_n which is uniform for all local fields of characteristic 0. This theorem was proven for the non-Archimedean case in [AGRS] and for the Archimedean case in [AG4] and [SZ]. We will not give all the details since this theorem was proven before. We will indicate the main steps and will give the details in the parts which are more essential. The proof that we present here is based on ideas from the previous proofs and uses our partial analog of the integrability theorem.

Let us first formulate the multiplicity one theorems for GL_n .

THEOREM 6.0.1: *Consider the standard embedding $GL_n(F) \hookrightarrow GL_{n+1}(F)$. We consider the action of $GL_n(F)$ on $GL_{n+1}(F)$ by conjugation. Then any $GL_n(F)$ -invariant distribution on $GL_{n+1}(F)$ is invariant with respect to transposition.*

It has the following corollary in representation theory.

THEOREM 6.0.2: *Let π be an irreducible admissible representation of $GL_{n+1}(F)$ and τ be an irreducible admissible representation of $GL_n(F)$. Then*

$$(3) \quad \dim \text{Hom}_{GL_n(F)}(\pi, \tau) \leq 1.$$

6.1. NOTATION.

- Let $V := V_n$ be the standard n -dimensional linear space defined over F .
- Let $\mathfrak{sl}(V)$ denote the Lie algebra of operators with zero trace.
- Denote $X := X_n := \mathfrak{sl}(V_n) \times V_n \times V_n^*$.
- Denote $G := G_n := GL(V_n)$.
- Denote $\mathfrak{g} := \mathfrak{g}_n := \text{Lie}(G_n) = \mathfrak{gl}(V_n)$.
- Let G_n act on G_{n+1} , \mathfrak{g}_{n+1} and on $\mathfrak{sl}(V_n)$ by $g(A) := gAg^{-1}$.
- Let G act on $V \times V^*$ by $g(v, \phi) := (gv, (g^{-1})^*\phi)$. This gives rise to an action of G on X .
- Let $\sigma : X \rightarrow X$ be given by $\sigma(A, v, \phi) = A^t, \phi^t, v^t$.
- We fix the standard trace form on $\mathfrak{sl}(V)$ and the standard form on $V \times V^*$.
- Denote $S := \{(A, v, \phi) \in X_n \mid A^n = 0 \text{ and } \phi(A^i v) = 0 \text{ for any } 0 \leq i \leq n\}$.
- Note that $S \supset \Gamma(X)$.
- Denote $S' := \{(A, v, \phi) \in S \mid A^{n-1}v = (A^*)^{n-1}\phi = 0\}$.
- Denote

$$\begin{aligned} \check{S} := & \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\} \\ & (A_i, v_j, \phi_j) \in S \text{ and } \forall \alpha \in \mathfrak{gl}(V), \alpha(A_1, v_1, \phi_1) \perp (A_2, v_2, \phi_2)\}. \end{aligned}$$

- Note that

$$\begin{aligned} \check{S} = & \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\} \\ & (A_i, v_j, \phi_j) \in S \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}. \end{aligned}$$

- Denote

$$\check{S}' := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in \check{S} \mid \forall i, j \in \{1, 2\} (A_i, v_j, \phi_j) \in S'\}.$$

6.2. REFORMULATION. A standard use of the Harish-Chandra descent method shows that it is enough to show that any $G(F)$ invariant distribution on $X(F)$ is invariant with respect to σ . Moreover, it is enough to show this under the assumption that this is true for distributions on $(X - S)(F)$. So it is enough to prove the following theorem:

THEOREM 6.2.1: *The action of G on X is special (and hence weakly linearly tame).*

Remark 6.2.2: One can show that this implies that the action of G_n on G_{n+1} is tame.

6.3. PROOF OF THEOREM 6.2.1. It is enough to show that any distribution $\xi \in \mathcal{S}^*(X(F))^{G(F)}$, such that $\xi, \mathcal{F}_{V \times V^*}(\xi), \mathcal{F}_{sl(V)}(\xi)$ and $\mathcal{F}_X(\xi)$ are supported on $S(F)$, is zero.

The proof is based on the following theorem:

THEOREM 6.3.1 (The geometric statement): *There are non empty $X \times X$ -weakly coisotropic subvarieties of \check{S}' .*

We will prove this theorem in the next subsection. Let us now explain why it implies Theorem 6.2.1. First we get the following corollary:

COROLLARY 6.3.2: *Any $X \times X$ -weakly coisotropic subvariety of \check{S} is a subset of*

$$(sl(V) \times (V \times 0 \cup 0 \times V^*)) \times (sl(V) \times (V \times 0 \cup 0 \times V^*)).$$

COROLLARY 6.3.3: *Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$ such that $\xi, \mathcal{F}_{V \times V^*}(\xi), \mathcal{F}_{sl(V^*)}(\xi)$ and $\mathcal{F}_X(\xi)$ are supported on $S(F)$ then both ξ and $\mathcal{F}_{V \times V^*}(\xi)$ are supported on $sl(V) \times (V \times 0 \cup 0 \times V^*)$.*

Theorem 6.2.1 now follows from the following lemma

LEMMA 6.3.4: *Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$ such that both ξ and $\mathcal{F}_{V \times V^*}(\xi)$ are supported on $sl(V) \times (V \times 0 \cup 0 \times V^*)$. Then $\xi = 0$.*

Proof. This is a direct computation using Propositions 2.1.3, 2.1.4, Theorems 2.1.5 and 2.1.10. ■

6.4. PROOF OF THE GEOMETRIC STATEMENT.

NOTATION 6.4.1: *Denote $\check{S}'' := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in \check{S}' \mid A_1^{n-1} = 0\}$.*

By Theorem 4.3.4 (and Example 4.3.3) there are no non-empty $X \times X$ -weakly coisotropic subvarieties of \check{S}'' . Therefore it is enough to prove the following Key proposition.

PROPOSITION 6.4.2 (Key proposition): *There are no non-empty $X \times X$ -weakly coisotropic subvarieties of $\check{S}' - \check{S}''$.*

NOTATION 6.4.3: *Let $A \in \mathfrak{sl}(V)$ be a nilpotent Jordan block. Denote*

$$R_A := (\check{S}' - \check{S}'')|_{\{A\} \times V \times V^*}.$$

By Proposition 3.0.11 the Key proposition follows from the following Key Lemma.

LEMMA 6.4.4 (Key Lemma): *There are no non-empty $V \times V^* \times V \times V^*$ -weakly coisotropic subvarieties of R_A .*

Proof. Denote $Q_A = \bigcup_{i=1}^{n-1} (Ker A^i) \times (Ker(A^*)^{n-i})$. It is easy to see that $R_A \subset Q_A \times Q_A$ and

$$Q_A \times Q_A = \bigcup_{i,j=0}^n (Ker A^i) \times (Ker(A^*)^{n-i}) \times (Ker A^j) \times (Ker(A^*)^{n-j}).$$

Denote $L_{ij} := (Ker A^i) \times (Ker(A^*)^{n-i}) \times (Ker A^j) \times (Ker(A^*)^{n-j})$.

It is easy to see that any weakly coisotropic subvariety of $Q_A \times Q_A$ is contained in $\bigcup_{i=1}^{n-1} L_{ii}$. Hence it is enough to show that for any $0 < i < n$, we have $\dim R_A \cap L_{ii} < 2n$.

Let $f \in \mathcal{O}(L_{ii})$ be the polynomial defined by

$$f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1},$$

where $(\cdot)_i$ means the i -th coordinate. It is enough to show that

$$f(R_A \cap L_{ii}) = \{0\}.$$

Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ii}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. Clearly, M is of the form

$$M = \begin{pmatrix} 0_{i \times i} & * \\ 0_{(n-i) \times i} & 0_{(n-i) \times (n-i)} \end{pmatrix}.$$

Note also that $M_{i,i+1} = f(v_1, \phi_1, v_2, \phi_2)$.

It is easy to see that any B satisfying $[A, B] = M$ is upper triangular. On the other hand, we know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i,i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.

To sum up, we have shown that $f(R_A \cap L_{ii}) = \{0\}$, hence $\dim(R_A \cap L_{ii}) < 2n$. Hence every coisotropic subvariety of R_A has dimension less than $2n$ and therefore is empty. ■

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