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THE RELATIONSHIP BETWEEN THE ZAMES REPRESENTATION AND LQG COMPENSATORS*

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ABSTRACT

Recently Zames [1] has introduced the so-called model reference transformation which can be used to conveniently parametrize the class of linear time-invariant multivariable compensators that lead to stable feedback control systems. A very popular design methodology for designing stable and robust multivariable control systems is that based on the Linear-Quadratic-Gaussian (LQG) design methodology with Loop Transfer Recovery (LTR). Thus, it is natural to inquire upon the relationship between the Zames representation and the LQG/LTR compensators. This paper summarizes this relationship.

1. INTRODUCTION

The Zames representation [1] has attracted a significant attention in the literature; see also [2]. A recent paper by Gustafson and Desoer [3] presents a design methodology which has its roots upon the Zames representation. On the other hand, the most popular design methodology for linear time-invariant multivariable systems remains the LQG-based design method [4],[7], with loop-transfer-recovery (LTR) [5],[6],[8]. The LQG/LTR method generates nominally stable compensators with superior robustness properties [6] in the absence of non-minimum phase zeroes. The design of LQG/LTR feedback control systems can now be routinely carried out by loop-shaping methods in the frequency domain, [5],[6],[8].

Only time will tell whether design methodologies based upon the Zames representation will approach the popularity of LQG/LTR methods. In the meantime it is of interest to relate these methodologies.

In Section 2 we summarize the Zames Representation. In Section 3 we start with the class of the so-called Model Based Compensators (MBC) and specialize them to the class of LQG/LTR compensators; many results not readily apparent in the literature are stated without proof. Next, we specify the Zames representation for the class of LQG/LTR designs.

2. THE ZAMES PARAMETRIZATION

In this section we summarize the Q-parametrization introduced by Zames [1]; see also [2].

Figure 1 illustrates a block diagram of a multivariable linear time-invariant feedback control system. The mxm transfer matrix $\underline{P}(s)$ represents the open-loop plant, and the mxm transfer matrix $\underline{C}(s)$ represents the dynamic compensator. Elementary algebra leads to the following closed-loop transfer matrix, $\underline{G}_{CL}(s)$, from the reference input $\underline{r}(s)$ to the output $\underline{y}(s)$, i.e.

$$\underline{y}(s) \overset{\Delta}{=} \underline{G}_{CL}(s)\underline{r}(s) \tag{1}$$

where

$$\begin{aligned}\underline{G}_{CL}(s) &= [\underline{I}+\underline{P}(s)\underline{C}(s)]^{-1}\underline{P}(s)\underline{C}(s) \\ &= \underline{P}(s)\underline{C}(s)[\underline{I}+\underline{P}(s)\underline{C}(s)]^{-1}\end{aligned} \tag{2}$$

The results of Zames [1] state that the closed-loop transfer matrix $\underline{G}_{CL}(s)$ can be expressed in the form

$$\underline{G}_{CL}(s) = \underline{P}(s)\underline{Q}(s) \tag{3}$$

where the mxm transfer matrix $\underline{Q}(s)$ is defined by the compensator $\underline{C}(s)$ and the open-loop plant $\underline{P}(s)$ as follows:

$$\begin{aligned}\underline{Q}(s) &= \underline{C}(s)[\underline{I}+\underline{P}(s)\underline{C}(s)]^{-1} \\ &= [\underline{I}+\underline{C}(s)\underline{P}(s)]^{-1}\underline{C}(s)\end{aligned} \tag{4}$$

It follows that the compensator $\underline{C}(s)$ is given by

$$\begin{aligned}\underline{C}(s) &= \underline{Q}(s)[\underline{I}-\underline{P}(s)\underline{Q}(s)]^{-1} \\ &= [\underline{I}-\underline{Q}(s)\underline{P}(s)]^{-1}\underline{Q}(s)\end{aligned} \tag{5}$$

the output sensitivity matrix $\underline{S}_o(s)$ by

$$\underline{S}_o(s) = [\underline{I}+\underline{P}(s)\underline{C}(s)]^{-1} = \underline{I}-\underline{P}(s)\underline{Q}(s) \tag{6}$$

and the input sensitivity matrix $\underline{S}_i(s)$

$$\underline{S}_i(s) = [\underline{I}+\underline{C}(s)\underline{P}(s)]^{-1} = \underline{I}-\underline{Q}(s)\underline{P}(s) \tag{7}$$

The recent interest [1],[2],[3] in this representation hinges on the fact that the sensitivity matrices $\underline{S}_o(s)$ and $\underline{S}_i(s)$, given by Eqs. (6) and (7), are linearly related to the matrix $\underline{Q}(s)$. Thus, by shaping $\underline{Q}(s)$, for any given $\underline{P}(s)$, one can obtain "good" sensitivity matrices, and then evaluate the dynamic compensator $\underline{C}(s)$ using Eq. (5). The resultant closed loop transfer matrix, given by Eq. (3), is then calculated by

$$\underline{G}_{CL}(s) = \underline{I}-\underline{S}_o(s) \tag{8}$$

Ideally, one would like to have $\underline{S}_o(j\omega)=\underline{0}$ for all ω. However, this is not possible, and approximations always take place.

3. THE CLASS OF LQG COMPENSATORS

In this section we examine the class of Linear-Quadratic-Gaussian (LQG) compensators with Loop-Transfer-Recovery (LTR) [4],[5],[6]. It is quite helpful to first define the class of Model-Based-Compensators (MBC); the MBC nonmeclature is non-standard.

3.1 The Open-Loop Plant

The time domain description of the open-loop plant

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$\underline{P}(s)$ is as follows:

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}\,\underline{x}(t) \tag{9}$$

with $\underline{x}(t)\in R^n$, $\underline{u}(t)\in R^m$, $\underline{y}(t)\in R^m$. Define the nxm matrix $\underline{\Phi}(s)$ by

$$\underline{\Phi}(s) \triangleq (s\underline{I}-\underline{A})^{-1} \tag{10}$$

Evidently, the open-loop plant transfer matrix $\underline{P}(s)$ is given by

$$\underline{P}(s) = \underline{C}(s\underline{I}-\underline{A})^{-1}\underline{B} = \underline{C}\,\underline{\Phi}(s)\underline{B} \tag{11}$$

3.2 Model Based Compensators (MBC)

In reference to the feedback system in Fig. 1 we define a model based compensator (MBC) with the transfer matrix

$$\underline{C}(s) = \underline{C}_{MBC}(s) = \underline{G}[s\underline{I}-\underline{A}+\underline{B}\,\underline{G}+\underline{H}\,\underline{C}]^{-1}\underline{H}$$
$$= \underline{G}[\underline{\Phi}^{-1}(s)+\underline{B}\,\underline{G}+\underline{H}\,\underline{C}]^{-1}\underline{H} \tag{12}$$

A block diagram of $\underline{C}_{MBC}(s)$ is shown in Fig. 2. The block diagram is drawn in such a way so that the relationship of the MBC to the LQG compensator is apparent; if we set $\underline{r}(s) = \underline{d}_o(s)=\underline{0}$ in Fig. 1, then $\underline{v}(s)$ would represent the innovations vector, and $\hat{\underline{x}}(s)$ the state estimate. Also, one can think of the MBC in Fig. 2 as been obtained by cascading a non-minimal stable Luenberger observer with a suitable control gain, $\underline{G}$, that may be obtained from a pole-placement algorithm [7].

Note that all parameters in $\underline{C}_{MBC}(s)$ in Eq. (12) and Fig. 2 are fixed, except for the mxn filter gain matrix $\underline{H}$, and the nxm control gain matrix $\underline{G}$. If $[\underline{A},\underline{B}]$ is stabilizable and $[\underline{A},\underline{C}]$ is detectable, then it is well-known (from non-minimal observer theory and pole-placement theory) that there exist gain matrices $\underline{H}$ and $\underline{G}$ such that the closed loop system of Fig. 1 is stable using the MBC of Eq. (12).

Next we present the closed-loop transfer function $\underline{G}_{CL}^{MBC}(s)$, $\underline{y}(s) = \underline{G}_{CL}^{MBC}(s)\underline{r}(s)$ associated with the feedback control system of Fig. 1 using the MBC of Fig. 2. Tedious algebra and the use of the matrix inversion lemma lead to

$$\underline{G}_{CL}^{MBC}(s)=\underline{C}\,\underline{\Phi}(s)\underline{B}[\underline{I}+\underline{G}\,\underline{\Phi}(s)\underline{B}]^{-1}\underline{G}\,\underline{\Phi}(s)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(s)\underline{H}]^{-1}$$
$$= \underline{P}(s)[\underline{I}+\underline{G}\,\underline{\Phi}(s)\underline{B}]^{-1}\underline{G}\,\underline{\Phi}(s)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(s)\underline{H}]^{-1} \tag{13}$$

By comparing Eqs. (3) and (13) we can readily conclude that the $\underline{Q}(s)$ matrix in the Zames representation [1] is as follows for MBC derived control systems

$$\underline{Q}(s)=\underline{Q}^{MBC}(s)=[\underline{I}+\underline{G}\,\underline{\Phi}(s)\underline{B}]^{-1}\underline{G}\,\underline{\Phi}(s)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(s)\underline{H}]^{-1} \tag{14}$$

It is the selection of the constant filter gain matrix $\underline{H}$ and of the constant control gain matrix $\underline{G}$ that would shape $\underline{Q}(s)$.

3.3 LQG/LTR Compensators

The structure of an LQG compensator is identical to that of an MBC (see Figure 2). What distinguishes an LQG compensator from an MBC is the specific way that one calculates the gain matrices $\underline{H}$ and $\underline{G}$ via the appropriate algebraic Riccati equations for the LQG optimization problem [4],[7]. These will not be reviewed, since an arbitrary LQG compensator does not provide any more insight than that contained in Eq. (14).

A greater insight can be obtained for LQG designs with LTR [5],[8]. Suppose that the open-loop plant has no right-half plane (non-minimum phase) transmission zeros. Furthermore, suppose that the control gain matrix $\underline{G}$ in Fig. 2 and in Eq. (12) is computed as follows:

$$\underline{G} = \frac{1}{\rho}\underline{B}^{-1}\underline{K}\,; \qquad \rho>0 \tag{15}$$

where the matrix $\underline{K}$ is the solution of the algebraic Riccati equation

$$\underline{0} = -\underline{K}\,\underline{A}-\underline{A}'\underline{K} - \beta\underline{C}'\underline{C} + \frac{1}{\rho}\,\underline{K}\,\underline{B}\,\underline{B}'\underline{K} \tag{16}$$

In particular in the LTR procedure we let $\beta\to\infty$. Then, the results in Refs. [5],[8] indicate that as $\beta\to\infty$, the control gain matrix $\underline{G}$ has the property that

$$\frac{1}{\sqrt{\beta}}\,\underline{G}\to\underline{W}\,\underline{C}\,; \quad \underline{W} = \text{orthonormal matrix} \tag{17}$$

and that the LQG compensator transfer matrix $\underline{C}_{LQG}(s)$— see Eq. (12)- approaches

$$\underline{C}(s) = \underline{C}_{LQG}(s) \to [\underline{C}\,\underline{\Phi}(s)\underline{B}]^{-1}\underline{C}\,\underline{\Phi}(s)\underline{H}$$
$$= \underline{P}^{-1}(s)\underline{C}\,\underline{\Phi}(s)\underline{H} \tag{18}$$

Thus, the LQG/LTR compensator $\underline{G}_{LQG}(s)$ performs an approximate inversion of the plant $\underline{P}(s)$; the assumption that $\underline{P}(s)$ has no non-minimum phase zeros guarantees that there are no right-half plane pole-zero cancellations.

The resulting closed-loop transfer matrix $\underline{G}_{CL}^{LQG}(s)$ tends to

$$\underline{G}_{CL}^{LQG}(s)\to\underline{C}\,\underline{\Phi}(s)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(s)\underline{H}]^{-1} \tag{19}$$

which is the closed-loop transfer of the feedback system shown in Figure 3.

The corresponding Zames parametrization $\underline{Q}(s)=\underline{Q}^{LQG}(s)$ can now be readily calculated. From Eq. (4)

$$\underline{Q}^{LQG}(s) = \underline{C}_{LQG}(s)[\underline{I}+\underline{P}(s)\underline{C}_{LQG}(s)]^{-1} \tag{20}$$

As $\beta\to\infty$, we deduce from Eqs. (18) and (20) that

$$\underline{Q}^{LQG}(s) \to \underline{P}^{-1}(s)\underline{C}\,\underline{\Phi}(s)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(s)\underline{H}]^{-1} \tag{21}$$

The interpretation of $\underline{Q}^{LQG}(s)$ is that it corresponds to the closed-loop transfer matrix of the system shown in Fig. 3 followed by the plant inverse $\underline{P}^{-1}(s)$.

In the LQG/LTR design procedure the filter gain matrix $\underline{H}$ is calculated by suitable solution of a Kalman Filter design problem so that the loop transfer matrix $\underline{C}\,\underline{\Phi}(s)\underline{H}$ in Fig. 3 and the closed-loop transfer function

$$\underline{C}\,\underline{\Phi}(s)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(s)\underline{H}]^{-1} \tag{22}$$

meet the posed performance and robustness specifications in the frequency domain. Indeed, at low frequencies, it is possible to select $\underline{H}$ so that

$$\underline{C}\,\underline{\Phi}(j\omega)\underline{H}[\underline{I}+\underline{C}\,\underline{\Phi}(j\omega)\underline{H}]^{-1}\approx\underline{I} \tag{23}$$

resulting in

$$\underline{Q}^{LQG}(j\omega)\approx\underline{P}^{-1}(j\omega) \tag{24}$$

and output sensitivity which is almost zero -see Eq. (6)-

$$\underline{S}_o(j\omega)\approx\underline{0} \tag{25}$$

4. CONCLUDING REMARKS

We have calculated the Zames $\underline{Q}$ matrix for the class of LQG/LTR designs. In the opinion of the author, the Zames decomposition offers no particular insights into the LQG/LTR problem, except that it confirms (in a round-about way) that in the absence of non-minimum phase zeros the LQG/LTR procedure can result in excellent feedback loops. However, this has been known anyway.

5. REFERENCES

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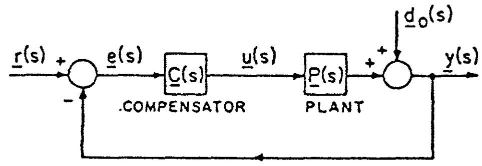

Figure 1: Block diagram of multivariable control system.

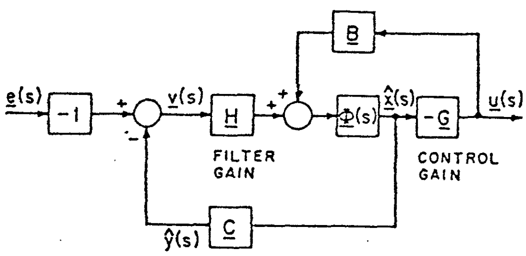

Figure 2: Block diagram of MBC and LQG compensator, driven by the error signal $\underline{e}(s)$ and generating the control $\underline{u}(s)$ in Figure 1.

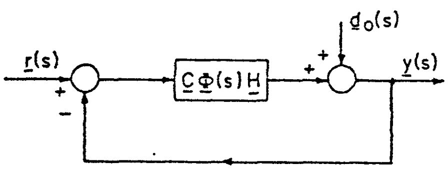

Figure 3: Block diagram of the limiting feedback control system obtained by the LQG/LTR procedure. The constant matrix $\underline{H}$ is the Kalman Filter gain matrix.