# *Action of longest element on a Hecke algebra cell module*

# ACTION OF LONGEST ELEMENT ON A HECKE ALGEBRA CELL MODULE

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*Dedicated to the memory of Robert Steinberg*

## Introduction

**0.1.** The Hecke algebra $\mathcal{H}$ (over $\mathcal{A}=\mathbf{Z}[v,v^{-1}]$, $v$ an indeterminate) of a finite Coxeter group $W$ has two bases as an $\mathcal{A}$-module: the standard basis $\{T_x; x\in W\}$ and the basis $\{C_x; x\in W\}$ introduced in [KL]. The second basis determines a decomposition of $W$ into two-sided cells and a partial order for the set of two-sided cells, see [KL]. Let $l:W\to\mathbf{N}$ be the length function, let $w_0$ be the longest element of $W$ and let $\mathbf{c}$ be a two-sided cell. Let $a$ (resp. $a'$) be the value of the $\mathbf{a}$-function [L3, 13.4] on $\mathbf{c}$ (resp. on $w_0\mathbf{c}$). The following result was proved by Mathas in [MA].

(a) *There exists a unique permutation $u\mapsto u^*$ of $\mathbf{c}$ such that for any $u\in\mathbf{c}$ we have $T_{w_0}(-1)^{l(u)}C_u=(-1)^{l(w_0)+a'}v^{-a+a'}(-1)^{l(u^*)}C_{u^*}$ plus an $\mathcal{A}$-linear combination of elements $C_{u'}$ with $u'$ in a two-sided cell strictly smaller than $\mathbf{c}$. Moreover, for any $u\in\mathbf{c}$ we have $(u^*)^*=u$.*

A related (but weaker) result appears in [L1, (5.12.2)].

A result similar to (a) which concerns canonical bases in representations of quantum groups appears in [L2, Cor. 5.9]; now, in the case where $W$ is of type $A$, (a) can be deduced from *loc.cit.* using the fact that irreducible representations of the Hecke algebra of type $A$ (with their canonical bases) can be realized as 0-weight spaces of certain irreducible representations of a quantum group with their canonical bases.

As R. Bezrukavnikov pointed out to the author, (a) specialized for $v=1$ (in the group algebra of $W$ instead of $\mathcal{H}$) and assuming that $W$ is crystallographic can be deduced from [BFO, Prop. 4.1] (a statement about Harish-Chandra modules), although it is not explicitly stated there.

In this paper we shall prove a generalization of (a) which applies to the Hecke algebra associated to $W$ and any weight function assumed to satisfy the properties P1-P15 in [L3,§14], see Theorem 2.3; (a) corresponds to the special case where the weight function is equal to the length function. As an application we show that

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the image of $T_{w_0}$ in the asymptotic Hecke algebra is given by a simple formula (see 2.8).

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**0.2.** *Notation.* $W$ is a finite Coxeter group; the set of simple reflections is denoted by $S$. We shall adopt many notations of [L3]. Let $\le$ be the standard partial order on $W$. Let $l : W \to \mathbf{N}$ be the length function of $W$ and let $L : W \to \mathbf{N}$ be a weight function (see [L3, 3.1]) that is, a function such that $L(ww') = L(w) + L(w')$ for any $w, w'$ in $W$ such that $l(ww') = l(w) + l(w')$; we assume that $L(s) > 0$ for any $s \in S$. Let $w_0, \mathcal{A}$ be as in 0.1 and let $\mathcal{H}$ be the Hecke algebra over $\mathcal{A}$ associated to $W, L$ as in [L3, 3.2]; we shall assume that properties P1-P15 in [L3, §14] are satisfied. (This holds automatically if $L = l$ by [L3,§15] using the results of [EW]. This also holds in the quasisplit case, see [L3,§16].) We have $\mathcal{A} \subset \mathcal{A}' \subset K$ where $\mathcal{A}' = \mathbf{C}[v, v^{-1}], K = \mathbf{C}(v)$. Let $\mathcal{H}_K = K \otimes_{\mathcal{A}} \mathcal{H}$ (a $K$-algebra). Recall that $\mathcal{H}$ has an $\mathcal{A}$-basis $\{T_x; x \in W\}$, see [L3, 3.2] and an $\mathcal{A}$-basis $\{c_x; x \in W\}$, see [L3, 5.2]. For $x \in W$ we have $c_x = \sum_{y \in W} p_{y,x} T_y$ and $T_x = \sum_{y \in W} (-1)^{l(xy)} p_{w_0 x, w_0 y} c_y$ (see [L3, 11.4]) where $p_{x,x} = 1$ and $p_{y,x} \in v^{-1}\mathbf{Z}[v^{-1}]$ for $y \ne x$. We define preorders $\le_{\mathcal{L}}, \le_{\mathcal{R}}, \le_{\mathcal{LR}}$ on $W$ in terms of $\{c_x; x \in W\}$ as in [L3, 8.1]. Let $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$ be the corresponding equivalence relations on $W$, see [L3, 8.1] (the equivalence classes are called left cells, right cells, two-sided cells). Let $^- : \mathcal{A} \to \mathcal{A}$ be the ring involution such that $\overline{v^n} = v^{-n}$ for $n \in \mathbf{Z}$. Let $^- : \mathcal{H} \to \mathcal{H}$ be the ring involution such that $\overline{fT_x} = \bar{f} T^{-1}_{x^{-1}}$ for $x \in W, f \in \mathcal{A}$. For $x \in W$ we have $\overline{c_x} = c_x$. Let $h \mapsto h^\dagger$ be the algebra automorphism of $\mathcal{H}$ or of $\mathcal{H}_K$ given by $T_x \mapsto (-1)^{l(x)} T^{-1}_{x^{-1}}$ for all $x \in W$, see [L3, 3.5]. Then the basis $\{c^\dagger_x; x \in W\}$ of $\mathcal{H}$ is defined. (In the case where $L = l$, for any $x$ we have $c^\dagger_x = (-1)^{l(x)} C_x$ where $C_x$ is as in 0.1.) Let $h \mapsto h^\flat$ be the algebra antiautomorphism of $\mathcal{H}$ given by $T_x \mapsto T_{x^{-1}}$ for all $x \in W$, see [L3, 3.5]; for $x \in W$ we have $c^\flat_x = c_{x^{-1}}$, see [L3, 5.8]. For $x, y \in W$ we have $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$, $c^\dagger_x c^\dagger_y = \sum_{z \in W} h_{x,y,z} c^\dagger_z$, where $h_{x,y,z} \in \mathcal{A}$. For any $z \in W$ there is a unique number $\mathbf{a}(z) \in \mathbf{N}$ such that for any $x, y$ in $W$ we have

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} + \text{strictly smaller powers of } v$$

where $g_{x,y,z^{-1}} \in \mathbf{Z}$ and $g_{x,y,z^{-1}} \ne 0$ for some $x, y$ in $W$. We have also

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{-\mathbf{a}(z)} + \text{strictly larger powers of } v.$$

Moreover $z \mapsto \mathbf{a}(z)$ is constant on any two-sided cell. The free abelian group $J$ with basis $\{t_w; w \in W\}$ has an associative ring structure given by $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$; it has a unit element of the form $\sum_{d \in \mathcal{D}} n_d t_d$ where $\mathcal{D}$ is a subset of $W$ consisting of certain elements with square 1 and $n_d = \pm 1$. Moreover for $d \in \mathcal{D}$ we have $n_d = \gamma_{d,d,d}$.

For any $x \in W$ there is a unique element $d_x \in \mathcal{D}$ such that $x \sim_{\mathcal{L}} d_x$. For a commutative ring $R$ with 1 we set $J_R = R \otimes J$ (an $R$-algebra).

There is a unique $\mathcal{A}$-algebra homomorphism $\phi : \mathcal{H} \to J_{\mathcal{A}}$ such that $\phi(c_x^\dagger) = \sum_{d\in\mathcal{D}, z\in W; d_z=d} h_{x,d,z} n_d t_z$ for any $x \in W$. After applying $\mathbf{C}\otimes_{\mathcal{A}}$ to $\phi$ (we regard $\mathbf{C}$ as an $\mathcal{A}$-algebra via $v \mapsto 1$), $\phi$ becomes a $\mathbf{C}$-algebra isomorphism $\phi_{\mathbf{C}} : \mathbf{C}[W] \xrightarrow{\sim} J_{\mathbf{C}}$ (see [L3, 20.1(e)]). After applying $K\otimes_{\mathcal{A}}$ to $\phi$, $\phi$ becomes a $K$-algebra isomorphism $\phi_K : \mathcal{H}_K \xrightarrow{\sim} J_K$ (see [L3, 20.1(d)]).

For any two-sided cell $\mathbf{c}$ let $\mathcal{H}^{\le \mathbf{c}}$ (resp. $\mathcal{H}^{<\mathbf{c}}$) be the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by $\{c_x^\dagger, x \in W, x \le_{\mathcal{LR}} x'$ for some $x' \in \mathbf{c}\}$ (resp. $\{c_x^\dagger, x \in W, x <_{\mathcal{LR}} x'$ for some $x' \in \mathbf{c}\}$). Note that $\mathcal{H}^{\le \mathbf{c}}, \mathcal{H}^{<\mathbf{c}}$ are two-sided ideals in $\mathcal{H}$. Hence $\mathcal{H}^{\mathbf{c}} := \mathcal{H}^{\le \mathbf{c}}/\mathcal{H}^{<\mathbf{c}}$ is an $\mathcal{H},\mathcal{H}$ bimodule. It has an $\mathcal{A}$-basis $\{c_x^\dagger, x \in \mathbf{c}\}$. Let $J^{\mathbf{c}}$ be the subgroup of $J$ spanned by $\{t_x; x \in \mathbf{c}\}$. This is a two-sided ideal of $J$. Similarly, $J_{\mathbf{C}}^{\mathbf{c}} := \mathbf{C} \otimes J^{\mathbf{c}}$ is a two-sided ideal of $J_{\mathbf{C}}$ and $J_K^{\mathbf{c}} := K \otimes J^{\mathbf{c}}$ is a two-sided ideal of $J_K$.

We write $E \in \mathrm{Irr} W$ whenever $E$ is a simple $\mathbf{C}[W]$-module. We can view $E$ as a (simple) $J_{\mathbf{C}}$-module $E_\spadesuit$ via the isomorphism $\phi_{\mathbf{C}}^{-1}$. Then the (simple) $J_K$-module $K \otimes_{\mathbf{C}} E_\spadesuit$ can be viewed as a (simple) $\mathcal{H}_K$-module $E_v$ via the isomorphism $\phi_K$. Let $E^\dagger$ be the simple $\mathbf{C}[W]$-module which coincides with $E$ as a $\mathbf{C}$-vector space but with the $w$ action on $E^\dagger$ (for $w \in W$) being $(-1)^{l(w)}$ times the $w$-action on $E$. Let $\mathbf{a}_E \in \mathbf{N}$ be as in [L3, 20.6(a)].

## 1. Preliminaries

**1.1.** Let $\sigma : W \to W$ be the automorphism given by $w \mapsto w_0 w w_0$; it satisfies $\sigma(S) = S$ and it extends to a $\mathbf{C}$-algebra isomorphism $\sigma : \mathbf{C}[W] \to \mathbf{C}[W]$. For $s \in S$ we have $l(w_0) = l(w_0 s) + l(s) = l(\sigma(s)) + l(\sigma(s)w_0)$ hence $L(w_0) = L(w_0 s) + L(s) = L(\sigma(s)) + L(\sigma(s)w_0) = L(\sigma(s)) + L(w_0 s)$ so that $L(\sigma(s)) = L(s)$. It follows that $L(\sigma(w)) = L(w)$ for all $w \in W$ and that we have an $\mathcal{A}$-algebra automorphism $\sigma : \mathcal{H} \to \mathcal{H}$ where $\sigma(T_w) = T_{\sigma(w)}$ for any $w \in W$. This extends to a $K$-algebra isomorphism $\sigma : \mathcal{H}_K \to \mathcal{H}_K$. We have $\sigma(c_w) = c_{\sigma(w)}$ for any $w \in W$. For any $h \in \mathcal{H}$ we have $\sigma(h^\dagger) = (\sigma(h))^\dagger$. Hence we have $\sigma(c_w^\dagger) = c_{\sigma(w)}^\dagger$ for any $w \in W$. We have $h_{\sigma(x),\sigma(y),\sigma(z)} = h_{x,y,z}$ for all $x, y, z \in W$. It follows that $\mathbf{a}(\sigma(w)) = \mathbf{a}(w)$ for all $w \in W$ and $\gamma_{\sigma(x),\sigma(y),\sigma(z)} = \gamma_{x,y,z}$ for all $x, y, z \in W$ so that we have a ring isomorphism $\sigma : J \to J$ where $\sigma(t_w) = t_{\sigma(w)}$ for any $w \in W$. This extends to an $\mathcal{A}$-algebra isomorphism $\sigma : J_{\mathcal{A}} \to J_{\mathcal{A}}$, to a $\mathbf{C}$-algebra isomorphism $\sigma : J_{\mathbf{C}} \to J_{\mathbf{C}}$ and to a $K$-algebra isomorphism $\sigma : J_K \to J_K$. From the definitions we see that $\phi : \mathcal{H} \to J_{\mathcal{A}}$ (see 0.2) satisfies $\phi\sigma = \sigma\phi$. Hence $\phi_{\mathbf{C}}$ satisfies $\phi_{\mathbf{C}}\sigma = \sigma\phi_{\mathbf{C}}$ and $\phi_K$ satisfies $\phi_K\sigma = \sigma\phi_K$.

We show:

(a) For $h \in \mathcal{H}$ we have $\sigma(h) = T_{w_0} h T_{w_0}^{-1}$.

It is enough to show this for $h$ running through a set of algebra generators of $\mathcal{H}$. Thus we can assume that $h = T_s^{-1}$ with $s \in S$. We must show that $T_{\sigma(s)}^{-1} T_{w_0} = T_{w_0} T_s^{-1}$: both sides are equal to $T_{\sigma(s)w_0} = T_{w_0 s}$.

**Lemma 1.2.** *For any $x \in W$ we have $\sigma(x) \sim_{\mathcal{LR}} x$.*

From 1.1(a) we deduce that $T_{w_0} c_x T_{w_0}^{-1} = c_{\sigma(x)}$. In particular, $\sigma(x) \le_{\mathcal{LR}} x$. Replacing $x$ by $\sigma(x)$ we obtain $x \le_{\mathcal{LR}} \sigma(x)$. The lemma follows.

**1.3.** Let $E \in \mathrm{Irr} W$. We define $\sigma_E : E \to E$ by $\sigma_E(e) = w_0 e$ for $e \in E$. We have $\sigma_E^2 = 1$. For $e \in E, w \in W$, we have $\sigma_E(we) = \sigma(w)\sigma_E(e)$. We can view $\sigma_E$ as a vector space isomorphism $E_\spadesuit \xrightarrow{\sim} E_\spadesuit$. For $e \in E_\spadesuit, w \in W$ we have $\sigma_E(t_w e) = t_{\sigma(w)}\sigma_E(e)$. Now $\sigma_E : E_\spadesuit \to E_\spadesuit$ defines by extension of scalars a vector space isomorphism $E_v \to E_v$ denoted again by $\sigma_E$. It satisfies $\sigma_E^2 = 1$. For $e \in E_v, w \in W$ we have $\sigma_E(T_w e) = T_{\sigma(w)}\sigma_E(e)$.

**Lemma 1.4.** *Let $E \in \mathrm{Irr} W$. There is a unique (up to multiplication by a scalar in $K - \{0\}$) vector space isomorphism $g : E_v \to E_v$ such that $g(T_w e) = T_{\sigma(w)} g(e)$ for all $w \in W, e \in E_v$. We can take for example $g = T_{w_0} : E_v \to E_v$ or $g = \sigma_E : E_v \to E_v$. Hence $T_{w_0} = \lambda_E \sigma_E : E_v \to E_v$ where $\lambda_E \in K - \{0\}$.*

The existence of $g$ is clear from the second sentence of the lemma. If $g'$ is another isomorphism $g' : E_v \to E_v$ such that $g'(T_w e) = T_{\sigma(w)} g'(e)$ for all $w \in W, e \in E_v$ then for any $e \in E_v$ we have $g^{-1}g'(T_w e) = g^{-1}T_{\sigma(w)}g'(e) = T_w g^{-1}g'(e)$ and using Schur's lemma we see that $g^{-1}g'$ is a scalar. This proves the first sentence of the lemma hence the third sentence of the lemma.

**1.5.** Let $E \in \mathrm{Irr} W$. We have

(a) $$\sum_{x \in W} \mathrm{tr}(T_x, E_v)\mathrm{tr}(T_{x^{-1}}, E_v) = f_{E_v} \dim(E)$$

where $f_{E_v} \in \mathcal{A}'$ is of the form

(b) $$f_{E_v} = f_0 v^{-2\mathbf{a}_E} + \text{ strictly higher powers of } v$$

and $f_0 \in \mathbf{C} - \{0\}$. (See [L3, 19.1(e), 20.1(c), 20.7].)

From Lemma 1.4 we see that $\lambda_E^{-1} T_{w_0}$ acts on $E_v$ as $\sigma_E$. Using [L4, 34.14(e)] with $c = \lambda_E^{-1} T_{w_0}$ (an invertible element of $\mathcal{H}_K$) we see that

(c) $$\sum_{x \in W} \mathrm{tr}(T_x \sigma_E, E_v)\mathrm{tr}(\sigma_E^{-1} T_{x^{-1}}, E_v) = f_{E_v} \dim(E).$$

**Lemma 1.6.** *Let $E \in \mathrm{Irr} W$. We have $\lambda_E = v^{n_E}$ for some $n_E \in \mathbf{Z}$.*

For any $x \in W$ we have

$$\mathrm{tr}(\sigma_E c_x^\dagger, E_v) = \sum_{d \in \mathcal{D}, z \in W; d = d_z} h_{x,d,z} n_d \mathrm{tr}(\sigma_E t_z, E_\spadesuit) \in \mathcal{A}'$$

since $\mathrm{tr}(\sigma_E t_z, E_\spadesuit) \in \mathbf{C}$. It follows that $\mathrm{tr}(\sigma_E h, E_v) \in \mathcal{A}'$ for any $h \in \mathcal{H}$. In particular, both $\mathrm{tr}(\sigma_E T_{w_0}, E_v)$ and $\mathrm{tr}(T_{w_0}^{-1}\sigma_E, E_v)$ belong to $\mathcal{A}'$. Thus $\lambda_E \dim E$ and $\lambda_E^{-1} \dim E$ belong to $\mathcal{A}'$ so that $\lambda_E = bv^n$ for some $b \in \mathbf{C} - \{0\}$ and $n \in \mathbf{Z}$. From the definitions we have $\lambda_E|_{v=1} = 1$ (for $v = 1$, $T_{w_0}$ becomes $w_0$) hence $b = 1$. The lemma is proved.

**Lemma 1.7.** *Let $E \in \text{Irr}W$. There exists $\epsilon_E \in \{1, -1\}$ such that for any $x \in W$ we have*

(a) $$\text{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) = \epsilon_E (-1)^{l(x)} \text{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v).$$

Let $(E_v)^\dagger$ be the $\mathcal{H}_K$-module with underlying vector space $E_v$ such that the action of $h \in \mathcal{H}_K$ on $(E_v)^\dagger$ is the same as the action of $h^\dagger$ on $E_v$. From the proof in [L3, 20.9] we see that there exists an isomorphism of $\mathcal{H}_K$-modules $b : (E_v)^\dagger \xrightarrow{\sim} (E^\dagger)_v$. Let $\iota : (E_v)^\dagger \to (E_v)^\dagger$ be the vector space isomorphism which corresponds under $b$ to $\sigma_{E^\dagger} : (E^\dagger)_v \to (E^\dagger)_v$. Then we have $\text{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) = \text{tr}(\iota T_x, (E_v)^\dagger)$. It is enough to prove that $\iota = \pm\sigma_E$ as a $K$-linear map of the vector space $E_v = (E_v)^\dagger$ into itself. From the definition we have $\iota(T_w e) = T_{\sigma(w)}\iota(e)$ for all $w \in W, e \in (E_v)^\dagger$. Hence $(-1)^{l(w)}\iota(T_{w^{-1}}^{-1} e) = (-1)^{l(w)} T_{\sigma(w^{-1})}^{-1}\iota(e)$ for all $w \in W, e \in E_v$. It follows that $\iota(he) = (-1)^{l(w)} T_{\sigma(h)}\iota(e)$ for all $h \in \mathcal{H}, e \in E_v$. Hence $\iota(T_w e) = T_{\sigma(w)}\iota(e)$ for all $w \in W, e \in E_v$. By the uniqueness in Lemma 1.4 we see that $\iota = \epsilon_E \sigma_E : E_v \to E_v$ where $\epsilon_E \in K - \{0\}$. Since $\iota^2 = 1$, $\sigma_E^2 = 1$, we see that $\epsilon_E = \pm 1$. The lemma is proved.

**Lemma 1.8.** *Let $E \in \text{Irr}W$. We have $n_E = -\mathbf{a}_E + \mathbf{a}_{E^\dagger}$.*

For $x \in W$ we have (using Lemma 1.4, 1.6)

(a) $$\text{tr}(T_{w_0 x}, E_v) = \text{tr}(T_{w_0} T_{x^{-1}}^{-1}, E_v) = v^{n_E} \text{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v).$$

Making a change of variable $x \mapsto w_0 x$ in 1.5(a) and using that $T_{x^{-1} w_0} = T_{w_0 \sigma(x)^{-1}}$ we obtain

$$\begin{aligned} f_{E_v} \dim(E) &= \sum_{x \in W} \text{tr}(T_{w_0 x}, E_v) \text{tr}(T_{w_0 \sigma(x)^{-1}}, E_v) \\ &= v^{2n_E} \sum_{x \in W} \text{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v) \text{tr}(\sigma_E T_{\sigma(x)}^{-1}, E_v). \end{aligned}$$

Using now Lemma 1.7 and the equality $l(x) = l(\sigma(x^{-1}))$ we obtain

$$\begin{aligned} f_{E_v} \dim(E) &= v^{2n_E} \sum_{x \in W} \text{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) \text{tr}(\sigma_{E^\dagger} T_{\sigma(x^{-1})}, (E^\dagger)_v) \\ &= v^{2n_E} \sum_{x \in W} \text{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) \text{tr}(T_{\xi^{-1}} \sigma_{E^\dagger}, (E^\dagger)_v) \\ &= v^{2n_E} f_{(E^\dagger)_v} \dim(E^\dagger). \end{aligned}$$

(The last step uses 1.5(c) for $E^\dagger$ instead of $E$.) Thus we have $f_{E_v} = v^{2n_E} f_{(E^\dagger)_v}$. The left hand side is as in 1.5(b) and similarly the right hand side of the form

$$f_0' v^{2n_E - 2\mathbf{a}_{E^\dagger}} + \text{strictly higher powers of } v$$

where $f_0, f_0' \in \mathbf{C} - \{0\}$. It follows that $-2\mathbf{a}_E = 2n_E - 2\mathbf{a}_{E^\dagger}$. The lemma is proved.

**Lemma 1.9.** *Let $E \in \text{Irr}W$ and let $x \in W$. We have*

(a) $$\text{tr}(T_x, E_v) = (-1)^{l(x)} v^{-\mathbf{a}_E} \text{tr}(t_x, E_\spadesuit) \mod v^{-\mathbf{a}_E+1}\mathbf{C}[v],$$

(b) $$\text{tr}(\sigma_E T_x, E_v) = (-1)^{l(x)} v^{-\mathbf{a}_E} \text{tr}(\sigma_E t_x, E_\spadesuit) \mod v^{-\mathbf{a}_E+1}\mathbf{C}[v].$$

For a proof of (a), see [L3, 20.6(b)]. We now give a proof of (b) along the same lines as that of (a). There is a unique two sided cell $\mathbf{c}$ such that $t_z|_{E_\spadesuit} = 0$ for $z \in W - \mathbf{c}$. Let $a = \mathbf{a}(z)$ for all $z \in \mathbf{c}$. By [L3, 20.6(c)] we have $a = \mathbf{a}_E$. From the definition of $c_x$ we see that $T_x = \sum_{y \in W} f_y c_y$ where $f_x = 1$ and $f_y \in v^{-1}\mathbf{Z}[v^{-1}]$ for $y \ne x$. Applying $^\dagger$ we obtain $(-1)^{l(x)} T_{x^{-1}}^{-1} = \sum_{y \in W} f_y c_y^\dagger$; applying $^-$ we obtain $(-1)^{l(x)} T_x = \sum_{y \in W} \bar{f}_y c_y^\dagger$. Thus we have

$$(-1)^{l(x)} \text{tr}(\sigma_E T_x, E_v) = \sum_{y \in W} \bar{f}_y \text{tr}(\sigma_E c_y^\dagger, E_v)$$
$$= \sum_{y,z \in W, d \in \mathcal{D}; d = d_z} \bar{f}_y h_{y,d,z} n_d \text{tr}(\sigma_E t_z, E_\spadesuit).$$

In the last sum we can assume that $z \in \mathbf{c}$ and $d \in \mathbf{c}$ so that $h_{y,d,z} = \gamma_{y,d,z^{-1}} v^{-a}$ mod $v^{-a+1}\mathbf{Z}[v]$. Since $\bar{f}_x = 1$ and $\bar{f}_y \in v\mathbf{Z}[v]$ for all $y \ne x$ we see that

$$(-1)^{l(x)} \text{tr}(\sigma_E T_x, E_v) = \sum_{z \in \mathbf{c}, d \in \mathcal{D} \cap \mathbf{c}} \gamma_{x,d,z^{-1}} n_d v^{-a} \text{tr}(\sigma_E t_z, E_\spadesuit) \mod v^{-a+1}\mathbf{C}[v].$$

If $x \notin \mathbf{c}$ then $\gamma_{x,d,z^{-1}} = 0$ for all $d, z$ in the sum so that $\text{tr}(\sigma_E T_x, E_v) = 0$; we have also $\text{tr}(\sigma_E t_x, E_\spadesuit) = 0$ and the desired formula follows. We now assume that $x \in \mathbf{c}$. Then for $d, z$ as above we have $\gamma_{x,d,z^{-1}} = 0$ unless $x = z$ and $d = d_x$ in which case $\gamma_{x,d,z^{-1}} n_d = 1$. Thus (b) holds again. The lemma is proved.

**Lemma 1.10.** *Let $E \in \text{Irr}W$. Let $\mathbf{c}$ be the unique two sided cell such that $t_z|_{E_\spadesuit} = 0$ for $z \in W - \mathbf{c}$. Let $\mathbf{c}'$ be the unique two sided cell such that $t_z|_{(E^\dagger)_\spadesuit} = 0$ for $z \in W - \mathbf{c}'$. We have $\mathbf{c}' = w_0\mathbf{c}$.*

Using 1.8(a) and 1.7(a) we have

(a) $$\text{tr}(T_{w_0x}, E_v) = v^{n_E} \text{tr}(\sigma_E T_{x^{-1}}^{-1}, E_v) = v^{n_E} \epsilon_E (-1)^{l(x)} \text{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v).$$

Using 1.9(a) for $E$ and 1.9(b) for $E^\dagger$ we obtain

$$\text{tr}(T_{w_0x}, E_v) = (-1)^{l(w_0x)} v^{-\mathbf{a}_E} \text{tr}(t_{w_0x}, E_\spadesuit) \mod v^{-\mathbf{a}_E+1}\mathbf{C}[v],$$

$$\text{tr}(\sigma_{E^\dagger} T_x, (E^\dagger)_v) = (-1)^{l(x)} v^{-\mathbf{a}_{E^\dagger}} \text{tr}(\sigma_{E^\dagger} t_x, E^\dagger_\spadesuit) \mod v^{-\mathbf{a}_{E^\dagger}+1}\mathbf{C}[v].$$

Combining with (a) we obtain

$$(-1)^{l(w_0x)}v^{-\mathbf{a}_E}\mathrm{tr}(t_{w_0x}, E_\spadesuit) + \text{strictly higher powers of } v$$
$$= v^{n_E}\epsilon_E v^{-\mathbf{a}_{E^\dagger}}\mathrm{tr}(\sigma_{E^\dagger}t_x, E^\dagger_\spadesuit) + \text{strictly higher powers of } v.$$

Using the equality $n_E = -\mathbf{a}_E + \mathbf{a}_{E^\dagger}$ (see 1.8) we deduce

$$(-1)^{l(w_0x)}\mathrm{tr}(t_{w_0x}, E_\spadesuit) = \epsilon_E\mathrm{tr}(\sigma_{E^\dagger}t_x, E^\dagger_\spadesuit).$$

Now we can find $x \in W$ such that $\mathrm{tr}(t_{w_0x}, E_\spadesuit) \neq 0$ and the previous equality shows that $t_x|_{(E^\dagger)_\spadesuit} \neq 0$. Moreover from the definition we have $w_0x \in \mathbf{c}$ and $x \in \mathbf{c}'$ so that $w_0\mathbf{c} \cap \mathbf{c}' \neq \emptyset$. Since $w_0\mathbf{c}$ is a two-sided cell (see [L3, 11.7(d)]) it follows that $w_0\mathbf{c} = \mathbf{c}'$. The lemma is proved.

**Lemma 1.11.** *Let $\mathbf{c}$ be a two-sided cell of $W$. Let $\mathbf{c}'$ be the two-sided cell $w_0\mathbf{c} = \mathbf{c}w_0$ (see Lemma 1.2). Let $a = \mathbf{a}(x)$ for any $x \in \mathbf{c}$; let $a' = \mathbf{a}(x')$ for any $x' \in \mathbf{c}'$. The $K$-linear map $J^{\mathbf{c}}_K \to J^{\mathbf{c}}_K$ given by $\xi \mapsto \phi(v^{a-a'}T_{w_0})\xi$ (left multiplication in $J_K$) is obtained from a $\mathbf{C}$-linear map $J^{\mathbf{c}}_{\mathbf{C}} \to J^{\mathbf{c}}_{\mathbf{C}}$ (with square 1) by extension of scalars from $\mathbf{C}$ to $K$.*

We can find a direct sum decomposition $J^{\mathbf{c}}_{\mathbf{C}} = \oplus_{i=1}^m E^i$ where $E^i$ are simple left ideals of $J_{\mathbf{C}}$ contained in $J^{\mathbf{c}}_{\mathbf{C}}$. We have $J^{\mathbf{c}}_K = \oplus_{i=1}^m K \otimes E^i$. It is enough to show that for any $i$, the $K$-linear map $K \otimes E^i \to K \otimes E^i$ given by the action of $\phi(v^{a-a'}T_{w_0})$ in the left $J_K$-module structure of $K \otimes E^i$ is obtained from a $\mathbf{C}$-linear map $E^i \to E^i$ (with square 1) by extension of scalars from $\mathbf{C}$ to $K$. We can find $E \in \mathrm{Irr}W$ such that $E^i$ is isomorphic to $E_\spadesuit$ as a $J_{\mathbf{C}}$-module. It is then enough to show that the action of $v^{a-a'}T_{w_0}$ in the left $\mathcal{H}_K$-module structure of $E_v$ is obtained from the map $\sigma_E : E \to E$ by extension of scalars from $\mathbf{C}$ to $K$. This follows from the equality $v^{a-a'}T_{w_0} = \sigma_E : E_v \to E_v$ (since $\sigma_E$ is obtained by extension of scalars from a $\mathbf{C}$-linear map $E \to E$ with square 1) provided that we show that $-n_E = a - a'$. Since $n_E = -\mathbf{a}_E + \mathbf{a}_{E^\dagger}$ (see Lemma 1.8) it is enough to show that $a = \mathbf{a}_E$ and $a' = \mathbf{a}_{E^\dagger}$. The equality $a = \mathbf{a}_E$ follows from [L3, 20.6(c)]. The equality $a' = \mathbf{a}_{E^\dagger}$ also follows from [L3, 20.6(c)] applied to $E^\dagger, \mathbf{c}' = w_0\mathbf{c}$ instead of $E, \mathbf{c}$ (see Lemma 1.10). The lemma is proved.

**Lemma 1.12.** *In the setup of Lemma 1.11 we have for any $x \in \mathbf{c}$:*

(a) $$\phi(v^{a-a'}T_{w_0})t_x = \sum_{x' \in \mathbf{c}} m_{x',x}t_{x'}$$

(b) $$\phi(v^{2a-2a'}T^2_{w_0})t_x = t_x$$

*where $m_{x',x} \in \mathbf{Z}$.*

Now (b) and the fact that (a) holds with $m_{x',x} \in \mathbf{C}$ is just a restatement of Lemma 1.11. Since $\phi(v^{a-a'}T_{w_0}) \in J_{\mathcal{A}}$ we have also $m_{x',x} \in \mathcal{A}$. We now use that $\mathcal{A} \cap \mathbf{C} = \mathbf{Z}$ and the lemma follows.

**Lemma 1.13.** *In the setup of Lemma 1.11 we have for any $x \in \mathbf{c}$ the following equalities in $\mathcal{H}^{\mathbf{c}}$:*

(a) $$v^{a-a'} T_{w_0} c_x^\dagger = \sum_{x' \in \mathbf{c}} m_{x',x} c_{x'}^\dagger,$$

(b) $$v^{2a-2a'} T_{w_0}^2 c_x^\dagger = c_x^\dagger$$

*where $m_{x',x} \in \mathbf{Z}$ are the same as in Lemma 1.12. Moreover, if $m_{x',x} \neq 0$ then $x' \sim_{\mathcal{L}} x$.*

The first sentence follows from Lemma 1.12 using [L3, 18.10(a)]. Clearly, if $m_{x',x} \neq 0$ then $x' \leq_{\mathcal{L}} x$ which together with $x' \sim_{\mathcal{LR}} x$ implies $x' \sim_{\mathcal{L}} x$.

## 2. The main results

**2.1.** In this section we fix a two-sided cell $\mathbf{c}$ of $W$; $a, a'$ are as in 1.11. We define an $\mathcal{A}$-linear map $\theta : \mathcal{H}^{\leq \mathbf{c}} \to \mathcal{A}$ by $\theta(c_x^\dagger) = 1$ if $x \in \mathcal{D} \cap \mathbf{c}$, $\theta(c_x^\dagger) = 0$ if $x \leq_{\mathcal{LR}} x'$ for some $x' \in \mathbf{c}$ and $x \notin \mathcal{D} \cap \mathbf{c}$. Note that $\theta$ is zero on $\mathcal{H}^{<\mathbf{c}}$ hence it can be viewed as an $\mathcal{A}$-linear map $\mathcal{H}^{\mathbf{c}} \to \mathcal{A}$.

**Lemma 2.2.** *Let $x, x' \in \mathbf{c}$. We have*

(a) $$\theta(c_{x^{-1}}^\dagger c_{x'}^\dagger) = n_{d_x} \delta_{x,x'} v^a + \text{ strictly lower powers of } v.$$

The left hand side of (a) is

$$\sum_{d \in \mathcal{D} \cap \mathbf{c}} h_{x^{-1},x',d} = \sum_{d \in \mathcal{D} \cap \mathbf{c}} \gamma_{x^{-1},x',d} v^a + \text{ strictly lower powers of } v$$
$$= n_{d_x} \delta_{x,x'} v^a + \text{ strictly lower powers of } v.$$

The lemma is proved.

We now state one of the main results of this paper.

**Theorem 2.3.** *There exists a unique permutation $u \mapsto u^*$ of $\mathbf{c}$ (with square 1) such that for any $u \in \mathbf{c}$ we have*

(a) $$v^{a-a'} T_{w_0} c_u^\dagger = \epsilon_u c_{u^*}^\dagger \mod \mathcal{H}^{<\mathbf{c}}$$

*where $\epsilon_u = \pm 1$. For any $u \in \mathbf{c}$ we have $\epsilon_{u^{-1}} = \epsilon_u = \epsilon_{\sigma(u)} = \epsilon_{u^*}$ and $\sigma(u^*) = (\sigma(u))^* = ((u^{-1})^*)^{-1}$.*

Let $u \in \mathbf{c}$. We set $Z = \theta((v^{a-a'} T_{w_0} c_u^\dagger)^\flat v^{a-a'} T_{w_0} c_u^\dagger)$. We compute $Z$ in two ways, using Lemma 2.2 and Lemma 1.13. We have

$$Z = \theta(c_{u^{-1}}^\dagger v^{2a-2a'} T_{w_0}^2 c_u^\dagger) = \theta(c_{u^{-1}}^\dagger c_u^\dagger) = n_{d_u} v^a + \text{ strictly lower powers of } v,$$

$$Z=\theta((\sum_{y\in\mathbf{c}}m_{y,u}c^\dagger_y)^\flat(\sum_{y'\in\mathbf{c}}m_{y',u}c^\dagger_{y'}))=\sum_{y,y'\in\mathbf{c}}m_{y,u}m_{y',u}\theta(c^\dagger_{y^{-1}}c^\dagger_{y'})$$
$$=\sum_{y,y'\in\mathbf{c}}m_{y,u}m_{y',u}n_{d_y}\delta_{y,y'}v^a+\text{ strictly lower powers of } v$$
$$=\sum_{y\in\mathbf{c}}n_{d_y}m^2_{y,u}v^a+\text{ strictly lower powers of } v$$
$$=\sum_{y\in\mathbf{c}}n_{d_u}m^2_{y,u}v^a+\text{ strictly lower powers of } v$$

where $m_{y,u}\in\mathbf{Z}$ is zero unless $y\sim_{\mathcal{L}}u$ (see 1.13), in which case we have $d_y=d_u$. We deduce that $\sum_{y\in\mathbf{c}}m^2_{y,u}=1$, so that we have $m_{y,u}=\pm1$ for a unique $y\in\mathbf{c}$ (denoted by $u^*$) and $m_{y,u}=0$ for all $y\in\mathbf{c}-\{u^*\}$. Then (a) holds. Using (a) and Lemma 1.13(b) we see that $u\mapsto u^*$ has square 1 and that $\epsilon_u\epsilon_{u^*}=1$.

The automorphism $\sigma:\mathcal{H}\to\mathcal{H}$ (see 1.1) satisfies the equality $\sigma(c^\dagger_u)=c^\dagger_{\sigma(u)}$ for any $u\in W$; note also that $w\in\mathbf{c}\leftrightarrow\sigma(w)\in\mathbf{c}$ (see Lemma 1.2). Applying $\sigma$ to (a) we obtain

$$v^{a-a'}T_{w_0}c^\dagger_{\sigma(u)}=\epsilon_u c^\dagger_{\sigma(u^*)}$$

in $\mathcal{H}^{\mathbf{c}}$. By (a) we have also $v^{a-a'}T_{w_0}c^\dagger_{\sigma(u)}=\epsilon_{\sigma(u)}c^\dagger_{(\sigma(u))^*}$ in $\mathcal{H}^{\mathbf{c}}$. It follows that $\epsilon_u c^\dagger_{\sigma(u^*)}=\epsilon_{\sigma(u)}c^\dagger_{(\sigma(u))^*}$ hence $\epsilon_u=\epsilon_{\sigma(u)}$ and $\sigma(u^*)=(\sigma(u))^*$.

Applying $h\mapsto h^\flat$ to (a) we obtain

$$v^{a-a'}c^\dagger_{u^{-1}}T_{w_0}=\epsilon_u c^\dagger_{(u^*)^{-1}}$$

in $\mathcal{H}^{\mathbf{c}}$. By (a) we have also

$$v^{a-a'}c^\dagger_{u^{-1}}T_{w_0}=v^{a-a'}T_{w_0}c^\dagger_{\sigma(u^{-1})}=\epsilon_{\sigma(u^{-1})}c^\dagger_{(\sigma(u^{-1}))^*}$$

in $\mathcal{H}^{\mathbf{c}}$. It follows that $\epsilon_u c^\dagger_{(u^*)^{-1}}=\epsilon_{\sigma(u^{-1})}c^\dagger_{(\sigma(u^{-1}))^*}$ hence $\epsilon_u=\epsilon_{\sigma(u^{-1})}$ and $(u^*)^{-1}=(\sigma(u^{-1}))^*$. Since $\epsilon_{\sigma(u^{-1})}=\epsilon_{u^{-1}}$, we see that $\epsilon_u=\epsilon_{u^{-1}}$. Replacing $u$ by $u^{-1}$ in $(u^*)^{-1}=(\sigma(u^{-1}))^*$ we obtain $((u^{-1})^*)^{-1}=(\sigma(u))^*$ as required. The theorem is proved.

**2.4.** For $u\in\mathbf{c}$ we have

(a) $$u\sim_{\mathcal{L}}u^*,$$

(b) $$\sigma(u)\sim_{\mathcal{R}}u^*.$$

Indeed, (a) follows from 1.13. To prove (b) it is enough to show that $\sigma(u)^{-1}\sim_{\mathcal{L}}(u^*)^{-1}$. Using (a) for $\sigma(u)^{-1}$ instead of $u$ we see that it is enough to show that $(\sigma(u^{-1}))^*=(u^*)^{-1}$; this follows from 2.3.

If we assume that

(c) *any left cell in* $\mathbf{c}$ *intersects any right cell in* $\mathbf{c}$ *in exactly one element*

then by (a),(b), for any $u \in \mathbf{c}$,

(d) $u^*$ *is the unique element of* $\mathbf{c}$ *in the intersection of the left cell of* $u$ *with right cell of* $\sigma(u)$.

Note that condition (c) is satisfied for any $\mathbf{c}$ if $W$ is of type $A_n$ or if $W$ is of type $B_n$ ($n \ge 2$) with $L(s) = 2$ for all but one $s \in S$ and $L(s) = 1$ or 3 for the remaining $s \in S$. (In this last case we are in the quasisplit case and we have $\sigma = 1$ hence $u^* = u$ for all $u$.)

**Theorem 2.5.** *For any* $x \in W$ *we set* $\vartheta(x) = \gamma_{w_0 d_{w_0 x^{-1}}, x, (x^*)^{-1}}$.

*(a) If* $d \in \mathcal{D}$ *and* $x, y \in \mathbf{c}$ *satisfy* $\gamma_{w_0 d, x, y} \ne 0$ *then* $y = (x^*)^{-1}$.

*(b) If* $x \in \mathbf{c}$ *then there is a unique* $d \in \mathcal{D} \cap w_0 \mathbf{c}$ *such that* $\gamma_{w_0 d, x, (x^*)^{-1}} \ne 0$, *namely* $d = d_{w_0 x^{-1}}$. *Moreover we have* $\vartheta(x) = \pm 1$.

*(c) For* $u \in \mathbf{c}$ *we have* $\epsilon_u = (-1)^{l(w_0 d)} n_d \vartheta(u)$ *where* $d = d_{w_0 u^{-1}}$.

Appplying $h \mapsto h^\dagger$ to 2.3(a) we obtain for any $u \in \mathbf{c}$:

(d) $$v^{a-a'}(-1)^{l(w_0)} \overline{T_{w_0}} c_u = \sum_{z \in \mathbf{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \mathbf{c}} \mathcal{A} c_{z'}.$$

We have $T_{w_0} = \sum_{y \in W} (-1)^{l(w_0 y)} p_{1,w_0 y} c_y$ hence $\overline{T_{w_0}} = \sum_{y \in W} (-1)^{l(w_0 y)} \overline{p_{1,w_0 y}} c_y$. Introducing this in (d) we obtain

$$v^{a-a'} \sum_{y \in W} (-1)^{l(y)} \overline{p_{1,w_0 y}} c_y c_u = \sum_{z \in \mathbf{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \mathbf{c}} \mathcal{A} c_{z'}$$

that is,

$$v^{a-a'} \sum_{y,z \in W} (-1)^{l(y)} \overline{p_{1,w_0 y}} h_{y,u,z} c_z = \sum_{z \in \mathbf{c}} \delta_{z,u^*} \epsilon_u c_z \mod \sum_{z' \in W - \mathbf{c}} \mathcal{A} c_{z'}.$$

Thus, for $z \in \mathbf{c}$ we have

(e) $$v^{a-a'} \sum_{y \in W} (-1)^{l(y)} \overline{p_{1,w_0 y}} h_{y,u,z} = \delta_{z,u^*} \epsilon_u.$$

Here we have $h_{y,u,z} = \gamma_{y,u,z^{-1}} v^{-a} \mod v^{-a+1}\mathbf{Z}[v]$ and we can assume than $z \le_{\mathcal{R}} y$ so that $w_0 y \le_{\mathcal{R}} w_0 z$ and $\mathbf{a}(w_0 y) \ge \mathbf{a}(w_0 z) = a'$.

For $w \in W$ we set $s_w = n_w$ if $w \in \mathcal{D}$ and $s_w = 0$ if $w \notin \mathcal{D}$. By [L3, 14.1] we have $p_{1,w} = s_w v^{-\mathbf{a}(w)} \mod v^{-\mathbf{a}(w)-1}\mathbf{Z}[v^{-1}]$ hence $\overline{p_{1,w}} = s_w v^{\mathbf{a}(w)} \mod v^{\mathbf{a}(w)+1}\mathbf{Z}[v]$. Hence for $y$ in the sum above we have $\overline{p_{1,w_0 y}} = s_{w_0 y} v^{\mathbf{a}(w_0 y)} \mod v^{\mathbf{a}(w_0 y)+1}\mathbf{Z}[v]$. Thus (e) gives

$$v^{a-a'} \sum_{y \in \mathbf{c}} (-1)^{l(y)} s_{w_0 y} \gamma_{y,u,z^{-1}} v^{\mathbf{a}(w_0 y) - a} - \delta_{z,u^*} \epsilon_u \in v\mathbf{Z}[v]$$

and using $\mathbf{a}(w_0 y) = a'$ for $y \in \mathbf{c}$ we obtain

$$\sum_{y \in \mathbf{c}} (-1)^{l(y)} s_{w_0 y} \gamma_{y,u,z^{-1}} = \delta_{z,u^*} \epsilon_u.$$

Using the definition of $s_{w_0 y}$ we obtain

(f) $$\sum_{d \in \mathcal{D} \cap w_0 \mathbf{c}} (-1)^{l(w_0 d)} n_d \gamma_{w_0 d,u,z^{-1}} = \delta_{z,u^*} \epsilon_u.$$

Next we note that

(g) *if $d \in \mathcal{D}$ and $x, y \in \mathbf{c}$ satisfy $\gamma_{w_0 d,x,y} \neq 0$ then $d = d_{w_0 x^{-1}}$.*

Indeed from [L3,§14, P8] we deduce $w_0 d \sim_{\mathcal{L}} x^{-1}$. Using [L3, 11.7] we deduce $d \sim_{\mathcal{L}} w_0 x^{-1}$ so that $d = d_{w_0^{-1} x^{-1}}$. This proves (g).

Using (g) we can rewrite (f) as follows.

(h) $$(-1)^{l(w_0)} (-1)^{l(d)} n_d \gamma_{w_0 d,u,z^{-1}} = \delta_{z,u^*} \epsilon_u$$

where $d = d_{w_0 u^{-1}}$.

We prove (a). Assume that $d \in \mathcal{D}$ and $x, y \in \mathbf{c}$ satisfy $\gamma_{w_0 d,x,y} \neq 0$, $y \neq (x^*)^{-1}$. Using (g) we have $d = d_{w_0 x^{-1}}$. Using (h) with $u = x, z = y^{-1}$ we see that $\gamma_{w_0 d,x,y} = 0$, a contradiction. This proves (a).

We prove (b). Using (h) with $u = x, z = x^*$ we see that

(i) $$(-1)^{l(w_0 d)} n_d \gamma_{w_0 d,x,(x^*)^{-1}} = \epsilon_u$$

where $d = d_{w_0 x^{-1}}$. Hence the existence of $d$ in (b) and the equality $\vartheta(x) = \pm 1$ follow; the uniqueness of $d$ follows from (g).

Now (c) follows from (i). This completes the proof of the theorem.

**2.6.** In the case where $L = l$, $\vartheta(u)$ (in 2.5(c)) is $\geq 0$ and $\pm 1$ hence 1; moreover, $n_d = 1$, $(-1)^{l(d)} = (-1)^{a'}$ for any $d \in \mathcal{D} \cap w_0 \mathbf{c}$ (by the definition of $\mathcal{D}$). Hence we have $\epsilon_u = (-1)^{l(w_0)+a'}$ for any $u \in \mathbf{c}$, a result of [MA].

Now Theorem 2.5 also gives a characterization of $u^*$ for $u \in \mathbf{c}$; it is the unique element $u' \in \mathbf{c}$ such that $\gamma_{w_0 d,u,u'^{-1}} \neq 0$ for some $d \in \mathcal{D} \cap w_0 \mathbf{c}$.

We will show:

(a) *The subsets $X = \{d^*; d \in \mathcal{D} \cap \mathbf{c}\}$ and $X' = \{w_0 d'; d' \in \mathcal{D} \cap w_0 \mathbf{c}\}$ of $\mathbf{c}$ coincide.*

Let $d \in \mathcal{D} \cap \mathbf{c}$. By 2.5(b) we have $\gamma_{w_0 d',d,(d^*)^{-1}} = \pm 1$ for some $d' \in \mathcal{D} \cap w_0 \mathbf{c}$. Hence $\gamma_{(d^*)^{-1},w_0 d',d} = \pm 1$. Using [L3, 14.2, P2] we deduce $d^* = w_0 d'$. Thus $X \subset X'$. Let $Y$ (resp. $Y'$) be the set of left cells contained in $\mathbf{c}$ (resp. $w_0 \mathbf{c}$). We have $\sharp(X) = \sharp(Y)$ and $\sharp(X') = \sharp(Y')$. By [L3, 11.7(c)] we have $\sharp(Y) = \sharp(Y')$. It follows that $\sharp(X) = \sharp(X')$. Since $X \subset X'$, we must have $X = X'$. This proves (a).

**Theorem 2.7.** *We have*

$$\phi(v^{a-a'}T_{w_0})=\sum_{d\in\mathcal{D}\cap\mathbf{c}}\vartheta(d)\epsilon_d t_{d^*}\mod \sum_{u\in W-\mathbf{c}}\mathcal{A}t_u.$$

We set $\phi(v^{a-a'}T_{w_0})=\sum_{u\in W}p_ut_u$ where $p_u\in\mathcal{A}$. Combining 1.12(a), 1.13(a), 2.3(a) we see that for any $x\in\mathbf{c}$ we have

$$\phi(v^{a-a'}T_{w_0})t_x=\epsilon_x t_{x^*},$$

hence

$$\epsilon_x t_{x^*}=\sum_{u\in\mathbf{c}}p_ut_ut_x=\sum_{u,y\in\mathbf{c}}p_u\gamma_{u,x,y^{-1}}t_y.$$

It follows that for any $x,y\in\mathbf{c}$ we have

$$\sum_{u\in\mathbf{c}}p_u\gamma_{u,x,y^{-1}}=\delta_{y,x^*}\epsilon_x.$$

Taking $x=w_0d$ where $d=d_{w_0y}\in\mathcal{D}\cap w_0\mathbf{c}$ we obtain

$$\sum_{u\in\mathbf{c}}p_u\gamma_{w_0d_{w_0y},y^{-1},u}=\delta_{y,(w_0d_{w_0y})^*}\epsilon_{w_0d_{w_0y}}$$

which, by 2.5, can be rewritten as

$$p_{((y^{-1})^*)^{-1}}\vartheta(y^{-1})=\delta_{y,(w_0d_{w_0y})^*}\epsilon_{w_0d_{w_0y}}.$$

We see that for any $y\in\mathbf{c}$ we have

$$p_{\sigma(y^*)}=\delta_{y,(w_0d_{w_0y})^*}\vartheta(y^{-1})\epsilon_{w_0d_{w_0y}}.$$

In particular we have $p_{\sigma(y^*)}=0$ unless $y=(w_0d_{w_0y})^*$ in which case

$$p_{\sigma(y^*)}=p_{(\sigma(y))^*)}=\vartheta(y^{-1})\epsilon_y.$$

(We use that $\epsilon_{y^*}=\epsilon_y$.) If $y=(w_0d_{w_0y})^*$ then $y^*\in X'$ hence by 2.6(a), $y^*=d^*$ that is $y=d$ for some $d\in\mathcal{D}$. Conversely, if $y\in\mathcal{D}$ then $w_0y^*\in\mathcal{D}$ (by 2.6(a)) and $w_0y^*\sim_{\mathcal{L}}w_0y$ (since $y^*\sim_{\mathcal{L}}y$) hence $d_{w_0y}=w_0y^*$. We see that $y=(w_0d_{w_0y})^*$ if and only if $y\in\mathcal{D}$. We see that

$$\phi(v^{a-a'}T_{w_0})=\sum_{d\in\mathcal{D}\cap\mathbf{c}}\vartheta(d^{-1})\epsilon_d t_{(\sigma(d))^*}+\sum_{u\in W-\mathbf{c}}p_ut_u.$$

Now $d\mapsto\sigma(d)$ is a permutation of $\mathcal{D}\cap\mathbf{c}$ and $\vartheta(d^{-1})=\vartheta(d)=\vartheta(\sigma(d))$, $\epsilon_{\sigma(d)}=\epsilon_d$. The theorem follows.

**Corollary 2.8.** *We have*

$$\phi(T_{w_0})=\sum_{d\in\mathcal{D}}\vartheta(d)\epsilon_d v^{-\mathbf{a}(d)+\mathbf{a}(w_0d)}t_{d^*}\in J_{\mathcal{A}}.$$

**2.9.** We set $\mathfrak{T}_{\mathbf{c}}=\sum_{d\in\mathcal{D}\cap\mathbf{c}}\vartheta(d)\epsilon_d t_{d^*}\in J^{\mathbf{c}}$. We show:
(a) $\mathfrak{T}_{\mathbf{c}}^2=\sum_{d\in\mathcal{D}\cap\mathbf{c}}n_d t_d$;
(b) $t_x\mathfrak{T}_{\mathbf{c}}=\mathfrak{T}_{\mathbf{c}}t_{\sigma(x)}$ for any $x\in W$.
By 2.7 we have $\phi(v^{a-a'}T_{w_0})=\mathfrak{T}_{\mathbf{c}}+\xi$ where $\xi\in J_K^{W-\mathbf{c}}:=\sum_{u\in W-\mathbf{c}}Kt_u$. Since $J_K^{\mathbf{c}},J_K^{W-\mathbf{c}}$ are two-sided ideals of $J_K$ with intersection zero and $\phi_K:\mathcal{H}_K\to J_K$ is an algebra homomorphism, it follows that

$$\phi(v^{2a-2a'}T_{w_0}^2)=(\phi(v^{a-a'}T_{w_0}))^2=(\mathfrak{T}_{\mathbf{c}}+\xi)^2=\mathfrak{T}_{\mathbf{c}}^2+\xi'$$

where $\xi'\in J_K^{W-\mathbf{c}}$. Hence, for any $x\in\mathbf{c}$ we have $\phi(v^{2a-2a'}T_{w_0}^2)t_x=\mathfrak{T}_{\mathbf{c}}^2t_x$ so that (using 1.12(b)): $t_x=\mathfrak{T}_{\mathbf{c}}^2t_x$. We see that $\mathfrak{T}_{\mathbf{c}}^2$ is the unit element of the ring $J_K^{\mathbf{c}}$. Thus (a) holds.

We prove (b). For any $y\in W$ we have $T_yT_{w_0}=T_{w_0}T_{\sigma(y)}$ hence, applying $\phi_K$,

$$\phi(T_y)\phi(v^{a-a'}T_{w_0})=\phi(v^{a-a'}T_{w_0})\phi(T_{\sigma(y)})$$

that is, $\phi(T_y)(\mathfrak{T}_{\mathbf{c}}+\xi)=(\mathfrak{T}_{\mathbf{c}}+\xi)\phi(T_{\sigma(y)})$. Thus, $\phi(T_y)\mathfrak{T}_{\mathbf{c}}=\mathfrak{T}_{\mathbf{c}}\phi(T_{\sigma(y)})+\xi_1$ where $\xi_1\in J_K^{W-\mathbf{c}}$. Since $\phi_K$ is an isomorphism, it follows that for any $x\in W$ we have $t_x\mathfrak{T}_{\mathbf{c}}=\mathfrak{T}_{\mathbf{c}}t_{\sigma(x)}\mod J_K^{W-\mathbf{c}}$. Thus (b) holds.

**2.10.** In this subsection we assume that $L=l$. In this case 2.8 becomes

$$\phi(T_{w_0})=\sum_{d\in\mathcal{D}}(-1)^{l(w_0)+\mathbf{a}(w_0d)}v^{-\mathbf{a}(d)+\mathbf{a}(w_0d)}t_{d^*}\in J_{\mathcal{A}}.$$

(We use that $\vartheta(d)=1$.)

For any left cell $\Gamma$ contained in $\mathbf{c}$ let $n_\Gamma$ be the number of fixed points of the permutation $u\mapsto u^*$ of $\Gamma$. Now $\Gamma$ carries a representation $[\Gamma]$ of $W$ and from 2.3 we see that $\mathrm{tr}(w_0,[\Gamma])=\pm n_\Gamma$. Thus $n_\Gamma$ is the absolute value of the integer $\mathrm{tr}(w_0,[\Gamma])$. From this the number $n_\Gamma$ can be computed for any $\Gamma$. In this way we see for example that if $W$ is of type $E_7$ or $E_8$ and $\mathbf{c}$ is not an exceptional two-sided cell, then $n_\Gamma>0$.

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