A Linear Parameter Varying Control Methodology for Reduction of Helicopter Higher Harmonic Vibration

by

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B.S. Aerospace Engineering, West Virginia University (2012)
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Submitted to the Department of Aeronautics and Astronautics in partial fulfillment of the requirements for the degree of Master of Science in Aeronautics and Astronautics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2016

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Abstract

Vibration in helicopters can have a significant impact on their utility, maintenance, weight, performance, and cost of operation. This thesis focuses on the design of a vibration reduction controller that is effective throughout the flight envelope of the helicopter, in spite of variations in vibration levels and dynamics with flight condition. Analysis of a UH-60 Black Hawk helicopter modeled in the Rotorcraft Comprehensive Analysis System (RCAS) aeromechanical simulation environment indicates that the steady state vibration levels and the helicopter dynamics depend primarily on the advance ratio. Two baseline vibration controllers are developed, specifically the Continuous Time Higher Harmonic Controller (CTHHC) and an $\mathcal{H}_\infty$ based controller, over a range of advance ratios. The unique controller developed in this thesis uses the Linear Parameter Varying (LPV) synthesis method, which provides performance and stability guarantees over the advance ratio parameter space. The three controllers are evaluated in the RCAS environment at fixed and maneuvering flight conditions. The results indicate that the full envelope controller designed using the LPV method exhibits increased performance over the CTHHC and $\mathcal{H}_\infty$ controllers.

Thesis Supervisor: Steven R. Hall
Title: Professor of Aeronautics and Astronautics
Acknowledgments

This thesis represents the culmination of the influence and work of many people in my life — this section will fail to mention everyone.

First and foremost, I would like to thank the tireless work and input from my adviser, Dr. Steven Hall. His experience and strong engineering skills guided my project along, when sometimes I felt like I had no path forward.

There were many other folks who made the technical aspect of my project succeed. I would like to thank Dr. Robert Ormiston of the U.S. Army AMRDEC, who guided me on the results from my aeromechanical simulation code. Dr. Hossein Saberi and Matthew Hasbun of the Advanced Rotorcraft Technologies set me up with the RCAS code and willingly answered my many questions. I would like to thank Dr. Peter Seiler and Dr. Harald Pfifer from the University of Minnesota who provided key notes on the linear parameter varying control synthesis method.

I would like to thank my office mates, Michael Klinker, Justin Miller, and Jack Quindlen, who provided me with hours of conversation on topics of work and many that varied all spectrums of life. Furthermore, I would like to thank my two closest friends at MIT, Brett Lopez and Ming Qing Foo. Brett and I shared many laughs, hundreds of running miles around Boston, and hundreds of hours in the Zesiger Center. Ming and I ate many lunches from the Couscous Kitchen On Wheels food truck together while sharing our quirky culture norms. Thanks for making my time at MIT memorable and enjoyable.

I would like to thank my parents, Arnold and Sheryl, for giving me the direction and guidance to become the person I am today — I am forever indebted to you both.

Lastly, this section would be remiss if it did not thank my loving and beautiful wife, Victoria, who did the many household chores I failed to do and sacrificed many hours as I pursued this all-consuming endeavor. Thank you so much for moving up to Boston and letting me to pursue my dream of studying at MIT.

It is also necessary to acknowledge the financial support that enabled my educational pursuits at MIT. This material is based upon work supported by the National
Science Foundation Graduate Research Fellowship under Grant No. 1122374. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors(s) and do not necessarily reflect the views of the National Science Foundation. The National Science Foundation supported the majority of my time at MIT. In addition, I would like to acknowledge the Raymond L. Bisplinghoff Fund for providing me with a research assistantship position.
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Chapter 1

Introduction

The ability of a helicopter to take-off vertically from any location has led to the helicopter becoming a vital asset to many stakeholders, including off-shore drilling companies, executive transport, search-and-research, and military cargo transport. While the helicopter has become an invaluable asset for its current users, a lingering problem for the helicopter is vibration, which has limited wider application of the technology. One consequence of excessive vibration is a direct increase in the operating and service costs, due to the increased maintenance requirements. Because of vibration, helicopter structural components experience fatigue and the life of electrical components is reduced. Since vibration problems persist in the industry, further research must be conducted in this area.

Over the last three decades, researchers in academia and industry have developed technologies to eliminate the persistent vibration problem. These techniques range from passive to active controllers. The goal of this thesis is to design an active vibration controller and demonstrate it in a nonlinear aeromechanical helicopter simulator over the entire flight envelope. The technique used to design the controller is the Linear Parameter Varying synthesis method, which provides stability and performance guarantees over the flight envelope. Further acceptance of vibration reducing technologies rests on providing stability and performance guarantees of the vibration controller throughout the flight envelope.

This chapter provides a discussion of the helicopter vibration problem, methods
to alleviate the vibration, an overview of available aeromechanical simulation codes, and the research approach to solve this problem is discussed. Finally, an outline of this thesis is provided.

1.1 Helicopter Vibration Problem

Helicopter vibration originates from several sources, such as the main rotor gearbox, tail rotor, transmission, rotor imbalance, and manufacturing variations. However, the largest contributor to helicopter vibration is due to the periodic variation in the aerodynamic loads as the rotor blades precess around the main rotor head, as shown in Figure 1-1. The unsteady aerodynamics produce vibratory loads that enter the hub and travel down the mast to the gearbox, which finally transfer to the airframe. These vibratory loads are greatest during forward flight, when the asymmetric loading is most significant [10]. In more detail, the unsteady aerodynamics in forward flight

![Figure 1-1: Unsteady aerodynamics of helicopter in forward flight](58)
are caused by reversed flow, dynamic stall, retreating blade stall, blade vortex interaction, and supercritical flow.

Helicopter forward speeds are characterized by the advance ratio, a non-dimensional coefficient represented as

$$\mu = \frac{V \cos(\alpha_s)}{\Omega R}$$  \hspace{1cm} (1.1)

where $V$ is the helicopter airspeed velocity, $\alpha_s$ is the shaft angle (positive when tilted aft), $\Omega$ is the rotation rate of the rotor, and $R$ is the radius of the main rotor. The rotor dynamics of a helicopter vary as a function of the advance ratio as shown in Figure 1-2 [24, 27]. It can be seen that as the helicopter transitions from hover to forward flight ($\mu \approx 0.1$) there is a high level of vibration, which can be attributed to blade-vortex interaction. The vibration level increases again at high speeds due to compressibility and stall as previously discussed.

Traditionally, the largest contributor to the vibration is the $N\Omega$ frequency, where $N$ is the number of blades. This is usually stated as the $N$ per rev frequency. It has also been shown that all of the per rev frequencies (1 per rev, 2 per rev,...) can contribute to the overall main rotor vibration [18], but these additional frequencies are not traditionally targeted. To affectively reduce the costs and increase the passenger

![Figure 1-2: Helicopter vibration as a function of forward speed](image-url)
comfort level, the largest contributors of the vibration, namely the $N$ per rev frequencies, must be reduced.

1.2 Vibration Reduction Methods

The method to reduce helicopter vibration can be broken into two major groups, passive and active control methods. Active control methods largely achieve vibration reduction by influencing the helicopter rotor properties, such as the blade twist or pitch. Even though active control methods have been demonstrated, most production aircraft still employ passive methods to attenuate rotor vibrations. This section provides a brief overview of key passive and active vibration techniques to solve the helicopter vibration problem.

1.2.1 Passive Methods

There are a few passive vibration methods that have been employed by helicopter engineers. The reduction of vibration largely begins with the design of the rotor, where aerodynamic, inertial, and aeroelastic loads all act to create the fundamental problem. Therefore designers largely attempt to minimize vibration by selection of the blade airfoils, the distribution of blade airfoils along the blade, and the selection of the blade twist [34]. In addition to the typical rotor property selections, vibration reduction has been passively managed by optimizing the blades with geometry tips. An example of this type of rotor design is the BERP blade, whose optimization parameters included reduced vibration, higher cruise speed, and additional lift capability [1].

After designing the rotor system, a common technique is to minimize vibration by adjusting the structural properties (stiffness and mass distribution) of the fuselage. This is done to ensure that the resonant frequencies of the helicopter fuselage structure do not coincide with the frequencies of the exciting forces. Furthermore, the fuselage must be designed to avoid resonances with the harmonics of the rotor speed (i.e., the $N$ per rev frequency). Dampers and weights are placed throughout the helicopter.
that are tuned to the primary rotor angular rate. Through years of experience, effective dampeners, specified weights, and weight locations have been identified for commercial and military helicopters. While aerelastic models are improving, in order to predict the weights, the location and total weights are generally finalized during helicopter ground testing.

When the mass distribution and stiffness method cannot provide further vibration reduction, vibration absorbers are placed in order to achieve either (1) overall vibration reduction in the helicopter fuselage (2) provide reduction in local areas of the fuselage, such as the pilot cab which also has the benefit of minimizing the vibration experienced by the vehicle’s electronics [10]. Another vibration absorber example is a centrifugal pendulum bifilar absorber, which places weights on the rotating side of the rotor head and is able to reduce the in-plane forces.

The problem with the passive technique is that it is not robust to cases outside the tuned conditions. Furthermore, passive control techniques treat the vibration loads after they have been generated, rather than removing them before transferring throughout the helicopter. Due to this shortcoming, research has been done over the years in designing actuation and control techniques for active vibration reduction.

1.2.2 Active Methods

In contrast to the passive control method, active control methods minimize vibration by modifying the rotor blade parameters, such as the blade pitch and the blade twist. Traditionally, the goal is to modify the blade parameters to produce a sinusoidal input that counteracts the sinusoidal motion of the vibration. Specifically, how must the system adjust the magnitude and phase of a sinusoidal input to eliminate the vibration. The principle relationship that active control methods consider is the Fourier series expansion of the blade pitch represented as

\[ \theta = \theta_0 + \theta_1 r + \theta_{1s} \sin(\Omega t) + \theta_{1c} \cos(\Omega t) + \theta_{2s} \sin(2\Omega t) + \theta_{2c} \cos(2\Omega t) + \ldots \]  

(1.2)
where $\theta_0$ is the collective pitch angle, $\theta_1$ is the built-in linear blade twist rate (function of the radius location), $r$ is the radial location, $\theta_{1c}$ is the lateral cyclic pitch angle, and $\theta_{1s}$ is the longitudinal cyclic pitch angle. Higher order terms are represented by $\theta_{2s}$ and $\theta_{2c}$. Active control methods can be designed to adjust any of the coefficients in Equation 1.2 depending on the actuator type.

The relative location of the actuator on the classic swashplate (rotating or fixed side of the swashplate) in Figure 1-3 breaks active control methods into two types: Higher Harmonic Control (HHC) and Individual Blade Control (IBC). Active control techniques have been proven in simulation [16, 23, 64, 28], wind tunnel testing, [18, 53, 49, 37, 38, 22, 54], and flight testing [32, 3].

![Articulated rotor hub control system](image)

Figure 1-3: Articulated rotor hub control system [34]

In the HHC method, the additional control system is on the fixed side of the swashplate and works through the same inputs as the primary controls. In the IBC method, the control system is on the rotating side of the swashplate. A visual comparison of HHC and IBC actuation techniques is shown in Figure 1-4. Three common methods of IBC are active pitch link, active rotor twist/morphing, and active trailing edge flap. Table 1.1 provides a brief list of research programs that have demonstrated active control methods. The scale of each program is provided. The final IBC method shown in Table 1.1, Hub Mounted Vibration Suppressor (HMVS), represents a relatively new technique in actively rejecting vibrations on the rotating side of the
swashplate and was flight tested in 2014. The actuator nullifies all fuselage $N$ per rev vibrations due to \textit{in-plane} vibratory hub loads \cite{3,56,59}. The active isolator replaces a passive bifilar that sits atop the mast. In hover, the two masses are placed exactly
180 degrees apart so that the loads cancel each other. When in forward flight, the masses are moved closer to each other so that they produce a load that counters the load that the rotor is producing.

1.3 Simulation Options

A key aspect of this current thesis is the evaluation of the vibration reducing controllers in a nonlinear simulation environment. Within the helicopter community, simulators typically fall into two categories: flight simulation/dynamics codes and aeromechanical code. The latter category can also be described in literature as comprehensive analysis simulation codes. It is important to compare the different capabilities that each of these types of codes provide.

Flight dynamic codes are time domain simulation environments based on a mathematical model that determines the dynamical response of the helicopter due to pilot inputs or disturbances, such as wind gusts. The math models have six degrees of freedom and calculate the force balance of the helicopter equations of motions. During a typical time based simulation, the primary calculation of the simulator is the orientation of the flight vehicle due to the input. In addition to the helicopter equations of motion, many popular helicopter flight dynamics codes also incorporate the calculation of blade rotor and lag dynamics, nonlinear blade aerodynamics, and nonlinear fuselage aerodynamic forces and moments. Flight dynamics codes are also used for real-time simulation, including large dome simulators. Two examples of flight dynamics codes used in the helicopter industry is the Sikorsky General Helicopter (GENHEL) Flight Dynamics Simulation \cite{21} and the Boeing Helicopters Simulation (BHSIM) \cite{33}. It is important to note that GENHEL is distributed by the U.S. Army Aeroflightdynamics Directorate (AFDD) for U.S. citizens pursuing academic endeavors.

Aeromechanical codes differ from flight dynamics codes by their increased complexity and their expansive analysis capabilities. On the topic of complexity, aeromechanical codes are by definition comprehensive in the various disciplines that are
combined to accurately calculate helicopter dynamics. These codes combine advanced models in helicopter geometry, structure, dynamics, aeroelastic, and aerodynamics. Two example modeling capabilities that are typically available in an aeromechanical code, but not in a flight dynamics code, is advanced modeling of the rotor air inflow and elastic (rather than rigid) rotor blades. In addition to advanced modeling capabilities, comprehensive analysis codes also provide the ability to build and evaluate numerous types of helicopter rotor configurations. Furthermore, these codes can perform multiple analyses including trim, blade motion, air loading, structural loads, vibration, noise, aeroelastic stability, and flight dynamics. These comprehensive codes are not a familiar tool for helicopter control engineers because such a high level of modeling detail is not typically required. In addition, aeromechanical codes are computationally intense and do not fit into the real-time simulation environment that control engineers are accustomed to using. However, for analyses that require accurate calculation of blade motion and vibration throughout the helicopter, advanced simulation tools must be used. Examples of aeromechanical codes in industry are TECH-02 used by Boeing Helicopters, CAMRAD II developed by Johnson Aeronautics, FLIGHTLAB developed by Advanced Rotorcraft Technology (ART), UMARC used by the University of Maryland, and RCAS used by the U.S. Army [26]. As with the flight dynamics code GENHEL, RCAS is distributed by the U.S. Army Aeroflightdynamics Directorate (through Advanced Rotorcraft Technology, Inc.) for U.S. citizens pursuing academic endeavors.

1.4 Research Approach

The approach to eliminate the $N$ per rev frequency is to actively control it using feedback control theory. The fundamental approach to this problem is shown in Figure [1-5] In this framework, the $N$ per rev frequency is viewed as a disturbance, $d$, to the plant model of the helicopter, $G(s; \rho(t))$. Therefore, the problem of alleviating the $N$ per rev frequency is similar to the classic disturbance rejection problem, as noted by Hall and Wereley [19]. The output, $y$, of the framework is then used as a
feedback signal to the controller, $K(s; \rho(t))$. The reference signal, $r$, is set to zero in this thesis. The $\rho(t)$ variable represents the parameters that vary as a function of time, but are also measureable during real-time operation. Both the controller and the plant are a function of a time varying parameter.

The first step in demonstrating a vibration rejecting controller is the development of a nonlinear helicopter model. The nonlinear simulator used in this thesis is the Rotorcraft Comprehensive Analysis System (RCAS) aeromechanical code, which was developed for the U.S. Army Aeroflightdynamics Directorate (AFDD) by Advanced Rotorcraft Technology, Inc. (ART) [26, 44]. The helicopter model developed in the RCAS environment is a UH-60 Black Hawk helicopter.

Using the RCAS nonlinear simulator, dynamics of the UH-60 helicopter are determined using frequency based system identification methods at various flight conditions using a wind tunnel trim (no flapping and constant thrust coefficient). The frequency based transfer functions are referred to as the Equivalent Transfer Function Estimates (ETFE). The primary transfer functions of interest are the relationship between the fixed-side rotor hub forces and moments and the primary control inputs (collective, lateral cyclic, and longitudinal cyclic). The focus in this thesis is the relationship between the out-of-plane fixed-side rotor hub force, $F_z$, and the collective input, since the input of the collective has the largest impact on the out-of-plane hub force. The ETFEs are then used to create linear time-invariant (LTI) state space models using

Figure 1-5: Disturbance rejection framework for helicopter vibration problem
optimization based system identification methods.

This thesis develops three different vibration controllers, specifically, the Hall and Wereley Continuous Time Higher Harmonic Controller (CTHHC) [19], the $H_\infty$ controller [16], and a controller developed in the Linear Parameter Varying (LPV) synthesis framework for varying values of the advance ratio. The CTHHC and $H_\infty$ controllers are controllers that have been developed by other researchers and represent the baseline controllers. The design of a helicopter vibration reducing controller in the LPV framework using traditional system identification techniques is the unique contribution of this work. The LPV framework [60] determines a parameter-dependent Lyapunov function to assure stability and performance guarantees over the flight envelope. The LPV design method differs from classic control methods, where gain-scheduling is used to develop a controller for the full envelope. In the classic technique, the discussion of stability is ignored or passively assumed by making the argument that the variations in the helicopter dynamics are slow enough such that the design of the controller at various flight conditions is maintained [43]. The three different controllers are evaluated in the RCAS aeromechanical simulation environment at fixed and maneuvering flight conditions.

1.5 Thesis Outline

The remainder of this thesis is organized as follows:

Chapter 2 covers the development of the UH-60 helicopter model used in the RCAS simulation environment, including a detailed discussion of the structural model and the impact of aerodynamic modeling options. The chapter includes the calculation of the baseline vibration levels, frequency based ETFEs, and the LTI state space models that are used for the design of a vibration controller.

Chapter 3 provides a theoretical background of the plant model assumptions that have been used to solve the helicopter vibration problem, specifically a linear quasi-steady model and a linear time-invariant model. The theory and design of the two baseline controllers, the CTHHC and $H_\infty$ controllers, is provided for the LTI state
space models developed in Chapter 2.

Chapter 4 discusses the theory of the Linear Parameter Varying control synthesis method. The synthesis method is then applied to the UH-60 model to develop an LPV controller.

Chapter 5 uses the two baseline controllers and the LPV controller in the RCAS simulation environment. Specifically, the controllers are evaluated and compared at fixed and maneuvering flight conditions. There are five different maneuvering flight conditions considered that range from mild to aggressive maneuvers.

Chapter 6 provides a summary of the thesis, main contributions, and areas for future research.
Chapter 2

Helicopter Model Development

The first step for designing an effective helicopter vibration controller is the development of a representative and accurate model. Effective simulation of a helicopter for the vibration problem must be performed using an aeromechanical code, such as Rotorcraft Comprehensive Analysis System (RCAS). Further information about RCAS’ history, modeling structure, analysis techniques, and time-integration method is provided in Appendix A. This chapter provides a detailed overview of the UH-60 helicopter model constructed and simulated in the RCAS environment. Baseline vibration levels are then developed for a set of flight simulation test points. Following the baseline vibration data is the development of the Equivalent Transfer Function Estimate (ETFE) of the helicopter plant models. The final section of this chapter develops a linear time-invariant (LTI) state space model that is used for the synthesis design of a vibration rejecting controller.

2.1 Nonlinear Simulation Model Development

The major tool used for the evaluation of helicopter vibration controllers is RCAS, which is an aeromechanical simulation code that provides the user with the essential modeling capabilities. This section provides an overview of the helicopter modeled in the RCAS environment. The primary simulation analysis capability used is the nonlinear transient maneuver option, as this provides the most realistic framework by
which development of a vibration reducing controller for a full-scale helicopter would be developed. Therefore, the process follows that which would be completed on a full-scale helicopter.

2.1.1 Overview of Model

The helicopter modeled for assessment of helicopter reducing technologies is the Sikorsky UH-60 Black Hawk medium-lift utility helicopter. The UH-60 is a four-bladed rotor with a full articulated rotor head, where the blade moves in the flapping, lead-lag, and pitch axes. The flap and lead-lag axes use an elastomeric bearing. The rotor head also features a lead-lag damper, which is used to prevent excess forward and aft movement of each blade. Table 2.1 lists key specifications of the UH-60 that are used to model the helicopter.

The UH-60 Black Hawk was the chosen helicopter for multiple reasons. The first reason is that the UH-60 helicopter has been used heavily by the U.S. Army since its inception in 1979 and is planned to be part of the helicopter fleet until the mid-2030s when the Future Vertical Lift advanced rotorcraft is planned to replace it [57]. The

<table>
<thead>
<tr>
<th>Parameter</th>
<th>UH-60 Specification</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor Diameter</td>
<td>53.67</td>
<td>feet</td>
</tr>
<tr>
<td>Rotor Head Diameter</td>
<td>6.76</td>
<td>feet</td>
</tr>
<tr>
<td>Rotor Rate</td>
<td>27</td>
<td>rad/s</td>
</tr>
<tr>
<td>Main Rotor Blade Chord</td>
<td>1.73</td>
<td>feet</td>
</tr>
<tr>
<td>Main Rotor Blade Twist</td>
<td>9.5</td>
<td>degrees</td>
</tr>
<tr>
<td>Tail Rotor Diameter</td>
<td>11</td>
<td>feet</td>
</tr>
<tr>
<td>Tail Rotor Rate</td>
<td>125</td>
<td>rad/s</td>
</tr>
<tr>
<td>Fuselage Length</td>
<td>50.625</td>
<td>feet</td>
</tr>
<tr>
<td>Wheel Base</td>
<td>29</td>
<td>feet</td>
</tr>
<tr>
<td>Geometric Shaft Angle</td>
<td>-14.7 (forward)</td>
<td>degrees</td>
</tr>
</tbody>
</table>
second major reason is that the UH-60 helicopter has been the choice helicopter by researchers for several decades. Therefore, a significant body of literature exists that can be used to validate the modeling. A key research program is the UH-60 Airloads Flight Project, which was jointly sponsored by NASA and the U.S. Army, was carried out from 1984 to 1994. This flight test program provided airloads data over the entire flight envelope of the UH-60 at multiple flight conditions, such as steady level flight and maneuvering, with a total of over 900 flight conditions. Some example data developed from this key program is blade structural loads, blade motion, control loads, and aircraft states [29].

2.1.2 RCAS UH-60 Model

Models are built in the RCAS environment as a finite element model. The user must specify the location of nodes and define the interaction between the nodes. RCAS has the capability to define the links between the nodes as inelastic (rigid) bars or elastic bars. The full UH-60 modeled in RCAS is shown in Figure 2-1.
The model completely represents the rotor fuselage, vertical tail, horizontal stabilizer, tail rotor, head rotor, and the front and rear landing gear. The fuselage, vertical tail, horizontal stabilizer, and tail rotor blades are modeled using rigid bars. The rotorhead includes complete modeling of the swashplate controls. Further detail of the finite element model of the rotor hub is shown in Figure 2-2.

It is worth comparing the modeled RCAS rotor head to the actual rotor head of the UH-60. Figures 2-3 and 2-4 display the actual UH-60 rotor head and the RCAS version of the rotor head elements, respectively. It can be seen that the RCAS version of the rotor head captures all elements of the actual UH-60 rotor head. The blade root hinge is offset from the axis of rotation and the majority of the bars are rigid. One of the key modeling benefits of using RCAS is the ability to model the blade as a nonlinear beam. Another modeling tool of RCAS is the ability to include springs, dampers, hinge, and sliding elements. The pitch link of each blade includes both spring and damping elements. The pitch control of the rotor model is represented by a slide element, which is where all of the commands from the pilot are ported through. The lead-lag damper is modeled as a damping element. The lag, flap, and pitch hinges all coincide at the same point and the modeling order is lag, flap, and pitch. The order of the hinges is important for accurate modeling of rotor dynamical
responses. RCAS has the capability to include spring and damper constants for each of the hinge elements. None of the hinge elements have spring constants, and only the pitch hinge has a damper constant.

A key aspect of modeling the rotor head to be comparable to an actual UH-60 is the selection of the constants for the spring and damping element in Figure 2-4. Ormiston [39] conducted an extensive investigation of the proper settings for a UH-60
rotorhead in order to match flight test data. The flight test data that was used in Ormiston’s investigation was from the UH-60 Airloads Flight Project. Table 2.2 lists the rotor head damper and spring constants, which come from the work of Ormiston.

A top view of the UH-60 rotorhead and the blade is shown in Figure 2-5. There are twelve nonlinear beam elements and ten aerodynamic panels used to represent each of the elastic blades. The final beam on the far right represents the tip of the helicopter blade.

Table 2.2: UH-60 RCAS model constants

<table>
<thead>
<tr>
<th>Element</th>
<th>Coefficient</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flap Hinge</td>
<td>Damper Constant</td>
<td>20</td>
<td>ft-lb-sec/rad</td>
</tr>
<tr>
<td>Pitch Link Spring</td>
<td>Spring Constant</td>
<td>62,631</td>
<td>lb/ft</td>
</tr>
<tr>
<td>Pitch Link Damper</td>
<td>Damper Constant</td>
<td>240</td>
<td>lb-sec/ft</td>
</tr>
<tr>
<td>Lead-Lag Damper</td>
<td>Damper Constant</td>
<td>2,500</td>
<td>lb-sec/ft</td>
</tr>
</tbody>
</table>

Figure 2-5: Top view of RCAS UH-60 rotorhead and blade

2.1.3 Comparison of Structural Model to Flight Test Data

A key aspect of all model development efforts is the validation of the model against a truth dataset. The goal is to develop a model with the closest possible similarity to the actual UH-60 helicopter. The validation of the helicopter structural model is conducted using a fan plot, which displays the blade modal frequencies as a function of the rotor rotation rate. The aerodynamics model is excluded in this analysis. This
comparison therefore validates the structural model, specifically, the location of the model nodes, rigid bars, and elastic rotor blade models. To produce a fan plot in RCAS the periodic solution analysis method is used and the blade modal frequencies are output for multiple rotor rates. The fan plot for the RCAS model is shown in Figure 2-6. It is important that the blade modal frequencies do not coincide with vibration modes, represented by the black dashed lines, since that can cause large amplification of the hub vibration loads.

The results of the fan plot in Figure 2-6 are presented in Table 2.3 for the key blade modal frequencies. Also presented in the table are the blade modal frequencies as determined by Ormiston [39] and a NASA modal analysis effort [20]. The results

![Figure 2-6: RCAS UH-60 blade modal frequencies on fan plot](image)
Table 2.3: UH-60 RCAS blade modal frequencies

<table>
<thead>
<tr>
<th>Blade Mode</th>
<th>RCAS Model (per rev)</th>
<th>Ormiston 2004 (per rev)</th>
<th>NASA 1990 (per rev)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lag 1P</td>
<td>0.27</td>
<td>0.25</td>
<td>-</td>
</tr>
<tr>
<td>Flap 1P</td>
<td>0.95</td>
<td>1.05</td>
<td>1.12</td>
</tr>
<tr>
<td>Flap 2P</td>
<td>2.63</td>
<td>2.8</td>
<td>3.02</td>
</tr>
<tr>
<td>Torsion 1P</td>
<td>4.07</td>
<td>3.8</td>
<td>-</td>
</tr>
<tr>
<td>Lag 2P</td>
<td>4.81</td>
<td>4.8</td>
<td>-</td>
</tr>
<tr>
<td>Flap 3P</td>
<td>5.81</td>
<td>5.3</td>
<td>6.05</td>
</tr>
<tr>
<td>Flap 4P</td>
<td>7.69</td>
<td>7.8</td>
<td>9.93</td>
</tr>
</tbody>
</table>

from Ormiston are from an RCAS model of the UH-60 and the NASA results are directly from the NASA Airloads Flight Project. In general, the two analyses are in good agreement.

2.1.4 Impact of Aerodynamic Modeling Selections

The previous section has shown that the structural modeling of the RCAS model performs similarly to other RCAS modeling efforts and real flight test data of the UH-60. Another key decision is the type of modeling used for the aerodynamic model of the main rotor. The aerodynamic options include linear or nonlinear aerodynamics, uniform or dynamic inflow model, and steady or unsteady aerodynamics. Another important modeling selection is the choice of inelastic (rigid) or elastic rotor blades. Figure 2-4 and 2-5 uses elastic beams to model the rotor blade. As noted by Abraham et al. [29], in order to have accurate prediction of the $N$ per rev hub loads, the model must have a dynamic inflow and have elastic rotor blades. This section will demonstrate the impact of aerodynamic modeling choices on the baseline $N$ per rev forces as a function of the advance ratio, which will culminate in the final model selections.

Figure 2-7 shows the impact of various aerodynamic modeling choices on the
Figure 2-7: Impact of aerodynamic modeling on thrust 4/rev force

fixed-side thrust hub four per rev force, $F_2$, amplitude for a trim-to-target solution. The model was trimmed for $C_T/\sigma = 0.075$ and zero blade flapping ($\beta_{1s} = \beta_{1c} = 0^\circ$). All of the results have elastic blades and linear aerodynamics ($C_{L\alpha} = 5.71/\text{deg}, C_{L0} = 0, C_M = 0.01, C_D = 0.008$). Note that the uniform inflow model is a three state inflow model and the dynamic inflow model is the 36 state Peters and He Inflow model.

The most critical modeling option is the inflow model choice, where the uniform inflow model is not able to capture the variation in the vertical force when transitioning from hover to forward flight, as predicted in Figure 1-2. Furthermore, modeling unsteady aerodynamics has an impact on high-speed flight vibration levels. Note that most flight dynamics codes do not include either modeling option, further indicating the necessity of using an aeromechanical code while working on the helicopter vibration problem.

The final aerodynamic modeling selections are elastic rotor blades, linear and unsteady aerodynamics, and a 36 state Peters and He dynamic inflow model.
2.2 Plant Data from RCAS Simulation

As part of developing a controller over the helicopter flight envelope, it is important to characterize the vibration levels. This section is concerned with the steps to develop the baseline vibration levels and the Equivalent Transfer Function Estimates of the helicopter plant model for various flight conditions. The ETFEs are frequency based representations of the plant model.

This thesis considers the impact of varying advance ratio, $\mu$, thrust coefficient to rotor solidity ratio (blade loading), $C_T/\sigma$, and the total rotor inflow ratio, $\lambda$, on the rotor dynamics and baseline vibration levels. Rotor solidity, $\sigma$, is defined as the ratio of the rotor blade area to the area of the rotor. The total rotor inflow ratio (positive down through the rotor disk) is

$$\lambda = \frac{V\sin(\alpha_s) + \nu}{\Omega R} \quad (2.1)$$

where $\nu$ is the component of the induced velocity that is normal to the rotor tip-path plane.

A diagram of the vibration control scheme implemented in RCAS is shown in Figure [2-8]. The vibration controller output is summed with the pilot collective control, passed through the control mixer, and input to the helicopter through the helicopter fixed-side hub links. The primary vibration considered is the thrust hub force, $F_z$, located on the fixed-side of the swashplate. Note that there is no cross channel mixing in the control mixer modeled in RCAS. A collective pilot input adjusting the final pedal command to the tail rotor is an example of cross channel mixing.
2.2.1 Modeling Setup

The UH-60 model shown in Figure 2-1 is set up in the simulation as a wind tunnel model, meaning the helicopter is fixed in space at a reference node (center of gravity) with no active translational or rotational degrees of freedom. The RCAS analysis method used is the nonlinear (maneuver) analysis. Therefore, the process replicates a wind tunnel test, where only time domain data is collected.

The range of flight conditions examined is shown in Table 2.4. The range of advance ratio setting is calculated from the range of airspeed settings, which are the actual inputs into RCAS. The range of airspeed settings is $V_{AS} \in [100, 270]$ feet per second. A detailed list of flight conditions examined is provided in Table 2.5. The same geometric shaft angle, $\alpha_s = -14.7^\circ$, is used for all test points.

<table>
<thead>
<tr>
<th>Flight Parameter</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advance Ratio, $\mu$</td>
<td>0.13 — 0.36</td>
</tr>
<tr>
<td>Blade Loading, $C_T/\sigma$</td>
<td>0.06 — 0.09</td>
</tr>
<tr>
<td>Total Inflow Ratio, $\lambda$</td>
<td>0.022 — 0.030</td>
</tr>
</tbody>
</table>
Table 2.5: RCAS simulation baseline flight test points

<table>
<thead>
<tr>
<th>Test Number</th>
<th>Advance Ratio $\mu$</th>
<th>Blade Loading $C_T/\sigma$</th>
<th>Total Inflow Ratio $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.13</td>
<td>0.060</td>
<td>Varies</td>
</tr>
<tr>
<td>2</td>
<td>0.20</td>
<td>0.060</td>
<td>Varies</td>
</tr>
<tr>
<td>3</td>
<td>0.26</td>
<td>0.060</td>
<td>Varies</td>
</tr>
<tr>
<td>4</td>
<td>0.36</td>
<td>0.060</td>
<td>Varies</td>
</tr>
<tr>
<td>5</td>
<td>0.13</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>6</td>
<td>0.16</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>7</td>
<td>0.20</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>8</td>
<td>0.23</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>9</td>
<td>0.26</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>10</td>
<td>0.30</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>11</td>
<td>0.33</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>12</td>
<td>0.36</td>
<td>0.075</td>
<td>Varies</td>
</tr>
<tr>
<td>13</td>
<td>0.13</td>
<td>0.090</td>
<td>Varies</td>
</tr>
<tr>
<td>14</td>
<td>0.20</td>
<td>0.090</td>
<td>Varies</td>
</tr>
<tr>
<td>15</td>
<td>0.26</td>
<td>0.090</td>
<td>Varies</td>
</tr>
<tr>
<td>16</td>
<td>0.33</td>
<td>0.090</td>
<td>Varies</td>
</tr>
<tr>
<td>17</td>
<td>0.13</td>
<td>Varies</td>
<td>0.022</td>
</tr>
<tr>
<td>18</td>
<td>0.20</td>
<td>Varies</td>
<td>0.022</td>
</tr>
<tr>
<td>19</td>
<td>0.26</td>
<td>Varies</td>
<td>0.022</td>
</tr>
<tr>
<td>20</td>
<td>0.36</td>
<td>Varies</td>
<td>0.022</td>
</tr>
<tr>
<td>21</td>
<td>0.13</td>
<td>Varies</td>
<td>0.025</td>
</tr>
<tr>
<td>22</td>
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<td>Varies</td>
<td>0.025</td>
</tr>
<tr>
<td>23</td>
<td>0.26</td>
<td>Varies</td>
<td>0.025</td>
</tr>
<tr>
<td>24</td>
<td>0.36</td>
<td>Varies</td>
<td>0.025</td>
</tr>
<tr>
<td>25</td>
<td>0.13</td>
<td>Varies</td>
<td>0.030</td>
</tr>
<tr>
<td>26</td>
<td>0.20</td>
<td>Varies</td>
<td>0.030</td>
</tr>
<tr>
<td>27</td>
<td>0.26</td>
<td>Varies</td>
<td>0.030</td>
</tr>
</tbody>
</table>

40
The first task for developing baseline vibrations levels and ETFEs of the UH-60 in RCAS is to calculate the pilot control settings for the various flight test points. The method to perform this in RCAS is to use the trim-to-target analysis option, which finds the trim variable setting for a given trim target. For this analysis, RCAS determines the lateral cyclic, longitudinal cyclic, and the collective setting for the desired blade loading (or inflow ratio) with the flapping angles set to zero ($\beta_{1s} = \beta_{1c} = 0^\circ$). The choice of blade loading or inflow ratio depends on the test point number in Table 2.5. For each test point in Table 2.5, there is a corresponding lateral and longitudinal cyclic and collective setting.

This trim technique is consistent with the method used by the SMART wind tunnel test program [54, 53] and it is a recommended method in conversations with Dr. Robert Ormiston of the U.S. Army Aeroflightdynamics Directorate. Other trim methods do exist including, (1) Adjusting the cyclic and collective settings to achieve a desired rotor lift, hub pitch moment, and hub roll moment (control propulsive force through shaft angle) [38] and (2) Adjusting the cyclic and collective settings to achieve a desired rotor lift, propulsive force, and hub roll moment [23].

Figures 2-9, 2-10, and 2-11 show the pilot collective, lateral cyclic, and longitudinal cyclic settings for all of the simulation test points. These parameters are used as the initial conditions of each respective simulation run to ensure that the specified blade loading, total inflow ratio, and flapping angles are maintained.
Figure 2-9: Collective initial conditions in RCAS simulation

Figure 2-10: Lateral cyclic initial conditions in RCAS simulation
2.2.2 Baseline Vibration

The first simulation analysis is the baseline vibration test, which is simply an open-loop test that is used to characterize the baseline hub forces of the helicopter. The baseline hub vibration forces are determined by performing a nonlinear maneuver simulation lasting 30 seconds for each test point. The final ten seconds of the maneuver are used to calculate the baseline forces, which allows any transients to settle. Below is a detailed step-by-step process to calculate the baseline hub forces:

1. Collect maneuver data from RCAS sampled at 256 samples per rotor revolution \( (T_s = 9.09025 \times 10^{-4} \text{ seconds}) \). The sampling rate, in samples per revolution, must be an integer number. If this is not achievable, the data must be regularized.

2. Segment the time series data to achieve an integer number of sinusoidal passes. This is performed by finding the index at which the rotor passes over the \( \psi = 0^\circ \) azimuthal position.
3. Perform a Fast Fourier transform (FFT) on the segmented vibration data. Note that the FFT must be performed for all data points from the data start to \( n - 1 \), where \( n \) is the number of segmented data points.

4. Convert the result from arbitrary units to the units of the input data (pounds force). Convert to the root mean square (RMS) pounds force by

\[
\text{RMS} = \frac{\sqrt{\Phi^T \Phi}}{n}
\]

where \( \Phi \) is the FFT of the segmented data. The FFT data can also be converted to the amplitude of the vibration sinusoidal wave.

5. Output the converted data at the harmonic frequencies. The output data includes the frequency content in \( \omega \in [N - 0.05, N + 0.05] \) per rev, where \( N \) is the harmonic number.

Without careful attention to the sample rate, data segmentation, and using \( n - 1 \) datapoints, the FFT of the baseline vibrations will spread over multiple frequencies on both sides of the key harmonic frequency, known as spectral leakage.

Results from the baseline vibration force analysis are shown in Table 2.6 and Figures 2-12 and 2-13. The results indicate that the harmonics contributing the most to the overall hub force vibration levels are the 4 and 8 per rev frequency, with the largest being the 4 per rev frequency. As is visible in Figures 2-12 and 2-13, there are harmonics at the even per rev frequencies for all the cases shown, but only the 4 and 8 per rev frequencies contribute significantly. Another noticeable trend is that increasing advance ratio is met by an increase in the harmonic force levels. The largest force at an advance ratio of 0.13 and 0.36 is the longitudinal in-plane force, \( F_x \). The largest force at an advance ratio of 0.26 is the out-of-plane force, \( F_z \). While this analysis indicates that it would be best to focus on designing a controller for in-plane forces, as will be later shown, using the collective input has the biggest impact on the vertical hub force. Therefore, the goal is to reduce the out-of-plane hub force.
Table 2.6: Baseline vibration amplitude data at $\mu = 0.13, 0.26, 0.36$ ($C_T/\sigma = 0.075$)

<table>
<thead>
<tr>
<th>Harmonic #</th>
<th>$\mu = 0.13$</th>
<th>$\mu = 0.26$</th>
<th>$\mu = 0.36$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F_x$ (lb)</td>
<td>$F_y$ (lb)</td>
<td>$F_z$ (lb)</td>
</tr>
<tr>
<td></td>
<td>$F_x$ (lb)</td>
<td>$F_y$ (lb)</td>
<td>$F_z$ (lb)</td>
</tr>
<tr>
<td></td>
<td>$F_x$ (lb)</td>
<td>$F_y$ (lb)</td>
<td>$F_z$ (lb)</td>
</tr>
<tr>
<td>0</td>
<td>14.5</td>
<td>245.8</td>
<td>14385.0</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>157.5</td>
<td>148.8</td>
<td>150.8</td>
</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>7</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>8</td>
<td>8.0</td>
<td>5.3</td>
<td>2.4</td>
</tr>
</tbody>
</table>

Figure 2-12: FFT amplitude data for hub $F_z$ forces for $\mu = 0.13$ ($C_T/\sigma = 0.075$)
Figure 2-13: FFT amplitude data for hub $F_z$ forces for $\mu = 0.36 \ (C_T/\sigma = 0.075)$

Figure 2-14 displays the 4 per rev out-of-plane amplitude force as a function of varying advance ratio. The constant lines correspond to constant blade loading coefficient and total inflow ratio. A key trend line to point out in Figure 2-14 is the case of $C_T/\sigma = 0.075$, which follows a similar trend as predicted in Figure 1-2. There is an increase as the helicopter progresses from hover to forward flight, then the force reduces when the helicopter levels out, and then the force further increases as the helicopter approaches its maximum airspeed. The trend line of $C_T/\sigma = 0.09$ is similar, but the magnitude is higher. The lowest blade loading coefficient trend line has a lower force magnitude and the lowest point occurs at a lower advance ratio. The two lowest trend lines of constant inflow ratio exhibit the expected response for varying advance ratio. The inflow ratio of $\lambda = 0.030$ has the largest force.

Figure 2-15 shows another representation (varying $C_T/\sigma$) of the baseline data and plots the trend lines in Figure 2-14 that have a constant $C_T/\sigma$. The results indicate that increasing the blade loading coefficient increases the $N$ per rev force.
Figure 2-14: Baseline 4 per rev thrust hub force amplitude for varying advance ratio

Figure 2-15: Baseline 4 per rev thrust hub force amplitude for varying blade loading
Figure 2-16 displays the baseline vibration data as a function of varying λ and plots the trend lines in 2-14 that have a constant λ. The results are similar to the varying $C_T/\sigma$ analysis, but the highest inflow ratio has greater vibration levels.

These results indicate that the blade loading coefficient and the total inflow ratio have the same effect on the baseline vibration data. This is as expected because the total inflow ratio in Equation 2.1 can be reduced and approximated [21] for high forward speeds by

$$\lambda = \frac{V \sin(\alpha_s) + \nu}{\Omega R}$$

$$= \mu \tan(\alpha_s) + \lambda_i$$

$$= \mu \tan(\alpha_s) + \frac{C_T}{2\sqrt{\mu^2 + \lambda^2}}$$

$$\approx \mu \tan(\alpha_s) + \frac{C_T}{2\mu}$$

Figure 2-16: Baseline 4 per rev thrust hub force amplitude for varying inflow ratio
where $\lambda_i$ is the induced inflow ratio. This reduction indicates that increasing the blade loading coefficient is similar to increasing the total inflow ratio.

### 2.2.3 Development of Equivalent Transfer Function Estimates

Following the characterization of the baseline loads of the UH-60 in the RCAS nonlinear simulation environment, the next step is to characterize the relationship between the collective input and the out-of-plane hub force vibration, defined as the Equivalent Transfer Function Estimate. This transfer function must be developed using traditional frequency based system identification methods, specifically a linear sinusoidal frequency sweep \[31\]. The ETFEs are developed in RCAS by inputting sinusoidal frequency sweeps into the pilot collective using a nonlinear maneuver analysis method. Details of the frequency signal input into the RCAS simulation environment are shown in Table 2.7. Note that the frequency sweep was split into two segments because of data size restrictions. The duration of each frequency sweep was chosen such that the frequency coherence is acceptable, where the coherence function provides a measure of the system identification accuracy and input-output linearity.

Table 2.7: RCAS linear sinusoidal frequency sweep parameters

<table>
<thead>
<tr>
<th>Sweep Segment</th>
<th>Staring Frequency (per rev)</th>
<th>Ending Frequency (per rev)</th>
<th>Duration (seconds)</th>
<th>Amplitude (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>6.965</td>
<td>85</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>6.035</td>
<td>13</td>
<td>85</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The process of developing a frequency-based representation of the out-of-plane vibration to collective input plant is similar to the process outlined for calculating the baseline vibration forces, as shown in Section 2.2.2. Below is a detailed step-by-step process to develop a frequency based representation of the helicopter plant:

1. Collect frequency sweep maneuver data from RCAS sampled at 256 samples per rotor revolution ($T_s = 9.09025 \times 10^{-4}$ seconds).
2. Subtract out the baseline input (determined from trim-to-target pilot control settings) and baseline force data from the sinusoidal sweep input and sweep force data, respectively. The baseline force data should be subtracted from the sweep force data as a function of the rotor azimuth position.

3. Segment the input and force data to achieve an integer number of sinusoidal passes.

4. Perform an FFT on the segmented vibration data and input sinusoidal signal.

5. Calculate the autospectral density of the input, \( \hat{\Phi}_{UU}(j\omega) \), and the cross spectral density of the input and output, \( \hat{\Phi}_{UY}(j\omega) \). These parameters are found by multiplying the respective FFT datasets. Note that a frequency based smoothing filter is included for the results shown herein.

6. The ETFE is calculated as the ratio of the previously calculated autospectral density estimates \([31]\) at each frequency by

\[
\hat{G}(j\omega) = \frac{\hat{\Phi}_{UY}(j\omega)}{\hat{\Phi}_{UU}(j\omega)} \tag{2.4}
\]

Figure 2-17 displays the ETFEs for the case of \( C_T/\sigma = 0.075 \) with varying advance ratios, which was developed using the previously outlined process. A key attribute to point out in these transfer function estimates is the presence of rotor dynamics and expected blade modal frequencies. The magnitude plot of the ETFE clearly shows the following blade modal frequencies: Flap 1P (0.95 per rev), Flap 2P (2.63 per rev), Lag 2P (4.81 per rev), and Flap 4P (7.69 per rev).
ETFEs have been developed for all of the simulation test points shown in Table 2.5. All of the ETFEs are included at the conclusion of this chapter in Figures 2-22 to 2-35. Many of the test points in Table 2.5 are repeated in the ETFE figures in order to display the impact of varying advance ratio, blade loading, and total inflow ratio. The results in general show that the largest impact on the rotor dynamics are changes in the advance ratio. This is clearly evident in Figure 2-17 where the rotor dynamics change significantly at 4.7 per rev and in the 6 to 8 per rev range. Note that at the key harmonic frequencies (4 and 8 per rev), the variations in the plant estimates are not as significant, but variations do exist. For example at $\mu = 0.13$, the magnitude at the 4 per rev is 556.1 lb/deg, whereas for $\mu = 0.36$, the magnitude at the 4 per rev is 847.4 lb/deg.

Another key trend to note in the ETFEs is the impact of the blade loading and total inflow ratio, which are evident in the figures with constant blade loading or total inflow ratio (Figures 2-24 to 2-28, Figures 2-32 to 2-35). The results show that these parameters have an impact on the rotor dynamics at high frequencies, specifically
greater than 6 per rev. Furthermore, the variations in the rotor dynamics for the blade loading and total inflow ratio parameters are similar.

The results from the development of the ETFEs indicate that a full envelope vibration controller should consider both the advance ratio and some sort of rotor inflow parameter, which could be represented by some combination of the blade loading coefficient and total inflow ratio. It is the contention of this work that at the dominant baseline load harmonic vibration (4 per rev), as shown in Table 2.6, the rotor dynamics are primarily a function of the advance ratio.

To further illustrate the importance of using aeromechanical codes for vibration analysis, as opposed to flight dynamics codes, Figure 2-18 displays the ETFE of a UH-60 model using the GENHEL simulation code. Note that there is a slight variation in the magnitude units. GENHEL is not able to capture any high frequency rotor dynamics, except for the Flap 1P, which is evident in the magnitude plot and the phase drop off in the phase plot. However, at the key harmonic (4 and 8 per rev) frequencies, no rotor dynamics are evident. The advanced modeling capability in RCAS is necessary for vibration analysis.

Figure 2-18: ETFE using GENHEL at $\mu = 0.26$
2.3 Development of LTI State Space Models

The equivalent transfer function estimates provide a frequency based representation of the helicopter plant. In order to use advanced synthesis control methods, an LTI state space model representation of the helicopter plant must be developed. Furthermore, a state space representation can be easily developed from an LTI transfer function (input-output) model representation. This section reviews the process to develop the LTI state space models.

2.3.1 LTI Model Representation

Prior to discussing the LTI model development, it is important to demonstrate that the helicopter plant can effectively be represented by a linear time-invariant model. One method to determine if the helicopter model plant is LTI, as opposed to linear time periodic (LTP), is to develop a set of plots that display the force level for sinusoidal inputs with various phases. These plots are developed by inputting an open loop sinusoid into each of the pilot inputs (collective, lateral cyclic, and longitudinal cyclic) at a rate of $4\Omega$ with phase settings in the range of $0^\circ \leq \phi \leq 360^\circ$. The input to the pilot controls is

$$\theta_{4P} = A \sin(4\Omega t + \phi)$$ (2.5)

where $A$ is the amplitude specified by the user. Figure 2-19 displays the result of varying the input phase for the RCAS model at $\mu = 0.26$ where the amplitude is set to $0.1^\circ$. Each of the hub forces is analyzed using the FFT analysis outlined in Section 2.2.2 to determine the $4\text{ per rev}$ harmonic force level.

Each of the plots in Figure 2-19 has a baseline point, which corresponds to the baseline vibration force, and twelve phase settings. Specific phase settings are highlighted by color and line type, which indicates that certain phase settings can reduce the baseline hub force. For example, for the $F_z$ output with collective input case, vibration is reduced when $\phi = 210^\circ$. Open-loop control can be used to eliminate
vibration with a phase setting of $\phi = 210^\circ$. A larger amplitude would be required to reject the 4 per rev vibration. This control strategy is the primary method that previous research efforts using aeromechanical codes have used to reduce vibration.

Another key observation from Figure 2-19 is that the collective input has the largest impact on a hub force, specifically the vertical hub force, $F_z$. Figure 2-19 demonstrates that the most effective means to reduce the $N$ per rev harmonic is to focus on reducing the out-of-plane hub force using the collective input.

Figure 2-19 also includes a solid red line that runs through each of phase setting points. The red line is determined by calculating the elements of the $T$ matrix from the quasi-steady plant model discussed in Chapter 3. Then sinusoidal pilot inputs are passed through the quasi-steady plant model to determine the corresponding force

![Figure 2-19: Open-loop 4 per rev sinusoidal inputs with varying phases at $\mu = 0.26$. $\phi = 0^\circ$ (solid), $\phi = 90^\circ$ (dash), $\phi = 180^\circ$ (dash-dot), $\phi = 270^\circ$ (dot)]
levels. The quasi-linear plant variables can be represented by

\[ z_n = \begin{bmatrix} F_{x,c} & F_{y,c} & F_{z,c} \end{bmatrix}^T \] (2.6)

\[ z_0 = \begin{bmatrix} F_{x,c,0} & F_{y,c,0} & F_{z,c,0} \end{bmatrix}^T \] (2.7)

\[ \theta_n = \begin{bmatrix} \theta^c_{\text{Lat}} & \theta^c_{\text{Lon}} & \theta^s_{\text{Lon}} & \theta^s_{\text{Col}} \end{bmatrix}^T \] (2.8)

where \( z_0 \) is the baseline vibration level, \( z_n \) is the vibration due to the baseline and input, and \( \theta_n \) is the sinusoidal input. Note that the elements of \( z_0 \) are the baseline points in Figure 2-19. This shows that the linear assumption made by the \( T \) matrix method results in a generally good representation of the helicopter plant. In addition to work conducted by Fan [16] and Hall and Wereley [19], this current analysis demonstrates that the periodicity of the helicopter forces can largely be ignored in developing an effective vibration reducing controller.

### 2.3.2 LTI State Space Model Development

The LTI state space models of the helicopter plant are found using an optimization based system identification method [31] that develops an input-output model representation from the ETFE data sets. The goal of this section is to develop an input-output model of the ETFE, which is assumed to take the form

\[ \hat{G}(s; \theta) = \sum_{i=1}^{M} \frac{b_{1i}s + b_{2i}}{s^2 + a_{1i}s + a_{2i}} + c \] (2.9)

where \( M \) is the number of second order systems, \( b_{1,2} \) are numerator coefficients, \( a_{1,2} \) are denominator coefficients, and \( c \) is an additional model coefficient. \( \theta \) is the vector of coefficients ordered as

\[ \theta = [b_{11} b_{12} 1 a_{11} a_{12} \ldots, b_{M1} b_{M2} 1 a_{M1} a_{M2} c] \] (2.10)

The task is to find the numerator and denominator coefficients, and additional model coefficient such that the input-output model matches the ETFE. Therefore, this prob-
lem can be setup as a constrained optimization problem, formulated as

$$\min_{\theta} J$$  \hspace{1cm} (2.11)

such that

$$a_{ij} > 0$$  \hspace{1cm} (2.12)

$$a_{\text{lower}} < a_{ij} < a_{\text{upper}}$$  \hspace{1cm} (2.13)

$$b_{\text{lower}} < b_{ij} < b_{\text{upper}}$$  \hspace{1cm} (2.14)

$$c_{\text{lower}} < c < c_{\text{upper}}$$  \hspace{1cm} (2.15)

where $J$ is the cost function. The cost function is chosen to be the weighted squared output error, defined as

$$J = \|E(j\omega; \theta)\|^2_{W(j\omega)} = \int_{\omega_{\text{start}}}^{\omega_{\text{end}}} |W(j\omega)E(j\omega; \theta)|^2 d\omega$$  \hspace{1cm} (2.16)

where $E$ is the output given as

$$E(j\omega; \theta) = Y(j\omega) - \hat{G}(j\omega; \theta)U(j\omega)$$  \hspace{1cm} (2.17)

where $U$ and $Y$ are the input and output data, respectively, collected from the RCAS simulation and are represented by the ETFEs. Note that the cost function is evaluated at each frequency in the range of $0 \leq \omega \leq 12$ per rev. $W$ is a frequency dependent weighting function defined as

$$W(j\omega) = \hat{G}_{\text{ETFE}}^{-1}$$  \hspace{1cm} (2.18)

The first constraint shown in Equation 2.12 is a stability constraint, which requires that the input-output model be stable. This constraint is permitted because it is known from RCAS simulation that the plant is stable. The constraints on $a_{ij}$ and $b_{ij}$ in Equations 2.13 and 2.14 are introduced to keep the coefficients close to those
found in the initialization step. It is found in practice that the coefficients calculated during initialization perform well, but can be further refined by optimization. In this work, the upper and lower constraints are specified to be within a percentage of the initialization coefficients.

To facilitate better performing optimization, the optimization solvers should be given the gradient of the cost function, $\partial J / \partial \theta$. However, since the cost function in Equation 2.16 is written as a computer program, due to evaluating the cost function at a discrete set of frequencies, the gradient of the cost function is not trivially found. The gradient can be found using automatic differentiation software, which takes the computer program for the cost function and develops a corresponding computer program for the cost function gradient.

The optimization framework previously outlined requires initial parameters for the model coefficients. The strategy for finding the initial parameters [16] is to define a model, over a specified frequency range, for each peak ($M$ peaks) in the magnitude of the ETFE, such that each model takes the form

$$G_{IC} = \frac{\beta_i 2}{s^2 + \alpha_i 1 s + \alpha_i 2}, \ i = 1, \ldots, M \quad (2.19)$$

where each of the peak models is fitted to the ETFE models using the least squares method. Note that each of the models is fitted over a unique range of frequencies that is defined by the user. The input to the initialization routine by the user is a vector of lower and upper frequency ranges to develop each of the $M$ models. The result from the initialization step is a set of coefficients (the coefficients in Equation 2.19) for each of the peak models. The coefficients in the input-output model in Equation 2.9 can be related to the initial parameters by

$$b_{i1} = 0, \ b_{i2} = \beta_i 2, \ a_{i1} = \alpha_i 1, \ a_{i2} = \alpha_i 2, \ c = 0 \quad (2.20)$$

In general, the results from the initialization find a close solution and can be refined further using optimization. If the model developed by the optimization performs poorly, the user can adjust the frequencies specified in the input vector or add more
models (i.e., a larger $M$ value). One drawback to the model outlined in Equation 2.9 is that the result is not strictly proper (due to the $c$ term). For more successful controller synthesis design, the model in Equation 2.9 can become strictly proper by adding an additional pole at $s = 50$, which is included in the model by multiplying the model by

$$\frac{50}{s + 50} \tag{2.21}$$

This additional pole has little effect on the identified transfer function in the frequency range of interest. The identified input-output model is converted to state space form using the built-in MATLAB function, `canon(G,'modal')`, which places the poles of the model on the diagonal of the state matrix.

The LTI state space models have been developed for key ETFEs, namely the case of $C_T/\sigma = 0.075$ with varying advance ratios. All of the state space models are 15th order. Figures 2-20 and 2-21 compare the ETFEs and the identified state space models for the case of $\mu = 0.13$ and 0.36, respectively. The identified models closely match the ETFEs, and are able to capture the blade modes evident in the ETFE. Furthermore, the LTI model is able to capture the notch in the transfer function for the case of $\mu = 0.36$. More importantly, the state space models perform well at the key harmonics of interest, 4 and 8 per rev frequencies..
Figure 2-20: ETFE and LTI state space model comparison for $\mu = 0.13$

Figure 2-21: ETFE and LTI state space model comparison for $\mu = 0.36$
2.4 Summary

This chapter provides a detailed overview of the UH-60 model constructed in the RCAS environment. The model includes the fuselage, vertical tail, horizontal tail, tail rotor, main rotor, and detailed modeling of the swashplate control. The results from the modal analysis indicates that the structural model of the helicopter performs similarly to flight test data. A review of the aerodynamic modeling options is provided, where it is shown that the use of elastic rotor blades, linear and unsteady aerodynamics, and a 36 state Peters and He dynamic inflow model effectively capture the vibration of a helicopter.

The remainder of the chapter discusses the baseline vibration forces and the development of the Equivalent Transfer Function Estimates for all of the simulation flight test points. Analysis of these forces indicates that a helicopter vibration controller should consider both the advance ratio and a rotor inflow parameter. However, at the dominant baseline load harmonic vibration frequency (4 per rev), the controller should be a function of the advance ratio. Based on analysis in this section, the controller should focus on reducing the vertical hub force, $F_z$, using a collective input. The final portion of this chapter develops an LTI state space model of the ETFE for $C_T/\sigma = 0.075$ and varying $\mu$. In general, the optimization process presented herein results in an acceptable model, as compared to the corresponding ETFE. The state space model developed in this chapter provides the framework for the next step, which is the design of the baseline and LPV controllers to address the helicopter vibration problem.
Figure 2-22: ETFE for varying $\mu$ at $C_T/\sigma = 0.060$

Figure 2-23: ETFE for varying $\mu$ at $C_T/\sigma = 0.090$
Figure 2-24: ETFE for varying $C_T/\sigma$ at $\mu = 0.13$

Figure 2-25: ETFE for varying $C_T/\sigma$ at $\mu = 0.20$
Figure 2-26: ETFE for varying $C_T/\sigma$ at $\mu = 0.26$

Figure 2-27: ETFE for varying $C_T/\sigma$ at $\mu = 0.33$
Figure 2-28: ETFE for varying $C_T/\sigma$ at $\mu = 0.36$

Figure 2-29: ETFE for varying $\mu$ at $\lambda = 0.022$
Figure 2-30: ETFE for varying $\mu$ at $\lambda = 0.025$.

Figure 2-31: ETFE for varying $\mu$ at $\lambda = 0.030$. 
Figure 2-32: ETFE for varying $\lambda$ at $\mu = 0.13$

Figure 2-33: ETFE for varying $\lambda$ at $\mu = 0.20$
Figure 2-34: ETFE for varying $\lambda$ at $\mu = 0.26$

Figure 2-35: ETFE for varying $\lambda$ at $\mu = 0.36$
Chapter 3

Baseline Vibration Controllers

This chapter discusses the theoretical background and design of the baseline vibration controllers. The first vibration controller is the Hall and Wereley Continuous Time Higher Harmonic Controller (CTHHC) \[19\], which has been developed from the work of Shaw \[47\]. The second controller uses the full-order $\mathcal{H}_\infty$ synthesis framework outlined by Fan \[16\] for the helicopter vibration problem. These controllers are used to compare to the Linear Parameter Varying (LPV) controller in the RCAS nonlinear simulation environment. This chapter begins with a discussion of the types of plant assumptions that have been used for vibration reduction. The Shaw framework assumes a linear quasi-steady plant and the CTHHC controller assumes a linear time invariant plant. The remainder of the chapter outlines the design of the CTHHC and $\mathcal{H}_\infty$ controller using the LTI state space models developed in Chapter 2.

3.1 Vibration Controller Model Types

3.1.1 Linear Quasi-Steady Plant

The primary model assumption that has been used for the helicopter vibration problem is the linear quasi-steady plant

\[ z_n = z_0 + T\theta_n \]  (3.1)
where $z_0$ is the baseline vibration level, $z_n$ is the vibration due to the baseline and input, $\theta_n$ is the input, $T$ relates the input to vibration levels, and $n$ is the harmonic number. It is the goal in this model to eliminate the baseline vibration level, $z_0$. This model assumes a global perspective on the vibrations, that is, the model is linear over the entire range of the control. The vibration and input parameters in Equation 3.1 are vectors and can represent multiple harmonics. An expanded form of Equation 3.1 for three harmonics is

$$
\begin{bmatrix}
    z_{2c} \\
    z_{2s} \\
    z_{3c} \\
    z_{3s} \\
    z_{4c} \\
    z_{4s}
\end{bmatrix}
= 
\begin{bmatrix}
    \frac{\partial z_{2c}}{\partial \theta_{2c}} & \frac{\partial z_{2c}}{\partial \theta_{2s}} & \frac{\partial z_{2c}}{\partial \theta_{3c}} & \frac{\partial z_{2c}}{\partial \theta_{3s}} & \frac{\partial z_{2c}}{\partial \theta_{4c}} & \frac{\partial z_{2c}}{\partial \theta_{4s}} \\
    \frac{\partial z_{2s}}{\partial \theta_{2c}} & \frac{\partial z_{2s}}{\partial \theta_{2s}} & \frac{\partial z_{2s}}{\partial \theta_{3c}} & \frac{\partial z_{2s}}{\partial \theta_{3s}} & \frac{\partial z_{2s}}{\partial \theta_{4c}} & \frac{\partial z_{2s}}{\partial \theta_{4s}} \\
    \frac{\partial z_{3c}}{\partial \theta_{2c}} & \frac{\partial z_{3c}}{\partial \theta_{2s}} & \frac{\partial z_{3c}}{\partial \theta_{3c}} & \frac{\partial z_{3c}}{\partial \theta_{3s}} & \frac{\partial z_{3c}}{\partial \theta_{4c}} & \frac{\partial z_{3c}}{\partial \theta_{4s}} \\
    \frac{\partial z_{3s}}{\partial \theta_{2c}} & \frac{\partial z_{3s}}{\partial \theta_{2s}} & \frac{\partial z_{3s}}{\partial \theta_{3c}} & \frac{\partial z_{3s}}{\partial \theta_{3s}} & \frac{\partial z_{3s}}{\partial \theta_{4c}} & \frac{\partial z_{3s}}{\partial \theta_{4s}} \\
    \frac{\partial z_{4c}}{\partial \theta_{2c}} & \frac{\partial z_{4c}}{\partial \theta_{2s}} & \frac{\partial z_{4c}}{\partial \theta_{3c}} & \frac{\partial z_{4c}}{\partial \theta_{3s}} & \frac{\partial z_{4c}}{\partial \theta_{4c}} & \frac{\partial z_{4c}}{\partial \theta_{4s}} \\
    \frac{\partial z_{4s}}{\partial \theta_{2c}} & \frac{\partial z_{4s}}{\partial \theta_{2s}} & \frac{\partial z_{4s}}{\partial \theta_{3c}} & \frac{\partial z_{4s}}{\partial \theta_{3s}} & \frac{\partial z_{4s}}{\partial \theta_{4c}} & \frac{\partial z_{4s}}{\partial \theta_{4s}}
\end{bmatrix}
\begin{bmatrix}
    \theta_{2c} \\
    \theta_{2s} \\
    \theta_{3c} \\
    \theta_{3s} \\
    \theta_{4c} \\
    \theta_{4s}
\end{bmatrix}
$$

(3.2)

where the 2, 3, and 4 subscripts represent the second, third, and fourth per rev harmonic components. The relationship in Equation 3.2 is common in HHC literature and represents the formulation for a three bladed rotor.

In the framework outlined by Shaw [47], the baseline vibrations, $z_0$, can be eliminated by applying the discrete control of

$$
\Delta \theta_n = -T^{-1}z_n
$$

(3.3)

In Shaw’s work, since the controller is discrete, the actual input coefficient is

$$
\theta_{ni} = \theta_{ni-1} + \Delta \theta_{ni}
$$

(3.4)

$$
= \theta_{ni-1} - T^{-1}z_{ni}
$$

$$
= \begin{bmatrix}
    \theta_{nc} \\
    \theta_{ns}
\end{bmatrix}_{i-1} - T^{-1}
\begin{bmatrix}
    z_{nc} \\
    z_{ns}
\end{bmatrix}_i
$$

where $i$ is the discrete time steps. Figure 3-1 shows the schematic of Shaw’s discrete
HHC algorithm, where $\Omega$ is the rotor angular rate. In this algorithm, the vibration levels are modulated by cosine and sine in order to determine the contribution of each component to the total vibration. The demodulated vibration signal is then integrated over one sampling period, $T$, which must be an integer number. The control adjustment, $\Delta \theta_c$ and $\Delta \theta_s$, is simply found by solving Equation 3.3. The control adjustments are sampled at a rate of $T$ and the current input coefficient is calculated as in Equation 3.4. The signals are then demodulated and summed together to produce the input to the plant, $G_p$.

The final input to the plant is

$$\theta_n = \theta_{nc_i}\cos(N\Omega t) + \theta_{ns_i}\sin(N\Omega t) \quad (3.5)$$

which can be written in equivalent form as

$$\theta_n = M_i \sin(N\Omega t + \phi_i) \quad (3.6)$$
where the magnitude, $M_i$, and phase, $\phi_i$, terms are

$$M_i = \sqrt{\theta_{nc_i}^2 + \theta_{ns_i}^2} \quad (3.7)$$

$$\phi_i = \arctan \left( \frac{\theta_{nc_i}}{\theta_{ns_i}} \right) \quad (3.8)$$

The quasi-steady plant method assumes perfectly linearity over the entire range of the control input. However, the vibration is a function of rotor dynamics, which change based on the flight condition. For example, the $T$ matrix and $z_0$ in Equations 3.1 and 3.2 change based on the advance ratio of the helicopter. Therefore, when using the quasi-steady plant assumption to control vibration, the $T$ matrix must be identified for varying flight conditions. Equation 3.3 can be rewritten as

$$\Delta \theta_n = -\hat{T}^{-1}z_n \quad (3.9)$$

where the estimated transfer function matrix, $\hat{T}$, must either be calculated apriori (offline) or must be identified actively (online) by the algorithm. In the case of offline identification, the $T$ matrix can be estimated using the least-squares method, but this method assumes constant parameters for a given vibration measurement. Many early methods of vibration reduction used the least-squares method for identification[25]. A majority of current applications using the quasi-steady plant estimate the transfer function matrix online, which is called an adaptive vibration controller.

Shaw’s original algorithm estimates the $T$ matrix online using the Kalman filter algorithm, a recursive least-squares method. An estimate of the $T$ matrix [25] is

$$\hat{T}_i = \hat{T}_{i-1} + K_i(z_{ni} - \theta_{n_i}^T \hat{T}_{i-1}) \quad (3.10)$$

where $K_i$ is the gain vector obtained by

$$K_i = \frac{P_i \theta_{n_i}}{R_i} \quad (3.11)$$
and $P_i$ is the covariance matrix after the measurement of the vibration level and is

$$P_i = P_{i-1} - \frac{P_{i-1}\theta_n^T \theta_n P_{i-1}}{R_i + \theta_n^T P_{i-1} \theta_n}$$

(3.12)

where $R_i$ is the process noise covariance.

As noted by Molusis et al. [36], an engineer must determine whether to implement an adaptive controller or a gain-scheduled controller for the vibration reduction problem. In the case of a gain-scheduled controller, the $T$ matrix is determined offline for multiple flight conditions, and the $T$ matrix is interpolated online based on the measured flight condition. Molusis et al. provide three advantages of using an adaptive controller over a gain-scheduled controller: (1) The adaptive controller does not require measurement of the current flight condition (such as the forward speed) (2) Adaptive controllers have the potential for achieving lower vibration levels (3) Adaptive controllers do not require storage of $T$ matrices that can be large and vary as a function of flight condition. Molusis et al. also note that gain-scheduled controllers are easier to implement.

As will be discussed in the following section, Hall and Wereley [19] have shown that gain-scheduled controllers can be found to achieve favorable vibration levels at a fixed flight condition (thereby negating Molusis et al.’s second point) and the controllers can effectively respond to variations in flight conditions.

It is worth noting that the quasi-steady plant assumption for helicopter vibration reduction is a method that is used for many current vibration reduction research efforts, such as the SMART program [53], the LORD/Sikorsky HMVS technology [3], and the Boeing MRRAP program [12].
3.1.2 Linear Time Invariant Plant

An alternate model assumption for helicopter vibration reduction is the linear time-invariant plant which is defined as

\[ \dot{x}_p(t) = A_p x_p(t) + B_p u(t) \]
\[ y_p(t) = C_p x_p(t) + D_p u(t) \]
\[ z(t) = y_p(t) + d(t) \]  

(3.13)

where \( x_p \in \mathbb{R}^{n_p} \) is the state vector of the plant, \( u \in \mathbb{R}^{n_u} \) is the control input, \( y_p \in \mathbb{R}^{n_y} \) is the output of the plant, \( z \in \mathbb{R}^{n_y} \) is the measurable vibration output (desired to be zero), \( d \in \mathbb{R}^{n_y} \) is the harmonic disturbance with frequency \( N \Omega \), and \( A_p, B_p, C_p, \) and \( D_p \) are matrices with corresponding dimensions. The state space representation of Equation 3.13 can be represented in input-output transfer function form by using

\[ G(s) = \frac{z(s)}{u(s)} = C_p (sI - A_p)^{-1} B_p + D_p \]  

(3.14)

where the transfer function relates the commanded input signal, \( u \), and the measurable vibration output, \( z \). The overarching assumption with this model is that the periodicity effects on the rotor dynamics are small. This fact has been proven in both simulation [16], whirl test stand [42], and wind tunnel tests [18, 50], validating the assumption of a linear time invariant model for the helicopter vibration problem.

The focus of this section is on the work of Hall and Wereley [19], who begin with the quasi-steady plant assumption and discrete-time algorithm developed by Shaw, and transform the problem to the linear time invariant plant assumption.

The first step in Hall and Wereley’s transformation of Shaw’s work to a continuous time controller is noting that the integration over one sampling period can be replaced by a continuous integrator and the sample and hold results from the Equation 3.4 can be sampled continuously. This realization transforms Shaw’s algorithm in Figure 3-1 to the continuous time representation in Figure 3-2 which is herein referred to as the modulation/demodulation scheme.
The continuous time version of Shaw’s algorithm can be applied to a single input/single output (SISO) LTI system \( G(s) \) by noting that the \( T \) matrix of helicopter rotors is skew symmetric where, assuming a SISO system, the \( T \) matrix takes the form

\[
T = \begin{bmatrix}
T_{cc} & T_{cs} \\
T_{sc} & T_{ss}
\end{bmatrix}
\quad (3.15)
\]

where

\[
T_{cc} = T_{ss} = \text{Real} \{G(jN\Omega)\} \quad (3.16)
\]
\[
T_{cs} = -T_{sc} = \text{Imag} \{G(jN\Omega)\} \quad (3.17)
\]

The inverse relationship of the \( T \) matrix, \( T^{-1} \), is

\[
T^{-1} = \begin{bmatrix}
a & b \\
-b & a
\end{bmatrix}
\quad (3.18)
\]
where

\[ a = \frac{\text{Real} \{G(jN\Omega)\}}{|G(jN\Omega)|^2} = \text{Real} \left\{ \frac{1}{G(jN\Omega)} \right\} \quad (3.19) \]

\[ b = -\frac{\text{Imag} \{G(jN\Omega)\}}{|G(jN\Omega)|^2} = -\text{Imag} \left\{ \frac{1}{G(jN\Omega)} \right\} \quad (3.20) \]

As derived in [19], the linear transfer function of the continuous time higher harmonic controller in Figure 3-2 is

\[ K(s) = \frac{-u(s)}{z(s)} = 2k \frac{as + bN\Omega}{s^2 + (N\Omega)^2} \quad (3.21) \]

where \( k \) is the controller gain. The continuous time controller places two poles on the imaginary axis at the \( N \) per rev frequency and a zero on the \( -\frac{bN\Omega}{a} \) location of the real axis. The structure of this controller is the same that would be chosen for the classic narrow band disturbance rejection. For direct comparison to Shaw’s discrete algorithm, the gain should be chosen as

\[ k = \frac{1}{T} \quad (3.22) \]

In actual application of this method, for favorable gain and phase margins, the desired gain has been found to be

\[ k \in \left[ \frac{1}{2T}, \frac{1}{5T} \right] \quad (3.23) \]

A state-space form of the controller in Equation 3.21 is represented by

\[ \dot{x}_c(t) = A_c x_c(t) + B_c u(t) \quad (3.24) \]

\[ y_c(t) = C_c x_c(t) + D_c u(t) \]
where the state space matrices for a SISO system are

\[
A_c = \begin{bmatrix}
0 & n\Omega \\
-n\Omega & 0
\end{bmatrix}
\]  

(3.25)

\[
B_c = [a \ b]^T
\]  

(3.26)

\[
C_c = [1 \ 0]
\]  

(3.27)

\[
D_c = 0
\]  

(3.28)

By forming the controller as a traditional LTI controller in Equation 3.21, the block diagram takes the traditional closed loop feedback form, as shown in Figure 3-3.

Figure 3-3: Hall and Wereley’s linear HHC controller

It is possible to extend the controller outlined in Equation 3.21 to reduce vibration at different harmonic frequencies. Shin, Cesnik, and Hall [50] developed a multi-frequency rejecting controller, which is simply just the summation of the individual continuous time linear controllers represented as

\[
K(s) = \sum_i k_i \frac{a_i s + b_i N_i \Omega}{s^2 + (N_i \Omega)^2}
\]

(3.29)

By formulating the discrete controller into a continuous controller, it is possible to use traditional LTI analysis and synthesis techniques to achieve complete disturbance rejection. In the traditional analysis of disturbance rejection, the designer must look
at the magnitude of the sensitivity transfer function in the frequency domain to determine if the undesired disturbance frequency is minimized. The sensitivity transfer function takes the form of

\[ S(s) = \frac{1}{1 + L(s)} \]  

(3.30)

where

\[ L(s) = G(s)K(s) \]  

(3.31)

is the loop gain transfer function. The goal is to design a controller such that the sensitivity function is zero at the disturbance frequency. Therefore, in the framework of the current helicopter problem, the goal is for the sensitivity transfer function at the \( N \) per rev frequency to be

\[ S(jN\Omega) = 0 \]  

(3.32)

Due to the framework outlined by Hall and Wereley in Equation 3.21, complete rejection of the harmonic disturbance at the \( N \) per rev frequency is guaranteed. This is ensured because the poles of the controller, \( K(s) \), are located at \( s = \pm jN\Omega \), therefore the controller gain at the harmonic frequency is infinite.

### 3.2 CTHHC Controller Development

The first baseline controller considered is the Hall and Wereley CTHHC controller [19], which has been demonstrated in simulation [16], whirl test stand [42], and wind tunnel tests [18, 50]. The RCAS baseline loads analysis in Chapter 2 indicates that a vibration controller should be designed to attenuate the dominant baseline harmonic load, which varies primarily as a function of the advance ratio. Therefore, the primary focus of this section is designing the CTHHC at fixed advance ratios in \( \mu \in [0.13, 0.36] \) for \( C_T/\sigma = 0.075 \) to attenuate the 4 per rev harmonic frequency.
The procedure for designing a controller using the CTHHC framework is relatively straightforward, and is a useful controller method because of its satisfactory performance and ease of implementation. The first step is to create a frequency based representation of the helicopter plant, which is accomplished using the Equivalent Transfer Function Estimate (ETFE) process outlined in Chapter 2. The controller is then formed as in Equation 3.21 by determining the \( a \) and \( b \) coefficients. The CTHHC controller can be represented as a linear controller, as shown in Figure 3-3, or in the modulation/demodulation scheme, as shown in Figure 3-2. The benefit of using the modulation/demodulation scheme is it allows for matching the controller poles to the rotor frequency, which is useful for a full-scale helicopter where the rotor rate can vary. The results presented in this thesis use the modulation/demodulation scheme.

Figures 3-4, 3-5, and 3-6 display the loop gain bode plot, sensitivity transfer function magnitude plot, and the Nichols plot for the CTHHC controller, respectively, at \( \mu = 0.13, 0.20, 0.26, 0.33, \) and \( 0.36. \) These transfer functions are calculated for each frequency in the range of interest, \( 0 \leq \omega \leq 12 \text{ per rev}. \)

Figure 3-4 displays the loop gain Bode plot for each of the controllers designed at a single condition. The notable result from this figure is the gain is large at the frequency of interest (4 per rev), which is followed by a phase shift of 180 degrees. This response is typical of systems with poles on the \( j\omega \) axis.
Figure 3-4: CTHHC loop gain Bode plot for multiple $\mu$ ($C_T/\sigma = 0.075$)

Figure 3-5 displays the absolute value of the sensitivity transfer function magnitude plot, where a notch is formed at the 4 per rev frequency ($|S(jN\omega)| = 0$). This response predicts that there will be near complete disturbance rejection of the 4 per rev frequency content. Note that there is a slight increase in the magnitude on either side of the notch. This feature will be explained below.

Figure 3-6 displays the Nichols plot of the loop gain transfer function. The information is the same as that shown in Figure 3-4, but the representation in the Nichols plot provides a convenient depiction of multiple control engineering parameters. Similar to the Bode plot, movement along the Nichols plot represents an increase in the frequency. The movement of this Nicholas plot is indicated by the frequency points markers (3.9/rev, 3.95/rev, 4.1/rev, 4.3/rev, 4.7/rev), where 0/rev is off the right edge of the plot and 12/rev is off the left edge of the plot. The dark lines on the background of the Nichols plot indicate the sensitivity transfer function at a given frequency. The closer to the center of the plot the loop gain is, the smaller the margins are, and therefore a magnification in the sensitivity plot magnitude. This is evident
Figure 3-5: CTHHC sensitivity transfer function for multiple $\mu$ ($C_T/\sigma = 0.075$)

Figure 3-6: CTHHC Nichols plot for multiple $\mu$ ($C_T/\sigma = 0.075$)
in this Nichols plot where at a frequency of $\sim 4.6$ per rev, the loop gain is closest to the center of the plot. This location corresponds to the largest value in the sensitivity transfer function plot in Figure 3-5.

The vertical location (gain margin) of the Nichols plot can be modified by adjusting the gain ($k$) in Equation 3.21. The gain used in the design results above is $k = 1/(4T)$, where $T$ is the period of the rotor rate in units of second per revolution. The Nichols plot can be adjusted horizontally (phase margin) by adjusting the phase, $\phi$, of the plant dynamics that the $a$ and $b$ coefficients are generated. If it is desired to adjust the phase margin, the $a$ and $b$ coefficients should be generated from a plant defined by

$$G_{\text{shifted}}(j\omega) = G(j\omega)e^{-j\phi} \quad (3.33)$$

which is the technique used by Shin, Cesnik, and Hall [50]. No phase shift was required for the design of the CTHHC controller in this thesis, since the controller is centered around $-180^\circ$.

The performance of the CTHHC controller can be predicted by the maximum sensitivity value ($S_{\text{max}}$) and the width of the notch at $|S(j\omega)| = 1$. More discussion of these parameters is included for design of a controller using the $\mathcal{H}_\infty$ control synthesis technique. The maximum sensitivity of the current CTHHC controllers is 1.145 and the notch width is 0.06 per rev for $\mu = 0.26$.

### 3.3 $\mathcal{H}_\infty$ Controller Development

The CTHHC controller has been proven to be a successful technique to eliminate helicopter vibration. However, the technique does not allow the choice in the shape of the notch at the dominant frequency of interest. The only control parameters that the CTHHC controller can impact are the gain and phase margin through the control gain, $k$. It is desirable to specify all parameters that impact the shape of the notch in the sensitivity transfer function. This desire leads to the use of modern frequency
based synthesis techniques. This section discusses the design of the second baseline vibration controller which follows the work of Fan [16] who adapted the $\mathcal{H}_\infty$ control synthesis method to the helicopter vibration problem. This controller is applied to the LTI state space models developed in Chapter 2.

The performance of the vibration controllers can be characterized by the shape of the sensitivity transfer function. Figure 3-7 displays the ideal linear notch shape and the corresponding key parameters, namely the peak sensitivity, $S_{\text{max}}$, and the bandwidth, $\omega_B$. The bandwidth of the controller is defined as the half width of the ideal linear notch at $|S(j\omega)| = 1$. The width of the notch impacts the response of controller to changes in the harmonic disturbance. It is desirable for the notch to be as wide as possible because a greater bandwidth corresponds to a faster response.

Another key parameter in defining the notch shape is the peak sensitivity of the sensitivity transfer function, $S_{\text{max}}$, which is defined as

$$S_{\text{max}} = \sup_{\omega} |S(j\omega)| = ||S||_{\infty}$$

![Figure 3-7: Ideal linear notch shape and parameters](image)

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The peak sensitivity does not impact the disturbance rejection capabilities of the vibration controllers at the key harmonic frequency (4 per rev). However, the peak sensitivity does predict the maximum amplification of vibration at frequencies other than the key harmonic. In addition, a lower peak sensitivity can be related to improved gain and phase margins [16]. The gain margin (GM) and phase margin (PM) are bounded by

\[
\begin{align*}
GM & \geq \frac{S_{\text{max}}}{S_{\text{max}} - 1} \\
PM & \geq 2 \arcsin \left( \frac{1}{2S_{\text{max}}} \right)
\end{align*}
\quad (3.34) \quad (3.35)
\]

By way of example, if the peak sensitivity is \( S_{\text{max}} = 1.1 \), the bounds on the margins are \( GM \geq 11 \) and \( PM \geq 54.1^\circ \).

It is desirable to minimize the peak sensitivity and increase the bandwidth as much as possible. In the case of the CTHHC controller, only the gain and phase margins are impacted through the choice of the controller gain, \( k \). The goal is to use the \( \mathcal{H}_\infty \) frequency based controller synthesis technique to find a controller such that the desired bandwidth is achieved with the smallest possible peak sensitivity.

### 3.3.1 Linear Notch Weighing Filter

The \( \mathcal{H}_\infty \) synthesis framework requires that the desired shape of the inverse of key transfer functions, such as the sensitivity transfer function, be specified in the frequency domain. This current section discusses the development of an \( \mathcal{H}_\infty \) performance weighting filter based on the desired bandwidth and peak sensitivity of the ideal linear notch.

In the \( \mathcal{H}_\infty \) framework, the vibration controller meets the desired peak sensitivity and bandwidth specifications if and only if

\[
|S(j\omega)| \leq \mathcal{V}(\omega; \omega_B, S_{\text{max}}), \quad \forall \omega \in \mathbb{R}
\quad (3.36)
\]

where \( \mathcal{V}(\omega; \omega_B, S_{\text{max}}) \) is the continuous frequency based representation of the linear
notch filter. Equation 3.36 can be equivalently written as

$$|\mathcal{W}(j\omega; \omega_B, S_{\text{max}})S(j\omega)| \leq 1, \ \forall \omega \in \mathbb{R} \quad (3.37)$$

where $\mathcal{W}(j\omega; \omega_B, S_{\text{max}})$ is the performance weighting filter used by the $\mathcal{H}_\infty$ synthesis method and is

$$\mathcal{W}(j\omega; \omega_B, S_{\text{max}}) = \mathcal{V}^{-1}(\omega; \omega_B, S_{\text{max}}) \quad (3.38)$$

The shape of the ideal notch filter for multiple harmonics can be expressed as

$$\mathcal{V}(\omega; \omega_B, S_{\text{max}}) = S_{\text{max}} \prod_{i=1}^{M} \left| H_m \left( \frac{j\omega - jN_i \Omega}{\omega_B S_{\text{max}}} \right) \right| \cdot \left| H_m \left( \frac{j\omega + jN_i \Omega}{\omega_B S_{\text{max}}} \right) \right| \quad (3.39)$$

where $M$ is the number of harmonic frequencies of the linear notch filter [16]. The product on the right hand side of Equation 4.11 produces two notches (one for each side of the ideal notch), for each of the harmonics of interest, that are approximations of the ideal linear notch. The product of these two notches, defined in terms of the Laplace variable ($s = j\omega$), are calculated as

$$H_m(s)H_m(-s) = \frac{-s^2 + s^4 - \ldots + (-1)^m s^{2m}}{1 - s^2 + s^4 - \ldots + (-1)^m s^{2m}} \quad (3.40)$$

where $m$ is the order of ideal linear notch approximation. Figure 3-8 displays the ideal linear notch in red and five notch approximations at $N\Omega$. It can be seen that increasing the approximation order leads to a closer estimate of the ideal linear notch. Furthermore, the notch approximation with order one performs significantly worse than the higher order approximations. The order of the notch approximation must be chosen such that the approximation performs closely to the ideal linear notch filter. While increasing the order of the approximation improves the notch filter, it comes at a cost of increased controller and computational complexity. The order selection of the linear notch approximation impacts the overall order of the $\mathcal{H}_\infty$ controller.
The finite-dimensional representation of the ideal notch filter at multiple harmonics is given by

\[ V(s) = S_{\text{max}} \prod_{i=1}^{M} H_m \left( \frac{s - jN_i \Omega}{\omega_B S_{\text{max}}} \right) \cdot \left| H_m \left( \frac{s + jN_i \Omega}{\omega_B S_{\text{max}}} \right) \right| \]  \hspace{1cm} (3.41)

Similarly, the infinite-dimensional performance weighting filter, \( W(j\omega; \omega_B, S_{\text{max}}) \), can be approximated by

\[ W_S(s) = V^{-1}(s) \]  \hspace{1cm} (3.42)

The approximated performance weighting filter, \( W_S(s) \), has poles on the imaginary axis at each of the \( M \) harmonic notch filters. As for the CTHHC controller, in Equation [3.21] poles must be located at the \( jN\Omega \) locations to reject a harmonic disturbance.
The $\mathcal{H}_\infty$ analysis problem is then to determine the controller, $K(s)$, such that

$$||Tzd||_\infty = \sup_{u(t)\neq 0} \frac{||z||_2}{||d||_2} = ||W_S||_\infty \leq 1 \quad (3.43)$$

where $z$ is the performance output and $d$ is the harmonic disturbance input. The infinity norm in the analysis problem can be interpreted as the energy gain from the input disturbance, $d$, to the output performance metric, $z$. The energy gain is also known in literature as the induced $\mathcal{L}_2$ norm.

The analysis problem in Equation 3.43 converts the helicopter vibration problem into the traditional $\mathcal{H}_\infty$ problem framework [65, 14], as shown in Figure 3-9. Furthermore, the state space realization of the $\mathcal{H}_\infty$ problem is a combination of the performance weighting filter, $W_S(s)$, and the original plant, $G(s)$, represented by the generalized augmented plant, $P$, which is given by

$$\begin{align*}
\dot{x} &= Ax + B_1d + B_2u \\
z &= C_1x + D_{11}d + D_{12}u \\
y &= C_2x + D_{21}d + D_{22}u
\end{align*} \quad (3.44)$$

where the state and output matrices are for the augmented plant, $P$. Traditionally, the $D_{11}$ and $D_{22}$ matrices are set to zero. The augmented plant represents the interconnection of the performance weighting filters and the original plant shown in Figure 3-9. Equation 3.44 can be equivalently written compactly as

$$\begin{bmatrix}
\dot{x} \\
z \\
y
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
x \\
d \\
u
\end{bmatrix} \quad (3.45)$$
3.3.2 $\mathcal{H}_\infty$ Synthesis Problem Setup

The goal of the $\mathcal{H}_\infty$ synthesis problem is to find the controller, $K(s)$, such that the closed-loop system is stable and $||T_{zd}||_\infty < \gamma = 1$. The process to perform this synthesis is to iterate over $\gamma$ and find the smallest $\gamma$ such that a solution satisfies a set of necessary and sufficient analysis conditions. The necessary and sufficient analysis conditions [14, 13] such that a controller exists and $||T_{zd}||_\infty < \gamma$ are

1. $\exists X \geq 0$ such that solves

$$A^TX +XA + C^T_1C_1 + X(\gamma^{-2}B_1^TB_1^T - B_2^TB_2^T)X = 0 \quad (3.46)$$

and

$$\Re\{\lambda_i[A + (\gamma^{-2}B_1^TB_1^T - B_2^TB_2^T)X]\} < 0 \quad (3.47)$$

2. $\exists Y \geq 0$ that solves

$$AY + YA^T + B_1B_1^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y = 0 \quad (3.48)$$

and

$$\Re\{\lambda_i[A + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)]\} < 0 \quad (3.49)$$
3. $\rho(XY) < \gamma^{-2}$, where $\rho$ is the spectral radius ($\rho(A) = \max_i |\lambda_i(A)|$) \hfill (3.50)

Once the synthesis method finds the smallest possible $\gamma$ (i.e., the algebraic Riccati equations exist and $\gamma$ is satisfied), the $\mathcal{H}_\infty$ controller is reconstructed \cite{14, 13} using the Riccati matrices, $X, Y$, and is given by

$$K(s) = \begin{bmatrix} A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)X - ZYC_2^TC_2 & ZYC_2^T \\ -B_2^TX & 0 \end{bmatrix} \hfill (3.51)$$

where $Z = (I - \gamma^{-2}YX)^{-1}$. An equivalent analysis test for the two algebraic Riccati equations in Equations 3.46 and 3.48 is to examine the location of the eigenvalues of the $\mathcal{H}_\infty$ Hamiltonian matrix. The Hamiltonian matrix corresponding to the algebraic Riccati equation in Equation 3.46, written more compactly, is defined as

$$H = \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix} \hfill (3.52)$$

A test equivalent to finding the $X \geq 0$ that satisfies the algebraic Riccati equation condition in Equation 3.46 is to ensure that no eigenvalues of the Hamiltonian matrix, $H$, are on the $j\omega$ axis. Consequently, the $\mathcal{H}_\infty$ synthesis method will fail if any poles of the augmented plant are on the imaginary axis. This restriction presents a difficulty for the current thesis, as the performance weighting filter, $W_S(s)$, has poles at the $jN\Omega$ location of the imaginary axis.

Zhou and Doyle \cite{65} proposed a technique to use the $\mathcal{H}_\infty$ synthesis method for problems where the performance weighting filters has poles on the $j\omega$ axis. Fan \cite{16} adapted the technique of Zhou and Doyle to the helicopter vibration problem. The technique is to split the weighting filter into two separate transfer functions as

$$W_S(s) = W_1(s)W_2(s) \hfill (3.53)$$

where the poles of the $W_1(s)$ transfer function are on the $j\omega$ axis and $W_2(s)$ contains the remaining poles of $W_S(s)$. Note that transfer functions must be proper but are
not required to be strictly proper. The selection of the zeros for $W_1(s)$ transfer function is not unique. The $\mathcal{H}_\infty$ synthesis is conducted as if the helicopter plant were $W_1(s)G(s)$ and the weighting performance filter were $W_2(s)$. The controller acting on this adjusted plant is $\hat{K}_\infty(s)$. The controller acting on the original plant is then

$$K_\infty(s) = \hat{K}_\infty(s)W_1(s) \quad (3.54)$$

By splitting the original performance weighting filter in two, it is possible for the traditional $\mathcal{H}_\infty$ control synthesis technique to calculate a suitable controller, $\hat{K}_\infty(s)$, and achieve complete rejection of the harmonic disturbances with poles on the $j\omega$ axis. A block diagram that displays the splitting of the performance weighting filter is shown in Figure 3-10.

### 3.3.3 $\mathcal{H}_\infty$ Synthesis Performance Results

This section discusses the results of the $\mathcal{H}_\infty$ control synthesis method using the LTI state space models developed in Chapter 2. The models considered here has an advance ratio of $\mu \in [0.13, 0.33]$ and a blade loading coefficient of $C_T/\sigma = 0.075$.

The $\mathcal{H}_\infty$ control synthesis is conducted using the built-in MATLAB function `hinfsyn`, which requires the generalized augmented matrix, $P$, and the desired $\gamma$ range. The interconnections of the augmented plant is formed in MATLAB using the `iconnect` function, which allows the user to connect the various plants and performance weighting filters using algebraic representations.

![Figure 3-10: $\mathcal{H}_\infty$ synthesis block diagram](image)
A key discussion when using the $\mathcal{H}_\infty$ control synthesis method is the order of the controller, $n_c$. By definition of the controller in Equation 3.51, the order of the controller is the same order as the augmented plant. In this current problem, the order of the controller is found by

$$n_c = n_p + n_h \times 2m$$  \hspace{1cm} (3.55)$$

where $n_p$ is the order of the state space plant, $n_h$ is the number of harmonics to be attenuated, and $m$ is the order of the ideal linear notch approximation. The order of all the LTI models is $n_p = 15$. The number of harmonics attenuated is one, which focuses on only the largest harmonic vibration disturbance. It is the task of this current section to determine the ideal value of $m$. The selection of $m$ can not solely be done using Figure 3-8 as increasing the notch order directly increases the order of the controller.

Figure 3-8 displays the performance results of the $\mathcal{H}_\infty$ control synthesis for various bandwidth values and linear notch approximations at the key harmonic frequency. Figure 3-11 displays the peak sensitivity as a function of the bandwidth. The strategy in developing the Pareto frontiers is to set the desired bandwidth, run the $\mathcal{H}_\infty$ control synthesis, and determine the corresponding peak sensitivity. The results show that increasing the bandwidth results in an increase in the peak sensitivity and increasing the order of the notch approximation reduces the peak sensitivity. As the order of the notch approximation increases, there is a point ($m = 4$) where a further increase in the order does not result in a significant improvement in the peak sensitivity. The largest improvement occurs when the order of the approximation increases from $m = 1$ to $m = 2$. The corresponding controller orders range $17 \leq n_c \leq 25$. Selecting the order of the notch approximation is a trade-off between computational complexity and performance. As shown in Figure 3-11, the order approximation of $m = 2$ performs similarly to the highest order approximations ($m = 3, 4, 5$), and has the lowest controller order ($n_c = 19$). This approximation order is used throughout this thesis for the $\mathcal{H}_\infty$ and LPV controllers.
Figure 3-12 displays the sensitivity transfer function at two advance ratios over the frequency range of interest for a bandwidth of 0.06 and an approximation order of $m = 2$. The red dashed line represents the inverse of the weighting filter, $W_S(s)$, for each specific condition. The peak sensitivity of the of the sensitivity transfer function matches the minimum of the weighting filter inverse. Furthermore, the peak sensitivities match the corresponding points in Figure 3-11.

3.3.4 Implementing $\mathcal{H}_\infty$ in RCAS

When incorporating $\mathcal{H}_\infty$ controllers in the RCAS simulation environment, the frequency warping that occurs when converting the controller from continuous to discrete must be considered. To overcome this issue, the key harmonic frequencies are pre-warped [17]. Specifically, the poles of the $W_1(s)$ weighting filter must be prewarped by

$$\omega_d = \frac{2}{T_s} \tan \left( \frac{\omega_c T_s}{2} \right)$$

where $\omega_d$ is the discrete frequency (to implement in RCAS), $\omega_c$ is the continuous frequency, and $T_s$ is the sampling period. As it relates to the current problem, given a sampling rate of 256 samples per rotor revolution ($9.09025 \times 10^{-4}$ seconds) and a continuous frequency (poles) of $4\Omega$, the discrete frequency is $108.0868 \text{ rad/s}$. This frequency must replace the harmonic poles in the state matrix of the $W_1(s)$ weighting filter in order to eliminate the frequency warping. This frequency pre-warping must also be completed when implementing the LPV controllers in RCAS.

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Figure 3-11: Pareto frontier for $\mathcal{H}_\infty$ controllers for multiple $\mu$ ($C_T/\sigma = 0.075$)
Figure 3-12: Sensitivity transfer function for $m = 2$, $BW = 0.06$ per rev for multiple $\mu$ ($C_T/\sigma = 0.075$)
3.4 Summary

This chapter begins with an overview on the relevant helicopter model plant assumptions used by researchers, which are used by Shaw [47] and Hall and Wereley [19], respectively. The discussion focuses on the linear quasi-steady and linear time invariant plant. The second portion of the chapter provides an overview of the design and results using the LTI state space models for the two baseline controllers in this thesis. The CTHHC represents the classic technique for vibration reduction. The $\mathcal{H}_\infty$ controller represents a more modern approach to the helicopter vibration problem. The two baseline controllers are developed for comparison to the LPV controller in RCAS.

The framework established by Fan in the $\mathcal{H}_\infty$ synthesis method serves as the starting point of the LPV controller design. The linear notch performance weighting filter with an approximation order of $m = 2$ and the parsing of the weighting filter into two parts is used in the LPV design synthesis.
Chapter 4

LPV Controller Synthesis

This chapter presents the design of a controller using the Linear Parameter Varying (LPV) control synthesis method for the helicopter vibration problem. The LPV controller builds off of the $\mathcal{H}_\infty$ synthesis method discussed in Chapter 3, where an appropriate performance weighting filter is created for vibration rejection. This chapter begins by discussing the theory of the LPV synthesis method. The second portion of the chapter applies the technique to the helicopter vibration problem and uses the LTI state space models developed from the RCAS simulation environment in Chapter 2. The design focuses on attenuating the dominant baseline harmonic load for varying advance ratio.

4.1 Motivation for LPV Synthesis

The traditional method to design a controller over the envelope is to design controllers at set operating points, using LTI models, and then combine the controllers through some interpolation method in a technique called gain-scheduling [45]. The interpolation scheme is a function of measurable gain scheduling parameters which are generalized as $\rho(t)$. Some common interpolation schemes are (1) interpolate the poles, zeros, and gains of the controller transfer functions (2) interpolate the solutions of Riccati equations for $\mathcal{H}_\infty$ controllers or (3) interpolate the output signal of controllers in parallel (controller blending) [51]. Due to many successful applica-
tions of this technique, gain-scheduling is a popular technique in industry. However, while these interpolation methods perform satisfactory in most applications, these interpolation methods are *ad hoc* and do not provide any sort of global stability or performance guarantees [61]. Furthermore, stability is assumed by arguing that the variations in the gain scheduling parameters, $\rho(t)$, are sufficiently slow and remain in the neighborhood of the operating conditions that the LTI models were developed [43, 46]. The framework by which a gain-scheduled controller is developed is visualized in Figure 4-1.

![Figure 4-1: Gain-scheduling design method](image)

The drawbacks of the classic gain-scheduling method and the ascent of modern control synthesis techniques led to the development of the LPV control synthesis techniques. Unlike classic gain-scheduling methods, the LPV control synthesis technique designs a set of controllers that are able to guarantee stability, performance, and robustness over the entire parameter space, $\mathcal{P}$, which is a continuous time representation of the discrete scheduling space shown in Figure 4-1. The controllers developed under the LPV framework use measurable real-time parameters, $\rho(t)$, to adjust the controller. It is not required for the trajectory of LPV systems to be known *a priori*, as the varying parameter which impacts the system matrices is measurable real-time. A block diagram of current problem in the LPV framework is shown in Figure 4-2.
It is important to note that the use of the LPV control synthesis framework has been applied to the helicopter vibration problem in work by Bittanti et al. [8, 9]. However, their work uses a low-order helicopter model (not an aeromechanical code) and assumes that the system matrices have polytopic (affine) dependency on the advance ratio for the LPV synthesis. The work in this thesis uses helicopter plants that are identified from traditional frequency based system identification techniques. Specifically, the LPV synthesis uses system matrices identified from the ETFEs.

4.2 LPV Theory and Synthesis Procedure

4.2.1 Overview

A linear parameter varying system is a special class of linear time-varying (LTV) systems and is represented by

\[
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \quad \text{(4.1)}
\]

\[
y(t) = C(\rho(t))x(t) + D(\rho(t))u(t)
\]

where the state and output matrices of the system are a function of the time-varying parameters, \(\rho(t)\). Unlike LTV systems, for LPV systems the trajectory must not be known \textit{a priori}, since the time-varying parameters are measurable. The measurable time-varying parameters are a set, of size \(p\), in a predefined parameter space, \(\mathcal{P}\), such that \(\rho(t) \in \mathcal{P}\). The parameter space can consist of multiple parameters. This thesis only considers a single parameter \((p = 1)\), namely the advance ratio. The time
variation of $\rho(t)$ is constrained by

$$\nu \leq \dot{\rho}(t) \leq \nu \tag{4.2}$$

where $\nu$ is a predefined rate constraint and the bars represent the upper and lower bounds of the rate. The parameter and parameter rate space can be compactly written by defining the parameter $\nu$-variation set as

$$\mathcal{F}_\nu = \{ \rho(t) \in \mathcal{P}, |\dot{\rho}(t)| \leq \nu \} \tag{4.3}$$

The goal of the LPV output-feedback synthesis problem is to design a controller, $K(s; \rho(t))$, that ensures that the closed-loop system is exponentially stable and the performance of the induced $\mathcal{L}_2$ norm of $d$ to $z$ is less than a desired value, $\gamma$, that is

$$||T_{zd}||_{\infty} = \sup_{\rho \in \mathcal{P}, ||d||_2 \neq 0} \frac{||z||_2}{||d||_2} \leq \gamma \tag{4.4}$$

over the entire parameter space. The generalized LPV plant can be written in the form

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t)) \\ C_1(\rho(t)) & D_{11}(\rho(t)) & D_{12}(\rho(t)) \\ C_2(\rho(t)) & D_{21}(\rho(t)) & D_{22}(\rho(t)) \end{bmatrix} \begin{bmatrix} \dot{x} \\ d \\ u \end{bmatrix} \tag{4.5}$$

which describes the augmented plant, $P$, for the LPV synthesis. As in the $\mathcal{H}_\infty$ control method, the augmented plant represents the interconnection of the performance weighting filter, $W_S(s)$, and the original plant, $G(s)$.

The resulting LPV controller has the form

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c(\rho(t), \dot{\rho}(t)) & B_c(\rho(t), \dot{\rho}(t)) \\ C_c(\rho(t), \dot{\rho}(t)) & D_c(\rho(t), \dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{4.6}$$

where $x_c$ are the controller states and the controller matrices are a function of both
the parameter and the parameter rate. Combining the LPV plant in Equation 4.5 and
the LPV controller in Equation 4.6 produces the LPV closed-loop system dynamics,
given by
\[
\begin{bmatrix}
\dot{x}_{cl} \\
z
\end{bmatrix} =
\begin{bmatrix}
A_{cl}(\rho(t), \dot{\rho}(t)) & B_{cl}(\rho(t), \dot{\rho}(t)) \\
C_{cl}(\rho(t), \dot{\rho}(t)) & D_{cl}(\rho(t), \dot{\rho}(t))
\end{bmatrix}
\begin{bmatrix}
x_{cl} \\
y
\end{bmatrix}
\] (4.7)

where \( x_{cl} \) is the closed-loop states that represent a combination of the plant and controller states. The closed-loop states in terms of the plant and controller states are \( x_{cl} = [x^T \ x_c^T]^T \). The closed-loop matrices of Equation 4.7 are written as
\[
\begin{bmatrix}
A_{cl} & B_{cl} \\
C_{cl} & D_{cl}
\end{bmatrix} =
\begin{bmatrix}
A + B_2 D_c C_2 & B_2 C_c & B_1 + B_2 D_c D_{21} \\
B_c C_2 & A_c & B_c D_{21} \\
C_1 + D_{12} D_c C_2 & D_{12} C_c & D_{11} + D_{12} D_c D_{21}
\end{bmatrix}
\] (4.8)

with the parameter and parameter rate dependencies omitted. It is important to note
that in general it is not possible to accurately measure the rate of the time varying parameter, such as the advance ratio, which is required for the LPV controller in Equation 4.6. The parameter rate dependency of the LPV controller is removed prior to implementing the controller in the RCAS environment. The parameter rates are necessary for the analysis step of the LPV synthesis procedure [40].

As with the \( \mathcal{H}_\infty \) method, a set of analysis conditions must be developed in order for the synthesis process to determine if a candidate controller satisfies the design requirements.

4.2.2 Analysis

The first analysis to consider is stability. The goal is to use an analysis criteria that determines if a candidate controller satisfies the exponential stability requirement for the closed-loop system. In traditional gain-scheduling, stability is assumed using the argument that the variation of the system is slow and the system remains close enough to the equilibrium points. For LPV analysis, it is desirable to include information
about the rate so that the time-varying parameter can vary at any rate within the rate bounds. Theorem 4.1 provides the stability analysis criteria for the LPV synthesis method [43].

**Theorem 4.1.** Suppose there exists a continuously differentiable \( X(\rho) = X(\rho)^T > 0 \) such that

\[
X(\rho)A_{cl}(\rho, \dot{\rho}) + A_{cl}^T(\rho, \dot{\rho})X(\rho) + \sum_{i=1}^{p} \nu_i \frac{\partial X(\rho)}{\partial \rho_i} < 0 \tag{4.9}
\]

for all \( \rho \in \mathcal{P} \), where \( p \) is the number parameters in the time-varying parameter set. Then the closed-loop system in Equation 4.7 is exponentially stable for all parameter trajectories within the specified parameter \( \nu \)-variation set, \( \mathcal{F}_\nu' \).

Note that Theorem 4.1 represents two analysis requirements. The first is for the lower rate bound, \( \nu \), and the second is for the upper rate bound, \( \bar{\nu} \). The two requirements are compactly written in Theorem 4.1 using the notation of \( \nu \). Theorem 4.1 assures that a classic quadratic Lyapunov function defined as

\[
V(x, \rho) = x^T X(\rho) x \tag{4.10}
\]

is formed, and verified for the closed-loop system in Equation 4.7.

The second analysis to consider is performance. The performance goal of the LPV output-feedback synthesis problem is to find a controller that achieves a performance of \( \gamma \) that satisfies the \( \mathcal{L}_2 \) norm written as in Equation 4.4. Theorem 4.2, which is a generalization of the bounded real lemma, provides a sufficient condition for achieving a closed-loop performance of \( \gamma \) [43]. Note that Theorem 4.2 also includes the general exponential stability requirement defined in Theorem 4.1.
Theorem 4.2. A controller achieves a performance level of $\gamma$ if there exists a continuously differentiable $X(\rho) = X(\rho)^T > 0$ such that

$$\begin{bmatrix}
X(\rho)A_{cl}(\rho, \dot{\rho}) + A^T_{cl}(\rho, \dot{\rho})X(\rho) + \sum_{i=1}^{p} \Xi_i \frac{\partial X(\rho)}{\partial \rho_i} X(\rho)B_{cl}(\rho, \dot{\rho}) & \gamma^{-1}C^T_{cl}(\rho, \dot{\rho}) \\
B^T_{cl}(\rho, \dot{\rho})X(\rho) & -I \\
\gamma^{-1}C_{cl}(\rho, \dot{\rho}) & \gamma^{-1}D^T_{cl}(\rho, \dot{\rho}) - I
\end{bmatrix} < 0 \quad (4.11)$$

for all parameter trajectories within the specified parameter $\nu$-variation set, $\mathcal{F}_\nu^\nu$.

Theorem 4.2 differs from the $\mathcal{H}_\infty$ analysis requirements in that the LPV analysis requirement is a linear matrix inequality (LMI), rather than an equality statement. Equation 4.11 represents two LMIs, one for the lower rate bound, $\nu$, and one for the upper rate bound, $\tilde{\nu}$. The two LMIs are compactly written using the notation of $\Xi_i$.

It is important to note that the LPV control analysis procedure used in this thesis uses a parameter-dependent quadratic Lyapunov matrix function, namely $X(\rho)$. Therefore, this technique does not find a single quadratic Lyapunov function that spans the entire parameter space, but instead finds a continuously differentiable Lyapunov function that is a function of the scheduling parameter. The parameter-dependent method is used because the single quadratic Lyapunov analysis method measures the performance against arbitrarily fast variations in the scheduling parameter, which introduces conservatism. Better performance is achieved by including bounds on the rate of parameter variation $[61]$. If the parameter dependency is removed and $X$ is constant, the $\frac{\partial X(\rho)}{\partial \rho_i}$ term in Theorem 4.1 and 4.2 is eliminated and the single quadratic Lyapunov function LPV analysis procedure is recovered. The parameter-dependent quadratic Lyapunov function LPV analysis procedure $[60]$ was developed from the single quadratic Lyapunov function LPV analysis procedure developed by Becker and Packard $[6, 7, 5]$.

4.2.3 LPV Synthesis

In the LPV synthesis framework, the task is to numerically find the smallest $\gamma$ subject to Theorem 4.2 over all of the $\nu$-variation set, $\mathcal{F}_\nu^\nu$, and then reconstruct the controllers. The synthesis procedure presented here follows the results of Wu $[60, 62, 63]$. 
It is possible to numerically search for the $X(\rho)$ term using efficient convex optimization algorithms, but Theorem 4.2 must be transformed for convex optimization. To this end, it is possible to simplify the LPV plant in Equation 4.5 by making the following assumptions:

(A1) $D_{22}(\rho(t)) = 0$,

(A2) $D_{12}(\rho(t)) = [0 \ I]^T$ is full column rank for all $\rho \in \mathcal{P}$,

(A3) $D_{21}(\rho(t)) = [0 \ I]$ is full row rank for all $\rho \in \mathcal{P}$,

With Assumptions A1–A3, the generalized LPV system in Equation 4.5 assumes the form of

$$\begin{bmatrix}
\dot{x} \\
\dot{z}_1 \\
\dot{z}_2 \\
y
\end{bmatrix} = \begin{bmatrix}
A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_{2}(\rho(t)) \\
C_{11}(\rho(t)) & 0 & 0 & 0 \\
C_{12}(\rho(t)) & 0 & 0 & I \\
C_2(\rho(t)) & 0 & I & 0
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
d_1 \\
d_2 \\
u
\end{bmatrix} \quad (4.12)$$

The transformed analysis criteria for the convex optimization method using the generalized LPV plant in Equation 4.12 is given in Theorem 4.3. Further details and proofs of this theorem can be found in [60].

**Theorem 4.3.** A controller achieves a performance level of $\gamma > 0$ for the LPV system in Equation 4.12 if and only if there exists the continuously differentiable functions $X(\rho) = X(\rho)^T > 0$ and $Y(\rho) = Y(\rho)^T > 0$, for all parameter trajectories within the specified parameter $\nu$-variation set, $\mathcal{F}_\nu^p$, such that

$$\begin{bmatrix}
X(\rho)\dot{A}(\rho) + \dot{A}(\rho)X(\rho) - \sum_{i=1}^p \frac{\partial X(\rho)}{\partial \nu_i} - B_2(\rho)B_2^T(\rho) & X(\rho)C_{11}(\rho) & \gamma^{-1}B_1(\rho) \\
C_{11}(\rho)X(\rho) & -I & 0 \\
\gamma^{-1}B_1^T(\rho) & 0 & -I
\end{bmatrix} < 0 \quad (4.13)$$

$$\begin{bmatrix}
\dot{A}(\rho)Y(\rho) + \dot{A}(\rho)Y(\rho) + \sum_{i=1}^p \frac{\partial Y(\rho)}{\partial \nu_i} - C_2^T(\rho)C_2(\rho) & Y(\rho)B_{11}(\rho) & \gamma^{-1}C_{11}(\rho) \\
B_{11}^T(\rho)Y(\rho) & -I & 0 \\
\gamma^{-1}C_1(\rho) & 0 & -I
\end{bmatrix} < 0 \quad (4.14)$$

$$\begin{bmatrix}
X(\rho) & \gamma^{-1}I \\
\gamma^{-1}I & Y(\rho)
\end{bmatrix} \geq 0 \quad (4.15)$$
where

\[ A(\rho) = A(\rho) - B_2(\rho)C_{12}(\rho) \]  \hspace{1cm} (4.16)
\[ \tilde{A}(\rho) = A(\rho) - B_{12}(\rho)C_2(\rho) \]  \hspace{1cm} (4.17)

If the conditions of Theorem 4.3 are satisfied, namely both an \( X(\rho) \) and \( Y(\rho) \) are numerically found, then the matrices of the LPV controller in Equation 4.6 are

\[
A_c(\rho, \dot{\rho}) = A(\rho) + B_2(\rho)F(\rho) + Q^{-1}(\rho)Y(\rho)L(\rho)C_2(\rho) - \gamma^{-2}Q^{-1}(\rho)M(\rho, \dot{\rho})
\]
\[
B_c(\rho) = -Q^{-1}(\rho)Y(\rho)L(\rho)
\]
\[
C_c(\rho) = F(\rho)
\]
\[
D_c(\rho) = 0
\]  \hspace{1cm} (4.18)

where \( F(\rho), L(\rho), \) and \( M(\rho, \dot{\rho}) \) are respectively found by

\[
F(\rho) = -[B_2^T(\rho)X^{-1}(\rho) + D_{12}^TC_1(\rho)]
\]  \hspace{1cm} (4.19)
\[
L(\rho) = -[Y^{-1}(\rho)C_2^T(\rho) + B_1(\rho)D_{21}^T]
\]  \hspace{1cm} (4.20)
\[
M(\rho, \dot{\rho}) = H(\rho, \dot{\rho}) + \gamma^2Q(\rho) [-Q^{-1}(\rho)Y(\rho)L(\rho)D_{21} - B_1(\rho)] B_1^T(\rho)X^{-1}(\rho)
\]  \hspace{1cm} (4.21)

Furthermore, the \( H(\rho, \dot{\rho}) \) matrix is defined by

\[
H(\rho, \dot{\rho}) = -\left[ X^{-1}(\rho)A_F(\rho) + A_F^T(\rho)X^{-1}(\rho) + \sum_{i=1}^{n} \left( \dot{\rho}_i \frac{\partial X^{-1}(\rho)}{\partial \rho_i} \right) \right]
\]
\[
+ \left[ C_F^T(\rho)C_F(\rho) + \gamma^{-2}X^{-1}(\rho)B_1(\rho)B_1^T(\rho)X^{-1}(\rho) \right]
\]  \hspace{1cm} (4.22)

where \( A_F(\rho) \) and \( C_F(\rho) \) are respectively given by

\[
A_F(\rho) = A(\rho) + B_2(\rho)F(\rho)
\]  \hspace{1cm} (4.23)
\[
C_F(\rho) = C_1(\rho) + D_{12}F(\rho)
\]  \hspace{1cm} (4.24)
4.2.4 LPV Synthesis Numeric Considerations

The solutions to the LMI constraints in Theorem 4.3 specifically $X(\rho)$ and $Y(\rho)$, are continuous functions, because of the differentiability requirement. In addition, there are an infinite number of LMI constraints because of the continuity of the parameter $\nu$-variation set, $F_\nu^\nu$. In order to make this problem tractable, it is possible to replace Equations 4.13 and 4.14 with a finite collection of LMIs by discretizing the continuous parameter space, $\mathcal{P}$. In addition, $X(\rho)$ and $Y(\rho)$ can be represented by a finite number of basis functions as in

$$X(\rho) = \sum_{k=1}^{N_x} f_k(\rho)X_k$$

$$Y(\rho) = \sum_{k=1}^{N_y} g_k(\rho)Y_k$$

(4.25)

(4.26)

where $N_x, N_y$, are the number of the respective basis functions and $f_k(\rho)$ and $g_k(\rho)$ are continuously differentiable basis functions.

A difficulty with the LPV method is the lack of guidance on the selection of the $f_k(\rho)$ and $g_k(\rho)$ basis functions. In general, it is sensible to choose basis functions that relate to the driving function of a given problem. For example, in [30] and [60], a problem is given where the basis functions are $f_k = g_k = [1, \cos(\rho), \sin(\rho)]$, and the nonlinear terms of the system dynamics are sine and cosine terms. Another rule of thumb is to use a polynomial basis function [40]. In this thesis, the number of the basis functions are chosen such that $n_b = N_x = N_y$ where $n_b$ is the number of basis functions. Furthermore, a polynomial basis function form is assumed, where the order $(n_b - 1)$ of the basis functions is free to vary. For example in this thesis, a basis function order of two implies that the basis functions are

$$f(\mu) = g(\mu) = [1, \mu, \mu^2]$$

(4.27)

which are a function of the time-varying parameter, $\mu$.

The size of the discrete optimization problem is proportional to $N_G(2^{p+1} + 1)$,
where \( p \) is the number of time-varying parameters and \( N_G \) is the size of the grid space. Furthermore, the number of decision variables (sum of the elements in \( X(\rho) \) and \( Y(\rho) \)) is \((N_x+N_y)[n_p(n_p+1)/2]\), where \( n_p \) is the plant state order \([48]\). Discretizing a large parameter space with multiple parameters in the parameter set can exclude the LPV synthesis method from use for some control problems. However, in this thesis, there is only one time-varying parameter and the parameter space is discretized over five points \((\mu = 0.13, 0.20, 0.26, 0.33, 0.36)\), making the LPV method a tractable technique.

The LPV literature provides little guidance as to how to form and solve the LPV control synthesis problem. This thesis adds to LPV literature by providing a pseudo-algorithm in Algorithm \([1]\) that outlines the LPV synthesis process. The LMIs are built and solved using MATLAB’s \texttt{limiter} and \texttt{mincx} solver functions \([15]\).

### 4.3 LPV Performance Synthesis Results

This section discusses the results of the LPV control synthesis design method using the LTI state space models developed in Chapter 2. The order of the models developed in Chapter 2 are all of order \( n_p = 15 \) because the LPV method requires that the order of the models in the parameter space, \( \mathcal{P} \), be the same \([40]\). The models considered here have an advance ratio in \( \mu \in [0.13, 0.36] \) and a constant blade loading coefficient of \( C_T/\sigma = 0.075 \). The weighting performance filter is the same as that used by the \( \mathcal{H}_\infty \) controller. Specifically, the approximation order is \( m = 2 \) and the filter is split into two parts so that the LPV synthesis does not fail. The order of the LPV controller, \( n_{LPV} \), is the same as the generalized augmented plant, \( P \), and is

\[
n_{LPV} = n_p + n_h \times 2m
\]  

(4.28)

where \( n_p \) is the order of the state space plant, \( n_h \) is the number of harmonics to be attenuated, and \( m \) is the order of the ideal linear notch approximation. The order of the LPV controller in this section is \( n_{LPV} = 19 \).
Algorithm 1 LPV Control Synthesis Framework

Require: Specify the basis functions, $f_k, g_k$

1: \[\triangleright\] Build the LMIs in MATLAB using \texttt{lmiterm}:
2: \hspace{1em} for $i = 1 \rightarrow 2^{N_G}$ do
3: \hspace{2em} for $k = 1 \rightarrow 2^{N_x}$ do
4: \hspace{3em} if $k$ is odd then
5: \hspace{4em} Form the lower bound, $\nu$, $X$ LMI in Equation 4.13 for $f_k$
6: \hspace{3em} else if $k$ is even then
7: \hspace{4em} Form the upper bound, $\nu$, $X$ LMI in Equation 4.13 for $f_k$
8: \hspace{3em} end if
9: \hspace{2em} end for
10: \hspace{1em} for $k = 1 \rightarrow 2^{N_y}$ do
11: \hspace{2em} if $k$ is odd then
12: \hspace{3em} Form the lower bound, $\nu$, $Y$ LMI in Equation 4.14 for $g_k$
13: \hspace{3em} else if $k$ is even then
14: \hspace{4em} Form the upper bound, $\nu$, $Y$ LMI in Equation 4.14 for $g_k$
15: \hspace{3em} end if
16: \hspace{2em} end for
17: Form the LMI for $X_{lb} \leq X \leq X_{ub}$
18: Form the LMI for $Y_{lb} \leq Y \leq Y_{ub}$
19: end for
20: \[\triangleright\] Solve set of LMIs to get $X, Y$ using \texttt{mincx}:
21: \hspace{1em} $\min c^T x_{\text{decision}}$ subject to LMI constraints
22: \hspace{1em} Form $X, Y$ from $x_{\text{decision}}$ solution
23: \[\triangleright\] Reconstruct Controller:
24: \hspace{1em} Form controller matrices, $A_c, B_c, C_c, D_c$ by Equations 4.18 to 4.24
25: \[\triangleright\] Remove Controller Parameter Rate Dependency:
26: \hspace{1em} Find controller at each $\rho_i$ by interpolating between solution at $\nu_i$ and $\nu_i$
Prior to looking at the performance of the LPV controllers implemented in RCAS, it is necessary to demonstrate the rationale for two key LPV synthesis parameter selections, specifically the number of basis functions, $n_b$, and the parameter rate bounds, $\nu$. One drawback to the LPV method is that a solution may simply not exist, meaning the constraints of the LMIs do not overlap such that a solution satisfies all of the constraints. This section demonstrates which settings for a given grid space of $N_G$ are feasible for the current helicopter vibration problem.

The first step is to determine the impact of the parameter rate bound on the feasibility of a solution set for one harmonic notch. Table 4.1 displays a feasibility matrix for various bandwidth and advance ratio rate bounds (given in velocity) for the single model ($N_G = 1$) of $\mu = 0.33$. This assumes that the number of basis functions is $n_b = 3$. A check mark implies that the LPV synthesis found a solution and an x-mark implies that a solution for the LMI constraints is not feasible. The case of $\nu = \pm \infty$ forms the problem of finding a single, constant, quadratic Lyapunov function matrix. The finite rate bounds form the problem of finding parameter-dependent quadratic Lyapunov function matrices. The results from Table 4.1 demonstrate that for all of the bandwidths and finite rate bounds examined, a solution is feasible. In the case of an infinite rate bound, a solution was only found for the smallest bandwidth. This demonstrates the benefit of formulating the LPV synthesis problem using the parameter-dependent Lyapunov functions. As the number of grid point increases ($N_G > 1$), using the parameter-dependent formulation is even more important for finding a solution that satisfies all of the LMI constraints.

Table 4.1: Feasibility matrix for various bandwidths and rate bounds for $\mu = 0.33$

<table>
<thead>
<tr>
<th>Bandwidth (per rev)</th>
<th>Rate Bound, $\nu$, (fps/sec)</th>
<th>$\pm 1$</th>
<th>$\pm 10$</th>
<th>$\pm 1000$</th>
<th>$\pm \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>0.04</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>0.06</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>0.08</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
</tbody>
</table>
While Table 4.1 demonstrates the feasibility of a solution, it does not demonstrate the impact of the rate bounds on the performance of the controller in terms of the peak sensitivity. Figure 4-3 displays the peak sensitivity for the various bandwidth and rate bound settings for the $\mu = 0.13$ and $\mu = 0.33$ cases. Note that each of these solutions are only solving for one grid point in the parameter space. In general, the parameter rates do not have an impact on the performance of the LPV controller. In addition, the performance at higher bandwidths is better for the smaller advance ratios. It was determined that restricting the rate bounds to low magnitudes did not result in better performance and the problem is still feasible for high finite value rate bounds. The remaining results in this section use a rate bound of $\nu = \pm 1000$ fps/sec. This rate bound is essentially an infinite bound. However, the choice of this bound is due to a desire for the synthesis to find a solution without a restrictive bound.

The next step is to determine what the order of the basis functions should be as the number of discrete points in the parameter space increases for one harmonic notch. Table 4.2 displays a feasibility matrix for various grid space sizing and order of basis functions ($n_b - 1$) for a bandwidth of 0.04 per rev. The strategy for determining

<table>
<thead>
<tr>
<th>Number of Models, $N_G$</th>
<th>$X(\rho), Y(\rho)$ Order ($n_b - 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3 4 5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>1 ($\mu = 0.13$)</td>
<td>✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓</td>
</tr>
<tr>
<td>2 ($\mu = 0.13, 0.20$)</td>
<td>✓ x x x x ✓ ✓ ✓ ✓ ✓ ✓ ✓</td>
</tr>
<tr>
<td>3 ($\mu = 0.13, 0.20, 0.26$)</td>
<td>x x x x x x ✓ ✓ ✓ ✓ ✓ ✓</td>
</tr>
<tr>
<td>4 ($\mu = 0.13, 0.20, 0.26, 0.33$)</td>
<td>x x x x x x x x x x x x</td>
</tr>
<tr>
<td>5 ($\mu = 0.13, 0.20, 0.26, 0.33, 0.36$)</td>
<td>x x x x x x x x x x x x</td>
</tr>
</tbody>
</table>
Figure 4-3: Impact of rate bounds on performance for various advance ratios
the appropriate basis function is to increase the order of the basis function until a solution is found. Higher order values above the found solution are not tested. In general, as the number grid points increases, the order of the basis function must be increased to find a solution. For the case $N_G = 5$, a solution was not found up to a basis function order of 11. An order greater than 11 was not examined for $N_G = 5$ because the number of decision variables to solve for becomes large and the likelihood of a solution decreases. For example, for the case of $N_G = 1$, the number of decision variables is 720. For the case of $N_G = 5$, for the highest order tested, the number of decision variables is 2880. While a solution was not found for the entire grid space of interest, a solution for $N_G = 4$ was found, which covers 90% of the parameter space. Furthermore, the parameter space from $\mu = 0.33$ to $\mu = 0.36$ is a region to be unlikely traveled by helicopters, as retreating blade stall is imminent for the UH-60.

The final consideration in the design of a controller in the LPV synthesis framework is the feasibility of solutions for various controller bandwidths with the identified basis function order for each respective grid sizing. Table 4.3 displays the feasibility matrix for various bandwidths and the parameter space grid sizing for a performance weighting filter with one notch. The order of the basis functions used for each grid sizing case is also provided. The strategy in this test is to increase the number of models until a solution is infeasible. Once a model number is found infeasible, higher model numbers are not tested. A solution exists for the two highest bandwidths up to a grid size of $N_G = 4$ for one notch filter. As before, $N_G = 4$ is the largest grid size that has a feasible solution. The computational time in hours to find each of the feasible solutions is included in the parentheses in Table 4.3. Finding a solution for four model numbers increases the computational time. The parentheses in the number of models column indicates the order of the Lyapunov matrices, which correspond with the results in Table 4.2.
Table 4.3: Feasibility matrix for various bandwidths and number of models for $\nu = \pm 1000 \text{ fps}^2$ and weighting filter at 4 per rev

<table>
<thead>
<tr>
<th>Number of Models, $N_G$ $(X(\rho), Y(\rho)$ Order)</th>
<th>Bandwidth, (per rev)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.02</td>
</tr>
<tr>
<td>$1^{(2)}$</td>
<td>✓</td>
</tr>
<tr>
<td>$2^{(5)}$</td>
<td>✓</td>
</tr>
<tr>
<td>$3^{(5)}$</td>
<td>✓</td>
</tr>
<tr>
<td>$4^{(8)}$</td>
<td>✓</td>
</tr>
<tr>
<td>$5^{(8)}$</td>
<td>✓</td>
</tr>
</tbody>
</table>

The results of the synthesis are shown in the sensitivity transfer function plots in Figure 4-4. The sensitivity transfer functions performs as expected and the peak sensitivity value is less than the minimum of the inverted performance weighting filter. Furthermore, the desired bandwidth is achieved. Across the different grid points, the shape of the sensitivity transfer functions are relatively similar at the harmonic frequency but they differ at the non-harmonic frequencies, especially at high frequencies. This is not a concern, as the high frequency content is still less than the peak sensitivity. Figure 4-5 displays the peak sensitivity value for each of models in Figure 4-4. An increase in the bandwidth requirement increases the overall peak sensitivity, but the increase is small.
Figure 4-4: Sensitivity transfer function for various bandwidths and $N_G = 4$
Another consideration is to determine if the LPV synthesis method can find a solution for two harmonic notches, namely 4 and 8 per rev. Table 4.4 displays the feasibility matrix for two bandwidths and the parameter space grid sizing for a performance weighting filter with two notches. The maximum grid size to find a feasible solution for both bandwidths is three, which only covers 60% of the parameter space. Therefore, the solutions for the single notch are examined in the RCAS simulation environment.

Figure 4-5: Peak sensitivity for each LPV controller with $N_C = 4$
Table 4.4: Feasibility matrix for various bandwidths and number of models for $\nu = \pm 1000$ fps$^2$ and weighting filter at 4 and 8 per rev

<table>
<thead>
<tr>
<th>Number of Models, $N_G$</th>
<th>Bandwidth, (per rev)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.04</td>
</tr>
<tr>
<td>1$^{(2)}$</td>
<td>✔</td>
</tr>
<tr>
<td>2$^{(5)}$</td>
<td>✔</td>
</tr>
<tr>
<td>3$^{(5)}$</td>
<td>✔</td>
</tr>
<tr>
<td>4$^{(8)}$</td>
<td>✗</td>
</tr>
<tr>
<td>5$^{(8)}$</td>
<td></td>
</tr>
</tbody>
</table>

### 4.4 Summary

In conventional control engineering, controllers are designed over the flight envelope by the gain-scheduling technique, but this method provides no global stability or performance guarantees. This shortcoming leads to the focus of this chapter, which is the theoretical background and design of a controller using the linear parameter varying control synthesis framework.

The LPV controller builds off of the baseline $\mathcal{H}_\infty$ controller and allows the designer to specify the desired controller bandwidth. The controller is found by numerically determining a set of parameter-dependent Lyapunov function matrices. While the LPV method provides stability and performance guarantees over the flight envelope, the technique is not guaranteed to find a solution. Despite these shortcomings, a 19 order LPV controller has been determined by the LPV method for the helicopter vibration problem using the UH-60 modeled in RCAS. Applying the LPV controller synthesis framework to the helicopter vibration problem using a state space model found by common system identification techniques represents a unique contribution of this work.

The next step is to evaluate the LPV controller against the baseline CTHHC and $\mathcal{H}_\infty$ controllers in the RCAS nonlinear simulation environment for fixed and maneuvering flight conditions.
Chapter 5

Evaluation of Vibration Controllers

The final step in the design of a vibration reducing controller is to demonstrate the performance in a simulation environment. This chapter demonstrates the capabilities of three controllers, specifically the Continuous Time Higher Harmonic Controller (CTHHC), the $\mathcal{H}_\infty$ controller, and the Linear Parameter Varying (LPV) controller in the RCAS aeromechanical simulation environment. The first two controllers represent the baseline controllers with which the LPV controller is compared. These controllers are first evaluated at fixed-flight conditions. The controllers are then evaluated at a variety of maneuvering flight conditions with varying advance ratio.

5.1 Controller Setup in RCAS

All of the controllers are connected to the UH-60 model discussed in Chapter 2 using the RCAS CSGE tool, which is similar to MATLAB’s Simulink environment. Further discussion of the RCAS CSGE environment is provided in Appendix A. This simulation environment allows the user to connect linear and nonlinear block elements to form a control law. RCAS allows the user to input any node from the finite element model of the helicopter and output to any of the desired control inputs. The user can input control inputs before or after the standard helicopter control mixer. The CSGE model for this thesis inputs the vertical hub force, $F_2$, and outputs a control command to the pilot collective input before the control mixer.
Figure 5-2 displays the CTHHC controller in the CSGE environment. The CTHHC controller uses the classic gain-scheduling technique to combine the controllers designed at set advance ratio operating conditions. Specifically, the coefficients of the transfer function numerator and denominator are interpolated. In the modulation/demodulation scheme, the interpolation is done for the $a$ and $b$ coefficients of the $T^{-1}$ control matrix. The control gain used by the CTHHC controller is $k = 1/(4T)$ where $T$ is the period of a rotor revolution. It is important to point out that a pseudo-derivative (high pass filter) element is included in the modulation/demodulation model. The pseudo-derivative (PD) is defined as

$$PD(s) = \frac{s}{s + 5} \tag{5.1}$$

where the cut-off frequency of the lag component is five radians per second. The pseudo-derivative is included to avoid transients when the controller is turned on and ensure that the same steady-state vibration level as the baseline vibration is maintained. The cut-off frequency is significantly lower than the rotor rate ($\Omega = 27$ rad/s) and does not impact the response of the system as the frequency response of the system is unaffected at the key frequencies of interest.

The interpolation scheme for the $\mathcal{H}_\infty$ and LPV controllers differs from the CTHHC controller. Both of these controllers use the controller blending technique, which combines the output of each controller by

$$u(\mu) = \sum_{k=1}^{N} \gamma_k(\mu) u_k \tag{5.2}$$

where $N$ is the number of controllers, $u_k$ is the control output for the $k^{th}$ controller,
and $\gamma_k$ is the blending function for the $k^{th}$ controller. The blending function is

$$
\gamma_1(\mu) = \begin{cases} 
1, & \mu < \mu_1 \\
\frac{\mu_2 - \mu}{\mu_2 - \mu_1}, & \mu_1 \leq \mu < \mu_2 \\
0, & \mu \geq \mu_2 
\end{cases} \quad \text{(5.3)}
$$

$$
\gamma_k(\mu) = \begin{cases} 
0, & \mu < \mu_{k-1} \\
\frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}, & \mu_{k-1} \leq \mu < \mu_k \\
\frac{\mu_{k+1} - \mu}{\mu_{k+1} - \mu_k}, & \mu_k \leq \mu < \mu_{k+1} \\
0, & \mu \geq \mu_{k+1} 
\end{cases} \quad \text{(5.4)}
$$

$$
\gamma_N(\mu) = \begin{cases} 
0, & \mu < \mu_{N-1} \\
\frac{\mu - \mu_{N-1}}{\mu_N - \mu_{N-1}}, & \mu_{N-1} \leq \mu < \mu_N \\
1, & \mu \geq \mu_N 
\end{cases} \quad \text{(5.5)}
$$

where $k = 2, 3, 4, ..., N - 1$. A visualization of the output of this blending function for a linearly increasing advance ratio flight profile is shown in Figure 5-1 for $N = 4$, where the control design points are $\mu = 0.13, 0.20, 0.26, 0.33$. Notice that the blending functions are only active when the advance ratio is in between the control design points.

The block diagram of the LPV controller in the CSGE environment is shown in Figure 5-3. The $\mathcal{H}_\infty$ controller CSGE block structure is the same as the LPV controller. The blending functions, $\gamma_k(\mu)$, are included in the block diagram. The pseudo-derivative element is also included in both of the $\mathcal{H}_\infty$ and LPV controllers. The $\mathcal{H}_\infty$ and LPV controllers must include an additional negative sign term in the CSGE environment, as both of the synthesis methods assume a positive feedback. The LPV controller blending technique is similar to that used by Balas et al. \[48\], where a controller blending interpolation scheme was developed for an F-16 controller designed under the LPV framework.
The primary metric of interest is the root mean square (RMS) of the 4 per rev vertical hub force, which is calculated using the FFT technique used for the baseline vibration forces outlined in Chapter 3. In the case of maneuvering flight, the FFT method will not result in accurate calculation of the vibration force values. An FFT of the maneuvering flight conditions will result in the spread of the vibration magnitude around the 4 per rev frequency. To calculate the RMS value in this case, the time-series force data is processed through a fourth order Butterworth bandpass filter with a lower and upper cutoff frequency of 2.5 and 5.5 per rev, respectively. The final reported RMS value for the maneuvering flight conditions includes the frequency content in $\omega \in [2, 6]$ per rev. The FFT method from Chapter 3 is used for the fixed-flight conditions, where the reported RMS value includes the frequency content in $\omega \in [3.95, 4.05]$ per rev. The duration of the fixed flight conditions is 30 seconds and the last ten seconds are used for the RMS calculation.

The following sections report the results of the three different controllers in fixed-flight and maneuvering flight conditions.
Figure 5-2: CTHHC modulation/demodulation model in CSGE
Figure 5-3: LPV model in CSGE
5.2 Fixed-Flight Conditions

The first set of results to consider is the performance of the controllers at fixed-flight conditions. Table 5.1 lists the static flight conditions and controllers evaluated. The primary advance ratio is \( \mu \in [0.13, 0.33] \), as this is the range in which the LPV control synthesis produced a controller. Five controller are examined: the CTHHC controller, \( \mathcal{H}_\infty \) controller with bandwidth of 0.04 per rev, \( \mathcal{H}_\infty \) controller with bandwidth of 0.06 per rev, and LPV controllers with \( N_G = 4 \) and designed bandwidths of 0.04 and 0.06 per rev.

Table 5.1: Fixed-flight conditions in RCAS

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Advance Ratio, ( \mu )</th>
<th>Controller Type</th>
<th>Designed Bandwidth (per rev)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.13</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0.13</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>0.13</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.06</td>
</tr>
<tr>
<td>4</td>
<td>0.13</td>
<td>LPV ((N_G = 4))</td>
<td>0.04</td>
</tr>
<tr>
<td>5</td>
<td>0.13</td>
<td>LPV ((N_G = 4))</td>
<td>0.06</td>
</tr>
<tr>
<td>6</td>
<td>0.20</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>0.20</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.04</td>
</tr>
<tr>
<td>8</td>
<td>0.20</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.06</td>
</tr>
<tr>
<td>9</td>
<td>0.20</td>
<td>LPV ((N_G = 4))</td>
<td>0.04</td>
</tr>
<tr>
<td>10</td>
<td>0.20</td>
<td>LPV ((N_G = 4))</td>
<td>0.06</td>
</tr>
<tr>
<td>11</td>
<td>0.26</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>12</td>
<td>0.26</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.04</td>
</tr>
<tr>
<td>13</td>
<td>0.26</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.06</td>
</tr>
<tr>
<td>14</td>
<td>0.26</td>
<td>LPV ((N_G = 4))</td>
<td>0.04</td>
</tr>
<tr>
<td>15</td>
<td>0.26</td>
<td>LPV ((N_G = 4))</td>
<td>0.06</td>
</tr>
<tr>
<td>16</td>
<td>0.33</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>0.33</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.04</td>
</tr>
<tr>
<td>18</td>
<td>0.33</td>
<td>( \mathcal{H}_\infty )</td>
<td>0.06</td>
</tr>
<tr>
<td>19</td>
<td>0.33</td>
<td>LPV ((N_G = 4))</td>
<td>0.04</td>
</tr>
<tr>
<td>20</td>
<td>0.33</td>
<td>LPV ((N_G = 4))</td>
<td>0.06</td>
</tr>
</tbody>
</table>
per rev, LPV controller with bandwidth of 0.04 per rev, and the LPV controller with bandwidth of 0.06 per rev. The LPV controllers examined in RCAS correspond with the controllers developed for $N_G = 4$.

The final results of the static flight condition analysis is shown in Figure 5-4 which displays the vertical RMS hub force, $F_z$, for all of the conditions in Table 5.1. The trend of the baseline data clearly indicates the typical trend of the vertical hub force as the advance ratio increases. Specifically, there is an initial increase in the force, followed by a decrease as the helicopter levels out, and a final increase as the helicopter approaches retreating blade stall. All of the controllers show a significant decrease in the hub thrust force for all of the fixed advance ratio conditions. Note that without prewarping, the $\mathcal{H}_\infty$ and LPV controllers would not demonstrate rejection of the vertical hub force. For example, without the prewarping, the vertical hub RMS force would still be approximately 14 lb. While excluding the prewarping produces a large decrease, it does not represent the decrease that the sensitivity transfer function predicts (i.e., zero vibration), and therefore, prewarping must be included.

![Figure 5-4: RMS performance of controllers at fixed-flight conditions](image-url)
Table 5.2 shows the detailed results of the trend lines in Figure 5-4 for the slowest and fastest flight conditions. The best performing controller for both flight conditions is the CTHHC controller, where a reduction of up to 99.91% is achieved. For the \( \mathcal{H}_\infty \) and LPV controllers, the best performing controllers have the higher bandwidth. Even though the CTHHC controller does outperform the other controllers, the other controllers all perform well, especially the controllers with a bandwidth of 0.06 per rev. For example, for the case of \( \mu = 0.13 \), the LPV controller (BW = 0.06 per rev) has a final force that is 0.3 pounds greater than the CTHHC controller, which is small considering the baseline force of 106.6 pounds.

The results for the \( \mathcal{H}_\infty \) and LPV controllers in Table 5.2 can achieve further reductions by increasing the sampling rate. For example, by increasing the sampling rate from 256 to 512 samples per revolution, the LPV controller at \( \mu = 0.33 \) with a bandwidth of 0.06 per rev reduces the RMS vibration to 0.432 lb, which is a 99.69% reduction. The results presented use a sampling rate of 256 samples per revolution.

While the static flight conditions indicate how each of the controllers are working, they are not the primary flight condition of concern. This work is more concerned with the performance of the controllers for maneuvering flight conditions.

Table 5.2: Flight controller results for fixed-flight conditions in RCAS

<table>
<thead>
<tr>
<th>Controller Type</th>
<th>( \mu = 0.13 )</th>
<th>( \mu = 0.33 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS Force (lb)</td>
<td>Percent Reduction</td>
<td>RMS Force (lb)</td>
</tr>
<tr>
<td>Baseline</td>
<td>106.610</td>
<td>-</td>
</tr>
<tr>
<td>CTHHC</td>
<td>0.093</td>
<td>99.91</td>
</tr>
<tr>
<td>( \mathcal{H}_\infty ), BW = 0.04</td>
<td>1.559</td>
<td>98.54</td>
</tr>
<tr>
<td>( \mathcal{H}_\infty ), BW = 0.06</td>
<td>0.863</td>
<td>99.19</td>
</tr>
<tr>
<td>LPV, BW = 0.04</td>
<td>1.301</td>
<td>98.78</td>
</tr>
<tr>
<td>LPV, BW = 0.06</td>
<td>0.396</td>
<td>99.62</td>
</tr>
</tbody>
</table>
## 5.3 Maneuvering Flight Conditions

The second set of results to consider is the performance of the controllers for maneuvering flight conditions. Table 5.3 lists the maneuvering flight conditions and controllers evaluated in RCAS. This thesis considers five different maneuvers. Table 5.3 provides the details of each of the maneuvering conditions. The “Step” maneuvers begin at $\mu = 0.13$ and linearly increases to $\mu = 0.36$ over 30 seconds. The maneuvers

<table>
<thead>
<tr>
<th>Case No</th>
<th>Maneuver Type</th>
<th>Starting $\mu$</th>
<th>Ending $\mu$</th>
<th>Maneuver Duration (sec)</th>
<th>Controller Type</th>
<th>Designed BW (per rev)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Step On</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>Step On</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>Step On</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.06</td>
</tr>
<tr>
<td>4</td>
<td>Step On</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>LPV</td>
<td>0.04</td>
</tr>
<tr>
<td>5</td>
<td>Step On</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>LPV</td>
<td>0.06</td>
</tr>
<tr>
<td>6</td>
<td>Step Off</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>Step Off</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.04</td>
</tr>
<tr>
<td>8</td>
<td>Step Off</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.06</td>
</tr>
<tr>
<td>9</td>
<td>Step Off</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>LPV</td>
<td>0.04</td>
</tr>
<tr>
<td>10</td>
<td>Step Off</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>LPV</td>
<td>0.06</td>
</tr>
<tr>
<td>11</td>
<td>Linear</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>12</td>
<td>Linear</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.06</td>
</tr>
<tr>
<td>13</td>
<td>Linear</td>
<td>0.13</td>
<td>0.36</td>
<td>30</td>
<td>LPV</td>
<td>0.06</td>
</tr>
<tr>
<td>14</td>
<td>Fast Linear</td>
<td>0.13</td>
<td>0.36</td>
<td>15</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>15</td>
<td>Fast Linear</td>
<td>0.13</td>
<td>0.36</td>
<td>15</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.06</td>
</tr>
<tr>
<td>16</td>
<td>Fast Linear</td>
<td>0.13</td>
<td>0.36</td>
<td>15</td>
<td>LPV</td>
<td>0.06</td>
</tr>
<tr>
<td>17</td>
<td>Doublet</td>
<td>0.13</td>
<td>0.13</td>
<td>20</td>
<td>CTHHC</td>
<td>-</td>
</tr>
<tr>
<td>18</td>
<td>Doublet</td>
<td>0.13</td>
<td>0.13</td>
<td>20</td>
<td>$\mathcal{H}_\infty$</td>
<td>0.06</td>
</tr>
<tr>
<td>19</td>
<td>Doublet</td>
<td>0.13</td>
<td>0.13</td>
<td>20</td>
<td>LPV</td>
<td>0.06</td>
</tr>
</tbody>
</table>
have three plateau (step) regions that last two seconds each. The steps occur at \( \mu = [0.20, 0.26, 0.33] \) for the “Step On” maneuver, which represent the design points of the controllers. The “Step Off” maneuver has steps at \( \mu = [0.16, 0.23, 0.30] \), which represent operating conditions in between the controller design points. The “Linear” maneuver begins at \( \mu = 0.13 \) and linearly increases to \( \mu = 0.36 \) over 30 seconds. The “Fast Linear” maneuver begins at \( \mu = 0.13 \) and linearly increases to \( \mu = 0.36 \) over 15 seconds. The “Doublet” maneuver begins at \( \mu = 0.13 \) and linearly increases to \( \mu = 0.36 \) over ten seconds and then linearly decreases to \( \mu = 0.36 \) over ten seconds for a total time of 20 seconds. Various flight profiles were tested in order to draw out differences in the controllers, such as the aggressive “Doublet” maneuver. Figure 5-5 shows all of the maneuvering flight profiles as a function of time and the corresponding baseline vertical force response. Only the 4 per rev content is shown in the force plots.

Figures 5-6 to 5-10 show the flight profile, the controller output, and the baseline and closed-loop vibration levels for each of the flight maneuvers. The response of only one controller is provided for each of the flight conditions. In general, it is evident that the controllers are effective at reducing the overall 4 per rev vibration. For the “Step” maneuvers, it is apparent that the controllers are able to further reduce the vibration at the plateau regions, as compared to the pure transient regions. For the “Linear” maneuver, the CTHHC controller significantly reduces the vibration level but it does not perform well as the advance ratio changes, and actually increases as the maneuver progresses. When the maneuver reaches the final value, the CTHHC controller is again able to perform well, since the controller is at a fixed-flight condition. For the “Fast Linear” and “Doublet” maneuvers, the LPV controller is able to reduce the vibration as the maneuver is occurring but it does not completely reject the vibration. At the inflection point of the “Doublet” maneuver, the vibration level significantly reduces.
Figure 5-5: Flight profiles and baseline response of maneuvering flight conditions
Figure 5-6: LPV (BW = 0.06) controller performance for on-design step maneuver
Figure 5-7: $\mathcal{H}_\infty$ ($BW = 0.06$) controller performance for off-design step maneuver
Figure 5-8: CTHHC controller performance for linear maneuver
Figure 5-9: LPV (BW = 0.06) controller performance for fast linear maneuver
Figure 5-10: LPV (BW = 0.06) controller performance for doublet maneuver
The next analysis of the maneuvering flight evaluation is to compare the results of the controllers. Table 5.4 displays the vibration reduction as a result of the various controllers for the “Step” maneuvers. In general, the LPV controllers are able to reduce the RMS force the most of all the controllers. The LPV controller with a bandwidth of 0.06 per rev is able to reduce the RMS vibration by approximately 4.5 pounds, for both the on and off design cases, as compared to the CTHHC controller. Each respective controller performs similarly for both the on and off maneuver cases, with controllers for the on-design maneuver performing marginally better. As expected, increasing the bandwidth results in further reduction of the vibration levels.

Table 5.4: Flight controller results for step maneuvering in RCAS

<table>
<thead>
<tr>
<th>Controller Type</th>
<th>On-Design RMS Force (lb)</th>
<th>Percent Reduction</th>
<th>Off-Design RMS Force (lb)</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>130.070</td>
<td>-</td>
<td>182.354</td>
<td>-</td>
</tr>
<tr>
<td>CTHHC</td>
<td>7.054</td>
<td>94.57</td>
<td>7.124</td>
<td>94.47</td>
</tr>
<tr>
<td>( \mathcal{H}_\infty ), BW = 0.04</td>
<td>4.239</td>
<td>96.74</td>
<td>4.288</td>
<td>96.67</td>
</tr>
<tr>
<td>( \mathcal{H}_\infty ), BW = 0.06</td>
<td>3.077</td>
<td>97.63</td>
<td>3.084</td>
<td>97.61</td>
</tr>
<tr>
<td>LPV, BW = 0.04</td>
<td>3.497</td>
<td>97.31</td>
<td>3.515</td>
<td>97.27</td>
</tr>
<tr>
<td>LPV, BW = 0.06</td>
<td>2.518</td>
<td>98.06</td>
<td>2.501</td>
<td>98.06</td>
</tr>
</tbody>
</table>

Tables 5.5 and 5.6 display the vibration reduction for each respective controller for the remaining maneuvers. The LPV controller reduces the vibration the most across all of the additional maneuver types. In the “Linear” maneuver case, the LPV controller reduces the RMS force by 4.5 pounds over the CTHHC controller. The last two maneuvers considered represent the aggressive maneuvering flight conditions. It can be seen that the performance of the CTHHC controller drops off as the maneuver becomes more aggressive, especially for the “Doublet” maneuver. The LPV controller reduces the vibration over the CTHHC controller by 7.1 and 9.5 pounds for the “Fast Linear” and “Doublet” maneuvers, respectively. The LPV controller outperforms the
$\mathcal{H}_\infty$ controller but not as much as compared to the CTHHC controller.

Figures 5-11 to 5-15 compare the controllers against each other for one maneuver. Only the controllers with a bandwidth of 0.06 per rev are shown. The results previously discussed in Tables 5.4, 5.5, and 5.6 are visually apparent in these sets of figures. In general, it can be seen that the controller order of performance, from smallest to largest residual force, is LPV, $\mathcal{H}_\infty$, and CTHHC. For the flight profiles that end at $\mu = 0.36$, the CTHHC controller reduces the vibration the most when $\mu > 0.33$. This is expected because the LPV and $\mathcal{H}_\infty$ controllers are designed up to $\mu = 0.33$, whereas the CTHHC has a controller designed for $\mu = 0.36$.

Table 5.5: Flight controller results for linear maneuver types in RCAS

<table>
<thead>
<tr>
<th>Controller Type</th>
<th>Linear RMS Force (lb)</th>
<th>Linear Percent Reduction</th>
<th>Fast Linear RMS Force (lb)</th>
<th>Fast Linear Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>129.710</td>
<td>-</td>
<td>133.270</td>
<td>-</td>
</tr>
<tr>
<td>CTHHC</td>
<td>6.641</td>
<td>94.88</td>
<td>10.998</td>
<td>91.75</td>
</tr>
<tr>
<td>$\mathcal{H}_\infty$, BW = 0.06</td>
<td>2.676</td>
<td>97.94</td>
<td>4.883</td>
<td>96.34</td>
</tr>
<tr>
<td>LPV, BW = 0.06</td>
<td>2.112</td>
<td>98.37</td>
<td>3.938</td>
<td>97.04</td>
</tr>
</tbody>
</table>

Table 5.6: Flight controller results for doublet maneuver in RCAS

<table>
<thead>
<tr>
<th>Controller Type</th>
<th>Doublet RMS Force (lb)</th>
<th>Doublet Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>121.460</td>
<td>-</td>
</tr>
<tr>
<td>CTHHC</td>
<td>16.645</td>
<td>86.29</td>
</tr>
<tr>
<td>$\mathcal{H}_\infty$, BW = 0.06</td>
<td>8.289</td>
<td>93.17</td>
</tr>
<tr>
<td>LPV, BW = 0.06</td>
<td>7.165</td>
<td>94.09</td>
</tr>
</tbody>
</table>
Figure 5-11: Performance of controllers for on-design step maneuver

Figure 5-12: Performance of controllers for off-design step maneuver
Figure 5-13: Performance of controllers for linear maneuver

Figure 5-14: Performance of controllers for fast linear maneuver
5.4 Summary

This chapter demonstrates the performance of the three different vibration controllers in the nonlinear RCAS simulation environment. The controllers are first evaluated at fixed-flight conditions. All of the controllers perform well at the fixed-flight conditions, with the CTHHC controller outperforming the other controllers using the RMS vibration metric. Note that the worst vibration reduction for the controllers with a bandwidth of 0.06 per rev is 98.8%, indicating that the other controllers perform favorably. The LPV controller performs similarly to the CTHHC controller for the fixed-flight condition of $\mu = 0.13$.

The primary result of this thesis is the comparison of the controllers for maneuvering flight conditions with time-varying advance ratio. Five maneuvering conditions are considered: “Step On”, “Step Off”, “Linear”, “Fast Linear”, and “Doublet”. The latter two conditions represent the most aggressive maneuvers. In general, the LPV
controller outperforms the $\mathcal{H}_\infty$ and CTHHC controllers across all maneuvers. The superior performance capabilities of the LPV controller is evident in the most aggressive maneuvers, where the LPV controller reduces the baseline vibration 7.8\% more than the CTHHC controller for the “Doublet” maneuver.
Chapter 6

Conclusions

The topic of this thesis is the development of a helicopter model, and the design and evaluation of a controller to solve the helicopter vibration problem over the entire flight envelope.

The first task of this thesis was the development of the UH-60 Black Hawk helicopter model for the Rotorcraft Comprehensive Analysis System (RCAS) aeromechanical simulation environment. The helicopter was modeled using elastic rotor blades, linear and unsteady aerodynamics, and a 36 state Peters and He dynamic inflow model. The UH-60 model was used to characterize the baseline vibration loads and frequency based transfer function models of the helicopter for varying advance ratio, blade loading coefficient, and total rotor inflow ratio. The frequency based transfer function models were then used to develop linear time-invariant (LTI) state space models, which were used for the design synthesis of the LPV and $\mathcal{H}_\infty$ controllers.

There were three controllers considered for the helicopter vibration problem. The first was the Hall and Wereley [19] Continuous Time Higher Harmonic Controller (CTHHC). The second was a controller designed using the $\mathcal{H}_\infty$ synthesis method that followed the work of Fan [16] who adapted the $\mathcal{H}_\infty$ framework to the helicopter vibration problem. The $\mathcal{H}_\infty$ controller was designed using a performance weighting filter formed as an ideal linear notch characterized by the bandwidth and the peak sensitivity. The first two controllers represent the baseline controllers. The unique controller of this work was the Linear Parameter Varying (LPV) controller. The
LPV synthesis method provided a framework to design a controller over a varying parameter space with guaranteed performance and stability. This synthesis method contrasted with traditional gain-scheduling techniques, which do not provide any guarantees. The three controllers are evaluated in the RCAS simulation environment for fixed and maneuvering flight conditions. The performance of the controllers were compared using the root mean square (RMS) 4 per rev $F_z$ vibration force. For the maneuvering flight conditions, the order of performance, from smallest to largest residual force, was LPV, $\mathcal{H}_\infty$, and CTHHC. The benefit of designing a controller in the LPV framework was further demonstrated with the two most aggressive maneuvering flight conditions, “Fast Linear” and “Doublet”.

### 6.1 Contributions

The contributions of this thesis fall into two categories. The two areas are the helicopter dynamics in a simulation environment and the design of a controller for the entire helicopter flight envelope.

The conclusions drawn from this thesis in the area of helicopter dynamics in the simulation environment are:

1. The design of a vibration controller in a simulator must use an aeromechanical simulation environment. As was demonstrated in this thesis, flight dynamics codes do not display any rotor dynamics above the one per rev frequency.

2. The steady state vibration levels and rotor dynamics vary as a function of the advance ratio, blade loading coefficient, and the total inflow ratio. At the largest harmonic vibration, the steady state vibration levels and rotor dynamics depended on the advance ratio. Furthermore, the advance ratio is a measurable real-time parameter, as compared to the inflow parameters which cannot be easily measured.

3. The transfer function between the out-of-plane hub force vibration and the collective pilot input can be represented as a linear time-invariant model. Fur-
thermore, for the design of the LPV controller, the flight envelope of a helicopter can be represented as a linear parameter varying model. It was also shown that periodicity effects of the rotor are small using open-loop sinusoidal inputs to the helicopter.

The conclusions drawn from this thesis in the area of the design of a controller over the entire flight envelope are:

1. A helicopter vibration controller can be developed over the entire flight envelope using the Linear Parameter Varying control synthesis method. The input to the synthesis is the generalized augmented plant model that is an interconnection of the helicopter plant and the performance filter. As with the $\mathcal{H}_\infty$ framework, the performance filter was characterized by the bandwidth and the peak sensitivity.

2. The LPV synthesis method can be setup using a set of LTI state space models that characterize the parameter space at discretized points, called the Jacobi linearization method. Furthermore, the LPV synthesis can determine controllers from LTI state spaces models found using standard frequency based system identification techniques.

3. An algorithm that outlines the setup of the LPV synthesis method, specifically the formulation of the analysis linear matrix inequalities (LMI), was provided. This algorithm contributes to LPV literature by providing a clear way to set up the problem, which is largely missing in common LPV references.

4. A drawback to the LPV synthesis method is that the number of decision variables to be numerically determined is proportional to $n_p^2$, where $n_p$ is the order of the plant. Therefore, this method may not be appropriate for high order plants. Furthermore, the size of the discretization space impacts the size of the optimization problem. In this thesis, a viable controller was found for one varying parameter, four discrete points, and a plant order of $n_p = 15$.

5. The LPV controller performed favorability at fixed and maneuvering flight conditions. The LPV controller exhibited near complete disturbance rejection
at fixed-flight conditions. Furthermore, the LPV controller outperformed the CTHHC and $\mathcal{H}_\infty$ controllers in all of the maneuvering flights considered, which are designed for fixed-flight conditions. The improvements of the LPV controller over the other controllers were apparent throughout the transient stages of each maneuver. In the case of the most aggressive maneuver considered, the “Doublet” maneuver, the LPV controller reduced the baseline RMS vibration by 94.1%. In contrast, the CTHHC and $\mathcal{H}_\infty$ controllers reduced the baseline RMS vibration by 86.3% and 93.2%, respectively.

### 6.2 Future Research

The vibration problem is an important area of research for further acceptance of the helicopter. There are several areas to build off of this thesis to further address the vibration problem:

1. It is possible to characterize the LPV model of the helicopter in another way. There are three classic ways to represent an LPV model. The first, as used in this thesis, is the Jacobian linearization method, which finds a set of LTI state space models at each operating point of a parameter space [32]. The second method is a linear fractional transformation (LFT) description, which forms the LPV model as an interconnection of two blocks [4]. The first block represents the nominal dynamics as LTI and the second block contains the time varying parameters. This method is traditionally used when the dynamics of the helicopter model is fully known and characterized in ordinary differential equation form. The third method is an affine parameter-dependent (polytopic) representation of the LPV model. In the polytopic representation, the LPV model is characterized as

$$
G(\rho(\theta)) = 
\begin{bmatrix}
A_0 + \rho(\theta)A_1 & B_0 + \rho(\theta)B_1 \\
C_0 + \rho(\theta)C_1 & D_0 + \rho(\theta)D_1
\end{bmatrix}
$$

(6.1)
where

$$\rho(\theta) = \rho_0 + \rho_1 \theta + \rho_2 \theta^2 + \ldots + \rho_r \theta^r$$  \hspace{1cm} (6.2)

where $r$ is the order of the polynomial function and $\theta$ is the time-varying measurable parameter. In this current problem, $\theta$ would be represented by the advance ratio. The $\rho$ coefficients can be compactly written as

$$\rho = [\rho_0, \rho_1, \rho_2, \ldots, \rho_r]$$  \hspace{1cm} (6.3)

The task in characterizing the system in the affine-parameter dependent form is to find the unknown system matrices ($A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1$) and $\rho$ coefficients [11]. The Jacobian linearization method is effective, but the parameter space between the discrete points is not characterized. The affine parameter-dependent technique is a useful alternative to the Jacobian linearization method because the unknown matrices and $\rho$ can be found by nonlinear optimization. Furthermore, the benefit of characterizing the model in this manner is that the parameter dependency of the advance ratio can be fully known. Therefore, a controller can be designed that uses the real-time measurement of the advance ratio to adjust the control instead of gain-scheduled type methods.

2. The LPV controller in this work combined four different controllers, $K_\infty(s)$, using the output blending method at the advance ratios of $\mu \in [0.13, 0.20, 0.26, 0.33]$. One drawback with this method is that there is no stability guarantee of the oscillator component ($W_1(s)$) at points in between the design advance ratios. In Chapter 4, the LPV synthesis determines the controller, $\hat{K}_\infty(s)$, and then the controller is multiplied by the weighting filter, $W_1(s)$, to form $K_\infty(s)$. However, all controllers across the different advance ratios have the same weighting filter, $W_1(s)$. Therefore, rather than blending the output of $K_\infty(s)$, these controllers should be evaluated in an aeromechanical environment where $\hat{K}_\infty(s)$ is output blended. Then the output blended solution must be multiplied by the oscillator,
$W_1(s)$, to form the final control command. Figure 6-1 displays the format that future implementation of an LPV controller should use. By combining the LPV controllers as in Figure 6-1, the oscillator component, $W_1$, of the controller is at least marginally stable. It is also possible to form the oscillator, $W_1(s)$, using the modulation/demodulation scheme.

Figure 6-1: Future implementation of LPV controllers

3. Further research must be conducted in the area of characterizing the transient response of the helicopter dynamics and vibration levels in wind tunnel or flight test. Better characterization of real vibration levels will provide the necessary datasets to compare simulation and develop effective controllers. Specifically, maneuvers similar to the ones examined in this thesis should be considered.

4. While the focus of this thesis is on the development of a controller for the helicopter vibration problem, an equally important area of research is further research on HHC or IBC actuation methods. Despite several decades of research, neither HHC nor IBC methods have bought their way onto a production helicopter program. It is expected that further advances in materials science will aid this adoption, but advances can be made in the near term by focusing on risk reduction of current IBC and HHC technologies.
Appendix A

RCAS Overview

A.1 History

The nonlinear simulator used in this thesis is the Rotorcraft Comprehensive Analysis System (RCAS) aeromechanical code. The origin of RCAS began with an aeromechanical code developed in the late 1970s called 2GCHAS, which was the collaboration of the U.S. Army, research organizations, and universities [26]. The first public release of 2GCHAS was in 1990. In the mid-1990s, the U.S. Army Aeroflightdynamics Directorate (AFDD) sought to improve the capabilities of 2GCHAS, which had limitations. Two example limitations included, 2GCHAS did not allow for large rigid body motions nor small elastic structural deformations. The Advanced Rotorcraft Technology, Inc. (ART) improved the limitations of 2GCHAS for the U.S. Army from 1998 to 2000 [26]. Furthermore, ART incorporated an interactive environment for analysis called Rscope. The first formal release of RCAS was in 2003 [44].

A.2 RCAS Tool Overview

RCAS is a comprehensive, multi-disciplinary code that has the ability to model multiple rotorcraft configurations, such as tandem and tilt-rotor helicopters. RCAS is capable of performing a variety of analyses, including helicopter performance, aerodynamics and rotor loads, structural vibration, control system analysis, aeroelastic
stability, flight dynamics, and flight simulation (similar to conventional flight dynamics codes). It can perform these analyses in a range of operating conditions including hover, forward flight, and maneuvering flight. A structural model in RCAS is built by a hierarchy of finite element components, beginning with the largest component and progressing to the smallest element (system → subsystem → primitive structure). In addition to modeling structural components as rigid beams, RCAS has the ability to model helicopter rotor blade as nonlinear beams. Similar to the hierarchy of the structural model, the aerodynamic model is also built in a hierarchy (model → supercomponent → component). Note that the structural and aerodynamics model are separate in RCAS. RCAS includes multiple aerodynamic modeling options, including linear steady and unsteady rotor aerodynamics, non-linear steady and unsteady look-up tables, advanced dynamic stall models, yawed flow effects, and compressibility effects. In addition, RCAS has the ability to model various rotor inflow models, ranging from no inflow, uniform inflow, to the Peters and He dynamic inflow model.

The helicopter model is typically built in a text script file and is easily navigated in the RCAS simulation environment using the interactive Rscope environment. Rscope is a key environment for verifying the correct construction of a helicopter model and addressing any simulation failures. RCAS automatically defaults to the Rscope environment upon a critical simulation failure. There also exists a graphical user interface version of RCAS, called GRCAS, that is distributed by ART. GRCAS is not distributed with the free version of RCAS available to U.S. citizens. While the graphical interface version of RCAS is helpful for new users, it is possible to effectively generate models and verify their construction in the Rscope environment.

Another key aspect of RCAS is its control system modeling capabilities. RCAS has the ability to model simple control systems through its script input interface or complex flight control systems (i.e., nonlinear elements) through an advanced graphical user interface, which is called the Control System Graphical Editor (CSGE). A view of the CSGE environment, including library blocks, is shown in Figure A-1. The CSGE environment includes a library of linear and nonlinear elements, such as transfer functions, state space models, gains, limit blocks, summing junctions, lookup
tables, and signal switches. The CSGE control environment can be interfaced with any node in the Rscope model, therefore it is easily possible to measure a vibration level at any point in a helicopter model and provide a corresponding control response.

A key discussion for simulation codes is an understanding of the analysis capabilities and the methods by which the analysis is performed. RCAS provides three basic analysis capabilities: trim, linear analysis (stability), and nonlinear analysis (maneuvering). Figure A-2 shows a diagram of the analysis capabilities of RCAS.

Within the trim capability, RCAS provides three different trim techniques: trim-to-target, periodic solution, and static equilibrium. A trim-to-target procedure is similar to typical trim procedures for simulation codes, meaning the code calculates the pilot inputs required for a specified vehicle orientation or state such that the free forces and moments are balanced. Helicopter trim-to-target falls into two broad categories: free flight conditions and wind tunnel conditions. In free flight trim, all six degrees of freedom of the helicopter are active and all three moments and forces must be balanced. In the case of wind tunnel trim, typically there are only two free forces and one moment to balance. A typical force and moment combination for wind tunnel trim is shown in Figure A-2. Many other types of wind tunnel trim targets exist for helicopters.
The second and third trim topics are periodic solution and static equilibrium. A periodic solution is found when the rotor blade states converge to the same values over a specified number of rotor revolutions. This trim technique is sufficient for analyses where the analysis is only concerned with rotor response due to pilot inputs. A static equilibrium solution is found by solving the system equations, but with time dependent terms removed. This trim technique is useful for operating conditions where the helicopter motion is not time varying, such as hover.

The second major analysis capability of RCAS is a linear analysis (stability). In this analysis technique, the equations of motions are numerically linearized about a steady state or a periodic equilibrium state. This linearization is performed internally by RCAS for each of the two options using the Constant Coefficient Equations Model (CCE) or the Periodic Coefficient Equations Model (PCE), respectively. The user can then use the linearized model to develop linear transfer functions or frequency responses for a given input signal.

The third major analysis capability of RCAS is the nonlinear transient method, which is effectively a maneuvering helicopter. The nonlinear transient analysis is a time domain based and is found by numerically integrating the equations of motion in a time marching scheme. RCAS uses the Hilbert-Hughes-Taylor (HHT) integration
method, which is based on the Newmark-Beta integration method, to perform the transient analysis. The HHT integration method is used because it provides damping of high frequency modes without impacting low frequencies.

### A.3 RCAS Time Integration Method

Discussion of the integration method is common when reviewing results from aeromechanical codes. The integration of the equations of motion from one time step to the next using the Hilbert-Hughes-Taylor (HHT) method can be written as

\[
M \ddot{X}_{n+1} + C \dot{X}_{n+1} + K X_{n+1} = F_{n+1} + \alpha_{\text{HHT}} \left( F_{n+1} - F_n \right) - \alpha_{\text{HHT}} C \left( \dot{X}_{n+1} - \dot{X}_n \right) - \alpha_{\text{HHT}} K \left( X_{n+1} - X_n \right)
\]  

(A.1)

where \( M, C, \) and \( K \) are the mass, damping, and stiffness matrices, respectively. \( X \) is the vector of helicopter states and \( F \) is the sum of inertial and aerodynamic forces acting on the helicopter. \( n \) is the time step number. Note that the HHT integration method includes additional terms on the right hand side of the equation, which have the effect of damping high frequency modes. The RCAS user has the capability to adjust the \( \alpha_{\text{HHT}} \) parameter to impact the amount of damping on high frequency modes. Equation (A.1) can be written in terms of residuals as

\[
\dot{Q} = F_{n+1} - M \ddot{X}_{n+1} - C \dot{X}_{n+1} - K X_{n+1} + \alpha_{\text{HHT}} \left( F_{n+1} - C \dot{X}_{n+1} - K X_{n+1} \right) - \alpha_{\text{HHT}} \left( F_n - C \dot{X}_n - K X_n \right)
\]

(A.2)

where \( \dot{Q} \) is the residual and is zero when the equation is in equilibrium. It is the goal for a given time step to iteratively calculate the helicopter states such that residuals are zero. This iterative calculation is performed using the Newton-Raphson integration method. After a series of manipulations, as shown in the RCAS Theory
Manual [2], Eq. [A.2] becomes

\[ \dot{Q}^i = -[(1 - \alpha_{\text{HHT}})M + Ca_0 + Ka_1] \ddot{X}^i_{n+1} + F_{n+1} + \alpha_{\text{HHT}} Q_{n+1} \quad (A.3) \]

\[ - (a_2 C + a_4 K + \alpha_{\text{HHT}} M) \ddot{X}_n - (C + a_3 K) \dot{X}_n - KX_n - \alpha_{\text{HHT}} Q_n \]

where \( i \) is the Newton-Raphson iteration number within a given time step, and \( a_1, a_2, a_3, a_4 \) are the Newmark-Beta constants defined as

\[ a_0 = \delta \Delta t \quad (A.4) \]
\[ a_1 = \alpha \Delta t^2 \quad (A.5) \]
\[ a_2 = (1 - \delta) \Delta t \quad (A.6) \]
\[ a_3 = \Delta t \quad (A.7) \]
\[ a_0 = \left( \frac{1}{2} - \alpha \right) \Delta t^2 \quad (A.8) \]

and the \( \delta \) and \( \alpha \) parameters are defined in the HHT method using the \( \alpha_{\text{HHT}} \) as

\[ \delta = \frac{1}{2} - \alpha_{\text{HHT}} \quad (A.9) \]
\[ \alpha = \frac{1}{4}(1 - \alpha_{\text{HHT}})^2 \quad (A.10) \]

Note that \( \Delta t \) is the step size of the time integration and is controlled by the user. It is possible to output data from every time step, which is the method used in this thesis. Equation [A.3] is further simplified by replacing all terms from the previous time step by \( g(n) \) and lumping the next time step state acceleration vector coefficient into an effective mass matrix, \( \hat{M} \), and is represented by

\[ \dot{Q}^i = -\hat{M} \ddot{X}^i_{n+1} + F_{n+1} + \alpha_{\text{HHT}} Q_{n+1} + g(n) \quad (A.11) \]
In order to find convergence within a given time step, it is desirable to find $\ddot{X}_{n+1}^{i+1}$ such that $\dot{Q}^{i+1}$ is zero. Equation A.11 is rewritten as

$$\dot{Q}^{i+1} = 0 = -\dot{M}\ddot{X}_{n+1}^{i+1} + F_{n+1} + \alpha_{\text{HHT}}Q_{n+1} + g(n)$$

(A.12)

By subtracting Equation A.12 from Equation A.11, it is possible to relate the residual to the state acceleration vector

$$\dot{Q}^i = \dot{M}\delta\ddot{X}_{n+1}^i$$

(A.13)

It is possible to use this result to find the next Newton-Raphson step of the state acceleration vector by

$$\ddot{X}_{n+1}^{i+1} = \ddot{X}_{n+1}^i + \delta\ddot{X}_{n+1}^i$$

$$= \ddot{X}_{n+1}^i + \beta\dot{M}^{-1}\dot{Q}^i$$

(A.14)

where $\beta$ is an under-relaxation parameter that can be specified by the RCAS user. The states and first derivative can then be found using the Newmark-Beta integration relations shown as

$$\dot{X}_{n+1}^{i+1} = \dot{X}_n + a_0\ddot{X}_{n+1}^{i+1} + a_2\ddot{X}_n$$

$$X_{n+1}^{i+1} = X_n + a_1\dot{X}_{n+1}^{i+1} + a_3\ddot{X}_{n+1}^{i+1} + a_4\dot{X}_n$$

(A.15)

(A.16)

A summary of the computation procedure to perform the nonlinear transient maneuver is shown in Algorithm 2. It is important to note that the inverse of the effective mass matrix can be computed every time step, as indicated in Algorithm 2. By default, RCAS does not recalculate this matrix every time step. However, in conversations with Dr. Hossein Saberi and Dr. Robert Ormiston, it is recommended to reassemble the effective mass matrix for large variations in the transient maneuver.
Algorithm 2: RCAS Nonlinear Transient Maneuver Integration

Require: Initial Conditions (from trim): $\tilde{X}_n, \dot{X}_n, X_n$

1: for $i = 1 \rightarrow \# \text{ of Time Steps}$ do
2:  \hspace{1cm} Compute $\hat{M}^{-1}$
3:  \hspace{1cm} for $j = 1 \rightarrow \text{Maximum Number of Newton-Raphson Iterations}$ do
4:  \hspace{2cm} Calculate residual force: $\hat{Q}^i$
5:  \hspace{2cm} Calculate acceleration vector delta: $\delta X_{n+1}^i = \hat{M}^{-1}\hat{Q}^i$
6:  \hspace{2cm} Compute $\dot{X}_{n+1}^{i+1} = \dot{X}_{n+1}^i + \beta \hat{M}^{-1}\hat{Q}^i$
7:  \hspace{2cm} Compute $\ddot{X}_{n+1}^{i+1} = \ddot{X}_{n+1}^i + a_0 \ddot{X}_{n+1}^i + a_2 \dddot{X}_n$
8:  \hspace{2cm} Computer $X_{n+1}^{i+1} = X_n + a_1 \dddot{X}_{n+1} + a_3 \dddot{X}_{n+1} + a_4 \dddot{X}_n$
9:  \hspace{1cm} Is convergence achieved? ($\hat{Q}^i < \text{tolerance}$)
10: end for
11: end for
Bibliography


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