THERMAL STRATIFICATION IN ENCLOSED FLUIDS DUE TO NATURAL CONVECTION

by

Edward S. Matulevicius

B. Eng., McGill University, 1964
S.M., Massachusetts Institute of Technology, 1966

Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Science at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY January, 1970

Signature of Author:

Department of Chemical Engineering,

Certified by: Thesis Supervisors

Accepted by: Chairman, Departmental Committee on Graduate Thesis

Archives

mass. inst. tech.

APR 16 1970
DISCLAIMER NOTICE

Due to the condition of the original material, there are unavoidable flaws in this reproduction. We have made every effort possible to provide you with the best copy available.

Thank you.

Despite pagination irregularities, this is the most complete copy available.
The following page numbers are missing from this thesis: pp 24, 25, 26, 154, 155, 156, 157, 158 and 176.

However this is an error in pagination and there are, in fact, no pages missing from the thesis. Two additional pages 64a and 80a have been numbered in a non-standard manner.

Signature Redacted

Lawrence B. Evans
Thesis Supervisor
THERMAL STRATIFICATION IN ENCLOSED FLUIDS
DUE TO NATURAL CONVECTION

by

Edward Matulevicius

Submitted to the Department of Chemical Engineering on February 2, 1970, in partial fulfillment of the requirements for the degree of Doctor of Science.

ABSTRACT

A study was undertaken of the natural convection phenomena which results in thermal stratification of a fluid enclosed in a heated container. A combined theoretical and experimental investigation was carried out.

In the theoretical work, the partial differential equations describing natural convection boundary layer flow near a vertical flat plate exposed to a non-isothermal core fluid were solved numerically using a finite difference technique. Solutions were obtained for the case where a uniformly heated vertical wall was placed in surroundings whose temperature increased linearly with height. It was found that after a sufficient distance from the bottom of the wall, the steady-state temperature and velocity profiles could be represented in dimensionless form by simple asymptotic equations.

In the experimental work, a constant heat flux was applied to the sidewalls of a rectangular container filled with liquid. The resulting temperature and velocity fields were then observed. The transient temperature field was measured by an array of fine wire thermocouples placed near the central plane of the container. Fluid motion was observed by taking time-exposure photographs of neutrally buoyant spheres immersed in the fluid. A quantitative determination of the transient velocity field and the stream-
lines were obtained from the streak photographs. Water and glycerin were used as the test fluids. Experiments were made for a range of Prandtl Number between 7 and 12000, Modified Rayleigh Number from $10^6$ to $10^{13}$, and the Aspect Ratio of the enclosure height to width from 1 to 3.

Thermal stratification of the fluid resulted from the interaction of two regions: The boundary layer region where the warm fluid rose towards the top of the container, and a core region where the displaced cooler fluid settled to the bottom. In this core region, the temperature was isothermal in the horizontal plane while it varied axially. The non-isothermal axial temperature profile determined the amount of fluid flowing in the boundary layer at a given height for any set of fluid conditions.

A theoretical model describing the fluid behaviour throughout the enclosure was developed by coupling the mathematical description of the boundary layer flow with the partial differential equations describing core fluid behaviour. This model forms the basis of a computer program which predicts the axial temperature profile and the volumetric flow rate of the fluid in the core as a function of time and depth. It was found that the axial temperature and velocity distribution could be correlated by dimensionless parameters independent of Prandtl Number for values of Prandtl Number greater than seven.

THESIS SUPERVISORS:
Professor R. C. Reid, Professor of Chemical Engineering
Professor L. B. Evans, Associate Professor of Chemical Engineering
Dear Professor E. Neal Hartley,

I herewith submit a thesis, entitled "Thermal Stratification in Enclosed Fluids due to Natural Convection", in partial fulfillment of the requirements for the degree of Doctor of Science in Chemical Engineering at the Massachusetts Institute of Technology.

Respectfully submitted,

Signature redacted

Edward S. Matulevicius
To my parents
ACKNOWLEDGEMENT

The subject of this work was suggested to the author by his supervisors, Professor R. E. Reid and Professor L. B. Evans. Especial thanks are extended to them for their continued interest and moral support during the period of this work.

Certainly, no one could have worked in the Fuels Research Laboratory without becoming indebted to Herbert Passler. His aid in the construction of the experimental equipment and his friendship are sincerely appreciated.

In the course of this thesis the author was fortunate to have had the association of his fellow doctoral candidates in the Fuels Research Laboratory. Sincere thanks for their interest and the seemingly interminable bull sessions are offered to Alvin Witt, Peter Jones, Gary Mellinger, Makato Ohara, Luis Lema, Iacovos Vasalos, Peter Pan, Professor William Dalzell, and Professor Jack Howard. Heartfelt appreciation is also proferred to Professor James Noble and Lloyd Clomburg for their many helpful suggestions and discussions especially in the computational phase of this work.

Economic support for this work was provided by the Center for Space Research at Massachusetts Institute of Technology and is gratefully acknowledged. Likewise, the author wishes to acknowledge the use of the computer facilities at the M.I.T. Computation Center.

Lastly, the author is indebted to Mrs. Barbara Gendron for her patient and fastidious effort in typing the manuscript.
# TABLE OF CONTENTS

**ACKNOWLEDGEMENT**

I SUMMARY

1.1 Solution of the Boundary Layer Equations for the Case of a Uniformly Heated Vertical Wall in Non-Isothermal Surroundings

1.1.1 Introduction

1.1.2 Boundary Layer Equations

1.1.3 A Special Solution

1.1.4 Dimensionless Equations

1.1.5 Boundary Layer Solution Results

1.1.6 Discussion and Conclusions

1.2 Model for Thermal Stratification of Contained Fluid Subjected to a Constant Sidewall Heat Flux

1.2.1 Introduction

1.2.2 Prior Investigations

1.2.3 Present Approach

1.2.4 Experimental

1.2.5 Test Results

1.2.6 Formulation of a Model

1.2.6.1 Boundary Layer

1.2.7 Conclusions

II INTRODUCTION

2.1 Thermal Stratification Phenomenon

2.2 Applications of the Work

2.3 Natural Convection Equations

2.3.1 The Boussinesq Approximation

2.3.2 The Vorticity Transport Equation

2.3.3 The Two-Dimensional Equations

2.3.4 Boundary Conditions

2.3.5 Boundary Conditions for the Present Study

2.3.6 Dimensionless Equations
5.4.1.3 The Velocity and Temperature as a Function of Height 159

5.4.2 Surroundings Where the Temperature Increases Linearly with Height 164

5.4.2.1 The Effect of Prandtl Number 164

5.4.2.2 Velocity and Temperature Profiles as a Function of Plate Height 168

5.4.3 The Effect of a Linear Bulk Temperature Gradient on the Natural Convection from a Uniformly Heated Vertical Plate 168

VI A MODEL FOR THERMAL STRATIFICATION IN ENCLOSED FLUIDS 173

6.1 Introduction 173

6.2 Thermal Stratification Model 173

6.2.1 Boundary Layer Momentum and Energy Flows 174

6.2.1.1 High Prandtl Number Case 175

6.2.2 Thermal Stratification Simulation 181

6.2.2.1 Boundary Layer Material and Energy Balances 185

6.2.2.2 Core Material and Energy Balances 187

6.2.2.3 Dimensionless Equations 188

6.2.2.4 Finite Difference Approximations 190

6.3 Thermal Stratification Model 192

6.3.1 Convergence 192

6.3.2 The Effect of Conduction 193

6.3.3 The Effect of $C_f$ on the Solution 194

6.4 Data Correlation 194

6.4.1 Temperature Correlation 194
J. Experimental Data 349
K. Thermal Stratification Simulation Data 356
L. Generalization of Drake's Model 381
M. Finite Difference Equations for the Uniformly Heated Vertical Plate Boundary Layer Equations 385
N. Data Reduction Procedures 450
O. Biographical Note 454
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Coordinates for Boundary Layer Solution</td>
<td>28</td>
</tr>
<tr>
<td>1.2</td>
<td>Uniformly Heated Vertical Plate in Isothermal Surroundings: The Effect of Prandtl Number</td>
<td>33</td>
</tr>
<tr>
<td>1.3</td>
<td>Uniformly Heated Vertical Plate in Non-Isothermal Surroundings: The Effect of Prandtl Number on the Temperature Profile</td>
<td>34</td>
</tr>
<tr>
<td>1.4</td>
<td>Uniformly Heated Vertical Plate in Non-Isothermal Surroundings: The Effect of Prandtl Number on the Velocity Profile</td>
<td>35</td>
</tr>
<tr>
<td>1.5</td>
<td>Uniformly Heated Vertical Plate in Non-Isothermal Surroundings: The Effect of Prandtl Number on the Horizontal Velocity Profile</td>
<td>36</td>
</tr>
<tr>
<td>1.6</td>
<td>Force Diagram for Momentum Equation: Pr=10, Ra=10^5</td>
<td>38</td>
</tr>
<tr>
<td>1.7</td>
<td>Experimental Container</td>
<td>44</td>
</tr>
<tr>
<td>1.8</td>
<td>Axial Temperature Profiles: Pr=7, Ra=1.81(10^{12})</td>
<td>47</td>
</tr>
<tr>
<td>1.9</td>
<td>Streamlines for Test G-3-Hi, τ=1.33(10^{-2})</td>
<td>48</td>
</tr>
<tr>
<td>1.10</td>
<td>Streamlines for Test G-3-Hi, τ=3.72(10^{-2})</td>
<td>49</td>
</tr>
<tr>
<td>1.11</td>
<td>Streamlines for Test G-3-Hi, τ=9.63(10^{-2})</td>
<td>50</td>
</tr>
<tr>
<td>1.12</td>
<td>Thermal Stratification Model</td>
<td>56</td>
</tr>
<tr>
<td>1.13</td>
<td>Integrated Energy Parameter as a Function of Height</td>
<td></td>
</tr>
<tr>
<td>1.14</td>
<td>Differential Element Used for Development of Stratification Model</td>
<td></td>
</tr>
<tr>
<td>1.15</td>
<td>Division of the Container for the Stratification Model</td>
<td>61</td>
</tr>
<tr>
<td>1.16</td>
<td>The Effect of Conduction on the Thermal Stratification Simulation</td>
<td>63</td>
</tr>
<tr>
<td>1.17</td>
<td>Core Temperature Profiles</td>
<td>64</td>
</tr>
<tr>
<td>1.18</td>
<td>The Effect of C_f on the Temperature Solution</td>
<td>66</td>
</tr>
<tr>
<td>1.19</td>
<td>Maximum Streamline Profiles for Test G-3-Hi</td>
<td>67</td>
</tr>
<tr>
<td>2.1</td>
<td>Two-Dimensional Container Coordinate System</td>
<td>80a</td>
</tr>
<tr>
<td>3.1</td>
<td>Experimental Apparatus</td>
<td>99</td>
</tr>
<tr>
<td>FIGURE</td>
<td>TITLE</td>
<td>PAGE</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>3.2</td>
<td>Experimental Container</td>
<td>101</td>
</tr>
<tr>
<td>3.3</td>
<td>Temperature Measurement System</td>
<td>103</td>
</tr>
<tr>
<td>3.4</td>
<td>Arrangement of Thermocouples</td>
<td>105</td>
</tr>
<tr>
<td>3.5</td>
<td>Lighting System</td>
<td>106</td>
</tr>
<tr>
<td>4.1</td>
<td>Temperature Vs. Time Readings&lt;br&gt;Pr = 7, $Ra = 10^{13}$</td>
<td>112</td>
</tr>
<tr>
<td>4.2</td>
<td>Axial Temperature Profiles&lt;br&gt;Pr = 7, $Ra = 10^{13}$</td>
<td>113</td>
</tr>
<tr>
<td>4.3</td>
<td>Axial Temperature Profiles&lt;br&gt;Pr = 7, $Ra = 1.81(10^{12})$</td>
<td>114</td>
</tr>
<tr>
<td>4.4</td>
<td>Axial Temperature Profiles&lt;br&gt;Pr = 12500, $Ra = 6.9 (10^{10})$</td>
<td>115</td>
</tr>
<tr>
<td>4.5</td>
<td>Streak Photographs, Test W-2-Hi&lt;br&gt;Pr = 7, $Ra = 1.81(10^{12})$&lt;br&gt;(a) 120 sec., (b) 360 sec.</td>
<td>117</td>
</tr>
<tr>
<td>4.6</td>
<td>Streak Photographs, Test W-2-Hi&lt;br&gt;Pr = 7, $Ra = 1.81 (10^{12})$&lt;br&gt;(a) 940 sec., (b) 2250 sec.</td>
<td>118</td>
</tr>
<tr>
<td>4.7</td>
<td>Streak Photographs, Test C-3-Hi&lt;br&gt;Pr = 12500, $Ra = 6.9 (10^{10})$&lt;br&gt;(a) 180 sec., (b) 245 sec.</td>
<td>119</td>
</tr>
<tr>
<td>4.8</td>
<td>Streak Photographs, Test C-3-Hi&lt;br&gt;Pr = 12500, $Ra = 6.9 (10^{10})$&lt;br&gt;(a) 525 sec., (b) 690 sec.</td>
<td>120</td>
</tr>
<tr>
<td>4.9a</td>
<td>Streamlines for Test G-3-Hi, $\tau = 1.33(10^{-2})$</td>
<td>123</td>
</tr>
<tr>
<td>4.9b</td>
<td>Streamlines for Test G-3-Hi, $\tau = 3.72(10^{-2})$</td>
<td>124</td>
</tr>
<tr>
<td>4.9c</td>
<td>Streamlines for Test G-3-Hi, $\tau = 9.63(10^{-2})$</td>
<td>125</td>
</tr>
<tr>
<td>4.10a</td>
<td>Streamlines for Test W-2-Hi, $\tau = 5.73(10^{-1})$</td>
<td>126</td>
</tr>
<tr>
<td>4.10b</td>
<td>Streamlines for Test W-2-Hi, $\tau = 8.58(10^{-1})$</td>
<td>127</td>
</tr>
<tr>
<td>5.1</td>
<td>Uniformly Heated Vertical Plate Coordinate System</td>
<td>135</td>
</tr>
<tr>
<td>5.2</td>
<td>Uniformly Heated Vertical Plate in Isothermal Surroundings: The Effect of Prandtl Number</td>
<td>145</td>
</tr>
<tr>
<td>5.3</td>
<td>The Effect of Prandtl Number on the Horizontal Velocity</td>
<td>146</td>
</tr>
<tr>
<td>FIGURE</td>
<td>TITLE</td>
<td>PAGE</td>
</tr>
<tr>
<td>--------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>5.4a</td>
<td>Force Diagram for Momentum Equation</td>
<td>148</td>
</tr>
<tr>
<td></td>
<td>( Pr = 1, \  R \alpha = 10^7 )</td>
<td></td>
</tr>
<tr>
<td>5.4b</td>
<td>Force Diagram for Momentum Equation</td>
<td>149</td>
</tr>
<tr>
<td></td>
<td>( Pr = 10, \  R \alpha = 10^6 )</td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>Force Diagram for Momentum Equation</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td>( Pr = 100, \  R \alpha = 10^6 )</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>Uniformly Heated Vertical Plate in Isothermal Surroundings: The Effect of Rayleigh Number</td>
<td>152</td>
</tr>
<tr>
<td>5.7</td>
<td>The Effect of Modified Rayleigh Number on the Horizontal Velocity</td>
<td>153</td>
</tr>
<tr>
<td>5.8</td>
<td>Force Diagram for Momentum Equation</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>( Pr = 10, \  R \alpha = 10^9 )</td>
<td></td>
</tr>
<tr>
<td>5.9</td>
<td>Force Diagram for Momentum Equation</td>
<td>161</td>
</tr>
<tr>
<td></td>
<td>( Pr = 10, \  R \alpha = 10^{12} )</td>
<td></td>
</tr>
<tr>
<td>5.10</td>
<td>Vertical Velocity Profiles as a Function of Plate Height</td>
<td>162</td>
</tr>
<tr>
<td>5.11</td>
<td>Temperature Profiles as a Function of Plate Height</td>
<td>163</td>
</tr>
<tr>
<td>5.12</td>
<td>Uniformly Heated Vertical Plate in Non-Isothermal Surroundings: The Effect of Prandtl Number on the Temperature Profile</td>
<td>165</td>
</tr>
<tr>
<td>5.13</td>
<td>Uniformly Heated Vertical Plate in Non-Isothermal Surroundings: The Effect of Prandtl Number on the Velocity Profiles</td>
<td>166</td>
</tr>
<tr>
<td>5.14</td>
<td>Uniformly Heated Vertical Plate in Non-Isothermal Surroundings: The Effect of Prandtl Number on the Horizontal Velocity Profile</td>
<td>167</td>
</tr>
<tr>
<td>5.15</td>
<td>Temperature as a Function of Plate Height</td>
<td>169</td>
</tr>
<tr>
<td>5.16</td>
<td>Velocity as a Function of Plate Height</td>
<td>170</td>
</tr>
<tr>
<td>6.1</td>
<td>Integrated Energy Parameter as a Function of Height</td>
<td>178</td>
</tr>
<tr>
<td>6.2</td>
<td>Comparison of the Energy Parameter Calculated with the High Prandtl Number Assumption to the Solution Calculated for Prandtl Number = 10</td>
<td>179</td>
</tr>
<tr>
<td>6.3</td>
<td>Division of the Container for the Stratification Model</td>
<td>182</td>
</tr>
<tr>
<td>6.4</td>
<td>Differential Element Used for Development of Stratification Model Equations</td>
<td>186</td>
</tr>
<tr>
<td>FIGURE</td>
<td>TITLE</td>
<td>PAGE</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>6.5</td>
<td>The Effect of Grid Spacing on the Temperature Solution</td>
<td>192</td>
</tr>
<tr>
<td>6.6</td>
<td>The Effect of Conduction on the Thermal Stratification Simulation</td>
<td>195</td>
</tr>
<tr>
<td>6.7</td>
<td>The Effect of Conduction on the Thermal Stratification Simulation</td>
<td>196</td>
</tr>
<tr>
<td>6.8</td>
<td>The Effect of $C_p$ on the Temperature Solution</td>
<td>197</td>
</tr>
<tr>
<td>6.9a</td>
<td>Core Temperature Profiles</td>
<td>198</td>
</tr>
<tr>
<td>6.9b</td>
<td>Core Temperature Profiles</td>
<td>199</td>
</tr>
<tr>
<td>6.10</td>
<td>Maximum Streamlines for Test G-3-Hi</td>
<td>205</td>
</tr>
<tr>
<td>6.11</td>
<td>Maximum Streamlines for Test W-2-Hi</td>
<td>206</td>
</tr>
<tr>
<td>6.12a</td>
<td>Core Velocity Profile</td>
<td>208</td>
</tr>
<tr>
<td>6.12b</td>
<td>Core Velocity Profile</td>
<td>209</td>
</tr>
<tr>
<td>6.12c</td>
<td>Core Velocity Profile</td>
<td>210</td>
</tr>
<tr>
<td>6.12d</td>
<td>Core Velocity Profile</td>
<td>211</td>
</tr>
<tr>
<td>7.1</td>
<td>Example Container System</td>
<td>219</td>
</tr>
<tr>
<td>C.1</td>
<td>Vertical Plate Problem with a Non-Isothermal Core</td>
<td>256</td>
</tr>
<tr>
<td>E.1</td>
<td>Block Diagram - PLATE Program</td>
<td>268</td>
</tr>
<tr>
<td>F.1</td>
<td>Axial Temperature Profiles</td>
<td>286</td>
</tr>
<tr>
<td>F.2</td>
<td>Front and Rear Walls</td>
<td>286</td>
</tr>
<tr>
<td>F.3</td>
<td>Heat Flux Through Front Wall</td>
<td>286</td>
</tr>
<tr>
<td>F.4</td>
<td>Bottom Surface Temperature</td>
<td>294</td>
</tr>
<tr>
<td>F.5</td>
<td>Heat Flux Through Container Bottom</td>
<td>294</td>
</tr>
<tr>
<td>F.6</td>
<td>Sidewall Temperature Distribution</td>
<td>296</td>
</tr>
<tr>
<td>F.7</td>
<td>Heat Loss Through Sidewalls</td>
<td>297</td>
</tr>
<tr>
<td>G.1</td>
<td>Block Diagram of STRAT</td>
<td>307</td>
</tr>
<tr>
<td>G.2</td>
<td>Sample Temperature Output</td>
<td>318</td>
</tr>
<tr>
<td>G.3</td>
<td>Sample Velocity Output</td>
<td>319</td>
</tr>
<tr>
<td>G.4</td>
<td>Sample Temperature Output</td>
<td>320</td>
</tr>
<tr>
<td>G.5</td>
<td>Sample Velocity Output</td>
<td>321</td>
</tr>
<tr>
<td>H.1</td>
<td>Refraction Effects Determining Particle Positions</td>
<td>322</td>
</tr>
<tr>
<td>FIGURE</td>
<td>TITLE</td>
<td>PAGE</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>H.2</td>
<td>Differences Between Observed and Actual Positions Due to Refraction Effects</td>
<td>325</td>
</tr>
<tr>
<td>H.3</td>
<td>Determination of True Length of Streak Line</td>
<td>325</td>
</tr>
<tr>
<td>L.1</td>
<td>Generalized Cylinder</td>
<td>382</td>
</tr>
<tr>
<td>M.1</td>
<td>Grid System for the Finite Difference Scheme</td>
<td>386</td>
</tr>
<tr>
<td>M.2</td>
<td>The Effect of IMAX on the Boundary Layer Solution when $T_{\infty} = T_0$</td>
<td>398</td>
</tr>
<tr>
<td>M.3</td>
<td>The Effect of JMAX on the Boundary Layer Solution when $T_{\infty} = T_0$</td>
<td>399</td>
</tr>
<tr>
<td>M.4</td>
<td>The Effect of YMAX on the Boundary Layer Solution when $T_{\infty} = T_0$</td>
<td>401</td>
</tr>
<tr>
<td>M.5</td>
<td>Comparison of the Computed Velocity and Temperature Profiles to the Classic Similar Solution, $Pr=1$, $Ra=10$</td>
<td>403</td>
</tr>
<tr>
<td>M.6</td>
<td>Comparison of the Computed Velocity and Temperature Profiles to the Classic Similar Solution, $Pr=10$, $Ra=10^6$</td>
<td>404</td>
</tr>
<tr>
<td>M.7</td>
<td>Comparison of the Computed Velocity and Temperature Profiles to the Classic Similar Solution, $Pr=100$, $Ra=10^{12}$</td>
<td>405</td>
</tr>
<tr>
<td>M.8</td>
<td>Comparison of the Computed Velocity and Temperature Profiles to the Classic Similar Solution, $Pr=10$, $Ra=10^6$</td>
<td>406</td>
</tr>
<tr>
<td>M.9</td>
<td>Comparison of the Computed Horizontal Velocity to the Calculated Value from the Classic Similar Solution - The Effect of Prandtl Number</td>
<td>407</td>
</tr>
<tr>
<td>M.10</td>
<td>Comparison of the Horizontal Velocity to the Calculated Value from the Classic Similar Solution - Effect of Rayleigh Number</td>
<td>408</td>
</tr>
<tr>
<td>M.11</td>
<td>The Effect of Time Step on the Transient Temperature Solution</td>
<td>409</td>
</tr>
<tr>
<td>M.12</td>
<td>The Effect of Time Step on the Transient Velocity Solution</td>
<td>410</td>
</tr>
<tr>
<td>M.13</td>
<td>The Effect of $c_0$ on the Boundary Layer Solution when $T_{\infty} = T_0$</td>
<td>415</td>
</tr>
<tr>
<td>M.14</td>
<td>Generalized Plot of $c_2$ on the Boundary Layer Solution when $T_{\infty} = T_0$</td>
<td>417</td>
</tr>
<tr>
<td>M.15</td>
<td>The Effect of Time Step Size on the Transient Temperature Solution</td>
<td>423</td>
</tr>
<tr>
<td>FIGURE</td>
<td>TITLE</td>
<td>PAGE</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>M.16</td>
<td>The Effect of Time Step Size on the Transient Velocity Solution</td>
<td>424</td>
</tr>
<tr>
<td>M.17</td>
<td>The Effect of JMAX on the Temperature Solution</td>
<td>426</td>
</tr>
<tr>
<td>M.18</td>
<td>The Effect of JMAX on the Velocity Solution</td>
<td>427</td>
</tr>
<tr>
<td>M.19</td>
<td>The Effect of YMAX on the Temperature Solution</td>
<td>429</td>
</tr>
<tr>
<td>M.20</td>
<td>The Effect of YMAX on the Velocity Solution</td>
<td>430</td>
</tr>
<tr>
<td>M.21</td>
<td>The Effect of the Time Step Size on the Transient Temperature Solution</td>
<td>431</td>
</tr>
<tr>
<td>M.22</td>
<td>The Effect of the Time Step Size on the Transient Velocity Solution</td>
<td>432</td>
</tr>
<tr>
<td>M.23</td>
<td>Generalized Plot of $c_o$ for a Uniformly Heated Vertical Plate in Surroundings with a Linear Temperature Gradient</td>
<td>434</td>
</tr>
<tr>
<td>N.1</td>
<td>Horizontal Velocity Vs. Width</td>
<td>453</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Range of Parameters Used in Experimental Program</td>
<td>97</td>
</tr>
<tr>
<td>5.1</td>
<td>Constants in the Integral Momentum and Energy Equation for Various Types of Functions of Velocity and Temperature</td>
<td>141</td>
</tr>
<tr>
<td>5.2</td>
<td>Comparison of Variables for $Pr = 10$, $Z = 1$, $Ra = 10^6$ for Isothermal Surroundings and Surroundings with a Linear Temperature Gradient</td>
<td>172</td>
</tr>
<tr>
<td>7.1</td>
<td>Design Conditions for Sample Problem</td>
<td>218</td>
</tr>
<tr>
<td>B.1</td>
<td>Scaling Parameters for Various Conditions when $T_s$ is Based on the Plate Height</td>
<td>254</td>
</tr>
<tr>
<td>B.2</td>
<td>Scaling Parameters for Various Conditions when $T_s$ is Based on the Heat Flux Condition</td>
<td>255</td>
</tr>
<tr>
<td>E.1</td>
<td>PLATE Computer Code</td>
<td>263</td>
</tr>
<tr>
<td>F.1</td>
<td>Pertinent Physical Properties for Test W-3-Hi</td>
<td>284</td>
</tr>
<tr>
<td>F.2</td>
<td>Heat Losses and Mean Fluid Temperature as a Function of Time for Test W-3-Hi</td>
<td>287</td>
</tr>
<tr>
<td>F.3</td>
<td>Heat Flux Through the Front and Rear Walls of the Container</td>
<td>292</td>
</tr>
<tr>
<td>F.4</td>
<td>Heat Loss Through the Front and Rear Container Walls</td>
<td>292</td>
</tr>
<tr>
<td>F.5</td>
<td>Heat Loss Through the Container Bottom</td>
<td>293</td>
</tr>
<tr>
<td>F.6</td>
<td>Heat Loss Through the Sidewalls</td>
<td>295</td>
</tr>
<tr>
<td>F.7</td>
<td>Heat Absorbed by the Container Heaters</td>
<td>298</td>
</tr>
<tr>
<td>F.8</td>
<td>Heat Distribution in Test W-3-Hi</td>
<td>299</td>
</tr>
<tr>
<td>G.1</td>
<td>STRAT Computer Code</td>
<td>300</td>
</tr>
<tr>
<td>H.1</td>
<td>Optical Parameters Used in Determining Velocity</td>
<td>326</td>
</tr>
<tr>
<td>M.1</td>
<td>Range of Parameters Tested for Stability of IAD Scheme</td>
<td>396</td>
</tr>
<tr>
<td>M.2</td>
<td>Parameters Involved in Finite Difference Computation of Natural Convection from a Uniformly Heated Vertical Plate in Non-Isothermal Surroundings</td>
<td>420</td>
</tr>
<tr>
<td>TABLE</td>
<td>TITLE</td>
<td>PAGE</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>M.3</td>
<td>Stability Criteria for the Explicit Form of the Energy and the Momentum Equations</td>
<td>440</td>
</tr>
<tr>
<td>M.4</td>
<td>Stability Criteria for the Explicit Form of the Energy and Momentum Equations</td>
<td>441</td>
</tr>
<tr>
<td>M.5</td>
<td>Stability Criteria for the Semi-Implicit Form of the Energy and The Momentum Equations</td>
<td>442</td>
</tr>
<tr>
<td>M.6</td>
<td>Stability Criteria for the Semi-Explicit Form of the Energy and the Momentum Equations</td>
<td>443</td>
</tr>
<tr>
<td>M.7</td>
<td>Values of the Parameters in the Finite Difference Equations</td>
<td>444</td>
</tr>
<tr>
<td>M.8</td>
<td>Maximum Time Increment Possible for Stability</td>
<td>445</td>
</tr>
<tr>
<td>M.9</td>
<td>Stability Criteria for the IAD Form of the Energy and Momentum Equations</td>
<td>446</td>
</tr>
</tbody>
</table>
THERMAL STRATIFICATION IN ENCLOSED FLUIDS
DUE TO NATURAL CONVECTION
I. SUMMARY

The purpose of this work was to gain a better understanding of the thermal stratification phenomenon which occurs when the sidewalls of a container partially filled with a fluid are subjected to a constant sidewall heat flux. It can intuitively be reasoned that since the fluid adjacent to the wall increases in temperature, a buoyancy force results which causes the warm boundary layer fluid to rise until it reaches the surface and is deflected towards the center. The cooler fluid is displaced and sinks down the central core of the container. This replacement of cold fluid with warm fluid results in a thermally stratified fluid.

While such a qualitative description of the thermal stratification phenomenon can easily be made, the detailed mechanisms which cause this type of behavior have not as yet been clearly described nor understood. For example, the effect of a non-isothermal axial temperature on the boundary layer velocity and temperature profile has received little attention. Similarly, the processes by which the boundary layer interacts with the central core of the containers have not been investigated sufficiently. This work attempted to gain an understanding of the thermal stratification phenomenon so that a model which quantitatively describes not only the temperature fields but also models more realistically the fluid dynamics of the system.

In order to achieve this goal, a two-fold study was undertaken. In the first part, the boundary layer equations were solved. for the case of a uniformly heated wall immersed in surroundings whose temperature increased linearly with height. The second part of this work experimentally examined the thermal and dynamical behaviour of a fluid in a rectangular container whose sidewalls were subjected to a constant heat flux. In this part, the
modified Rayleigh number, Prandtl number and Aspect ratio were varied over wide ranges. From this work, a model for the thermal stratification phenomenon was developed.

Although both studies contributed to the understanding of the mechanisms involved in the thermal stratification phenomenon, the two studies were sufficiently independent so that the results did not need to be integrated in order to be meaningful. In fact, a more logical description of the findings of this investigation can be presented by considering each part as a separate independent entity. It is for this reason that the results of this work are presented in two parts; the first dealing with the theoretical examination of the boundary layer equations and the second part dealing with the development of the thermal stratification model.

1.1 Solution of the Boundary Layer Equations for the Case of a Uniformly Heated Vertical Wall in Non-Isothermal Surroundings

1.1.1 Introduction

In many technical applications, heat is transferred from a vertical surface to what was initially a quiescent, isothermal fluid. The natural convection boundary layer flows which result at such surfaces have been the subject of many studies \((27, 41, 55, 56)\). However, most solutions have been limited to cases involving heat transfer between a vertical wall and isothermal surroundings. Studies with non-isothermal surroundings have indeed been very limited. With the increasing interest in the storage of cryogenic fluids, especially in cryogenic propellant tanks in the aerospace industry, knowledge of the behaviour of the boundary layer flows in the presence of
non-isothermal environs has become necessary. This is because investigations have shown that an axial temperature gradient develops in the bulk fluid due to ambient heat flow from the vessel's sides. This present study addressed itself to delineating the effect of non-isothermal environs on the boundary layer solutions. In particular, the case of a uniformly heated wall placed in surroundings whose temperature increased linearly with height was studied.

The two natural convection problems which have received the most attention involve (a) an isothermal vertical surface in an isothermal surrounding and (b) a uniformly heated vertical wall in an isothermal surrounding. A solution to the former problem has been obtained by Ostrach (41) using a similarity transformation of the boundary equations. Also, the latter case was solved by Sparrow and Gregg (56) using the similarity transformation technique, and by Sparrow (55) and Siegel (54) using the Kerman - Pohlhausen integral method (43, 52). In the integral method, the boundary layer velocity and temperature profiles were chosen to be of the form:

\[ w = \frac{w^*}{b^*} (1 - y/b^*)^2 \quad 1.1.1a \]

\[ \frac{d}{2k} \left(1 - y/b^*\right)^2 \quad 1.1.1b \]

where \( w^*, b^* \) are the velocity and boundary layer thickness-seeking parameters and are functions only of the plate height \( z \). The results of these solutions show that, for the case of a uniformly heated plate in isothermal surroundings, the plate temperature varies as the 1/5th power of the plate height, but the Nusselt number varies as the 1/4th power of the modified Grahof number. These predictions agree well with the experimental data of Dotson (15) and Goldstein (26). However, it was shown by Reid, et al. (46)
that large discrepancies between the integral method solution and the similarity solution result for other boundary layer parameters such as boundary layer flow rate or average boundary layer temperature. Thus it is necessary to choose more accurate boundary layer velocity and temperature profiles in order to predict the flow rate in the boundary layer.

Yang (64) derived a generalized procedure to obtain similarity solutions for natural convection problems. He showed that several vertical wall boundary conditions were possible. No solutions were obtained for non-isothermal surroundings. However, Cheesewright (9) showed that Yang's method could be modified to include just such non-isothermal surroundings. In particular, it was found that similarity solutions existed for the case of an isothermal surface in surroundings whose temperature varied as \( zn \). Also, Cheesewright pointed out that a similarity solution existed for natural convection from a non-isothermal surface maintained at constant temperature differential with its surroundings. It will be shown later in this work that this case corresponds to one for a uniformly heated plate in surroundings with a linear fluid temperature gradient. Kohlar (30) also derived a particular similarity solution for a uniformly heated vertical plate in surroundings with a temperature which varied as \( z^{1/5} \).

Hellums (27) developed a finite difference scheme to solve the boundary layer equations for the case of an isothermal wall in an isothermal surrounding. His scheme was particularly well suited to this type of problem since it was unconditionally stable and converged in a relatively short time to the solution obtained by a similarity transform technique. This technique was modified in the present work to solve the cases of uniformly heated walls in an isothermal surrounding and uniformly heated walls in surroundings with a linear temperature gradient.
1.1.2 Boundary Layer Equations

The classical boundary layer equations, containing the Boussinesq approximation that all the physical properties are constant except for the density in the body force term, can be written to describe the natural convection boundary layer behaviour as follows:

\[
\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \quad 1.1.2a
\]

\[
\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = g \beta T + \frac{\partial^2 w}{\partial y^2} \quad 1.1.2b
\]

\[
\frac{\partial T}{\partial t} + w \frac{\partial T}{\partial z} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} - w \frac{\partial T}{\partial z} \quad 1.1.2c
\]

where \( \bar{T} = (\theta - \theta_\infty) \) and the coordinates are given in Figure 1.1. These equations along with the appropriate boundary conditions completely describe the natural convection boundary layer flow.

\[ \begin{align*}
& q \\
& w \\
& z \\
& y \\
\end{align*} \]

**Figure 1.1 COORDINATES FOR BOUNDARY LAYER SOLUTION**

For the case of a uniformly heated vertical wall in surroundings whose temperature increases linearly with height, the boundary conditions become:

\[ \text{at } y = 0, \quad w = v = 0, \quad \frac{\partial T}{\partial y} = -\frac{q}{k} \quad 1.1.3a \]
1.1.3 **A Special Solution**

Assume that a steady state asymptotic solution of the boundary layer equations exists. That is, in the asymptotic region,

\[ \frac{\partial \bar{T}}{\partial z} = \frac{\partial \bar{w}}{\partial z} = 0 = v \]

For the boundary layer conditions of Equation 1.1.3, a solution of the boundary layer equations (Equation 1.1.2) can then be obtained. Defining

\[
T_0 = \sqrt{2} \frac{\Theta_0 - \Theta_\infty}{(q_L/\kappa)} \frac{Ra^*}{Ra}^{1/4}
\]

\[
W_0 = \frac{\sqrt{2}}{2} \frac{wl}{a} \frac{Ra^*}{Ra}^{3/4}
\]

\[
Y_0 = \frac{\sqrt{2}}{2} \frac{V}{L} \frac{Ra^*}{Ra}^{1/4}
\]

\[
Ra^* = \frac{\rho \beta L^2}{\alpha} \frac{\partial (\Theta_\infty)}{\partial z}
\]

the solution can be shown to be

\[
T_0 = 2 \exp(-Y_0) \cos(Y_0)
\]

\[
W_0 = \exp(-Y_0) \sin(Y_0)
\]

A new parameter, \( Ra^* \), results from the solution. It will be referred to as the **core Rayleigh number** in this work.

For such an asymptotic solution to exist, the amount of heat required to raise the boundary layer temperature at the same rate as that of the surrounding fluid is
exactly that supplied by the wall heat flux.

The solution given by Equation 1.1.6 is independent of height. Therefore, the temperature difference between the wall and the surrounding fluid remains constant. This, then, is identical to the similarity transformation solution which Cheesewright (2) suggested in his work.

Several qualitative phenomena can be deduced from Equation 1.1.6. The equation predicts a region of reverse flow in the boundary layer. Such reverse flow regions are not obtained in the case of a uniformly heated plate in isothermal surroundings. Also, there is a prediction of a dependence of Prandtl number on the solution for the latter case, while the solution for a uniformly heated vertical wall in surroundings with a linear thermal gradient is independent of Prandtl number. Therefore, the effect of a non-isothermal surrounding is seen to affect markedly the boundary layer flow conditions.

The questions remaining are: (a) Under what circumstances will an asymptotic solution exist? and (b) What is the form of both the boundary layer velocity and temperature profile before the asymptotic solution develops? These questions will be considered in the following sections.

1.1.4 Dimensionless Equations

If three more parameters are defined:

$$\overline{Z} = \frac{\sqrt{2} z}{L} \frac{Ra}{Ra^*} \frac{5}{4}$$  

$$\overline{V} = \sqrt{\frac{L}{a}} \frac{Ra^*}{Ra} {1/4}$$  

$$\overline{\tau} = \frac{at}{2L^2} \frac{Ra^*}{1/2}$$  

the boundary layer equations, Equation 1.1.2, and the
boundary conditions, Equation 1.1.2, can then be written in terms of the parameters given by Equations 1.1.5 and 1.1.7. These are:

\[
\frac{\partial W}{\partial z} + \frac{\partial V}{\partial y} = 0 \quad 1.1.8a
\]

\[
\frac{\partial W}{\partial \tau} + W_0 \frac{\partial W}{\partial z} + V_0 \frac{\partial W}{\partial y} = Pr(T_0 + \frac{\partial^2 T}{\partial y^2}) \quad 1.1.8b
\]

\[
\frac{\partial T}{\partial \tau} + W_0 \frac{\partial T}{\partial z} + V_0 \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} - 4W_0 \quad 1.1.8c
\]

with the boundary conditions

at \( y_0 = 0 \), \( \frac{\partial T_0}{\partial y} = -2 \), \( W_0 = V_0 = 0 \) \( 1.1.8d \)

as \( y_0 \to \infty \), \( T_0 = W_0 = 0 \) \( 1.1.8e \)

at \( z_0 = 0 \), \( T_0 = W_0 = 0 \) \( 1.1.8f \)

Therefore, the solution of Equation 1.1.8 for only one parameter, viz., the Prandtl number, will yield a complete solution as to the velocity and temperature fields.

The equations were solved using a finite difference procedure. These are described in Appendix M. Also, the equations for a uniformly heated wall in an isothermal surrounding was solved numerically. However, for this case, the variables were redefined as follows:

\[
W = \frac{wL}{\alpha} Ra^{-1/2} \quad T = \frac{q - \infty}{qLk} \quad 1.1.9
\]

\[
V = \frac{vL}{\alpha} Ra^{-1/4} \quad \tau = \frac{at}{L^2} Ra^{1/2} \quad 1.1.9
\]

\[
Z = z/L \quad \gamma = \frac{y/L}{Ra^{1/4}}
\]

Suffice it to say that for the solutions presented in the following sections are those which were found to be convergent and stable. The finite difference equations
were unconditionally stable for the case of a uniformly heated vertical wall in an isothermal surrounding while there was a limitation on the time step for the case of a uniformly heated wall in surroundings with a linear temperature gradient (Appendix M).

1.1.5 **Boundary Layer Solution Results**

Figure 1.2 shows the predicted vertical temperature and velocity profiles as a function of Prandtl number for the case of a uniformly heated wall in an isothermal surrounding. There is a strong Prandtl number dependence on the profiles. No such dependence was found for the case of a uniformly heated plate in surroundings with a linear temperature gradient.

The predicted vertical velocity and temperature profiles for the case of a uniformly heated plate in surroundings whose temperature increases with height are presented in Figures 1.3 and 1.4. It can be seen that the boundary layer thickness and the flow rate are less at a lower height, but the shapes of the velocity and temperature profiles are essentially the same as those found by solving the asymptotic case, i.e. Equation 1.1.6.

As can be seen from Figure 1.5, the asymptotic solution becomes valid at a dimensionless plate height of

\[ Z_0 \geq 5 \]

1.1.10

This height is also independent of Prandtl number.
FIGURE 1.2 UNIFORMLY HEATED VERTICAL PLATE IN ISOTHERMAL SURROUNDINGS:
- THE EFFECT OF PRANDTL NUMBER -
FIGURE 1.3
UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS: THE EFFECT OF PRANDTL NUMBER ON THE TEMPERATURE PROFILE

- Pr = 1
- Pr = 10
- Pr = 100

Asymptotic solution
FIGURE 1.4
UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS: THE EFFECT OF PRANDTL NUMBER ON THE VELOCITY PROFILE

- $\Delta$ - Pr = 1
- - $\bullet$ - Pr = 10
- - $\square$ - Pr = 100

asympotic solution

VERTICAL VELOCITY, $w_0$

WIDTH $y_0$

0.3

0.2

0.1

0.0

0.0 1.0 2.0 3.0 4.0 5.0 6.0
FIGURE 1.5 UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS: THE EFFECT OF PRANDTL NUMBER ON THE HORIZONTAL VELOCITY PROFILE
1.1.6 Discussion and Conclusions

A non-isothermal environment is seen to affect markedly the natural convection temperature and velocity profiles in the boundary layer. This results from the fact that the balance of forces in the case of an uniformly heated wall in an isothermal surrounding and a uniformly heated wall in surroundings with a linear temperature gradient are markedly different. In the former case, the boundary flow is accelerating and the fluid is constantly entrained. This results in dragging the entrained fluid back near the edge of the boundary layer thus establishing the characteristic boundary layer profiles. As seen in Figure 1.6, there are, effectively, two regions in the boundary layer to be formed, one wherein the viscous and buoyancy forces dominate, and another wherein the viscous and inertial forces dominate. On the other hand, in the case where the temperature of the surroundings increases linearly with height, the buoyancy and viscous forces are always dominant—the inertial forces are always small in comparison. This, therefore, leads to the reverse flow regions in the boundary layer and to the independence of Prandtl number.

Another interesting result of this analysis is that the height at which the boundary layer profiles become asymptotic is independent of Prandtl number. This was not apparent from the equations describing the flow (Equation 1.8). This height increases in proportion to the wall heat flux but decreases as the 5/4th power of the linear gradient in the core.

The presence of a linear temperature gradient in the core can drastically alter the boundary layer velocity and temperature profiles. This indicates that gross errors can be caused by using assumed velocity and temper-
FIGURE 1.6  FORCE DIAGRAM FOR MOMENTUM EQUATION  
- $Pr = 10, Ra = 10^6$ -
ature profiles in solving boundary layer problems with an integral method. Since systems which have non-isothermal environs are common in many applications, it is believed that a thorough study of the boundary layer equations in non-isothermal surroundings should be made.

1.2 Model for Thermal Stratification of Contained Fluids Subjected to a Constant Sidewall Heat Flux

1.2.1 Introduction

Vessels containing quiescent and isothermal fluids will tend to stratify thermally when the sidewalls are heated. Predictions of axial temperature profiles within a given vessel as a function of time is much more difficult than might be expected as the physical processes leading to stratification are quite complex. However, one can easily visualize situations where it is imperative to be able to estimate such profiles and, in particular, to determine the variation of axial temperature near the surface. For example, a non-vented tank of liquid will, when heated, thermally stratify and show a concomitant pressure rise in excess of the vapor pressure naively estimated by assuming the tank contents to be well mixed. Also, a pump taking suction from such a tank may vapor-lock due to insufficient NPSH when the warm upper fluid reaches the pump intake. Practical examples of a similar nature may be found in cryogenic storage tank systems, water-cooled hollow gas turbine blades, and, in fact, any fluid vessel subjected to wall temperatures in excess of the bulk contents. Conversely, inverse stratification may result when the tank walls are maintained colder than the bulk liquid.
Qualitatively, it is not difficult to visualize the process; difficulties arise when one attempts to quantify the thermal stratification phenomenon. It can intuitively be reasoned that the fluid near the hot walls becomes buoyant and rises along the walls in a relatively thin boundary layer. The warm fluid is deflected at the surface and spreads over it, thereby displacing cooler fluid into the central liquid core. Thus, the fluid becomes thermally stratified.

Experimental data from many studies (2, 16, 36, 47) have shown that there are essentially no reactant temperature gradients in the vessel except, of course, in the thin boundary layer region. This has lead previous investigators to model the natural convection process for the case of uniform wall heating and an adiabatic bottom as consisting of three well-defined zones, i.e., the boundary layer along the wall, the central core, and a well-mixed surface layer (3, 16). It turns out that this type of modeling is an oversimplification and a preferable analysis scheme would eliminate the surface layer concept and permit a greater interaction between the boundary layer and core. In this section, a theoretical analysis of the natural convection phenomenon in vessels with uniformly heated vertical sidewalls and an adiabatic bottom is presented. The predicted flow patterns and temperature distributions are then compared with those obtained experimentally.

1.2.2 Prior Investigations

A number of experimental studies have been carried out and temperature distributions reported (2, 4, 36, 60); however, only a few (2, 5, 12, 16) have attempted to model the phenomenon theoretically. These few attempts
were based in large measure on the model of Bailey (3) wherein the boundary layer entered a thermally stratified upper layer and mixed in some complex manner. This stratified homogeneous layer increased in volume with time at a rate which was calculated simply from the boundary layer flow.

Drake (16) was the first to deviate from this concept. She also employed a constant wall heat flux though she used a vertical cylinder. In that investigation, it was postulated that the boundary layer entered a constant volume (i.e., thickness) "mixing region" at the surface. The well-mixed fluid then descended in plug-flow into the core region. It was further assumed, and attested to experimentally, that a thermal gradient was eventually established in the core. She also studied the effect of the linear core temperature gradient. Boundary layer enthalpy and mass flow rates were estimated using the Karman - Poulhausen integral method (42, 53). The boundary layer velocity temperature profiles used were those assumed by Sparrow and Gregg (56). The predicted core temperature was:

\[
\frac{(k/q)}{dz} = 4 \frac{Fo^{4/9}}{Ra^{1/9}}
\]

where the factor 4 was empirical.

In an excellent experimental study, Schwind and Vliet (52), used a rectangular tank with constant heat flux with \(Ra^+ \approx 10^{12}\). They found that below the stratified region the boundary layer increased in thickness. However, at the boundary between the stratified layer and isothermal fluid, some of the boundary layer liquid separated from the wall and moved out and down into the core. The cause of this "reverse-shear" region was an adverse boundary temperature gradient that formed after a core temperature distribution had been established.
However, since only temperatures were determined in this study, an estimate of neither the amount of fluid that separated from the walls nor the rates of flow in the core and boundary layer could be obtained.

Another analytical treatment has been reported by Clark and Barakat (5, 12) for the constant wall heat flux case. Numerical techniques were used but computational times were excessive and instability and convergence difficulties were encountered at relatively low Grashof numbers (Gr<sub>b</sub>~10<sup>6</sup>). The findings of this study agreed with experimental observation, i.e., that there was no horizontal thermal gradient in the core, and the axial core thermal gradient was linear after sufficient time had elapsed.

All previous work while having provided much information on the thermal stratification phenomenon, does not provide an understanding of the mechanisms involved nor does it give an adequate indication of the flows involved. This latter point must be clarified if the thermal stratification process is to be sufficiently understood so that vessels can be intelligently designed to minimize stratification.

1.2.3 Present Approach

An experimental and analytical study was conducted with the aim of developing improved practical techniques for predicting transient, natural convection temperature and velocity fields in enclosed fluids and to gain an insight into the mechanisms causing thermal stratification. In the experimental portion of the study, a rectangular container, partially filled with liquid was subjected to a uniform sidewall heating. A range of Prandtl numbers from 7 to 12000, aspect ratios from 1 to 3, and modified
Rayleigh numbers from $10^7$ to $10^{13}$ was studied.

Temperature measurements and streakline observations from eighteen tests were used as guidelines in determining a system model. Modified boundary layer equations were used to solve the problem of boundary layer interaction with the core fluid in the presence of a non-isothermal core. From this a model was developed with predicted transient temperature profiles for a wide range of system parameters and predicted the total amount of fluid flowing from the boundary layer to the core reasonably well. In the analysis, it was assumed that no heat or mass transfer occurred across the liquid-vapor interface, and this condition was imposed on the experimental system.

1.2.4 Experimental

The experimental test apparatus was designed to facilitate measurement of the temperature distribution and fluid velocities; most measurements were taken in the top 20 percent of the fluid to delineate more clearly the boundary layer-surface zone interactions.

The test tank is shown in Figure 1.7. The side walls were Pyrex glass panels coated with a thin, electrical conducting film. The coated panel side was not in contact with the fluid. Silver screens bonded to the coated side allowed power to be fed from buss-bars. Independent heating control was possible for the 0-8, 8-16, and 16-24 inch vertical sections. The sides and bottom of the test tank were insulated with foamed polyurethane and temperature measurements in this insulation allowed an estimation of the heat losses.

Both of the narrow ends were made from double panes of Pyrex glass separated by an insulation air-gap. Photo-
Figure 1.7
Experimental Container
graphs were taken through these windows. To illuminate a single, thin test plane within the fluid, an intense, collimated light source was located above the 0.25-inch slit. The light was a G.E. 240 Par 56 VSNP lamp rated at 110,000 candlepower with an angular divergence of less than 5°. This beam was further collimated in a cylindrical lens with a focal length of 110 mm. Most of the IR frequencies were filtered by the lens and little heating of the fluid resulted from the illumination.

The fluids used were water, glycerine, and an 85 weight percent water in glycerine mixture. In all tests the liquid level was maintained 2 inches below vessel top. Different liquid depths were attained by using false bottoms. The liquid vessel, at maximum depth, was, therefore, 24 x 24 x 8 inches. With even wall heating on the two large sides, two-dimensional flow pattern resulted.

Temperatures were measured with 34 3-mil iron constantan thermocouples placed in the illuminated test plane and cantilevered from a thin glass support rod. During various periods in the test, each temperature was scanned at 30 second intervals and recorded on a dual-channel Sanborn recorder.

The fluid motion was deduced from photographs taken in the illuminated test plane. Neutrally buoyant, aluminumized polystyrene spheres (diameter of 35μ) were added to the fluid so as to produce streaks on the time exposures. The streak length could then be related to the local velocity; also, by advancing the f-stop during exposure, the decrease in admitted light produced a streak of nonuniform intensity, i.e., the streak "thinned out" in the direction of motion. All photography was made with a 35-mm reflex camera with Kodak Tri-x film. Approximately fifteen pictures were taken during a run.
1.2.5 Test Results

Temperature profiles from a typical test are shown in Figure 1.8. The increase in temperature is plotted as a function of dimensionless height for various times within the test. Temperature data can be found in Appendix J. Velocity data were obtained from the streak photographs and typical results are shown in Figures 1.9 to 1.11. Analysis of such data from the runs of this experiment have shown that

(a) Sidewall heating results in a thin boundary layer along the walls. The thickness of the boundary layer depended on the fluid used and the degree of thermal stratification.

(b) Boundary layer flow was discharged into the core over the entire length of the stratified region. The greatest amount of fluid discharge occurred approximately at the interface of the stratified region and the unstratified region. This result agrees with the observations of Schwind and Vliet (52).

(c) Horizontal temperature gradients in the fluid are small. The warmer fluid deposited into the core settles downward as cooler fluid from the lower regions is fed into the boundary layer. Except for the region near the top, the magnitude of the vertical velocity in the core is much greater than the horizontal component. A "plug-flow" assumption, therefore, appears adequate.

(d) Except for the top part of the core and a region near the bottom, the axial temperature profile is linear after an initial transient period.

(e) An instability in the flow behaviour was noted. Initially, the flow appears to be symmetrical about the central plane (Figure 1.9). However,
Figure 1.8: Axial temperature profiles, $Pr = 7$, $Ra = 1.81 \times 10^{12}$
Figure 1.9 Streamlines for Test G-3-H1, $\tau=1.33\times10^{-2}$
FIGURE 1.10 STREAMLINES FOR TEST G-3-H1, $\tau = 3.72 \times 10^{-2}$
FIGURE 1.11 STREAMLINES FOR TEST G-3-H1, \( \tau = 9.63 \times 10^{-2} \)
with the passage of time, the streamlines become skewed and the symmetry destroyed (Figures 1.10, 1.11). This observation was noted in all tests. Also, dye studies by Drake (16) indicated the same type of behaviour. The reason for such an instability is yet unknown. It should be noted however, that the temperature field was not appreciably affected by this type of behaviour.

1.2.6 Formulation of a Model

On the basis of the experimental data the system was modeled in terms of two regions: a boundary layer rising at the heated wall and a central core as shown in Figure 1.12.

1.2.6.1 Boundary Layer

Since the boundary-layer thickness is small relative to the container width, the wall was treated as a vertical plate. A provision for having non-isothermal surroundings, \( T_\infty(z) \), was added. In the overall model, \( T_\infty \) is synonymous with the core temperature. A detailed analysis of the free convection equations from a vertical plate to a non-isothermal fluid is described elsewhere (16), only the results essential to the model will be summarized here.

The integrated forms of the momentum and energy equations for steady state boundary-layer flow are

\[
\begin{align*}
\frac{3}{2} \int w^2 dy &= \beta g \int (\theta - \theta_\infty) dy - rw/\rho \quad 1.2.2a \\
\frac{3}{2} \int w(\theta - \theta_\infty) dy &= \frac{a}{\rho C_p} \int w \frac{dT_\infty}{dz} dy \quad 1.2.2b
\end{align*}
\]
FIGURE 1.12 THERMAL STRATIFICATION MODEL
If the surrounding fluid is isothermal, the last term in Equation 1.2.2b is zero.

For laminar flow, the following functional forms for the velocity and temperature profiles have appeared in the literature (16, 55)

\[
\begin{align*}
    w &= \frac{w^* y}{\delta^*} (1 - y/\delta^*)^2 \quad 1.1.1.a \\
    \Theta - \Theta_\infty &= \frac{a \delta^*}{2k} (1 - y/\delta^*)^2 \quad 1.1.1.b
\end{align*}
\]

Also, it has been shown in Section 1.1 that for the case of a uniformly heated vertical plate in surroundings with a linear temperature gradient, the following functional forms seem to apply:

\[
\begin{align*}
    w &= w^* e^{-y/\delta^*} \sin y/\delta^* \quad 1.2.3a \\
    \Theta - \Theta_\infty &= \frac{a \delta^*}{k} e^{-y/\delta^*} \cos y/\delta^* \quad 1.2.3b
\end{align*}
\]

Outside the boundary layer, \((y \to \delta^*)\) \(w = 0, \Theta = \Theta_\infty\). Finally, the wall shear stress is given by

\[
\tau_w = -\mu \frac{\partial w}{\partial y} \quad 1.2.4
\]

When Equations 1.2.4 and either 1.1.1 or 1.2.3 are substituted into Equation 1.2.2, a pair of simultaneous ordinary differential equations are obtained which define the unknown functions \(\delta^*(z)\) and \(w^*(z)\). It is convenient to define new dimensionless variables:

\[
\begin{align*}
    M^* &= \frac{w^* \delta^*}{\frac{a}{k} e_{CM}} (E^g)^{1/4} \quad 1.2.5a \\
    E^* &= \frac{w^* \delta^*}{C_E} (E^g)^{1/4} \quad 1.2.5b
\end{align*}
\]

where \(C_E = 2/A_1\)
\[ C_M = \frac{A_4}{A_1} \left( \frac{1}{\frac{14}{5} A_3 \text{Fr} + 1} \right)^{3/5} \left( \frac{2}{A_1} \right)^{4/5} \]  \hspace{1cm} 1.2.6b

\( A_1, A_2, A_3, \) and \( A_4 \) are constants which depend on the functional forms of the assumed velocity and temperature profiles. In terms of these new dimensionless variables, the pair of simultaneous ordinary differential equations are

\[ \frac{dE^*}{dz/L} = \frac{Pr \text{Ra}^{1/4}}{A_4} \left[ A_2 \frac{C_{E}^{1/3 \text{Pr} \text{Ra}}}{M^{5/3}} - \frac{1}{C_{E} E^{4/3}} \right] \]  \hspace{1cm} 1.2.7b

\[ \frac{dM^*}{dz/L} = \frac{\text{Ra}^{1/4}}{z} \left[ 1 - A_2 \frac{C_{E}^{1/3 \text{Pr} \text{Ra}}}{M^{5/3}} \frac{\text{Ra}^{1/3 \text{Pr} \text{Ra}}}{M^{1/3}} \right] \]  \hspace{1cm} 1.2.7a

For the case of a uniformly heated vertical wall in isothermal surroundings, the solution of Equation 1.2.7 has been found to be

\[ E^* = \frac{\text{Ra}^{1/4}}{z} \]  \hspace{1cm} 1.2.8a

\[ M^* = \frac{\text{Ra}^{7/20}}{z} \]  \hspace{1cm} 1.2.8b

Now, as \( Pr \to \infty \),

\[ C_M = \left( \frac{A_4}{A_1} \right)^{3/5} \left( \frac{2}{A_1} \right)^{4/5} \]  \hspace{1cm} 1.2.9

and Equation 1.2.7b simplifies to

\[ \frac{A_4}{2A_3} \frac{C_{E}^{4/3}}{C_{M}^{5/3}} \frac{E^{1/3 \text{Pr} \text{Ra}}}{M^{5/3}} = \frac{1}{C_{E} A_3} \frac{M^*}{E^*} \]  \hspace{1cm} 1.2.10

or, using Equations 1.2.5 and 1.2.9, Equation 1.2.10 can be simplified to

\[ M^* = E^*7/5 \]  \hspace{1cm} 1.2.11

Then, Equation 1.2.7a can be simplified to
\[
\frac{dE^+}{d(z/L)} = Ra^+ 1/4 (1 - C_r 1/5 E^+ 4/5) \tag{1.2.12}
\]

where

\[
C_r = (A_2 E R M C_1)^{1/3} \tag{1.2.3}
\]

and

\[
E^+ = \frac{R_{Af}}{R_{A^+}} E^* \tag{1.2.14}
\]

Integrating,

\[
C_r \frac{R_{Af}}{R_{A^+}} \frac{5}{R_{Af}} = \frac{5}{4} \ln \frac{1 + C_r 1/5 E^+ 1/5}{1 - C_r 1/5 E^+ 1/5} + 2 \tan^{-1} \left( C_r 1/5 E^+ 1/5 \right) - 4 C_r 1/5 E^+ 1/5 \tag{1.2.15}
\]

This solution for the positive values of

\[
z^* = C_r \frac{R_{Af}}{R_{A^+}} \frac{5}{R_{Af}} z^+ \tag{1.2.16}
\]

is given in Figure 1.13. This solution was found to be valid for Pr > 5.

Now, from Equation 1.3.17, it can be seen that

\[
\frac{dE^+}{dz} = f(E^+) \tag{1.2.16}
\]

Assuming that the boundary layer can always be described by Equation 1.2.16, it is obvious that

\[
\frac{E^+}{f(E^+)} = \int dE^+ \tag{1.2.17}
\]

Rewriting Equation 1.2.17

\[
\begin{align*}
E_2^+ & = E_1^+ \\
\int \frac{dE^+}{f(E^+)} & = \int dZ \\
E_2^+ & = E_1^+ \\
\end{align*}
\tag{1.2.18}
\]

Since the value of \(E_1^+\) is known, the integral
\[ C_f \frac{R_a^{5/4}}{R_a^{1/4}} \text{ as a function of height.} \]
\[
\int_0^{E_1^+} \frac{dE^+}{f(E^+)} = Z_1(\text{eff})
\]
1.2.19

can be calculated from Figure 1.16. Therefore,

\[
\int_0^{E_2^+} \frac{dE^+}{f(E^+)} = Z_2 - Z_1 + Z_1(\text{eff}) = Z_2(\text{eff})
\]
1.2.20

Then, again using Figure 1.13, \(E_2^+\) and, hence, \(E_2\) can be found. That is, knowing the energy parameter at any height, the energy flow parameter can be found at any other height by use of Equation 1.2.20 and Figure 1.13 provided that the core Rayleigh number is constant between the two points.

Now, take a differential element in the container such as shown in Figure 1.14. Then, assuming

(a) the core fluid is isothermal along any isothermal plane;

(b) the fluid moves down the central core in essentially "plug" flow;

(c) the boundary layer temperature and velocity profiles assume their steady state shape for any temperature distribution, and using the result of Drake (16) that

\[
\int w dy = \alpha C_{f}^{4/5} E_{*}^{4/5}
\]
1.2.21a

\[
\int w (\Theta - \Theta_\infty) dy = \frac{qL}{k} \alpha \frac{1}{\text{Ra}^{1/4}} E_{*}
\]
1.2.21b

material and energy balances can be made on the differential element so that

\[
\alpha C_{f}^{4/5} \frac{dE_{*}^{4/5}}{dz} = - Q_f
\]
1.2.22

and

\[
q - \frac{qL}{\text{Ra}^{1/4}} \frac{dE_{*}}{dz} = \rho C_p \alpha C_{f}^{4/5} \frac{d(\Theta E_{*}^{4/5})}{dz} = \eta
\]
1.2.23
FIGURE 1.14 DIFFERENTIAL ELEMENT USED FOR DEVELOPMENT OF STRATIFIED MODEL EQUATIONS
where $Q_f$ is the boundary layer volume flow rate and $\varepsilon$ is the boundary layer enthalpy flow rate.

Now, consider the differential element in the core. Assuming that the boundary layer is much smaller than the width of the container, a material and energy balance can be written on the element so that

$$2Q_f = - D \frac{dw}{dz} \quad 1.2.24$$

and

$$2\varepsilon + \rho C_p D \frac{dW_{T_0}}{dz} + kD \frac{d^2 T_0}{dz^2} = \rho C_p D \frac{dT_0}{t} \quad 1.2.25$$

Now, substitution of Equations 1.2.22 and 1.2.23 into Equations 1.2.24 and 1.2.25 and simplification yields

$$\rho C_p D \frac{dT_0}{dt} = 2q - \frac{2qL}{Ra^{1/4}} \frac{\partial E^*}{\partial z} + kD \frac{\partial^2 T_0}{\partial z^2} \quad 1.2.26$$

Finally, define a set of dimensionless parameters so that

$$W = \frac{wD}{a} \frac{1}{Ra^{1/5}} \quad 1.2.27$$

$$T = \frac{T_0}{(qL/k)Ra^{1/5}} \quad 1.2.27$$

$$Z = z/L \quad 1.2.27$$

$$\tau = \frac{at}{LD} Ra^{1/5} \quad 1.2.27$$

Then, Equation 1.2.24 after integration becomes

$$W = 2C_f^{1/5} E_c^{4/5} \quad 1.2.28$$

and, Equation 1.2.26 can be written in these parameters as

$$\frac{dT}{dt} = 2 - \frac{2qL}{Ra^{1/4}} \frac{\partial E^*}{\partial z} + \left( \frac{D}{L} \right) \frac{1}{Ra^{1/5}} \frac{\partial^2 T}{\partial z^2} \quad 1.2.29$$

From Equation 1.2.21a and 1.2.13, it can be seen that

$$W = 2 - \frac{2qL}{Ra^{1/4}} \frac{\partial E^*}{\partial z} \quad 1.2.30$$
so that Equation 1.2.29 can be rewritten as

\[
\frac{\partial T}{\partial \tau} = W \frac{\partial T}{\partial Z} + \left( \frac{D}{L} \right) \frac{1}{R^a} \frac{1}{1/3} \frac{\partial^2 T}{\partial Z^2}
\]

1.2.31

The solution of the thermal stratification of the fluid was achieved using Equation 1.2.28 and a finite difference analog of Equation 1.2.29. A grid system given in Figure 1.1 was used for the finite difference scheme. The, Equation 1.2.31 can be written in finite difference form as

\[
\frac{T_{n+1}^{i} - T_{n}^{i}}{\Delta \tau} = 2.0 - 2.0 \frac{1}{R^a} \frac{1}{1/4} \frac{E_{n-1}^{i} - E_{n+1}^{i}}{\Delta Z} + \frac{T_{n+1}^{i} - 2T_{n}^{i} + T_{n-1}^{i}}{\Delta Z^2} \frac{1}{R^a} \frac{1}{1/5}
\]

1.2.32

At the top of the container, all of the energy in the boundary layer entered the core, so that

\[
\frac{T_{n+1}^{Nzones} - T_{n}^{Nzones}}{\Delta \tau} = 2.0 + \frac{2E_{n}^{Nzones}}{\Delta Z} \frac{1}{R^a} \frac{1}{1/4}
\]

+ \left( \frac{D}{L} \right) \frac{1}{R^a} \frac{1}{1/5} \frac{T_{n}^{Nzones} - T_{n-1}^{Nzones}}{2} \frac{1}{R^a} \frac{1}{1/5}
\]

1.2.33

and

\[
W_{n+1}^{1} = 2c^{4/5} \frac{E_{1}^{i}}{R^a} \frac{1}{1/5}
\]

1.2.34

The procedure for solving these finite difference equations was as follows: first, the enthalpy flows in the boundary layer, \( E_i \), were determined at each level \( i \), using the core temperature distribution at \( T^n \) using the technique outlined above. (it was assumed that the core temperature gradient was linear between the \( i \)th and \( i+1 \)th levels); then, the core temperatures were advanced in time to \( T^{n+1} \) using Equations 1.2.32 and 1.2.33; finally, the velocity was calculated using 1.2.34. This procedure was repeated through successive time intervals until the
FIGURE 11.5 DIVISION OF THE CONTAINER FOR THE STRATIFICATION MODEL
desired time level had been reached.

It was found that the solution was convergent for Nzones greater than 50.

Figure 1.16 shows the effect of conduction on the thermal stratification solution. It is seen that for the range of system parameters used in this study, conduction was negligible. Therefore, Equation 1.2.34 can be written as

$$\frac{dT}{dr} = W \frac{dT}{dZ} \quad 1.2.35$$

to a good approximation. The results are independent of the Aspect ratio, Prandtl number (Pr > 5), and modified Rayleigh number. This substantiated the assumption made by Drake (16) in her work. The validity of Equation 1.2.35 was confirmed by experimental data, as seen from Figure 1.17. The entire range of experimental data was correlated equally well by this model.

In order to correlate the data, a value of $C_f = 5.66$ had to be used. However, if the theoretical value of $C_f$, which would be obtained by using the velocity and temperature profiles given by Equation 1.2.4 ($C_f = 2.83$), serious deviation from the experimental data would result. Figure 1.18 shows the effect of $C_f$ on the solution. Even greater deviation was encountered when the velocity and temperature profiles assumed by Sparrow (55) (Equation 1.2.3) were used ($C_f = 1.71$).

Reasons for this discrepancy are not readily apparent. Possible reasons can be advanced but further study of this problem is required before an adequate explanation can be found. One possible explanation is that because, as will be presently seen, the core is not in perfect "plug" flow. Therefore, $C_f$ may not only contain the velocity scale but also some averaging parameter which cannot be accounted for by theory. Also, the fact that
FIGURE 1.16 THE EFFECT OF CONDUCTION ON THE THERMAL STRATIFICATION SIMULATION
FIGURE 1.17a  CORE TEMPERATURE PROFILES

\[ Z = \frac{z}{L} \]

\[ \frac{(T_\infty - T_p)}{(qL/k)} R_a^{1/5} \]

**LEGEND**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Test</th>
<th>Modified Rayleigh Number</th>
<th>Prandtl Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>○</td>
<td>W-3-Hi</td>
<td>(1.0 \times 10^{13})</td>
<td>70</td>
</tr>
<tr>
<td>▲</td>
<td>W-3-Lo</td>
<td>(1.1 \times 10^{11})</td>
<td>70</td>
</tr>
<tr>
<td>▪</td>
<td>W-2-Hi</td>
<td>(3.1 \times 10^{12})</td>
<td>70</td>
</tr>
<tr>
<td>■</td>
<td>G-3-Hi</td>
<td>(5.9 \times 10^{10})</td>
<td>12500</td>
</tr>
<tr>
<td>■</td>
<td>G-3-Lo</td>
<td>(3.1 \times 10^{10})</td>
<td>12500</td>
</tr>
<tr>
<td>✓</td>
<td>G-2-Hi</td>
<td>(1.5 \times 10^{10})</td>
<td>12500</td>
</tr>
<tr>
<td>◆</td>
<td>G-2-Lo</td>
<td>(1.4 \times 10^{8})</td>
<td>12500</td>
</tr>
<tr>
<td>▼</td>
<td>G-1-Lo</td>
<td>(5.0 \times 10^{7})</td>
<td>12500</td>
</tr>
<tr>
<td>●</td>
<td>Drake(15)</td>
<td>(1.1 \times 10^{11})</td>
<td>70</td>
</tr>
<tr>
<td>●</td>
<td>Drake(16)</td>
<td>(2.9 \times 10^{12})</td>
<td>70</td>
</tr>
<tr>
<td>△</td>
<td>Drake(17)</td>
<td>(8.3 \times 10^{13})</td>
<td>70</td>
</tr>
</tbody>
</table>

---

Drake's Equation
FIGURE 1.7b CORE TEMPERATURE PROFILES

See Figure 1.7a for Legend
there is a vertical velocity in the core, it may be possible that another set of velocity and temperature profiles result which have a value of $C_f = 5.66$ rather than the 2.83 calculated from theory.

Also, it was found that Drake's Equation, viz.,

$$\frac{\partial T}{\partial \tau} = 2 \tau^{4/9} \quad 1.2.36$$

fits the data and the model developed in this work for

$$\tau \geq 0.625$$

The model can also predict the maximum streamline, i.e., the volume flow rate, in the system. Figure 1.19 compares the calculated values with the experimental values. It is seen that the agreement is good despite the fact that the flow seemed to be unstable as seen from Figures 1.9 to 1.11.

1.2.7 Conclusions

Thermal stratification of an enclosed fluid subjected to a constant sidewall heat flux is the result of boundary layer--core interactions. The mechanism is one of a feedback system where the core temperature gradient determines the amount of fluid and enthalpy flowing in the boundary layer, and also the amount of fluid which reaches the surface. This, in turn, determines the shape of the core gradient and the degree of thermal stratification. The so-called "mixing region" postulated by Drake (16) does not exist. The top portion of the container differs from the main portion only in the fact that there is a much higher horizontal velocity component in the fluid.

The flow in the core is unstable, i.e., it is not
FIGURE 1.18 The Effect of $C_f$ on the Temperature Solution
FIGURE 1.19 MAXIMUM STREAMLINE PROFILES FOR TEST G-3-H1
symmetrical about the central plane. While the reasons for this are not clear, the fluid asymmetry does not, however, seem to influence the temperature gradient in the core.

Conduction in the fluids used plays a negligible role in determining the axial temperature gradient in the core. Therefore, convection is the primary mode of heat transfer in the system. Figure 1.17 can, therefore, be used to predict the transient temperature profiles for all fluid of Prandtl number greater than 5. Similarly, the amount of fluid flowing axially in the core can be reasonably estimated from the thermal stratification model developed in this study.
II. INTRODUCTION

2.1 Thermal Stratification Phenomenon

Heating the walls of a container partially filled with fluid results in a natural convection circulation which tends to stratify the fluid thermally. The fluid heated at the wall rises toward the surface where it is deflected horizontally displacing the cooler fluid downward in the central core of the container. In this way, a vertical thermal gradient is established in the fluid.

Qualitatively, three regions must be considered to fully describe the thermal stratification process. The first is the boundary layer flow up along the walls of the container. The wall heat flux increases the temperature and decreases the fluid density near the wall. Thus, there results an upward buoyancy force on the fluid near the wall. This buoyancy force interacts with the inertial and viscous forces present to move the fluid up the wall in a relatively thin boundary layer. As the fluid moves up the wall, it reaches an area where it becomes influenced by the surface. In this surface-influenced region, the boundary layer fluid is decelerated in vertical direction, separates from the wall, and spreads across the surface in a complex manner and displaces the cooler fluid downward into the third region, viz., the container core. In this region, the fluid slowly moves downward establishing a thermal gradient in the core. The core axial temperature gradient is then approximately linear (16), and increases with time but there is virtually no radial variation in the temperature.

In order to understand fully the thermal stratification process, it is necessary not only to understand the fluid behaviour in these three regions, but also to define the interactions among these regions. For example, it is reasonable to envision the magnitude of the thermal gradient in the core
influences the boundary layer fluid flow rate. For in
the presence of a linear core gradient, the rising boundary
layer fluid encounters a steadily increasing ambient temper-
ature. Therefore, some of the wall heat flux is required
to heat the boundary layer fluid so that the rate at which
the average boundary layer temperature increases is at
least as great as the rate at which the temperature in the
core increases. This diminishes the buoyancy force and
causes a decrease in both the maximum fluid velocity and
boundary layer thickness. In a similar manner, arguments
can be presented to show that one might expect interactions
between the boundary layer and surface-influenced region,
and the core and the surface-influenced region.

It was the purpose of the study to describe quanti-
tatively these regions and to examine the interactions
between them. In particular, the effort was concentrated
on examining the fluid behaviour during the thermal strati-
fication process. Previous studies, with few exceptions,
have treated the problem in an ad hoc manner, with more
attention being given to predicting the temperature field
in existing vessels rather than attempting to understand
the fluid dynamics of the system. As a result, the liter-
ature contains voluminous temperature data for various
systems (2, 4, 26, 60). However, very little information
is available which can explain the physics involved in
thermally stratifying the fluid. Recently, the works of
Drake (16) and Schwind and Vliet (52) have attempted to
describe the convection processes in a more theoretical
and general manner. The present work extends the knowledge
gained from these works.

The present investigation was undertaken for two main
reasons. The primary purpose was to provide thermal and
velocity data for a wide variety of system parameters which
will be useful in engineering design of storage vessels in
which thermal stratification occurs. It not only augments
the existing studies, but, by presenting the data in the form of a model which approaches the actual situation more realistically, it also provides guidelines for predicting the expected fluid behaviour in new system designs. To best achieve this, it was also the purpose of this study to examine the fluid dynamics of the system. Because the temperature and velocity fields are coupled in natural convection situations, an understanding of the fluid behaviour is essential if one is to gain a knowledge of the processes causing the fluid to stratify thermally. At best, empirical models for predicting temperature fields can predict the thermal stratification in relatively few vessels under similar conditions. If, however, an understanding of the fluid physics were available, a means to minimize the degree of thermal stratification for various vessel configurations might be developed. As mentioned above, there are complex interactions between the various regions in the vessel. Modification of the boundary conditions can drastically alter these interactions thereby changing the temperature field in the tank. A knowledge of the factors which govern these interactions would provide a good basis for the design of vessels which would minimize the degree of stratification and thus give optimum design. It was toward this end that this work has been directed.

The method of approach used in this investigation was to combine a theoretical study of the boundary layer-core interactions with an experimental study of the surface influenced region to develop a model which relates the fluid behaviour to various system parameters. In order to make the problem more tractable, both the theoretical and experimental programs were confined to a two-dimensional case. The experimental system chosen for study was a rectangular container partially filled with fluid. A positive heat flux was imposed on the sidewalls, while the bottom was adiabatic. Velocities were determined using particle streak photography
techniques and thermocouples were used to measure the temperature field. In conjunction with this, a theoretical study of the boundary layer equations was undertaken for the case of a constant wall heat flux and a non-isothermal core temperature. A model was developed which not only predicts the temperature behaviour but also offers a description of the fluid dynamics of the system for a wide range of system parameters.

2.2 Applications of the Work

An understanding of the thermal stratification phenomenon would be beneficial for several applications. As was pointed out previously, a knowledge of the spatial temperature history of the fluid for a given system of parameters is required in order to design vessels storing fluids at temperatures different from ambient. That is, if one knows the degree of thermal stratification which will result for a given set of conditions, one can then determine the container wall stresses, the venting losses if a maximum tank pressure is not to be surpassed, and the pump sizes and/or pressurization requirements to move the fluid out of the vessel. However, a knowledge of the fluid dynamics is necessary if one is to attempt to alter the degree of stratification. Attempting to reduce the degree of thermal stratification by the installation of baffles or by the modification of the vessel configuration is an empirical art unless some knowledge of the fluid dynamics of the system is available.

By far the most important area requiring a model for thermal stratification processes is in the design of cryogenic propellant tanks. The amount of pressurant required to expel the cryogen is a function of the extent of stratification. Also, the design of the tank depends on the level of stratification. For example, in a hydrogen propellant
tank, a one degree rise in surface temperature raises the vapor pressure several atmospheres. This necessitates use of a thicker tank wall resulting in a reduction of the payload weight allowable or venting a significant portion of the fuel in order to avoid exceeding maximum safe tank pressures.

Storage tanks of volatile liquids have similar problems. Also, in this case, the maximum temperature of the fluid must be known so that the NPSH of the associated pumps can be determined. Minimization of the extent of thermal stratification also would be beneficial in reducing the vapor losses from such tanks.

In recent years, hollow gas turbine blades have been used in many designs so that they could be water cooled. Water flows into the hollow section of the blade by natural convection circulations. Hence, the heat transfer characteristics of the blade are enhanced permitting the use of much higher operating temperatures. A better understanding of the circulation patterns could lead to a more propitious design of the turbine blade, thereby maximizing the turbine efficiency.

2.3 Natural Convection Equations

In this section, the governing partial differential equations for natural convection flow of a viscous incompressible fluid are considered. The classic Boussinesq modification of the Navier-Stoke's equation for natural convection is presented. This approximation is usually used to describe flows in which the density differences due to the temperature variation in the fluid are assumed to be important only insofar as they affect the body force. Also, several other assumptions are made in the derivation of these equations. These are outlined as they are incorporated
in the derivation. Finally, the equations for the two-dimensional convection case in Cartesian coordinates are obtained from the general vector equations.

2.3.1. The Boussinesq Approximation

The Boussinesq equations for natural convection in an incompressible viscous fluid may be stated in vector notation as:

The Continuity Equation

$$\nabla \cdot \mathbf{V} = 0$$

2.3.1a

The Equation of Motion

$$\rho \frac{D \mathbf{V}}{D t} = \rho g - \nabla P + \mu \nabla (\nabla \cdot \mathbf{V})$$

2.3.1b

The Equation of Energy

$$\frac{D T}{D t} = \alpha \nabla \cdot (T \nabla \Theta)$$

2.3.1c

The Linearized Equation of State

$$\rho = \rho_0 + \delta \rho = \rho_0 \left(1 - \beta(\Theta - \Theta_0)\right)$$

2.3.1d

where the substantial derivative operator is defined as:

$$\frac{D}{D t} = \frac{\partial}{\partial t} \mathbf{V} \cdot \nabla$$

These equations are restricted to the case where density variations due to the temperature are small compared to the density, $\rho_0$, such that

$$\frac{\delta \rho}{\rho_0} = \beta(\Theta - \Theta_0) > 1$$

2.3.2
Finally, the pressure, $P$, in Equation 2.3.1b is defined as the total pressure, $P'$, less the static pressure head, i.e.,

$$P = P' - \rho_0 \sigma_g$$

where $\sigma_g$ is defined as the gravitational potential

$$\sigma_g = \nabla \sigma_g$$

Equations 2.3.1 are derived subject to the following assumptions (Appendix A):

1. The fluid is Newtonian.
2. All the fluid properties except the density are constant.
3. Density variations are important only insofar as they affect the gravity force.
4. Viscous dissipation is negligible compared to the conduction and convection terms. This can be shown to be true from an order to magnitude analysis provided that the Brinkman number,

$$\text{Br} = \mu \frac{U^2}{C} < 1$$

is less than unity (38).
5. The effect of pressure on density is negligible compared to that of temperature.

Furthermore, it can be shown thermodynamically (31) that the Equation of Motion (Equation 2.3.1b) is equivalent to the requirement that the local Mach number be very much less than unity. Subject to these restriction, the Boussinesq buoyancy-induced incompressible viscous flows.

2.3.2 The Vorticity Transport Equation

Equations 2.3.1 represent four equations in the four unknowns $V$, $P$, $\Theta$, and $\delta P$. Specification of the initial and
the boundary conditions completes the statement of the convection problem. However, by noting that the curl grad P vanishes identically, the pressure terms can be completely eliminated from the problem statement by taking the curl of Equation 2.3.1b. Introducing the vorticity vector \( \omega \),

\[
\omega = \text{curl} \, V \quad 2.3.5
\]

and recognizing the identity

\[
\text{curl} \left( V \cdot V \right) = V \cdot \omega - \omega \cdot V + \omega (\text{del} \, V) \quad 2.3.6
\]

the curl of Equation 2.3.1b can be taken and with the aid of Equation 2.3.1a the following equation is obtained:

\[
\frac{D\omega}{Dt} - \omega \cdot V = -\omega \text{curl} \left( \text{grad} \frac{\delta p}{\rho_0} \right) + \gamma \text{grad} (\text{del} \, \omega) \quad 2.3.7
\]

Equation 2.3.7 is the vorticity transport equation. Since the vorticity is related to the angular momentum of a fluid particle \( (I) \), Equation 2.3.7 is also a statement of the conservation of angular momentum.

One can see from Equation 2.3.1a that the velocity field is solenoidal. Therefore, the velocity can be expressed in terms of some vector potential \( \varphi \), such that

\[
V = \text{curl} \, \varphi \quad 2.3.8
\]

Then, the vorticity may be defined in terms of this potential using Equation 2.3.5.

\[
\omega = \text{curl} \left( \text{curl} \, \varphi \right) \quad 2.3.9
\]

Finally, assuming that the vector potential is also solenoidal, Equation 2.3.9 reduces to

\[
\omega = -\text{grad} \left( \text{del} \, \varphi \right) \quad 2.3.10
\]
2.3.3 The Two-Dimensional Equations

Consider a plane in the y-z plane normal to the x-axis of a Cartesian coordinate system. Then, the two-dimensional velocity vector can be defined as:

\[ \mathbf{V} = v_1 \mathbf{i} + w \mathbf{k} \]  

From Equation 2.3.10, one can see that \( \mathbf{g} \) and \( \mathbf{q} \) lie in the same direction--this direction, as seen from Equations 2.3.8 and 2.3.5, is normal to the y-z plane, i.e., along the x-axis. Identifying the potential with the stream function, \( \sigma \), the vorticity and the vector potential can be expressed as scalars by

\[ \omega = \sigma_1 \]  
\[ \mathbf{q} = q_1 \]  

Finally, the gravity vector is assumed to act parallel to the z-axis so that

\[ \mathbf{g} = -g \mathbf{k} \]  

Using these relations, the two-dimensional analogues of the three dimensional equations can be written. These are:

**Velocity**

\[ w = \frac{\partial \sigma}{\partial y}, \quad v = \frac{\partial \sigma}{\partial z} \]  

**Continuity**

\[ \frac{\partial w}{\partial z} + \frac{\partial \sigma}{\partial y} = 0 \]  

**Equation of Motion**

\[ \frac{\partial w}{\partial t} + w \frac{\partial \sigma}{\partial z} + v \frac{\partial \sigma}{\partial y} = \beta g (\theta - \theta_o) - \frac{1}{\rho_o} \frac{\partial P}{\partial z} + \gamma \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial y^2} \]
\[
\frac{\partial v}{\partial t} + w \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho_0} \frac{\partial p}{\partial y} + \gamma \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial y^2}
\]

**Equation of Energy**

\[
\frac{\partial \theta}{\partial t} + w \frac{\partial \theta}{\partial z} + v \frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial y^2}
\]

Further, the vorticity, \( \omega \), can be shown to be:

\[
\omega = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}
\]

or,

\[
\omega = - \frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial y^2}
\]

Finally, the vorticity equation, Equation 2.3.7, can be rewritten as shown in Equation 2.3.14h.

**Conservation of Angular Momentum**

\[
\frac{\partial \omega}{\partial t} + w \frac{\partial \omega}{\partial z} + v \frac{\partial \omega}{\partial y} = - \beta \frac{\partial \theta}{\partial y} + \gamma \frac{\partial^2 \omega}{\partial z^2} + \frac{\partial^2 \omega}{\partial y^2}
\]

### 2.3.4 Boundary Conditions

To complete the statement of the hydrodynamical problem, the initial conditions and the boundary conditions must be defined. The initial conditions require the complete specification of the velocity and temperature field, i.e., at \( t = 0 \),

\[
w = w_1(y, z, 0) \\
v = v_1(y, z, 0) \\
\theta = \theta_1(y, z, 0)
\]

In the usual experimental studies, a completely quiescent isothermal fluid is usually assumed for an initial condition, i.e., at \( t = 0 \)

\[
w = v = 0 \\
\theta = \theta_0
\]
Referring to Equations 2.3.14, it can be seen that a boundary condition is required at each boundary for \( w, v, \theta \). Various conditions are possible; the ones encountered most often in practice will now be considered.

**Hydrodynamical "No-Slip" Condition**

This condition is encountered where a solid wall is present. Both the tangential velocity, \( v_t \), and the normal velocity, \( v_n \), are zero at the solid wall, i.e.,

\[
v_t = v_n = 0 \quad 2.3.17
\]

**"Porous" Wall Condition**

In this case, a normal velocity component is permitted, but a no-slip condition is still imposed on the tangential velocity. Then,

\[
v_t \neq 0
\]

\[
\frac{\partial v_n}{\partial n} = 0 \quad 2.3.18
\]

It should be noted that this condition is also equivalent to the condition often encountered in situations where a vertical plate is placed in infinite surroundings. If a thermal perturbation is applied to the plate, fluid motion develops near the wall. If the velocity parallel to the plate remains zero then continuity dictates that there will be a velocity component normal to the wall, i.e., Equation 2.3.18 is the boundary condition far from the plate.

**Free Surface**

In the case of a free surface, the shear force at the surface is negligible compared to the shear in the body of the fluid. Therefore, a no-shear condition is a good approximation at the free surface. Then,
\[ v_n = 0 \]
\[ \frac{\partial v}{\partial n} = 0 \]  
2.3.19

**Isothermal Surface**

This condition is stated mathematically by the condition:

At the surface,
\[ \Theta = \Theta_s \]  
2.3.10

**Adiabatic Surface**

An adiabatic surface is one where no heat flow occurs across it. Then,
\[ q_o = 0 \]  
2.3.21a

which is equivalent to saying
\[ \frac{\partial \Theta}{\partial n} = 0 \]  
2.3.21b

**Constant Heat Flux Surface**

Mathematically, this condition is:
\[ q = -k \frac{\partial \Theta}{\partial n} = q_o \]  
2.3.22

2.3.5 **Boundary Conditions for the Present Study**

The present study was designed to investigate the thermal stratification phenomenon in a two-dimensional flow such as that shown in Figure 2.1. Initially, the fluid is quiescent and isothermal so that, at \( t = 0 \),
\[ w = v = 0 \]
\[ \Theta = \Theta_o \]  
2.3.23

For \( t > 0 \), a constant heat flux is imposed on the sidewalls so that
\[ q(0, z, t) = -q(D, z, t) = q_o \]  
2.3.24

It is further assumed that the surface and the bottom
Note: Nos. in parentheses are the $(y,z)$ coordinates

FIGURE 2.1 Two-Dimensional Container Coordinate System
are adiabatic, so that
\[ \frac{\partial \theta}{\partial z}(y, 0, t) = \frac{\partial \theta}{\partial z}(y, L, t) = 0 \] 2.3.25

No-slip conditions are applied at the solid walls, and the free-surface condition is imposed at \( z = L \), so that

At \( z = 0 \)
\[
\begin{align*}
y &= 0 \\
y &= D \\
z &= L
\end{align*}
\]
\[
\begin{align*}
w &= v = 0 \\
\frac{\partial \omega}{\partial z} &= 0 \\
w &= 0
\end{align*}
\] 2.3.26a
2.3.26b

The natural convection equations (Equations 2.3.14), the initial conditions (Equations 2.3.24), and the boundary conditions (Equations 2.3.24 to 2.3.26) completely specify the problem mathematically.

2.3.6 Dimensionless Equations

Normalization of the natural convection equations is particularly useful since it yields dimensionless groups which enable one to correlate results with the minimum number of physical parameters. In particular, an order of magnitude analysis of the system equations enables one to choose scaling parameters which are physically correct and which yield correct asymptotic solutions.

Order of magnitude analyses for the system shown in Figure 2.1 result in various scaling parameters depending on the relative magnitude of the forces present. Appendix B presents a list of parameters arising from various assumed force balances. The correct parameters must be determined by experimentation. Also, the choice of scaling parameters in an enclosure is rather difficult since they may not be the same for all regions in the container. For example,
near the walls, a boundary layer may have formed where the horizontal length scale is much less than the vertical length scale. However, the core is characterized by a slow motion where the horizontal and vertical length scales may be of the same order. Hence, scaling parameters chosen for the core may differ from those chosen for the boundary layer. This problem is discussed in more detail by Noble (33).

In order to illustrate the usefulness of normalization, an order of magnitude analysis of Equations 2.3.14 was used to obtain the following scaling parameters:

\[
\begin{align*}
\omega_s &= \frac{\alpha}{L}(Ra^+)^{1/2} \\
\sigma_s &= \alpha(Ra^+)^{1/4} \\
\phi_s &= q_0L/k \\
\rho^+ &= \frac{\rho_0\alpha^2Ra^+}{L^2} \\
1/t_s &= \frac{\alpha L(Ra^+)^{1/2}}{2.3.27} \\
y_s &= L(Ra^+)^{-1/4}
\end{align*}
\]

In this analysis, it was assumed that the viscous forces were of the same order as the bouyancy forces in Equation 2.3.14c and the inertial forces of the same order as the conduction terms in Equation 2.3.14e. Then, one can define a set of dimensionless parameters such that

\[
\begin{align*}
W &= \frac{w}{w_s} \\
\psi &= \frac{\sigma}{\sigma_s} \\
\omega^+ &= \frac{\omega}{\omega_s} \\
T &= \frac{\phi}{\phi_s} \\
P^+ &= \frac{P}{P_s} \\
\tau &= \frac{t}{t_s} \\
Y &= \frac{y}{y_s} \\
Z &= \frac{z}{L}
\end{align*}
\]

Then, Equations 2.3.14 can be rewritten in terms of these dimensionless constants.
Velocity

\[ W = -\frac{\partial V}{\partial Y} \quad V = -\frac{\partial W}{\partial Z} \] 2.3.29a

Continuity

\[ \frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y} = 0 \] 2.3.29b

Equation of Motion

\[ \frac{\partial W}{\partial \tau} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = Pr \frac{\partial T}{\partial \tau} + Pr \left( \frac{\partial^2 W}{\partial Z^2} + \frac{1}{(Ra^+)^{1/2}} \frac{\partial^2 W}{\partial Y^2} \right) \] 2.3.29c

\[ \frac{\partial V}{\partial \tau} + W \frac{\partial V}{\partial Z} + V \frac{\partial V}{\partial Y} = -\frac{1}{(Ra^+)^{1/2}} \frac{\partial P^+}{\partial Y} \]
\[ + Pr \left( \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} \frac{1}{(Ra^+)^{1/2}} \right) \] 2.3.29d

Equation of Energy

\[ \frac{\partial T}{\partial \tau} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} = \frac{\partial^2 T}{\partial Y^2} + \frac{1}{(Ra^+)^{1/2}} \frac{\partial^2 T}{\partial Z^2} \] 2.3.29e

Equation of Angular Momentum

\[ \frac{\partial \omega^+}{\partial \tau} + W \frac{\partial \omega^+}{\partial Z} + V \frac{\partial \omega^+}{\partial Y} = -Pr \frac{\partial T}{\partial Y} + Pr \frac{\partial^2 \omega^+}{\partial Y^2} + \frac{1}{(Ra^+)^{1/2}} \frac{\partial^2 \omega^+}{\partial Z^2} \] 2.3.29f

The initial conditions are

\[ V = W = 0 \]
\[ T = T_0 \] 2.3.29g

Finally, the boundary conditions, Equations 2.3.24 to 2.3.26, become

\[ \frac{\partial T}{\partial Z}(Y, 0, \tau) = \frac{\partial T}{\partial Z}(Y, 1, \tau) = 0 \] 2.3.29h
\[
\frac{\partial T}{\partial Y}(0, Z, \tau) = -\frac{\partial T}{\partial Y}(D/y_s, Z, \tau) = -1/(Ra)^{1/4}
\]

At

\[
\begin{align*}
Z &= 0, \text{all } Y \\
Y &= 0, \text{all } Z \\
Y &= D/y_s, \text{all } Z \\
Z &= 1, \text{all } Y \\
\frac{\partial V}{\partial Y} &= W = 0
\end{align*}
\]

From these equations, it can be seen that the flow depends only on the Prandtl Number, Pr, the Modified Rayleigh Number, Ra\textsuperscript{+}, and the Aspect Ratio, L/D. The dependance on the Aspect Ratio arises from intuitive reasoning. Note that the scaling parameters predict

\[
\frac{\lambda}{L} = (Ra)^{1/4}
\]

However, if there is interaction between boundary layers, or if the boundary layer cannot fully develop, Equation 2.3.30 may not be correct. Hence, the Aspect Ratio, L/D, may be another important scaling parameter.

2.4 Review of the Literature

2.4.1 The Vertical Plate Problem

In this section, the case of natural convection from a heated semi-infinite vertical plate immersed in infinite surroundings will be considered. This problem has relevance to the present work in that the resulting boundary layer flows along the plate bear many similarities to the boundary layer flows at the wall of a container with similar wall boundary conditions.

Using a Zehnder-Mach interferometer, Goldstein and Eckert (26) studied the development of the boundary layer
for the case of a constant wall heat flux. The temperature was found to develop initially as for heat conduction in a semi-infinite solid. Then, the buoyancy forces became sufficient to cause a boundary layer flow along the wall. The boundary layer thickness increases initially until it reaches a maximum and then decreases to a steady state value. Siegel (54) showed theoretically that Fourier Number, \( Fo \), relating the time required to reach steady state for this case was

\[
Fo = 4.78(0.8 + Pr)^{2/5} (Ra^+Pr)^{-2/5} Z^{2/5} \quad 2.4.1
\]

while, for the case of a constant wall temperature, steady state is reached in

\[
Fo = \frac{5.24(0.952 + Pr)^{1/4} + 7.10(0.377 + Pr)^{1/2}}{2.0}
\]

\[
(Ra^+ Pr)^{-1/2} Z^{1/2} \quad 2.4.2
\]

His analysis also confirmed Goldstein's observation of a maximum in the boundary layer thickness with time.

Sparrow and Gregg (56) solved the uniformly heated vertical plate problem for the steady state using the Karman-Pohlhausen integral method (52, 43), and also by using similarity transformations of the pertinent equations. The solutions obtained agreed well with the experimental results obtained by Dotson (15) and Goldstein (26). These results show that for laminar boundary layer flow, the plate temperature varies as the 1/5th power of the distance from the leading edge. The Nusselt Number, \( Nu \), varies as the 1/4th power of the modified Grashof Number, \( Gr^+ \).

It is interesting to note that the transition point to turbulent boundary layer flow occurs at a modified Grashof Number of about \( 10^{11} \). Saunders (49) and Eckert (12) experimentally found that when a constant wall temperature was applied to a vertical plate, transition to turbulent flow
occurred at a Grashof Number of about $7.0 \times 10^9$. However, it can be shown that

$$\text{Gr}^+_z = \text{Gr}_z \text{ Nu}_z$$  \hspace{1cm} 2.4.3

and using Sparrow and Gregg's (56) heat transfer results

$$\text{Nu}_z = 78 \text{ at } \text{Gr}^+_z = 10^{11}$$  \hspace{1cm} 2.4.4

one finds that the corresponding Grashof Number for turbulence to occur for the uniformly heated wall case is

$$\text{Gr}_z = 1.3 \times 10^9$$  \hspace{1cm} 2.4.5

Thus it can be concluded that the transition point seems to occur at approximately the same Grashof Number for the uniformly heated wall as for the isothermal wall.

At Grashof Numbers greater than $10^9$, the flow in the boundary layer becomes turbulent. Jakob (29) used data obtained by Mull and Reiher (35) and others and found that the Nusselt Number varied as the $1/3$rd power of the Grashof Number. Eckert and Carlson (18) in their study of natural convection between one heated and one cooled vertical wall found the variation of the Nusselt Number with Grashof Number to be as the $1/3$rd power also even though the flow was laminar rather than the expected $1/4$th power dependence. This unusual dependance was attributed to a linear vertical temperature gradient which was present in the mid-region between the plates.

Chang, Akins, etal. have shown that the similarity solutions for the uniformly heated plate in isothermal surroundings are actually zeroth-order perturbation solutions of the boundary layer equations. They developed a first-order perturbation solution which accounts for the possibility of a horizontal component of velocity at the edge of the boundary layer. It was found that at small distances
from the leading edge of the plate the maximum velocity is appreciably less than that represented by the boundary layer solution even though the average velocity for the entire field is increased considerably. As the height increases, both the velocity and temperature profiles tend to the values given by boundary layer theory. Also, they found that higher Grashof Numbers and Prandtl Numbers give better agreement with boundary layer theory.

Gill (25) presents an exact solution for the case where the temperature of the plate is held at a value which differs by a constant amount, B, from the temperature at infinity. The fluid temperature at infinity varies uniformly with height—the uniform temperature gradient, G, being positive. Then,

\[ \theta = B \exp(-y/\lambda) \cos(y/\lambda) + Gz \quad 2.4.5 \]
\[ w = (\beta g \alpha G / \gamma)^{1/2} B \exp(-y/\lambda) \sin(y/\lambda) \quad 2.4.6 \]

Equation 2.4.6 indicates that the thermal gradient far from the plate causes the vertical velocity to change sign and undergo a weak reverse flow. Also, the same oscillating behaviour is shown by the temperature which, at a fixed height, undershoots its asymptotic value to reach a minimum at some value \( y_0 / \lambda \).

Investigation of the effect of a non-isothermal environment on the solution of the boundary layer equations also has been done by Eichorn (20) and Cheesewright (2). Both of these investigators studied the boundary layer flow from an isothermal plate in non-isothermal surroundings. The former used a series method to obtain a solution while the latter developed a solution by using similarity transformations of the boundary layer equations. Both authors found that there were regions of reverse flow in the boundary layer as well as regions where the boundary layer temperature was less than ambient. Cheesewright attributed an unstable
flow pattern which he encountered experimentally to the reverse flow development in the boundary layer. Eichorn found that an attempt to use the Karman-Pohlhausen technique and the assumed velocity and temperature profiles of Eckert and Drake (18) gave results which deviated from the series solution and the experimental results of Eckert and Carlson (18). This is undoubtedly due to the assumed velocity and temperature profiles. This is an interesting point to note for the model of Drake (16) which is described in Section 2.4.2 uses the same technique and velocity and temperature profiles. Therefore, it is reasonable to believe that the empirical constant in Drake's model was required to correct for the deviation of the assumed velocity and temperature profiles from the correct solution.

There have been many other studies attempting to solve the vertical plate problem theoretically. Several of these are mentioned below to show the wide number of techniques which have been used. Sparrow and Gregg (56) considered the problem for several thermal conditions at the wall using similarity transforms. Yang (64) presents a technique for solving the natural convection equations by a similarity transform. He also lists the possible cases of the boundary conditions that can be solved using the method. Kohlar (30), also using a similarity transformation technique, solved the case of a uniformly heated vertical plate immersed in a fluid whose bulk temperature varies as the 1/5th power of height. The Karman-Pohlhausen integral method has been successfully used by Sparrow (55), Sparrow and Gregg (56), and Ostrach (41) for solving the isothermal plate and constant heat flux cases with isothermal surroundings. Illingworth (28) obtained a solution of the isothermal plate problem using Bessel functions. A series of perturbation equations were used to obtain transient solutions to the isothermal plate problem by Chung and Anderson (10). Neumer (37) solved several cases where the
plate temperature varied either with position or time. He used a finite difference approximation of the equations of motion and energy. He solved this system of equations using an alternating direction technique similar to that of Peaceman-Rachford (42).

Scherberg (50, 52) theoretically studied the effect of various leading edge conditions on the flow behaviour. It was found that at a suitable distance above the leading edge of the plate, only the relative vertical position of the boundary layer and not the velocity or temperature profiles was affected.

In summary, natural convection of an isothermal, or, uniformly heated plate in an isothermal surrounding has been extensively studied both theoretically and experimentally. In general, the boundary layer flow is laminar up to Grashof Numbers of $10^9$ where transition to turbulent boundary layer flow occurs. In the laminar boundary layer flow regime, the Nusselt Number varies as the $1/4$th power of the Grashof Number. In the turbulent regime, the dependence of the Nusselt Number on the Grashof Number changes to the $1/3$rd power. Similarity transformation techniques are possible only in several special cases. These boundary layer solutions can be considered as zeroth-order perturbation solutions to the more general case where a horizontal component of velocity is permitted in the surroundings.

Transient solutions of the vertical plate problem are very sparse indeed. However, the data which are available indicate that a minimum in the Nusselt Number occurs during the transient. Reasons for this are obscure and probably merit further study.

While the problem of a plate in isothermal surroundings has been extensively studied very little data are available for the plate problem in non-isothermal surroundings. The results available indicate that the boundary layer flow is significantly different than those in the isothermal environ-
ment. In particular, use of the Karman-Pohlhausen technique is not feasible until a better approximation of the velocity and temperature cases can be made.

2.4.2 Enclosed Fluids

This section reviews investigations on natural convection circulations in contained fluids. Most of the studies have been made to elucidate the fluid behaviour when a temperature differential was impressed across the container. For example, a rectangular container where the temperature on one wall is higher than that of the opposite wall has been extensively investigated. Such a system has the advantage over the usual thermal stratification studies in that a steady state is achieved.

Analytical solutions of the container problem where one side is at a higher temperature than the opposite side have been attempted by Poots (44) and Batchelor (6). Both found that for Rayleigh Numbers greater than $10^4$, a boundary layer was formed at the wall while the core remained isothermal and of constant vorticity. In actual fact, as has been shown by the experimental observations of Eckert and Carlson (18) and Elder (22, 21), the central region of the cavity was thermally stratified and characterized by a linear vertical gradient.

In a most recent theoretical study, Gill (25) has used asymptotic methods to study the rectangular container with unequal wall temperatures. He obtained results for the limiting case of very large Prandtl Numbers which agree well with Elder's (21) experimental work.

Although several investigators have attempted to treat the problem analytically, the most prevalent technique has been the use of finite difference approximations to solve the system of partial differential equations. Hellums (27)
91 has treated the problem of heat transfer to a fluid confined in an infinite horizontal cylinder. The cylinder was heated on one vertical half and cooled on the other. He obtained convergent results up to $Gr = 6.15 \times 10^5$ and $Pr = 10$. At higher Grashof and Prandtl Numbers the solutions became unstable. Clark and Barakat (5, 12) have devised an explicit finite difference scheme to solve the differential equations and have applied it to an enclosure with an insulated top and bottom and a constant heat flux on two sides. Although center line temperatures could be predicted using this method, large computational times were required. Also, at higher Grashof Numbers, convergence was not possible in regions near the wall, in the corners, and at short distances below the surface. Wilkes (63) used an implicit finite difference scheme based on the Peaceman-Rachford alternating direction implicit method (42) to solve the two dimensional problem with unequal isothermal sidewalls. Although computation times were considerably shorter (≤ 5 minutes on the IBM 7090), severe restrictions on the magnitude of the Grashof and Prandtl Numbers were required to assure stable solutions. Deardorff (13), Azziz (2), and Samuels and Churchill (48) also obtained solutions for similar natural convection systems using various finite difference schemes. These studies all have indicated that an important limitation on finite difference solutions is that at higher Grashof Numbers, the scale of the actual boundary layer width and local eddy sizes may be considerably smaller than the smallest practical computation grid sizes.

Most recently, Noble (38) has solved the rectangular cavity problem investigated by Wilkes. He posed the problem in an implicit finite difference scheme using the conservative vorticity and energy transport equations as suggested by Fromm (24). It was found that stability of the equations depended not only on the finite difference scheme used but also on the way the boundary conditions were posed; especially
the vorticity boundary conditions. To overcome the "boundary induced" instabilities, a smoothing parameter was applied to the estimate of the boundary vorticity at the advanced time level. This "boundary smoothing" weighed the previous value of the wall vorticity much more heavily than the newly computed value. The net effect of this is that large changes in wall vorticity are prevented. In this way, many of the stability restrictions which plagued the other methods were eased considerably.

In addition to the above, there is another class of problems which do not have a steady state solution. These problems, which will be called thermal stratification problems, have a positive heat flux applied to the walls so that the average enthalpy of the fluid continues to increase with time. Natural convection flows cause the fluid to become thermally stratified. It is this problem with which this work is directly concerned.

Bailey et al. (4) developed a model for thermal stratification in confined fluids which was based on the integration of the liquid mass flow in the natural convection boundary layer along the heated tank wall. This model is the one which has been the basis for subsequent models used by virtually every other investigator. The assumptions made by Bailey were:

1. The initial temperature in the tank was uniform.
2. All the heat input to the tank appears as sensible heat in the boundary layer.
3. All of the fluid in the boundary layer goes into a warm stratum and remains there.
4. The warm stratum is well mixed.
5. There is no mixing between the warm stratum and the lower unheated liquid.

The model distinguishes three main regions in the container. A boundary layer flow at the container wall brings warm fluid to the surface where it mixes in a stratified region. The
warm fluid then forces the cool fluid down into the central core.

Several discrepancies of this basic model are apparent from experimental observation. First, the model predicts a step change in temperature between the thermally stratified region and the core. This does not occur in practice. The stratified region is not isothermal but has an axial thermal gradient (2), (4), (16). This temperature variation was taken into account by Ruder (47) who empirically assumed a temperature profile of the form

\[ \theta - \theta_o = (\theta_{sur} - \theta_o) \exp(-c(L-z)^2) \quad 2.4.7 \]

Also, provision was made for the fact that a portion of the heat added at the wall was used to heat the core. When these modifications were added to Bailey's model, the experimental data correlate moderately well. Use of the velocity and temperature profiles obtained from the solution of the uniformly heated vertical plate in an isothermal surrounding implies that no convection of enthalpy into the core occurs. The experimental data indicate that a radial temperature gradient is not present in the container. Therefore, conduction could not have been the main mode of heat transfer to the core. Hence, while an attempt was made to account for heat transfer to the core, the empirical approach did not permit any suitable explanation of the mechanism of heat transfer.

Tellup and Harper (60) assumed a dimensionless temperature

\[ T_d = (1 - z/b_{st})^n \quad 2.4.8 \]

in the stratified region. The growth of the stratified layer was determined by a simple mass balance. For turbulent boundary layer flow, the thickness of the stratified region was found to be:
While for the laminar boundary layer

$$\frac{\delta_{st}}{L} = 1 - (1 - 0.616(2L/D)(Gr^+/Pr)(Gr^+/Pr)^{2/7}Fo)^{1/5}$$

Experimental data fell into a region bounded by the two lines representing Equations 2.4.9 and 2.4.10.

Tatom (57, 58, 59) studied the effect of bottom and side-wall heating on the thermal stratification in confined liquids. His studies were confined to the turbulent region. Two constants were used to correlate the experimental data; one was a measure of the portion of side-wall heat used to raise the bulk temperature, the other was related to the maximum depth to which thermal stratification occurs. Good correlation was achieved for the cases where both bottom and side-wall heating were present. Poor correlations were obtained when only side-wall heating was used. Visual observations indicated that a significant portion of the side-wall heat "diffused" into the bulk fluid by shedding eddies into the core from the boundary layer.

Review articles by Neff (36) and Clark (11) indicate that very little effort has been expended to examine the fluid dynamics of the system. Analytical models have treated the stratification process in a lump-wise or integral approach with very little attention being given to details. The pertinent system parameters have not been tested extensively but have been restricted to cryogenic fluids.

Probably the first qualitative description of the stratification process was given by Schwind and Vliet (52). A Schlieren technique was used to study the temperature development in a rectangular tank subject to a constant side-wall heat flux in the range of $Ra^+ = 10^{12}$. Salient features of the fluid behaviour are:
1. A starting transient where the fluid near the wall was heated by conduction.

ii. When buoyancy forces were sufficient, a boundary layer flow up the walls resulted.

iii. The warm boundary layer fluid mixed at the top of the tank, thereby forming a stratified layer whose thickness increased with time.

iv. Once a stratified layer was formed, a "reverse shear" region at the stratified layer--core interface developed where boundary layer fluid separated from the wall and moved downward due to an adverse temperature gradient. This resulted in a thinning of the boundary layer thickness in the stratified region.

v. The thermally stratified region was characterized by a linear axial temperature gradient in the central part of the container. While qualitative descriptions of the fluid behaviour was possible, the technique used could not give an indication of the fluid velocities in various portions of the container.

Vliet (61) developed a computerized method for predicting the temperature profile in the container. It gives good correlation of the experimental temperature data, but it does not give any indication of the fluid dynamics.

Drake (16) studied the case of a constant heat flux imposed on the side of a vertical cylinder. System parameters were varied over a wide range. In all cases tested, a linear temperature gradient formed in the bulk of the fluid. However, the temperature profile deviated from this linear gradient in the top ten percent of the fluid. She attributed this to the complex mixing process which occurs there. Using the Karman-Poulhausen integral method the boundary layer equations were solved for the case of a constant side-wall heat flux and linear temperature gradient. The model
predicted that the temperature gradient can be described by:

For laminar flow:

$$\frac{k}{q} \frac{d\theta}{dz} = 4 \text{ Fo}^{4/9} / \text{ Ra}^{1/9}$$  \hspace{1cm} 2.4.11

For turbulent flow:

$$\frac{k}{q} \frac{d\theta}{dz} = 11.5 \text{ Fo}^{8/15} \text{ Pr}^{7/45} / \text{ Ra}^{2/15}$$  \hspace{1cm} 2.4.12

Drake's model also predicts a constant boundary thickness in the stratified region—a result which is in conflict with Schwind and Vliet's observations where appreciable reduction in the boundary layer thickness occurred in the presence of a non-isothermal core. This, in part, may be due to Drake's choice of polynomials for the velocity and temperature profiles.

Attempts to modify the degree of thermal stratification in vessels has received little consideration. Dickey et al. (14) placed baffles along the walls of the Drake cylinder. It was found that baffles had to be extended to at least $3/4$ of the cylinder radius before any appreciable reduction in the degree of thermal stratification was affected. Marx and Suraiya (33) found that altering the relative magnitude of the heat flux to various areas of the cylinder caused the degree of thermal stratification and the shape of the temperature profiles to be markedly altered even though the same total enthalpy was impressed on the cylinder.
III. EXPERIMENTAL PROCEDURE AND APPARATUS

3.1 Scope of the Experimental Work

The experimental program provided data to facilitate the development of a model suitable for describing the thermal stratification phenomenon in partially enclosed fluids. Previous studies in this area have concerned themselves primarily with obtaining the spatial temperature history of the fluid. The present work was initiated to supplement this knowledge with both velocity and temperature data.

The experimental container was a narrow rectangular cavity. Two opposite sidewalls were heated with a constant heat flux; the other two sides were adiabatic. With this arrangement, the problem became two-dimensional though the results are useful for other geometries.

As shown in Section 2.3, the parameters affecting the fluid behaviour are the Aspect Ratio, the Prandtl Number, and the Modified Rayleigh Number. In this study, these parameters were varied over large ranges as shown in Table 3.1. In order to obtain this wide range in fluid parameters, water, glycerine, and a mixture of 85 percent glycerine and 15 percent water were used as test fluids.

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>MINIMUM</th>
<th>MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspect Ratio</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Prandtl Number</td>
<td>7</td>
<td>12000</td>
</tr>
<tr>
<td>Modified Rayleigh No.</td>
<td>(10^7)</td>
<td>(10^{13})</td>
</tr>
</tbody>
</table>

Fixed dimension: width between heated plates = 8 inches
The maximum times for the test runs were as long as 48 hours for the low heat flux experiments to as short as 1 hour for the high heat flux runs. This has considerably extended the range of Fourier Numbers thus far investigated. Because of the rapid recording of the temperature output, meaningful temperature data was obtained at low Fourier Numbers, i.e., at the onset of thermal stratification. These data were used with those of several other investigators, especially those of Drake (16), in the development of the thermal stratification model.

Previous studies have developed reasonable models which correlate temperature data for the core region of the container. Furthermore, the fluid in this core region is believed to descent in a plug flow manner with very little interaction with the boundary layer. However, the surface-influenced region behaves in a much more complex manner which is as yet poorly understood. Therefore, the flow visualization and temperature measurements were taken mainly in this region, i.e., in the region extending from the surface to about 20 percent into the test fluid.

3.2. **Experimental Apparatus**

A photograph of the apparatus is shown in Figure 3.1. It can be described most easily by describing separately the experimental cavity, the temperature sensors, and the velocity measuring equipment.

3.2.1 **The Experimental Container**

The experimental cavity was designed to give two-dimensional flow in the central plane of the container normal to the two heated sidewalls. The boundary conditions
FIGURE 3.1
EXPERIMENTAL APPARATUS
were chosen so as to facilitate a theoretical description. 
Essentially, the cavity was 24 inches long by 24 inches by 
8 inches wide as shown in Figure 3.2. The two 24 inches by 
24 inches sidewalls were modified Corning Glass Radiant 
Heaters. Such heaters consist of 1/4 inch glass panels 
having the outside coated with an electrically conducting 
film of constant resistivity. Half-inch silver screening 
was bonded to the heater at 8 inch spacings across the 
heater so that bus-bar connections could be made to permit 
power application at various Aspect Ratios.

To prevent an excessive heat loss to the surroundings, 
the heaters were backed with 4 inches of polyurethane 
insulation (see Appendix F for heat loss calculations). 
Since the thermal conductivity in the glass was much less 
than the heat transfer coefficient to the fluid, an 
essentially constant heat flux boundary condition resulted 
after the first few minutes of a test run.

To minimize the heat leaks from the fluid to the 
surroundings, the non-heated front and back walls consisted 
of two 1/4 inch Pyrex glass plates separated by a 1/2 inch 
air gap. The bottom of the container was a Masonite board 
backed with two inches of polyurethane foam insulation. 
A small ullage vapor space, two inches deep, above the 
fluid was covered by plywood flaps to minimize the heat 
loss from the surface. A half-inch slit remained between 
the flaps to permit sufficient illumination of the central 
test plane for photographs to be made of the tracer parti-

cles.

To measure the heat loss from the container, several 
thermocouples were embedded into the polyurethane insulation 
at a distance of 1/2 inch and 1 inch from the back of the 
heater. Readings from 10 of these were recorded on a 16-
point Brown recorder (see Figure 3.1) while the others 
were monitored manually using a potentiometer. Heat losses 
were calculated as shown in Appendix F.
FIGURE 3.2
EXPERIMENTAL CONTAINER
Various Aspect Ratios were studied by raising a false bottom to the desired heights. Power was then applied to the cavity through the top buss-bar and the one at the false bottom level. This reduced the possibility of overheating the portion of the enclosure not in contact with the test fluid.

3.2.2 Temperature Measurement System

Temperature measurements were made with 28 to 34 iron-constantan thermocouples placed in the test plane. This test plane was the central portion of the container normal to the heated sidewalls. These thermocouples were manufactured from 3-mil wire so as to minimize any disturbance of the fluid. The glass thermocouple support was placed 3/4 of an inch away from the test plane to prevent its affecting the fluid dynamics. Visual observation of tracer particles in the fluid showed that the flow paths had not been altered by this apparatus in the region of interest.

The thermocouple leads were attached to two motorized switching circuits and referenced to an ice-water constant-temperature bath. The output was then monitored on a Sanborn two-channel recorder. The speed of the switching circuit was such that each thermocouple was scanned once every thirty seconds. A schematic of the thermocouple circuit is given in Figure 3.3.

Before the test program was initiated, each thermocouple was calibrated by comparing its output to that of a "standard" thermocouple at two temperatures. The "standard" thermocouple was calibrated using a constant-temperature bath and a thermometer which was accurate to 0.05°C. For all of the thermocouples tested, it was found that, while the signal for a given temperature varied by as much as 0.5°C from thermocouple to thermocouple, the
FIGURE 3.3
TEMPERATURE MEASUREMENT SYSTEM
slope of the temperature-potential curve was virtually the same as that of the "standard" thermocouple for the range of temperatures of interest in this work. Since only differences in temperature were required, no correction was necessary for individual thermocouples.

The thermocouples were arranged in the tank as shown in Figure 3.4. Note that most of the temperature measurements were made in the top section of the tank. This is because previous work (16) and preliminary work showed that there were no horizontal temperature gradients in the core region.

3.2.3 Velocity Measurements

The fluid dynamics of the system were studied using a "particle streak" photography technique. In this method time-exposure photographs of polystyrene spheres were taken in a thin illuminated section in the central portion of the tank (see Figure 3.2). The length of the resulting streaks is proportional to the velocity of the fluid. In this way, it was possible to calculate the streamlines for the test at any given time.

The equipment which was associated with obtaining the streak photographs was: (i) a light system which produces a thin plane of light in the central plane of the cavity parallel to the front wall; (ii) particles which can be suspended in the fluid and subsequently photographed; and (iii) the photographic equipment.

(1) Lighting System

A collimated curtain of intense light was generated using the apparatus schematically shown in Figure 3.5. The light source was a General Electric 240 Par 56 VSNP lamp. This lamp produced a beam of light with a rating of 110,000 peak initial candlepower and an angle of
NOTE: This is a typical arrangement—actual position was determined from photographs.

Figure 3.4 Arrangement of Thermocouples
To 12 volt transformer

HOUSING

SPOT LIGHT

1/4 in. SLIT

CYLINDRICAL LENS

FIGURE 3.5 LIGHTING SYSTEM
divergence of less than 5 degrees. This light was passed through a 1/4 inch slit placed about 1/2 inch from the lamp. The light emitting from this slit was then further collimated using a cylindrical lens which had a 110 mm focal length. A copper cooling water coil was placed next to the slit to prevent overheating of the lamp. Furthermore, the lens succeeded in absorbing most of the infrared radiation from the lamp, thereby preventing unwanted heating of the fluid from above.

(ii) Polystyrene Test Particles

The particles used in this study were 35μ nominal diameter polystyrene particles available from Dow Chemical Company. Since the density of the particles was 1.05 gm/cc, the terminal velocity in all fluids was small compared to the fluid velocities encountered. (See Appendix G.)

One disadvantage of the polystyrene spheres is that they do not produce isotropic reflection of the incident light. In fact, maximum reflection seems to occur at 45 degrees from the direction of incident light. To overcome this, a thin film of aluminum was vapor-deposited on these particles. Since this film was only several monolayers thick, the particle density was not markedly affected (Appendix G). The resulting particles then reflected light isotropically as was required for this study.

(iii) Photographic System

A 35 mm single reflex camera (Miranda Model F) was used to take streak photographs through the front wall of the container (see Figure 3.1). When Kodak Tri-X film was used, f-stops ranging from f-1.9 to f-8.0 were used to obtain photographs of sufficiently good quality.

The time exposures were taken in two ways. For one second exposures, the automatic timer which was part of the camera was used. For longer exposure times, the
exposure time was measured manually and the shutter was operated remotely.

3.3 Procedure

3.3.1 Test Preparation

The tank was filled with the test fluid about 15 hours before the start of the run. About five hours were allowed for entrapped air to rise to the surface. When water was used, it was necessary to remove the air mechanically. It was found that a Cenco water filter adequately trapped and removed the air bubbles when the test fluid was recirculated through it for several hours. At least ten hours before starting the experiment, 5 cc's of the polystyrene solution (approximately 0.05 percent solid by volume) was dispersed by stirring throughout the fluid. Then the fluid was allowed to virtually attain a completely quiescent state.

3.3.2 Test Procedure

Immediately before starting the test, a ten-second exposure photograph was taken. Also, a base reading of the temperatures was obtained. Then, the power to the side-wall heaters was turned on, the recorder measuring the heat loss through the sides of the container was turned on, and the timer started. Time exposure photographs and temperature measurements were taken throughout the test run at various intervals depending upon the Modified Rayleigh Number being run. In order that the direction of the fluid flow might be determined from the streak
photographs, the f-stop of the camera was advanced by a stop while the film was being exposed. This permitted less light to strike the film during the latter part of the exposure thereby making the streak thinner in the direction of motion.

To determine the magnification of the streak photographs a grid was placed in the test plane after each test and then photographed. The length between two points on the grid measured from the photograph divided by the actual distance between these same points gave the magnification. Errors due to refraction were slight when velocities were calculated in this manner. This was observed from both the grid photographs and from calculations (Appendix H).
IV. EXPERIMENTAL DATA

4.1 Introduction

In this chapter, experimental results obtained from tests using the rectangular cavity described in Chapter 3 are presented. The first part presents the temperature data. Then, the velocity data for several tests are presented. Finally, a brief description of the procedure for reducing the data is given. A more detailed description of this procedure can be found in Appendix N.

Test runs in this work were identified by a three symbol code, e.g., W-3-HI. The first symbol indicated the fluid used—W = water, G = glycerine, M = 85 percent glycerine—water solution; the second symbol gave the Aspect Ratio of test; and the third symbol indicated the heat flux used—HI = high heat flux, LO = low heat flux. Therefore, the code W-3-HI indicates that the test was run with water as a test fluid in a container with an Aspect Ratio of 3 and a high heat flux. For each fluid and Aspect Ratio, a high and low heat flux was used. The high heat flux usually had a current load of 10 amperes imposed to each side wall heater, while the low heat flux test had 1 ampere imposed on each side wall.

4.2 Experimental Data Reduction

The raw experimental data consisted of visual and photographic observation of the movement of polystyrene particles in the test fluid and the measurement of temperatures within a plane perpendicular to the heated
sidewalls at specific times during the test. Also, the power input to the sidewalls was recorded from readings of the current through each panel and from the known resistance of the panel. The reduction of this data into the final form are discussed below. A detailed procedure of the experimental apparatus and procedure is given in Chapter 3.

4.2.1 Temperature Data

The temperature data was converted from chart readings to actual temperature rise, \(T - T_0\), for each of the thermocouples as a function of time. These were plotted as functions of time as in Figure 4.1. The resulting thermal history of the fluid was then cross-plotted as a function of position at several selected times. The results of this procedure are illustrated in Figures 4.2 to 4.4.

4.2.2 Velocity Data

The motion of the fluid was observed using a "particle streak" photography technique which is described in detail in Chapter 3 and Appendix H. Briefly, time exposure photographs of neutrally buoyant particles immersed in the fluid were taken throughout the length of the experiment. The time elapsed since the start of the experiment and the length of each exposure was recorded as each photograph was taken. Prior to the start of the experiment, a photograph of a ruled grid placed in the plane where the neutrally buoyant particles are to be observed during the course of the experiment was taken. Considerable care was taken throughout the entire experiment not to
$\text{FIGURE 4.1 TEMPERATURE VS. TIME READING, P_f = 7, Ra} = 10^{13}$
Figure 4.2: Axial temperature profiles, $Pr = 7$, $Ra = 10^{13}$.
FIGURE 4.3 AXIAL TEMPERATURE PROFILES, Pr = 7, R* = 1.81x10^{12}
FIGURE 4.4 AXIAL TEMPERATURE PROFILES, Pr = 12500, $R\theta = 6.9 \times 10^{10}$
move the camera from its position at this time. From this
grid reference photograph, the magnification of the photo-
graphs could then be easily determined.

Figures 4.5 through 4.8 give examples of the streak
photographs obtained. Because the f-stop was reduced
during the time of exposure (see Chapter 3), the streak
width decreases in the direction of motion (Figure 4.5).
Therefore, the direction of the fluid motion can be
ascertained from the "streak" photograph.

Examination of these "streak" photographs was used
to obtain a qualitative understanding of the fluid dyna-
mics encountered during the thermal stratification process.
The conclusions so obtained are discussed in a later
section (Section 4.3).

Quantitative velocity data was obtained from several
of these photographs in order to substantiate the validity
of the thermal stratification simulation which is developed
in Chapter 6. A transparent sheet of graph paper was
placed over the "streak" photograph to be analyzed. The
coordinates of the top corner and several points on the
surface and wall were noted. Then, the coordinates of
the top and bottom tips of several hundred streaks were
recorded. Care was given to the selection of the streaks
so that a random sample of velocities might be obtained.
These data were then used to calculate the horizontal and
vertical velocities as a function of position. Finally,
the velocities so obtained were grouped in increments of
height, usually a \((z/L)\) of 0.02, and plotted. Stream-
lines were then calculated by integrating the velocity
profiles. When the vertical velocities were used, the
streamlines were calculated according to

\[
\begin{align*}
= D \int_{y/D=0}^{y/D} w \, dy/D
\end{align*}
\]

4.2.1
FIGURE 4.5 STREAK PHOTOGRAPHS FOR TEST W-2-H1, Pr=7, Ra=1.81(10^{12})

(a) 120 sec.

(b) 360 sec.
(a) 940 sec.
(b) 2250 sec.

FIGURE 4.6 STREAK PHOTOGRAPHS FOR TEST W-2=Hi, Pr=7, Ra*=1.81(10^{12})
FIGURE 4.7 STREAK PHOTOGRAPHS FOR TEST G-3-Hi., Pr=12500, $Ra=6.9 \times 10^{10}$
(a) 525 sec.
(b) 690 sec.

FIGURE 4.8 STREAK PHOTOGRAPHS FOR TEST G=3-H1, Pr=12500, $Ra=6.9(10^{10})$
while
\[
\frac{(z/L)}{z/L} \int \text{d}(Z/L) = L
\]

was used to calculate the stream functions when the horizontal velocities were used. It was this latter formula which was used more frequently in this work. This is because the magnitude of the vertical velocity and its position in the boundary layer was very difficult to measure from the photographs due to the thinness of the boundary layer. Therefore, considerable difficulty and error would be encountered in attempting to calculate the stream function using Equation 4.2.1. On the other hand, no such problem existed when horizontal velocities were used. Hence, the stream functions calculated from Equation 4.2.2 tend to be the more accurate. The results obtained from the integration of the velocity data are presented in Section 4.3.

4.3 Description of Velocity Data

In the rest of this chapter, quantitative analysis of several "streak" photographs are presented. In particular, two tests were analyzed, viz., G-3-H1 and W-2-H1. The first was a high heat flux test with glycerine as the test fluid and Aspect Ratio of 3. The nominal Modified Rayleigh Number for this test was \(6.91 \times 10^{10}\). The other test was a high heat flux test with water as the test medium and an Aspect Ratio of 2. The nominal Modified Rayleigh Number for this test was \(1.81 \times 10^{12}\). The purpose of these analyses was to determine whether the thermal stratification simulation model could also be used to predict the flow behaviour in the central core of the container. It is obvious from the "streak" photographs
given in Figures 4.5 to 4.8 that there is a substantial horizontal velocity in the top part of the container. This means that the "plug flow" assumption cannot be exactly valid in this region. However, the model may still be useful in predicting the flow rate in the core.

One interesting flow phenomenon is evident from the streamline data. This is that the flow is unstable in the system. Consider the G-3-HI test: at $\tau = 1.33 \times 10^{-2}$ (Figure 4.9a), the streamlines are regular. The streamlines indicate that for $Z < 0.8$, the flow in the core actually seems to be in "plug flow". Also, the flow is symmetrical about the central plane of the container, i.e., at $y/d = 0.5$

$$\Psi = 0 \text{ and } \frac{\partial W}{\partial y} = v = 0.$$ 

However, as time progresses, this symmetry condition begins to disappear. Figure 4.9b shows that at a later time, $\tau = 3.72 \times 10^{-2}$, considerable disturbance of the flow occurs. This perturbation grows with time, Figure 4.9c and so the flow of fluid in the core is said to be unstable. As noted in Figure 4.9d, the $\Psi = 0$ centerline boundary condition is entirely distorted. This type of core instability was present in all of the tests of this experiment. As seen from Figures 4.10a and 4.10b, the unstable flow behaviour seems to be much more severe at lower Prandtl Numbers.

The reason for this instability is unclear. Since this type of behaviour was noticed in all of the experiments in this work, it is clearly apparent that the growth of the perturbation is independent of fluid type or heat flux. However, the source of the perturbation cannot be easily ascertained. Also, since only one container was used in the experiments, it cannot be established whether the distortion of the streamlines is a function of the apparatus used, or, is, in fact, common to all thermal stratification experiments. The fact that Drake (16) experienced a diffusion of dye into the entire core of
FIGURE 4.9a STREAMLINES FOR TEST G-3-H1, $\tau=1.33 \times 10^{-2}$
FIGURE 4.9b STREAMLINES FOR TEST G-3-H1, $\tau = 3.72 \times 10^{-2}$
FIGURE 4.9o STREAMLINES FOR TEST G-3-H1, \( \tau = 9.63 \times 10^{-2} \)
FIGURE 4.10: STREAMLINES FOR TEST W-2-H1, $\tau = 5.73 \times 10^{-1}$
FIGURE 4.10b STREAMLINES FOR TEST W-2-H1, $\tau = 8.58 \times 10^{-1}$
her cylinder while injecting dye at only one point strongly suggests that this type of instability existed in that work also. However, the cause of the instability is not evident and can be resolved only when more flow behaviour data will be obtained from several different geometrical configurations.

Several possibilities for this unstable behaviour might be put forth. Firstly, the electrically conducting film used for the heaters may not have been perfectly uniform over the entire surface of the glass heater. Although the overall resistances matched well for both heaters, and the process whereby the film was attached to the glass an even coating, there is a good possibility that the slight fluctuations in resistivity over the surface of the heater did not match perfectly. This would slightly alter the boundary layer flow characteristics for each heater. Initially, since most of the flow reaches the surface, these slight differences in flow may not be readily apparent. However, as time progresses, the amount of fluid flowing, and the amount reaching the top of the container diminishes. Therefore, the slight variations in the resistance may cause perturbations which might be amplified and grow with time. Since Drake's apparatus was built using the same electrically conductive coating, the same type of perturbations would be expected to be found in her experiments also. Secondly, the tests of Drake and those of this work both had a free surface. Therefore, the convection patterns which were set up in the vapor space above might have caused some perturbation which was transferred to the core of the fluid and was amplified with time. Finally, it might be possible that vibrations from the surroundings might have caused the perturbations. The experimental apparatus was insulated from high frequency noise which is most often found in the laboratory, but certainly not all of the vibrations could be eliminated.
On the other hand, Drake's apparatus did not have any such precautions. The fact that both experimental works had similar instabilities tends to discredit this possibility as being the primary source of the perturbations.

Even though the unstable flow behaviour in the core boundary layer fluid was present, the thermal gradient was not affected. This is believed due to the fact that the fluid which is at any one level is in thermal equilibrium with the surrounding fluid. That is, in the core, the fluid at any one level is hotter than the fluid below it and cooler than the fluid above it. Hence, while a perturbation in the fluid motion may exist, a thermal perturbation did not.

As can be seen from Figures 4.9 and 4.10, the height at which the boundary layer separation begins decreases as the fluid becomes more stratified. This point continues to decrease throughout the entire length of the experiment. This is due to the fact that when the boundary layer fluid reaches the thermally stratified fluid, it can no longer support the entire amount of fluid at this higher core temperature. Consider a small differential boundary layer fluid element such as that shown in Figure 4.11. At some height $z$, the average temperature of the boundary layer fluid is

$$T_{ave} = \frac{w(T - T_o)dy}{wdy} z \quad 4.3.1$$

at some height $z + z$, the average temperature becomes

$$T_{ave} = \frac{w(T - T_o)dy}{wdy} z + z \quad 4.3.2$$

and

$$T_{ave} - T_{ave} = \frac{gZ}{\rho C_p} \quad 4.3.3$$

If, however, the fluid reaches some thermally stratified fluid, it is possible that

$$T_{ave} - T_{ave} = T - T \quad 4.3.4$$
FIGURE 4.11 DIFFERENTIAL ELEMENT OF VERTICAL CONTAINER
i.e., the average boundary layer temperature is increased by less than the increase in core temperature. This would result in deceleration of the boundary layer and some separation of the fluid. Hence, since the degree of thermal stratification and depth of stratification increases with time, the depth at which separation occurs decreases with time. It should be noted that the position of the point where separation begins is dependant on both the degree and extent of thermal stratification.

Drake (16), in her model of thermal stratification, postulated a complex "mixing region" in the top 10 percent of the container. It was into this "mixing region" where all of the boundary layer fluid entered and was mixed in some complex manner before settling down the central core of the vertical cylinder in "plug" flow. As seen from Figures 4.9 and 4.10, no such "mixing region" exists. The "mixing region" concept does not have any actual physical correspondence. The fluid at the top is deflected by the surface and flows out towards the center of the container. Then, it moves down the core of the container. No severe mixing of the fluid implied by Drake exists. The only distinguishing feature of the top 10 percent of the container is that since a greater amount of boundary layer fluid is ejected in this region, the horizontal velocities encountered here are much higher than in other parts of the container. The reason that the thermal gradient in this region is not linear, as is approximately true in the lower portion of the container, is that, in this region, fluid is ejected from the boundary layer which is at a considerably higher average temperature than the core temperature, while in the lower region, the boundary layer fluid ejected is at approximately the same temperature as the core temperature (see model development in Chapter 6). Therefore, the core temperature in the upper region increases at a much higher rate than that in the
lower region and so deviates from a linear gradient.

A listing of the temperature data and the location of the original results found in Appendix J.
5.1 Introduction

The primary mode of heat transfer in enclosed unstirred fluids is by natural convection. Heat is transferred to the core by convection of the warm fluid at the top of the container downwards and presumably by interaction of the warm boundary layer directly with the core. It is this latter mode which is the subject of this chapter.

For fluids of Prandtl Number greater than 1, the boundary layer thickness, especially the thermal boundary layer, becomes very thin. This makes experimental investigation of the boundary layer difficult. Temperature measurements using thermocouples is not possible since the diameter of even the finest thermocouple is of the same magnitude as the boundary layer thickness. Therefore, both temperature and velocity measurements require an elaborate optical system which would magnify the boundary layer region so that meaningful observations might be obtained. Such experimental study was beyond the scope of this investigation.

An alternative method to study the boundary layer--core interactions is by a theoretical analysis of the boundary layer equations with boundary conditions similar to those existing in enclosed thermally stratified fluids. It is this method which was chosen for this study.

Because of the complex nature of the Navier-Stoke's equation and energy equation, an analytical solution is not presently possible except for some special situations where similarity transformations of the equations can be obtained. Therefore, numerical solution of the pertinent
boundary layer equations appears to be the best method of solution. As seen in Chapter 2, many of the finite difference schemes which have been developed to solve the general equations thus far suffer from rather restrictive stability requirements and require lengthy computational times to achieve steady state solutions. However, simplification of the general equations afforded by the boundary layer assumptions (55) reduces the complexity of the equations so that solution of these equations using an implicit alternating direction scheme requires relatively small computational times.

This chapter is divided into roughly three sections: (i) the formulation of the basic equations which describe the natural convection process and the derivation of their finite difference analogs; (ii) the solution of the problem of natural convection from a uniformly heated vertical plate in isothermal surroundings; (iii) the solution of the problem of natural convection from a uniformly heated vertical plate in surroundings where the temperature increases linearly with height. The results obtained are compared to existing solutions wherever possible to illustrate the stability and convergence properties of the finite difference scheme used. Finally, the results for the two cases are compared to illustrate the differences in fluid behaviour when a linear temperature gradient is present in the bulk.

5.2 Modification of the General Equations Describing Natural Convection

5.2.1 The Boundary Layer Equations

Consider the situation shown in Figure 5.1. A uniformly heated vertical plate is placed in an infinite
surrounding of some temperature distribution $T_\infty$. Free convection results on the plate in such a way that a boundary layer results. Then, following the procedure of Schlichting (55), it can be shown that the vertical viscous forces, $\gamma \frac{\partial^2 p}{\partial z^2}$, are negligible to the other forces in Equation 2.3.14c. Similarly, the $z$-direction conduction, $\alpha \frac{\partial^2 \theta}{\partial z^2}$, is negligible to the other terms in Equation 2.3.14e. Also, invocation of the above boundary layer assumptions precludes the use of the $y$-direction momentum equation, Equation 2.3.20, so that the continuity equation, $z$-direction momentum equation, energy equation, the boundary conditions and initial conditions constitute a complete description of the problem. For this problem, these are:

\[
\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \quad 5.2.1a
\]

\[
\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = g\beta (\theta - \theta_0) - \frac{1}{\rho_0} \frac{\partial P}{\partial z} + \frac{\partial^2 w}{\partial y^2} \quad 5.2.1b
\]

\[
\frac{\partial \theta}{\partial t} + w \frac{\partial \theta}{\partial z} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \quad 5.2.1c
\]

\[
w = v = 0 \quad \{t = 0, \forall y, z\} \quad 5.2.1d
\]

\[
\theta = \theta_1
\]
\[ w = v = 0 \] 
\[ q = -k\frac{\partial \theta}{\partial y} \] 
\[ y = 0, \text{all } z \] \hfill 5.2.1e

\[ w = 0, \theta = \theta_0, y \to \infty, \text{all } z \] 
\[ w = v = 0, \theta = \theta_0, z = 0, \text{all } y \] 

At distances far from the plate, \( w = 0 \) for a pure free convection problem. Therefore, Equation 2.3.19 can be written for the ambient fluid as:

\[ \frac{1}{\rho_0} \frac{\partial P}{\partial z} = g\beta(\theta - \theta_0) \] \hfill 5.2.2

From an order of magnitude analysis (Appendix B), it can be shown that the pressure gradient due to fluid motion is negligible to the other forces in the \( z \)-direction momentum equation. Therefore, it can be assumed that

\[ P = P_\infty \] \hfill 5.2.3

with little error. Using this relation and Equation 5.2.2, \( \frac{1}{\rho_0} \frac{\partial P}{\partial z} \) can be eliminated from Equation 4.2.1b so that:

\[ \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = g\beta(\theta - \theta_0) + \frac{\partial^2 w}{\partial y^2} \] \hfill 5/2/4

Further modification can be made by defining a new temperature variable such that

\[ \tilde{T} = \theta - \theta_\infty \] \hfill 5.2.5

Then, Equation 5.2.1 and 5.2.4 can be rewritten as:

\[ \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \] \hfill 5.2.6a

\[ \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = g\beta \tilde{T} + \frac{\partial^2 w}{\partial y^2} \] \hfill 5.2.6b

\[ \frac{\partial \tilde{T}}{\partial t} + w \frac{\partial \tilde{T}}{\partial z} + v \frac{\partial \tilde{T}}{\partial y} = \frac{\alpha}{\gamma} \frac{\partial^2 \tilde{T}}{\partial y^2} - \frac{\partial \tilde{F}_{tg}}{\partial z} \] \hfill 5.2.6c

\[ w = v = 0, \tilde{T} = T_1, t = 0, \text{all } y, z \] \hfill 5.2.6d
For \( t = 0 \),
\[
\begin{align*}
w &= 0 = T = 0, y \quad \text{all } z \\
w &= v = 0, \frac{\partial T}{\partial y} = -\frac{q_0}{k}, y = 0 \text{ all } z \quad 5.2.6e \\
w &= v = T = 0, z = 0 \text{ all } y
\end{align*}
\]

5.2.2 Normalization of Equations

Using the scaling parameters in Equation 2.3.27, Equations 5.2.6 can be rewritten as:

\[
\begin{align*}
\frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y} &= 0 \quad 5.2.7a \\
\frac{\partial W}{\partial \tau} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} &= Pr T + Pr \frac{\partial^2 W}{\partial Y^2} \quad 5.2.7b \\
\frac{\partial T}{\partial \tau} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} &= \frac{\partial^2 T}{\partial Y^2} - W \frac{\partial T}{\partial Z} \quad 5.2.7c \\
W &= V = 0, T = T_1, \tau = 0 \text{ all } Y, Z \quad 5.2.7d
\end{align*}
\]

For \( \tau \to 0 \)
\[
\begin{align*}
w &= V = 0, \frac{\partial T}{\partial Y} = -1/Ra \quad 1/4, Y = 0 \text{ all } Z \\
w &= V = 0, n = 0, Z = 0 \text{ all } Y \quad 5.2.7e \\
w &= 0, T = 0, Y \to \infty \text{ all } Z \\
\text{where } T &= -\frac{n - q_0}{q_0 L/k}.
\end{align*}
\]

Note that the Equations depend upon only two parameters, viz., the Prandtl number, \( Pr \), and the modified Rayleigh number, \( Ra \).

5.3 A Special Analytical Solution

In this section a steady state asymptotic solution of the boundary layer equations (Equation 5.2.6) is derived. By an asymptotic solution, it is meant that the dependent variables, in this case the velocity, \( W \), and the temper-
ature, T, both reach a limit with respect to one of the independent variables, in this case the height, Z. That is, for this case,

$$\frac{\partial T}{\partial Z} = \frac{\partial W}{\partial Z} = 0$$  \hspace{1cm} 5.3.1

From continuity (Equation 5.2.6a) and Equation 5.3.1, it can be shown (Appendix C) that

$$v = 0$$  \hspace{1cm} 5.3.2

Using these relations, and assuming that the temperature of the surrounding fluid increases linearly with height from the leading edge of the plate, i.e.,

$$\frac{dT}{dZ} = B$$  \hspace{1cm} 5.3.3

where B is a positive constant, Equations 5.2.6b and 5.2.6c can be rewritten as

$$g\beta T + \gamma \frac{\partial^2 T}{\partial y^2} = 0$$  \hspace{1cm} 5.3.4a

and,

$$a\frac{\partial^2 T}{\partial y^2} - WB = 0$$  \hspace{1cm} 5.3.4b

Therefore, one has a set of ordinary differential equations which is readily amenable to solution. Furthermore, if one applies the pertinent boundary conditions given in Equation 5.2.6e, viz.,

$$- k \frac{\partial T}{\partial y} = q, \quad y = 0$$

$$W = 0 \quad y = 0$$

$$T = 0 \quad y \to \infty$$

$$W = 0 \quad y \to \infty$$  \hspace{1cm} 5.3.4c

it can be shown (Appendix C) that the solution is
\[ T_0 = 2 \exp(-Y_0) \cos(Y_0) \quad 5.3.5a \]
\[ W_0 = \exp(-Y_0) \sin(Y_0) \quad 5.3.5b \]

where
\[ T_0 = \sqrt{2} \frac{(\Theta - \Theta)}{\Theta L/k} \text{Ra}^{1/4} \]
\[ W_0 = \frac{\sqrt{2} \nu L}{\alpha} \text{Ra}^{3/4} \text{Ra}^{+} \]
\[ Y_0 = \frac{\sqrt{2} \nu L}{\alpha} \text{Ra}^{1/4} \]

and \( \text{Ra}^{+} = \frac{g \beta L^4 (dT_0/dz)}{\alpha \gamma} \)

A new parameter, \( \text{Ra}^{+} \), results from this analysis. This parameter gives a measure of the bulk temperature gradient. It will be referred to as the Core Rayleigh Number in the remainder of this work.

Under what condition does such an asymptotic solution exist? To answer this, one may calculate the amount of heat required to raise the boundary layer fluid temperature from some point \( Z_0 \) to the base temperature some distance up the wall, \( Z_0 + 6Z \). The amount of heat required is

\[ \int_0^\infty \rho c_p W_0 (dT_\infty/dz) dY_0. \]

Using Equation 4.3.5b, and integrating, one finds that

\[ \int_0^\infty \rho c_p W_0 (dT/dz) dY_0 = q_0 \]

This means that for an asymptotic solution to exist, the amount of heat required to raise the average boundary layer temperature at the same rate as that of the surrounding fluid is exactly that supplied by the wall heat flux, \( q_0 \).

Both in Drake's work (16) and in the present study (Chapter 4), it was found that at long times the core
temperature gradient in enclosed fluids whose sidewalls are heated at a uniform rate tended to be approximately linear with height. Therefore, an asymptotic boundary layer flow might be present under these circumstances. Using an integral form of the boundary layer equations and assuming a polynomial form for the boundary layer velocity and temperature of the form

\[ \frac{\theta - \theta_0}{\theta_f - \theta_0} = \frac{q}{2k} \delta^* (1 - y/\delta^*)^2 \]  

\[ w = w^* y/\delta^* (1 - y/\delta^*)^2 \]

Drake derived a set of ordinary differential equations for the dimensionless boundary layer thickness, \( \Delta \), and boundary layer velocity scale, \( \mathcal{A} \). These equations were:

\[ \frac{d}{dz} (\Delta^2 \mathcal{A}) = \frac{2A_1}{A_f} \frac{1}{Pr} - \frac{2A_2}{A_1} \mathcal{A} \Delta (Ra/Rd) \sqrt{Gr}^{1/4} \]

\[ \frac{d}{dz} (\mathcal{A}^2 \Delta) = \frac{A_4}{2A_3} \Delta^2 - \frac{1}{A_3} \mathcal{A} \Delta \sqrt{Gr}^{1/4} \]

where \( A_1, A_2, A_3, A_4 \) are constants which depend on the type of polynomial expressions chosen for the velocity and temperature. Comparing Equations 5.3.7 and 5.3.5, it is seen that the functional forms of the velocity and temperature functions are significantly different. However, if Drake's integral technique were re-done using Equations 5.3.5, it is interesting to note that the resulting differential equations for \( \mathcal{A} \) and \( \Delta \) would be identical to those in Equation 5.3.8 except for the values of the constants. Table 5.1 gives the values for these constants for the two cases. From this, it is seen that the general conclusions reached by Drake remain in this asymptotic region although quantitatively there may be some error. For example, the Nusselt Number predicted by Drake for
the case of a uniformly heated vertical plate in surroundings which increase linearly with temperature was

\[ \text{Nu} = 0.686 \text{ Ra}^{1/4} \]  

in this asymptotic region. On the other hand, using the correct asymptotic solution (Equation 5.3.5), the Nusselt Number was found to be

\[ \text{Nu} = 0.707 \text{ Ra}^{1/4} \]

**TABLE 5.1**

<table>
<thead>
<tr>
<th>Function</th>
<th>(A_1)</th>
<th>(A_2)</th>
<th>(A_3)</th>
<th>(A_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial (Eq'n 4.3.7)</td>
<td>1/30</td>
<td>1/12</td>
<td>1/105</td>
<td>1/3</td>
</tr>
<tr>
<td>Asymtotic (Eq'n 4.3.5)</td>
<td>1/4</td>
<td>1/2</td>
<td>1/8</td>
<td>1</td>
</tr>
</tbody>
</table>

Since the asymptotic solution is independent of height, the wall temperature must increase linearly with height, i.e.,

\[ Q_w = Bz + C \]  

Therefore, for a uniformly heated vertical plate in surroundings where the vertical temperature gradient is constant, the wall temperature also must increase linearly with the distance from the leading edge and must remain in a constant amount greater than the surrounding fluid temperature at the same height. This solution is equivalent to the one obtained by Gill (25) for the boundary conditions given in Equation 5.3.11. It is also a solution of the similarity equations which were posed but not solved by
Cheesewright (2).

From this, the following conclusion may be drawn:

Where the asymptotic solution is valid,

1. The boundary layer thickness is constant;
2. There is no entrainment of the surrounding fluid and hence no acceleration of the fluid up the wall;
3. The heat added at the wall is used to raise the average boundary layer temperature at the same rate as that of the surrounding fluid;
4. The thermal boundary layer thickness and velocity boundary layer thickness are approximately the same width;
5. The velocity, temperature, and boundary layer thicknesses are independent of the Prandtl number;
6. The boundary layer thickness and temperature both depend on a new parameter, the Core Rayleigh Number, while the velocity depends not only on this parameter but also on the Modified Rayleigh Number.

In an enclosed container, the asymptotic solution cannot exist throughout the entire height of the container since fluid must be entrained into the boundary at the bottom. In order to determine the height at which the asymptotic solution can be considered valid, a solution of Equations 5.2.6 was undertaken using finite difference approximations.

Considerable reduction in the computational effort can be achieved by redefining the dimensionless parameters as follows:

\[ T_\Theta = \sqrt{2} \frac{(\Theta - \Theta) R^\infty}{q L/k} \]  
\[ W_\Theta = \frac{\sqrt{2}}{2} \frac{\text{\textit{W}}L}{a} \frac{R^\infty}{R^\text{d}} \frac{3/4}{R^\text{d}} \]
\[ Y_\delta = \frac{\sqrt{2} V L}{R\alpha^{1/4}} \quad 5.3.12 \]
\[ Z_\delta = \frac{\sqrt{2} Z L}{R\alpha^{5/4}} \quad \frac{R\alpha^{-1/4}}{R\alpha^{1/4}} \]
\[ V_\delta = \sqrt{\frac{2}{\alpha}} R L \quad 5.3.13 \]
\[ \tau_\delta = \frac{a t_2}{2L} \quad R\alpha^{1/2} \]

Then, Equations 5.2.6 can be rewritten (Appendix D) as:

\[ \frac{\partial W_\delta}{\partial Y_\delta} + \frac{\partial V_\delta}{\partial Z_\delta} = 0 \quad 5.3.13a \]

\[ \frac{\partial W_\delta}{\partial T_\delta} + W_\delta \frac{\partial W_\delta}{\partial Z_\delta} + V_\delta \frac{\partial W_\delta}{\partial Y_\delta} = PrT_\delta + Pr \frac{\partial^2 W_\delta}{\partial Y_\delta^2} \quad 5.3.13b \]

\[ \frac{\partial T_\delta}{\partial T_\delta} + W_\delta \frac{\partial T_\delta}{\partial Z_\delta} + V_\delta \frac{\partial T_\delta}{\partial Y_\delta} = \frac{\partial^2 T_\delta}{\partial Y_\delta^2} - 4W_\delta \quad 5.3.13c \]

with boundary conditions

\[ \left\{ \begin{array}{l}
\frac{\partial T_\delta}{\partial Y_\delta} = -2 \\
W_\delta = V_\delta = 0
\end{array} \right. \quad 5.3.13d \]

\[ T_\delta = W_\delta = 0 \quad Y_\delta \to \infty \]

\[ T_\delta = W_\delta = 0 \quad Z_\delta = 0 \]

Therefore, the solution of Equations 5.3.13 for various Prandtl Numbers will yield complete information as to the velocity and temperature fields. The finite difference equations used to solve these equations are developed in Appendix M. Also, criteria for obtaining stable and convergent solutions for both the case of a uniformly heated vertical plate in isothermal surroundings and the case of a uniformly heated vertical plate in surroundings...
with a linear temperature gradient are developed in the same appendix. The computed results are presented and discussed in the next section.

5.4 Uniformly Heated Vertical Plate Computed Results

5.4.1 Isothermal Surroundings

5.4.1.1 Effect of Prandtl Number

As the Prandtl Number is increased, the ratio of viscous diffusivity to thermal diffusivity increases. This is equivalent to saying that a greater amount of heat is carried away by convection than by conduction with increasing Prandtl Number. The effect of the Prandtl Number for the case of a uniformly heated vertical plate in isothermal surroundings is described in this section.

Figure 5.2 shows the velocity and temperature fields at a height $Z = 1$ and Modified Rayleigh Number of $10^6$ as a function of Prandtl Number. For increasing Prandtl Number, the maximum value of $W$ increases, $T_w$ decreases, and the ratio of thermal boundary layer to momentum boundary layer decreases. This is compatible with the analytical results of Chang (7) and Sparrow and Gregg (47). Also, as the Prandtl Number increases the dimensionless horizontal velocity, $V$, increases as seen in Figure 5.3. The amount of fluid entrained by the boundary layer is a weak function of height at low Prandtl Numbers but becomes a strong function of height at higher Prandtl Numbers.

In terms of the physics of the problem, the temperature and velocity profiles are determined by a balance of forces given by Equation 5.2.7. In the momentum equation there is a balance between the inertial, viscous, and buoyancy forces while in the energy equation there is a
Figure 5.2 Uniformly Heated Vertical Plate in Isothermal Surroundings: The Effect of Prandtl Number
FIGURE 5.3 THE EFFECT OF PRANDTL NUMBER ON THE HORIZONTAL VELOCITY

\[ V = \frac{\nu L}{a} \left( \frac{R_a^+}{10^6} \right)^{1/4} \]

\[ R_a^+ = 10^6 \]
balance between convection and conduction.

For all Prandtl Numbers, the convection and conduction terms must be of the same order of magnitude. However, in the momentum equation, one of the forces may be small relative to the other forces. Force diagrams of the momentum equation for a plate height of \( z = 1 \), and a Modified Rayleigh Number of \( 10^6 \), and Prandtl Numbers of 1, 10, and 100 are shown in Figures 5.4 to 5.5. There are basically three distinguishable regions in the boundary layer. In Region 1, near the wall, the buoyancy forces are balanced by the viscous forces. In Region 3, near the bulk, the viscous forces are balanced by the inertial forces. Finally, in Region 2, all the forces are of the same order of magnitude. As the Prandtl Number increases, Region 1 remains at about the same thickness. However, the region becomes better defined, i.e., the inertial forces are negligible for a much longer distance from the plate. Also, Region 2 tends to be eliminated as the Prandtl Number increases. Therefore, for Prandtl Numbers equal to or greater than 10, the boundary layer is effectively composed of two regimes—one where viscous and buoyancy forces predominate, and the other where the viscous and inertial forces predominate. Physically, the wall heat flux is conducted into the first region thereby setting up a buoyancy force which tends to accelerate the fluid. This tendency is opposed by viscous forces tending to retard the fluid. It is in this region that most of the wall heat flux is removed by convection. Therefore, in the region far from the wall, the buoyancy force is small and the fluid is retarded by viscous forces. Mathematically, for high Prandtl Numbers, in Region 1,

\[
\frac{\partial^2 w}{\partial y^2} = -T 
\]

\[
\frac{\partial w T}{\partial z} + \frac{\partial v T}{\partial y} = \frac{\partial^2 T}{\partial y^2}
\]
FIGURE 5.4a FORCE DIAGRAM FOR MOMENTUM EQUATION
- $Pr = 1$, $Ra = 10^6$ -
FIGURE 5.4b FORCE DIAGRAM FOR MOMENTUM EQUATION

\[ \text{Pr} = 10, \text{Ra} = 10^6 \]
FIGURE 5.5  FORCE DIAGRAM FOR MOMENTUM EQUATION
- \( PR = 100, Ra = 10^6 \) -
and in Region 3,
\[ W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = \text{Pr} \left( \frac{\partial^2 W}{\partial Y^2} \right) \]
\[ T = 0 \]

In summary, increasing the Prandtl Number reduces the thermal boundary layer thickness, and increases the momentum boundary layer thickness. For Prandtl Numbers greater than 10, the momentum boundary layer can be divided into two force regions—the region near the wall where the viscous and buoyancy forces are of the same order of magnitude and the region far from the wall where the viscous and inertial forces are of the same order. The division between the two regions for high Prandtl Numbers occurs at a distance from the wall where
\[ \frac{\partial^2 W}{\partial Y^2} = 0 \]

5.4.1.2 The Effect of the Modified Rayleigh Number

The Rayleigh Number can be thought of as a ratio of forces favoring fluid circulation to the forces retarding circulation. This section will consider the effect of increasing the Rayleigh Number on the velocity and temperature fields.

Figure 5.6 shows the velocity, W, and temperature, T, fields for Modified Rayleigh Numbers from $10^6$ to $10^{12}$, a Prandtl Number of 10 and a plate height $Z = 1$. Figure 5.7 shows the corresponding horizontal velocity field, V. The dimensionless temperature and velocity are shown to decrease with increasing Rayleigh Number. The thermal boundary layer thickness decreases with Modified Rayleigh Number also. Finally, while the horizontal velocity field, V,
VELOCITY

- $R_a = 10^6$
- $R_a = 10^9$
- $R_a = 10^{12}$

$Pr = 10$, $Z=1.0$

FIGURE 5.6 UNIFORMLY HEATED VERTICAL PLATE IN ISOTHERMAL SURROUNDINGS:
- THE EFFECT OF RAYLEIGH NUMBER -
FIGURE 5.7 THE EFFECT OF MODIFIED RAYLEIGH NUMBER ON THE HORIZONTAL VELOCITY
decreases with increasing Modified Rayleigh Number, its basic shape remains the same.

As in the case of the Prandtl Number dependence, let us examine these results in terms of the forces acting in the momentum equation. Figures 5.4, 5.8 and 5.9 represent the forces acting in the momentum equation at \( Z = 1 \), Prandtl Number = 10, and Modified Rayleigh Numbers of \( 10^6 \), \( 10^9 \), and \( 10^{12} \) respectively. As in Section 5.4.1.1, there are effectively three distinct force regions. By increasing the Rayleigh Number, one has increased the buoyancy force. Hence, the region required for the viscous forces to overcome this force, i.e., the region where viscous forces are of the same order of magnitude as the buoyancy forces, is extended. Increasing the Modified Rayleigh Number also has the effect of extending the region where all of the forces are of the same magnitude. Finally, in the last region, as in the previous section, the buoyancy forces are small compared to the viscous and inertial forces.

Therefore, it may be concluded that the Rayleigh Number basically determines the region where the buoyancy forces exert a strong influence while the Prandtl Number determines the "sharpness" or distinctness of the various force regions.

5.4.1.3 The Velocity and Temperature as a Function of Height

The velocity and temperature profiles as a function of height are represented in Figures 5.10 and 5.11 respectively. It is seen that the boundary layer thickness for both the velocity and temperature fields increase with height. Also, the magnitudes of the maximum velocity and temperature increase with height. The position of the maximum velocity
FIGURE 5.8 FORCE DIAGRAM FOR MOMENTUM EQUATION
- PR = 10, Ra = 10^9 -
FIGURE 5.9 FORCE DIAGRAM FOR MOMENTUM EQUATION
- PR = 10, Ra = 10^{12}-
\[ Pr = 10 \]
\[ Ra = 10^6 \]

FIGURE 5.10 VERTICAL VELOCITY PROFILES
AS A FUNCTION OF PLATE HEIGHT
FIGURE 5.11 TEMPERATURE PROFILES AS A FUNCTION OF PLATE HEIGHT

\[ T = \left( \frac{\theta - \theta_i}{Q / L} \right) \]

\[ y = \frac{Y}{L} (Ra)^{1/4} \]

- \( Pr=10 \)
- \( Ra=10^6 \)
- \( \bigcirc \) - \( Z=1.00 \)
- \( \square \) - \( Z=0.67 \)
- \( \triangle \) - \( Z=0.33 \)
moves from the wall as the distance up from the leading edge increases. Since the profiles have the same shape, it can be concluded that the behaviour of the force regions is similar at all heights of the plate.

5.4.2 Surroundings Where the Temperature Increases Linearly with Height

5.4.2.1 The Effect of Prandtl Number

The effect of the Prandtl Number on the temperature and velocity fields are shown in Figures 5.12 and 5.13 respectively. Equation 5.3.5 indicates that the temperature and velocity fields are independent of Prandtl Number when the boundary layer reaches its asymptotic limit. As seen from Figure 5.12, this is so for the temperature field. However, at Pr = 1, the velocity is slightly too high. The reason for this is not readily apparent. Tests varying the finite difference parameters were carried out to check that the solution converged. All of these tests agreed quite well. Therefore, the reason for this difference in velocity is probably due to the fact that the inertial terms affect the momentum equation more seriously the lower the Prandtl Number. Hence, a small error in the temperature may influence the momentum equation to a much greater extent through the buoyancy force term the lower the Prandtl Number.

Figure 5.14 indicates that the steady state horizontal velocity, $V_0$, is independent of the Prandtl Number. The greatest amount of fluid is entrained at the bottom of the plate. At $Z_0 = 5$, there is virtually no further entrainment of the fluid into the boundary layer. This means that the inertial forces are always small compared to the buoyancy and viscous forces in the momentum equation.
FIGURE 5.12
UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS: THE EFFECT OF PRANDTL NUMBER ON THE TEMPERATURE PROFILE

\[ \text{Pr} = 1 \]
\[ \text{Pr} = 10 \]
\[ \text{Pr} = 100 \]
FIGURE 5.13
UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS: THE EFFECT OF PRANDTL NUMBER ON THE VELOCITY PROFILE

-Pr = 1
-Pr = 10
-Pr = 100

ASYMPTOTIC SOLUTION
FIGURE 5.14 UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS: THE EFFECT OF PRANDTL NUMBER ON THE HORIZONTAL VELOCITY PROFILE
Hence, the Prandtl Number has no influence on the velocity.

In summary, variations in the Prandtl Number produces no effect on the steady state temperature or velocity profiles. However, the numerical solution is more sensitive to truncation errors the lower the Prandtl Number. The velocity and temperature profiles can be assumed to have reached steady state values at a dimensionless plate height, $Z_b$, greater than 5.

5.4.2.2 Velocity and Temperature Profiles as a Function of Plate Height

At $Z_b$ greater than 5, the velocity and temperature fields have reached an asymptotic limit and can be described by Equation 5.3.5. At values of $Z_b$ less than 5, the boundary layer is growing. However, as seen from Figures 5.15 and 5.16, the shape of the temperature and velocity profiles are similar. Thus, as the bulk temperature gradient diminishes, or at a lower portion of the plate, the limiting values of the velocity and temperature diminish while maintaining the same type of velocity and temperature form as in Equation 5.3.5.

5.4.3 The Effect of a Linear Bulk Temperature Gradient on the Natural Convection from a Uniformly Heated Vertical Plate

Section 5.6.1 concerned itself with convection from a uniformly heated vertical plate in isothermal surroundings, while Section 5.6.2 directed its attention to natural convection in surroundings with a linear temperature gradient. This section compares the two conditions and examines the changes which occur in the presence of a
FIGURE 5.15
TEMPERATURE AS A FUNCTION OF PLATE HEIGHT

- $Z_0 = 1.67$
- $Z_0 \gg 5.0$
FIGURE 5.16
VELOCITY AS A FUNCTION OF PLATE HEIGHT

- $z_b = 1.67$
- $z_b \geq 5.0$

VELOCITY $w$

WIDTH $y_b$
bulk temperature gradient.

For the sake of discussion, the natural convection flows will be examined under the following conditions:

i. The same height up the plate, viz., \( Z = z/L = 1 \);

ii. A Prandtl Number of 10;

iii. A Modified Rayleigh Number of \( 10^6 \).

In order to have reached an asymptotic solution for the above conditions, the Core Rayleigh Number, \( Ra \), must be greater than \( 5.25 \times 10^5 \). Table 5.2 gives various system variables for a uniformly heated plate in isothermal surroundings and in surroundings with a linear temperature gradient when the Core Rayleigh Number is equal to \( 5.25 \times 10^6 \). One can see that the thermal boundary layer thickness is not greatly affected, however, the momentum boundary layer is reduced by about 62 percent. Also, while the wall temperature is reduced by 51 percent, the maximum velocity is reduced by 83 percent. The presence of a bulk thermal gradient, therefore, affects the velocity more severely.

The decrease in the size of the above parameters is relatively greater for high Prandtl Numbers. This is because these parameters increase with increasing Prandtl Number when isothermal surroundings are present while they are independent of Prandtl Number when a bulk temperature gradient is present. Furthermore, one can rationalize that the variables in Table 5.2 cannot exceed the value at \( Ra = 0 \), i.e., in isothermal surroundings. For this to be true, as the Core Rayleigh Number decreases, \( Z_0 \), must also decrease, thereby maintaining these variables at values smaller than the isothermal case. Obviously, from the definition of the variables in Equation 5.3.12, the asymptotic solution must fail at some value of \( Ra \). This limit cannot be ascertained from the numerical solution.

There is also a basic difference in the force regions
in the two cases. While, as was seen in Section 5.4.1.1, the momentum boundary layer region could be divided into at least two regions for the case of a uniformly heated vertical plate in isothermal surroundings—one where buoyancy and viscous forces predominate, the other where viscous and inertial forces are dominant. This is not the case in the "non-isothermal" surroundings case. Here, the buoyancy and viscous forces are dominant throughout the entire boundary layer. This is, in fact, what causes the reverse flow region. Similarly, in the energy equation, while the wall heat flux is swept away by convection in the isothermal surroundings case, the wall heat flux is used to maintain the average temperature of the boundary layer increasing at the same rate as the bulk temperature.

In summary, a linear temperature gradient in the bulk fluid results in a decrease of all the pertinent system variables. The velocity and momentum boundary layer thickness are most markedly affected with increasing Core Rayleigh Number and with increasing Prandtl Number.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Isothermal Surroundings</th>
<th>Surroundings where Ra = 5.25x10^5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal Boundary Layer Thickness, ( \delta/L )</td>
<td>0.25</td>
<td>0.24</td>
</tr>
<tr>
<td>Momentum Boundary Layer Thickness</td>
<td>0.78</td>
<td>0.30</td>
</tr>
<tr>
<td>Wall Temperature, ((\Theta - \Theta_0)/(qL/k))</td>
<td>0.11</td>
<td>0.05</td>
</tr>
<tr>
<td>Maximum Velocity, (wL/a)</td>
<td>200.</td>
<td>23.5</td>
</tr>
</tbody>
</table>
VI. A MODEL FOR THERMAL STRATIFICATION IN ENCLOSED FLUIDS

6.1 Introduction

In this section, a model is developed for predicting the degree of thermal stratification in enclosed fluids when a constant sidewall heat flux is imposed on the sidewalls. A procedure for integrating the boundary layer equations is developed so that the energy flow into the core region of the container can be determined. Then, a procedure for determining the core velocity and temperature profile is derived. Finally, the model is compared with the experimental results of Chapter 4.

6.2 Thermal Stratification Model

The imposition of a constant heat flux on the sidewalls of a container partially filled with fluid causes thermal stratification to occur. Qualitatively, as can be seen from Chapter 4, stratification occurs due to the ejection of warm boundary layer fluid into the core - the warmer fluid being ejected higher up the wall. The question to be resolved is how much of the boundary layer fluid is ejected at any given height and how does this effect the core temperature gradient?

In this section, a technique for predicting the amount of fluid ejected into the core for any core temperature profile (Section 6.2.1) is developed. A model which simulates the thermal stratification phenomenon is developed (Section 6.2.2).
6.2.1 Boundary Layer Momentum and Energy Flows

Drake (16) integrated the boundary layer equations to obtain Equations 5.3.8. The constants $A_1$, $A_2$, $A_3$, $A_4$ were found to be those given in Table 5.1. However, in Chapter 5, it was found that the velocity and temperature profiles were of the form

\[ w = w^* \exp(-y/\delta^*) \sin(y/\delta^*) \quad 6.2.1a \]

\[ \theta - \theta_\infty = T^* \exp(-y/\delta^*) \cos(y/\delta^*) \quad 6.2.1b \]

where $w^*$, $T^*$, $\delta^*$ are functions of the height $(z)$ only, rather than the forms assumed by Drake (Equations 5.3.7) in the presence of a linear temperature gradient in the surroundings. However, the form of Equation 5.3.8 using these profiles remains the same except for the values of the constants (see Table 5.1). Following a procedure similar to that used by Drake, energy and momentum parameters are defined as follows:

\[ E = C_E E^* = \Delta \Pr^{5/4} \quad 6.2.2a \]

\[ M = C_M M^* = \Delta \Pr^{7/4} \quad 6.2.2b \]

where $C_E = 2/A_1$, and

\[ C_M = \left[ \frac{A_4}{A_1} \left( \frac{1}{4} \frac{A_3}{A_1 \Pr} + 1 \right) \right]^{3/5} (2/A_1)^{4/5} \quad 6.2.2c \]

Then, Equation 5.3.8 can be rewritten in terms of $E^*$ and $M^*$ as follows:

\[ \frac{dE^*}{d(z/L)} = \text{Ra}^{1/4} \left( 1 - A_2 C_E^{1/3} C_M^{1/3} \frac{\text{Re}}{\text{Ra}} E^*^{1/3} M^*^{1/3} \right) \quad 6.2.3a \]

\[ \frac{dM^*}{d(z/L)} = \text{PrRa}^{1/4} \left[ \frac{A_4}{2A_3} \frac{C_E^4}{3} \frac{M^*^{4/3}}{M^{2/3}} - \frac{1}{C^4} \frac{M^*}{M^{1/3}} \right] \quad 6.2.3b \]
For the case of an isothermal core, the solution of Equations 6.2.3 has been found (55) to be

\[ E^* = Ra_z^{1/3} \quad 6.2.4a \]
\[ M^* = Ra_z^{7/20} \quad 6.2.4b \]

### 6.2.1.1 High Prandtl Number Case

In the case of high Prandtl Numbers, the solution of Equations 6.2.3 is considerably simplified. As \( Pr \rightarrow \infty \)

\[ \frac{A_4}{A_1} \frac{C}{E} = \frac{2}{A_1} \quad 6.2.5 \]

Also, Equation 6.2.3b simplifies to

\[ \frac{A_4}{2A_3} \frac{C}{E}^{4/3} \frac{E^{4/3}}{C_{M}^{4/3}} = \frac{1}{CEA_3^{4/3} E^*} \quad 6.2.6a \]

Using Equations 6.2.5 and 6.2.2c, Equation 6.2.6a can be rearranged to yield the following equation:

\[ M^* = E^{7/5} \quad 6.2.6b \]

Therefore, Equation 6.2.3a can be rewritten as

\[ \frac{dE^*}{d(z/L)} = Ra_+^{1/4} (1 - A_2 C_{E}^{1/3} C_{M}^{1/3} \frac{Ra}{Ra_+}) E^{4/5} \quad 6.2.7a \]

Define

\[ C_f = (A_2 C_{E}^{1/3} C_{M}^{1/3})^{5/4} \quad 6.2.7b \]

and

\[ E^+ = (\frac{Ra}{Ra_+})^{5/4} E^* \quad 6.2.7c \]

Then, Equation 6.2.7a becomes
\[
\frac{dE^+}{d(z/L)} = \left(\frac{Ra}{Ra^*}\right)^{5/4} \left(1 - C_f^{4/5} E^+^{4/5}\right)
\]

Equation 6.2.8 can be integrated to yield

\[
c_f \left(\frac{Ra}{Ra^*}\right)^{5/4} \frac{Ra^*}{Ra} z^{1/4} = \frac{5}{4} \ln \left(\frac{1 + C_f^{1/5} E^+^{1/5}}{1 - C_f^{1/5} E^+^{1/5}}\right) + 2\tan^{-1}\left(C_f^{1/5} E^+^{1/5}\right) - 4C_f^{1/5} E^+^{1/5}
\]

6.2.9

The solution of Equation 6.2.9 for positive values of

\[
c_f \left(\frac{Ra}{Ra^*}\right)^{5/4} \frac{Ra^*}{Ra} z^{1/4}
\]

is given in Figure 6.1. However, note that mathematically a solution is possible for negative values of

\[
c_f \left(\frac{Ra}{Ra^*}\right)^{5/4} \frac{Ra^*}{Ra}
\]

Redefining

\[Z^* = c_f \left(\frac{Ra}{Ra^*}\right)^{5/4} \frac{Ra^*}{Ra} z^{1/4},\]

the following observations can be made:

1. The solution for Equation 6.2.9 is valid for the range \(-\infty < Z^* < \infty\).

2. Allowing only positive values of \(E^+\), the solution is single-valued from \(-\infty < Z^* < 0\), then it becomes double-valued from \(0 < Z^* < 2\). For values of \(Z^*\) greater than 2, \(E^+ = 0.35355\).

3. For \(E^+ = 0.35355\) and \(Z^* < 10^{-2}\), the solution approximates the one for the case of a uniformly heated vertical surface in the presence of isothermal surroundings.

It should be remembered that this solution is only valid at high Prandtl numbers. Figure 6.2 compares the solution of Equation 6.2.3 for a Prandtl number of 10 and \(Ra^*/Ra^* = 10^{-2}\) obtained by numerical integration to the high Prandtl number solution (Equation 6.2.9). It is seen
FIGURE 5.1 INTEGRATED ENERGY PARAMETER AS A FUNCTION OF HEIGHT
FIGURE 6.2 COMPARISON OF THE ENERGY PARAMETER CALCULATED WITH THE HIGH PRANDTL NUMBER ASSUMPTION TO THE SOLUTION CALCULATED FOR Pr=10.
that agreement is quite good. Therefore, the high Prandtl number solution can be used for all of the test fluids used in this work, viz., water (Pr = 7.05), glycerine (Pr = 12500), and the glycerine-water mixture (Pr = 200).

Finally, a procedure must be developed which will predict the value of $E^*$ for any initial condition of $E^*$. For example, consider the case of a uniformly heated vertical plate immersed in surroundings where

$$0 < z/L < Z_1, \quad T_\infty = 0$$

$$z/L > Z_1, \quad \partial T_\infty / \partial z = \phi$$

where $\phi$ is a constant. The problem is to find the value of $E^*$ at some $Z_2 > Z_1$. Equation 6.2.8 can be rewritten as

$$\frac{dE^+}{dZ^*} = f(E^+) \quad 6.2.10$$

Now, assuming that the boundary layer always obeys Equation 6.2.10, the solution of interest is

$$E_2^+ - E_1^+ = \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} = \int_{Z_1^*}^{Z_2^*} dZ^* \quad 6.2.11$$

Now, from the definition of $Ra^o$ and $Ra^+$, it can be seen that

$$\frac{Ra^o}{Ra^+} = \frac{k}{q} \frac{dT_\infty}{dz} \quad 6.2.12$$

Since $T_\infty = 0$ up to $z/L = Z_1$,

$$E^+ = Ra^+ 1/4 \text{ at } Z_1 \quad 6.2.13a$$

Also

$$E^+ = \left(\frac{Ra^o}{Ra^+}\right)^{5/4} \frac{Ra^+ 1/4}{Z_1} \quad 6.2.13b$$

$$Z_1^* = \left(\frac{Ra^o}{Ra^+}\right)^{5/4} \frac{Ra^+ 1/4}{Z_1} \quad 6.2.13c$$
\[ z_2^* = \left( \frac{Ra}{Ra_1} \right)^{5/4} Ra_1^{1/4} \]

Then one can modify Equation 6.2.11 as follows:

\[ \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} = \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} - \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} = z_2^* - z_1^* \quad 6.2.14 \]

Since the value of \( E_1^+ \) is known, the value of \( \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} \) can be found from Equation 6.2.9. Let

\[ \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} = Z_1^{\text{eff}} \quad 6.2.15 \]

Therefore,

\[ \int_{E_1^+}^{E_2^+} \frac{dE^+}{f(E^+)} = z_2^* - z_1^* + Z_1^{\text{eff}} = Z_2^{\text{eff}} \quad 6.2.16 \]

Then, using Figure 6.1, the value of \( E_2^+ \) and \( E_2^\ast \) can be determined.

The use of the above technique enables one to determine the value of \( E^\ast \) for any core temperature profile where \( \frac{\partial T_0}{\partial z} \) is positive. As will be seen below, this will prove to be of great use in the thermal stratification process simulation.

6.2.2 Thermal Stratification Simulation

In this section, a model which simulates the thermal stratification process is developed. Essentially, the model divides the container into \( n \) horizontal zones as shown in Figure 6.3. Then a material and energy balance is made on each zone, thereby advancing the temperature
FIGURE 4.3 DIVISION OF THE CONTAINER FOR THE STRATIFICATION MODEL
field in time. In order to make the problem tractible, the following simplifications are made in the model:

1. In the bulk of the fluid, the temperature is isothermal along any horizontal plane except in the boundary layer. This assumption is justified by the qualitative observations of the temperature profiles found in this work and that of Drake (16). In both studies, it was found that the temperature was invariant in a horizontal plane in the bulk of the fluid. However, near the top of the container, there is some variation in the horizontal temperature (see Section 4.2). This variation in temperature diminishes with time in all of the tests. This is a reflection of the fact that the entire boundary layer fluid which reaches the top of the container is ejected into the bulk with sufficient momentum to maintain its identity for some time, thereby giving a horizontal thermal gradient. However, with passing time the amount of fluid and its momentum diminishes and so the thermal variation also diminishes. The horizontal temperature distribution in a horizontal plane is small even in the topmost region when compared with the axial temperature gradient. Therefore, the "isothermal horizontal plane" assumption is believed to be valid for this case.

11. The fluid moves down the central core in essentially "plug" flow. Again, this was experimentally verified in this work (see Section 4.2). The "streak" photography results indicate that while a horizontal velocity component is present, it is relatively small except at the top of the container and at the edge of the boundary layer. In the central core region, the streamlines become nearly evenly spaced, indicating that the "plug flow" assumption is approximately true. In view of this, the assumption that the fluid moves down the core in "plug" flow seems reasonable.

111. The boundary layer temperature and velocity profiles
instantaneously assume their steady state shape for any
temperature distribution. This can be considered true if
the time constant for the boundary layer development is
much smaller than the time constant in the core. This
ratio can be estimated by comparing the time it takes a
hypothetical "particle" of fluid to move from the bottom
of the container to the top to the time required for the
fluid to return down the central core to the bottom of the
container. The average velocity in the boundary layer is

\[
\frac{C_{B/L}}{C_{c}} = \frac{L}{L} \int_{0}^{\infty} \frac{w_{B/L} dy}{\delta} \, dz
\]

Assuming plug flow in the core, the average fluid velocity
is

\[
C_{c} = \frac{L}{L} \int \frac{w_{c} dy}{D} \, dz = \frac{L}{L} \int w_{c} \, dz
\]

Now, for the instantaneous steady-state boundary layer
assumption to be true it is required that

\[
\frac{C_{B/L}}{C_{c}} \gg 1
\]

Since,

\[
\int_{0}^{\infty} w_{B/L} dy = \int_{0}^{w_{c} dy}
\]

therefore,

\[
\frac{C_{B/L}}{C_{c}} = \frac{D}{\delta} \gg 1
\]

Therefore, if the distance between the heated walls is
much greater than the boundary layer thickness, the
assumption is valid.

**iv.** If the core axial temperature gradient is equal
to zero, the boundary layer temperature and velocity can
be described by the Equations 5.3.7, viz.,

\[
\theta - \theta_{\infty} = \frac{\theta_{\infty}^4}{2k} \left(1 - \frac{y}{\delta}\right)^2
\]
\[ w = w^* \frac{y}{\delta^*}(1 - \frac{y}{\delta^*})^2 \]

On the other hand, if there is an axial thermal gradient in the core, the boundary layer temperature and velocity are described by

\[ \Theta - \Theta_\infty = T^* \exp(-y/\delta^*) \cos(y/\delta^*) \quad 6.2.1a \]
\[ w = w^* \exp(-y/\delta^*) \sin(y/\delta^*) \quad 6.2.1b \]

In Chapter 5, it was seen that the boundary layer velocity and temperature profiles were of the form given in Equation 6.2.1 even when the boundary layer was forming so long as a linear thermal gradient is present in the core. If the core gradient is not linear, the gradient can be approximated by a series of linear gradient approximations over small segments. Therefore, it is reasonable to assume that Equation 6.2.1 is valid for any thermal core profile so long as the gradient is always positive.

6.2.2.1 Boundary Layer Material and Energy Balances

Consider a differential element in Figure 6.4. In the boundary layer, a material balance can be written as follows

\[ \int_0^\infty w \, dy \bigg|_{z + \Delta z} \bigg|_z + A \int w \, dz = -Q \Delta z \quad 6.2.15 \]

Similarly, an energy balance can be written on this same differential element, viz.,

\[ q \Delta z - \rho c_p \left( \int_0^\infty w(T - T) \, dy \bigg|_{z + \Delta z} \bigg|_z - \int_0^\infty w(T - T) \, dy \bigg|_z \right) \]
\[ - \varepsilon \Delta z - \rho c_p \left( \int_0^\infty w(T - T_o) \, dy \bigg|_{z + \Delta z} \bigg|_z - \int_0^\infty w(T - T_o) \, dy \bigg|_z \right) \]
\[ = 0 \quad 6.2.16 \]
FIGURE 6.4 DIFFERENTIAL ELEMENT USED FOR DEVELOPMENT OF STRATIFIED MODEL EQUATIONS
Now, Drake (16) has shown that
\[
\int_{0}^{\infty} w dy = A_2 \alpha \left[ \frac{2}{A_1} \frac{A_4}{A_1} - \frac{2}{A_1} \right] E^{4/5} 1/3 3/5 = \alpha C_f^{4/5} E^{4/5} 6.2.17
\]
where
\[
C_f = \left\{ A_2 \left[ \frac{2}{A_1} \frac{A_4}{A_1} - \frac{2}{A_1} \right] \right\}^{5/4}
\]
and
\[
\int_{0}^{\infty} w(T - T_0) dy = (\frac{qL}{K} \alpha/Ra^1/4) E^{4/5} 6.2.18
\]
Therefore, Equations 6.2.15 and 7.2.16 can be rewritten in differential form using Equation 6.2.17 and 6.2.18 as
\[
\alpha C_f^{4/5} \frac{dE^{4/5}}{dz} = -Q_f 6.2.19
\]
and,
\[
q = \frac{qL}{Ra^1/4} \frac{dE^{4/5}}{dz} = \rho \sigma p \frac{C_f^{4/5}}{z} \frac{d(T_0 E^{4/5})}{dz} = \varepsilon 6.2.20
\]

6.2.2.2 Core Material and Energy Balances

Now, consider the same differential element in the core. Then, a material balance on the core can be written as:
\[
2Q_f \Delta z + D(w|w + \Delta z - w|z) = 0 6.2.21
\]
Similarly, an energy balance on the core can be written as follows:
\[
2\varepsilon \Delta z + \rho c_p D(wT_\infty|w + \Delta z - wT_\infty|z) + kD(T_\infty|z + \Delta z - T_\infty|z - \frac{T_\infty|z - T_\infty|z}{z}) = \rho c_p Dz/T_\infty 6.2.22
\]
Equations 6.2.21 and 6.2.22 can be rewritten in differential form as:

$$2Q_f = - D \frac{dw}{dz} \quad \text{6.2.23}$$

and

$$2\varepsilon + \rho pD \frac{\partial W_T}{\partial z} + kD \frac{\partial^2 T}{\partial z^2} = \rho c D \frac{\partial T}{\partial t} \quad \text{6.2.24}$$

Equation 6.2.19 can be substituted into Equation 6.2.23 and the resulting equation can be integrated to yield

$$2q - \frac{2qL}{R_d} \frac{\partial E^*}{\partial z} = 2\rho p \rho c \frac{\partial T}{\partial z} + kD \frac{\partial^2 T}{\partial z^2} + \rho pD \frac{\partial W_T}{\partial z}$$

$$= \rho pD \frac{\partial T}{\partial t} \quad \text{6.2.25}$$

However, from Equations 6.2.17 and 6.2.23 it can be seen that

$$\rho pD \frac{\partial W_T}{\partial z} = 2\rho p \rho c \frac{5/4}{\partial z} \frac{\partial T}{\partial z} \quad \text{6.2.26}$$

Therefore, Equation 6.2.25 can be further simplified to give

$$\rho pD \frac{\partial T}{\partial t} = 2q - \frac{2qL}{R_d} \frac{\partial E^*}{\partial z} + kD \frac{\partial^2 T}{\partial z^2} \quad \text{6.2.27}$$

### 6.2.2.3 Dimensionless Equations

Define a set of dimensionless equations such that

$$Q_f = \frac{Q_f L}{\alpha} \quad \varepsilon = \varepsilon / q$$

$$W = \frac{wD}{\alpha} \quad T = \frac{T}{T_c / (qL/k)} \quad \text{6.2.28}$$

$$Z = \frac{z}{L} \quad \tau = \frac{at}{kL}$$

Then, Equation 6.2.23, after integration, becomes

$$W = 2C_f^{4/5} E^*^{4/5} \quad \text{6.2.29}$$
and Equation 6.2.27 becomes

\[ \frac{\partial T}{\partial \tau} = 2 - \frac{2}{Ra^{1/4}} \frac{\partial E^*}{\partial Z} + \left( \frac{D}{L} \right) \frac{\partial^2 T}{\partial Z^2} \]  

6.2.30

From Equation 6.2.7, it can be seen that

\[ \frac{dE^*}{Ra^{1/4}dZ} = 1 - C_{4/5}^{4/5} \frac{Re}{Ra^{4/5}} \]

\[ = 1 - \frac{1}{2} \frac{Re}{Ra} W \]  

6.2.31

Finally, from the definition of the modified Rayleigh number and the core Rayleigh number, it can be seen that

\[ 2 - \frac{2\partial E^*}{Ra^{1/4}\partial Z} = W \frac{\partial T}{\partial Z} \]  

6.2.32

so that an alternate formulation of Equation 6.2.30 can be written as:

\[ \frac{\partial T}{\partial \tau} = W \frac{\partial T}{\partial Z} + \frac{D}{L} \frac{\partial^2 T}{\partial Z^2} \]  

6.2.33

In the case of an isothermal core, it can be seen from Equation 6.2.3a that

\[ E^* = Ra^{1/4} \]  

6.2.4a

Since the value of \( E^* \) decreases for the same modified Rayleigh number as the core Rayleigh number increases, one can easily see that the maximum velocity possible is

\[ W = 2C_{4/5}^{4/5} Ra^{1/5} \]  

6.2.34

Therefore, one can further modify Equation 6.2.33 by redefining the velocity as

\[ W = W/Ra^{1/5} \]  

6.2.35a

Then, by redefining the temperature and the time as
\[ T = T \text{Ra}^{1/5} \]  
\[ \text{Ra} = \text{Fo} \text{Ra}^{1/5} \]  
Equation 6.2.33 can be rewritten as

\[ \frac{\partial T}{\partial \tau} = W \frac{\partial T}{\partial Z} + \left( \frac{D}{L} \right) \frac{1}{\text{Ra}^{1/5}} \frac{\partial^2 T}{\partial Z^2} \]  

It should be noted that the variables in Equation 6.2.36 are meant to be the same as those defined in Equation 6.2.35. However, the underline has been omitted for simplicity. In the remaining discussion, the variables will refer to the dimensionless variables defined in Equation 6.2.35 unless otherwise noted. Then, Equation 6.2.30 can be written in terms of the new variables as

\[ \frac{\partial T}{\partial \tau} = 2 - \frac{2}{\text{Ra}^{1/4}} \frac{\partial E^*}{\partial Z} + \frac{\partial^2 T}{\partial Z^2} \frac{1}{\text{Ra}^{1/5}} \]  

### 6.2.2.4 Finite Difference Approximations

The basic procedure used for the thermal stratification simulation was as follows. Firstly, the enthalpy flows, \( E_i^* \), were determined at each level, \( i \), of Figure 6.3 using the known core temperatures at time level \( \tau^n \) and Equations 6.2.9, 6.2.15 and Figure 6.1. It was assumed that the core temperature gradient was linear between the two levels \( i \) and \( i-1 \). Then the core temperatures were advanced in time to \( \tau^{n+1} \) using a finite difference approximation of Equation 6.2.37 given in Equation 6.2.38

\[
\frac{T_i^{n+1} - T_i^n}{\Delta \tau} = 2.0 - 2.0 \frac{1}{\text{Ra}^{1/4}} \frac{E_{i+1}^* - E_i^*}{\Delta Z} \\
+ \frac{T_i^{n+1} - 2T_i^n + T_{i-1}^n}{\Delta Z^2}.
\]  

At the top, it was assumed that all of the energy at level
Nzones entered the core so that

\[
\frac{T_{n+1}^{Nzones} - T_n^{Nzones}}{\Delta t} = 2.0 + \frac{E^*_n}{\Delta Z} - \frac{1}{\text{Ra}^{1/4}} + \left(\frac{D}{L}\right) \frac{1}{\text{Ra}^{1/5}}
\]

\[
\frac{T_n^{Nzones} - T_n^{Nzones}}{\Delta Z^2} = 6.2.39
\]

Note that the formulation of the conduction term presumed no heat loss at the surface. Finally, the velocity was calculated using Equation 6.2.29, i.e.,

\[
W_{n+1}^i = 2C_f \frac{E^*_n}{\text{Ra}^{1/5}}
\]

This procedure was repeated through successive time levels until the desired time level has been reached. The computer program used for this model simulation is given in Appendix G.

Since a temperature at level \(i\) cannot be less than at \(i - 1\), in actual fact, a check was made after each iteration procedure for such an anomaly. If one were found, that temperature was mixed with the two adjacent ones. This was found to give excellent stability.

6.3 Thermal Stratification Model

6.3.1 Convergence

In order to ensure that the finite difference solution was convergent for the test run made when Nzones = 50, another run was made using 100 grid lines keeping the other parameters constant. As seen from Figure 6.5, the two solutions agreed very well. Therefore, the results obtained with Nzones = 50 were considered to be adequate in representing the true solution.
Figure C.5 Effect of Grid Spacing on Temperature Solution

\[ \tau = 0.174 \]

\[ c_f = 5.66 \]

- ΔZ = 0.01
- ΔZ = 0.02
6.3.2 The Effect of Conduction

Figures 6.6 and 6.7 show the effect of conduction on the thermal stratification solution. Two runs were made using the same parameters except that conduction was eliminated from one of the runs. From these results, it can be seen that conduction is negligible. Energy convection downward in the core is by far the dominant mode of energy transport in the central core of the container. Therefore, to a very good approximation, Equation 6.2.36 can be written

$$\frac{\partial T}{\partial \tau} = W \frac{\partial T}{\partial Z}$$  \hspace{1cm} \text{6.3.1}

The above equation is independent of the Rayleigh number, Prandtl number (for $Pr > 7$), and Aspect Ratio. Therefore, a single solution, the dimensionless time, $\tau$, as a parameter completely describes the thermal stratification problem. For this reason, the solution given in the following discussions is one where the conduction term has been eliminated.

This result substantiates the assumption of convection dominated heat transfer in the core which was used by Drake [16] in the development of her model. Furthermore, the thermal stratification process is thermal convection dominated throughout the entire time scale of interest. Therefore, this seems to negate the assumption of conduction dominant heat transfer in the core made by Fan, Chu, and Scott [23] in the development of their model. The fact that the equation which they solved, i.e., Fourier's heat conduction equation, fits their data remarkably well, strongly indicates that there may be some basis for seeking a similar solution for the case when convection is assumed dominant. This was not pursued further in this work, but it is believed that this line of investigation merits further study.
6.3.3 The Effect of $C_f$ on the Solution

From Equation 6.2.34, it can be seen that the magnitude of the velocity is directly influenced by the magnitude of $C_f$. Since convection is the dominant mode of energy transfer, the value of $C_f$ must markedly influence the temperature profile also. It is expected that the temperature gradient in the core of the fluid would decrease with increasing values of $C_f$. Figure 6.8 shows this to be so.

The three values of $C_f$ chosen in Figure 6.8 have the following significance. When $C_f = 1.72$, the boundary layer velocity and temperature profiles have the form given by Equation 5.3.7. However, if the velocity and temperature profiles are of the form given by Equation 6.2.1, i.e., the boundary layer velocity and temperature profiles correspond to the case of a uniformly heated vertical wall in surroundings whose temperature increases linearly with height, the value of $C_f$ is found to be 2.83. Finally, as will be seen in the next section, the experimental data could best be correlated with the thermal stratification correlation when a value of $C_f$ equal to 5.66 was used.

6.4 Data Correlation

6.4.1 Temperature Correlation

In Figure 6.9, the experimental data obtained from this work and that of Drake (16) are correlated with the solution obtained from the thermal stratification simulation. It is seen that very good correlation is obtained when $C_f = 5.66$. Also, the data for all Prandtl numbers greater than 7.0, i.e., all of the conditions tested in this work,
FIGURE 6.6 THE EFFECT OF CONDUCTION ON THE THERMAL STRATIFICATION SIMULATION
FIGURE 6.7 The Effect of Conduction on the Thermal Stratification Simulation
FIGURE 6.8 The Effect of $C_f$ on the Temperature Solution

\[ \frac{z}{L} = 0.5 \]

\[ \frac{T-T_0}{qL/k} \cdot Ra^{1/5} \]

- $C_f = 1.72$
- $C_f = 5.66$

\[ \tau = Fo Ra^{1/5} = 0.175 \]
FIGURE 6.9a CORE TEMPERATURE PROFILES
FIGURE 6.9b  CORE TEMPERATURE PROFILES

See Figure 6.9a for Legend
seem to agree equally well with the simulated solution for \( C_f = 5.66 \).

There are two points to be noted in the presentation of Figure 6.9. Firstly, the modified Rayleigh number used in the correlation was calculated at the mean temperature of the fluid. That is, the variation of the fluid properties are considered only in the sense of variation of the average system temperature level. The spatial variation of the fluid properties as a result of a non-uniform temperature field is not considered. This simplification is justified since this mean temperature represents a reasonable average temperature level for both the core and the boundary layer fluid. Furthermore, Drake (16) has achieved success in the correlation of similar results using this simplification.

Secondly, the model described by Drake (16) and the temperature data obtained for the core fluid is dependent upon the system geometry. Appendix L generalizes Drake's model so that it accounts for all system geometries which have a uniform heat flux generated on vertical sidewalls. This generalization assumes that Drake's postulated "mixing region" (Appendix L) remains the same regardless of the shape of the cross-sectional area of the container. Since it will be shown in the next section that the "mixing region" does not in fact exist but must be considered as an empirical factor which correlates the data, the above assumption seems to be reasonable. From this, it can be shown that the temperature in Drake's work is twice that in this work. This has been taken into account in presenting Drake's data in Figure 6.9. Also, for laminar boundary layer flow, the equation which Drake developed for predicting the core temperature gradient in a cylinder with a uniformly heated vertical wall was

\[
\frac{\partial T}{\partial y} = 4 \tau^{4/9}
\]
However, as seen from Appendix L, for a rectangular container with uniformly heated sidewalls, Drake's Equation becomes

\[ \frac{\partial T}{\partial \tau} = 2 \tau^{4/9} \]

It is this latter equation which is plotted in Figure 6.9.

The value of \( C_f \) which correlates the data, i.e., \( C_f = 5.66 \), is much larger than that which would be predicted either by the boundary layer equations assumed by Sparrow (55), Equation 5.3.7, for the case of a uniformly heated vertical plate in isothermal surroundings or by the boundary layer equations developed in this work for the case of a uniformly heated vertical plate in surroundings whose temperature increases linearly with height. This seems to pose a serious question as to the validity of the model as far as it having relevance to the actual physical situation. For, if initially the situation is analogous to two uniformly heated vertical plates isothermal infinite surroundings, the value of \( C_f \) would be 1.72 and at long times, e.g., \( \tau \approx 10 \), the situation is very similar to the case of two widely separated uniformly heated vertical plates in surroundings with a linear temperature gradient (\( C_f \approx 2.83 \)), it is reasonable to believe that the value of \( C_f \) would be somewhere between 1.73 and 2.83 during the length of the test and would probably vary with time. A value of \( C_f \) greater than 2.83 does not seem probable.

However, it has been shown (46) that the assumed forms of the velocity and temperature profiles for the case of a uniformly heated vertical plate in isothermal surroundings (Equation 5.3.7) are, in fact, adequate only in predicting the Nusselt number. Serious error is introduced by using Equation 5.3.7 in predicting most other quantities. For example, in calculating the volume flow rate in the boundary layer using Equation 5.3.7, it was found that
\[
\frac{\int wdy}{\alpha \frac{1}{\text{Ra}^{1/5}}} = 1.517
\]

for a Prandtl number of 10 and 1.538 for a Prandtl number of 100. On the other hand, if the similarity solution of the boundary layer equations (56) is used to calculate the same parameter, it was found (46) that

\[
\frac{\int wdy}{\alpha \frac{1}{\text{Ra}^{1/5}}} = 3.949
\]

for a Prandtl number of 10 and 9.883 for a Prandtl number of 100. Thus, it can be seen that, in actual fact, the value of \( C_f \) initially is much larger than that predicted by Equations 5.3.7 and 5.3.1 for larger Prandtl numbers and may be greater than \( C_f = 5.66 \). Therefore, attempting to attach physical significance to the value of \( C_f = 5.66 \) seems to be more plausible in light of the above discussion.

The question which remains is why does the same value of \( C_f \) correlate the experimental data equally well? That is, a value of \( C_f = 5.66 \) seems to correlate all of the data over a wide range of Prandtl numbers (7 to 12500) with about the same degree of confidence. That this should be so is not intuitively obvious. Since the value of \( C_f \) varies with the Prandtl number, at least when the environs are isothermal, e.g., for \( Pr = 10, C_f = 5.45 \), and for \( Pr = 100, C_f = 17.5 \), one would expect the value of \( C_f \) to vary with the degree of stratification also. For, if the explanation lies in the fact that the boundary layer profiles have not yet reached the shape of those given by Equation 5.3.1 but lie somewhere intermediate between those assumed for the isothermal surroundings and those for surroundings with a linear temperature gradient, then it is reasonable to assume that the value of \( C_f \) would vary with Prandtl number. Also, one would expect \( C_f \) to vary with time. This is because as time progresses, the core temperature gradient tends to become linear and increase more slowly with time.
Assuming Drake's Equation (Equation 6.4.2) to be correct, it can be seen that the rate of change of the temperature gradient increases as $r^{-5/9}$. Hence, one can expect $C_f$ to decrease with time to a limiting value of 2.83 (i.e., to a value corresponding to a linear thermal gradient in the core). However, this does not occur as can be seen from Figure 6.9. A single value of $C_f = 5.66$ is valid for the entire time range of the experiment. The true explanation of this result would require a much more intensive study of the effect of the boundary conditions on the solution of the boundary layer equations than the scope of this work permits. However, several possible avenues of investigation could be pursued, some of which are considered below.

One possibility is that the value of $C_f$ takes into effect the fact that the fluid motion in the core is not truly in "plug flow". As can be seen from the "streak" photographs (Figures 5.5 to 5.8), there is some horizontal motion of the fluid especially near the surface. Hence, the core velocity which is used in the model is, in reality, some sort of average. The value of $C_f$, therefore, could be considered as also containing the averaging parameter as well as the true value of $C_f$ which could be predicted from the boundary layer solution.

One of the more intriguing possibilities is that a unique velocity and temperature profile may exist for the exact boundary conditions of this study. Many reasons for this profile differing from that given by Equation 6.2.1 could be given. One would be due to the fact that there is a vertical velocity in the core. Since the asymptotic solution (Equation 5.3.5) depends upon a balance between viscous and buoyancy forces, the core velocity might be sufficient to upset this balance so as to alter the velocity and temperature profiles significantly. Another factor could be the presence of the other wall. Although the separation between boundaries in all of the experiments
was sufficient to prevent physical interaction of the boundary layers, longer acting forces such as pressure gradient could be sufficient to alter the boundary layer velocity and temperature profiles. The classical formulation of the natural convection boundary layer equations assumes the pressure force terms due to fluid motion are negligible (see Appendix A and Chapter 5). However, in the case where there are two walls present, this may no longer be true. For now, there no longer is an infinite supply of fluid to be fed into the boundary layer and so pressure gradients in the surroundings may result. Therefore, the boundary layer velocity and temperature profiles may be affected.

In this work, the maximum modified Rayleigh number was $10^{13}$. This was, however, not sufficiently high to result in a turbulent boundary layer. In fact, because the thermal gradient in the core tends to stabilize the boundary layer (16), the limiting value of $\text{Ra}^*$ at which the boundary layer becomes turbulent is not known. Since one would not expect the core behaviour to be markedly altered, one could use the model developed in this work to predict the degree of stratification and the severity of stratification for very high modified Rayleigh numbers. However, some experimentation would be required to determine the value of $C_f$ which must be used under turbulent boundary layer conditions.

In Figures 6.10 and 6.11, the maximum streamlines are plotted for tests G-3-HI and W-2-HI respectively and are compared with the results calculated from the thermal stratification simulation. Note that the maximum streamline is actually half the dimensionless flow rate in the core. As such, it can be calculated from the results of the thermal stratification simulation as being

$$\Upsilon_{\text{max}} = W/2$$
FIGURE 6.10 MAXIMUM STREAMLINE PROFILES FOR TEST C-3-H1
FIGURE 6.11 MAXIMUM STREAMLINE PROFILES FOR W-2-H1

\[ \tau = 5.73 \times 10^{-1} \]

\[ \tau = 8.58 \times 10^{-1} \]
Deviation of the model results from the experimental data can arise from several sources. Velocity data is accurate to only ± 25 percent because of the techniques used (Appendix H). Hence, considerable deviation may be expected from this source alone. Also, the model is really concerned only with thermal convection flows, i.e., only with $w_T^3 \frac{\partial T}{\partial Z}$. It implies that the flow from the boundary layer entering the core at any one point is at the temperature of the core fluid. This seems to be a reasonably good assumption since no horizontal thermal gradient can be detected experimentally. However, slight variations, ± 1°F, might be possible. Also, the core fluid does not move down the core in truly "plug" flow. These deviations of the model from the physical situation might be accounted for by some averaging constant which would be lumped in with the constant $C_f$ as described in Section 6.4.2. Therefore, when only the velocity is calculated from the model, some additional error might be obtained.

However, from Figures 6.10 and 6.11, it can be concluded that the core velocities calculated from the thermal stratification model can be used to estimate at least the order of magnitude of the flow in the core due to natural convection flows in vertical, enclosed fluids resulting in thermal stratification. Also, the flow variation with height and time can be assumed to be relatively correct.

This flow variation with height and time is shown in Figures 6.12a through 6.12d. The flow in the thermally unstratified region has been left out in Figure 6.12a since it depends upon the type of fluid used. To calculate this some solution such as the similarity transformation solution of the boundary layer equations for a uniformly heated plate in isothermal surroundings calculated by Sparrow and Gregg (56) should be used.
Figure 6.12a  Core Velocity Profile

\[ \frac{wD}{a} \left( \frac{1}{R \alpha^{1/5}} \right) \]
FIGURE 6.12b  CORE VELOCITY PROFILE
FIGURE 6.19: CORE VELOCITY PROFILE
FIGURE 6.12d  CORE VELOCITY PROFILE
7.1 Application of the Thermal Stratification Model

In this chapter, an attempt is made to evaluate the utility of the theoretical and experimental findings described in Chapters 5 and 6 in the light of application to practical problems. First, some general considerations are presented. Then, limitations of the experimental and theoretical work are stated, but some possible ways of circumventing these limitations are suggested.

The design engineer can utilize correlations developed in this thesis to estimate the time and spatial temperature distribution within a vertical container of fluid subjected to sidewall heating by:

1. noting that the horizontal temperature gradient is negligible;

2. using Figure 6.9 to predict the axial core temperature field for any given time after the start of heating. It should be noted that the temperature distribution calculated from Figure 6.9 refers to a rectangular cavity with constant sidewall heating. These results can be easily modified for a uniformly heated vertical cylinder with constant sidewall heating. As seen in Appendix L, the temperature rise at any one point in a vertical cylinder is twice that given in Figure 6.9. Similarly, Figure 6.9 can be used for any other cross-sectional area with slight modification, by using methods developed in Appendix L.

3. taking care that the system being used is not in the turbulent boundary layer flow regime. If it is in this region, modification of $C_f$ would be required to obtain an accurate solution. The limiting value of the modified Rayleigh number at which transition to turbulent boundary
layer flow occurs in the presence of a non-isothermal core temperature is not now available in the literature. This work has obtained results up to $Ra = 10^{13}$. Even at this Rayleigh number, the boundary layer was still laminar. Therefore, Figure 6.9 can be used with confidence up to this value of modified Rayleigh number. Otherwise, some other estimation of the limiting value of Rayleigh number at which the transition occurs must be used. For example, the criterion developed by Drake (16) (Figure 4 in that work) might provide a reasonable estimate of the maximum limit of the modified Rayleigh number at which the transition of the boundary layer occurs.

4. using Figure 6.9 only for systems where there is no heat loss to the vapor space above the fluid. Slight heat losses may not affect the accuracy of the solution seriously. However, in situations where surface evaporation occurs, Figure 6.9 may differ significantly from the actual situation. A possible modification of the thermal stratification model which would account for this is presented below.

Also, the design engineer can obtain some estimate of the movement of the fluid in the system by using Figure 6.12. Again, the results presented in that figure refer to a rectangular cavity only and must be modified for other cross-sectional areas. This, however, is not difficult.

One of the calculations of interest to the designer may be the level of stratification at any time, i.e., it may be necessary to predict the depth of warm fluid at any time. The level of stratification may of course be obtained from Figure 6.9. However, an approximate analytical solution which predicts the depth of stratification can be derived from the model. Assume that is the thickness of the stratified level. Then, from the model,
At the bottom of the stratified level, the temperature gradient is approximately equal to zero, i.e.,

\[ w_D = \frac{2c_f^{4/5}}{R_d^{4/5}} \]  

Therefore,

\[ E^* = Ra_d^{1/4} \]  

Then, at \( L-\Delta \), the velocity can be calculated as

\[ w_D = \frac{2a c_f^{4/5}}{D} \]  

The warm fluid moves towards the bottom of the container at a velocity of \( w \). Therefore, the thermally stratified fluid thickness increases at a rate

\[ \frac{d(\Delta D)}{dt} = w_D \]  

The height of the stratified level from the bottom, \( z \), is equal to

\[ z = L - \Delta \]  

Therefore,

\[ \frac{d}{dt} z = - \frac{2a}{D} c_f^{4/5} \]  

Integrating Equation 7.1.5, one obtains

\[ 5z^{1/5} = \frac{2a}{D} c_f^{4/5} \left( \frac{P_H}{\alpha k} \right)^{1/5} \]  

Now at \( t = 0 \), the stratified level \( z = L \). Therefore, the constant in Equation 7.1.6 is equal to \( 5L^{1/5} \). Rearranging Equation 7.1.6 into the variables of Equation 5.3.35, an expression for the level of stratification is obtained. It is
From this work, it was found that the results are best correlated when

\[ C_f = 5.66 \]

Therefore, the thickness of the stratified fluid is found to be

\[ \frac{\Delta}{L} = 1 - (1 - 1.6 \tau)^5 \]

This equation provides a convenient expression for estimating the thickness of the equation which is thermally stratified at any time in a fluid contained in a rectangular enclosure with uniformly heated sidewalls. Modification of this equation can be made to account for other cross-sectional area shapes.

### 7.2 Limitations of the Model

In using the results from this thesis, the designer should keep in mind the following limitations:

1. The solution is valid only for fluids of Prandtl number greater than 5. No investigation of its validity was made for lower Prandtl numbers. The reason for this restriction is that the assumption that the infinite Prandtl number solution of the boundary layer enthalpy flow is no longer valid for fluids of Prandtl number less than 5. In a solution for a system where the fluid has a Prandtl number less than 5, account must be made for the deviation from the solution for \( E^* \) obtained in Section 6.2 or the enthalpy flow equation (Equation 6.2.3) must be integrated by numerical means. Otherwise, the
model is correct for all fluids.

2. Care should be taken to account for the geometrical cross-sectional area shape. As mentioned above, the results obtained from this work can be applied to all vertical enclosures with uniform sidewall heating if account is made of the cross-sectional geometry. Appendix L affords a general procedure for modifying Drake's model. Similar analysis can be used for the results in this thesis.

3. The solution is not valid for cases where a turbulent boundary layer is present. As a criterion for turbulent boundary layer transition in the presence of a non-uniform core temperature is not available, strict guidelines as to when this solution cannot be used cannot be given. However, for $Ra \leq 10^{13}$, the solution is believed to be valid. Above this, the criteria developed by Drake (16) might prove useful. If the boundary layer is, in fact, turbulent, modification of $C_f$ must be made. Until experimental data for this situation are obtained, such modification is not possible.

4. The thermal stratification model does not account for heat transfer from the surface. However, since it was found that the surface temperature determined only the maximum temperature while the gradient was really established by boundary layer ejection of the fluid into the core, it is believed that Equation 6.2.39 was modified to

$$\frac{T_{n+1}^{Nzones} - T_n^{Nzones}}{\Delta \tau} = 2.0 + \frac{E^{Nzones}}{\Delta Z} \frac{1}{Ra^{1/4}}$$

$$+ \frac{D}{L} \frac{1}{Ra^{1/5}} \frac{T_n^{Nzones} - T_n^{Nzones} - 1}{\Delta Z^2} = Q_{sur}$$

where $Q_{sur}$ is the surface heat loss, reasonably accurate results might be obtained. It should also be mentioned that, if because of this procedure,
\[ T_{N\text{zones}} < T_{N\text{zones}} - 1 \]

then, these two temperatures should be mixed, i.e., averaged. Otherwise, the computer program STRAT (Appendix G) can be used directly for predicting the temperature profiles in the core of the container.

7.3 Example

Applications for this work fall into two classes. One is the storage of fluids in cryogenic containers. Here the temperature difference is great and the heat transfer to the fluid is relatively small because of wall insulation. The second is the storage of fluids at temperatures slightly above or below ambient temperature. Thus, while heat transfer is relatively small, the heat flux changes markedly with temperature (due to a small \( T \)).

In the latter case, results of this thesis cannot be directly used. Modification must be made to allow for a variable heat flux. This can easily be accomplished by using STRAT but feeding in a new value of \( \text{Ra}^* \) after each time step. Or, if the heat transfer coefficient for the system is sufficiently high, then, the boundary conditions at the wall approximate a constant wall temperature. Therefore, the equations must be modified to account for this and the stratification model must be resolved.

However, in the former case, the temperature between ambient and the container is relatively large while the heat transfer coefficient is relatively low so that the constant wall heat flux condition is a good approximation to the actual situation. For this case, the model is very well suited. An example solution is, therefore,
presented for this type of situation.

Consider the hypothetical case where it is necessary to store a liquid at a low temperature in order to ensure against degradation of the fluid. In order to do this, a Dewar must be designed which can hold the fluid for a period of time before degradation occurs. Such a vessel is shown in Figure 7.1. The conditions which will be used in the design are given in Table 7.1. It would be desirable for the designer to be able to calculate the length of time it would be possible to maintain the temperature below $-80^\circ F$ if a vacuum were applied between the walls of the container. This calculation might proceed as follows:

**TABLE 7.1**

Design Conditions for Sample Problem

**Fluid Properties**

- $Pr = 7.0$
- $\frac{\alpha \beta}{\alpha yk} = 4.92 \times 10^5 \text{ hr Btu ft}^2$
- $\alpha = 5.5 \times 10^{-3} \text{ ft}^2/\text{hr}$
- $k = 0.35 \text{ Btu/hr/ft}/^\circ F$
- Degradation Temperature $= -80^\circ F$
- Vapor Pressure $\sim 0$

**System Properties**

- Ambient Temperature, $T_a = 80^\circ F$
- Initial Fluid Temperature, $T_o = -100^\circ F$
- Dewar Height, $L = 10 \text{ ft}$ (fluid level)
- Dewar Width, $D = 5 \text{ ft}$
- Emissivity of Dewar walls $= 0.04$
- Wall Insulation = double wall with $\frac{1}{2}''$ space at vacuum
FIGURE 7.1 EXAMPLE CONTAINER SYSTEM
7.3.1 Problem Solution

Assuming that the Dewar walls are grey, i.e., the emissivity equals the absorptivity, then, from the Stefan-Boltzmann law,

\[ q = 0.173 \epsilon \left( \frac{T_1}{100} \right)^4 - \left( \frac{T_2}{100} \right)^4 \]

where \( \epsilon \) is the emissivity, \( T_2 \) and \( T_1 \) are the temperature of the inside and outside walls respectively.

For the first trial, assume the boundary layer on the outside causes the outer wall to be \( 9^\circ F \) less than ambient, and the boundary layer on the inside of the tank raises that wall temperature by \( 1^\circ F \). Then,

\[ q = 0.173 (0.040) (5.36^4 - 3.59^4) \]
\[ = 4.53 \text{ Btu/hr/ft}^2 \]

Or, if \( -80^\circ F \) were used as the tank inside temperature

\[ q = 0.173 (0.040) (5.36^4 - 3.81^4) \]
\[ = 4.27 \text{ Btu/hr/ft}^2 \]

Since these differ by less than 6 percent, an average is probably sufficient for this calculation, i.e.,

\[ q = 4.4 \text{ Btu/hr/ft}^2 \]

Before proceeding further, the assumptions of the temperature drop at the walls will be checked.

Because the diameter of the Dewar is large, the walls can be considered as flat plates. From McAdams (34)

\[ k = 0.27 T^{0.25} \]

for air. Since

\[ q = h \Delta T \]

the outside wall temperature difference is
\[ T = \left( \frac{4.4}{0.27} \right)^{0.8} \]
\[ = 9.3^\circ F \]

which is very close to the assumed one (90°F). Similarly using Sparrow and Gregg's (56) estimate of heat transfer coefficients from a uniformly heated plate

\[ N_u = \frac{2}{(360)^{1/5}} \frac{Pr^2}{0.8 + Pr} \frac{1}{G_{\infty}^{1/5}} 7.3.4 \]

for this study

\[ N_u = \frac{2}{(360)^{1/5}} \left( \frac{49.0}{7.8} \right)^{1/5} \left( 4.92 \times 10^5 \times 4.4 \times 10^4 \right)^{1/5} \]
\[ = 103.8 \]

From Equation 7.3.3 and the definition of the Nusselt number,

\[ T = \frac{qL}{k} \frac{1}{N_u} \]
\[ 7.3.5 \]

for the inside wall. For this case,

\[ T = \frac{4.4(10)}{0.35} \frac{1}{(103.8)} \]
\[ = 1.2^\circ F \]

which compares well with the 1°F assumption. Therefore, the estimate of \( q = 4.4 \) Btu/hr/ft\(^2\) seems adequate.

From this, pertinent parameters can now be calculated:

\[ Ra = 4.92(10^5)(4.4)10^4 \]
\[ = 2.16 \times 10^{10} \]

\[ Fo = \frac{at}{LD} \]
\[ = \frac{5.5(10^{-3})}{10(5)} t = 1.1 \times 10^{-4} \text{ t(hrs)} \]

\[ \frac{qL}{k} = \frac{4.4(10)}{0.35} = 125.7^\circ F \]

Now, assuming that, as above, the temperature increase in the boundary is 1 or 2°F, then
\[ T - T_0 = -78 - (-100) = 18^\circ F \]

and
\[ \frac{T - T_0}{q L/k} R_d^{1/5} = \frac{18}{125.7} \frac{2.16 \times 10^{10}}{2.0} \]
\[ = 8.35 \]

From Figure 6.9, the surface temperature will be at \(-78^\circ F\) when
\[ \frac{T - T_0}{q L/k} R_d^{1/5} = 8.35 \]

(Division by 2.0 is necessary because the Dewar is a cylinder (see Section 7.2).) This occurs at
\[ Fo R_d^{1/5} = 2.9 \]

which corresponds to a time, \( t \)
\[ t = \frac{2.9}{(a/LD) R_d^{1/5}} = 226.0 \text{ hrs} \]
\[ = 9.4 \text{ days} \]

Therefore, the fluid will not degrade for 9 days. If the holding time required is less than this, the problem can be recalculated assuming some air is present in the vapor space. On the other hand, redesign of the container is essential if the hold-up time is longer.

It should be noted that no vaporization or heat loss at the surface was permitted. Hence, the answer obtained is conservative. However, for most cases where vaporization is small, it is believed that this model gives the best estimate of the axial temperature profile in the enclosure.
VIII. CONCLUSIONS AND RECOMMENDATIONS

8.1 Conclusions

8.1.1 Solution of the Boundary Layer Equations

1. For the case of a uniformly heated wall in surroundings whose temperature varies linearly with height, a steady state, asymptotic solution of the boundary layer equations exists. This solution (Equation 4.3.5) is a function of the core Rayleigh number and the modified Rayleigh number. This asymptotic solution occurs when the amount of heat entering the fluid is just sufficient to raise the average boundary layer temperature at the same rate as that of the surrounding fluid.

2. A finite difference analog of the boundary layer equations using a backward difference representation of the vertical convective terms and a forward difference representation of the horizontal convective terms was used to solve the case of natural convection from a uniformly heated vertical plate. The following conclusions about this method were made.

   i. An implicit alternating direction scheme was required to provide stability of the finite difference equations for the case of a uniformly heated vertical plate in surroundings whose temperature increases linearly with height.

   ii. The finite difference equations used in conjunction with the implicit alternating direction scheme were unconditionally stable for the case of a uniformly heated vertical plate in isothermal surroundings.

   iii. For the case of a uniformly heated vertical plate in surroundings with a linear temperature gradient, there was a maximum time interval which permitted a stable solution. This limiting time
step was decreasing a function of the Prandtl number.

iv. In order for the equations to be convergent, it was necessary to have the horizontal grid size sufficiently small so that at least one grid point lies between the maximum velocity and the wall.

v. It was necessary to specify arbitrarily the wall heat flux conditions. Curves were developed which provide the correct parameter to give correct steady state results.

vi. Because of the arbitrary specification of the wall heat flux, the transient solution was in considerable error. This was due to the fact that the grid size could not resolve the initial temperature gradient.

3. The steady state solution for a uniformly heated vertical wall in isothermal surroundings is a strong function of Prandtl number. The higher the Prandtl number, the momentum boundary layer increases while the thermal boundary layer thickness decreases. However, there is no such dependence for the case of a uniformly heated vertical plate in surroundings with a linear temperature gradient.

4. The height at which an asymptotic solution is achieved for a uniformly heated vertical plate in surroundings whose temperature increases linearly with height was found to be

\[ z_0 = \frac{2}{4} \frac{Z}{L} \frac{R_m^{5/4}}{R_a} \approx 5.0 \]

5. The presence of a non-isothermal core temperature distribution limits the boundary layer growth as well as the amount of fluid being carried in the boundary layer and the average temperature, \( (\Theta - \Theta_m) \), in the boundary layer.
8.1.2 Natural Convection in Enclosed Fluids

1. The thermal stratification process consists of a thin boundary layer of warm fluid rising towards the surface where it is deflected towards the center of the container and then sinking down the central portion of the container as still warmer fluid replaces it. As the boundary layer fluid rises, additional fluid is entrained in the boundary layer or fluid is detrained from the boundary layer depending on the axial core temperature gradient.

2. As a rule, detrainment of boundary layer fluid occurs when it meets an adverse temperature gradient, i.e., when the core temperature increases with height at a rate greater than the average boundary layer temperature increases.

3. The fluid which leaves the boundary layer due to an adverse temperature gradient is at a temperature approximately equal to the core temperature at that height.

4. The maximum temperature of the core fluid is determined by the fluid which is deflected from the boundary layer by the surface. Because the enthalpy of this fluid is greater than the core fluid it displaces, it increases the core thermal gradient with time.

5. The temperature gradient in the central portion of the container (0.15 < Z < 0.8) is approximately linear at long times. The model derived by Drake (16) which predicts the axial temperature gradient was found to be valid for times

\[
\text{Fo Ra}^{1/5} > 0.625
\]

6. For Pr > 7, thermal convection is the dominant mode of heat transfer in the core of the container.

7. An exact solution of the integrated form of the boundary layer equations is possible for a given assumed
velocity and temperature boundary layer profiles for a fluid of infinite Prandtl number. This solution was found to be valid for all fluids of Prandtl number greater than 5.

8. A "mixing region" postulated by Drake where radial mixing occurs does not, in fact, exist. The only distinguishing characteristics of the top 20 percent of the container is that there is a significantly greater horizontal velocity than in other regions of the container.

9. The time and spatial temperature of a fluid enclosed in a rectangular container with uniformly heated sidewalls can be predicted using Figure 6.9.

10. Even though the core temperature increases linearly with height at long times, the correlating parameter, $C_f$, which determines the slope of the temperature gradient, calculated from the solution of the boundary layer equations for the case of a uniformly heated wall in surroundings whose temperature increases linearly with height obtained in Chapter 5, cannot be used to correlate the results. However, a single empirical constant, $C_f = 5.66$, correlated all of the experimental data well. This constant was independent of Prandtl number and time.

11. Velocities calculated from "streak" photographs agreed reasonably well with the velocities computed from the thermal stratification modelo.

12. The core fluid behaviour was unstable. The cause of this instability is unknown. However, it did not affect the temperature results.

8.2 Recommendations

8.2.1 Solution of the Boundary Layer Equations

1. A detailed study of the stability of the finite
difference scheme used in this work is required.

The finite difference scheme used in this work was stable for the case of a uniformly heated vertical plate in non-isothermal surroundings only within certain stability limits (Appendix M). However, the criteria for stability derived from a von Karman analysis did not apply. Because of the utility of solving partial differential equations using numerical means, it would be very profitable to evaluate the various techniques available for predicting stability criteria with the view of determining accurately the criteria for the scheme used in this work. Also, a development of methods to predict stability for partial differential equations with variable coefficients would be extremely useful.

2. Development of new finite difference schemes would facilitate the study of the boundary layer equations.

A more thorough investigation of the formulation of finite difference analogs for the boundary layer equations would prove extremely useful. If a scheme were developed which had better stability characteristics and adaptability to different boundary conditions than that used in this work, the investigation of the boundary layer solutions would be greatly facilitated.

Several methods might be investigated. One is the so-called "conservative formulation" of the finite difference equations. Noble (38) used such a formulation in solving the complete momentum and energy equations. Modification of this method so that it could be used for the solution of the boundary layer equations might prove to be useful.

Another method which looks promising from a stability point of view is a "conservative formulation" where the direction of the vertical velocities is recorded. Then, two sets of equations, one using a backward difference formulation of the vertical convective term, and another
using forward differencing are written. The direction of the velocity determines which set of equations should be used in advancing the temperature and velocity fields. This results in an effectively unconditionally stable scheme provided some scheme could be derived to account for the condition at the top of the grid. This work did not require such a development of a technique because only backward differences were used for the vertical convective terms and so a condition on the top grid was not needed.

3. An investigation of the boundary layer equations is required.

Several questions have arisen from this work about the form of the boundary layer temperature and velocity profiles present during the thermal stratification process. Since these profiles determine the degree of stratification, the availability of accurate solutions of the boundary layer equations might shed some light on why an empirical constant, \( C_f = 5.66 \), was able to correlate all of the results. A theoretical study of the effect of various boundary conditions on the boundary layer solutions would be warranted especially if some experimental verification of the theory would be attempted.

8.2.2 Natural Convection in Enclosed Fluids

1. A study of the boundary conditions on natural convection flows is warranted.

Several investigations have now been completed where the sidewalls of the vertical enclosure have been uniformly heated. This thesis has indicated that there is a significant boundary layer-core interaction in the presence of a temperature gradient in the core. In light of this, it might be expected that this interaction may be altered,
if the wall heat flux boundary condition is changed. Therefore, it may be possible to alter the core temperature distribution by this method. Therefore, criteria as to when the results of this thesis are applicable as well as means of reducing the degree of thermal stratification may be afforded by an investigation of the wall boundary conditions on the thermal stratification phenomenon.

2. A study of the thermal stratification phenomenon in the presence of turbulent boundary layers is recommended.

3. A study is also recommended to determine the cause of the instability in the flow of the core fluid.

Such a study would require several different apparatus so that the cause of the instability might be isolated.
IX. BIBLIOGRAPHY


15. Dotson, J.P., "Heat Transfer from a Vertical Plate by
Free Convection" M.S. Thesis, Purdue University, 1954


32. Leibmann, G., Trans. ASME 78C 655 (1958)


35. Mull, W., Reiher, H., Gesund-Ing. Beihifte 1 20 (1930)

36. Neff, R., Advances in Cryogenic Engineering 5 460 (1960)


39. Noble, J., personal communication


43. Pohlhausen, K., ZAMM, 1 252 (1921)


46. Reid, R.C., Evans, L.B., Matulevicius E.S., "A Comparison of Various Solutions for Laminar Free Convection from a Vertical Plate with Uniform Surface Heat Flux" to be published

47. Ruder, J.M., AIAA J. 2 135 (1964)


54. Siegel, R., Trans. ASME 80 347 (1958)


### X. NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$, $A_2$, $A_3$, $A_4$</td>
<td>Constants in integrated form of the boundary layer equations</td>
</tr>
<tr>
<td>$A_j$</td>
<td>Matrix coefficient</td>
</tr>
<tr>
<td>$B_j$</td>
<td>Constant</td>
</tr>
<tr>
<td>$B_j$</td>
<td>Matrix coefficient</td>
</tr>
<tr>
<td>$c$</td>
<td>Constant in thermal stratification expression</td>
</tr>
<tr>
<td>$c_o$</td>
<td>Constant in heat flux boundary condition</td>
</tr>
<tr>
<td>$c_1$</td>
<td>Constant in heat flux boundary condition</td>
</tr>
<tr>
<td>$c_2$</td>
<td>Constant in heat flux boundary condition - as specified from Appendix M</td>
</tr>
<tr>
<td>$C_E$</td>
<td>Constant in energy equation (Equation 6.2.2)</td>
</tr>
<tr>
<td>$C_f$</td>
<td>Velocity scale constant (Equation 6.2.7)</td>
</tr>
<tr>
<td>$C_M$</td>
<td>Constant in energy equation (Equation 6.2.2)</td>
</tr>
<tr>
<td>$C_p$</td>
<td>Specific heat</td>
</tr>
<tr>
<td>$C_j$</td>
<td>Matrix coefficient</td>
</tr>
<tr>
<td>$C$</td>
<td>Width of the container</td>
</tr>
<tr>
<td>$D_j$</td>
<td>Matrix coefficient</td>
</tr>
<tr>
<td>$E$</td>
<td>Energy function (Equation 6.2.2)</td>
</tr>
<tr>
<td>$E$</td>
<td>Dimensionless energy flow parameter (Equation 6.2.2)</td>
</tr>
<tr>
<td>$E^+$</td>
<td>Dimensionless group used in calculating $E$ (Equation 6.2.2)</td>
</tr>
<tr>
<td>$F$</td>
<td>Similarity velocity</td>
</tr>
<tr>
<td>$f$</td>
<td>Force vector</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational constant</td>
</tr>
<tr>
<td>$\vec{g}$</td>
<td>Gravity vector</td>
</tr>
<tr>
<td>$G$</td>
<td>Temperature gradient</td>
</tr>
<tr>
<td>$H$</td>
<td>Container width</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>Unit vector in $x$-direction</td>
</tr>
<tr>
<td>$i$</td>
<td>Space index in $z$-direction</td>
</tr>
<tr>
<td>$\text{IMAX}$</td>
<td>Maximum grid points in $z$-direction</td>
</tr>
<tr>
<td>$\hat{j}$</td>
<td>Unit vector in $y$-direction</td>
</tr>
<tr>
<td>$j$</td>
<td>Space index in $y$-direction</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>---------------------------------------------------------------------------</td>
</tr>
<tr>
<td>JMAX</td>
<td>maximum grid points in Y-direction</td>
</tr>
<tr>
<td>k</td>
<td>unit vector in z-direction</td>
</tr>
<tr>
<td>k</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>L</td>
<td>container height</td>
</tr>
<tr>
<td>M</td>
<td>momentum function (Equation 6.2.2)</td>
</tr>
<tr>
<td>( M^* )</td>
<td>dimensionless momentum flow parameter</td>
</tr>
<tr>
<td>( n )</td>
<td>time index</td>
</tr>
<tr>
<td>Nzones</td>
<td>maximum grids for thermal stratification model</td>
</tr>
<tr>
<td>P</td>
<td>pressure</td>
</tr>
<tr>
<td>( P^+ )</td>
<td>dimensionless pressure (Equation 2.3.38)</td>
</tr>
<tr>
<td>p</td>
<td>container perimeter</td>
</tr>
<tr>
<td>q</td>
<td>wall heat flux</td>
</tr>
<tr>
<td>( Q_f )</td>
<td>boundary layer flow rate</td>
</tr>
<tr>
<td>( Q_f^\prime )</td>
<td>dimensionless boundary layer flow rate</td>
</tr>
<tr>
<td>( T )</td>
<td>temperature</td>
</tr>
<tr>
<td>( T )</td>
<td>dimensionless temperature (Equation 2.3.38)</td>
</tr>
<tr>
<td>( \bar{T} )</td>
<td>dimensionless temperature (Equation 6.2.35)</td>
</tr>
<tr>
<td>( T^+ )</td>
<td>(in summary and in later discussion = ( T ))</td>
</tr>
<tr>
<td>( \Theta = \Theta_{\infty} )</td>
<td></td>
</tr>
<tr>
<td>( \Theta = \Theta_{\infty} )</td>
<td></td>
</tr>
<tr>
<td>( T_0 )</td>
<td>core temperature</td>
</tr>
<tr>
<td>( T_0^* )</td>
<td>dimensionless temperature (Equation 4.3.5)</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( U_c )</td>
<td>critical velocity in Brinkman Number</td>
</tr>
<tr>
<td>U</td>
<td>internal energy</td>
</tr>
<tr>
<td>( w )</td>
<td>vertical velocity</td>
</tr>
<tr>
<td>( W )</td>
<td>dimensionless vertical velocity (Equation 2.3.28)</td>
</tr>
<tr>
<td>( W )</td>
<td>dimensionless vertical velocity (Equation 6.3.28)</td>
</tr>
<tr>
<td>( W )</td>
<td>dimensionless vertical velocity (Equation 6.3.35)</td>
</tr>
<tr>
<td>( W_0^\prime )</td>
<td>(in summary and later discussion = ( W ))</td>
</tr>
<tr>
<td>( w^* )</td>
<td>dimensionless velocity (Equation 4.3.5)</td>
</tr>
<tr>
<td>( w^* )</td>
<td>velocity scaling parameter in integral equation</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$\nu$</td>
<td>volume</td>
</tr>
<tr>
<td>$v$</td>
<td>horizontal velocity</td>
</tr>
<tr>
<td>$V$</td>
<td>dimensionless horizontal velocity (Equation 2.3.28)</td>
</tr>
<tr>
<td>$V_0$</td>
<td>dimensionless horizontal velocity (Equation 4.3.12)</td>
</tr>
<tr>
<td>$\mathbf{V}$</td>
<td>velocity vector</td>
</tr>
<tr>
<td>$y$</td>
<td>horizontal direction</td>
</tr>
<tr>
<td>$Y$</td>
<td>dimensionless distance from wall (Equation 2.3.38)</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>dimensionless horizontal distance (Equation 4.3.5)</td>
</tr>
<tr>
<td>$Y_{\text{MAX}}$</td>
<td>boundary condition in finite difference scheme for $y$</td>
</tr>
<tr>
<td>$z$</td>
<td>vertical direction</td>
</tr>
<tr>
<td>$Z$</td>
<td>$z/L$</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>dimensionless height (Equation 4.3.12)</td>
</tr>
<tr>
<td>$Z^*$</td>
<td>dimensionless group $= C_f \frac{Ra^{5/4}}{Ra + Ra^*} \frac{1}{4}$</td>
</tr>
<tr>
<td>$Z_{\text{MAX}}$</td>
<td>maximum height in finite difference scheme</td>
</tr>
</tbody>
</table>

**Greek Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>thermal diffusivity</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>constant in stability analysis</td>
</tr>
<tr>
<td>$\beta$</td>
<td>thermal expansion coefficient</td>
</tr>
<tr>
<td>$\beta$</td>
<td>constant in stability analysis</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>constant in stability analysis</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>constant in stability analysis</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>kinematic viscosity</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>thermally stratified height</td>
</tr>
<tr>
<td>$\delta^*$</td>
<td>dimensionless boundary layer thickness of Drake (16)</td>
</tr>
<tr>
<td>$\delta^*$</td>
<td>boundary layer scaling parameter used in integral expression</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>energy flow out of the boundary layer</td>
</tr>
<tr>
<td>$\xi$</td>
<td>dimensionless energy flow out of the boundary layer</td>
</tr>
</tbody>
</table>
layer (equation 6.2.28)

amplification factor in stability analysis

similarity transform distance

temperature

boundary layer thickness

viscosity

density

gravitational potential

stream function

vector potential

stress tensor

dimensionless time (Equation 5.3.35)

(in summary and later discussion = τ)

dimensionless time (Equation 2.3.28)

dimensionless time (Equation 4.3.12)

similarity temperature

streamline

dimensionless streamline (Equation 2.3.8)

dimensionless velocity scale of Drake (16)

vorticity vector

dimensionless vorticity (Equation 2.3.38)

\[ \text{Dimensionless Groups} \]

\[
\begin{align*}
\text{As} & \quad \text{aspect ratio} = L/D \\
\text{Br} & \quad \text{Brinkman number} = \mu U^2/\kappa \theta \\
\text{Fo} & \quad \text{Fourier number} = \alpha t/LD \\
\text{Gr} & \quad \text{Grashof number} = g \beta L^3 \Delta T/\gamma^2 \\
\text{Gr} & \quad \text{Modified Grashof number} = g \beta L^4 q/\alpha \gamma^2 \\
\text{Nu} & \quad \text{Nusselt number} = h L/k \\
\text{Pr} & \quad \text{Prandtl number} = \gamma/\alpha \\
\text{Ra} & \quad \text{Modified Rayleigh number} = g \beta L^4 q/\alpha \gamma k \\
\text{Rc} & \quad \text{Core Rayleigh number} = g \beta L^4 (dT_c/dz)/\alpha \gamma
\end{align*}
\]
<table>
<thead>
<tr>
<th>Subscripts</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>depth</td>
</tr>
<tr>
<td>i</td>
<td>vertical space index</td>
</tr>
<tr>
<td>j</td>
<td>horizontal space index</td>
</tr>
<tr>
<td>L</td>
<td>overall height</td>
</tr>
<tr>
<td>Nzones</td>
<td>top of the grid</td>
</tr>
<tr>
<td>o</td>
<td>base temperature</td>
</tr>
<tr>
<td>s</td>
<td>surface</td>
</tr>
<tr>
<td>sur</td>
<td>surface</td>
</tr>
<tr>
<td>s</td>
<td>scale</td>
</tr>
<tr>
<td>st</td>
<td>stratified</td>
</tr>
<tr>
<td>w</td>
<td>wall</td>
</tr>
<tr>
<td>z</td>
<td>point height</td>
</tr>
<tr>
<td></td>
<td>core or surroundings</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Superscripts</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>time increment</td>
</tr>
<tr>
<td>i</td>
<td>intermediate</td>
</tr>
</tbody>
</table>
APPENDIX A
DERIVATION OF THE NATURAL CONVECTION EQUATIONS

The mathematical description of a moving fluid is affected by means of functions which give the distribution of the fluid velocity, \( \mathbf{V} = \mathbf{V}(x, y, z, t) \), and of any two thermodynamic quantities pertaining to the fluid, e.g., the temperature, \( \Theta = \Theta(x, y, z, t) \), and density, \( \rho = \rho(x, y, z, t) \).

A.1 Conservation of Mass

The law of conservation of mass will now be derived. Consider a volume, \( V_0 \), of fluid. The mass of fluid in this volume is \( \rho dV \), the integral being taken over \( V_0 \). The mass of fluid flowing out of this volume is \( \int \rho \mathbf{V} \cdot d\mathbf{n} \). Finally, the decrease in mass of the fluid in \( V_0 \) is

\[
- \frac{\partial}{\partial t} \int \rho dV
\]

Equating these, one obtains

\[
- \frac{\partial}{\partial t} \int \rho dV = \int \rho \mathbf{V} \cdot d\mathbf{n} \quad \text{(A.1.1)}
\]

or, using Green's Theorem

\[
- \frac{\partial}{\partial t} \int \rho dV = \int \text{div}(\rho \mathbf{V})dV \quad \text{(A.1.2)}
\]

Rewriting Equation A.1.2, one obtains

\[
\int \left( \frac{\partial}{\partial t} \rho + \text{div}(\rho \mathbf{V}) \right)dV = 0 \quad \text{(A.1.2a)}
\]

Since Equation A.1.2a is independent of the size of \( V_0 \), the integrand must vanish, i.e.,
\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{V}) = 0 \]  \hspace{2cm} A.1.3

This is equivalent to

\[ \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \text{grad} \rho = -\rho \text{div} \mathbf{V} \]  \hspace{2cm} a.1.3a

Defining the substantial derivative as

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \text{grad}, \]

equation A.1.2a becomes

\[ \frac{D\rho}{Dt} = -\rho \text{div} \mathbf{V} \]  \hspace{2cm} A.1.2b

For an incompressible fluid, Equation A.1.2b becomes

\[ \text{div} \mathbf{V} = 0 \]  \hspace{2cm} A.1.4

A.2 Equation of Motion

As before, consider a volume of fluid \( V_0 \). The total force acting on this volume is equal to the sum of the external and internal forces on the volume element. Let the external forces be represented by \( f \). Then, the sum of the external forces acting on \( V_0 \) is

\[ \iiint \rho f \, dV \]

The sum of the internal forces acting on \( V_0 \) is

\[ \iiint \mathbf{T} \cdot \mathbf{n} \, dS \]

The stress tensor \( \mathbf{T} \) can be divided into a pressure force \( P \) and a shear force \( \tau \) so that
Using Newton's law

\[ F = ma \]  

one can write a force balance on the volume \( V_0 \) as follows

\[
\iiint \rho \frac{dV}{dt} \, dV = \iiint \rho \, \delta \, dV + \iiint (-P \delta + \tau) \cdot n \, dS \]  

By Green's Theorem, Equation A.2.3 is equivalent to

\[
\iiint \rho \frac{dV}{dt} \, dV = \iiint \rho \, \delta \, dV + \iiint (\text{div}(P \delta) + \text{div} \tau) \, dV \]  

This can only be true if

\[
\frac{dV}{dt} = \rho \delta - \text{grad} \, P + \text{div} \, \tau \]  

Note that this expression (Equation A.2.3c) does not denote the rate of change of fluid velocity at a fixed point in space, but the rate of a given particle as it moves about in space. To find an equivalent expression referred to a fixed point in space, note that \( \frac{dV}{dt} \) is composed of two parts, viz.,

\[
\frac{dV}{dt} = \frac{\delta V}{t} + (\frac{\delta V}{\delta t} \cdot \text{grad})V \]  

or,

\[
\frac{dV}{dt} = \frac{\delta V}{t} + (V \cdot \text{grad})V \]

Equation A.2.3c then becomes

\[
\frac{\delta V}{\delta t} + (V \cdot \text{grad})V = \frac{1}{\rho} \text{grad} \, P + \frac{1}{\rho} \cdot \text{div} \, \tau \]

Again, introducing the substantial derivative \( \frac{D}{Dt} \), one obtains
Following the argument of Landau and Lifshitz (28), note that

\[ \tau = 0, \text{ when } \mathbf{V} = 0 \]
\[ \tau = 0, \text{ when } \mathbf{V} = \mathbf{0} \times \nabla \]  

Hence, using standard tensor summation nomenclature, \( \tau_{ik} \) must depend on \( \frac{\partial \mathbf{V}}{\partial x_k} \). For small velocity gradients, terms greater than first order may be neglected for the momentum transfer due to viscosity. Furthermore, assuming that the fluid is Newtonian, then \( \tau_{ik} \) is a linear function of \( \frac{\partial \mathbf{V}}{\partial x_k} \). Now, \( \left( \frac{\partial \mathbf{V}}{\partial x_k} + \frac{\partial \mathbf{V}}{\partial x_l} \right) \) are linear combinations of \( \frac{\partial \mathbf{V}}{\partial x_k} \), and vanish when \( \mathbf{V} = \mathbf{0} \times \nabla \). A tensor of rank two satisfying the above conditions is

\[ \tau_{ik} = \mu \left( \frac{\partial \mathbf{V}}{\partial x_k} + \frac{\partial \mathbf{V}}{\partial x_l} - \frac{2}{3} \delta_{ik} \frac{\partial \mathbf{V}}{\partial x_1} \right) + \mu^{+} \delta_{ik} \frac{\partial \mathbf{V}}{\partial x_1} \]  

where \( \mu > 0 \) and \( \mu^{+} > 0 \). Then,

\[ \frac{\partial \tau_{ik}}{\partial x_k} = \text{div} \left[ \tau \right] \]

\[ = \frac{\partial}{\partial x_k} \left( \mu \left( \frac{\partial \mathbf{V}}{\partial x_k} + \frac{\partial \mathbf{V}}{\partial x_l} - \frac{2}{3} \delta_{ik} \frac{\partial \mathbf{V}}{\partial x_1} \right) + \frac{\partial \mu^{+}}{\partial x_1} \right) \]

Note that \( \mu \) and \( \mu^{+} \) are functions of \( P \) and \( Q \), hence they cannot be taken outside the gradient operator. However, for most cases, these viscosity coefficients do not change noticeably in the fluid and may be regarded as constant. Then, Equation A.2.9 becomes

\[ \frac{\partial \tau_{ik}}{\partial x_k} = \mu \frac{\partial^2 \mathbf{V}}{\partial x_k^2} + \left( \mu^{+} + \frac{1}{3} \mu \right) \frac{\partial}{\partial x_1} \frac{\partial \mathbf{V}}{\partial x_1} \]  

But

\[ \frac{\partial^2 \mathbf{V}}{\partial x_k^2} = \frac{\partial^2 \mathbf{V}}{\partial x_1^2} \text{ and } \frac{\partial \mathbf{V}}{\partial x_1} = \text{div} \mathbf{V} \]
so that Equation A.2.6b becomes

\[
\frac{DV}{Dt} = -\frac{1}{\rho} \text{grad} P + \gamma \nabla^2 V + \frac{u^+ + (1/3)u}{\rho} \text{grad} (\text{div } V) + f \quad \text{A.2.11}
\]

For an \textbf{incompressible fluid},

\[
\text{div } V = 0 \quad \text{A.1.4}
\]

so that Equation A.2.11 becomes

\[
\frac{DV}{Dt} = -\frac{\text{grad } P}{\rho} + \gamma \nabla^2 V + f \quad \text{A.2.12}
\]

Now, in most systems, including the one under consideration, the gravity force is the only external force acting on the fluid, i.e.,

\[
f = g \quad \text{A.2.13}
\]

Now, if the Bousinesq approximations are made, i.e., all the physical properties are assumed constant except for the density in the body force term. Here, density varies as

\[
\rho = \rho_o + \frac{\partial \rho_o}{\partial \theta} (\theta - \theta_o) + \frac{1}{2} \frac{\partial^2 \rho_o}{\partial \theta^2} (\theta - \theta_o)^2 + \ldots .
\]

\[
\text{A.2.14}
\]

Assuming that terms of second order and greater can be neglected in Equation A.2.14, then

\[
\rho - \rho_o = \frac{\partial \rho_o}{\partial \theta} (\theta - \theta_o)
\]

\[
\text{A.2.15}
\]

But, from thermodynamics, the volumetric expansion coefficient is defined as

\[
\beta = -\frac{1}{\rho_o} \frac{\partial \rho_o}{\partial \theta}
\]

so that

\[
\rho/\rho_o = 1 - \beta (\theta - \theta_o)
\]

\[
\text{A.2.16}
\]
Since \( \partial p / \partial \theta \) is assumed to be negligible in all but the body force terms, then

\[
\beta(\theta - \theta_0) \ll 1
\]

Further, the pressure in the fluid can be divided into a hydrostatic pressure and a pressure, \( p \), due to temperature gradients and fluid motion, i.e.,

\[
P = p + (p_{\text{atm}} + \rho g \cdot \mathbf{z})
\]

Equation A.2.12 can now be rewritten as:

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{\nabla p}{\rho} - \frac{\rho g}{\rho_0} + \gamma \nabla^2 \mathbf{v} + \mathbf{g}
\]

or,

\[
\frac{\partial \mathbf{v}}{\partial t} = \beta g (\theta - \theta_0) - \frac{\nabla p}{\rho} + \gamma \nabla^2 \mathbf{v}
\]

Equation A.2.18b is the basic equation of motion describing the natural convection process.

If the vorticity, \( \omega \), is defined as

\[
\omega = \text{curl} \, \mathbf{v}
\]

then, Equation A.2.18 becomes

\[
\frac{\partial \omega}{\partial t} = \text{curl} \left( \beta g (\theta - \theta_0) \right) + \gamma \nabla^2 \omega
\]

A.3 **Equation of Energy**

Again, consider a volume element \( V_0 \). For our purpose, only changes in the kinetic energy, \( 1/2 \rho v^2 \), and the internal energy, \( \rho U \), will be considered. Then, by the First Law of Thermodynamics,

\[
dE = \delta Q - \delta W
\]
or,
\[
\frac{d}{dt} \iiint \rho (\frac{1}{2}v^2 + U) d\mathbf{V} = - \iiint \mathbf{q} \cdot \mathbf{n} d\mathbf{S} - \iiint \mathbf{P} \cdot \mathbf{v} d\mathbf{S} + \iiint \mathbf{q}_f \cdot \mathbf{v} d\mathbf{V} + \iiint \mathbf{P} \cdot \mathbf{v} d\mathbf{S} + \iiint \mathbf{q}_f \cdot \mathbf{v} d\mathbf{V}
\]
\[\text{A.3.2}\]

As in the case of the Equation of Motion, this relates the rate of energy change to a moving axis in space. To refer this rate to a fixed point in space, a procedure similar to the one used above can be incorporated. Then, by using the divergence theorem, Equation A.3.2 becomes

\[
\iiint \frac{d}{dt} \rho (\frac{1}{2}v^2 + U) d\mathbf{V} + \iiint \mathbf{V} \cdot \nabla \rho (\frac{1}{2}v^2 + U) d\mathbf{V} = - \iiint \mathbf{div} \mathbf{q} d\mathbf{V} - \iiint \mathbf{div}(\mathbf{P} \mathbf{v}) d\mathbf{V} + \iiint \mathbf{div} \left( \mathbf{\tau} \right) \mathbf{V} d\mathbf{V} + \iiint \mathbf{rho} \cdot \mathbf{V} d\mathbf{V}
\]
\[\text{A.3.3}\]

In order for Equation A.3.3 to be valid for all sizes of \(V_0\)

\[
(\frac{1}{2}v^2 + U) \frac{D\rho}{Dt} + \rho \frac{DU}{Dt} (\frac{1}{2}v^2 + U) = - \ mathbf{div} \mathbf{q} - \mathbf{div}(\mathbf{P} \mathbf{v}) + \mathbf{div} \left( \mathbf{\tau} \right) \mathbf{V} + \mathbf{rho} \cdot \mathbf{V}
\]
\[\text{A.3.4a}\]

Using Equation A.1.4, Equation A.3.4 can be modified to

\[
\rho \frac{D}{Dt} (\frac{1}{2}v^2) + \rho \frac{DU}{Dt} = - \mathbf{div} \mathbf{q} - \mathbf{div}(\mathbf{P} \mathbf{v}) + \mathbf{div} \left( \mathbf{\tau} \right) \mathbf{V} + \mathbf{rho} \cdot \mathbf{V}
\]
\[\text{A.3.4b}\]

Now, multiply Equation A.2.6b by \(\mathbf{V}\) to give

\[
\rho \frac{D}{Dt} (\frac{1}{2}v^2) = \mathbf{rho} \cdot \mathbf{V} - \mathbf{V} \cdot \mathbf{\nabla} P + \mathbf{V} \cdot (\mathbf{div} \mathbf{\tau})
\]
\[\text{A.3.5}\]

Noting that

\[
\mathbf{div} \left( \mathbf{\tau} \right) \mathbf{V} = \mathbf{V} \cdot (\mathbf{div} \mathbf{\tau}) + \mathbf{\tau} \cdot (\mathbf{div} \mathbf{V})
\]

and substituting Equation A.3.5 into A.3.4b, one obtains

\[
\rho \frac{DU}{Dt} = - \mathbf{div} \mathbf{q} - P \mathbf{div} \mathbf{V} + \mathbf{\tau} \cdot (\mathbf{div} \mathbf{V})
\]
\[\text{A.3.6a}\]
or
\[ \rho \frac{DU}{Dt} = - \text{div} \, q - P \text{div} \, \mathbf{v} + \tau : \nabla \nabla \mathbf{v} \quad \text{A.3.6b} \]

The term \[ \tau : \nabla \nabla \mathbf{v} \] is the viscous dissipation term. It is comprised of terms such as \( (\partial v_i / \partial x_k)^2 \). Thus, for small \( \partial v_i / \partial x_k \), the viscous dissipation term can be neglected. Such is the case for our system, so that

\[ \rho \frac{DU}{Dt} = - \text{div} \, q - P \text{div} \, \mathbf{v} \quad \text{A.3.7} \]

For a pure fluid, the internal energy is a function of \( \phi, P \). Then,

\[ dU = \left( \frac{\partial U}{\partial \phi} \right)_P \phi d\phi + \left( \frac{\partial U}{\partial P} \right)_\phi dP \quad \text{A.3.8a} \]

From the definition of enthalpy,

\[ H = U + PV \quad \text{A.3.8b} \]

and the definition of specific heat, \( c_p \)

\[ c_p = \left( \frac{\partial H}{\partial \phi} \right)_P \quad \text{A.3.8c} \]

one obtains

\[ \left( \frac{\partial U}{\partial \phi} \right)_P = c_p - P \left( \frac{\partial V}{\partial \phi} \right)_P \quad \text{A.3.8d} \]

Substituting this into Equation A.3.8a, one obtains for a constant pressure process

\[ dU = c_p \phi d\phi - P dV \quad \text{A.3.8e} \]

Then,

\[ \frac{DU}{Dt} = c_p \frac{D\phi}{Dt} - P \frac{DV}{Dt} \quad \text{A.3.8f} \]

Noting that

\[ \rho \frac{DV}{Dt} = -1/\rho \frac{DP}{Dt} = \text{div} \, \mathbf{v} \quad \text{A.3.8g} \]

one can obtain from Equation A.3.7 that
\[ \rho c_p \frac{D \theta}{Dt} = - \text{div} \, q \tag{A.3.9} \]

Assuming that Fourier's Law holds, then for an isotropic medium,

\[ q = - k \text{ grad } \theta \tag{A.3.10} \]

Substituting this into Equation A.3.9, one obtains the required Equation of Energy.

\[ \frac{D \theta}{Dt} = \nabla^2 \theta \tag{A.3.11} \]
APPENDIX B
ORDER OF MAGNITUDE ANALYSIS OF BOUNDARY LAYER EQUATIONS FOR NATURAL CONVECTION

The Navier-Stokes equation is a balance among the viscous, inertial, and buoyancy forces. Similarly, the energy equation is a balance of energy transported by convection and conduction. It is entirely possible that some of the forces are small relative to the other forces in the equation. An order of magnitude analysis allows one to properly scale the pertinent equations in such a way that the resulting dimensionless equations will yield correct asymptotic solutions. Further, correct choice of scaling parameters afforded by an order of magnitude analysis maintains the values of the various terms in the equations within the range of values permitted by the computational device used. This minimizes the truncation error. This appendix illustrates this method of non-dimensionalizing equations and affords some insight as to the type of dependent variables which result when certain terms can be neglected in the equations.

The pertinent equations for this work are:

1. **Continuity**
   \[ \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \]  \hspace{2cm} B.1

2. **Momentum (Navier-Stokes Equation)**
   \[ \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = g\beta \theta + \frac{\partial^2 w}{\partial y^2} \]  \hspace{2cm} B.2

3. **Energy**
   \[ \frac{\partial \theta}{\partial t} + w \frac{\partial \theta}{\partial z} + v \frac{\partial \theta}{\partial y} = c \frac{\partial^2 \theta}{\partial y^2} \]  \hspace{2cm} B.3

where \( \theta = (T - T_o) \).

Scaling parameters \( W_s, V_s, L_s, \delta_s, T_s \) are chosen for \( w, v, z, y, \theta \) respectively. By a scaling parameter,
it is meant that the resulting dimensionless terms, e.g. the vertical velocity
\[ W = \frac{W}{W_s} \]
are of the order 1 in size. Then, for the continuity equation
\[ \left[ \frac{W_s}{L_s} \right] \sim \left[ \frac{V_s}{\delta_s} \right] \]
where \([f]\) denotes "order of magnitude".

Now, let us look at the steady state case only of Equations B.2 and B.3, i.e., \( \frac{\partial W}{\partial t} = \frac{\partial W}{\partial t} = 0 \). In the momentum equation, the following forces are present:

- **inertial** \( \left[ \frac{W_s^2}{L_s} \right] \sim \left[ \frac{V_s W_s}{\delta_s} \right] \sim 1 \)
- **buoyancy** \( \left[ g \beta T_s \right] \sim B \)
- **viscous** \( \left[ \frac{W_s}{\delta_s^2} \right] \sim V \)

Similarly, in the energy equation,

- **convection** \( \left[ \frac{W_s T_s}{L_s} \right] \sim \left[ \frac{V_s T_s}{\delta_s} \right] \sim C_0 \)
- **conduction** \( \left[ \frac{\alpha T_s}{\delta_s^2} \right] \sim C_{02} \)

Therefore, there are five unknown scaling parameters and only three equations (Equations B.1, B.2, B.3). It is therefore necessary that two scaling parameters be known or assumed. For this section, it is assumed that

\[ L_s \sim L \] \hspace{1cm} \text{B.5}
\[ T_s \sim qL/k \] \hspace{1cm} \text{B.6}
Then, the various scaling parameters can be ascertained by assuming that certain of the forces are dominant, e.g.,
assume
\[ \text{convection } \sim \text{ conduction} \]
and
\[ \text{buoyancy } \sim \text{ viscous} \]

From Equation B.3,
\[
\left[ \frac{W_s T_s}{L_s} \right] \sim \left[ \frac{gT_s}{\theta_s^2} \right]
\]
B.7

and from Equation B.2
\[
\left[ \frac{W_s}{\theta_s^2} \right] \sim \left[ g\theta T_s \right]
\]
B.8

Combining Equations B.7, B.8 and using B.5 and B.6, the following relations are obtained
\[
\frac{L}{\theta_s} \sim \text{Ra}^{1/4}
\]
B.9
\[
W_s \sim \frac{a}{L} \text{Ra}^{1/2}
\]
B.10

and from Equation B.4,
\[
V_s \sim \frac{a}{L} \text{Ra}^{1/4}
\]
B.11

In order for these scaling parameters to be physically correct it is necessary that
\[
\frac{I}{B} = \frac{\text{inertial forces}}{\text{buoyancy forces}} = \frac{I}{V} \sim 1
\]
B.12

Using the above scaling parameters, and Equation B.2, one can see that
\[
\frac{I}{B} \sim \frac{1}{\text{Pr}}
\]
B.13

This means that these scaling parameters will give
correct asymptotic solutions at high Prandtl numbers.
Table B.1 gives the scaling parameters for various other combinations of forces. It should be noted that at Prandtl number equal to one all of the forces are of equal magnitude, i.e., at $\text{Pr} = 1$,

$$I \sim V \sim B$$

$$C_0 \sim C_{02}$$

Finally, if one defines a set of dimensionless parameters as follows

$$W = w/W_s, \ V = v/V_s, \ T = \theta/T_s$$

$$Z = z/L, \ \text{and} \ Y = y/\delta_s$$

the dimensionless equations for each of these cases are as follows:

1. $C_0 \sim C_{02}; \ B \sim V$

$$\frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y}$$

$$\frac{\partial W}{\partial \tau} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = \text{Pr} T + \text{Pr} \frac{\partial^2 W}{\partial Y^2}$$

$$\frac{\partial T}{\partial \tau} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} = \frac{\partial^2 T}{\partial Y^2}$$

where $\tau = \frac{at}{L^2} R_a^{1/2} = Fo R_a^{1/2}$

and $Fo$ is the Fourier number.

2. $C_0 \sim C_{02}; \ I \sim B$

$$\frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y}$$

$$\frac{\partial W}{\partial \tau} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = T + \text{Pr} \frac{\partial^2 W}{\partial Y^2}$$
\[ \frac{\partial T}{\partial \tau} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} = \frac{\partial^2 T}{\partial Y^2} \]  

where \( \tau = \frac{\alpha t}{L^2} \, \text{Gr}^{1/2} = \text{Pr} \, \text{Fo} \, \text{Gr}^{1/2} \)  

3. \( I \sim B \sim V \)  

\[ \frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y} = 0 \]  

\[ \frac{\partial W}{\partial \tau} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = T + \frac{\partial^2 W}{\partial Y^2} \]  

\[ \frac{\partial T}{\partial \tau} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} = \frac{1}{\text{Pr}} \frac{\partial^2 T}{\partial Y^2} \]  

where \( \tau = \frac{\text{Gr}^{1/2}}{L} \, \text{Gr}^{1/2} = \text{Pr} \, \text{Fo} \, \text{Gr}^{1/2} \)  

4. \( I. \sim V; \, C_5 \sim C_02 \)  

Here the Prandtl number must equal one so that all of the above equations become equivalent, viz.,  

\[ \frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y} = 0 \]  

\[ \frac{\partial W}{\partial \tau} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = T + \frac{\partial^2 W}{\partial Y^2} \]  

\[ \frac{\partial T}{\partial \tau} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} = \frac{\partial^2 T}{\partial Y^2} \]  

where \( \tau = \text{Fo} \, \text{Gr}^{1/2} \)  

The above order of magnitude analysis assumed scaling parameters for \( L \) and \( \theta \). However, since the present work is concerned with a uniformly heated vertical wall, one further relationship, viz., the wall heat flux condition, might be used to obtain a scaling parameter for the temperature.  

Using the wall heat flux condition,  

\[ q = -k \frac{\partial T}{\partial Y} \]
the following relationship results from an order of magnitude analysis

\[ T_s \sim \frac{q_{\delta s}}{k} \quad \text{B.29} \]

Then, a new set of scaling parameters result as given in Table B.2. The form of the dimensionless differential equations remain the same as those given above for the corresponding cases. However, the dimensionless time, \( \tau \), for the cases now become:

1. \( C_{\delta 2} \sim C_0; B \sim V \)

\[ \tau = \text{Fo} \, \text{Ra}^{2/5} \quad \text{B.30} \]

2. \( C_0 \sim C_{\delta 2}; I \sim B \)

\[ \tau = \text{Fo} \, \left( \text{Pr} \, \text{Ra} \right)^{2/5} \quad \text{B.31} \]

3. \( I \sim B \sim V \)

\[ \tau = \text{Fo} \, \text{Gr}^{2/5} \quad \text{B.32} \]

4. \( C_0 \sim C_{\delta 2}; I \sim V \)

\[ \tau = \text{Fo} \, \text{Gr}^{2/5} \quad \text{B.33} \]

Since the dimensionless form of the equations remain the same, the solutions are asymptotically correct for both choices of \( T_s \). Therefore, either set of dimensionless variables may be used without affecting the solution. The main differences between the two choices of scaling parameters is in the resulting magnitude of the dimensionless variables. However, even these are small, e.g., the ratio of the velocities calculated by the two sets of scaling parameters for case 1, is \( \text{Ra}^{1/20} \sim 2 \), and for the temperature it is about \( \text{Ra}^{1/5} \sim 100 \).
### TABLE B.1

SCALING PARAMETERS FOR VARIOUS CONDITIONS

WHEN $T_s$ IS BASED ON THE PLATE HEIGHT

<table>
<thead>
<tr>
<th>FORCE ASSUMPTIONS</th>
<th>$C_0 \sim C_{02}^*$</th>
<th>$B \sim V$</th>
<th>$C_0 \sim C_{02}^*$</th>
<th>$I \sim V$</th>
<th>$I \sim B \sim V$</th>
<th>$C_0 \sim C_{02}^*$</th>
<th>$I \sim V$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PARAMETERS</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_s$</td>
<td>$L_{s}$</td>
<td>$L$</td>
<td>$L$</td>
<td>$L$</td>
<td>$L$</td>
<td>$L$</td>
<td></td>
</tr>
<tr>
<td>$\delta_s$</td>
<td>$L/Ra^s 1/4$</td>
<td>$L/(Pr Ra^s)^{1/4}$</td>
<td>$L/Gd^s 1/4$</td>
<td>any previous $^e$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_s$</td>
<td>$qL/k$</td>
<td>$qL/k$</td>
<td>$qL/k$</td>
<td>$qL/k$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_s$</td>
<td>$\frac{a}{L}Ra^s 1/2$</td>
<td>$\frac{a}{L}Gd^s 1/2$</td>
<td>$\frac{a}{L}Gd^s 1/2$</td>
<td>any previous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_s$</td>
<td>$\frac{a}{L}Ra^s 1/4$</td>
<td>$\frac{a}{L}Gd^s 1/4$</td>
<td>$\frac{a}{L}Gd^s 1/4$</td>
<td>any previous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Ratio of neglected terms to dominant</strong></td>
<td>$\frac{I}{V} \sim \frac{1}{Pr}$</td>
<td>$\frac{I}{V} \sim \frac{1}{Pr}$</td>
<td>$\frac{C_0}{C_{02}} \sim Pr$</td>
<td>$\frac{I}{V} \sim 1 = Pr$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Range Valid</strong></td>
<td>high $Pr$</td>
<td>low $Pr$</td>
<td>all $Pr$</td>
<td>$Pr = 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^e$At $Pr = 1$, which is the necessary condition for this case, all of the terms for a given scaling factor are equivalent.
TABLE B.2  
SCALING PARAMETERS FOR VARIOUS CONDITIONS WHEN  
$T_s$ IS BASED ON THE HEAT FLUX CONDITION  

<table>
<thead>
<tr>
<th>FORCE ASSUMPTIONS</th>
<th>$C_6 \sim C_{82}; B \sim V$</th>
<th>$C_6 \sim C_{82}; I \sim B$</th>
<th>$I \sim B \sim V$</th>
<th>$C_6 \sim C_{82}; I \sim V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PARAMETERS</td>
<td>$L_s$</td>
<td>$L_s$</td>
<td>$L_s$</td>
<td>$L_s$</td>
</tr>
<tr>
<td></td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L_{c}G^*_L 1/5$</td>
<td>$L_{c}G^*_L 1/5$</td>
</tr>
<tr>
<td></td>
<td>$\delta_s$</td>
<td>$\delta_s$</td>
<td>$\delta_s$</td>
<td>$\delta_s$</td>
</tr>
<tr>
<td></td>
<td>$(qL/k)/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$(qL/k)/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$(qL/k)/G^*_L 1/5$</td>
<td>$(qL/k)/G^*_L 1/5$</td>
</tr>
<tr>
<td></td>
<td>$\omega_s$</td>
<td>$\omega_s$</td>
<td>$\omega_s$</td>
<td>$\omega_s$</td>
</tr>
<tr>
<td></td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
</tr>
<tr>
<td></td>
<td>$V_s$</td>
<td>$V_s$</td>
<td>$V_s$</td>
<td>$V_s$</td>
</tr>
<tr>
<td></td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
<td>$L/\dot{R}^*_{\dot{L}} 1/5$</td>
</tr>
<tr>
<td>Ratio of neglected terms to dominant</td>
<td>$I/V \sim 1/Pr$</td>
<td>$I/V \sim 1/Pr$</td>
<td>$I/V \sim 1/Pr$</td>
<td>$I/V \sim 1/Pr$</td>
</tr>
<tr>
<td>case, all of the terms for a given scaling factor are equivalent.</td>
<td>$I/V \sim 1/Pr$</td>
<td>$I/V \sim 1/Pr$</td>
<td>$I/V \sim 1/Pr$</td>
<td>$I/V \sim 1/Pr$</td>
</tr>
</tbody>
</table>

Range Valid  
| high Pr | low Pr | all Pr | Pr = 1 |

At Pr = 1, which is the necessary condition for this case, all of the terms for a given scaling factor are equivalent.
APPENDIX C

ASYMPTOTIC SOLUTION TO THE VERTICAL PLATE PROBLEM WITH A CONSTANT WALL HEAT FLUX AND A LINEAR AMBIENT TEMPERATURE GRADIENT

Consider a vertical plate with a constant heat flux, \( q_0 \), imposed on it placed in an infinite surrounding where the temperature, \( \Theta \), increases linearly with height, \( z \). Such a plate is shown in Figure C.1.

![Diagram of a vertical plate problem with a non-isothermal core](image)

**Figure C.1**
Vertical Plate Problem with a Non-isothermal Core

The boundary conditions for this problem are:

At \( y = 0 \)

\[
q_o = -k \frac{\partial \Theta}{\partial y} \quad \text{C.1a}
\]

\( w = 0 \) \quad \text{C.1b}
At \( y = \infty \)
\[
\begin{align*}
\theta &= C.1c \\
w &= 0 & C.1d
\end{align*}
\]

If one now defines a new temperature variable \( T^+ \) such that
\[
T^+ = \theta - \Theta
\]
then the general boundary layer equations can be written as
\[
\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \quad C.3a
\]
\[
\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + v \frac{\partial w}{\partial y} = \frac{\partial T^+}{\partial y} + \gamma \frac{\partial^2 w}{\partial y^2} \quad C.3b
\]
\[
\frac{\partial T^+}{\partial t} + w \frac{\partial T^+}{\partial z} + v \frac{\partial T^+}{\partial y} = \frac{\partial^2 T^+}{\partial y^2} - \frac{\partial \Theta_\infty}{\partial y} \quad C.3c
\]
Assuming that at steady state an asymptotic solution exists, then,
\[
\frac{\partial w}{\partial t} = \frac{\partial T^+}{\partial t} = \frac{\partial T^+}{\partial z} = \frac{\partial w}{\partial z} = 0 \quad C.4
\]
Therefore, by Equation C.3a
\[
\frac{\partial v}{\partial y} = 0 \quad C.5
\]
Finally, since at \( y = 0 \), a no-slip condition is imposed, the horizontal velocity, \( v \), at any \( y_o \) is
\[
v = \int_0^{y_o} \frac{\partial y}{\partial y} dy = 0 \quad C.6
\]
Then, Equation C.3b and C.3c reduce to
\[
\frac{\partial^2 w}{\partial y^2} = - \frac{\beta T^+}{\gamma} \quad C.7a
\]
\[
\frac{\partial^2 T^+}{\partial y^2} = \frac{w}{a} \frac{\partial \Theta_\infty}{\partial z} \quad C.7b
\]
Differentiating Equation C.7a twice and substituting into Equation C.7b, one obtains, using standard differential equation notation,

\[
\begin{align*}
D^4 + \frac{L^4 g \beta (\partial^2 \Theta_0 / \partial z^2)}{\alpha \gamma L^4} \end{align*}
\]

Then, if one defines a core Rayleigh number, Ra, as

\[
Ra = \frac{g \beta L^4 (\partial \Theta_0 / \partial z)}{\alpha \gamma}
\]

Equation C.8 can be rewritten as

\[
(D^4 + Ra^\infty / L^4) w = 0
\]

The roots of D are:

\[
\sqrt{\frac{1}{2}} Ra^\infty 1/4 / L (\pm 1 \pm i), \quad \sqrt{\frac{1}{2}} Ra^\infty 1/4 / L (\mp 1 \mp i)
\]

So that if one defines

\[
\sigma_a = \sqrt{\frac{1}{2}} Ra 1/4 / L
\]

a solution of Equation C.10 is

\[
\begin{align*}
w &= \exp(-\sigma_a y)(A \sin (\sigma_a y) + B \cos (\sigma_a y)) \\
&\quad + \exp(\sigma_a y)(C \sin (\sigma_a y) + D \cos (\sigma_a y))
\end{align*}
\]

Since, as seen from Equation C.1, w must be bounded at \( y = \infty \), then,

\[
C = D = 0
\]

Also, since at \( y = 0 \), \( w = 0 \) (Equation C.1)

\[
B = 0
\]
In order to solve for $T^+$, Equation C.12 is differentiated twice and substituted into Equation C.7a so that

$$T^+ = \frac{2\gamma}{\varepsilon \beta} \sigma_a^2 \exp(-\sigma_a y)(A \cos(\sigma_a y)) \quad \text{C.14}$$

Using the boundary conditions of Equation C.1, one obtains

$$T^+ = \frac{a \gamma \text{Ra}^+}{g \beta L \sigma_a} \exp(-\sigma_a y) \cos(\sigma_a y) \quad \text{C.15a}$$

$$w = \frac{\alpha \text{Ra}^+}{2L \sigma_a^3} \exp(-\sigma_a y) \sin(\sigma_a y) \quad \text{C.15b}$$

Since $\sigma_a$ is defined by Equation C.11, one can eliminate it from Equation C.15 so that

$$T^+ = \frac{2\sqrt{2}}{2} \frac{q L}{k} (\text{Ra}^*)^{-1/4} \exp(- \frac{\sqrt{2}}{2} (\text{Ra}^*)^{1/4} \frac{y}{L}) \cos(\frac{\sqrt{2}}{2} (\text{Ra}^*)^{1/4} \frac{y}{L}) \quad \text{C.16a}$$

$$w = \sqrt{2} \frac{q}{L} \text{Ra}^+(\text{Ra}^*)^{-2/3} \exp(- \frac{\sqrt{2}}{2} (\text{Ra}^*)^{1/4} \frac{y}{L}) \sin(\frac{\sqrt{2}}{2} (\text{Ra}^*)^{1/4} \frac{y}{L}) \quad \text{C.16b}$$

$T^+$ and $w$ are the required solutions.
APPENDIX D
NORMALIZATION OF THE NATURAL CONVECTION BOUNDARY LAYER EQUATIONS FOR A UNIFORMLY HEATED VERTICAL PLATE IN A NON-ISOTHERMAL SURROUNDING

It is desired to rewrite the following equations into a normalized form:

\[ \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \]  \hspace{1cm} D.1a

\[ \frac{Dw}{Dt} = \alpha \frac{\partial^2 q^+}{\partial y^2} - \frac{\partial e_\infty}{\partial z} \]  \hspace{1cm} D.1b

\[ \frac{DQ^+}{Dt} = \alpha \frac{\partial^2 q^+}{\partial y^2} - \frac{\partial e_\infty}{\partial z} \]  \hspace{1cm} D.1c

To accomplish this, a set of dimensionless parameters are defined as:

\[ \tau = t/t_s \quad Z = z/z_s \quad Y = y/y_s \]  \hspace{1cm} D.2

Then, Equation D.1 can be rewritten in dimensionless form as:

\[ \frac{w_s}{t_s} \frac{\partial W}{\partial \tau} + \frac{v_s}{y_s} \frac{\partial V}{\partial Y} = 0 \]  \hspace{1cm} D.3a

\[ \frac{w_s}{t_s} \frac{\partial W}{\partial \tau} + \frac{w_s^2}{t_s} \frac{\partial W}{\partial Z} + \frac{v_s}{y_s} \frac{w_s}{y_s} V \frac{\partial W}{\partial Y} = \alpha \frac{\partial q^+}{\partial y} + \frac{w_s}{y_s y_s} \frac{\partial^2 W}{\partial y^2} \]  \hspace{1cm} D.3b

\[ \frac{Q_s}{t_s} \frac{\partial T}{\partial \tau} + \frac{w_s}{t_s} \frac{Q_s}{w_s} W \frac{\partial T}{\partial Z} + \frac{v_s}{y_s} \frac{Q_s}{y_s} V \frac{\partial T}{\partial Y} = \frac{\partial Q_s}{y_s y_s} \frac{\partial^2 T}{\partial Y^2} - \frac{w_s}{y_s} \frac{\partial Q_s}{\partial Z} \frac{\partial e_\infty}{\partial Z} \]  \hspace{1cm} D.3c

From the asymptotic solution which was derived in Appendix C, one can see that the maximum values of the vertical velocity
and temperature would be of the order 1 if $Q_s$, $w_s$, and $y_s$ were

\[
Q_s = \frac{\sqrt{2}}{2} q_o L \sqrt{k(R_\infty)}^{-1/4}
\]

\[
w_s = \sqrt{2} \frac{a}{L} R_{\infty}^{3/4}
\]

\[
y_s = \sqrt{2} \frac{L}{(R_\infty)}^{1/4}
\]

Now, by equating the inertial terms in the energy equation to the conduction terms, one obtains

\[
\frac{w_s Q_s}{z_s} = \frac{Q_s}{y_s y_s}
\]

Substituting Equation D.4 into Equation D.5, one obtains

\[
z_s = 2 \sqrt{2} L R_{\infty}^{5/4}
\]

In order that there be no constants in front of the terms of the continuity equation, it is necessary that

\[
\frac{w_s}{z_s} = \frac{v_s}{y_s}
\]

Using Equations D.4 and D.6, one obtains from Equation D.7 that

\[
v_s = \frac{\sqrt{2}}{2} \frac{a}{L} (R_\infty)^{1/4}
\]

Similarly, equate

\[
\frac{Q_s}{t_s} = \frac{a}{y_s y_s}
\]

so that

\[
t_s = \frac{2L^2}{a} (R_\infty)^{-1/2}
\]

Using these scaling parameters, Equation D.3 can be rewritten in dimensionless form as:
\[ \frac{\partial W}{\partial Z} + \frac{\partial V}{\partial Y} = 0 \quad \text{D.11a} \]

\[ \frac{\partial W}{\partial T} = \text{Pr} \, T + \text{Pr} \, \frac{\partial^2 W}{\partial Y^2} \quad \text{D.11b} \]

\[ \frac{\partial T}{\partial T} = \frac{\partial^2 T}{\partial Y^2} - 4W \quad \text{D.11c} \]

Furthermore, the boundary conditions

\[ \frac{\partial \varrho^+}{\partial Y} = -\frac{q_o}{k} \quad \text{at } y = 0 \quad \text{D.12a} \]

\[ w = 0, \quad v = 0 \]

\[ \varrho^+ = 0 \quad \text{as } y \quad \text{D.12b} \]

\[ \frac{\partial \varrho_{\infty}}{\partial Z} = \text{constant} \]

can be written in dimensionless form as

\[ \begin{aligned} \frac{\partial T}{\partial Y} &= -2 \quad \text{at } Y = 0 \\ W &= 0, \quad V = 0 \quad \text{at } Y = 0 \quad \text{D.13a} \\ T &= 0 \quad \text{as } Y \to \infty \quad \text{D.13b} \end{aligned} \]
APPENDIX E.
PLATE COMPUTER PROGRAM

This section contains a definition of the variables used in PLATE, the computer program used to solve the uniformly heated vertical plate in a non-siothermal surrounding problem. This section is followed by a block diagram for the programs required to calculate the temperature and velocity fields, viz., MAIN, TEMP, WVEL and VVEL. Finally, a listing of the complete program is given.

The following table gives a list of the symbols used in PLATE and their definition, followed by the name the subroutine where the variable is calculated, and where it is used. Note that in the table "Input" means the value was supplied by the user while "INPUT" refers to the subroutine INPUT.

### TABLE E.1
PLATE COMPUTER CODE

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>DEFINITION</th>
<th>CALC.</th>
<th>USED</th>
</tr>
</thead>
<tbody>
<tr>
<td>A(J)</td>
<td>Matrix coefficient used in calculating temperature and velocity field</td>
<td>TEMP</td>
<td>WVEL</td>
</tr>
<tr>
<td>AA</td>
<td>Dummy variable used in z-sweep</td>
<td>TEMP</td>
<td>TEMP</td>
</tr>
<tr>
<td>AINT</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>AMULT</td>
<td>Multiplying factor for time increment after every ITI-th iteration</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>B(J)</td>
<td>Matrix coefficient used in calculating temperature and velocity field</td>
<td>TEMP</td>
<td>THOM</td>
</tr>
<tr>
<td>BL(I)</td>
<td>Dummy variable</td>
<td>THOM</td>
<td>THOM</td>
</tr>
<tr>
<td>C(J)</td>
<td>Matrix coefficient used in calculating temperature and velocity field</td>
<td>TEMP</td>
<td>THOM</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>CO</td>
<td>Generalized temperature constant for wall boundary condition</td>
<td>MAINTEMP</td>
<td></td>
</tr>
<tr>
<td>C2</td>
<td>Generalized temperature constant for wall boundary condition</td>
<td>Input TEMP</td>
<td></td>
</tr>
<tr>
<td>CONS</td>
<td>Dummy variable for Ζ-sweep</td>
<td>TEMP TEMP</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>WVEL WVEL</td>
<td></td>
</tr>
<tr>
<td>CORE</td>
<td>(0 Θ0/0Z) Ra+</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td>CTIME</td>
<td>Time that the method of calculating DT is altered</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td>CW1</td>
<td>Generalized velocity constant (optional)</td>
<td>-- --</td>
<td></td>
</tr>
<tr>
<td>CW2</td>
<td>Generalized velocity constant</td>
<td>-- --</td>
<td></td>
</tr>
<tr>
<td>D(J)</td>
<td>Matrix coefficient used in calculating temperature and velocity field</td>
<td>TEMP THOM</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>WVEL WVEL</td>
<td></td>
</tr>
<tr>
<td>DT</td>
<td>Time interval functional field is advanced</td>
<td>Input MAIN</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>MAIN SET</td>
<td></td>
</tr>
<tr>
<td>DTM</td>
<td>Minimum time interval permitted</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td>DTMAX</td>
<td>Maximum time interval permitted</td>
<td>Input MAIN</td>
<td></td>
</tr>
<tr>
<td>DY</td>
<td>Grid spacing in Y-direction</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>TEMP</td>
<td></td>
</tr>
<tr>
<td>DZ</td>
<td>Grid spacing in Z-direction</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td>E(I,J)</td>
<td>Dummy array = TI or WI</td>
<td>THOM THOM</td>
<td></td>
</tr>
<tr>
<td>G(I)</td>
<td>Dummy array</td>
<td>THOM THOM</td>
<td></td>
</tr>
<tr>
<td>HT(I)</td>
<td>Nusselt number</td>
<td>HTTRAN HTTRAN</td>
<td></td>
</tr>
<tr>
<td>HTTRAN</td>
<td>Subroutine for calculating heat transfer coefficients</td>
<td>-- --</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>Array subscript for grid pts. in Z-direction</td>
<td>-- ALL</td>
<td></td>
</tr>
<tr>
<td>ICH</td>
<td>No. of iterations since convergence criteria have last been applied</td>
<td>TEMP MAIN</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>TEMP</td>
<td></td>
</tr>
<tr>
<td>IDT</td>
<td>Control parameter which determines whether DT is increased or decreased</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td>IHT</td>
<td>No. of iterations since heat transfer coefficients were last printed</td>
<td>MAIN MAIN</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>Counter in DØ loop</td>
<td>-- TEMP</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>WVEL</td>
<td></td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>IM</td>
<td>IMAX-1</td>
<td>MAIN</td>
<td>ALL</td>
</tr>
<tr>
<td>IMAX</td>
<td>Maximum grid pts. in z-directional</td>
<td>Input</td>
<td>ALL</td>
</tr>
<tr>
<td>INPUT</td>
<td>Subroutine for imposing arbitrary temperature and velocity field</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>IP</td>
<td>Print iteration counter</td>
<td>--</td>
<td>MAIN</td>
</tr>
<tr>
<td>IPM</td>
<td>Print subroutine called when Ip=IPM</td>
<td>--</td>
<td>MAIN</td>
</tr>
<tr>
<td>IT</td>
<td>Vertical grid point at which T=0</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITER</td>
<td>Iteration number</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITER</td>
<td>Dummy variable = JM</td>
<td>THOM</td>
<td>THOM</td>
</tr>
<tr>
<td>ITEST</td>
<td>Identification no. of test being run</td>
<td>Input</td>
<td>SET</td>
</tr>
<tr>
<td>ITI</td>
<td>No. of iterations after which DT is changed</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITICH</td>
<td>No. of iterations since last change in DT</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITM</td>
<td>Max. no. of iterations (user's estimate of the number of iterations required to reach steady state)</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITPUN</td>
<td>Iteration No. at which T and W and V fields are punched</td>
<td>Input</td>
<td>INPUT</td>
</tr>
<tr>
<td>J</td>
<td>Array subscript for grid pts. in Y-direction</td>
<td>--</td>
<td>ALL</td>
</tr>
<tr>
<td>JJ</td>
<td>Counter in DØ loop</td>
<td>--</td>
<td>TEMP</td>
</tr>
<tr>
<td>JM</td>
<td>JMAX-1</td>
<td>MAIN</td>
<td>ALL</td>
</tr>
<tr>
<td>JT(J)</td>
<td>Heading numbers in output subroutines</td>
<td>MAIN</td>
<td>Output</td>
</tr>
<tr>
<td>JMAX</td>
<td>Max. no. of grid pts. in Y-direction+1</td>
<td>Input</td>
<td>ALL</td>
</tr>
<tr>
<td>L</td>
<td>Counter in DØ loop</td>
<td>--</td>
<td>TEMP</td>
</tr>
<tr>
<td>LCH</td>
<td>Counter for changing time control parameters</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>M</td>
<td>Counter in DØ loop</td>
<td>--</td>
<td>TEMP</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>MAIN</td>
<td>Main program: controls logic for program</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>N</td>
<td>Counter in DO loop</td>
<td>THOM</td>
<td>THOM</td>
</tr>
<tr>
<td>NCHA</td>
<td>Parameter controlling time interval</td>
<td>--</td>
<td>MAIN</td>
</tr>
<tr>
<td>NCONV</td>
<td>Logical variable for testing convergence of temperature field</td>
<td>TEMP</td>
<td>TEMP</td>
</tr>
<tr>
<td>NN</td>
<td>Logical variable for testing stability of finite difference equations</td>
<td>TEMP</td>
<td>TEMP</td>
</tr>
<tr>
<td>NN</td>
<td>Dummy variable = JM-1</td>
<td>THOM</td>
<td>THOM</td>
</tr>
<tr>
<td>NRUN</td>
<td>Total number of tests per computer run</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NRUNS</td>
<td>Number of tests being computed</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>NW</td>
<td>Logical variable for testing convergence of velocity field</td>
<td>WVEL</td>
<td>WVEL</td>
</tr>
<tr>
<td>Pr</td>
<td>Prandtl number</td>
<td>Input</td>
<td>WVEL</td>
</tr>
<tr>
<td>PRINTT</td>
<td>Subroutine for printing temperatures</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>PRINTV</td>
<td>Subroutine for printing horizontal velocities</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>PRINTW</td>
<td>Subroutine for printing vertical velocities</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>PUNCH</td>
<td>Subroutine for punching out temperature and velocity fields</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>RAM</td>
<td>Modified Rayleigh number or $= 0.0625$ when a linear temperature is present in the core</td>
<td>Input</td>
<td>Output</td>
</tr>
<tr>
<td>RAMQ</td>
<td>$(Ra^+)^{1/4}$</td>
<td>MAIN</td>
<td>TEMP</td>
</tr>
<tr>
<td>RØY</td>
<td>DT/DY</td>
<td>MAIN</td>
<td>ALL</td>
</tr>
<tr>
<td>RØ2Y</td>
<td>DT/DY/DY</td>
<td>MAIN</td>
<td>ALL</td>
</tr>
<tr>
<td>RØZ</td>
<td>DT/DZ</td>
<td>MAIN</td>
<td>ALL</td>
</tr>
<tr>
<td>RYZ</td>
<td>DY/DZ</td>
<td>VVEL</td>
<td>VVEL</td>
</tr>
<tr>
<td>SET</td>
<td>Subroutine for placing headings on output</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALO.</td>
<td>USED</td>
</tr>
<tr>
<td>----------</td>
<td>---------------------------------------------------------------------------</td>
<td>--------------------</td>
<td>---------------</td>
</tr>
<tr>
<td>T(I,J)</td>
<td>Temperature field array</td>
<td><strong>TEMP</strong></td>
<td><strong>ALL</strong></td>
</tr>
<tr>
<td>TCRIT</td>
<td>Convergence criteria for temp. field</td>
<td>Input</td>
<td><strong>TEMP</strong></td>
</tr>
<tr>
<td>TEMP</td>
<td>Subroutine for calculating temperatures</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>THOM</td>
<td>Subroutine for inverting and solving tri-diagonal matrix</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>TI(I,J)</td>
<td>Advanced temperature field array</td>
<td><strong>TEMP</strong></td>
<td><strong>TEMP</strong></td>
</tr>
<tr>
<td>TIARR(I,J)</td>
<td>Time increment control parameters</td>
<td>--</td>
<td><strong>MAIN</strong></td>
</tr>
<tr>
<td>TIMAX</td>
<td>Maximum time (user's estimate of the time required to reach steady state)</td>
<td>Input</td>
<td><strong>TEMP</strong></td>
</tr>
<tr>
<td>TIME</td>
<td>Dimensionless time to which temperature and velocities have advanced</td>
<td><strong>MAIN</strong></td>
<td><strong>SET</strong></td>
</tr>
<tr>
<td>TINF(I)</td>
<td>Temperature at infinity</td>
<td><strong>MAIN</strong></td>
<td><strong>MAIN</strong></td>
</tr>
<tr>
<td>TS</td>
<td>TSLØPE</td>
<td><strong>TEMP</strong></td>
<td><strong>TEMP</strong></td>
</tr>
<tr>
<td>TSLØPE</td>
<td>Slope of linear temperature gradient in core ($Y = \infty$)</td>
<td>Input</td>
<td><strong>MAIN</strong></td>
</tr>
<tr>
<td>TW(I)</td>
<td>Wall temperature</td>
<td><strong>TEMP</strong></td>
<td><strong>TEMP</strong></td>
</tr>
<tr>
<td>V(I,J)</td>
<td>Horizontal velocity field</td>
<td><strong>VVEL</strong></td>
<td><strong>ALL</strong></td>
</tr>
<tr>
<td>VVEL</td>
<td>Subroutine for calculating horizontal velocities</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>W(I,J)</td>
<td>Vertical velocity field</td>
<td><strong>WVEL</strong></td>
<td><strong>ALL</strong></td>
</tr>
<tr>
<td>W(I)</td>
<td>Variable</td>
<td><strong>THOM</strong></td>
<td><strong>THOM</strong></td>
</tr>
<tr>
<td>WCRIT</td>
<td>Convergence criteria for velocity field</td>
<td><strong>WVEL</strong></td>
<td><strong>WVEL</strong></td>
</tr>
<tr>
<td>WI(I,J)</td>
<td>Advanced vertical velocity field</td>
<td><strong>WVEL</strong></td>
<td><strong>WVEL</strong></td>
</tr>
<tr>
<td>WVEL</td>
<td>Subroutine for calculating vertical velocities</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>YMAX</td>
<td>Maximum distance from plate ($Y_0$)</td>
<td>Input</td>
<td><strong>MAIN</strong></td>
</tr>
<tr>
<td>Z</td>
<td>Height</td>
<td><strong>MAIN</strong></td>
<td><strong>MAIN</strong></td>
</tr>
<tr>
<td>ZMAX</td>
<td>Maximum height of plate</td>
<td>Input</td>
<td><strong>MAIN</strong></td>
</tr>
<tr>
<td>ZØ</td>
<td>Height at which $T \neq 0$</td>
<td><strong>MAIN</strong></td>
<td><strong>MAIN</strong></td>
</tr>
</tbody>
</table>
STARI

YES

EAPUNCH

OUTPUT

I

YES

AVE

YES

HAS THE LIMIT ON THE NUMBER OF ITERATIONS BEEN REACHED

READ INPUT physical & logical parameters

INITIALIZE TIME & ITERATION COUNTERS

INITIALIZE LOGICAL VARIABLES

COMPUTE SYSTEM CONSTANTS

INITIALIZE ALL ARRAYS TO 0.0

SPECIFY CORE TEMPERATURE

IS AN ARBITRARY TEMPERATURE & VELOCITY FIELD TO BE ADDED?

ADD ARBITRARY VELOCITY AND TEMPERATURE FIELDS

HAVE ALL THE TESTS BEEN RUN?

PUNCH OUTPUT

PUNCH OUTPUT?

HAS SOLUTION CONVERGED?

CHANGE TIME STEP

SHOULD TIME STEP BE CHANGED?

PRINT TEMPERATURE AND VELOCITY FIELDS

ADVANCE TIME

ADVANCE TEMPERATURES & check for convergence & check for stability

ADVANCE VELOCITY field & check for convergence

COMPUTE TEMPERATURE NUMBER

SHOULD TEMPERATURE NUMBER BE COMPUTED?

COMPUTE NUSSELT NUMBER

SHOULD NUSSELT NUMBER BE COMPUTED?

FIGURE E.1 BLOCK DIAGRAM – PLATE PROGRAM

NO

NO

NO

NO

YES

YES

YES

YES

NO

NO

NO

YES

YES

YES

YES

NO
C SOLUTION OF THE NATURAL CONVECTION BOUNDARY LAYER EQUATIONS FOR THE
C CASE OF A UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUND-
CINGS THIS PROGRAM USES AN IMPLICIT ALTERNATING DIRECTION FINITE
C DIFFERENCE SCHEME FOR SOLVING THE PERTINENT EQUATIONS
C
C DATA INPUT
C NOTE
C
C THE FIRST CARD GIVES THE NUMBER OF TESTS PER COMPUTER RUN (15)
C FOR EACH TEST DATA IS SUPPLIED IN THE FOLLOWING ORDER
C CARD 1- TEST NUMBER(15)
C CARD 2- PRANDTL NUMBER AND RAYLEIGH NUMBER(F10.0,D10.0)
C CARD 3- ITERATION INTERVAL BEFORE PRINT IS GIVEN(15)
C CARD 4- SIZE OF MATRIX(215)
C CARD 5- DIMENSIONS OF MATRIX(2F10.0)
C CARD 6- TEMPERATURE AND VELOCITY CONVERGENCE CRITERIA, TIME AT WHICH
C CONVERGENCE IS TO BE TESTED, MAXIMUM ITERATIONS(3F10.0,15)
C
C CARDS 7 AND 8 ALLOW FOR A VARIATION IN THE TIME STEP DURING THE
C CALCULATIONS. THE TIME INCREMENTS ARE MULTIPLIED BY A SPECIFIED
C AMOUNT AT SPECIFIED INTERVALS UNTIL IT REACHES THE MAXIMUM TIME
C INCREMENT - THEN IT IS DIVIDED BY THIS SPECIFIED AMOUNT UNTIL THE
C MINIMUM INCREMENT IS REACHED.
C
C CARD 7- NO OF CHANGES OF THE MAXIMUM TIME INTERVAL, NO. OF ITERATIONS
C AT WHICH TIME INTERVAL IS CHANGED(215)
C CARD 8 TO NCHA+8- MAXIMUM TIME INTERVAL, STARTING TIME INTERVAL,
C MULTIPLYING FACTOR FOR INTERVAL, TIME AT WHICH TIME INTERVAL
C IS CHANGED (4F10.0)
C CARD 9- READ IN C2 AND CW2(2F10.0)
C CARD 10- READ IN CORE TEMPERATURES, GRID HEIGHT WHERE T.GT.0, SLOPE OF
C THE TEMPERATURE GRADIENT (15,F10.0)
C CARD 11- IS INPUT DATA FURNISHED - 1-YES, 2-NO (11)
C CARD 12- ITERATION NUMBER OF INPUT, TIME (15,F10.0)
C CARD 13- OUTPUT DATA FROM PUNCH SUBROUTINE (INPUT CARDS)
C AFTER OUTPUT DATA FROM PREVIOUS RUNS GIVE ITERATION NUMBERS AT
C WHICH THE PUNCH SUBROUTINE WILL BE CALLED (15)
C END OF DATA INPUT
C
C MAIN PROGRAM LOGISTICS
C
C IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION TIARR(10,4)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,TI,W,WI,TI
COMMON TW
COMMON ROY,RODY,DY,DZ,DT,RO2Y,RYZ,TCRIT,WCRIT,ZMAX,YMAX,PR,TIME
COMMON RAM,RANQ,TIMAX,C0,C1,C2,CW1,CW2
COMMON JT
COMMON JMAX,JMAX,NCONV,NW,JM,IM,ITEST,NN,ITER
C READ IN INPUT ACCORDING TO SEQUENCE ABOVE
DO 705 I=1,50
   JT(I)=1
705 CONTINUE
NRUNS=1
READ(5,550)NRUN
550 FORMAT(15)
551 READ(5,100)ITEST,PR,PRAM
100 FORMAT(15/F10.0,D10.0)
   IP=2
READ(5,51)IPM
51 FORMAT(15)
ICH=1
ITICH=1
IHT=1
READ(5,101)IMAX,JMAX,ZMAX,YMAX,TCRIT,WCRIT,TIMAX,ITM
101 FORMAT(2I5/2F10.0/3F10.0/15)
READ(5,103)NCHA,ITI
103 FORMAT(215)
DO 104 I=1,NCHA
   READ(5,105)TIARR(I,J),J=1,4
105 FORMAT(4F10.0)
104 CONTINUE
READ(5,102)C2,CW2
102 FORMAT(2F10.0)
C CALCULATION OF CONSTANTS USED IN SUBROUTINES
LCH=1
DT=TIARR(1,2)
AMULT=TIARR(1,3)
DTMAX=TIARR(1,1)
CTIME=TIARR(1,4)
D=DT
IDT=1
C0=-1.0+C2
C1=1.0-2.0*C2
IM=IMAX
JM=JMAX
AINT=IM
DZ=ZMAX/AINT
AINT=JM
DY=YMAX/AINT
RAMQ=DSQRT(DSQRT(RAM))
C DATA INITIALIZATION
TIME=0.0
ITER=0
DO 600 I=1,50
   TW(I)=0.0
DO 600 J=1,50
   TI(I,J)=0.0D+00
   W(I,J)=0.0D+00
   V(I,J)=0.0D+00
   WI(I,J)=0.0D+00
   T(W(I,J))=0.0D+00
600 CONTINUE
C SPECIFICATION OF CORE TEMPERATURE
READ(5,350) IT,TSLOPE
350 FORMAT(5,F10.0)
C INITIALIZE THE CORE TEMPERATURE FIELD, I.E., SET THE CORE BOUNDARY
Z0=0.00+00
DO 300 I=1,IMAX
IF(I-IT)301,302,301
302 Z0=IT
Z0=(Z0-1.0)*DZ
Z=Z0
301 TINF(I)=TSLOPE*(Z-Z0)
IF(I.LT.IT)GO TO 300
Z=Z+DZ
300 CONTINUE
C CORE=TSLOPE*RAM
WRITE(6,500)Z0,CORE
500 FORMAT(1CORE TEMPERATURE FIELD'// HEIGHT AT WHICH THE CORE TEMPE
IRATURE VARIES FROM 0 =","F10.5"," THE CORE RA NO IS","D15.4)
DO 501 I=1,IMAX
WRITE(6,502)I,TINF(I)
501 CONTINUE
C IMPOSE BOUNDARY CONDITIONS, I.E., CONSTANT HEAT FLUX , AND NO-SLIP CONDI-
C TION
C INPUT DATA FROM ANY PREVIOUS RUN CAN BE ADDED HERE
CALL INPUT
READ(5,950)ITPUN
950 FORMAT(I5)
C LOOP FOR TIME ADVANCEMENT
C OPTIMIZE THE TIME INTERVAL
400 CONTINUE
IF(DT.GT.DTMAX)DT=DTMAX+0.001
IF(DT.LT.DTM)DT=DTM-0.001
TIME=TIME+DT
ITER=ITER+1
ROY=DT/DY
ROZ=DT/DZ
RO2YzROY/DY
CALL TEMP(ICH)
C CHECK FOR INSTABILITY - JOB IS TERMINATED IF TEMPERATURE IS UNSTABLE
IF(NN.EQ.2)GO TO 401
CALL WVEL
CALL VVEL
IF(IP.LT.IPM)GO TO 50
CALL SET
CALL PRINTT
CALL PRINTW
CALL PRINTV
IP=0
50 IP=IP+1
C CALCULATE NUSSELT NUMBERS FOR PLATE
IF(IHT.LT.3)GO TO 900
CALL HTTRAN
IHT=1
2 GO TO 901
900 IHT=IHT+1
901 CONTINUE
IF(ITER-ITPUN)951,952,951
952 READ(5,950)ITPUN
951 CONTINUE
C CHECK FOR PROGRAM STATUS, I.E., FOR CONVERGENCE AND ITERATIONS
IF(NW.EQ.1)GO TO 402
IF(TIME.GT.CTIME)GO TO 802
IF(ITICH.EQ.ITU)GO TO 700
ITICH=ITICH+1
GO TO 701
C RECALCULATE THE TIME INCREMENT IF NECESSARY
700 IF(IDT.EQ.1)GO TO 800
801 IF(DT.LT.DTM)GO TO 800
   IF(IDT.EQ.1)DT=DT+2.0*DTM/3.0
   DT=DT-DMT*AMULT
   IDT=2
   ITICH=1
   GO TO 701
800 IF(DT.GT.DTMAX)GO TO 801
   IF(IDT.EQ.2)DT=DT+2.0*DTM/3.0
   DT=DT+DTM*AMULT
   IDT=1
   ITICH=1
   GO TO 701
801 IF(LCH.EQ.NCHA)GO TO 801
   LCH=LCH+1
   DT=TIARR(LCH,2)
   DTM=DT
   AMULT=TIARR(LCH,3)
   DTMAX=TIARR(LCH,1)
   CTIME=TIARR(LCH,4)
   IDT=1
   GO TO 800
C CHECK PROGRAM STATUS
701 IF(ITER.LT.ITM)GO TO 400
402 CALL PUNCH
401 CONTINUE
   IF(NRUNS.EQ.NRUN)GO TO 552
   NRUNS=NRUNS+1
   GO TO 551
552 STOP
END

SUBROUTINE TEMP(ICH)
C
C ADVANCE TEMPERATURE FIELD
C
IMPLICIT REAL*(A-H,O-Z)
DIMENSION A(50),B(50),C(50),D(50)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,V,W,WI,TI
COMMON TW
**COMMON ROY, ROZ, DY, DZ, DT, RO2Y, RY2, TCRIT, WCRLT, ZMAX, YMAX, PR, TIME**
**COMMON RAM, RAMQ, TIMAX, CO, CI, C2, CW1, CW2**
**COMMON JT**
**COMMON IMAX, JMAX, NCONV, NW, JM, IM, ITEST, NN, IITER**

**C ADVANCE TEMPERATURES - IMPLICIT IN Y-DIRECTION**

NCONV=2
TS=TSLOPE
JM=JMAX-1
JMM=JM-1
DO 87 I=1, IMAX
DO 88 J=1, JM
A(J)=-RO2Y
B(J)=I.0-V(I,J)*ROY+2.0*RO2Y
C(J)=V(I,J)*ROY-RO2Y
IF(I.EQ.1)D(J)=T(1,J)-ROZ*W(1,J)*T(I#J)-W(1,J)*TS*DT
IF(I.EQ.1)GO TO 88
D(J)=T(I,J)-ROZ*W(I,J)*(T(I,J)-T(I-1,J))-W(I,J)*TS*DT
88 CONTINUE
B(I)=CO*B(I)-A(I)*C1
C(I)=C0*C(I)-A(I)*C2
D(I)=CO*D(I)+A(I)/RAMQ*DY
CALL THOM(A,B,C,D,T1,I,JM)
87 CONTINUE
DO 102 I=1, IMAX
TWfI)=(-DY/RAMQ-CI*T(I,1)-C2*T(I,2))/CO
102 CONTINUE

**C ADVANCE TEMPERATURES - IMPLICIT IN Z-DIRECTION**

DO 600 I=1, IMAX
DO 600 J=1, JMAX
AA=RO2Y*(TW(I)-2.0*TI(I,1)+TI(I,2))
GO TO 602
601 AA=R02Y*(TI(I,J-1)-2.0*TIfIJ)+TIf(I+1,J)
602 CONS=T(I,J)-ROY*V(I,J)*(TI(I,J+1)-TI(I,J))-W(I,J)*TS*DT+AA
IF(I.GT.1)CONS=CONS+W(I,J)*ROY*T(I-1,J)
T(I,J)=CONS/(1.0+W(I,J)*ROY)
600 CONTINUE
DO 603 I=1, IMAX
603 TW(I)=(-DY/RAMQ-CI*T(I,1)-C2*T(I,2))/CO

**C CHECK FOR STABILITY AND CONVERGENCE IS MADE EVERY THIRD ITERATION**

NN=1
ICH=ICH+1
IF(ICH.LT.4)GO TO 800
ICH=1

**C CHECK FOR STABILITY**

DO 700 I=1, IMAX
DO 700 J=1, JMAX
IF(NN.EQ.2) GO TO 700
IF(T(I,J).GT.T(I,1))GO TO 701
700 CONTINUE
WRITE(6,703)
703 FORMAT(' UNSTABLE SOLUTION - T.GT.1')
704 CALL PRINTT
NN=2
700 CONTINUE
IF(NN.EQ.2) GO TO 401

**C CONVERGENCE CHECK**
IF (TIME .GT. TIMAX) GO TO 402
NCONV = 2
GO TO 401

402 NCONV = 1
DO 101 II = 1, IMAX
 I = IMAX + 1 - II
 DO 101 JJ = 1, JMAX
 J = JMAX + 1 - JJ
 IF (NCONV .EQ. 2) GO TO 101
 IF (DABS (T(I, J)) / T(I, 1) .LT. 1.0D-05) GO TO 101
 C IF THE TEMPERATURE TESTED IS EQUAL TO ZERO, THE CONVERGENCE LIMITS ARE
 C NOT APPLIED SINCE DIVISION BY ZERO WOULD BE INVOLVED
 IF (T(I, J) .EQ. 0.0) GO TO 101
 IF (DABS (T(I, J) - T(I-1, J)) / T(I, J) .LT. TCRIT) GO TO 101
 C IF SOLUTION IS NON-CONVERGENT NCONV IS -2 - OTHERWISE IT IS 1
 NCONV = 2
101 CONTINUE
IF (NCONV .EQ. 2) GO TO 401
800 CONTINUE
401 RETURN
END

SUBROUTINE WVEL

IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION A(50), B(50), C(50), D(50)
DIMENSION T(50,50), W(50,50), V(50,50), TI(50,50), WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON ISLOPE
COMMON TINF
COMMON T, V, W, WI, TI
COMMON TW
COMMON ROY, ROZ, DY, DZ, DT, RO2Y, RYZ, TCRIT, WCRIT, ZMAX, YMAX, PR, TIME
COMMON RAM, RAMQ, TIMAX, CO, CL, C2, CW1, CW2
COMMON JT
COMMON IMAX, JMAX, NCONV, NW, JM, IM, ITEST, NN, ITER

C ADVANCE VELOCITY FIELD

JM = JMAX - 1
DO 87 J = 1, JM
 A(J) = -RO2Y*PR
 B(J) = 1.0 - V(I, J) * ROY + 2.0 * RO2Y * PR
 C(J) = V(I, J) * ROY - RO2Y * PR
 IF (I .EQ. 1) GO TO 88
88 CONTINUE
CALL THOM(ABCDWIIJM)
87 CONTINUE

C ADVANCE VELOCITY - IMPLICIT IN THE Z-DIRECTION

DO 600 I = 1, IMAX
 DO 600 J = 1, JM
 A(I) = -RO2Y*PR
 B(I) = 1.0 - V(I, J) * ROY + 2.0 * RO2Y * PR
 C(I) = V(I, J) * ROY - RO2Y * PR
 IF (I .EQ. 1) GO TO 88
88 CONTINUE
CALL THOM(ABCDWIIJM)
87 CONTINUE

C ADVANCE VELOCITY - IMPLICIT IN THE Y-DIRECTION

DO 87 J = 1, JM
 A(J) = -RO2Y*PR
 B(J) = 1.0 - V(I, J) * ROY + 2.0 * RO2Y * PR
 C(J) = V(I, J) * ROY - RO2Y * PR
 IF (I .EQ. 1) GO TO 88
88 CONTINUE
CALL THOM(ABCDWIIJM)
IF(J.GT.1)GO TO 601
AA=RO2Y*PR*(-2.0*WI(I,1)+WI(I,2))
GO TO 602
601 AA=PR*RO2Y*(WI(I,J-1)-2.0*WI(I,J)+WI(I,J+1))
602 CONS=WI(I,J)+PR*T(I,J)*DT-V(1,J)*ROY*(WI(I,J+1)-WI(I,J))+AA
IF(I.GT.1)CONS=CONS+WI(I-1,J)*ROZ
WI(I,J)=CONS/(1.0+ROZ*W(I,J))
600 CONTINUE
C CONVERGENCE CHECK
C NOTE VELOCITY CONVERGENCE WILL BE CHECKED ONLY IF TEMPERATURE
C CONVERGES
NW=2
*IF(NCONV.EQ.2) GO TO 200
NW=1
DO 102 JJ=1,JMAX
J=JMAX+1-JJ
DO 102 II=1,IMAX
I=IMAX+1-II
IF(NW.EQ.2) GO TO 102
IF(DABS(W(I,J)).LT.1.0D-05)GO TO 102
C IF THE VELOCITY POINT TESTED IS EQUAL TO ZERO, THE CONVERGENCE LIMITS
C ARE NOT TESTED SINCE DIVISION BY ZERO WOULD BE INVOLVED
IF(W(IJ).EQ.0.0)GO TO 102
IF(SABS(W(I,J)-WI(I,J))/WtIJ)).LT.WCRIT)GO TO 102
C IF SOLUTION IS NOT CONVERGENT NW=2
NW=2
102 CONTINUE
IF(NW.EQ.2)GO TO 401
GO TO 401
200 CONTINUE
401 RETURN
END

SUBROUTINE VVEL
C
C CALCULATE HORIZONTAL VELOCITIES
C
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T, V, W, WI, TI
COMMON TW
COMMON ROY, ROZ, DY, DZ, DT, RO2Y, RYZ, TCRT, WCRIT, ZMAX, YMAX, PR, TIME
COMMON RAM, RAMQ, TIMAX, CO, CI, C2, CW1, CW2
COMMON JT
COMMON IMAX, JMAX, NCONV, NW, JM, IM, ITEST, NN, ITER
RYZ=DY/DZ
V1,1=-RYZ*W(1,1)
DO 100 I=2,IMAX
VI(I,1)=-RYZ*(WI(I,1)-WI(I-1,1))
100 CONTINUE
DO 101 J=2,JMAX
V(1,J) = V(1,J-1) - RYZ*W(1,J)

101 CONTINUE
   DO 102 I=2,IMAX
   DO 102 J=2,JMAX
   V(I,J) = V(I,J-1) - RYZ*(W(I,J) - W(I-1,J))
   102 CONTINUE
   IF(NW.EQ.2)GO TO 103
C PRINT STEADY STATE RESULTS
   CALL SET
   CALL PRINTT
   CALL PRINTW
   CALL PRINTV
   CALL HYTRAN
103 RETURN
END

SUBROUTINE SET

C
C SET UP HEADINGS
C
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,V,W,WT
COMMON TW
COMMON ROY,ROZ,DY,DZ,DT,RO2Y,RYZ,TCRIT,WCRIT,ZMAX,ymax,PR,TIME
COMMON RAM,Q,TIMAX,CP,C1,C2,CW1,CW2
COMMON JT
COMMON IMAX,JMAX,NCONV,NW,JM,IM,ITEST,NN,ITER
WRITE(*,100)ITEST,PR,RAM,IMAX,JMAX,ZMAX,ymax,ITER,TIMEDT
100 FORMAT(*,ITEST,' PRANDTL NUMBER = ',F7.0,5X,' MODIFIED RAYLEIGH NUMBER = ',D8.2,' MATRIX SIZE = ',I3,' BY ',I3,' MAXIMUM HEIGHT = ',F5.1,' MAXIMUM WIDTH = ',F5.1,' ITERATION = ',I5,' TIME = ',D10.4,' TIME INCREMENT IS = ',D10.4) RETURN
END
SUBROUTINE PRINTW

PRINT VERTICAL VELOCITY FIELD

IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,V,W,VI,TI
COMMON TW
COMMON ROY,ROY2,DY,DZ,DT,ROY2Y,ROYZ,TCRIT,WCRIT,ZMAX,YMAX,PR,TIME
COMMON RAM,RAQ,TIMAX,C0,C1,C2,CW1,CW2
COMMON JT
COMMON IMAX,JMAX,NCONV,NW,JM,IM,ITEST,NNITER
IF(NCONV.EQ.2) GO TO 400
IF(NW.EQ.2) GO TO 400
WRITE(6,100)
100 FORMAT(* VELOCITY FIELD HAS CONVERGED'//)
GO TO 401
400 WRITE(6,101)
101 FORMAT(* VELOCITY FIELD HAS NOT CONVERGED'//)
401 WRITE(6,102)
102 FORMAT(* VERTICAL VELOCITY FIELD *'//, I,J='6X,'10X,'2X,'10X,'3X,'10X,'4X,'10X,'5X,'10X,'6X,'10X,'7X,'10X,'8X,'10X,'9X,'10X,'10X')
DO 200 I=1,IMAX
WRITE(6,103)I,(W(I,J),J=1,50)
103 FORMAT(14X,10D11.4)
200 CONTINUE
IF(JM.LT.11)GO TO 300
WRITE(6,301)(JT(I),I=1,10)
301 FORMAT(* I,J='6X,'10X,'10X,'10X,'10X,'10X,'10X,'10X,'10X,'10X,'10X')
DO 302 I=1,IMAX
WRITE(6,303)I,(W(I,J),J=1,20)
303 FORMAT(* I,J='6X,'10X,'10X,'10X,'10X,'10X,'10X,'10X,'10X,'10X,'10X')
302 CONTINUE
IF(JM.LT.21)GO TO 300
WRITE(6,301)(JT(I),I=1,30)
304 CONTINUE
IF(JM.LT.31)GO TO 300
WRITE(6,301)(JT(I),I=1,40)
305 CONTINUE
IF(JM.LT.41)GO TO 300
WRITE(6,301)(JT(I),I=1,50)
306 CONTINUE
RETURN
END
SUBROUTINE PRINTT

C PRINT TEMPERATURE FIELD
C
C IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION T(50,50), W(50,50), V(50,50), TI(50,50), WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T, W, WI, TI
COMMON TW
COMMON ROY, ROZ, DYZ, DT, RO2Y, RO2Z, TCRT, TCRIT, ZMAX, YM, PR, TIME
COMMON RAM, RAMQ, TIMAX, CO, CI, C2, CW1, CW2
COMMON JT
COMMON IMAX, JMAX, NCONV, NW, JM, IM, INTEST, NN, ITER
IF (NCONV .EQ. 2) GO TO 400
WRITE (6, 100)
100 FORMAT(' TEMPERATURE FIELD HAS CONVERGED')
GO TO 401
400 WRITE (6, 101)
101 FORMAT(' TEMPERATURE FIELD HAS NOT CONVERGED')
DO 500 I = 1, IMAX
WRITE (6, 102) I, TW(I)
102 FORMAT(' WALL TEMPERATURE', * I = 1, 10X, D11.4)
500 CONTINUE
WRITE (6, 103)
103 FORMAT('# TEMPERATURE FIELD', * I = 1, 10X, D11.4)
DO 200 I = 1, IMAX
WRITE (6, 301) I, (T(I, J), J = 1, 10)
200 CONTINUE
IF (JMAX .LT. 11) GO TO 300
WRITE (6, 302) (JT(I), I = 1, 11)
302 FORMAT('# J = 1, 10X, D11.4', * I = 1, 11, 9X)
DO 303 I = 1, IMAX
WRITE (6, 301) I, (T(I, J), J = 1, 11)
303 CONTINUE
IF (JMAX .LT. 21) GO TO 300
WRITE (6, 302) (JT(I), I = 1, 21)
304 CONTINUE
IF (JMAX .LT. 31) GO TO 300
WRITE (6, 301) (JT(I), I = 1, 31, 40)
305 CONTINUE
IF (JMAX .LT. 41) GO TO 300
WRITE (6, 301) (JT(I), I = 1, 41, 50)
306 CONTINUE
RETURN
END
SUBROUTINE PRINTV
C
C PRINT HORIZONTAL VELOCITY FIELD
C
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,V,W,WI,TI
COMMON TW
COMMON ROY,ROZ,DY,DZ,DT,RO2Y,RYZ,TCRIT,WCRIT,ZMAX,YMAX,PR,TIME
COMMON RAM,RAMQ,TIMAX,CO,C1,C2,CW1,CW2
COMMON JT
COMMON IMAX,JMAX,NCONV,NW,JM,IM,ITER
WRITE(6,102)
102 FORMAT(' HORIZONTAL VELOCITY FIELD'//I,4X,J=',6X,1I1,10X,2I4,
110X,3I4,10X,5I4,10X,6I4,10X,7I4,10X,8I4,10X,9I4,10X,10I4)DO
200 I=1,IMAX
WRITE(6,103)I,(V(I,J),J=1,10)
103 FORMAT(' ',14,5X,10D11.4)
200 CONTINUE
IF(JMAX.LT.11)GO TO 300
WRITE(6,301)(JT(I),I=11,20)
301 FORMAT(' I J=',5X,'1I12,9X)DO
302 I=1,IMAX
WRITE(6,303)I,(V(I,J),J=11,20)
303 FORMAT(' ',14,5X,10D11.4)
302 CONTINUE
IF(JMAX.LT.21)GO TO 300
WRITE(6,301)(JT(I),I=21,30)
DO 304 I=1,IMAX
WRITE(6,303)I,(V(I,J),J=21,30)
304 CONTINUE
IF(JMAX.LT.31)GO TO 300
WRITE(6,301)(JT(I),I=31,40)
DO 305 I=1,IMAX
WRITE(6,303)I,(V(I,J),J=31,40)
305 CONTINUE
IF(JMAX.LT.41)GO TO 300
WRITE(6,301)(JT(I),I=41,50)
DO 306 I=1,IMAX
WRITE(6,303)I,(V(I,J),J=41,50)
306 CONTINUE
300 RETURN
END
SUBROUTINE INPUT

READ IN INITIAL CONDITIONS WHEN DIFFERENT FROM ZERO

IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,V,W,WTI
COMMON TW
COMMON ROY,ROZ,ROZ,DT,RO2Y,RYZ,TCRIT,WCRIT,ZMAX,YMAX,PR,TIME
COMMON RAM, QMQ,TIMAX,CO,C1,C2,CW1,CW2
COMMON JT
COMMON IMAX, JMAX, NCONV, NW, JM, IM, ITEST, NN, ITER

INPUT DATA FURNISHED, 1-YES, 2-NO
READ(5,100)IN
100 FORMAT(II)
IF(IN.EQ.2) GO TO 400
READ(5,101)ITERTIME
101 FORMAT(15,D15.7)
DO 200 L=1,IMAX
DO 200 M=1,JMAX
READ(5,102)IJ,T(I,J),W(I,J),V(I,J)
102 FORMAT(212,3D20.13)
TI(I,J)=T(I,J)
WI(I,J)=W(I,J)
200 CONTINUE
400 RETURN
END

SUBROUTINE PUNCH

PUNCH OUT INTERMEDIATE DATA
THIS ROUTINE IS COMPATIBLE WITH THE INPUT SUBROUTINE

IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)
DIMENSION TINF(50)
DIMENSION TW(50)
DIMENSION JT(50)
COMMON TSLOPE
COMMON TINF
COMMON T,V,WTI
COMMON TW
COMMON ROY,ROZ,ROZ,DT,RO2Y,RYZ,TCRIT,WCRIT,ZMAX,YMAX,PR,TIME
COMMON RAM, QMQ,TIMAX,CO,C1,C2,CW1,CW2
COMMON JT
COMMON IMAX, JMAX, NCONV, NW, JM, IM, ITEST, NN, ITER
WRITE(7,200)ITERTIME
200 FORMAT(15,D15.7)
DO 300 I=1,IMAX
DO 300 J=1,JMAX
IF(DABS(T(I,J)).GT.0.1D-70)GO TO 300
T(I,J)=0.1D-70
300 IF(DABS(W(I,J)).GT.0.1D-70)GO TO 301
W(I,J)=0.1D-70
301 IF(DABS(V(I,J)).GT.0.1D-70)GO TO 302
V(I,J)=0.1D-70
302 RETURN
SUBROUTINE HTTRAN

C  
C CALCULATE THE HEAT TRANSFER COEFFICIENTS  
C  
IMPLICIT REAL*8 (A-H,O-Z)  
DIMENSION T(50,50),W(50,50),V(50,50),TI(50,50),WI(50,50)  
DIMENSION HT(50)  
DIMENSION TINF(50)  
DIMENSION TW(50)  
DIMENSION JT(50)  
COMMON TSLOPE  
COMMON TINF  
COMMON T,V,W,WI,TTI  
COMMON TW  
COMMON ROY,ROZ,DY,DZ,DT,RO2Y,RYZ,TCRIT,WCRIT,ZMAX,YMAX,PR,TIME  
COMMON RAM,RAQ,TIMAX,CO,C1,C2,CW1,CW2  
COMMON JT  
COMMON YMAX,JMAX,NCONV,NW,JM,IM,IINST,NN,ITER  
WRITE(6,400)TIME

400 FORMAT(' HEAT TRANSFER COEFFICIENTS ALONG CONSTANT HEAT FLUX WALL'  
1/" TIME='D15.5//' I'5X,'NU (T-TINF)' )

DO 100 I=1,IM  
HT(I)=1.0/TW(I)

100 CONTINUE  
DO 101 I=1,IM  
WRITE(6,401)I,HT(I)

401 FORMAT(' I','D15.5')  
101 CONTINUE  
RETURN  
END
SUBROUTINE THOM(A,B,C,D,E,LL,ITER)

C

C SOLVE THE TRIDIAGONAL MATRIX

C

IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION A(50),B(50),C(50),D(50),E(50,50),W(50),BL(50),G(50)
W(1)=B(1)
BL(1)=C(1)/W(1)
G(1)=D(1)/W(1)
NN=ITER-1
DO 1 N=2,NN
   W(N)=B(N)-A(N)*BL(N-1)
   BL(N)=C(N)/W(N)
   G(N)=(D(N)-A(N)*G(N-1))/W(N)
1 CONTINUE
N=ITER
W(N)=B(N)-A(N)*BL(N-1)
G(N)=(D(N)-A(N)*G(N-1))/W(N)
E(LL,N)=G(N)
DO 6 I=2,ITER
   N=N-1
   E(LL,N)=G(N)-BL(N)*E(LL,N+1)
6 CONTINUE
RETURN
END
APPENDIX F

HEAT LOSSES IN THE EXPERIMENTAL CONTAINER

The heat which was electrically generated on the sidewalls of the experimental container was apportioned as follows:

1. Into the test fluid itself;
2. Into the vapor space above the test fluid;
3. Into the heater panels;
4. Heat loss through the sidewalls;
5. Heat loss through the front and rear walls.

This appendix evaluates the magnitude of the heat which was allotted to each of the above regions. A heat balance for a typical run will be made by way of illustrating the distribution of heat.

F.1 Test Designation

All of the tests were designated by a three group title. In this appendix, test W-3-HI will be used to illustrate the procedure for data analysis. Test conditions could be ascertained as follows:

Fluid (W) Water is the test fluid
Aspect Ratio (3) The aspect ratio is three
Heat Flux (HI) Basically two power levels were used for each test fluid at each aspect ratio. The higher one (nominally 10 amperes) was designated by "HI" and the lower one (nominally 2 amperes) was designated by "LO".
F.2 System Geometry

The enclosure was a rectangular container. For Test W-3-HI, the system parameters were:

Heater Size: 2 ft. x 2 ft. x 1/4 in.;
Material: Pyrex

Front and rear wall (Each wall has 2 panels separated by a 1/2 in. gap) Size: 2 ft. x 8 in. x 1/4 in.; material: Pyrex

Sidewall Insulation Size: 2 ft. x 2 ft. x 4 in.; material: Polyurethane foam

Fluid level 2 ft.

Heat flux (nominal) 1232 btu/hr/ft²

F.3 Physical Properties

The physical materials of the materials at 70°F. in this test are given in Table F.1.

<table>
<thead>
<tr>
<th>TABLE F. 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PERTINENT PHYSICAL PROPERTIES FOR TEST W-3-HI</strong></td>
</tr>
<tr>
<td>MATERIAL PROPERTY</td>
</tr>
<tr>
<td>ρ (lb/ft³)</td>
</tr>
<tr>
<td>k (Btu/hr/ft²°F)</td>
</tr>
<tr>
<td>c_p (Btu/lb/°F)</td>
</tr>
<tr>
<td>α (ft²/hr)</td>
</tr>
<tr>
<td>γ (ft²/sec)</td>
</tr>
<tr>
<td>β (°F⁻¹)</td>
</tr>
<tr>
<td>λ (Btu/ft²)</td>
</tr>
</tbody>
</table>

Specific Heat of Water Vapor = 0.43 Btu/lb/°F.
F.4 Heat Generation Terms

The total heat generated at any time $t_1$ is

$$Q_{in} = \int_{0}^{t_1} 2qLWdt$$

For example, at $t = 10$ minutes:

$$Q_{in} = 2(1232)2(2)10/60 = 1643 \text{ Btu}$$

F.5 Fluid Enthalpy Terms

The amount of heat, $Q_f$, which has been absorbed by the fluid at a time, $t_1$, is:

$$Q_f = \int_{0}^{L} \rho c_p WD(T - T_0)dz = \rho c_p WD \int_{0}^{L}(T - T_0)dz$$

Since an axial temperature distribution is present in the fluid a graphical integration of the temperature field is required, i.e., the area under the curve given in Figure F.1 for time, $t_1$, is required.

Also, the mean temperature of the fluid, which will be required in later calculations is

$$T = \frac{\int_{0}^{L} (T - T_0)dz}{L}$$

The heat generated, $Q_{in}$, the heat absorbed, $Q_f$, the heat loss, and the mean temperature of the fluid are given in Table F.2.
FIGURE F.1 AXIAL TEMPERATURE profiles

Test W-3-H1

\( \text{Ra} = 10^{13} \)
TABLE F. 2
HEAT LOSSES AND MEAN FLUID TEMPERATURE AS
A FUNCTION OF TIME FOR TEST W-3-HI

<table>
<thead>
<tr>
<th>t</th>
<th>Q_in</th>
<th>Q_r</th>
<th>6Q</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>Btu</td>
<td>Btu</td>
<td>Btu</td>
<td>°F</td>
</tr>
<tr>
<td>10</td>
<td>1643</td>
<td>1480</td>
<td>160</td>
<td>9.0</td>
</tr>
<tr>
<td>30</td>
<td>4929</td>
<td>4660</td>
<td>269</td>
<td>28.4</td>
</tr>
<tr>
<td>50</td>
<td>8215</td>
<td>7455</td>
<td>760</td>
<td>44.7</td>
</tr>
<tr>
<td>70</td>
<td>11501</td>
<td>10246</td>
<td>1255</td>
<td>61.5</td>
</tr>
</tbody>
</table>

F.6 Heat Loss

F.6.1 Heat Loss in Vapor Space

Above the fluid level there is a two-inch high air gap. Heat is lost to this air gap in two ways:

1. Heating of the air in the gap to the fluid surface temperature, T_s. (It is assumed that the air gap is isothermal at this temperature.)

As a first approximation, the amount of heat Q_a necessary is:

\[ Q_a = \frac{2}{12} W D c_p (T_s - T_o) = \frac{2}{12} W D \frac{P V}{R T_s} (T_s - T_o) \] F.6.1

So that at a time of 10 minutes

\[ Q_a = \frac{2(2)(2118)(29)(28)}{12} \frac{1543}{558} = 0.5 \text{ Btu} \]

2. Heat loss due to vaporization of water vapor

Assuming that condensation does not occur, and that the air gap is filled with saturated water vapor at all times, then the heat required, Q'_a, is
\[ Q_a' = \delta m_w (\lambda + c_p (T - 70)) \]

where \( \delta m_w \) is the amount of water vaporized since the beginning of the test. At 10 minutes,

\[ Q_a' = ((5.9 - 2.61) \times 10^{-4}) (1054 + 0.43(28)) = 0.352 \text{ Btu} \]

Note that this does not take into account the air and the water vapor lost to the surroundings. However, since this term is small, the additional amount of heat is small also.

F.6.2 Heat Loss to the Front and Rear Walls

Heat which is lost through the front and rear walls can be attributed to two factors:

1. Heat flow through the walls

Because of the thermal stratification of the fluid, a vertical temperature gradient existed in the wall. Sparrow and Gregg (56) have shown that the use of the midpoint temperature in the isothermal vertical plate correlations gave good agreement for the constant wall heat flux conditions. Assuming that this is true for this case, the heat flux at the mean fluid temperature (which was close to the midpoint temperature in these tests except at the very beginning) was calculated and used to determine the total flux.

Also, since the time constants, \( ct/d^2 \), for the flow through the glass and air gap are small (about 7 sec. for air and 1 min. for the glass walls) compared to the time scale of the experiment, it was assumed that the steady state temperature profiles existed at any time \( t_1 \) through the glass and air.

From McAdams (34), if \( Gr_d < 2 \times 10^4 \), heat transfer
through the air gap is by conduction. For $2 \times 10^4 \leq Gr_d \leq 10^9$

$$\frac{h_c x}{k} = \frac{0.2}{(L/x)^{1/9}} (Ra)^{1/4}$$

In this case, for $T < 15^\circ F$, conduction is present, and for $T > 15^\circ F$,

$$h_c = 0.072 \text{Ra}^{1/4}$$

On the outside wall, the heat transfer coefficient, $h_o$, was found to be (29b)

$$h_o = 0.29(T_w - T_{ambient})^{1/4}$$

where the temperatures are defined in Figure F.2. The heat flux, $q$, is

$$q = U(T_f - T_{ambient})$$

where

$$U = \frac{1}{\frac{21}{k_g} + \frac{d}{k_a} + \frac{1}{h_o}}$$

when $T < 15^\circ F$. When $T \geq 15^\circ F$,

$$U = \frac{1}{\frac{2}{48(.065)} + \frac{1}{24(.015)} + \frac{1}{h_o}}$$

and $h_o$ has to be found by trial and error. Assuming that $T_w - T_{ambient} = 4^\circ F$, and using Equation F.6.5

$$U = 0.162 \text{Btu/hr/ft}^2/^\circ F$$

so that

$$q = 0.162(T_w - T_o) = 0.162(9) = 1.458 \text{Btu/ft}^2/\text{hr}$$
Figure F.2
Front and Rear Walls

Figure F.3
Heat Flux Through the Front Wall
Checking the assumption that $T_w - T_{ambient} = 4^\circ F$, one finds that since

$$ q = h_o(T_w - T_{ambient}), \quad F.6.8 $$

$$ (T_w - T_{ambient}) = 4.14^\circ F, $$

which is close enough to the assumed temperature difference. This calculation was repeated for several times. The results are plotted in Figure F.3.

The heat loss at $t_1$ is

$$ Q_w = \int_{0}^{1} 2qdt \quad F.6.9 $$

Therefore, graphical integration of Figure F.3 is required to obtain the amount of heat lost at any time. Table F.3 gives the temperature distribution and heat flux for given times.

2. Heat Absorbed by the Walls

Heat is also required to heat the walls to the average temperatures given in Table F.3. The heat for each panel and the air gap is:

$$ Q_{w1} = \rho c_p D((T_1 + T_f)/2 - T_o) \quad F.6.10a $$

$$ Q_{w2} = \rho c_p D((T_2 + T_w)/2 - T_o) \quad F.6.10b $$

$$ Q_{gap} = \rho_{air} c_p dD((T_1 + T_2)/2 - T_o) \quad F.6.10c $$

The heat required for each wall and the total heat flux is given in Table F.4.
### TABLE F.3
HEAT FLUX THROUGH THE FRONT AND REAR WALLS OF THE CONTAINER

<table>
<thead>
<tr>
<th>t (min.)</th>
<th>$T_f$ ($^\circ$F)</th>
<th>$T_1$ ($^\circ$F)</th>
<th>$T_2$ ($^\circ$F)</th>
<th>$T_w$ ($^\circ$F)</th>
<th>$T_a$ ($^\circ$F)</th>
<th>q (Btu/hr $/ft^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>79.0</td>
<td>78.85</td>
<td>74.15</td>
<td>74.10</td>
<td>70.0</td>
<td>1.7</td>
</tr>
<tr>
<td>30</td>
<td>98.4</td>
<td>98.2</td>
<td>81.5</td>
<td>81.3</td>
<td>70.0</td>
<td>6.0</td>
</tr>
<tr>
<td>50</td>
<td>114.7</td>
<td>114.3</td>
<td>90.4</td>
<td>90.0</td>
<td>70.0</td>
<td>12.3</td>
</tr>
<tr>
<td>70</td>
<td>131.8</td>
<td>131.2</td>
<td>98.3</td>
<td>97.7</td>
<td>70.0</td>
<td>18.3</td>
</tr>
</tbody>
</table>

### TABLE F.4
HEAT LOSS THROUGH THE FRONT AND REAR CONTAINER WALLS

<table>
<thead>
<tr>
<th>t (min.)</th>
<th>$Q_{\text{flux}}$ (Btu)</th>
<th>$Q_{w1}$ (Btu)</th>
<th>$Q_{w2}$ (Btu)</th>
<th>$Q_{\text{air}}$ (Btu)</th>
<th>$Q/\text{wall}$ (Btu)</th>
<th>$Q/2 \text{ walls}$ (Btu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.18</td>
<td>9.43</td>
<td>5.31</td>
<td>0.08</td>
<td>15.0</td>
<td>30.0</td>
</tr>
<tr>
<td>30</td>
<td>1.86</td>
<td>28.8</td>
<td>11.95</td>
<td>0.25</td>
<td>43.9</td>
<td>87.7</td>
</tr>
<tr>
<td>50</td>
<td>5.88</td>
<td>34.1</td>
<td>21.2</td>
<td>0.38</td>
<td>61.5</td>
<td>123.1</td>
</tr>
<tr>
<td>70</td>
<td>12.6</td>
<td>64.4</td>
<td>29.4</td>
<td>0.52</td>
<td>107.0</td>
<td>214.0</td>
</tr>
</tbody>
</table>
Figure F.4 gives the bottom temperature as a function of time. The slope of the bottom temperature, \( a \), for \( t > 10 \) minutes is 30°F/hr. The solution of the conduction equation for a semi-infinite solid whose surface temperature increases linearly with temperature can be shown to be (7).

\[
T - T_0 = 4at \text{erfc}(x/(2(\alpha t)^{1/2}))
\]  

F.6.11

From this it can be seen that

\[
\frac{dT}{dx} = -\frac{2at}{x(\alpha t)^{1/2}} \text{erfc}(x/(\alpha t)^{1/2})
\]  

F.6.12

so that at \( x = 0 \) (the container fluid-solid interface),

\[
q = 0.5642 \frac{2kat^{1/2}}{\alpha^{1/2}}
\]  

F.6.13

Noting that the time in Equation F.6.13 is actually \( t_{\text{actual}} - 10 \) minutes, the heat flux can be calculated. This is plotted in Figure F.5. Graphical integration of Figure F.5 yields the heat loss through the bottom. Table F.5 lists the heat loss through the container bottom.

**TABLE F.5**

<table>
<thead>
<tr>
<th>time (t) min.</th>
<th>Q Btu</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>0.11</td>
</tr>
<tr>
<td>50</td>
<td>0.36</td>
</tr>
<tr>
<td>70</td>
<td>0.69</td>
</tr>
</tbody>
</table>
Figure F.4
Bottom Surface Temperature

Figure F.5
Heat Flux Through the Container Bottom
F.6.4 **Heat Loss Through the Sidewalls**

Assuming a steady state profile through the polyurethane insulation

\[ q = -k \frac{\Delta T}{\Delta z} \]  

where \( \Delta T, \Delta z \) is the temperature difference and distance between two thermocouples inbedded in the polyurethane insulation. The position of the thermocouples and their temperature history are given in Figure F.6. The heat flux was found to be approximately constant with height. This heat flux for the two sidewalls is shown in Figure F.7. Graphical integration yields the heat flux through the sidewalls. This is given in Table F.6.

<table>
<thead>
<tr>
<th>TIME (min.)</th>
<th>HEAT (Btu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.9</td>
</tr>
<tr>
<td>30</td>
<td>6.5</td>
</tr>
<tr>
<td>50</td>
<td>16.5</td>
</tr>
<tr>
<td>70</td>
<td>30.7</td>
</tr>
</tbody>
</table>

F.6.5 **Heat Absorbed by the Heater**

The temperature of the heater must be raised to a level which will permit heat transfer to the fluid to occur. Assuming a linear gradient in the glass heater and a linear gradient in the core, the average glass temperature can be calculated. From Chapter 5,

\[ T_0 = (2)^{1/2} \frac{(T_w - T_\infty)}{qL/k} Ra^{1/4} = 2 \]
FIGURE F.6 SIDEWALL TEMPERATURE DISTRIBUTION
Heat Loss Rate (Btu/hr) vs. time (min)

Figure F.7
Heat Loss Through the Sidewalls
Also, 
\[ q = - k \frac{\Delta T_w}{\Delta z} \]  
F.6.16

where \( \frac{\Delta T_w}{\Delta z} \) is the gradient through the wall. The fluid temperature gradient is calculated for any time from Figure F.1. Using these values, \( R_a \) is calculated. Then, using Equation F.6.15 the wall temperature is calculated. The gradient through the wall is calculated using Equation F.6.16. Finally, the mean temperature of the glass is calculated, viz.,

\[ T_g = T_w + \frac{1}{2} \frac{\Delta T_w}{\Delta z} \]  
F.6.17

where \( l \) is the thickness of the heater. The heat absorbed by the two glass heaters is given by

\[ Q_h = 2p\rho_p(T_g - T_o)lw \]  
F.6.18

The values of these various parameters are given in Table F.7.

<table>
<thead>
<tr>
<th>(min.)</th>
<th>( T_w - T ) (°F)</th>
<th>( \frac{T_w}{z} )l (°F)</th>
<th>( T_g - T_o ) (°F)</th>
<th>( Q_h ) (Btu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>23.6</td>
<td>36.5</td>
<td>50.8</td>
<td>235</td>
</tr>
<tr>
<td>30</td>
<td>20.2</td>
<td>36.5</td>
<td>67.8</td>
<td>314</td>
</tr>
<tr>
<td>50</td>
<td>19.6</td>
<td>36.5</td>
<td>82.5</td>
<td>383</td>
</tr>
<tr>
<td>70</td>
<td>18.25</td>
<td>36.5</td>
<td>97.9</td>
<td>452</td>
</tr>
</tbody>
</table>

Table F.8 shows how the generated heat is apportioned. It can be seen that the heat balance closes fairly well.
TABLE F.8
HEAT DISTRIBUTION IN TEST W-3-HI

Heat Generated = 100 percent

<table>
<thead>
<tr>
<th>Time</th>
<th>Vapor Space</th>
<th>Front and Rear Walls</th>
<th>Sidewalls</th>
<th>Heater</th>
<th>Bottom</th>
<th>Fluid</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>30</td>
<td>50</td>
<td>70</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.93</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.82</td>
<td>1.78</td>
<td>1.50</td>
<td>1.86</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.13</td>
<td>0.30</td>
<td>0.27</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>14.30</td>
<td>6.37</td>
<td>4.66</td>
<td>3.93</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>90.07</td>
<td>94.54</td>
<td>90.74</td>
<td>89.09</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-6.3</td>
<td>-2.84</td>
<td>2.86</td>
<td>4.82</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Figures in the table are percentages of the total heat generated.
This section contains a definition of the variables used in STRAT, the computer simulation of the thermal stratification phenomenon. This is followed by a block diagram of the computer program used and a listing of the Fortran IV program. Note that subroutines PLOTER, GINT, and WRITE are an added facility to the main program which permits graphic representation of the temperature and velocity field for as many as 10 different Fourier numbers. The array SYM(I) contains the symbols for the plots; the first ten symbols can be any character which is accepted by Fortran IV - however, characters 11 to 14 must be specified as follows: 11 - blank, 12 - P, 13 - -, 14 - I. A sample output of the graphical facility is given in Figure G.2.

The following table gives a list of the symbols used in STRAT and their definition, followed by the name of the subroutine where the variable is calculated, and where it is used.

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>DEFINITION</th>
<th>CALC.</th>
<th>USED</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>The inverse of the Aspect Ratio (A = D/L)</td>
<td>MAIN</td>
<td>TCORE</td>
</tr>
<tr>
<td>AF</td>
<td>Dummy variable</td>
<td>ESTAR</td>
<td>ESTAR</td>
</tr>
<tr>
<td>AL</td>
<td>Height of the Container</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>B</td>
<td>(z^2)</td>
<td>ECALC</td>
<td>ECALC</td>
</tr>
<tr>
<td>C</td>
<td>Empirical velocity scaling factor</td>
<td>Input</td>
<td>TCORE</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>WCORE</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>MAIN</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>CI</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>CI</td>
<td>Empirical velocity scaling factor</td>
<td>MAIN</td>
<td>WCORE</td>
</tr>
<tr>
<td>CON</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>PARM</td>
</tr>
<tr>
<td>D</td>
<td>Width of the container</td>
<td>Input</td>
<td>MAIN</td>
</tr>
<tr>
<td>D</td>
<td>$Z^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DI</td>
<td>Error in heat balance</td>
<td>PRINT</td>
<td>PRINT</td>
</tr>
<tr>
<td>DT</td>
<td>Time step size</td>
<td>TCORE</td>
<td>TCORE</td>
</tr>
<tr>
<td>DX</td>
<td>Size of an increment in the x-direction on the graph</td>
<td>MAIN</td>
<td>WRITE</td>
</tr>
<tr>
<td>DY</td>
<td>Size of an increment in the y-direction on the graph</td>
<td>MAIN</td>
<td>WRITE</td>
</tr>
<tr>
<td>DZ</td>
<td>Step size increment</td>
<td>MAIN</td>
<td>TCORE</td>
</tr>
<tr>
<td>DZS</td>
<td>Dummy variable</td>
<td>PARM</td>
<td>PARM</td>
</tr>
<tr>
<td>E</td>
<td>Energy flow parameter</td>
<td>ESTAR</td>
<td>ESTAR</td>
</tr>
<tr>
<td>E</td>
<td>$Z^5$</td>
<td>ECALC</td>
<td>ECALC</td>
</tr>
<tr>
<td>E1</td>
<td>Energy flow parameter at Z1</td>
<td>PARM</td>
<td>PARM</td>
</tr>
<tr>
<td>E2</td>
<td>Energy flow parameter at Z2</td>
<td>PARM</td>
<td>PARM</td>
</tr>
<tr>
<td>ESTAR</td>
<td>Function for calculating the equivalent height</td>
<td>--</td>
<td>PARM</td>
</tr>
<tr>
<td>F</td>
<td>$Z^6$</td>
<td>ECALC</td>
<td>ECALC</td>
</tr>
<tr>
<td>FO</td>
<td>Fourier Number x 2.0</td>
<td>PRINT</td>
<td>PRINT</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>FO(I,J)</td>
<td>Array containing the Fourier and temperature and velocity data for plot</td>
<td>MAIN</td>
<td>WRITE</td>
</tr>
<tr>
<td>GINT</td>
<td>Subroutine for initializing the plotting facility</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>GRAPH</td>
<td>Array which constitutes the graphical representation of the data</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>(Points are plotted as (y, x). In this plot the height is the y-variable.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HEAT</td>
<td>Heat in the fluid</td>
<td>PRINT</td>
<td>PRINT</td>
</tr>
<tr>
<td>I</td>
<td>Counter</td>
<td>--</td>
<td>ALL</td>
</tr>
<tr>
<td>II</td>
<td>Counter</td>
<td>--</td>
<td>WRITE</td>
</tr>
<tr>
<td>ICH</td>
<td>Subroutine for determining which polynomial will be used in calculating</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>the energy flow parameter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ICOUNT</td>
<td>Polynomial reference number for calculating the energy flow parameter</td>
<td>ICH</td>
<td>ECALC</td>
</tr>
<tr>
<td>IDATA</td>
<td>Symbol reference number for plots</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>IPO(I)</td>
<td>Index for determining place on graph where Fourier number is written</td>
<td>MAIN</td>
<td>WRITE</td>
</tr>
<tr>
<td>IG</td>
<td>Integer which determines whether temperature or velocity field is printed</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>IMAX</td>
<td>Number of divisions in the x-direction</td>
<td>WRITE</td>
<td>WRITE</td>
</tr>
<tr>
<td>INPUT</td>
<td>Specification number which indicates whether a temperature field is</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td></td>
<td>specified in the input</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IPR</td>
<td>Number of iterations since last printout</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>IRK</td>
<td>Dummy variable which determines whether temperature and velocity field</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td></td>
<td>are to be plotted</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>---------------------------------------------------------------------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>ISTA</td>
<td>Number of grid points to be printed</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>IT</td>
<td>= IG</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITER</td>
<td>Number of iterations which have been calculated</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>ITIME</td>
<td>Number which specifies whether DT is to be modified</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>IX</td>
<td>Number of spaces from the origin in the x-direction</td>
<td>PLOTER</td>
<td>PLOTER</td>
</tr>
<tr>
<td>IXMAX</td>
<td>Maximum number of spaces in the x-direction</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>IY</td>
<td>Number of spaces from the origin in the y-direction</td>
<td>PLOTER</td>
<td>PLOTER</td>
</tr>
<tr>
<td>IYMAX</td>
<td>Maximum number of spaces in the y-direction</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>J</td>
<td>Counter</td>
<td>--</td>
<td>ALL</td>
</tr>
<tr>
<td>JJ</td>
<td>Counter</td>
<td>--</td>
<td>WRITE</td>
</tr>
<tr>
<td>JMAX</td>
<td>= IYMAX</td>
<td>WRITE</td>
<td>WRITE</td>
</tr>
<tr>
<td>K</td>
<td>Counter</td>
<td>--</td>
<td>ALL</td>
</tr>
<tr>
<td>KFO</td>
<td>Counter keeping track of the number of plots</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>LL</td>
<td>Counter for number of points to be plotted</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>LS</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>N</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>N</td>
<td>Index determining which grid points to print</td>
<td>PRINT</td>
<td>PRINT</td>
</tr>
<tr>
<td>MCHOS</td>
<td>Parameter indicating whether specific times are given for plotting</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td></td>
<td>temperatures and velocities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NDT</td>
<td>Number of iterations at which</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>NDT</td>
<td>time step is altered</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NDT$^*$</td>
<td>Number of iterations since time step has last been altered</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>NITER</td>
<td>Total number of iterations</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NMAP</td>
<td>Parameter indicating if previously calculated temperature and velocity fields are to be given</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NNN</td>
<td>Dummy variable</td>
<td>PRINT</td>
<td>PRINT</td>
</tr>
<tr>
<td>NPL</td>
<td>Number of plots which have been imposed</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>NPL$^*$</td>
<td>Total number of plots to be imposed</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NPRINT</td>
<td>Number of iterations at which printout is given</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NRUN</td>
<td>Total number of runs in test</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NRUNS</td>
<td>Number of run being run</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>NSP</td>
<td>Number of plots which have been previously calculated which are to be imposed</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>NSTA</td>
<td>Minimum grid point which printout is given</td>
<td>MAIN</td>
<td>PRINT</td>
</tr>
<tr>
<td>NZONES</td>
<td>Number of grid points</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>P</td>
<td>Dummy variable</td>
<td>ESTAR</td>
<td>ESTAR</td>
</tr>
<tr>
<td>PARM</td>
<td>Subroutine for calculating energy flows as a function of height</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>PLOT</td>
<td>Array of times at which plots are to be imposed</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>PLOT$^*$</td>
<td>Subroutine for plotting velocity and temperature fields</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>PR</td>
<td>Prandtl number</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>PRINT</td>
<td>Subroutine for printing out data</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>PUNCH</td>
<td>Subroutine for punching data cards</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>RAINF</td>
<td>Core Rayleigh number</td>
<td>PARM</td>
<td>PARM</td>
</tr>
<tr>
<td>RAM</td>
<td>Modified Rayleigh number</td>
<td>input</td>
<td>MAIN</td>
</tr>
<tr>
<td>RAMQ</td>
<td>$Ra^{1/4}$</td>
<td>MAIN</td>
<td>TCORE PARM</td>
</tr>
<tr>
<td>RSP</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>ESTAR PARM</td>
</tr>
<tr>
<td>STORE</td>
<td>Array for storing points to be plotted</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>SYM</td>
<td>Array containing symbols for plots</td>
<td>input</td>
<td>GINT PLOTER</td>
</tr>
<tr>
<td>SYMB</td>
<td>Dummy variable</td>
<td>PLOTER</td>
<td>PLOTER</td>
</tr>
<tr>
<td>T</td>
<td>Temperature array</td>
<td>TCORE</td>
<td>TCORE MAIN</td>
</tr>
<tr>
<td>TCORE</td>
<td>Subroutine for calculating temperature field</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>TEST</td>
<td>Dummy variable</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>TIME</td>
<td>Fourier number</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>TIMEM</td>
<td>Subroutine for altering DT</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>TN</td>
<td>Temperature at time level n+1</td>
<td>TCORE</td>
<td>TCORE</td>
</tr>
<tr>
<td>TO</td>
<td>Bottom temperature</td>
<td>TCORE</td>
<td>TCORE</td>
</tr>
<tr>
<td>W</td>
<td>Velocity array</td>
<td>WCORE</td>
<td>WCORE MAIN</td>
</tr>
<tr>
<td>WCORE</td>
<td>Subroutine for calculating velocity field</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>WMAX</td>
<td>Maximum velocity in array at time $\tau$</td>
<td>MAIN</td>
<td>MAIN TIMEM</td>
</tr>
<tr>
<td>WRITE</td>
<td>Subroutine for plotting graph</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>X</td>
<td>X-direction coordinate divisions</td>
<td>WRITE</td>
<td>WRITE</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>CALC.</td>
<td>USED</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------</td>
<td>-------</td>
<td>--------</td>
</tr>
<tr>
<td>XMAX</td>
<td>Maximum value of x-coordinate</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>XMIN</td>
<td>Minimum value of x-coordinate</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>YMAX</td>
<td>Maximum value of y-coordinate</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>YMIN</td>
<td>Minimum value of y-coordinate</td>
<td>MAIN</td>
<td>PLOTER</td>
</tr>
<tr>
<td>YSAVE</td>
<td>Array storing values of y-coordinate divisions</td>
<td>WRITE</td>
<td>WRITE</td>
</tr>
<tr>
<td>Z</td>
<td>Array storing values of height at each grid point</td>
<td>MAIN</td>
<td>MAIN</td>
</tr>
<tr>
<td>Z1</td>
<td>Height at grid point i - 1</td>
<td>PARM</td>
<td>PARM</td>
</tr>
<tr>
<td>Z2</td>
<td>Height at grid point i</td>
<td>PARM</td>
<td>PARM</td>
</tr>
<tr>
<td>ZEFI</td>
<td>Dummy variable</td>
<td>ESTAR</td>
<td>ESTAR</td>
</tr>
<tr>
<td>ZONE</td>
<td>Dummy variable</td>
<td>ESTAR</td>
<td>ESTAR</td>
</tr>
<tr>
<td>ZTWO</td>
<td>Dummy variable</td>
<td>ESTAR</td>
<td>ESTAR</td>
</tr>
</tbody>
</table>
FIGURE G.1
BLOCK DIAGRAM
OF STRAT

START
READ IN LOGICAL PARAMETERS
ARE ANY VEL & TEMP FIELDS TO BE ADDED TO PLOT?
yes
READ IN VEL & TEMP FIELDS
no
ARE SPECIFIC TIMES FOR PLOTS TO BE GIVEN?
yes
READ IN TIMES
no
CALCULATE ARBITRARY TIMES

READ IN INPUT DATA
IS AN INITIAL NON-ZERO VEL & TEMP FIELD TO BE SPECIFIED?
yes
SPECIFY VEL & TEMP FIELD
no
INITIALIZE ARRAYS
no
INITIALIZE PRINTING ROUTINE PARAMETERS

SUBROUTINE WRITE
Print out graph
SUBROUTINE PLOTER
Plot graphs

DO LOOP END

SUBROUTINE PUNCH
Punch vel & temp fields
CALCULATE GRAPH PARAMETERS

SUBROUTINE PRINT
Print results

IS IRK=2?
no
SUBROUTINE WCORE
Advance velocity field

PRINT ROUTINE PARAMETERS

no
SUBROUTINE TCORE
Advance temp field

SUBROUTINE PARM
Calculate Energy Parameters

DO LOOP
ALTER TIME STEP?
yes
no

END
HAVE ALL TESTS BEEN RUN?
yes
no

no

yes

SUBROUTINE PUNCH
Punch vel & temp fields
STORE VEL & TEMP FIELDS

IRK=1
yes
no

IRK=2
yes
no

PRINT VEL & TEMP FIELD

ALTER TIME STEP

no

yes
C THERMAL STRATIFICATION PROGRAM

C PLACEMENT OF CARDS
C READ IN 14 SYMBOLS FOR PLOTTING GRAPHS- NOTE THAT ONLY 10 CAN BE USED
C FOR PLOT SYMBOLS, THE LAST 4 BEING USED FOR SETTING UP THE GRAPH
C THESE LAST 4 SYMBOLS MUST BE -- BLANK,P++,I
C FORMAT (14A1)
C FORMAT (12)
C READ IN THE NUMBER OF TESTS TO BE RUN
C ****************************************
C NEXT CARD
C ARE TIMES FOR PLOTTING SPECIFIED? YES=2, NO=1
C ARE TEMPERATURE AND VELOCITY FIELDS TO BE ADDED FROM AN EXTERNAL
C SOURCE TO THE PLOT? YES=2, NO=1
C FORMAT(12)
C NEXT CARD
C IF TEM. AND VEL. FIELDS ARE TO BE ADDED, READ IN NO. OF FIELDS
C (NOTE THAT FIELDS MUST HAVE THE SAME FORMAT AS THE OUTPUT FROM PUNCH)
C OTHERWISE, CONTINUE
C NEXT CARD
C IF TIMES FOR PLOTTING ARE TO BE GIVEN, READ IN TIMES HERE
C FORMAT(7E10.0/3E10.0)
C NEXT CARD
C READ IN C (NOTE C= CF**0.8- SEE TEXT FOR CF TO BE USED)
C FORMAT(F10.0)
C NEXT CARD
C READ PARAMETER WITH GRID ZONES, MAX. NC. OF ITERATIONS, INTERVAL
C BETWEEN PRINT IS CALLED, IS INITIAL FIELD TO BE ADDED? (YES=1, NO=2),
C IS THE TIME STEP TO BE CHANGED? (YES=1, NO=2 - MUST BE 2 IF NO FIELD
C IS ADDED), ITERATION AT WHICH TIME STEP IS TO BE CHANGED,
C FORMAT(10.E10.0/2F10.0/6I5)
C NOTE- IN REALITY NO. PRANDTL NUMBER OR RAYLEIGH NUMBER IS REQUIRED,
C THEREFORE ANY ARBITRARY ONES MAY BE GIVEN, E.G., REMOVE C FROM THE
C FOLLOWING CARD:
C 10.0  1.0E+10
C NEXT CARD
C IF INITIAL FIELD IS TO BE ADDED, ADD HERE
C (NOTE THAT THE FORMAT MUST BE THE SAME AS THE OUTPUT FROM PUNCH)
C ****************************************
C REPEAT CARDS BETWEEN ***S FOR EACH RUN
C
C STRAT PROGRAM
C PROGRAM CONTAINING THE LOGISTICS

DIMENSION F(10,3),PLOT(10),IFO(11)
DIMENSION GRAPH(101,200),SYM(14),STORE(1000,2),ACCESS(100,2)
DIMENSION W(500),T(500),EST(500,2),Z(500),TN(500)
COMMON W,T,EST,Z,TN,RAMQ,A,CI,C,CON,C1,DZ,TIME,DT,TC
COMMON NZCNES,ITER,N
READ(5,908)(SYM(I),I=1,14)
908 FORMAT(14A1)
READ(5,507)RUN
507 FORMAT(12)
NRUNS=1

C **************************************** MAIN PROGRAM ****************************************
405 CONTINUE
NCN=1
LL=1
KFO=1
IT=1
IRK=1
C IF NCHOS=1, THEN PLOTS ARE GIVEN EVERY 10 PERCENT OF TOTAL ITERATIONS
C OTHERWISE IF NCHOS = 2 TIMES FOR PLOTTING ARE SPECIFIED
READ(5,930)NCHOS,NMAP
930 FORMAT(2I2)
IF(NMAP.EQ.1)GO TO 800
C ADD PLOTS HERE
READ(5,801)NSP
801 FORMAT(I2)
DO 802 J=1,NSP
READ(5,105)PR,RAM,NZCNES,TIME,CT,NSTA,ISTA
READ(5,103)(Z(I),W(I),T(I),I=1,NZCNES)
DO 803 I=1,NZONES
STC(RE(LL,1))=T(I)
STC(RE(LL,2))=W(I)
LL=LL+1
FO(KFO,1)=TIME
FO(KFO,2)=STC(RE(LL-1,1))
FO(KFO,3)=STC(RE(LL-1,2))
802 KFO=KFO+1
NPL=NSP+1
800 IF(NCHOS.EQ.1)GO TO 931
C SPECIFY TIMES AT WHICH PLOTS ARE TO BE GIVEN
READ(5,930)NPLCT
READ(5,933)(PLCT(I),I=1,NPLCT)
933 FORMAT(2E10.0/3E10.0)
C READ INPUT PARAMETERS
931 READ(5,100)C
100 FORMAT(F10.0)
W(1)=0.0
C IF INPUT EQUALS 1 THEN THERE IS SOME TEMPERATURE FIELD THAT IS APPLIED
READ(5,101)PR,RAM,NZCNES,NITER,PRINT,INPUT,IT1ME,N1
101 FORMAT(F10.0,E10.0/F10.0/6F10.0/F10.0/6F10.0)
C CALCULATE PROGRAM PARAMETERS
ITER=1
RAMC=RAM**0.25
N=NZONES-2
A=C/AL
TIME=0.0
CT=0.0
C1=(1.57*(PR**0.4))**0.8
CCN=C**1.25
C1=CN**0.2
DZ=1.0/NZCNES
1(I)=0.0
C ECHO CHECK
WRITE(6,102)RAMC,A,C1,C,CCN,C1,DZ,NZCNES,N
102 FORMAT(15,2E10.0,F10.0/6F10.0/2I10)
DO 200 I=2,NZONES
C INITIALIZE ARRAYS
200 Z(I)=Z(I-1)+DZ
IF(NCHOS.EQ.1)GO TO 940
IF(NMAP.EQ.2)GO TO 804
NPL=1
GO TO 941
940 NPL=NTTER/10
NPL=NPL
IF(NMAP.EQ.2)GO TO 804
C INITIALIZE PRINTING PARAMETERS
804 DO 202 I=1,NZONES
202 T(I)=0.0
ISTA=NZCNES/10
NSTA=NZONES-NZONES/10
IPR=NPRINT
DO 806 I=1,NZCNES
ACESS(I,2)=Z(I)+DZ/2.0
806 ACESS(I,1)=0.0
IF(INPUT.EQ.1)GO TO 301
GO TO 302
301 CONTINUE
C ADD INITIAL TEMPERATURE AND VELOCITY FIELD
DO 909 I=1,NZONES
ACESS(I,2)=Z(I)+DZ/2.0
909 ACESS(I,1)=0.0
READ(5,105)PR,RA,M,NZONES,TIME,DT,NSTA,ISTA
105 FORMAT(F10.0,E11.4,E11.4,E11.4,E11.4,E15)
READ(5,103)(Z(I),W(I),T(I),I=1,NZCNES)
103 FORMAT(5(2CA4/))
DO 400 I=1,NZCNES
EST(I,1)=Z(I)*RAMQ
400 EST(I,2)=0.0
IF(ITIME.EQ.2)GO TO 401
C CALCULATE NEW TIME STEP
WMAX=0.0
DO 402 I=2,NZONES
IF(W(I).GT.WMAX)WMAX=W(I)
402 CONTINUE
DT=0.9*DZ/(WMAX+2.0*A/DZ)
401 ITER=2
IPR=NPRINT
C START ADVANCING THERMAL STRATIFICATION MODEL
302 DO 201 I=1,NITER
NDTS=NDTS+1
C CHECK TO SEE IF MODIFICATION OF DT IS REQUIRED
IF(NDTS.LT.NCT)GO TO 700
CALL TIMEM(W,DT,NZCNES,DZ)
NDTS=1
C ADVANCE BOUNDARY LAYER ENERGY PARAMETER
700 CALL PARM
C ADVANCE TEMPERATURES
CALL TCCRE
IPR=IPR+1
C CHECK TO SEE IF POINTS ARE TO BE PLOTTED
IF(NCHCS.EQ.1)GO TO 934
IF(NPL.GT.NPLOT)GO TO 934
IF(TIME.LT.PLOT(NPL))GO TO 934
NPL=NPL+1
IRK=2
GO TO 935
C CHECK TO SEE IF PRINT OUT IS REQUIRED
934 IF(IPR.LT.NPRINT)GO TO 201
C ADVANCE VELOCITY FIELD
935 CALL WCCRE
IPR=1
C REVISE PRINT OUT PARAMETERS
IF(NSTA.EQ.1)GO TO 300
IF(ITER.EQ.1)NSTA=NSTA-NZONES/10
IF(ITER.GT.ISTA)ISTA=ISTA+NZONE/10
IF(NSTA.LT.1)NSTA=1

C PRINT OUT
300 CALL PRINT(PR,RA,NSTA,NITER)
IF(NCHOS.EQ.2)GO TO 936
    IF(NPL.GT.I)GC TO 201
    NPL=NPL+NPLCT
    IRK=2
936 IF(IRK.EQ.1)GO TO 201
WRITE(6,910)
910 FORMAT(* THESE DATA ARE PLOTTED*)
C PUNCH OUT FIELDS TO BE PLOTTED
CALL PUNCH(W,T,Z,PR,RA,NZONE,T,I,NSTA,ISTA)
C STORE PLOTTED POINTS
DO 911 J=1,NZONE
STORE(LL,1)=T(J)
STORE(LL,2)=W(J)
911 LL=LL+1
C KEEP TRACK OF TIMES PLOTS TAKEN
FC(KF,1)=TIME
    FO(KF,2)=STORE(LL-1,1)
    FC(KF,3)=STORE(LL-1,2)
    KF=KF+1
    IRK=1
201 ITER=ITER+1
C END OF THERMAL STRATIFICATION MODEL
LL=LL-1
C PUNCH OUT FINAL FIELDS
CALL PUNCH(W,T,Z,PR,RA,NZONE,T,I,NSTA,ISTA)
C DEVELOP PLOTTING PARAMETERS
LS=LL/NZONE
IXMAX=101
IYMAX=101
XMIN=0.0
YMIN=0.0
DC 912 T=1,2
    IDATA=1
    IF(I.EQ.2)GO TO 919
    XMAX=STORE(LLI)
    GO TO 918
919 XMAX=.0
WRITE(6,950)
950 FORMAT(*1X,20X,'LEGEND: */,21X,'SYMBOL',5X,'TIME',4X,'MAX. VEL.')
C TEMPERATURE FIELD IS PLOTTED FIRST, THEN VELOCITIES
C SET UP GRAPH
918 CALL GINT(Graph,SYM,IXMAX,IYMAX,IDATA)
    LL=1
    DO 913 J=1,LS
    DO 915 JJ=1,NZONE
    ACESS(JJ,1)=STORE(LL,1)
    915 LL=LL+1
    IF(IT.EQ.1)GO TO 920
    TEST=ACCESS(JJ,1)
    DO 922 K=1,NZONE
    922 IF(IT.EQ.1)ACCESS(K,1)TEST=ACCESS(K,1)
    DO 921 K=1,NZONE
    ACCESS(K,1)=ACCESS(K,1)/TEST
    WRITE(6,951)SYM(IDATA),FC(IDATA,1),TEST


C PLCT POINTS

920 CALL PLCTR(NRUNS,NZCNES,ACCESS,GRAPH,SYM,IXMAX,ITYMAX,IXMIN,YMIN,ICATA)

913 IDATA=IDATA+1

DX=XMAX/100.0

DY=0.01

916 CONTINUE

K=1+LS

917 IF(I.EQ.2)GO TO 917

IFC(K)=FC(K,2)/CX+1

GO TO 916

917 IFD(K)=FD(K,3)/DX+1

CONTINUE

960 IFC(LS+1)=IXMAX+2

PRINT OUT GRAPH

CALL WRITE(GRAPH,IXMAX,ITYMAX,IT,FO,IFC,DX,DY)

406 CONTINUE

STOP

END

SUBROUTINE PAR

C PROGRAM FOR ADVANCING BOUNDARY LAYER ENERGY FLOWS

DIMENSION W(500),T(500),EST(500,2),Z(500),TN(500)
COMMON \* T,NZCNES,ACJACNT,ACJACNP,ACJACQ,ACJACN,TIME,DT,TC
COMMON NZONES,ITER,N

ICOUNT=1

ZS=DZ*RAMC

IF(ITER.GT.1)GO TO 302

1=1,NZONES

EST(I,1)=Z(I)*RAMQ

200 EST(I,2)=0.0

GO TO 300

C CHECK TO SEE IF TEMPERATURE IS WARMER THAN TEMPERATURE BELOW

C -- IF NOT AVERAGE ADJACENT TEMPERATURES

302 DO 202 I=3,NZONES

1=I-1

IF(T(I).EQ.T(I-1))GO TO 202

IF(T(I).GT.T(I-1))GO TO 202

WRITE(6,100)ITER,T(I-1),T(I),T(I+1)

100 FORMAT(4 TEMPERATURE FAULT IN ITERATION*,I5,2X AT STEP *,*,I5,

15X,3E15.5)

T(I)=T(I)+T(I-1)+T(I+1))/3.0

T(I-1)=T(I)

T(I+1)=T(I)

202 CONTINUE

C CALCULATE NEW ENERGY FLOWS

300 DO 201 I=2,NZONES

1=I-1

IF(T(I).GT.(T(I-1),GT.1.0E-06))GO TO 303

305 EST(I,2)=EST(I-1,2)+DZ

GO TO 201

C CALCULATE THE CORE RAYLEIGH NUMBER DIVIDED BY THE MODIFIED RAYLEIGH

C NUMBER

303 RAINF=(T(I)-T(I-1))/DZ/(RAMC**.8)

306 RSP=RAINF**1.25

Z1=EST(I-1,1)*CCN

Z2=EST(I,1)*CON

E1=EST(I-1,2)*CON
C CHECK TO SEE THAT THE ENERGY IS GREATER THAN 0.0
IF(E1.LT.0.0)WRITE(6,400)E1,Z1,Z2,RSP
400 FORMAT(* ,E15.5)
IF(E1.LT.0.0)WRITE(6,101)(J,FST(J,1),EST(J,2),J=1,NZCNES)
101 FORMAT(101 ** I3,5X,E15.5,5X,E15.5/))
C CALCULATE EQUIVALENT HEIGHT
Z2=ESTAR(Z1,E1,Z2,RSP,RAINF,C1,CCN)
C CALCULATE NEW ENERGY FLOW
CALL ICH(E1,Z2,RSP,ICCN)
E2=ECALC(Z2,ICCN)/RSP/CCN
EST(I,2)=E2
CONTINUE
RETURN
END
SUBROUTINE WCPE
C PROGRAM TO ADVANCE VELOCITY FIELD
DIMENSION W(500),T(500),EST(500,2),Z(500),TN(500)
COMMON W,T,EST,Z,TN,RAmq,A,C1,CCN,C1,DZ,TIME,DT,TC
COMMON NZONES,ITER,N
301 DO 200 I=2,NZCNES
IF(T(I).GT.T(I-1))GO TO 302
W(I)=2.0*C*(EST(I,2)**0.8)/(RAMQ**0.8)
GO TO 200
302 W(I)=2.0*C*(EST(I,2)**0.8)/(RAMQ**0.8)
200 CONTINUE
RETURN
END
SUBROUTINE TCPP
C PROGRAM TO ADVANCE TEMPERATURE FIELD
DIMENSION W(500),T(500),EST(500,2),Z(500),TN(500)
COMMON W,T,EST,Z,TN,RAmq,A,C1,CCN,C1,DZ,TIME,DT,TC
COMMON NZONES,ITER,N
IF(ITER.GT.1)GO TO 300
C CALCULATE TIME STEP
DT=DZ/(2.0*C**2*A/RAMQ**0.8/CZ)
CT=0.8*DT
300 TN(NZONES)=2.0*EST(NZCNES,2)/RAMQ*CT/CZ+T(NZONES)+2.0*CT
1-(T(NZONES)-T(NZONES-1))/DZ/DZ*A/RAMQ**0.8
C ADVANCE TEMPERATURES
DO 200
T=1,NZONES-I
200 TN(IN)=T(IN)+(2.0*(EST(IN,2)-EST(IN+1,2))/RAMQ/DZ+2.0*A*(T(IN+1)-
12.0*T(IN)+T(IN-1))/DZ/DZ*RAMQ**0.8)*DT
TIME=TIME+DT
DO 201 I=1,NZONES
IF(T(NZONES).LT.1.0E-07)TN(I)=1.0
201 T(I)=TN(I)
RETURN
END
SUBROUTINE PRINT(PR,RAm,NSTA,ALTER)
C PROGRAM TO PRINT DATA
DIMENSION W(500),T(500),EST(500,2),Z(500),TN(500)
COMMON W,T,EST,Z,TN,RAmq,A,C1,CCN,C1,DZ,TIME,DT,TC
COMMON NZONES,ITER,N
WRITE(6,101) PR,RAm,NSTA,ALTER
101 FORMAT(*1PR NO=",FI10.5,",Z2X,RA NO=",E10.4,2X,FO NO=",FI12.4,2X,DZC
1NF=",E12.4/,",3X,",7X,*",12X,*")
N=(NZONES-NSTA)/10
IF(N.LT.3) N=1
IF(N.EQ.3) N=2
IF(N.GT.3) N=4
301 NNN=NZONES*N/100
   IF(NNN.LT.1) NNN=1
200 CONTINUE
   I=NZONES
   IF(N.GT.1) WRITE(6,100)(I,IT(I),W(I),Z(I),EST(I,1),EST(I,2))
   FO=2.0*TIME
   HEAT=0.0
   201 HEAT=HEAT+T(I)
   HEAT=HEAT*DZ
   DI=FC-HEAT
   WRITE(6,102) FC*HEAT/DI
   102 FORMAT(1HEAT BALANCE/1 HEAT IN=",E11.4,5X,HEAT ABSORBED=",E11.4", DIFFERENCE=",E11.4)
RETURN
END

SUBROUTINE PUNCH(W, T, Z, PR, RAM, NZC, NE, EST, TSTA, ISTA)
C
   PROGRAM TO PUNCH DATA
   DIMENSION Z(500), W(500), T(500)
   WRITE(7,100) PR, RAM, NZC, NE, EST, TSTA, ISTA
   100 FORMAT(F10.0, E11.4, E11.4, 2E11.4)
   WRITE(7,101)(Z(I), W(I), T(I), I=1,NZC)
   101 FORMAT(75(2CA4/))
RETURN
END

FUNCTION ESTAR(Z1, E1, Z2, RSP, RAINFC, CON)
C
   PROGRAM TO CALCULATE EQUIVALENT HEIGHT OF ENERGY PARAMETER
   ZONE=Z1*RSP
   E=E1*RSP
   ZTW=Z2*RSP
   P=E
   IF(P.EQ.1.0) GO TO 200
   AF=ABS((1.0+P)/(1.0-P))
   ZEF1=1.25 *(ALOG(AF)+2.0*ATAN(P)-4.0*P)
   ESTAR=ZTW-ZONE+ZEF1-
   200 IF(P.EQ.1.0) ESTAR=1.00E+10
RETURN
END

FUNCTION ECALC(Z, ICCUNTP, CCN)
C
   PROGRAM CONTAINING FIT OF ENERGY FUNCTION DIVIDED INTO M SEGMENTS
   B=7*Z
   C=B*Z
   D=C*Z
   E=D*Z
   F=E*Z
   GC TC (390, 391, 392, 393, 394, 395, 396, 397, ICCUNTP)
   300 ECALC=0.611657670-66C, 0.594488942-1.467225967*B+21.32132519*C-
   1459.32494*D+5853.820028*E-3103.45844*F
   GC TC 306
   301 E=1.0
   GC TC 306
   303 ECALC=1.818591851-0.6127256665*B+0.2186995937*A-4.58562377D-02*C+
   16.49712778D-03*D-3.2731449632D-04*E-1.34667783D-06*F
GO TO 306

302 ECALC=0.1754963571D+01*0.6642429381*Z+0.2016142507*D-0.3185973930D-03*E+0.1167840857D-07*F
GO TO 306

304 ECALC=1.818269075-0.6134708025*Z+0.2177383277*B-0.4652370930-01*C
1+6.127614760-03*D-4.4127868950-04*E+1.1754352960-05*F
GO TO 306

305 ECALC=0.10726734D-01+0.8925156472*Z-0.4273045517*B+0.1348613757*C-
10.02734834298*D+0.31341447620-02*F-1.513412194D-04*F
GO TO 306

307 ECALC=9.540464498D-05+0.988635367*Z-0.8812751405*B+1.898734983*C
1-5.7548472*D+11.1374636*E-1C.02511106*F
306 RETURN
END

SUBROUTINE ICH(E1,Z2,RSP,I)
C PROGRAM TO DETERMINE WHICH FIT TO USE IN CALCULATING ENERGY FLOW
E=E1*RSP
IF(Z2.GT.5.0)GO TO 300
IF(Z2.LT.-4.886)GO TO 301
IF(Z2.LT.-1.227)GO TO 302
IF(E.GT.1.0)GO TO 303
IF(Z2.GT.0.23811)GO TO 304
IF(Z2.GT.0.525)GO TO 306
I=1
GO TO 305
300 I=2
GO TO 305
301 I=3
GO TO 305
302 I=4
GO TO 305
303 I=5
GO TO 305
304 I=6
GO TO 305
306 I=7
305 RETURN
END

SUBROUTINE CINT(GRAPH,SYM,IXMAX,IYMAX,IG)
C PROGRAM TO INITIALIZE GRAPHS
DIMENSION GRAPH(101,200),SYM(14)
DO 500 I=1,IXMAX
GRAPH(I,1)=SYM(13)
GRAPH(I,YMAX)=SYM(13)
500 CONTINUE
DO 501 I=1,IXMAX
GRAPH(1,I)=SYM(14)
GRAPH(IXMAX,1)=SYM(14)
501 CONTINUE
DO 502 I=1,IXMAX,10
GRAPH(I,1)=SYM(14)
GRAPH(I,IXMAX)=SYM(14)
502 CONTINUE
DO 503 I=1,IXMAX,10
GRAPH(1,I)=SYM(13)
GRAPH(IXMAX,1)=SYM(13)
503 CONTINUE
IY=IYMAX-1
IX=IXMAX-1
DO 505 I=2,IX
505 RETURN
C PROGRAM TO PRINT GRAPH

DIMENSION GRAPH(101,200)
DIMENSION IFO(11),FO(10,3),YSAVE(11)
YSAVE(1)=0.0
DO 202 J=2,11
202 YSAVE(J)=YSAVE(J-1)+10.0*CY
II=10
X=-10.0*DX
JJ=1
WRITE(6,103)(YSAVE(J),J=1,11)
103 FORMAT(*11003.1,10(7XF3.1))
DO 200 I=1,IMAX
II=II+1
WRITE(6,101)(GRAPH(J,I),J=1,JMAX)
101 FORMAT(*11011*/*,11X,F3.1,10(7XF3.1))
IF(II.LT.11)GO TO 201
II=1
X=X+10.0*DX
WRITE(6,102)X
102 FORMAT(*,11X,FPEB.2)
200 IF(IG.EQ.2)GO TO 200
IF(IFOIDJ),GT,11GO TO 200
WRITE(6,104)FC(JJ,1)
104 FORMAT(*,112X,1PE8.2)
JJ=JJ+1
201 CONTINUE
WRITE(6,105)
105 FORMAT(*11)
400 RETURN
END

SUBROUTINE FLCTER(ITEM,S,T,GRAPH,SYM,IXMAX,IYMAX,IXMIN,IYMIN)

C PROGRAM TO PLOTT POINTS
C GRAPH TO BE PLOTTED MUST HAVE POINTS (X,Y), I.E., FOR THIS PROGRAM
C THE TEMPERATURE DATA IS FIRST
DIMENSION T(100,2)
DIMENSION GRAPH(101,200),SYM(14)
IYMAX=IYMAX+1
IXMAX=IXMAX+1
DY=(IYMAX-IYMIN)/IYMAX
DX=(IXMAX-IXMIN)/IXMAX
IYMAX=IYMAX+1
IXMAX=IXMAX+1

C IN THE DATA X IS FIRST AND Y IS SECOND
SYM=SYM(IDATA)
DO 200 J=1,IY
IX=TR(II,1)-XMIN)/DX
IY=TR(II,2)-YMIN)/DY
IF(IX.LT.0)IX=0
IF(IY.LT.0)IY=0
IX=IX+1
IY=IY+1
IF(IX.GT.IXMAX)GO TO 301
IF(IY.GT.IYMAX)GO TO 302
}
300 GRAPH(IY,IX)=SYMB
   GO TO 200
301 IX=IXMAX
   IF(IY.GT.IYMAX) IY=IYMAX
   GO TO 303
302 IY=IYMAX
303 GRAPH(IY,IX)=SYMB
200 CONTINUE
RETURN
END

SUBROUTINE TIME(W,D,T,ZCNES,DZ)
   C PROGRAM TO CALCULATE NEW TIME STEP
   DIMENSION W(500)
   WMAX=0.0
   DO 200 I=1,ZCNES
      200 IF(W(I).GE.WMAX) WMAX=W(I)
      DT=DZ/WMAX*0.9
      RETURN
END
FIG. G.4 SAMPLE TEMPERATURE OUTPUT
SAMPLE VELOCITY OUTPUT
APPENDIX H

ERRORS IN THE DETERMINATION OF FLUID VELOCITIES

H.1 Errors due to Refraction Effects

Due to the differences in refractive index between the air, glass, and the test fluid, the apparent position of a particle deviates from its actual position when the particle is not along the optical axis perpendicular to the container walls. This section considers the magnitude of the errors involved due to these refractive effects in determining the velocity field in this work.

Consider the situation given in Figure H.1. A particle G is viewed from a position F. Because of the differences in refractive index which are incurred along the line of sight, the particle appears to be at a position G'. Therefore, an error G-G' results in the estimation of the position of the particle. Letting $\mu_a$, $\mu_g$, $\mu_f$ represent the refractive index of air, glass, and the test fluid respectively, it can be seen from Snell's Law that

$$\mu_a \sin \alpha = \mu_g \sin \beta = \mu_f \sin \gamma \tag{H.1.1}$$

From geometric considerations, it can be seen that the true position of the particle, AG, is

$$AG = EF \tan \alpha + DE \tan \beta + CD \tan \alpha + BC \tan \beta + AB \tan \gamma \tag{H.1.2}$$

or

$$AG = (EF + CD) \tan \alpha + (DE + BC) \tan \beta + AB \tan \gamma \tag{H.1.3}$$

Further, define

$$g = \mu_a / \mu_g \tag{H.1.4a}$$

$$g = \mu_a / \mu_f \tag{H.1.4b}$$
Figure H.1

Refraction Effects in Determining Particle Positions
From Equations H.1.1 and H.1.4,

\[
\sin \beta = g \sin \alpha \quad \text{H.1.5a}
\]

\[
\sin \gamma = f \sin \alpha \quad \text{H.1.5b}
\]

Recognizing that

\[
\tan \theta = \frac{\sin \theta}{1 - \sin^2 \theta} \quad \text{H.1.6}
\]

and using Equations H.1.5, Equation H.1.2b can be arranged as follows:

\[
\frac{(EF + CD) \sin \alpha}{1 - \sin^2 \alpha} + \frac{(DE + BC) \frac{g}{2} \sin \alpha}{1 - \frac{2}{g} \sin^2 \alpha} + \frac{AB \frac{f}{2} \sin \alpha}{1 - \frac{f}{2} \sin^2 \alpha} = \overline{AG} \quad \text{H.1.7}
\]

From geometrical considerations,

\[
\sin \alpha = \frac{\overline{AG}'}{(\overline{AG}^2 + \overline{AF}^2)^{1/2}} \quad \text{H.1.8}
\]

Therefore, Equation H.1.7 can be rewritten as:

\[
\frac{(EF + CD) \overline{AG}'}{\overline{AF}} + \frac{(DE + BC) \frac{g}{2} \overline{AG}'}{(1 - \frac{2}{g} \overline{AG}^2 + \overline{AF}^2)^{1/2}} + \frac{AB \frac{f}{2} \overline{AG}'}{(1 - \frac{f}{2} \overline{AG}^2 + \overline{AF}^2)^{1/2}} = \overline{AG} \quad \text{H.1.9}
\]

The further away a particle is from the optical axis, \( \overline{AF} \), the greater the error. Figure H.2 shows the difference between the observed and the actual readings at various distances from the optical axis for the conditions used in this work (Table H.1).
Figure H.2
Differences Between Observed and Actual Positions Due to Refraction Effects

Figure H.3
Determination of True Length of Streak Line
TABLE H.1

OPTICAL PARAMETERS USED IN DETERMINING VELOCITY

<table>
<thead>
<tr>
<th>Distance (inches)</th>
<th>AB</th>
<th>BC</th>
<th>CD</th>
<th>DE</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.0</td>
<td>0.25</td>
<td>0.5</td>
<td>0.25</td>
<td>5.0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AIR</th>
<th>GLASS</th>
<th>WATER</th>
<th>GLYCERINE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.59</td>
<td>1.33</td>
<td>1.47</td>
</tr>
</tbody>
</table>

Because of the nature of the experimental system, it can be seen from Figure H.2 that the observed positions deviate from the correct positions linearly with distance from the optical axis of the system. This fortuitous property minimizes the number of calculations that need be performed in calculating the velocity. For, suppose one wished to determine the velocity represented by the streak $\overline{AB}$ in Figure H.3. From Figure H.2,

$$\overline{AG'} = \Delta^T \overline{AG} \quad \text{H.1.10}$$

where $\Delta^T$ is the slope of the pertinent curve in Figure H.2. Now,

$$\overline{AB} = \overline{OA} - \overline{OB} \quad \text{H.1.11}$$

or

$$\overline{AB} = \Delta^T (\overline{OA'} - \overline{OB'}) = \Delta^T \overline{A'B'} \quad \text{H.1.12}$$

where $A'B'$ is the true length of the curve. Therefore, only the slope $\Delta^T$ need be known in order to calculate the true position of the particle, and thus the repeated use of the rather long Equation H.1.9 can be avoided.

In this work, the magnification of the photograph was determined by measuring the distance between two points on a grid of known size which was immersed into the test fluid prior to the test. From this, an "apparent" magnification can be calculated, i.e.,
However, because of the effect of refraction, it can be seen that

\[ m_a = \frac{m}{\Delta r} \quad \text{H.1.13} \]

where \( m \) is the true magnification. Therefore, from this measurement, the true velocity at any point may be calculated. For, if \( ab \) is the measured length of \( AB \) in the magnified picture, then,

\[ AB = \frac{ab}{m_a} = \frac{AB}{m_a} \quad \text{H.1.14} \]

i.e., the apparent magnification includes the effect of refraction.

H.2 Errors due to Data Reduction

Fluid velocities are determined from a photograph of known magnification. The "apparent" magnification, \( m_a \), is determined by measuring the distance between two points on a grid which has been immersed in the test plane of the enclosure prior to the test run. In determining the "apparent" magnification, the distance between grid points which lie two inches apart are measured. For example, in Test G-3-HI, the distance between the grid points was 3.25 inches \( \pm 0.05 \) inches. Therefore, the "apparent" magnification was

\[ m_a = 1.62 \pm 0.03 \]

The coordinates of a streak required for obtaining the velocity were obtained by imposing a grid on the photograph and reading its coordinates. The coordinates could
be read in this way to approximately ± 0.025 inches. In order to determine the vertical velocity, the vertical distance between the top of the streak and the bottom of the streak, \( l \), was required. This distance was on the order of 0.40 inches for this test for an exposure time, \( t \), of 10 seconds ± 0.1 seconds. The vertical velocity was calculated to be

\[
w = \frac{1}{(t_m a)} \\
= \frac{0.3}{(10 \times 1.62)} \\
= 0.0108 \text{ in./sec. ±15 percent}
\]

Therefore, the streak photographs were able to determine velocities to better than ± 15 percent.

H.3 Suitability of Polystyrene Particles as Velocity Tracers

In order that a particle submerged in a fluid give an accurate representation of the fluid velocity and path, it is necessary that:

1. the particle terminal velocity, \( v_t \), be small with respect to the fluid velocity; (error due to density differences)

2. the particle inertia be small so that the particle follows the fluid streamlines when it is accelerated and decelerated. (Error due to particle size.)

In this work, polystyrene particles coated with aluminum were used as velocity tracers. Their suitability in light of the above criteria are examined in this section.
H.3.1  Density of the Particles

The density of the polystyrene beads, $\rho_s$, was 1.05 g/cc. However, aluminum was vapor deposited on the surface of these beads in order to provide the isotropic reflection characteristics necessary in this work. The extent that this deposition of aluminum increased the particle density is now considered.

The equivalent density, $\rho_e$, of a sphere with an inner core of density $\rho_s$ and an outer shell of density $\rho_{Al}$ is

$$\rho_e = \rho_{Al} - (\rho_{Al} - \rho_s)(d/D)^3 \quad \text{H.3.1}$$

where $d$ is the diameter of the inner core, and $D$ is the outer diameter. In this case, the inner sphere is polystyrene ($\rho_s = 1.05$ g/cc) and the outer shell is aluminum ($\rho_{Al} = 2.7$ g/cc). Vapor deposition of aluminum on the sphere leaves, at most, about 10 monolayers of aluminum on the surface. The atomic radius of an aluminum molecule is about 1.5 Å. Therefore, the shell thickness is about 15 Å. For the 35μ polystyrene particles used in this work, the density would have changed by

$$\frac{\rho_e - \rho_s}{\rho_s} = \frac{1.65(1 - \frac{350000}{350030})}{1.05} \approx 0 \quad \text{H.3.2}$$

Therefore, vapor deposition of aluminum does not affect the density of the test particles markedly. In fact, in order to change the density by 1 percent, a deposition of 7800 Å of aluminum would be required. This is far in excess of what was actually deposited.
H.3.2 Terminal Velocity of the Test Particles

Because of the difference in densities between the polystyrene particles ($\rho_s = 1.05$ g/cc) and the test fluids (water, $\rho_w = 1.00$ g/cc, glycerine, $\rho_g = 1.25$ g/cc), there will be a difference in velocity between the fluid and the sphere. From Stoke's Law, this difference in velocities, $v_t$, is

$$v_t = v_p - v_f = \frac{(\rho_s - \rho)gD^2}{18 \rho_f}$$

where $\rho$ is the fluid density, $v_p$ is the particle velocity, $v_f$ is the fluid velocity, and $D$ is the diameter of the particle. For this system,

$$D = 35\mu$$

kinematic viscosity of water, $\gamma_w = 1.083 \times 10^{-5}$ ft$^2$/sec.

kinematic viscosity of glycerine, $\gamma_g = 0.0127$ ft$^2$/sec.

Then, the terminal velocity of the particle in water is

$$v_t = \frac{(1.05 - 1.00)\times32.2\times(35)^2}{18(1.083)\times10^{-5}\times(30.48)^2} \times 10^{-8}$$

$$= 1.09(10^{-4}) \text{ ft/sec}$$

while for glycerine it is

$$v_t = 4.64 \times 10^{-7} \text{ ft/sec}$$

The fluid velocity in the bulk of the fluid is on the order of $10^{-3}$ ft/sec or greater. Therefore, the error in velocity due to density differences between the particle tracers and fluid is less than 15 percent.
H.3.3 Inertial Effects on the Particle Tracers

The question which remains is how well do the particles follow the flow. Inertial effects may cause the particle to deviate from the flow path in an accelerating fluid or decelerating fluid. The section is concerned with estimating the error which might be induced in this system due to inertial effects.

Assuming that "creeping" flow occurs around the sphere at all times, a force balance can be written around the sphere. By Newton's Law,

\[ \frac{\pi D^3}{6} \rho_s \frac{dv}{dt} p = \frac{2}{5} \left( \rho - \rho_s \right) - 3 \eta u D \left( v_p - v_f \right) \]

or, rearranging,

\[ \frac{dv}{dt} p = \frac{\left( \rho - \rho_s \right)}{\rho_s} g - \frac{18uv}{\rho_s D^2} p + \frac{18uv}{\rho_s D^2} f \]

In order to examine the inertial effects, suppose the fluid velocity varies as:

\[ v_f = v \sin(\omega t) \]

Further, define

\[ P_f = \frac{18u}{\rho_s D^2} \]

Therefore, the equation which is to be solved can be written as:

\[ \frac{dv}{dt} p = \frac{\rho - \rho_s}{\rho_s} g - P_f v_p + P_f v \sin(\omega t) \]

Using a LaPlace transformation of Equation H.3.8,

\[ s \bar{v}_p = \frac{\rho - \rho_s}{\rho_s} \bar{g} - P_f \bar{v}_s + \frac{P_f \omega}{s^2 + \omega^2} \]

Rearranging, one obtains
\[ v_p = \frac{\rho - \rho_s}{\rho_s} g \frac{1}{s(s + P_f)} + \frac{P_f v_0}{(s^2 + \omega^2)(s + P_f)} \]  \hspace{1cm} H.3.10

Inverting the Laplace transform, the equation for the velocity of the particle is:

\[ v_p = g \frac{\rho - \rho_s}{\rho_s} \frac{1 - \exp(-P_f t)}{P_f} + \frac{P_f v_0}{P_f^2 + \omega^2} \exp(-P_f t) - \cos(\omega t) + \frac{P_f^2 v}{P_f^2 + \omega^2} \sin(\omega t) \]  \hspace{1cm} H.3.11

At long times,

\[ \exp(-P_f t) \to 0 \]  \hspace{1cm} H.3.12

Therefore,

\[ v_p = g \frac{\rho - \rho_s}{\rho_s} + \frac{P_f v}{P_f^2 + \omega^2} P_f \sin(\omega t) - \omega \cos(\omega t) \]  \hspace{1cm} H.3.13

Now, combining Equations H.3.13, H.3.7 and H.3.3, one obtains

\[ v_p = - v_t + \frac{P_f v}{P_f^2 + \omega^2} (P_f \sin(\omega t) - \omega \cos(\omega t)) \]  \hspace{1cm} H.3.14

From Section H.3.2, it can be assumed that the terminal velocity is negligible so that

\[ v_p = \frac{P_f v}{P_f^2 + \omega^2} (P_f \sin(\omega t) - \omega \cos(\omega t)) \]  \hspace{1cm} H.3.15

When the fluid moves with a velocity of the form given in Equation H.3.6, inertial effects will cause the particle to be slightly out of phase and its amplitude to be smaller than that of the fluid, i.e.,

\[ v_p = v^+ \sin(\omega t + \phi) \]  \hspace{1cm} H.3.16

Combining this equation with Equation H.3.15, one obtains
\[ v^+ \sin(\omega t + \phi) = v^+ (\sin(\omega t) \cos \phi + \sin \phi \cos(\omega t)) \]
\[ = \frac{P_f v}{(P_f^2 + \omega^2)^{1/2}} \quad \frac{P_f \sin(\omega t)}{(P_f^2 + \omega^2)^{1/2}} - \frac{\omega \cos(\omega t)}{(P_f^2 + \omega^2)^{1/2}} \quad \text{H.3.17} \]

Identifying terms, one can see that

\[ \phi = \tan^{-1} \left( \frac{\omega}{P_f} \right) \quad \text{H.3.18} \]

and

\[ \frac{v^+}{v} = \frac{P_f}{(P_f^2 + \omega^2)^{1/2}} \quad \text{H.3.19} \]

For this work, the minimum value of \( P_f \) is for water. Hence, the most serious errors occur in the water case when the fluid is decelerating near the surface. As an order of magnitude, the maximum velocity decelerates from about 10^{-2} \text{ft/sec} to 0 \text{ ft/sec} in about two inches. This can be thought of as being about 1/4 of a cycle, so that,

\[ \omega = 2\pi \frac{10^{-2} \times 12}{2} \approx 1.5 \]

and

\[ P_f = \frac{18 \times 1 \times 1.083 \times 10^{-5} \times 30.48^2}{1 \times 1.05 \times 35^2 \times 10^{-8}} \approx 1.4 \times 10^4 \]

Therefore, for this work,

\[ \phi \approx \tan^{-1}(0) = 0 \]
\[ \frac{v^+}{v} \approx 1 \]

For this system, inertial effects play a negligible role and the particle follows the fluid motion closely. The most serious error is due to differences in the density which result in a difference in velocities between the fluid and the tracer particles. The terminal velocity is about 11 percent of the total velocity. Therefore, the streak photography technique can determine the fluid velocity in the bulk fluid to better than \( \pm 25 \) percent.
APPENDIX I

PLATE COMPUTER OUTPUT FOR THE CASE OF A UNIFORMLY HEATED VERTICAL PLATE IN SURROUNDINGS WITH A LINEAR TEMPERATURE GRADIENT

In this Appendix, the computer output from PLATE is presented. The temperature and velocity profiles for the case of a uniformly heated vertical wall in surroundings with a linear temperature gradient as a function of time are presented.

Firstly, the wall temperature as a function of time is listed. Then, the interior velocity and temperature profiles are presented.

For this test, IMAX=6. Therefore, ΔZ=1.67 since ZMAX=10.0.
<table>
<thead>
<tr>
<th>TIME</th>
<th>0.1</th>
<th>0.5</th>
<th>1.1</th>
<th>1.5</th>
<th>2.1</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.103</td>
<td>1.688</td>
<td>1.860</td>
<td>1.884</td>
<td>1.889</td>
<td>1.890</td>
</tr>
<tr>
<td>2</td>
<td>1.103</td>
<td>1.712</td>
<td>1.927</td>
<td>1.962</td>
<td>1.973</td>
<td>1.976</td>
</tr>
<tr>
<td>3</td>
<td>1.103</td>
<td>1.712</td>
<td>1.931</td>
<td>1.969</td>
<td>1.981</td>
<td>1.984</td>
</tr>
<tr>
<td>4</td>
<td>1.103</td>
<td>1.712</td>
<td>1.931</td>
<td>1.969</td>
<td>1.982</td>
<td>1.985</td>
</tr>
<tr>
<td>5</td>
<td>1.103</td>
<td>1.712</td>
<td>1.931</td>
<td>1.969</td>
<td>1.982</td>
<td>1.985</td>
</tr>
<tr>
<td>6</td>
<td>1.103</td>
<td>1.712</td>
<td>1.931</td>
<td>1.969</td>
<td>1.982</td>
<td>1.985</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TIME</th>
<th>3.1</th>
<th>3.5</th>
<th>4.1</th>
<th>4.5</th>
<th>5.1</th>
<th>5.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.890</td>
<td>1.890</td>
<td>1.890</td>
<td>1.890</td>
<td>1.890</td>
<td>1.890</td>
</tr>
<tr>
<td>2</td>
<td>1.976</td>
<td>1.976</td>
<td>1.976</td>
<td>1.976</td>
<td>1.976</td>
<td>1.976</td>
</tr>
<tr>
<td>3</td>
<td>1.984</td>
<td>1.985</td>
<td>1.986</td>
<td>1.985</td>
<td>1.985</td>
<td>1.985</td>
</tr>
<tr>
<td>4</td>
<td>1.985</td>
<td>1.986</td>
<td>1.985</td>
<td>1.986</td>
<td>1.985</td>
<td>1.986</td>
</tr>
<tr>
<td>5</td>
<td>1.985</td>
<td>1.986</td>
<td>1.985</td>
<td>1.986</td>
<td>1.986</td>
<td>1.986</td>
</tr>
<tr>
<td>6</td>
<td>1.985</td>
<td>1.986</td>
<td>1.986</td>
<td>1.986</td>
<td>1.986</td>
<td>1.986</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TIME</th>
<th>5.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.890</td>
</tr>
<tr>
<td>2</td>
<td>1.976</td>
</tr>
<tr>
<td>3</td>
<td>1.985</td>
</tr>
<tr>
<td>4</td>
<td>1.986</td>
</tr>
<tr>
<td>5</td>
<td>1.986</td>
</tr>
<tr>
<td>6</td>
<td>1.986</td>
</tr>
</tbody>
</table>
### TEMPERATURE FIELD

<table>
<thead>
<tr>
<th>J=</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5870</td>
<td>0.62650</td>
<td>0.14220</td>
<td>0.33333</td>
<td>0.71642</td>
<td>0.27170</td>
<td>0.49950</td>
<td>0.17473</td>
<td>0.17374</td>
<td>3.55393</td>
</tr>
<tr>
<td>2</td>
<td>0.25870</td>
<td>0.62650</td>
<td>0.14220</td>
<td>0.33333</td>
<td>0.71642</td>
<td>0.27170</td>
<td>0.49950</td>
<td>0.17473</td>
<td>0.17374</td>
<td>3.55393</td>
</tr>
<tr>
<td>3</td>
<td>0.25870</td>
<td>0.62650</td>
<td>0.14220</td>
<td>0.33333</td>
<td>0.71642</td>
<td>0.27170</td>
<td>0.49950</td>
<td>0.17473</td>
<td>0.17374</td>
<td>3.55393</td>
</tr>
<tr>
<td>4</td>
<td>0.25870</td>
<td>0.62650</td>
<td>0.14220</td>
<td>0.33333</td>
<td>0.71642</td>
<td>0.27170</td>
<td>0.49950</td>
<td>0.17473</td>
<td>0.17374</td>
<td>3.55393</td>
</tr>
<tr>
<td>5</td>
<td>0.25870</td>
<td>0.62650</td>
<td>0.14220</td>
<td>0.33333</td>
<td>0.71642</td>
<td>0.27170</td>
<td>0.49950</td>
<td>0.17473</td>
<td>0.17374</td>
<td>3.55393</td>
</tr>
<tr>
<td>6</td>
<td>0.25870</td>
<td>0.62650</td>
<td>0.14220</td>
<td>0.33333</td>
<td>0.71642</td>
<td>0.27170</td>
<td>0.49950</td>
<td>0.17473</td>
<td>0.17374</td>
<td>3.55393</td>
</tr>
</tbody>
</table>

### VERTICAL VELOCITY FIELD

<table>
<thead>
<tr>
<th>J=</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.46000</td>
<td>0.38880</td>
<td>0.26760</td>
<td>0.16560</td>
<td>0.13243</td>
<td>0.27272</td>
<td>0.38130</td>
<td>0.23370</td>
<td>0.14790</td>
<td>0.06900</td>
</tr>
<tr>
<td>2</td>
<td>0.46000</td>
<td>0.38880</td>
<td>0.26760</td>
<td>0.16560</td>
<td>0.13243</td>
<td>0.27272</td>
<td>0.38130</td>
<td>0.23370</td>
<td>0.14790</td>
<td>0.06900</td>
</tr>
<tr>
<td>3</td>
<td>0.46000</td>
<td>0.38880</td>
<td>0.26760</td>
<td>0.16560</td>
<td>0.13243</td>
<td>0.27272</td>
<td>0.38130</td>
<td>0.23370</td>
<td>0.14790</td>
<td>0.06900</td>
</tr>
<tr>
<td>4</td>
<td>0.46000</td>
<td>0.38880</td>
<td>0.26760</td>
<td>0.16560</td>
<td>0.13243</td>
<td>0.27272</td>
<td>0.38130</td>
<td>0.23370</td>
<td>0.14790</td>
<td>0.06900</td>
</tr>
<tr>
<td>5</td>
<td>0.46000</td>
<td>0.38880</td>
<td>0.26760</td>
<td>0.16560</td>
<td>0.13243</td>
<td>0.27272</td>
<td>0.38130</td>
<td>0.23370</td>
<td>0.14790</td>
<td>0.06900</td>
</tr>
<tr>
<td>6</td>
<td>0.46000</td>
<td>0.38880</td>
<td>0.26760</td>
<td>0.16560</td>
<td>0.13243</td>
<td>0.27272</td>
<td>0.38130</td>
<td>0.23370</td>
<td>0.14790</td>
<td>0.06900</td>
</tr>
</tbody>
</table>

### HORIZONTAL VELOCITY FIELD

<table>
<thead>
<tr>
<th>J=</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1380</td>
<td>-0.7545</td>
<td>-0.3645</td>
<td>-0.828</td>
<td>-0.4353</td>
<td>-0.7432</td>
<td>-0.4648</td>
<td>-0.06590</td>
<td>0.4551</td>
<td>0.4677</td>
</tr>
<tr>
<td>2</td>
<td>-0.0000</td>
<td>0.3000</td>
<td>0.2000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>-0.0000</td>
<td>0.3000</td>
<td>0.2000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>-0.0000</td>
<td>0.3000</td>
<td>0.2000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>5</td>
<td>-0.0000</td>
<td>0.3000</td>
<td>0.2000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>6</td>
<td>-0.0000</td>
<td>0.3000</td>
<td>0.2000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
### Temperature Field

<table>
<thead>
<tr>
<th>J=</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.78250</td>
<td>0.37400</td>
<td>0.46980</td>
<td>0.17400</td>
<td>-0.42720</td>
<td>-0.63600</td>
<td>-0.68600</td>
<td>0.70700</td>
<td>0.14640</td>
<td>0.01210</td>
</tr>
<tr>
<td>2</td>
<td>0.80510</td>
<td>0.37870</td>
<td>0.54710</td>
<td>0.26890</td>
<td>-0.43550</td>
<td>-0.61320</td>
<td>-0.62320</td>
<td>0.68700</td>
<td>0.13640</td>
<td>0.01290</td>
</tr>
<tr>
<td>3</td>
<td>0.82750</td>
<td>0.38230</td>
<td>0.54870</td>
<td>0.26810</td>
<td>-0.43880</td>
<td>-0.61850</td>
<td>-0.62950</td>
<td>0.65500</td>
<td>0.12980</td>
<td>0.01370</td>
</tr>
<tr>
<td>4</td>
<td>0.85050</td>
<td>0.38600</td>
<td>0.54880</td>
<td>0.26810</td>
<td>-0.43880</td>
<td>-0.62380</td>
<td>-0.63550</td>
<td>0.62600</td>
<td>0.12420</td>
<td>0.01460</td>
</tr>
<tr>
<td>5</td>
<td>0.85050</td>
<td>0.38600</td>
<td>0.54880</td>
<td>0.26810</td>
<td>-0.43880</td>
<td>-0.62810</td>
<td>-0.64150</td>
<td>0.59800</td>
<td>0.11860</td>
<td>0.01550</td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>J=</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20730</td>
<td>0.21610</td>
<td>0.16970</td>
<td>0.11740</td>
<td>0.07130</td>
<td>0.47990</td>
<td>0.29310</td>
<td>0.17440</td>
<td>0.01590</td>
<td>0.00630</td>
</tr>
<tr>
<td>2</td>
<td>0.21090</td>
<td>0.22610</td>
<td>0.17860</td>
<td>0.12470</td>
<td>0.08050</td>
<td>0.57590</td>
<td>0.31110</td>
<td>0.18850</td>
<td>0.01190</td>
<td>0.00680</td>
</tr>
<tr>
<td>3</td>
<td>0.21460</td>
<td>0.23640</td>
<td>0.18790</td>
<td>0.13140</td>
<td>0.09080</td>
<td>0.60780</td>
<td>0.31170</td>
<td>0.19900</td>
<td>0.01180</td>
<td>0.00680</td>
</tr>
<tr>
<td>4</td>
<td>0.21820</td>
<td>0.24660</td>
<td>0.19740</td>
<td>0.13740</td>
<td>0.10080</td>
<td>0.63970</td>
<td>0.31230</td>
<td>0.20930</td>
<td>0.01180</td>
<td>0.00680</td>
</tr>
<tr>
<td>5</td>
<td>0.22180</td>
<td>0.25660</td>
<td>0.20700</td>
<td>0.14330</td>
<td>0.11060</td>
<td>0.67160</td>
<td>0.31290</td>
<td>0.21920</td>
<td>0.01180</td>
<td>0.00680</td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J=</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.67700</td>
<td>0.12550</td>
<td>0.17640</td>
<td>0.21170</td>
<td>-0.32460</td>
<td>0.94280</td>
<td>0.52750</td>
<td>0.24800</td>
<td>0.07600</td>
<td>0.06270</td>
</tr>
<tr>
<td>2</td>
<td>0.69930</td>
<td>0.12960</td>
<td>0.18640</td>
<td>0.20680</td>
<td>-0.31650</td>
<td>0.92360</td>
<td>0.51750</td>
<td>0.23800</td>
<td>0.06600</td>
<td>0.05270</td>
</tr>
<tr>
<td>3</td>
<td>0.72130</td>
<td>0.13370</td>
<td>0.19650</td>
<td>0.20170</td>
<td>-0.30740</td>
<td>0.91350</td>
<td>0.50750</td>
<td>0.22800</td>
<td>0.05600</td>
<td>0.04270</td>
</tr>
<tr>
<td>4</td>
<td>0.74320</td>
<td>0.13780</td>
<td>0.20660</td>
<td>0.19670</td>
<td>-0.29830</td>
<td>0.90350</td>
<td>0.49750</td>
<td>0.21800</td>
<td>0.04600</td>
<td>0.03270</td>
</tr>
<tr>
<td>5</td>
<td>0.76510</td>
<td>0.14190</td>
<td>0.21670</td>
<td>0.19170</td>
<td>-0.28930</td>
<td>0.89350</td>
<td>0.48750</td>
<td>0.20800</td>
<td>0.03600</td>
<td>0.02270</td>
</tr>
</tbody>
</table>
### Temperature Field

<table>
<thead>
<tr>
<th>( j )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
<th>( 9 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0.034770</td>
<td>0.353520</td>
<td>0.031900</td>
<td>0.18700</td>
<td>0.0647700</td>
<td>0.074500</td>
<td>0.074500</td>
<td>0.074500</td>
<td>0.074500</td>
<td>0.074500</td>
</tr>
<tr>
<td>( y )</td>
<td>0.096000</td>
<td>0.171400</td>
<td>0.040600</td>
<td>0.078970</td>
<td>0.017080</td>
<td>0.008160</td>
<td>0.532100</td>
<td>0.057100</td>
<td>0.009900</td>
<td>0.085700</td>
</tr>
<tr>
<td>( z )</td>
<td>0.103900</td>
<td>0.174200</td>
<td>0.041500</td>
<td>0.082700</td>
<td>0.019200</td>
<td>0.008570</td>
<td>0.581600</td>
<td>0.068900</td>
<td>0.012100</td>
<td>0.071800</td>
</tr>
<tr>
<td>( v )</td>
<td>0.003900</td>
<td>0.174600</td>
<td>0.041500</td>
<td>0.082700</td>
<td>0.019200</td>
<td>0.008570</td>
<td>0.581600</td>
<td>0.068900</td>
<td>0.012100</td>
<td>0.071800</td>
</tr>
<tr>
<td>( w )</td>
<td>0.103900</td>
<td>0.174200</td>
<td>0.041500</td>
<td>0.082700</td>
<td>0.019200</td>
<td>0.008570</td>
<td>0.581600</td>
<td>0.068900</td>
<td>0.012100</td>
<td>0.071800</td>
</tr>
<tr>
<td>( j_x )</td>
<td>0.042100</td>
<td>0.274600</td>
<td>0.0419700</td>
<td>0.089900</td>
<td>0.018000</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
</tr>
<tr>
<td>( j_y )</td>
<td>0.042100</td>
<td>0.274600</td>
<td>0.0419700</td>
<td>0.089900</td>
<td>0.018000</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
</tr>
<tr>
<td>( j_z )</td>
<td>0.042100</td>
<td>0.274600</td>
<td>0.0419700</td>
<td>0.089900</td>
<td>0.018000</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
<td>0.0171200</td>
</tr>
<tr>
<td>( J )</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>( k )</td>
<td>0.723000</td>
<td>0.211200</td>
<td>0.017300</td>
<td>0.013050</td>
<td>0.008450</td>
<td>0.005500</td>
<td>0.003100</td>
<td>0.001900</td>
<td>0.001200</td>
<td>0.000700</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.295000</td>
<td>0.305000</td>
<td>0.012500</td>
<td>0.008300</td>
<td>0.005300</td>
<td>0.003100</td>
<td>0.001900</td>
<td>0.001200</td>
<td>0.000700</td>
<td>0.000400</td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>( j )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
<th>( 9 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0.074700</td>
<td>0.274800</td>
<td>0.017500</td>
<td>0.004999</td>
<td>0.0061000</td>
<td>0.015053</td>
<td>0.014480</td>
<td>0.019020</td>
<td>0.017660</td>
<td>0.014050</td>
</tr>
<tr>
<td>( y )</td>
<td>0.251900</td>
<td>0.274700</td>
<td>0.019480</td>
<td>0.0013700</td>
<td>0.0041500</td>
<td>0.014750</td>
<td>0.014850</td>
<td>0.015020</td>
<td>0.015030</td>
<td>0.015040</td>
</tr>
<tr>
<td>( z )</td>
<td>0.253100</td>
<td>0.276600</td>
<td>0.019780</td>
<td>0.0018990</td>
<td>0.0049890</td>
<td>0.019080</td>
<td>0.018720</td>
<td>0.018590</td>
<td>0.018580</td>
<td>0.018570</td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>( j )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
<th>( 9 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>-0.110100</td>
<td>-0.673100</td>
<td>-0.028800</td>
<td>-0.213700</td>
<td>-0.121300</td>
<td>-0.354433</td>
<td>-0.210200</td>
<td>-0.287600</td>
<td>-0.232400</td>
<td>-0.217000</td>
</tr>
<tr>
<td>( y )</td>
<td>-0.105900</td>
<td>-0.691400</td>
<td>-0.029000</td>
<td>-0.212300</td>
<td>-0.120300</td>
<td>-0.353433</td>
<td>-0.210200</td>
<td>-0.287600</td>
<td>-0.232400</td>
<td>-0.217000</td>
</tr>
<tr>
<td>( z )</td>
<td>-0.105900</td>
<td>-0.691400</td>
<td>-0.029000</td>
<td>-0.212300</td>
<td>-0.120300</td>
<td>-0.353433</td>
<td>-0.210200</td>
<td>-0.287600</td>
<td>-0.232400</td>
<td>-0.217000</td>
</tr>
<tr>
<td>( J )</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
</tbody>
</table>
### Temperature Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.958700</td>
<td>0.347400</td>
<td>0.149390-01</td>
<td>0.1870-01</td>
<td>0.9680-01</td>
<td>0.31263-01</td>
<td>0.60680-01</td>
<td>0.16900-01</td>
<td>0.2940-02</td>
<td>0.34100-02</td>
</tr>
<tr>
<td>2</td>
<td>0.103100</td>
<td>0.349400</td>
<td>0.45310-01</td>
<td>0.6990-01</td>
<td>0.10960-01</td>
<td>0.33990-01</td>
<td>0.62820-01</td>
<td>0.2050-01</td>
<td>0.4780-02</td>
<td>0.44870-02</td>
</tr>
<tr>
<td>3</td>
<td>0.103700</td>
<td>0.349300</td>
<td>0.46550-01</td>
<td>0.9010-01</td>
<td>0.11110-01</td>
<td>0.48460-01</td>
<td>0.49170-01</td>
<td>0.2020-01</td>
<td>0.4760-02</td>
<td>0.44920-02</td>
</tr>
<tr>
<td>4</td>
<td>0.103800</td>
<td>0.349300</td>
<td>0.46780-01</td>
<td>0.90160-01</td>
<td>0.11120-01</td>
<td>0.47540-01</td>
<td>0.46760-01</td>
<td>0.1970-01</td>
<td>0.4650-02</td>
<td>0.45270-02</td>
</tr>
<tr>
<td>5</td>
<td>0.108100</td>
<td>0.339500</td>
<td>0.46740-01</td>
<td>0.90170-01</td>
<td>0.11150-02</td>
<td>0.54820-01</td>
<td>0.49260-01</td>
<td>0.1970-01</td>
<td>0.4650-02</td>
<td>0.45270-02</td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.256000</td>
<td>0.261600</td>
<td>0.188500</td>
<td>0.108700</td>
<td>0.50720-01</td>
<td>0.17810-1</td>
<td>0.34990-02</td>
<td>0.47480-02</td>
<td>0.1190-02</td>
<td>0.36410-02</td>
</tr>
<tr>
<td>2</td>
<td>0.246000</td>
<td>0.291700</td>
<td>0.212900</td>
<td>0.122900</td>
<td>0.54640-01</td>
<td>0.18280-01</td>
<td>0.16030-02</td>
<td>0.24400-02</td>
<td>0.6190-03</td>
<td>0.2880-02</td>
</tr>
<tr>
<td>3</td>
<td>0.276000</td>
<td>0.294600</td>
<td>0.214000</td>
<td>0.124000</td>
<td>0.57090-01</td>
<td>0.18720-01</td>
<td>0.16250-02</td>
<td>0.2720-02</td>
<td>0.7820-03</td>
<td>0.2710-02</td>
</tr>
<tr>
<td>4</td>
<td>0.276900</td>
<td>0.294900</td>
<td>0.215200</td>
<td>0.125300</td>
<td>0.57150-01</td>
<td>0.18750-01</td>
<td>0.16270-02</td>
<td>0.2740-02</td>
<td>0.8060-03</td>
<td>0.2690-02</td>
</tr>
<tr>
<td>5</td>
<td>0.276900</td>
<td>0.294900</td>
<td>0.215200</td>
<td>0.125300</td>
<td>0.57160-01</td>
<td>0.18760-01</td>
<td>0.16270-02</td>
<td>0.2740-02</td>
<td>0.8060-03</td>
<td>0.2690-02</td>
</tr>
<tr>
<td>6</td>
<td>0.276700</td>
<td>0.294900</td>
<td>0.215200</td>
<td>0.125300</td>
<td>0.57160-01</td>
<td>0.18760-01</td>
<td>0.16270-02</td>
<td>0.2740-02</td>
<td>0.8060-03</td>
<td>0.2690-02</td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.75010-01</td>
<td>0.153500</td>
<td>0.210000</td>
<td>0.247600</td>
<td>0.257900</td>
<td>0.255300</td>
<td>0.246200</td>
<td>0.536400</td>
<td>0.265700</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.71730-01</td>
<td>0.16340-01</td>
<td>0.23590-01</td>
<td>0.27790-01</td>
<td>0.29510-01</td>
<td>0.29430-01</td>
<td>0.28320-01</td>
<td>0.27810-01</td>
<td>0.2750-01</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.69780-01</td>
<td>0.15550-01</td>
<td>0.23090-01</td>
<td>0.26540-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
</tr>
<tr>
<td>4</td>
<td>0.67630-01</td>
<td>0.15910-01</td>
<td>0.24040-01</td>
<td>0.26800-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
</tr>
<tr>
<td>5</td>
<td>0.63110-01</td>
<td>0.14190-01</td>
<td>0.26310-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.61870-01</td>
<td>0.14330-01</td>
<td>0.26540-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.58700-01</td>
<td>0.15070-01</td>
<td>0.27380-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.58700-01</td>
<td>0.15070-01</td>
<td>0.27380-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.58700-01</td>
<td>0.15070-01</td>
<td>0.27380-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.58700-01</td>
<td>0.15070-01</td>
<td>0.27380-01</td>
<td>0.29590-01</td>
<td>0.29460-01</td>
<td>0.29630-01</td>
<td>0.28990-01</td>
<td>0.27920-01</td>
<td>0.2660-01</td>
<td></td>
</tr>
</tbody>
</table>
### Temperature Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9430 00 0.9430 00 0.8176-03 0.8637-01 0.1008 03 0.7537-01 0.4720-01 0.1746-01 0.4933-02 0.1830-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1040 01 0.8920 00 0.4248-01 0.9540-01 0.1160 03 0.9150-01 0.1259-01 0.2360-01 0.5490-02 0.2440-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1040 01 0.9770 00 0.4390-01 0.9690-01 0.1180 03 0.9708-01 0.1516-01 0.2740-01 0.6198-02 0.2506-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.1040 01 0.9820 00 0.4414-01 0.9760-01 0.1140 33 0.4735-01 0.3799-01 0.2410-01 0.6170-02 0.2510-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1040 01 0.9880 00 0.4410-01 0.9760-01 0.1140 33 0.4735-01 0.3799-01 0.2410-01 0.6170-02 0.2510-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2490 33 0.2580 00 0.1870 00 0.1013 00 3.4013-01 0.5980-02 0.9860-02 0.1710-01 0.1097-01 0.9590-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2750 00 0.2900 00 0.2070 00 0.1140 00 3.4480-01 0.6080-02 0.1240-01 0.1640-01 0.1500-01 0.8540-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.2780 00 0.2930 00 0.2100 00 0.1150 00 3.4540-01 0.4870-02 0.1300-01 0.1730-01 0.1540-01 0.1170-01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9430 00 0.9430 00 0.8176-03 0.8637-01 0.1008 03 0.7537-01 0.4720-01 0.1746-01 0.4933-02 0.1830-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1040 01 0.8920 00 0.4248-01 0.9540-01 0.1160 03 0.9150-01 0.1259-01 0.2360-01 0.5490-02 0.2440-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1040 01 0.9770 00 0.4390-01 0.9690-01 0.1180 03 0.9708-01 0.1516-01 0.2740-01 0.6198-02 0.2506-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.1040 01 0.9820 00 0.4414-01 0.9760-01 0.1140 33 0.4735-01 0.3799-01 0.2410-01 0.6170-02 0.2510-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1040 01 0.9880 00 0.4410-01 0.9760-01 0.1140 33 0.4735-01 0.3799-01 0.2410-01 0.6170-02 0.2510-02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### TEMPERATURE FIELD

<table>
<thead>
<tr>
<th>I</th>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
<tr>
<td>0.44850</td>
<td>0.34940</td>
<td>0.37260</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.21760</td>
<td>0.23070</td>
<td>0.24080</td>
<td>0.24930</td>
<td>0.25660</td>
<td>0.25880</td>
</tr>
</tbody>
</table>

### VERTICAL VELOCITY FIELD

<table>
<thead>
<tr>
<th>I</th>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.21280</td>
<td>0.24730</td>
<td>0.18760</td>
<td>0.16780</td>
<td>0.14640</td>
<td>0.12340</td>
<td>0.09830</td>
<td>0.07170</td>
<td>0.04390</td>
<td>0.01510</td>
<td>0.00000</td>
<td></td>
</tr>
</tbody>
</table>

### HORIZONTAL VELOCITY FIELD

<table>
<thead>
<tr>
<th>I</th>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.75560</td>
<td>0.15460</td>
<td>0.21020</td>
<td>0.24212</td>
<td>0.25300</td>
<td>0.26500</td>
<td>0.26400</td>
<td>0.25930</td>
<td>0.25860</td>
<td>0.25880</td>
<td>0.00000</td>
<td></td>
</tr>
</tbody>
</table>

### VELOCITY FIELD HAS NOT CONVERGED
## Temperature Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.94660 0.34600 0.13200 0.43600 0.110000 0.843600 0.103600 0.743600 0.0143600 0.1943600 0.2819436</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.10430 0.36430 0.14230 0.45430 0.12230 0.854300 0.113400 0.754300 0.0154300 0.2054300 0.2920543</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.10230 0.36230 0.14030 0.45230 0.12030 0.852300 0.112400 0.752300 0.0152300 0.2052300 0.2920523</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.10130 0.36130 0.14020 0.45130 0.12000 0.851300 0.112000 0.751300 0.0151300 0.2051300 0.2920513</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.10130 0.36013 0.13401 0.44013 0.12000 0.850130 0.112000 0.750130 0.0150130 0.2050130 0.2920501</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.10020 0.35900 0.13300 0.43900 0.11900 0.849000 0.111000 0.749000 0.0149000 0.1949000 0.2819490</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## Vertical Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.250000 0.35000 0.184000 0.418000 0.101800 0.818000 0.1081800 0.7108180 0.017108180 0.1917108 0.2819171</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.24300 0.34700 0.181000 0.411000 0.098100 0.8081000 0.09808100 0.70988010 0.017098801 0.1910782 0.2819107</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.24130 0.34610 0.179000 0.409000 0.096100 0.7961000 0.09607900 0.70699809 0.017069980 0.1911177 0.2819112</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.24000 0.34500 0.177000 0.407000 0.094000 0.7840000 0.09407840 0.70586905 0.017058690 0.1911033 0.2819113</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.23860 0.34300 0.173000 0.404000 0.092000 0.7720000 0.09207720 0.70504907 0.017050490 0.1910099 0.2819112</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.23790 0.34200 0.172000 0.402000 0.091000 0.7610000 0.09107610 0.70492705 0.017049270 0.1910049 0.2819106</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.742100-0.13300 0.272300 0.327800 0.253300 0.212500 0.243500 0.234500 0.243500 0.234500 0.212500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.787000-0.12400 0.281400 0.314500 0.235000 0.214500 0.236500 0.226500 0.236500 0.226500 0.214500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.826000-0.11400 0.289400 0.314500 0.235000 0.214500 0.236500 0.226500 0.236500 0.226500 0.214500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.855000-0.10400 0.296000 0.314500 0.235000 0.214500 0.236500 0.226500 0.236500 0.226500 0.214500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.877000-0.09200 0.301400 0.314500 0.235000 0.214500 0.236500 0.226500 0.236500 0.226500 0.214500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.890000-0.08000 0.305000 0.314500 0.235000 0.214500 0.236500 0.226500 0.236500 0.226500 0.214500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The velocity field has not been Crawforded.
### Temperature Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.904009</td>
<td>0.294510</td>
<td>0.371832-0.247073</td>
<td>0.463945</td>
<td>0.529060</td>
<td>0.276102</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.104401</td>
<td>0.288685</td>
<td>0.371310-0.246860</td>
<td>0.462560</td>
<td>0.515640</td>
<td>0.274601</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.105220</td>
<td>0.283209</td>
<td>0.370810-0.246460</td>
<td>0.462309</td>
<td>0.515309</td>
<td>0.274202</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.105220</td>
<td>0.283209</td>
<td>0.370810-0.246460</td>
<td>0.462309</td>
<td>0.515309</td>
<td>0.274202</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.115701</td>
<td>0.276570</td>
<td>0.394010-0.265831</td>
<td>0.511840</td>
<td>0.457600</td>
<td>0.290402</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.135800</td>
<td>0.284560</td>
<td>0.394110-0.265930</td>
<td>0.511840</td>
<td>0.457600</td>
<td>0.290402</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.155000</td>
<td>0.292650</td>
<td>0.394210-0.266030</td>
<td>0.511840</td>
<td>0.457600</td>
<td>0.290402</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.174200</td>
<td>0.290740</td>
<td>0.394310-0.266130</td>
<td>0.511840</td>
<td>0.457600</td>
<td>0.290402</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.193400</td>
<td>0.288830</td>
<td>0.394410-0.266230</td>
<td>0.511840</td>
<td>0.457600</td>
<td>0.290402</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.212600</td>
<td>0.286920</td>
<td>0.394510-0.266330</td>
<td>0.511840</td>
<td>0.457600</td>
<td>0.290402</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.215009</td>
<td>0.273870</td>
<td>0.313870</td>
<td>0.105100</td>
<td>0.446040-0.115300</td>
<td>0.177500-0.522800-0.383700-0.118500</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.270409</td>
<td>0.253900</td>
<td>0.273200</td>
<td>0.110700</td>
<td>0.513000-0.177300-0.433400-0.279400-0.576200-0.200700</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.283509</td>
<td>0.246760</td>
<td>0.219500</td>
<td>0.121400</td>
<td>0.529800-0.172300-0.417700-0.262100-0.469500-0.224500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.752900-0.159300-0.296500-0.274130</td>
<td>-0.295700-0.258500-0.258400-0.255700-0.255300-0.255100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.752900-0.159300-0.296500-0.274130</td>
<td>-0.295700-0.258500-0.258400-0.255700-0.255300-0.255100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.666400-0.219500-0.371832</td>
<td>-0.136590-0.311369-0.31813659</td>
<td>-0.283718-0.325136</td>
<td>-0.283718-0.325136</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.571900-0.282600-0.394010</td>
<td>-0.370810-0.246460-0.246860</td>
<td>-0.370810-0.246460</td>
<td>-0.370810-0.246460</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-0.477400-0.345600-0.394110</td>
<td>-0.370910-0.246560-0.246660</td>
<td>-0.370910-0.246560</td>
<td>-0.370910-0.246560</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.383900-0.408700-0.394210</td>
<td>-0.371010-0.246660-0.246760</td>
<td>-0.371010-0.246660</td>
<td>-0.371010-0.246660</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.290400-0.472600-0.394310</td>
<td>-0.371110-0.246760-0.246860</td>
<td>-0.371110-0.246760</td>
<td>-0.371110-0.246760</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.197000-0.534500-0.394410</td>
<td>-0.371210-0.246860-0.246960</td>
<td>-0.371210-0.246860</td>
<td>-0.371210-0.246860</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.103600-0.596400-0.394510</td>
<td>-0.371310-0.246960-0.247060</td>
<td>-0.371310-0.246960</td>
<td>-0.371310-0.246960</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.000000</td>
<td>0.658300-0.394610</td>
<td>-0.371410-0.247060-0.247160</td>
<td>-0.371410-0.247060</td>
<td>-0.371410-0.247060</td>
<td>-0.371410-0.247060</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:** The velocity field values are not converged.
### Temperature Field

<table>
<thead>
<tr>
<th>( J )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td>0,0440</td>
<td></td>
</tr>
<tr>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td>0,0450</td>
<td></td>
</tr>
<tr>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td>0,0460</td>
<td></td>
</tr>
<tr>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td>0,0470</td>
<td></td>
</tr>
<tr>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td>0,0480</td>
<td></td>
</tr>
</tbody>
</table>

### Velocity Field

#### Vertical Velocity Field

<table>
<thead>
<tr>
<th>( J )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td></td>
</tr>
<tr>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td></td>
</tr>
<tr>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td></td>
</tr>
<tr>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td></td>
</tr>
<tr>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td></td>
</tr>
</tbody>
</table>

#### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>( J )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td>-0,0440</td>
<td></td>
</tr>
<tr>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td>-0,0450</td>
<td></td>
</tr>
<tr>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td>-0,0460</td>
<td></td>
</tr>
<tr>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td>-0,0470</td>
<td></td>
</tr>
<tr>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td>-0,0480</td>
<td></td>
</tr>
</tbody>
</table>
### Temperature Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.96470</td>
<td>0.24570</td>
<td>0.21650</td>
<td>0.56350</td>
<td>0.11080</td>
<td>0.0</td>
<td>0.36460</td>
<td>0.1</td>
<td>0.56170</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.10490</td>
<td>0.29700</td>
<td>0.16570</td>
<td>0.05700</td>
<td>0.11600</td>
<td>0.0</td>
<td>0.38670</td>
<td>0.1</td>
<td>0.52250</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.14590</td>
<td>0.30480</td>
<td>0.46400</td>
<td>0.11800</td>
<td>0.0</td>
<td>0.36450</td>
<td>0.1</td>
<td>0.33130</td>
<td>0.1</td>
<td>0.27540</td>
</tr>
<tr>
<td>4</td>
<td>0.15930</td>
<td>0.40200</td>
<td>0.54980</td>
<td>0.11580</td>
<td>0.0</td>
<td>0.34960</td>
<td>0.1</td>
<td>0.33140</td>
<td>0.1</td>
<td>0.27320</td>
</tr>
<tr>
<td>5</td>
<td>0.15930</td>
<td>0.46540</td>
<td>0.49600</td>
<td>0.11800</td>
<td>0.0</td>
<td>0.33140</td>
<td>0.1</td>
<td>0.33490</td>
<td>0.1</td>
<td>0.27320</td>
</tr>
<tr>
<td>6</td>
<td>0.15840</td>
<td>0.46540</td>
<td>0.49600</td>
<td>0.11800</td>
<td>0.0</td>
<td>0.33140</td>
<td>0.1</td>
<td>0.33490</td>
<td>0.1</td>
<td>0.27320</td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.29500</td>
<td>0.26140</td>
<td>0.18490</td>
<td>0.10640</td>
<td>0.04500</td>
<td>0.1</td>
<td>0.11600</td>
<td>0.0</td>
<td>0.25360</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.27200</td>
<td>0.26390</td>
<td>0.21160</td>
<td>0.05100</td>
<td>0.11970</td>
<td>0.1</td>
<td>0.52360</td>
<td>0.2</td>
<td>0.28600</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.28100</td>
<td>0.26700</td>
<td>0.21600</td>
<td>0.051390</td>
<td>0.11970</td>
<td>0.1</td>
<td>0.52360</td>
<td>0.2</td>
<td>0.28600</td>
<td>0.2</td>
</tr>
<tr>
<td>4</td>
<td>0.28700</td>
<td>0.26700</td>
<td>0.21600</td>
<td>0.051390</td>
<td>0.11970</td>
<td>0.1</td>
<td>0.52360</td>
<td>0.2</td>
<td>0.28600</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>0.17000</td>
<td>0.16100</td>
<td>0.21700</td>
<td>0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.16100</td>
<td>0.13900</td>
<td>0.21700</td>
<td>0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.13900</td>
<td>0.11700</td>
<td>0.21700</td>
<td>0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.11700</td>
<td>0.09500</td>
<td>0.21700</td>
<td>0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.09500</td>
<td>0.07300</td>
<td>0.21700</td>
<td>0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.07300</td>
<td>0.05100</td>
<td>0.21700</td>
<td>0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.76200</td>
<td>-0.51470</td>
<td>-0.27550</td>
<td>-0.14000</td>
<td>-0.61700</td>
<td>-0.2</td>
<td>-0.58200</td>
<td>-0.1</td>
<td>-0.58170</td>
<td>-0.1</td>
</tr>
<tr>
<td>2</td>
<td>-0.25300</td>
<td>-0.46300</td>
<td>-0.36000</td>
<td>-0.11600</td>
<td>-0.26300</td>
<td>-0.1</td>
<td>-0.28100</td>
<td>-0.1</td>
<td>-0.28600</td>
<td>-0.1</td>
</tr>
<tr>
<td>3</td>
<td>-0.47000</td>
<td>-0.30490</td>
<td>-0.44600</td>
<td>-0.11580</td>
<td>-0.26490</td>
<td>-0.1</td>
<td>-0.28690</td>
<td>-0.1</td>
<td>-0.28600</td>
<td>-0.1</td>
</tr>
<tr>
<td>4</td>
<td>-0.53490</td>
<td>-0.42900</td>
<td>-0.49600</td>
<td>-0.11800</td>
<td>-0.27290</td>
<td>-0.1</td>
<td>-0.27320</td>
<td>-0.1</td>
<td>-0.27320</td>
<td>-0.1</td>
</tr>
<tr>
<td>5</td>
<td>-0.33190</td>
<td>-0.15000</td>
<td>-0.21200</td>
<td>-0.23960</td>
<td>-0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.49500</td>
<td>-0.20110</td>
<td>-0.33900</td>
<td>-0.37350</td>
<td>-0.2</td>
<td>0.27320</td>
<td>0.2</td>
<td>0.27320</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.27320</td>
<td>-0.25290</td>
<td>-0.32790</td>
<td>-0.27320</td>
<td>0.2</td>
<td>0.27320</td>
<td>0.2</td>
<td>0.27320</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.22790</td>
<td>-0.27320</td>
<td>-0.32790</td>
<td>-0.27320</td>
<td>0.2</td>
<td>0.27320</td>
<td>0.2</td>
<td>0.27320</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.17000</td>
<td>-0.21700</td>
<td>-0.23960</td>
<td>-0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.11700</td>
<td>-0.21700</td>
<td>-0.23960</td>
<td>-0.23960</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td>0.25460</td>
<td>0.2</td>
<td></td>
</tr>
</tbody>
</table>

### Notes

- VERTICAL VELOCITY FIELD WAS NOT CONVERGED.
### Temperature Field

<table>
<thead>
<tr>
<th>J0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

### Vertical Velocity Field

<table>
<thead>
<tr>
<th>J0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

### Horizontal Velocity Field

<table>
<thead>
<tr>
<th>J0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

#### VELcITY

| 1  |
| 2  |
| 3  |
| 4  |
| 5  |
| 6  |
| 7  |
| 8  |
| 9  |
| 10 |

#### VELcITY FIELD NOT CONVERGED
### THERMOMETER FIELD

<table>
<thead>
<tr>
<th>J =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.6640</td>
<td>1.5670</td>
<td>1.3970</td>
<td>1.2690</td>
<td>1.1670</td>
<td>1.0890</td>
<td>1.0350</td>
<td>0.9950</td>
<td>0.9690</td>
<td>0.9460</td>
</tr>
<tr>
<td>2</td>
<td>1.5670</td>
<td>1.4860</td>
<td>1.3380</td>
<td>1.2160</td>
<td>1.1140</td>
<td>1.0360</td>
<td>0.9780</td>
<td>0.9330</td>
<td>0.8920</td>
<td>0.8550</td>
</tr>
<tr>
<td>3</td>
<td>1.3380</td>
<td>1.2160</td>
<td>1.1140</td>
<td>1.0360</td>
<td>0.9780</td>
<td>0.9330</td>
<td>0.8920</td>
<td>0.8550</td>
<td>0.8210</td>
<td>0.7900</td>
</tr>
<tr>
<td>4</td>
<td>1.1140</td>
<td>1.0360</td>
<td>0.9780</td>
<td>0.9330</td>
<td>0.8920</td>
<td>0.8550</td>
<td>0.8210</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
</tr>
<tr>
<td>5</td>
<td>0.9780</td>
<td>0.9330</td>
<td>0.8920</td>
<td>0.8550</td>
<td>0.8210</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
<td>0.7100</td>
<td>0.6870</td>
</tr>
<tr>
<td>6</td>
<td>0.9330</td>
<td>0.8920</td>
<td>0.8550</td>
<td>0.8210</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
<td>0.7100</td>
<td>0.6870</td>
<td>0.6660</td>
</tr>
<tr>
<td>7</td>
<td>0.8920</td>
<td>0.8550</td>
<td>0.8210</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
<td>0.7100</td>
<td>0.6870</td>
<td>0.6660</td>
<td>0.6470</td>
</tr>
<tr>
<td>8</td>
<td>0.8550</td>
<td>0.8210</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
<td>0.7100</td>
<td>0.6870</td>
<td>0.6660</td>
<td>0.6470</td>
<td>0.6300</td>
</tr>
<tr>
<td>9</td>
<td>0.8210</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
<td>0.7100</td>
<td>0.6870</td>
<td>0.6660</td>
<td>0.6470</td>
<td>0.6300</td>
<td>0.6140</td>
</tr>
<tr>
<td>10</td>
<td>0.7900</td>
<td>0.7610</td>
<td>0.7350</td>
<td>0.7100</td>
<td>0.6870</td>
<td>0.6660</td>
<td>0.6470</td>
<td>0.6300</td>
<td>0.6140</td>
<td>0.5990</td>
</tr>
</tbody>
</table>

### VERTICAL VELOCITY FIELD

<table>
<thead>
<tr>
<th>J =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.6640</td>
<td>2.5670</td>
<td>2.3970</td>
<td>2.2690</td>
<td>2.1670</td>
<td>2.0890</td>
<td>2.0350</td>
<td>1.9950</td>
<td>1.9690</td>
<td>1.9460</td>
</tr>
<tr>
<td>2</td>
<td>2.5670</td>
<td>2.4860</td>
<td>2.3380</td>
<td>2.2160</td>
<td>2.1140</td>
<td>2.0360</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
</tr>
<tr>
<td>3</td>
<td>2.3380</td>
<td>2.2160</td>
<td>2.1140</td>
<td>2.0360</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
</tr>
<tr>
<td>4</td>
<td>2.1140</td>
<td>2.0360</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
</tr>
<tr>
<td>5</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
</tr>
<tr>
<td>6</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
</tr>
<tr>
<td>7</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
</tr>
<tr>
<td>8</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
<td>1.6300</td>
</tr>
<tr>
<td>9</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
<td>1.6300</td>
<td>1.6140</td>
</tr>
<tr>
<td>10</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
<td>1.6300</td>
<td>1.6140</td>
<td>0.5990</td>
</tr>
</tbody>
</table>

### HORIZONTAL VELOCITY FIELD

<table>
<thead>
<tr>
<th>J =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.6640</td>
<td>2.5670</td>
<td>2.3970</td>
<td>2.2690</td>
<td>2.1670</td>
<td>2.0890</td>
<td>2.0350</td>
<td>1.9950</td>
<td>1.9690</td>
<td>1.9460</td>
</tr>
<tr>
<td>2</td>
<td>2.5670</td>
<td>2.4860</td>
<td>2.3380</td>
<td>2.2160</td>
<td>2.1140</td>
<td>2.0360</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
</tr>
<tr>
<td>3</td>
<td>2.3380</td>
<td>2.2160</td>
<td>2.1140</td>
<td>2.0360</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
</tr>
<tr>
<td>4</td>
<td>2.1140</td>
<td>2.0360</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
</tr>
<tr>
<td>5</td>
<td>1.9780</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
</tr>
<tr>
<td>6</td>
<td>1.9330</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
</tr>
<tr>
<td>7</td>
<td>1.8920</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
</tr>
<tr>
<td>8</td>
<td>1.8550</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
<td>1.6300</td>
</tr>
<tr>
<td>9</td>
<td>1.8210</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
<td>1.6300</td>
<td>1.6140</td>
</tr>
<tr>
<td>10</td>
<td>1.7900</td>
<td>1.7610</td>
<td>1.7350</td>
<td>1.7100</td>
<td>1.6870</td>
<td>1.6660</td>
<td>1.6470</td>
<td>1.6300</td>
<td>1.6140</td>
<td>0.5990</td>
</tr>
</tbody>
</table>
APPENDIX J

EXPERIMENTAL DATA

In the following Appendix, velocity data are presented in ft/sec $\times 10^3$ for test W-2-Hi. It is intended to give an estimate of the type of data and the variation of the data which is typical of the streamline analysis presented in Figures 4.9 and 4.10. The temperature data is presented in graphical form in Figure 6.9.

The original data for the temperature analysis and the velocity analysis for all of the tests is in the possession of the author.
<table>
<thead>
<tr>
<th>V</th>
<th>W</th>
<th>V</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9941</td>
<td>0.4125</td>
<td>-1.429</td>
<td>10.714</td>
</tr>
<tr>
<td>0.9933</td>
<td>0.6079</td>
<td>-0.000</td>
<td>13.910</td>
</tr>
<tr>
<td>0.9931</td>
<td>0.3293</td>
<td>-1.905</td>
<td>10.000</td>
</tr>
<tr>
<td>0.9970</td>
<td>0.7464</td>
<td>2.857</td>
<td>11.905</td>
</tr>
<tr>
<td>0.9851</td>
<td>0.9554</td>
<td>-9.524</td>
<td>3.571</td>
</tr>
<tr>
<td>0.9872</td>
<td>1.0196</td>
<td>-5.238</td>
<td>3.571</td>
</tr>
<tr>
<td>0.9870</td>
<td>0.2843</td>
<td>0.952</td>
<td>10.476</td>
</tr>
<tr>
<td>0.9838</td>
<td>0.9171</td>
<td>-7.143</td>
<td>2.957</td>
</tr>
<tr>
<td>0.9888</td>
<td>0.5507</td>
<td>0.476</td>
<td>13.810</td>
</tr>
<tr>
<td>0.9719</td>
<td>0.1464</td>
<td>-29.762</td>
<td>21.429</td>
</tr>
<tr>
<td>0.9790</td>
<td>0.3907</td>
<td>-4.762</td>
<td>10.952</td>
</tr>
<tr>
<td>0.9748</td>
<td>0.1754</td>
<td>-15.476</td>
<td>9.286</td>
</tr>
<tr>
<td>0.9745</td>
<td>1.0929</td>
<td>-2.857</td>
<td>-0.952</td>
</tr>
<tr>
<td>0.9780</td>
<td>1.1786</td>
<td>-9.524</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.9781</td>
<td>1.1936</td>
<td>-7.143</td>
<td>-4.786</td>
</tr>
<tr>
<td>0.9741</td>
<td>0.3464</td>
<td>-0.000</td>
<td>9.048</td>
</tr>
<tr>
<td>0.9693</td>
<td>0.1114</td>
<td>-14.524</td>
<td>11.429</td>
</tr>
<tr>
<td>0.9624</td>
<td>0.9193</td>
<td>-3.571</td>
<td>-2.381</td>
</tr>
<tr>
<td>0.9551</td>
<td>0.3571</td>
<td>-5.000</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.9501</td>
<td>0.4479</td>
<td>-3.810</td>
<td>6.190</td>
</tr>
<tr>
<td>0.9507</td>
<td>0.7714</td>
<td>-0.476</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.9543</td>
<td>0.8339</td>
<td>-1.905</td>
<td>-3.571</td>
</tr>
<tr>
<td>0.9530</td>
<td>0.9550</td>
<td>-3.333</td>
<td>-3.333</td>
</tr>
<tr>
<td>0.9516</td>
<td>0.8671</td>
<td>-1.905</td>
<td>-3.910</td>
</tr>
<tr>
<td>0.9530</td>
<td>0.8929</td>
<td>-2.333</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.9559</td>
<td>1.0257</td>
<td>-6.429</td>
<td>-2.857</td>
</tr>
<tr>
<td>0.9581</td>
<td>1.0518</td>
<td>-4.762</td>
<td>-1.905</td>
</tr>
<tr>
<td>0.9565</td>
<td>1.0589</td>
<td>-3.095</td>
<td>-1.905</td>
</tr>
<tr>
<td>0.9590</td>
<td>1.0671</td>
<td>-3.333</td>
<td>-1.905</td>
</tr>
<tr>
<td>0.9541</td>
<td>1.0929</td>
<td>-4.286</td>
<td>0.000</td>
</tr>
<tr>
<td>0.9590</td>
<td>1.1454</td>
<td>-7.143</td>
<td>-1.667</td>
</tr>
<tr>
<td>0.9539</td>
<td>1.1779</td>
<td>-3.810</td>
<td>0.476</td>
</tr>
<tr>
<td>0.9626</td>
<td>1.2071</td>
<td>-4.762</td>
<td>0.000</td>
</tr>
<tr>
<td>0.9430</td>
<td>1.1911</td>
<td>-5.238</td>
<td>1.190</td>
</tr>
<tr>
<td>0.9324</td>
<td>0.1554</td>
<td>-7.143</td>
<td>5.952</td>
</tr>
<tr>
<td>0.9350</td>
<td>0.5364</td>
<td>-4.286</td>
<td>4.286</td>
</tr>
<tr>
<td>0.9364</td>
<td>0.6368</td>
<td>-2.857</td>
<td>-3.095</td>
</tr>
<tr>
<td>0.9349</td>
<td>0.8954</td>
<td>-5.714</td>
<td>-4.048</td>
</tr>
<tr>
<td>0.9355</td>
<td>0.9304</td>
<td>-3.333</td>
<td>-3.571</td>
</tr>
<tr>
<td>0.9326</td>
<td>0.9796</td>
<td>-4.048</td>
<td>-3.095</td>
</tr>
<tr>
<td>0.9261</td>
<td>0.1232</td>
<td>-14.286</td>
<td>-1.190</td>
</tr>
<tr>
<td>0.9295</td>
<td>0.4114</td>
<td>-7.143</td>
<td>5.714</td>
</tr>
<tr>
<td>0.9230</td>
<td>0.5250</td>
<td>-3.810</td>
<td>4.762</td>
</tr>
<tr>
<td>0.9251</td>
<td>0.2696</td>
<td>-4.286</td>
<td>13.095</td>
</tr>
<tr>
<td>0.9269</td>
<td>0.9621</td>
<td>-2.857</td>
<td>-3.333</td>
</tr>
<tr>
<td>0.9235</td>
<td>0.7993</td>
<td>-3.095</td>
<td>4.286</td>
</tr>
<tr>
<td>0.9217</td>
<td>0.0821</td>
<td>-5.238</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.9233</td>
<td>0.7875</td>
<td>-3.571</td>
<td>3.571</td>
</tr>
<tr>
<td>0.9255</td>
<td>1.0282</td>
<td>-1.429</td>
<td>-2.143</td>
</tr>
<tr>
<td>0.9240</td>
<td>1.0407</td>
<td>-4.286</td>
<td>-2.381</td>
</tr>
<tr>
<td>0.9206</td>
<td>1.0757</td>
<td>-7.143</td>
<td>-2.857</td>
</tr>
<tr>
<td>0.9221</td>
<td>1.1418</td>
<td>-5.238</td>
<td>-1.667</td>
</tr>
<tr>
<td>0.9267</td>
<td>1.1857</td>
<td>-6.190</td>
<td>0.000</td>
</tr>
<tr>
<td>0.9293</td>
<td>1.2054</td>
<td>-6.190</td>
<td>1.190</td>
</tr>
<tr>
<td>0.9159</td>
<td>0.3164</td>
<td>-1.667</td>
<td>97.619</td>
</tr>
<tr>
<td>0.9194</td>
<td>0.1964</td>
<td>-4.286</td>
<td>4.762</td>
</tr>
<tr>
<td>0.9188</td>
<td>0.2543</td>
<td>-4.048</td>
<td>6.667</td>
</tr>
<tr>
<td>0.9119</td>
<td>0.2589</td>
<td>-1.190</td>
<td>8.333</td>
</tr>
<tr>
<td>0.9155</td>
<td>0.6529</td>
<td>-5.476</td>
<td>1.005</td>
</tr>
<tr>
<td>0.9145</td>
<td>1.0079</td>
<td>-4.286</td>
<td>-5.238</td>
</tr>
<tr>
<td>0.9179</td>
<td>0.7829</td>
<td>-5.238</td>
<td>3.810</td>
</tr>
<tr>
<td>0.9139</td>
<td>0.8050</td>
<td>-5.714</td>
<td>3.333</td>
</tr>
<tr>
<td>0.9100</td>
<td>0.8221</td>
<td>-3.571</td>
<td>4.286</td>
</tr>
<tr>
<td>0.9154</td>
<td>0.8800</td>
<td>-2.857</td>
<td>-3.810</td>
</tr>
<tr>
<td>0.9134</td>
<td>1.1429</td>
<td>-5.000</td>
<td>-0.952</td>
</tr>
<tr>
<td>0.9017</td>
<td>0.2093</td>
<td>-3.810</td>
<td>3.333</td>
</tr>
<tr>
<td>0.9076</td>
<td>0.3089</td>
<td>-7.143</td>
<td>1.190</td>
</tr>
<tr>
<td>0.9050</td>
<td>0.4771</td>
<td>-6.429</td>
<td>4.762</td>
</tr>
<tr>
<td>0.9074</td>
<td>0.5193</td>
<td>-2.857</td>
<td>6.190</td>
</tr>
<tr>
<td>0.8707</td>
<td>1.1339</td>
<td>-4.286</td>
<td>-5.952</td>
</tr>
<tr>
<td>0.8638</td>
<td>0.2821</td>
<td>#2.381</td>
<td>#7.143</td>
</tr>
<tr>
<td>0.8603</td>
<td>0.4007</td>
<td>-4.048</td>
<td>4.786</td>
</tr>
<tr>
<td>0.8678</td>
<td>0.4143</td>
<td>-4.762</td>
<td>4.762</td>
</tr>
<tr>
<td>0.8682</td>
<td>0.4429</td>
<td>-3.810</td>
<td>7.143</td>
</tr>
<tr>
<td>0.8676</td>
<td>0.6929</td>
<td>-2.381</td>
<td>4.762</td>
</tr>
<tr>
<td>0.8670</td>
<td>0.7143</td>
<td>-3.000</td>
<td>4.762</td>
</tr>
<tr>
<td>0.8694</td>
<td>1.0379</td>
<td>-0.238</td>
<td>4.333</td>
</tr>
<tr>
<td>0.8672</td>
<td>1.1839</td>
<td>-3.333</td>
<td>-5.952</td>
</tr>
<tr>
<td>0.8602</td>
<td>0.7421</td>
<td>-2.857</td>
<td>-4.286</td>
</tr>
<tr>
<td>0.8614</td>
<td>0.9293</td>
<td>-3.571</td>
<td>-4.286</td>
</tr>
<tr>
<td>0.8619</td>
<td>1.1464</td>
<td>-2.381</td>
<td>-6.190</td>
</tr>
<tr>
<td>0.8562</td>
<td>0.3554</td>
<td>-2.381</td>
<td>8.333</td>
</tr>
<tr>
<td>0.8533</td>
<td>0.1714</td>
<td>-4.762</td>
<td>9.524</td>
</tr>
<tr>
<td>0.8570</td>
<td>0.4925</td>
<td>-4.286</td>
<td>2.619</td>
</tr>
<tr>
<td>0.8562</td>
<td>0.7436</td>
<td>-2.381</td>
<td>-4.286</td>
</tr>
<tr>
<td>0.8591</td>
<td>1.1346</td>
<td>-2.381</td>
<td>-6.667</td>
</tr>
<tr>
<td>0.8588</td>
<td>0.9521</td>
<td>-3.095</td>
<td>-3.333</td>
</tr>
<tr>
<td>0.8554</td>
<td>0.9893</td>
<td>-4.286</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.8586</td>
<td>1.0589</td>
<td>-1.190</td>
<td>-5.952</td>
</tr>
<tr>
<td>0.8591</td>
<td>1.0836</td>
<td>-16.667</td>
<td>7.619</td>
</tr>
<tr>
<td>0.8568</td>
<td>1.1107</td>
<td>-8.310</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.8443</td>
<td>0.1443</td>
<td>-5.952</td>
<td>-5.714</td>
</tr>
<tr>
<td>0.8413</td>
<td>0.4043</td>
<td>-5.288</td>
<td>7.619</td>
</tr>
<tr>
<td>0.8408</td>
<td>0.4393</td>
<td>-9.524</td>
<td>11.905</td>
</tr>
<tr>
<td>0.8450</td>
<td>0.1929</td>
<td>-8.810</td>
<td>4.762</td>
</tr>
<tr>
<td>0.8465</td>
<td>0.3686</td>
<td>-3.810</td>
<td>7.619</td>
</tr>
<tr>
<td>0.8474</td>
<td>0.7571</td>
<td>-4.286</td>
<td>-2.381</td>
</tr>
<tr>
<td>0.8421</td>
<td>0.9032</td>
<td>-7.143</td>
<td>-4.574</td>
</tr>
<tr>
<td>0.8440</td>
<td>0.9554</td>
<td>-9.048</td>
<td>-5.952</td>
</tr>
<tr>
<td>0.8448</td>
<td>0.9900</td>
<td>-7.143</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.8438</td>
<td>1.0964</td>
<td>-4.762</td>
<td>-7.143</td>
</tr>
<tr>
<td>0.8309</td>
<td>0.3929</td>
<td>-5.714</td>
<td>9.524</td>
</tr>
<tr>
<td>0.8305</td>
<td>0.4621</td>
<td>-4.762</td>
<td>6.190</td>
</tr>
<tr>
<td>0.8302</td>
<td>0.9429</td>
<td>-7.857</td>
<td>-4.762</td>
</tr>
<tr>
<td>0.8317</td>
<td>1.0939</td>
<td>-5.476</td>
<td>-7.957</td>
</tr>
</tbody>
</table>

<p>| 0.8170 | 0.1832 | -6.190 | -5.476 |
| 0.8138 | 0.6304 | -3.571 | -3.571 |
| 0.8175 | 0.8943 | -5.238 | -3.810 |
| 0.8111 | 1.0636 | -6.429 | -4.286 |
| 0.8129 | 2.729 | -5.952 | 5.714 |
| 0.8110 | 0.9843 | -5.714 | -3.810 |
| 0.8198 | 1.0636 | -6.667 | -4.286 |
| 0.8187 | 0.3125 | -0.952 | 3.571 |
| 0.8076 | 0.5839 | -4.762 | -3.571 |
| 0.8076 | 0.9221 | -7.143 | -5.238 |
| 0.8000 | 0.2179 | -3.810 | 2.381 |
| 0.8030 | 0.2236 | -1.905 | 5.238 |
| 0.8023 | 0.2636 | -0.000 | 5.238 |
| 0.8034 | 0.4079 | -4.762 | 5.238 |
| 0.8050 | 0.2793 | 0.714 | 4.286 |
| 0.8011 | 0.3964 | -3.910 | 3.333 |
| 0.8002 | 0.3821 | -5.238 | 2.391 |
| 0.7957 | 0.1693 | -8.333 | 6.190 |
| 0.7905 | 0.2089 | -4.762 | 3.571 |
| 0.7905 | 0.3536 | -2.381 | 2.391 |
| 0.7949 | 0.2707 | -1.429 | 4.286 |
| 0.7903 | 0.1143 | -4.286 | 0.000 |
| 0.7908 | 0.3054 | -1.667 | 3.571 |
| 0.7819 | 0.2214 | -2.381 | 4.762 |
| 0.7890 | 0.3657 | -3.571 | 3.571 |
| 0.7821 | 0.3839 | -3.810 | 3.571 |
| 0.7932 | 0.4179 | -4.048 | -2.391 |
| 0.7863 | 0.4854 | -4.762 | -2.143 |
| 0.7730 | 0.2661 | -1.667 | 3.571 |
| 0.7745 | 0.4429 | -4.762 | -4.762 |
| 0.7773 | 1.0179 | -7.143 | -3.333 |
| 0.7709 | 1.0614 | -6.190 | -2.857 |
| 0.7727 | 1.0893 | -5.476 | -2.391 |
| 0.7741 | 1.1000 | -6.667 | -2.391 |
| 0.7700 | 0.7550 | -3.571 | -1.424 |
| 0.7630 | 0.1957 | -5.000 | 2.857 |
| 0.7633 | 0.3693 | -3.571 | -2.381 |
| 0.7690 | 0.3786 | -3.571 | -4.762 |</p>
<table>
<thead>
<tr>
<th>Z</th>
<th>Y</th>
<th>W</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9962</td>
<td>0.1118</td>
<td>0.014</td>
<td>0.150</td>
</tr>
<tr>
<td>0.9974</td>
<td>0.1371</td>
<td>-0.000</td>
<td>0.276</td>
</tr>
<tr>
<td>0.9930</td>
<td>0.2500</td>
<td>0.007</td>
<td>0.333</td>
</tr>
<tr>
<td>0.9906</td>
<td>0.4457</td>
<td>-0.019</td>
<td>0.305</td>
</tr>
<tr>
<td>0.9914</td>
<td>0.5264</td>
<td>-0.024</td>
<td>0.271</td>
</tr>
<tr>
<td>0.9968</td>
<td>0.5643</td>
<td>-0.014</td>
<td>0.238</td>
</tr>
<tr>
<td>0.9981</td>
<td>0.7607</td>
<td>-0.024</td>
<td>0.262</td>
</tr>
<tr>
<td>0.9928</td>
<td>0.8250</td>
<td>-0.029</td>
<td>0.262</td>
</tr>
<tr>
<td>0.9971</td>
<td>0.9214</td>
<td>-0.000</td>
<td>0.486</td>
</tr>
<tr>
<td>0.9953</td>
<td>1.0696</td>
<td>-0.002</td>
<td>0.226</td>
</tr>
<tr>
<td>0.9914</td>
<td>1.1272</td>
<td>-0.071</td>
<td>0.155</td>
</tr>
<tr>
<td>0.9967</td>
<td>0.9857</td>
<td>-0.095</td>
<td>0.143</td>
</tr>
<tr>
<td>0.9865</td>
<td>1.0421</td>
<td>-0.043</td>
<td>0.100</td>
</tr>
<tr>
<td>0.9898</td>
<td>0.1743</td>
<td>0.007</td>
<td>0.400</td>
</tr>
<tr>
<td>0.9825</td>
<td>0.9336</td>
<td>-0.062</td>
<td>0.014</td>
</tr>
<tr>
<td>0.9874</td>
<td>0.3764</td>
<td>-0.029</td>
<td>0.510</td>
</tr>
<tr>
<td>0.9840</td>
<td>0.3800</td>
<td>-0.048</td>
<td>0.438</td>
</tr>
<tr>
<td>0.9810</td>
<td>0.9750</td>
<td>-0.048</td>
<td>-0.062</td>
</tr>
<tr>
<td>0.9819</td>
<td>0.2814</td>
<td>-0.024</td>
<td>0.276</td>
</tr>
<tr>
<td>0.9810</td>
<td>1.0446</td>
<td>-0.057</td>
<td>-0.060</td>
</tr>
<tr>
<td>0.9874</td>
<td>0.5900</td>
<td>-0.019</td>
<td>0.305</td>
</tr>
<tr>
<td>0.9819</td>
<td>0.9611</td>
<td>-0.048</td>
<td>0.098</td>
</tr>
<tr>
<td>0.9850</td>
<td>0.6943</td>
<td>-0.052</td>
<td>0.238</td>
</tr>
<tr>
<td>0.9857</td>
<td>0.8429</td>
<td>-0.071</td>
<td>0.143</td>
</tr>
<tr>
<td>0.9802</td>
<td>0.2371</td>
<td>0.000</td>
<td>0.152</td>
</tr>
<tr>
<td>0.9869</td>
<td>1.1721</td>
<td>-0.090</td>
<td>0.014</td>
</tr>
<tr>
<td>0.9766</td>
<td>0.9454</td>
<td>-0.024</td>
<td>-0.040</td>
</tr>
<tr>
<td>0.9790</td>
<td>1.0171</td>
<td>-0.048</td>
<td>-0.067</td>
</tr>
<tr>
<td>0.9748</td>
<td>0.4982</td>
<td>-0.060</td>
<td>0.083</td>
</tr>
<tr>
<td>0.9728</td>
<td>0.5479</td>
<td>-0.062</td>
<td>0.005</td>
</tr>
<tr>
<td>0.9771</td>
<td>1.1125</td>
<td>-0.057</td>
<td>-0.107</td>
</tr>
<tr>
<td>0.9750</td>
<td>0.5957</td>
<td>-0.076</td>
<td>0.000</td>
</tr>
<tr>
<td>0.9767</td>
<td>1.1493</td>
<td>-0.083</td>
<td>-0.138</td>
</tr>
<tr>
<td>0.9739</td>
<td>1.1868</td>
<td>-0.033</td>
<td>-0.126</td>
</tr>
<tr>
<td>0.9770</td>
<td>0.6825</td>
<td>-0.079</td>
<td>0.045</td>
</tr>
<tr>
<td>0.9531</td>
<td>0.4018</td>
<td>0.171</td>
<td>-0.036</td>
</tr>
<tr>
<td>0.9509</td>
<td>0.3839</td>
<td>-0.057</td>
<td>-0.060</td>
</tr>
<tr>
<td>0.9649</td>
<td>0.4171</td>
<td>-0.052</td>
<td>-0.076</td>
</tr>
<tr>
<td>0.9593</td>
<td>0.4650</td>
<td>-0.052</td>
<td>-0.090</td>
</tr>
<tr>
<td>0.9501</td>
<td>0.4936</td>
<td>-0.038</td>
<td>-0.052</td>
</tr>
<tr>
<td>0.9543</td>
<td>0.1457</td>
<td>0.152</td>
<td>0.124</td>
</tr>
<tr>
<td>0.9514</td>
<td>0.5643</td>
<td>-0.071</td>
<td>-0.095</td>
</tr>
<tr>
<td>0.9502</td>
<td>0.6464</td>
<td>-0.064</td>
<td>-0.119</td>
</tr>
<tr>
<td>0.9538</td>
<td>1.2079</td>
<td>-0.083</td>
<td>0.052</td>
</tr>
<tr>
<td>0.9574</td>
<td>0.6750</td>
<td>-0.055</td>
<td>-0.119</td>
</tr>
<tr>
<td>0.9510</td>
<td>1.1929</td>
<td>-0.071</td>
<td>0.048</td>
</tr>
<tr>
<td>0.9575</td>
<td>0.7164</td>
<td>-0.100</td>
<td>-0.157</td>
</tr>
<tr>
<td>0.9592</td>
<td>0.9161</td>
<td>-0.076</td>
<td>-0.107</td>
</tr>
<tr>
<td>0.9524</td>
<td>0.9196</td>
<td>-0.095</td>
<td>-0.083</td>
</tr>
<tr>
<td>0.9595</td>
<td>0.7821</td>
<td>-0.083</td>
<td>-0.167</td>
</tr>
<tr>
<td>0.9514</td>
<td>0.7579</td>
<td>-0.071</td>
<td>-0.100</td>
</tr>
<tr>
<td>0.9591</td>
<td>1.0500</td>
<td>-0.098</td>
<td>-0.143</td>
</tr>
<tr>
<td>0.9562</td>
<td>1.0711</td>
<td>-0.095</td>
<td>-0.069</td>
</tr>
<tr>
<td>0.9568</td>
<td>1.1000</td>
<td>-0.071</td>
<td>-0.071</td>
</tr>
<tr>
<td>0.9468</td>
<td>0.1589</td>
<td>-0.098</td>
<td>0.060</td>
</tr>
<tr>
<td>0.9448</td>
<td>0.1836</td>
<td>-0.095</td>
<td>0.033</td>
</tr>
<tr>
<td>0.9421</td>
<td>0.2143</td>
<td>-0.067</td>
<td>0.000</td>
</tr>
<tr>
<td>0.9438</td>
<td>0.2443</td>
<td>-0.071</td>
<td>0.014</td>
</tr>
<tr>
<td>0.9404</td>
<td>0.2679</td>
<td>-0.071</td>
<td>0.024</td>
</tr>
<tr>
<td>0.9407</td>
<td>0.2836</td>
<td>-0.064</td>
<td>0.014</td>
</tr>
<tr>
<td>0.9434</td>
<td>0.2954</td>
<td>-0.062</td>
<td>-0.007</td>
</tr>
<tr>
<td>0.9462</td>
<td>0.3161</td>
<td>-0.083</td>
<td>-0.036</td>
</tr>
<tr>
<td>0.9488</td>
<td>0.3179</td>
<td>-0.090</td>
<td>0.167</td>
</tr>
<tr>
<td>0.9476</td>
<td>0.4429</td>
<td>-0.048</td>
<td>-0.071</td>
</tr>
<tr>
<td>0.9499</td>
<td>0.5404</td>
<td>-0.043</td>
<td>-0.040</td>
</tr>
<tr>
<td>0.9495</td>
<td>0.5836</td>
<td>-0.071</td>
<td>-0.129</td>
</tr>
<tr>
<td>0.9490</td>
<td>0.0929</td>
<td>-0.057</td>
<td>0.048</td>
</tr>
<tr>
<td>0.9486</td>
<td>0.6636</td>
<td>-0.071</td>
<td>-0.090</td>
</tr>
<tr>
<td>0.9496</td>
<td>1.1732</td>
<td>-0.048</td>
<td>0.036</td>
</tr>
<tr>
<td>0.9446</td>
<td>1.1679</td>
<td>-0.052</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9476</td>
<td>0.9643</td>
<td>-0.071</td>
<td>-0.048</td>
</tr>
<tr>
<td>0.9352</td>
<td>0.4014</td>
<td>-0.048</td>
<td>0.038</td>
</tr>
<tr>
<td>0.9385</td>
<td>0.8186</td>
<td>-0.052</td>
<td>-0.171</td>
</tr>
<tr>
<td>0.9390</td>
<td>0.9586</td>
<td>-0.057</td>
<td>0.038</td>
</tr>
<tr>
<td>0.9314</td>
<td>0.4571</td>
<td>-0.048</td>
<td>-0.095</td>
</tr>
<tr>
<td>0.9338</td>
<td>0.4375</td>
<td>-0.036</td>
<td>0.302</td>
</tr>
<tr>
<td>0.9329</td>
<td>0.5268</td>
<td>-0.060</td>
<td>0.060</td>
</tr>
<tr>
<td>0.9358</td>
<td>0.5500</td>
<td>-0.081</td>
<td>0.000</td>
</tr>
<tr>
<td>0.9290</td>
<td>0.1482</td>
<td>-0.083</td>
<td>0.060</td>
</tr>
<tr>
<td>0.9222</td>
<td>0.1864</td>
<td>-0.064</td>
<td>0.090</td>
</tr>
<tr>
<td>0.9202</td>
<td>0.2143</td>
<td>-0.052</td>
<td>0.095</td>
</tr>
<tr>
<td>0.9267</td>
<td>1.1579</td>
<td>-0.024</td>
<td>0.300</td>
</tr>
<tr>
<td>0.9200</td>
<td>0.2921</td>
<td>-0.049</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9215</td>
<td>1.1743</td>
<td>-0.057</td>
<td>0.076</td>
</tr>
<tr>
<td>0.9227</td>
<td>0.5607</td>
<td>-0.019</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9295</td>
<td>0.2493</td>
<td>-0.095</td>
<td>0.052</td>
</tr>
<tr>
<td>0.9289</td>
<td>1.0600</td>
<td>-0.088</td>
<td>0.076</td>
</tr>
<tr>
<td>0.9221</td>
<td>0.6579</td>
<td>-0.062</td>
<td>-0.005</td>
</tr>
<tr>
<td>0.9252</td>
<td>0.9071</td>
<td>-0.036</td>
<td>0.048</td>
</tr>
<tr>
<td>0.9221</td>
<td>0.9150</td>
<td>0.195</td>
<td>0.043</td>
</tr>
<tr>
<td>0.9216</td>
<td>1.0961</td>
<td>-0.064</td>
<td>0.098</td>
</tr>
<tr>
<td>0.9266</td>
<td>0.9643</td>
<td>-0.071</td>
<td>0.048</td>
</tr>
<tr>
<td>0.9286</td>
<td>0.5100</td>
<td>-0.071</td>
<td>0.067</td>
</tr>
<tr>
<td>0.9298</td>
<td>0.9879</td>
<td>-0.067</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9225</td>
<td>1.0157</td>
<td>-0.062</td>
<td>0.057</td>
</tr>
<tr>
<td>0.9238</td>
<td>1.0436</td>
<td>-0.071</td>
<td>0.100</td>
</tr>
<tr>
<td>0.9173</td>
<td>0.0571</td>
<td>-0.076</td>
<td>0.048</td>
</tr>
<tr>
<td>0.9190</td>
<td>0.1529</td>
<td>-0.050</td>
<td>0.067</td>
</tr>
<tr>
<td>0.9138</td>
<td>0.2214</td>
<td>-0.060</td>
<td>0.095</td>
</tr>
<tr>
<td>0.9157</td>
<td>0.2857</td>
<td>-0.060</td>
<td>0.095</td>
</tr>
<tr>
<td>0.9164</td>
<td>0.8793</td>
<td>-0.005</td>
<td>0.052</td>
</tr>
<tr>
<td>0.9114</td>
<td>0.5739</td>
<td>-0.024</td>
<td>0.088</td>
</tr>
<tr>
<td>0.9192</td>
<td>0.9107</td>
<td>-0.029</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9167</td>
<td>1.0536</td>
<td>-0.048</td>
<td>0.071</td>
</tr>
<tr>
<td>0.9162</td>
<td>1.0982</td>
<td>-0.029</td>
<td>0.060</td>
</tr>
<tr>
<td>0.9176</td>
<td>0.7707</td>
<td>-0.043</td>
<td>0.052</td>
</tr>
<tr>
<td>0.9117</td>
<td>0.9307</td>
<td>-0.055</td>
<td>0.081</td>
</tr>
</tbody>
</table>
APPENDIX K
THERMAL STRATIFICATION SIMULATION DATA

In this appendix, results obtained from the computer thermal stratification simulation are presented. The data presented here are an extension of Figures 17 and 22. Therefore, this appendix may be used in lieu of the above figures in calculating the thermal stratification in rectangular containers subjected to a uniform sidewall heat flux.

The symbols pertain to those defined in Equation 5.3.35, i.e.,

\[ T = \frac{\Theta - \Theta_0}{(1L/k)^{1/5}} \]

\[ W = \frac{wD}{a} \frac{-1}{R\dot{a}^{1/5}} \]

\[ \tau = \frac{\alpha t}{L D} R\dot{a}^{1/5} \]

The height can be obtained from the value of I by the following formula

\[ Z_t = (21/100) - 0.01 \] \hspace{1cm} K.1

\[ Z_w = (21/100) - 0.02 \] \hspace{1cm} K.2

where \( Z_t \) is the height of the temperature point and \( Z_w \) is the height of the velocity point.
<table>
<thead>
<tr>
<th>$\tau$ = 0.002</th>
<th>$\tau$ = 0.062</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$T$</td>
</tr>
<tr>
<td>45</td>
<td>0.045E C-1</td>
</tr>
<tr>
<td>46</td>
<td>C.00CE CC</td>
</tr>
<tr>
<td>47</td>
<td>C.00CE CC</td>
</tr>
<tr>
<td>48</td>
<td>0.0000E 00</td>
</tr>
<tr>
<td>49</td>
<td>C.00CE CC</td>
</tr>
<tr>
<td>50</td>
<td>C.20CE CC</td>
</tr>
</tbody>
</table>

$\tau$ = 0.028

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>C.4038E- C1</td>
<td>0.5951E C1</td>
</tr>
<tr>
<td>41</td>
<td>0.6229E- C1</td>
<td>0.5705E C1</td>
</tr>
<tr>
<td>42</td>
<td>C.8988E- C1</td>
<td>0.5387E C1</td>
</tr>
<tr>
<td>43</td>
<td>0.1285E CC</td>
<td>0.4919E C1</td>
</tr>
<tr>
<td>44</td>
<td>C.1611E 00</td>
<td>0.442E C1</td>
</tr>
<tr>
<td>45</td>
<td>C.2051E CC</td>
<td>0.3897E C1</td>
</tr>
<tr>
<td>46</td>
<td>C.2559E CC</td>
<td>0.3237E C1</td>
</tr>
<tr>
<td>47</td>
<td>C.3155E 00</td>
<td>0.2750E C1</td>
</tr>
<tr>
<td>48</td>
<td>C.3881E CC</td>
<td>0.2155E C1</td>
</tr>
<tr>
<td>49</td>
<td>C.4827E CC</td>
<td>0.1537E C1</td>
</tr>
<tr>
<td>50</td>
<td>C.6267E CC</td>
<td>0.872E C0</td>
</tr>
</tbody>
</table>

$\tau$ = 0.090

<table>
<thead>
<tr>
<th>$I$</th>
<th>$T$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>C.4720E- C1</td>
<td>0.3844E C1</td>
</tr>
<tr>
<td>26</td>
<td>C.5709E 00</td>
<td>0.3862E C1</td>
</tr>
<tr>
<td>27</td>
<td>C.7072E- C1</td>
<td>0.383E C1</td>
</tr>
<tr>
<td>28</td>
<td>C.8526E- C1</td>
<td>0.378C E C1</td>
</tr>
<tr>
<td>29</td>
<td>C.1013E CC</td>
<td>0.375CE C0</td>
</tr>
<tr>
<td>30</td>
<td>C.1184E CC</td>
<td>0.362E C1</td>
</tr>
<tr>
<td>31</td>
<td>C.1362E CC</td>
<td>0.3529E C1</td>
</tr>
<tr>
<td>32</td>
<td>C.155CE C0</td>
<td>0.3421E C1</td>
</tr>
<tr>
<td>33</td>
<td>C.1747E CC</td>
<td>0.3227E C1</td>
</tr>
<tr>
<td>34</td>
<td>0.1554E CC</td>
<td>0.3217E C1</td>
</tr>
<tr>
<td>35</td>
<td>C.2172E 00</td>
<td>0.3102E C1</td>
</tr>
<tr>
<td>36</td>
<td>C.2422E 00</td>
<td>0.2982E C1</td>
</tr>
<tr>
<td>37</td>
<td>C.2644E CC</td>
<td>0.2857E C1</td>
</tr>
<tr>
<td>38</td>
<td>C.2901E CC</td>
<td>0.272E C1</td>
</tr>
<tr>
<td>39</td>
<td>C.3174E 00</td>
<td>0.2590E C1</td>
</tr>
<tr>
<td>40</td>
<td>C.3465E CC</td>
<td>0.244E C1</td>
</tr>
<tr>
<td>41</td>
<td>0.3754E CC</td>
<td>0.2300E C1</td>
</tr>
<tr>
<td>42</td>
<td>C.4118E CC</td>
<td>0.2146E C1</td>
</tr>
<tr>
<td>43</td>
<td>C.4468E CC</td>
<td>0.1983E C1</td>
</tr>
<tr>
<td>44</td>
<td>C.4897E CC</td>
<td>0.1812E C1</td>
</tr>
<tr>
<td>45</td>
<td>C.5356E CC</td>
<td>0.1621E C1</td>
</tr>
<tr>
<td>46</td>
<td>C.589CCE CC</td>
<td>0.1437E C1</td>
</tr>
<tr>
<td>47</td>
<td>C.6457E CC</td>
<td>0.1229E C1</td>
</tr>
<tr>
<td>48</td>
<td>0.7253E CC</td>
<td>0.10C1E C1</td>
</tr>
<tr>
<td>49</td>
<td>0.8251E 00</td>
<td>0.7453E C0</td>
</tr>
<tr>
<td>50</td>
<td>C.5EC5E CC</td>
<td>0.4446E C0</td>
</tr>
<tr>
<td></td>
<td>$r=0.106$</td>
<td></td>
</tr>
<tr>
<td>----</td>
<td>-----------------</td>
<td>----</td>
</tr>
<tr>
<td>20</td>
<td>C.3157E-01</td>
<td>20</td>
</tr>
<tr>
<td>21</td>
<td>C.3963E-01</td>
<td>21</td>
</tr>
<tr>
<td>22</td>
<td>C.4833E-01</td>
<td>22</td>
</tr>
<tr>
<td>23</td>
<td>0.584CE-01</td>
<td>23</td>
</tr>
<tr>
<td>24</td>
<td>C.7C49E-01</td>
<td>24</td>
</tr>
<tr>
<td>25</td>
<td>C.E4C9E-01</td>
<td>25</td>
</tr>
<tr>
<td>26</td>
<td>C.9857E-01</td>
<td>26</td>
</tr>
<tr>
<td>27</td>
<td>C.1148E CC</td>
<td>27</td>
</tr>
<tr>
<td>28</td>
<td>C.1314E CC</td>
<td>28</td>
</tr>
<tr>
<td>29</td>
<td>C.1487E CC</td>
<td>29</td>
</tr>
<tr>
<td>30</td>
<td>C.1668E CC</td>
<td>30</td>
</tr>
<tr>
<td>31</td>
<td>0.1856E CC</td>
<td>31</td>
</tr>
<tr>
<td>32</td>
<td>C.2C53E CC</td>
<td>32</td>
</tr>
<tr>
<td>33</td>
<td>0.2259E CC</td>
<td>33</td>
</tr>
<tr>
<td>34</td>
<td>0.2474E CC</td>
<td>34</td>
</tr>
<tr>
<td>35</td>
<td>C.2659E CC</td>
<td>35</td>
</tr>
<tr>
<td>36</td>
<td>C.2936E CC</td>
<td>36</td>
</tr>
<tr>
<td>37</td>
<td>C.3168E CC</td>
<td>37</td>
</tr>
<tr>
<td>38</td>
<td>0.3456E CC</td>
<td>38</td>
</tr>
<tr>
<td>39</td>
<td>C.3723F CC</td>
<td>39</td>
</tr>
<tr>
<td>40</td>
<td>C.4C3CE CC</td>
<td>40</td>
</tr>
<tr>
<td>41</td>
<td>0.4352E CC</td>
<td>41</td>
</tr>
<tr>
<td>42</td>
<td>C.47CCE CC</td>
<td>42</td>
</tr>
<tr>
<td>43</td>
<td>C.5C3CE CC</td>
<td>43</td>
</tr>
<tr>
<td>44</td>
<td>C.55C0E CO</td>
<td>44</td>
</tr>
<tr>
<td>45</td>
<td>C.5971E CC</td>
<td>45</td>
</tr>
<tr>
<td>46</td>
<td>C.65C0E CC</td>
<td>46</td>
</tr>
<tr>
<td>47</td>
<td>C.7141E CC</td>
<td>47</td>
</tr>
<tr>
<td>48</td>
<td>C.7516E CC</td>
<td>48</td>
</tr>
<tr>
<td>49</td>
<td>0.8939E 00</td>
<td>49</td>
</tr>
<tr>
<td>50</td>
<td>C.1053E C1</td>
<td>50</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>----</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>1</td>
<td>C.1551E-01</td>
<td>0.1929F C1</td>
</tr>
<tr>
<td>2</td>
<td>C.2146E-01</td>
<td>C.2145E C1</td>
</tr>
<tr>
<td>3</td>
<td>C.5378E-01</td>
<td>C.7291E C1</td>
</tr>
<tr>
<td>4</td>
<td>C.8618E-01</td>
<td>0.2325E 01</td>
</tr>
<tr>
<td>5</td>
<td>C.122CE CC</td>
<td>C.229CE C1</td>
</tr>
<tr>
<td>6</td>
<td>C.1310F C1</td>
<td>0.2552E 01</td>
</tr>
<tr>
<td>8</td>
<td>C.1637E CC</td>
<td>0.2546E 01</td>
</tr>
<tr>
<td>9</td>
<td>C.1572E CC</td>
<td>C.2572E C1</td>
</tr>
<tr>
<td>10</td>
<td>0.2338E 00</td>
<td>0.2479E 01</td>
</tr>
<tr>
<td>11</td>
<td>C.2732E CC</td>
<td>0.2411E 01</td>
</tr>
<tr>
<td>12</td>
<td>C.3147E CC</td>
<td>C.2321E 01</td>
</tr>
<tr>
<td>13</td>
<td>C.4C43E CC</td>
<td>0.2153E 01</td>
</tr>
<tr>
<td>14</td>
<td>C.5457E CC</td>
<td>C.2045E C1</td>
</tr>
<tr>
<td>15</td>
<td>C.911E CC</td>
<td>C.1539E C1</td>
</tr>
<tr>
<td>16</td>
<td>C.5166E CC</td>
<td>C.1737E C1</td>
</tr>
<tr>
<td>17</td>
<td>0.1486E C1</td>
<td>0.70763E 00</td>
</tr>
<tr>
<td>18</td>
<td>C.1572E CC</td>
<td>C.1544E C1</td>
</tr>
<tr>
<td>19</td>
<td>C.5187E CC</td>
<td>C.1546E C1</td>
</tr>
<tr>
<td>20</td>
<td>0.144CC C1</td>
<td>0.7463E 00</td>
</tr>
<tr>
<td>21</td>
<td>C.1284E C1</td>
<td>C.3355E C0</td>
</tr>
<tr>
<td>22</td>
<td>0.1284E C1</td>
<td>C.3355E CC</td>
</tr>
<tr>
<td>23</td>
<td>0.1333E C1</td>
<td>0.3250E CC</td>
</tr>
</tbody>
</table>

\( \tau = 0.162 \)

<table>
<thead>
<tr>
<th>I</th>
<th>T</th>
<th>W</th>
<th>I</th>
<th>T</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C.613E3E-C3</td>
<td>0.000CEF C0</td>
<td>1</td>
<td>C.5363E-E3</td>
<td>C.000CEF CC</td>
</tr>
<tr>
<td>2</td>
<td>C.2335E-C2</td>
<td>C.6055E CC</td>
<td>3</td>
<td>C.5575E-C2</td>
<td>C.595SE CC</td>
</tr>
<tr>
<td>5</td>
<td>C.7299E-C2</td>
<td>C.1037E C1</td>
<td>7</td>
<td>C.2EC7E-C1</td>
<td>C.503CEF C1</td>
</tr>
<tr>
<td>7</td>
<td>C.2557E-C1</td>
<td>C.1367E C1</td>
<td>9</td>
<td>C.6571E-C1</td>
<td>0.1447E C1</td>
</tr>
<tr>
<td>11</td>
<td>C.1859E C0</td>
<td>0.1741E C1</td>
<td>11</td>
<td>C.11C6E C0</td>
<td>C.1956E C1</td>
</tr>
<tr>
<td>13</td>
<td>C.1529E C0</td>
<td>C.1859E C1</td>
<td>15</td>
<td>C.1419E C0</td>
<td>C.1729E C0</td>
</tr>
<tr>
<td>15</td>
<td>0.1321E CC</td>
<td>C.1583E C1</td>
<td>17</td>
<td>0.2074E C0</td>
<td>0.1912E C1</td>
</tr>
<tr>
<td>17</td>
<td>C.162CC CC</td>
<td>C.2645E C0</td>
<td>19</td>
<td>0.2434E OC</td>
<td>0.1972E C1</td>
</tr>
<tr>
<td>19</td>
<td>C.1542E CC</td>
<td>0.231CCE C1</td>
<td>21</td>
<td>0.217CE E0</td>
<td>C.1996E C1</td>
</tr>
<tr>
<td>21</td>
<td>C.2268E C0</td>
<td>0.217E C0</td>
<td>23</td>
<td>C.315SE CC</td>
<td>0.215CE C1</td>
</tr>
<tr>
<td>23</td>
<td>C.255ECE C0</td>
<td>0.225CE C1</td>
<td>25</td>
<td>0.3618E 00</td>
<td>0.1994E 01</td>
</tr>
<tr>
<td>25</td>
<td>C.3C21E CC</td>
<td>0.216C E1</td>
<td>27</td>
<td>0.2104E 01</td>
<td>0.1972E C1</td>
</tr>
<tr>
<td>27</td>
<td>C.5311E C0</td>
<td>0.214CE 01</td>
<td>29</td>
<td>0.4506E 00</td>
<td>0.1926E C1</td>
</tr>
<tr>
<td>29</td>
<td>C.4302E CC</td>
<td>0.237CE 01</td>
<td>31</td>
<td>0.493ECC C1</td>
<td>0.1883E 01</td>
</tr>
<tr>
<td>31</td>
<td>C.483E2E C0</td>
<td>0.1951E 01</td>
<td>33</td>
<td>0.54E3E CC</td>
<td>C.181E C1</td>
</tr>
<tr>
<td>33</td>
<td>C.5327E CC</td>
<td>1.186EC C1</td>
<td>35</td>
<td>0.6041E 00</td>
<td>0.1736E 01</td>
</tr>
<tr>
<td>35</td>
<td>C.5907E CC</td>
<td>C.1748E 01</td>
<td>37</td>
<td>0.6644E CC</td>
<td>0.1635E 01</td>
</tr>
<tr>
<td>37</td>
<td>C.5551E CC</td>
<td>C.1616E C0</td>
<td>39</td>
<td>0.7322E C0</td>
<td>0.1523E C1</td>
</tr>
<tr>
<td>39</td>
<td>C.726CE CC</td>
<td>C.1462E C1</td>
<td>41</td>
<td>0.8062E 00</td>
<td>0.1376E 01</td>
</tr>
<tr>
<td>41</td>
<td>C.8125E 00</td>
<td>C.1283E C1</td>
<td>43</td>
<td>C.6544E CC</td>
<td>C.1217E C1</td>
</tr>
<tr>
<td>43</td>
<td>C.5158E 00</td>
<td>C.1013E C1</td>
<td>45</td>
<td>0.1001E C1</td>
<td>0.1010E C1</td>
</tr>
<tr>
<td>45</td>
<td>C.105CE C1</td>
<td>C.82415E 00</td>
<td>47</td>
<td>0.114CE C1</td>
<td>0.773E 00</td>
</tr>
<tr>
<td>47</td>
<td>0.1255E 01</td>
<td>C.5036E 00</td>
<td>49</td>
<td>0.1350E C1</td>
<td>C.4774E CC</td>
</tr>
<tr>
<td>49</td>
<td>C.1435E C0</td>
<td>0.3048E OC</td>
<td>50</td>
<td>C.1536E 01</td>
<td>0.2877E 00</td>
</tr>
</tbody>
</table>

\( \tau = 0.174 \)

\( \tau = 0.202 \)

\( \tau = 0.230 \)
<table>
<thead>
<tr>
<th>I</th>
<th>I</th>
<th>W</th>
<th>I</th>
<th>I</th>
<th>W</th>
<th>I</th>
<th>I</th>
<th>W</th>
<th>I</th>
<th>I</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C.1661E-C2</td>
<td>C.CCCCE CC</td>
<td>1</td>
<td>C.3352E-C2</td>
<td>C.CCCCE CC</td>
<td>1</td>
<td>C.675CE-02</td>
<td>C.CCCCE CC</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1375E-C1</td>
<td>0.5833E 00</td>
<td>3</td>
<td>C.275CE-C1</td>
<td>0.5594E CC</td>
<td>3</td>
<td>C.4586E-01</td>
<td>0.5336E CC</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>C.4C79E-C1</td>
<td>C.9279E 00</td>
<td>5</td>
<td>C.6572E-C1</td>
<td>0.8679E CO</td>
<td>5</td>
<td>0.9291E-01</td>
<td>0.8190E 00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>C.7533E-C1</td>
<td>C.114E C1</td>
<td>7</td>
<td>C.1459E CC</td>
<td>C.1687E C1</td>
<td>7</td>
<td>C.134E CC</td>
<td>C.1024E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>C.1114E CO</td>
<td>0.1241E C1</td>
<td>9</td>
<td>C.155E CO</td>
<td>C.1402E C1</td>
<td>9</td>
<td>C.1823E CC</td>
<td>C.1194E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>C.1467E CC</td>
<td>C.1492E C1</td>
<td>11</td>
<td>C.2236E CO</td>
<td>C.1506E C1</td>
<td>11</td>
<td>C.226CE C0</td>
<td>C.1323E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>C.1824E CC</td>
<td>0.1613E C1</td>
<td>13</td>
<td>C.226CE CO</td>
<td>C.1506E C1</td>
<td>13</td>
<td>C.2263E CC</td>
<td>C.1426E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>C.202E CC</td>
<td>C.1694E 01</td>
<td>15</td>
<td>C.226CE CO</td>
<td>C.1506E C1</td>
<td>15</td>
<td>C.3126E CO</td>
<td>C.1504E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>C.255E CC</td>
<td>C.1751E C1</td>
<td>17</td>
<td>C.3602E CC</td>
<td>C.134E C1</td>
<td>17</td>
<td>C.3602E CC</td>
<td>C.134E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>C.2969E OC</td>
<td>0.1816E 01</td>
<td>19</td>
<td>C.3458E CC</td>
<td>C.1656E C1</td>
<td>19</td>
<td>0.4012E CO</td>
<td>C.1455E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>C.3355E OC</td>
<td>0.1859E 01</td>
<td>21</td>
<td>C.3456E 00</td>
<td>0.1746E 01</td>
<td>21</td>
<td>0.4436E 00</td>
<td>0.1537E 01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>C.3758E OC</td>
<td>0.1883E 01</td>
<td>23</td>
<td>C.4235E CC</td>
<td>C.1754E C1</td>
<td>23</td>
<td>C.4626E CC</td>
<td>C.1575E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>C.4157E O0</td>
<td>0.1873E 01</td>
<td>25</td>
<td>C.4787E CC</td>
<td>C.1766E C1</td>
<td>25</td>
<td>C.5373E CC</td>
<td>C.1481E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>C.4656E OC</td>
<td>0.1347E 01</td>
<td>27</td>
<td>C.5244E 00</td>
<td>0.1769E 01</td>
<td>27</td>
<td>C.5875E CC</td>
<td>0.1498E 01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>0.2512E CC</td>
<td>C.1E17E C1</td>
<td>29</td>
<td>C.5792E CC</td>
<td>C.1735E C1</td>
<td>29</td>
<td>0.6332E OC</td>
<td>C.1557E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>C.5631E CC</td>
<td>C.1772E C1</td>
<td>31</td>
<td>0.6266E 00</td>
<td>0.1690E 01</td>
<td>31</td>
<td>C.6830E CC</td>
<td>0.1534E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>0.6162E CC</td>
<td>C.1714E C1</td>
<td>33</td>
<td>0.6116E CC</td>
<td>C.1540E C1</td>
<td>33</td>
<td>C.7434E CC</td>
<td>0.1514E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.6721E CC</td>
<td>C.1734E C1</td>
<td>35</td>
<td>0.6858E CC</td>
<td>C.1576E C1</td>
<td>35</td>
<td>C.8C54E CC</td>
<td>C.1457E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>0.7365E 00</td>
<td>0.1457E 01</td>
<td>37</td>
<td>0.8266E CC</td>
<td>C.1371E C1</td>
<td>37</td>
<td>0.8732E 00</td>
<td>0.1379E 01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>C.8E38E CC</td>
<td>C.1442E C1</td>
<td>39</td>
<td>0.9579E 00</td>
<td>0.1241E 01</td>
<td>39</td>
<td>0.4575E CC</td>
<td>C.1282E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>C.8826E CC</td>
<td>C.1304E C1</td>
<td>41</td>
<td>0.1C52E C1</td>
<td>0.1089E 01</td>
<td>41</td>
<td>0.1302E 01</td>
<td>C.1167E C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>C.5738E CC</td>
<td>0.114E 01</td>
<td>43</td>
<td>0.1089E 01</td>
<td>0.1089E 01</td>
<td>43</td>
<td>0.1129E C1</td>
<td>0.1030E 01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>C.1084E C1</td>
<td>0.5567E CC</td>
<td>45</td>
<td>C.1132E 01</td>
<td>0.6998E 00</td>
<td>45</td>
<td>C.1245E C1</td>
<td>C.1256E CC</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>0.1227E C1</td>
<td>0.7335E 00</td>
<td>47</td>
<td>0.1534E C1</td>
<td>0.4327E CC</td>
<td>47</td>
<td>0.1396E 01</td>
<td>0.6673E 00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>C.1442E C1</td>
<td>0.4531E CC</td>
<td>49</td>
<td>0.1730E C1</td>
<td>0.2611E CC</td>
<td>49</td>
<td>C.1623E C1</td>
<td>C.1414E C0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>C.1634E C1</td>
<td>C.2733E C0</td>
<td>50</td>
<td>C.1730E C1</td>
<td>0.2611E CC</td>
<td>50</td>
<td>C.1824E C1</td>
<td>C.2504E C0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ t = 0.342 \]

<table>
<thead>
<tr>
<th>( I )</th>
<th>( T )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C.123CE-C1</td>
<td>C.000CE CC</td>
</tr>
<tr>
<td>3</td>
<td>0.6742E-C1</td>
<td>C.9568E CC</td>
</tr>
<tr>
<td>5</td>
<td>C.1217E 00</td>
<td>0.7774E CC</td>
</tr>
<tr>
<td>7</td>
<td>C.172CE CC</td>
<td>C.6779E 00</td>
</tr>
<tr>
<td>9</td>
<td>C.2202E OC</td>
<td>C.1134E C1</td>
</tr>
<tr>
<td>11</td>
<td>C.2676E CC</td>
<td>C.1256E C1</td>
</tr>
<tr>
<td>13</td>
<td>C.2157E CC</td>
<td>C.1345E C1</td>
</tr>
<tr>
<td>15</td>
<td>0.3621E 00</td>
<td>0.1422E C1</td>
</tr>
<tr>
<td>17</td>
<td>C.4C75E CC</td>
<td>C.1492E C1</td>
</tr>
<tr>
<td>19</td>
<td>C.4587E CC</td>
<td>C.154CE C1</td>
</tr>
<tr>
<td>21</td>
<td>C.5C11E CO</td>
<td>C.1569E F1</td>
</tr>
<tr>
<td>23</td>
<td>C.549CE CC</td>
<td>C.1606E C1</td>
</tr>
<tr>
<td>25</td>
<td>C.5549E OC</td>
<td>C.1519E C1</td>
</tr>
<tr>
<td>27</td>
<td>C.6465E CC</td>
<td>C.1548E C1</td>
</tr>
<tr>
<td>29</td>
<td>C.6531E FC</td>
<td>C.1404E C1</td>
</tr>
<tr>
<td>31</td>
<td>C.747CE OC</td>
<td>C.1425E C1</td>
</tr>
<tr>
<td>33</td>
<td>C.866GE CC</td>
<td>C.1404E C1</td>
</tr>
<tr>
<td>35</td>
<td>C.871CE CC</td>
<td>C.1360E C1</td>
</tr>
<tr>
<td>37</td>
<td>0.9407E 00</td>
<td>C.1299E C1</td>
</tr>
<tr>
<td>39</td>
<td>C.1C18E C1</td>
<td>C.1214E C1</td>
</tr>
<tr>
<td>41</td>
<td>C.1104E C1</td>
<td>C.1105E C1</td>
</tr>
<tr>
<td>43</td>
<td>C.1204E C1</td>
<td>C.9823E CC</td>
</tr>
<tr>
<td>45</td>
<td>0.1224E C1</td>
<td>C.8285E CC</td>
</tr>
<tr>
<td>47</td>
<td>C.1678E C1</td>
<td>C.5396E CC</td>
</tr>
<tr>
<td>49</td>
<td>C.1711E C1</td>
<td>C.398CE CC</td>
</tr>
<tr>
<td>50</td>
<td>0.1916E C1</td>
<td>C.2405E CC</td>
</tr>
</tbody>
</table>

\[ t = 0.370 \]

<table>
<thead>
<tr>
<th>( I )</th>
<th>( T )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C.2C12E-C1</td>
<td>C.000CE CO</td>
</tr>
<tr>
<td>3</td>
<td>C.5IC3E-C1</td>
<td>C.4842E CC</td>
</tr>
<tr>
<td>5</td>
<td>C.1520E CO</td>
<td>0.7428E CC</td>
</tr>
<tr>
<td>7</td>
<td>C.2C71E CC</td>
<td>0.9342E CO</td>
</tr>
<tr>
<td>9</td>
<td>0.2594E CC</td>
<td>0.1382E C1</td>
</tr>
<tr>
<td>11</td>
<td>0.3105E 00</td>
<td>0.1196E C1</td>
</tr>
<tr>
<td>13</td>
<td>0.26C1E CC</td>
<td>C.129CE 01</td>
</tr>
<tr>
<td>15</td>
<td>0.4083E CO</td>
<td>C.137CE C1</td>
</tr>
<tr>
<td>17</td>
<td>C.4573E CC</td>
<td>0.1427E C1</td>
</tr>
<tr>
<td>19</td>
<td>0.5068E OC</td>
<td>C.1467E C1</td>
</tr>
<tr>
<td>21</td>
<td>0.5563E 00</td>
<td>0.1265E C1</td>
</tr>
<tr>
<td>23</td>
<td>C.6C27E CC</td>
<td>C.1364E C1</td>
</tr>
<tr>
<td>25</td>
<td>C.65C8E CC</td>
<td>C.1428E C1</td>
</tr>
<tr>
<td>27</td>
<td>0.7C21E CC</td>
<td>C.1476E C1</td>
</tr>
<tr>
<td>29</td>
<td>C.7513E CC</td>
<td>C.1283E C1</td>
</tr>
<tr>
<td>31</td>
<td>0.892CE CC</td>
<td>C.1393E C1</td>
</tr>
<tr>
<td>33</td>
<td>C.871CE CC</td>
<td>C.1360E C1</td>
</tr>
<tr>
<td>35</td>
<td>0.9372E CC</td>
<td>C.1219E C1</td>
</tr>
<tr>
<td>37</td>
<td>C.1C09E C1</td>
<td>0.1255E F1</td>
</tr>
<tr>
<td>39</td>
<td>C.1C08E C1</td>
<td>C.1174E C1</td>
</tr>
<tr>
<td>41</td>
<td>C.117CE C1</td>
<td>C.1073E C1</td>
</tr>
<tr>
<td>43</td>
<td>0.1275E CE</td>
<td>C.95C6E CC</td>
</tr>
<tr>
<td>45</td>
<td>0.1C1E C1</td>
<td>C.8C12E CC</td>
</tr>
<tr>
<td>47</td>
<td>C.1559E CE</td>
<td>0.6137E CC</td>
</tr>
<tr>
<td>49</td>
<td>C.177F C1</td>
<td>C.3498E CC</td>
</tr>
<tr>
<td>50</td>
<td>C.20C6F C1</td>
<td>0.2325E CC</td>
</tr>
</tbody>
</table>

\[ t = 0.398 \]

<table>
<thead>
<tr>
<th>( I )</th>
<th>( T )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C.3C15E-C1</td>
<td>C.000CE CO</td>
</tr>
<tr>
<td>3</td>
<td>C.1164E CC</td>
<td>C.4645E CC</td>
</tr>
<tr>
<td>5</td>
<td>0.1837E 00</td>
<td>C.7124E CC</td>
</tr>
<tr>
<td>7</td>
<td>C.2434E CC</td>
<td>C.8953E CC</td>
</tr>
<tr>
<td>9</td>
<td>C.2995E CC</td>
<td>C.1035E C1</td>
</tr>
<tr>
<td>11</td>
<td>C.3532E 00</td>
<td>C.1150E 01</td>
</tr>
<tr>
<td>13</td>
<td>C.4E51E CC</td>
<td>C.1242E C1</td>
</tr>
<tr>
<td>15</td>
<td>C.45E1F CC</td>
<td>C.1305E 01</td>
</tr>
<tr>
<td>17</td>
<td>0.1C7E 00</td>
<td>0.1456E C1</td>
</tr>
<tr>
<td>19</td>
<td>C.55E2E CC</td>
<td>C.1425E C1</td>
</tr>
<tr>
<td>21</td>
<td>0.6088E 00</td>
<td>0.1456E C1</td>
</tr>
<tr>
<td>23</td>
<td>C.65E8E CC</td>
<td>C.1501E C1</td>
</tr>
<tr>
<td>25</td>
<td>C.7C74E CC</td>
<td>C.1529E C1</td>
</tr>
<tr>
<td>27</td>
<td>0.7591E CC</td>
<td>0.1490E C1</td>
</tr>
<tr>
<td>29</td>
<td>0.61C7E CC</td>
<td>C.1401E C1</td>
</tr>
<tr>
<td>31</td>
<td>C.8712E OC</td>
<td>0.1381E C1</td>
</tr>
<tr>
<td>33</td>
<td>C.9353E CC</td>
<td>C.1345E 01</td>
</tr>
<tr>
<td>35</td>
<td>C.10C3F C1</td>
<td>C.1256E C1</td>
</tr>
<tr>
<td>37</td>
<td>0.1C77F C1</td>
<td>0.1232E C1</td>
</tr>
<tr>
<td>39</td>
<td>C.1C7E C1</td>
<td>C.1158E C1</td>
</tr>
<tr>
<td>41</td>
<td>0.1249E C1</td>
<td>C.1047E C1</td>
</tr>
<tr>
<td>43</td>
<td>C.1353F C1</td>
<td>C.9245E CC</td>
</tr>
<tr>
<td>45</td>
<td>0.1478E C1</td>
<td>C.7782E CC</td>
</tr>
<tr>
<td>47</td>
<td>C.1E82E C1</td>
<td>C.3725E CC</td>
</tr>
<tr>
<td>49</td>
<td>0.2095F 01</td>
<td>C.2257E CC</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>---</td>
<td>-----------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>C.4239E-1</td>
<td>0.000CE</td>
</tr>
<tr>
<td>3</td>
<td>C.1432E</td>
<td>C.446CE</td>
</tr>
<tr>
<td>5</td>
<td>C.2166E</td>
<td>0.6852E</td>
</tr>
<tr>
<td>7</td>
<td>C.26C7E</td>
<td>C.868E</td>
</tr>
<tr>
<td>9</td>
<td>C.340E</td>
<td>C.597E</td>
</tr>
<tr>
<td>11</td>
<td>C.3964E</td>
<td>0.110CE</td>
</tr>
<tr>
<td>13</td>
<td>C.4515E</td>
<td>C.119CE</td>
</tr>
<tr>
<td>15</td>
<td>C.5657E</td>
<td>0.126CE</td>
</tr>
<tr>
<td>17</td>
<td>C.5553E</td>
<td>0.131CE</td>
</tr>
<tr>
<td>19</td>
<td>C.613CE</td>
<td>C.135CE</td>
</tr>
<tr>
<td>21</td>
<td>0.662CE</td>
<td>0.135CE</td>
</tr>
<tr>
<td>23</td>
<td>C.762E</td>
<td>0.143CE</td>
</tr>
<tr>
<td>25</td>
<td>C.762E</td>
<td>0.143CE</td>
</tr>
<tr>
<td>27</td>
<td>C.813E</td>
<td>C.144CE</td>
</tr>
<tr>
<td>29</td>
<td>C.872E</td>
<td>0.148CE</td>
</tr>
<tr>
<td>31</td>
<td>C.932E</td>
<td>0.146CE</td>
</tr>
<tr>
<td>33</td>
<td>0.986CE</td>
<td>0.138CE</td>
</tr>
<tr>
<td>35</td>
<td>C.106E</td>
<td>0.131CE</td>
</tr>
<tr>
<td>37</td>
<td>0.114CE</td>
<td>0.123CE</td>
</tr>
<tr>
<td>39</td>
<td>C.1226E</td>
<td>C.114CE</td>
</tr>
<tr>
<td>41</td>
<td>0.1319E</td>
<td>0.103CE</td>
</tr>
<tr>
<td>43</td>
<td>C.1426E</td>
<td>C.905CE</td>
</tr>
<tr>
<td>45</td>
<td>0.1554E</td>
<td>C.759CE</td>
</tr>
<tr>
<td>47</td>
<td>C.1718E</td>
<td>0.583CE</td>
</tr>
<tr>
<td>49</td>
<td>C.1565E</td>
<td>C.362CE</td>
</tr>
<tr>
<td>50</td>
<td>C.2183E</td>
<td>0.219CE</td>
</tr>
<tr>
<td>( t = 0.510 )</td>
<td>( t = 0.538 )</td>
<td>( t = 0.566 )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>I</td>
<td>( T )</td>
<td>W</td>
</tr>
<tr>
<td>1</td>
<td>C.29C4E-01</td>
<td>C.CCCCE CC</td>
</tr>
<tr>
<td>3</td>
<td>0.2312E CC</td>
<td>C.4037E CC</td>
</tr>
<tr>
<td>5</td>
<td>C.3212E CC</td>
<td>C.6194E CC</td>
</tr>
<tr>
<td>7</td>
<td>C.3570E CC</td>
<td>C.7796E CC</td>
</tr>
<tr>
<td>9</td>
<td>0.4651E CC</td>
<td>0.9056E CC</td>
</tr>
<tr>
<td>11</td>
<td>C.5252E CC</td>
<td>C.1CC7E C1</td>
</tr>
<tr>
<td>13</td>
<td>C.55C8E CC</td>
<td>C.1C85E C1</td>
</tr>
<tr>
<td>15</td>
<td>0.6504E 00</td>
<td>C.1157E C1</td>
</tr>
<tr>
<td>17</td>
<td>C.7C83E CC</td>
<td>C.1215E C1</td>
</tr>
<tr>
<td>19</td>
<td>0.7641E GC</td>
<td>0.1269E 01</td>
</tr>
<tr>
<td>21</td>
<td>C.8186E CC</td>
<td>C.1317E 01</td>
</tr>
<tr>
<td>23</td>
<td>C.9731E GC</td>
<td>0.1356E C1</td>
</tr>
<tr>
<td>25</td>
<td>C.9550E 00</td>
<td>0.1376E C1</td>
</tr>
<tr>
<td>27</td>
<td>C.55C1E CC</td>
<td>C.1359E C1</td>
</tr>
<tr>
<td>29</td>
<td>0.1053E C1</td>
<td>C.1336E C1</td>
</tr>
<tr>
<td>31</td>
<td>C.1119E C1</td>
<td>C.1303E C1</td>
</tr>
<tr>
<td>33</td>
<td>C.1188E C1</td>
<td>C.1259E C1</td>
</tr>
<tr>
<td>35</td>
<td>C.1261E C1</td>
<td>0.1206E 01</td>
</tr>
<tr>
<td>37</td>
<td>C.1342E C1</td>
<td>C.1123E C1</td>
</tr>
<tr>
<td>39</td>
<td>C.1211F C1</td>
<td>C.1047E C1</td>
</tr>
<tr>
<td>41</td>
<td>0.1529E C1</td>
<td>C.9483E 00</td>
</tr>
<tr>
<td>43</td>
<td>C.1642E C1</td>
<td>C.8344E CC</td>
</tr>
<tr>
<td>45</td>
<td>C.1778E C1</td>
<td>C.7006E CC</td>
</tr>
<tr>
<td>47</td>
<td>C.1951E C1</td>
<td>C.5394E CC</td>
</tr>
<tr>
<td>49</td>
<td>0.2211E 01</td>
<td>C.235CE CC</td>
</tr>
<tr>
<td>50</td>
<td>C.2440E C1</td>
<td>0.2029E 00</td>
</tr>
<tr>
<td>( \tau = 0.594 )</td>
<td>( \tau = 0.622 )</td>
<td>( \tau = 0.626 )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>1</td>
<td>0.1510E 00</td>
<td>0.1737E 00</td>
</tr>
<tr>
<td>3</td>
<td>0.3277E CC</td>
<td>0.3613E CC</td>
</tr>
<tr>
<td>5</td>
<td>0.4321E CC</td>
<td>0.4701E CC</td>
</tr>
<tr>
<td>7</td>
<td>0.5175E 00</td>
<td>0.5586E CO</td>
</tr>
<tr>
<td>9</td>
<td>0.5626E CC</td>
<td>0.6269E CC</td>
</tr>
<tr>
<td>11</td>
<td>0.6635E CC</td>
<td>0.7162E CO</td>
</tr>
<tr>
<td>13</td>
<td>0.7259E CO</td>
<td>0.7768E CO</td>
</tr>
<tr>
<td>15</td>
<td>0.7528E OC</td>
<td>0.8466E CO</td>
</tr>
<tr>
<td>17</td>
<td>0.8539E 00</td>
<td>0.9637E 00</td>
</tr>
<tr>
<td>19</td>
<td>0.9145E CC</td>
<td>0.9673E CC</td>
</tr>
<tr>
<td>21</td>
<td>0.5775E CC</td>
<td>0.1219E C1</td>
</tr>
<tr>
<td>23</td>
<td>0.1641E C1</td>
<td>0.1229E 01</td>
</tr>
<tr>
<td>25</td>
<td>0.1144E C1</td>
<td>0.1241E C1</td>
</tr>
<tr>
<td>27</td>
<td>0.1167E 01</td>
<td>0.1226E 01</td>
</tr>
<tr>
<td>29</td>
<td>0.1231E C1</td>
<td>0.1194E 01</td>
</tr>
<tr>
<td>31</td>
<td>0.1300E C1</td>
<td>0.1184E C1</td>
</tr>
<tr>
<td>33</td>
<td>0.1373E C1</td>
<td>0.1154E C1</td>
</tr>
<tr>
<td>35</td>
<td>0.1453E C1</td>
<td>0.1104E C1</td>
</tr>
<tr>
<td>37</td>
<td>0.1538E C1</td>
<td>0.1042E C1</td>
</tr>
<tr>
<td>39</td>
<td>0.1622E C1</td>
<td>0.9681E C0</td>
</tr>
<tr>
<td>41</td>
<td>0.1726E C1</td>
<td>0.8866E CC</td>
</tr>
<tr>
<td>43</td>
<td>0.1855E C1</td>
<td>0.7771E 00</td>
</tr>
<tr>
<td>45</td>
<td>0.1957E C1</td>
<td>0.6539E 00</td>
</tr>
<tr>
<td>47</td>
<td>0.2178E C1</td>
<td>0.5043E CC</td>
</tr>
<tr>
<td>49</td>
<td>0.2450E C1</td>
<td>0.3137E 00</td>
</tr>
<tr>
<td>50</td>
<td>0.2665E C1</td>
<td>0.1901E CC</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>----</td>
<td>--------</td>
<td>--------</td>
</tr>
<tr>
<td>1</td>
<td>C.2CC5E</td>
<td>00</td>
</tr>
<tr>
<td>3</td>
<td>C.4CC5E</td>
<td>CC</td>
</tr>
<tr>
<td>5</td>
<td>C.5142E</td>
<td>CC</td>
</tr>
<tr>
<td>7</td>
<td>C.6CC5E</td>
<td>CC</td>
</tr>
<tr>
<td>9</td>
<td>C.6E6E</td>
<td>CC</td>
</tr>
<tr>
<td>11</td>
<td>C.76C3E</td>
<td>CC</td>
</tr>
<tr>
<td>13</td>
<td>C.8254E</td>
<td>CC</td>
</tr>
<tr>
<td>15</td>
<td>C.8555E</td>
<td>CC</td>
</tr>
<tr>
<td>17</td>
<td>C.9611E</td>
<td>CC</td>
</tr>
<tr>
<td>19</td>
<td>C.1C27E</td>
<td>CC</td>
</tr>
<tr>
<td>21</td>
<td>C.1052E</td>
<td>CC</td>
</tr>
<tr>
<td>23</td>
<td>C.1158E</td>
<td>CC</td>
</tr>
<tr>
<td>25</td>
<td>C.1223E</td>
<td>CC</td>
</tr>
<tr>
<td>27</td>
<td>C.1250E</td>
<td>CC</td>
</tr>
<tr>
<td>29</td>
<td>C.1275E</td>
<td>CC</td>
</tr>
<tr>
<td>31</td>
<td>C.1430E</td>
<td>CC</td>
</tr>
<tr>
<td>33</td>
<td>C.15C7E</td>
<td>CC</td>
</tr>
<tr>
<td>35</td>
<td>C.155CE</td>
<td>CC</td>
</tr>
<tr>
<td>37</td>
<td>C.1678E</td>
<td>CC</td>
</tr>
<tr>
<td>39</td>
<td>C.1775E</td>
<td>CC</td>
</tr>
<tr>
<td>41</td>
<td>C.1EE2E</td>
<td>CC</td>
</tr>
<tr>
<td>43</td>
<td>C.2CC0E</td>
<td>CC</td>
</tr>
<tr>
<td>45</td>
<td>C.2151E</td>
<td>CC</td>
</tr>
<tr>
<td>47</td>
<td>C.2338E</td>
<td>CC</td>
</tr>
<tr>
<td>49</td>
<td>C.2617E</td>
<td>CC</td>
</tr>
<tr>
<td>50</td>
<td>C.2664E</td>
<td>CC</td>
</tr>
<tr>
<td>I</td>
<td>W</td>
<td>I</td>
</tr>
<tr>
<td>----</td>
<td>-----------</td>
<td>-----</td>
</tr>
<tr>
<td>0.730</td>
<td>C.2705E OC</td>
<td>C.0305E CC</td>
</tr>
<tr>
<td>0.758</td>
<td>C.4564E OC</td>
<td>C.3317E CC</td>
</tr>
<tr>
<td>0.776</td>
<td>C.6237E CC</td>
<td>C.5132E CC</td>
</tr>
</tbody>
</table>

Note: The table entries represent values, potentially in scientific notation, with the first column indicating a parameter 'I', and the second column indicating a corresponding value 'W'.
<table>
<thead>
<tr>
<th>I</th>
<th>T</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3438E 00</td>
<td>0.0000E 00</td>
</tr>
<tr>
<td>3</td>
<td>0.5528E 00</td>
<td>0.3154E 00</td>
</tr>
<tr>
<td>5</td>
<td>0.126E 00</td>
<td>0.4898E 00</td>
</tr>
<tr>
<td>7</td>
<td>0.8316E 00</td>
<td>0.6229E 00</td>
</tr>
<tr>
<td>9</td>
<td>0.5225E 00</td>
<td>0.734E 00</td>
</tr>
<tr>
<td>11</td>
<td>0.116E 01</td>
<td>0.8184E 00</td>
</tr>
<tr>
<td>13</td>
<td>0.116E 01</td>
<td>0.9902E 00</td>
</tr>
<tr>
<td>15</td>
<td>0.116E 01</td>
<td>0.5484E 00</td>
</tr>
<tr>
<td>17</td>
<td>0.116E 01</td>
<td>0.9948E 00</td>
</tr>
<tr>
<td>19</td>
<td>0.116E 01</td>
<td>0.1020E 00</td>
</tr>
<tr>
<td>21</td>
<td>0.116E 01</td>
<td>0.1055E 00</td>
</tr>
<tr>
<td>23</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>25</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>27</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>29</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>31</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>33</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>35</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>37</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>39</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>41</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>43</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>45</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>47</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>49</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>50</td>
<td>0.116E 01</td>
<td>0.1070E 00</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>-----</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>1</td>
<td>0.4238E C0</td>
<td>0.0000E CO</td>
</tr>
<tr>
<td>3</td>
<td>C.6945E C0</td>
<td>C.3013E CO</td>
</tr>
<tr>
<td>5</td>
<td>C.8373E C0</td>
<td>C.4652E CO</td>
</tr>
<tr>
<td>7</td>
<td>C.9488E C0</td>
<td>C.5973E CO</td>
</tr>
<tr>
<td>9</td>
<td>0.1045E C1</td>
<td>C.7CC3E C0</td>
</tr>
<tr>
<td>11</td>
<td>0.113C E01</td>
<td>0.7344E CO</td>
</tr>
<tr>
<td>13</td>
<td>C.1216E C1</td>
<td>C.8533E CO</td>
</tr>
<tr>
<td>15</td>
<td>0.1255E C1</td>
<td>C.9C52E CO</td>
</tr>
<tr>
<td>17</td>
<td>C.1373E C0</td>
<td>0.9535E CO</td>
</tr>
<tr>
<td>19</td>
<td>0.1449E C1</td>
<td>C.5872E CO</td>
</tr>
<tr>
<td>21</td>
<td>C.1525E C1</td>
<td>0.1011E C1</td>
</tr>
<tr>
<td>23</td>
<td>C.16CCE C1</td>
<td>0.1026E C1</td>
</tr>
<tr>
<td>25</td>
<td>C.1677E C1</td>
<td>0.1034E C1</td>
</tr>
<tr>
<td>27</td>
<td>C.1754E C1</td>
<td>0.1033E C1</td>
</tr>
<tr>
<td>29</td>
<td>C.1821E C1</td>
<td>0.1016E C1</td>
</tr>
<tr>
<td>31</td>
<td>C.1915E C1</td>
<td>C.9946E CO</td>
</tr>
<tr>
<td>33</td>
<td>C.2C2E C1</td>
<td>0.9641E CO</td>
</tr>
<tr>
<td>35</td>
<td>C.2C55E C1</td>
<td>C.9242E CO</td>
</tr>
<tr>
<td>37</td>
<td>C.2194E C1</td>
<td>0.8745E CO</td>
</tr>
<tr>
<td>39</td>
<td>C.2314E C1</td>
<td>0.8140E CO</td>
</tr>
<tr>
<td>41</td>
<td>C.2421E C1</td>
<td>C.7415E CO</td>
</tr>
<tr>
<td>43</td>
<td>0.2557E C1</td>
<td>0.6551E CO</td>
</tr>
<tr>
<td>45</td>
<td>C.2718E C1</td>
<td>C.5518E CO</td>
</tr>
<tr>
<td>47</td>
<td>C.2923E C1</td>
<td>C.4260E CO</td>
</tr>
<tr>
<td>49</td>
<td>C.3229E C1</td>
<td>0.2653E CO</td>
</tr>
<tr>
<td>50</td>
<td>C.35CCE C1</td>
<td>C.1611E C0</td>
</tr>
<tr>
<td></td>
<td>( t = 0.954 )</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>I</td>
<td>( t )</td>
<td>( C )</td>
</tr>
<tr>
<td>1</td>
<td>( 0.5555 )</td>
<td>( 0.0300 )</td>
</tr>
<tr>
<td>3</td>
<td>( 0.7958 )</td>
<td>( 0.2854 )</td>
</tr>
<tr>
<td>5</td>
<td>( 0.5470 )</td>
<td>( 0.4516 )</td>
</tr>
<tr>
<td>7</td>
<td>( 0.1064 )</td>
<td>( 0.5751 )</td>
</tr>
<tr>
<td>9</td>
<td>( 0.1166 )</td>
<td>( 0.6743 )</td>
</tr>
<tr>
<td>11</td>
<td>( 0.1250 )</td>
<td>( 0.7553 )</td>
</tr>
<tr>
<td>13</td>
<td>( 0.1345 )</td>
<td>( 0.8216 )</td>
</tr>
<tr>
<td>15</td>
<td>( 0.1428 )</td>
<td>( 0.8754 )</td>
</tr>
<tr>
<td>17</td>
<td>( 0.1509 )</td>
<td>( 0.9181 )</td>
</tr>
<tr>
<td>19</td>
<td>( 0.1588 )</td>
<td>( 0.9506 )</td>
</tr>
<tr>
<td>21</td>
<td>( 0.1667 )</td>
<td>( 0.9737 )</td>
</tr>
<tr>
<td>23</td>
<td>( 0.1746 )</td>
<td>( 0.9877 )</td>
</tr>
<tr>
<td>25</td>
<td>( 0.1825 )</td>
<td>( 0.9952 )</td>
</tr>
<tr>
<td>27</td>
<td>( 0.1905 )</td>
<td>( 0.9934 )</td>
</tr>
<tr>
<td>29</td>
<td>( 0.1985 )</td>
<td>( 0.9765 )</td>
</tr>
<tr>
<td>31</td>
<td>( 0.2072 )</td>
<td>( 0.9568 )</td>
</tr>
<tr>
<td>33</td>
<td>( 0.2162 )</td>
<td>( 0.9282 )</td>
</tr>
<tr>
<td>35</td>
<td>( 0.2258 )</td>
<td>( 0.8504 )</td>
</tr>
<tr>
<td>37</td>
<td>( 0.2360 )</td>
<td>( 0.8429 )</td>
</tr>
<tr>
<td>39</td>
<td>( 0.2471 )</td>
<td>( 0.7849 )</td>
</tr>
<tr>
<td>41</td>
<td>( 0.2594 )</td>
<td>( 0.7137 )</td>
</tr>
<tr>
<td>43</td>
<td>( 0.2734 )</td>
<td>( 0.6322 )</td>
</tr>
<tr>
<td>45</td>
<td>( 0.2556 )</td>
<td>( 0.5326 )</td>
</tr>
<tr>
<td>47</td>
<td>( 0.3111 )</td>
<td>( 0.4113 )</td>
</tr>
<tr>
<td>49</td>
<td>( 0.3425 )</td>
<td>( 0.2563 )</td>
</tr>
<tr>
<td>50</td>
<td>( 0.3763 )</td>
<td>( 0.1555 )</td>
</tr>
<tr>
<td>I</td>
<td>τ=1.032</td>
<td>τ=1.380</td>
</tr>
<tr>
<td>---</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>0.5959E 00</td>
<td>0.1029E 01</td>
</tr>
<tr>
<td>3</td>
<td>0.9055E 00</td>
<td>0.1412E 01</td>
</tr>
<tr>
<td>5</td>
<td>0.1065E 01</td>
<td>0.1666E 01</td>
</tr>
<tr>
<td>7</td>
<td>0.1189E 01</td>
<td>0.1755E 01</td>
</tr>
<tr>
<td>9</td>
<td>0.1296E 01</td>
<td>0.1882E 01</td>
</tr>
<tr>
<td>11</td>
<td>0.1392E 01</td>
<td>0.1997E 01</td>
</tr>
<tr>
<td>13</td>
<td>0.1463E 01</td>
<td>0.2105E 01</td>
</tr>
<tr>
<td>15</td>
<td>0.1569E 01</td>
<td>0.2211E 01</td>
</tr>
<tr>
<td>17</td>
<td>0.1654E 01</td>
<td>0.2303E 01</td>
</tr>
<tr>
<td>19</td>
<td>0.1736E 01</td>
<td>0.2368E 01</td>
</tr>
<tr>
<td>21</td>
<td>0.1818E 01</td>
<td>0.2453E 01</td>
</tr>
<tr>
<td>23</td>
<td>0.1963E 01</td>
<td>0.2536E 01</td>
</tr>
<tr>
<td>25</td>
<td>0.1962E 01</td>
<td>0.2677E 01</td>
</tr>
<tr>
<td>27</td>
<td>0.2065E 01</td>
<td>0.2772E 01</td>
</tr>
<tr>
<td>29</td>
<td>0.2149E 01</td>
<td>0.2871E 01</td>
</tr>
<tr>
<td>31</td>
<td>0.2239E 01</td>
<td>0.2972E 01</td>
</tr>
<tr>
<td>33</td>
<td>0.2322E 01</td>
<td>0.3076E 01</td>
</tr>
<tr>
<td>35</td>
<td>0.2431E 01</td>
<td>0.3187E 01</td>
</tr>
<tr>
<td>37</td>
<td>0.2536E 01</td>
<td>0.3305E 01</td>
</tr>
<tr>
<td>39</td>
<td>0.2651E 01</td>
<td>0.3434E 01</td>
</tr>
<tr>
<td>41</td>
<td>0.2777E 01</td>
<td>0.3575E 01</td>
</tr>
<tr>
<td>43</td>
<td>0.2921E 01</td>
<td>0.3735E 01</td>
</tr>
<tr>
<td>45</td>
<td>0.3052E 01</td>
<td>0.3925E 01</td>
</tr>
<tr>
<td>47</td>
<td>0.3308E 01</td>
<td>0.4165E 01</td>
</tr>
<tr>
<td>49</td>
<td>0.3631E 01</td>
<td>0.4520E 01</td>
</tr>
<tr>
<td>50</td>
<td>0.3916E 01</td>
<td>0.4837E 01</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>----</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>0.19CCE</td>
<td>0.000E</td>
</tr>
<tr>
<td>3</td>
<td>0.2387E</td>
<td>0.2024E</td>
</tr>
<tr>
<td>5</td>
<td>0.2633E</td>
<td>0.3186E</td>
</tr>
<tr>
<td>7</td>
<td>0.2819E</td>
<td>0.4062E</td>
</tr>
<tr>
<td>9</td>
<td>0.2976E</td>
<td>0.4777E</td>
</tr>
<tr>
<td>11</td>
<td>0.3116E</td>
<td>0.5356E</td>
</tr>
<tr>
<td>13</td>
<td>0.3249E</td>
<td>0.5771E</td>
</tr>
<tr>
<td>15</td>
<td>0.3369E</td>
<td>0.6215E</td>
</tr>
<tr>
<td>17</td>
<td>0.3484E</td>
<td>0.6638E</td>
</tr>
<tr>
<td>19</td>
<td>0.3596E</td>
<td>0.6908E</td>
</tr>
<tr>
<td>21</td>
<td>0.3707E</td>
<td>0.6876E</td>
</tr>
<tr>
<td>23</td>
<td>0.3823E</td>
<td>0.7098E</td>
</tr>
<tr>
<td>25</td>
<td>0.3932E</td>
<td>0.7114E</td>
</tr>
<tr>
<td>27</td>
<td>0.4046E</td>
<td>0.6929E</td>
</tr>
<tr>
<td>29</td>
<td>0.4154E</td>
<td>0.6620E</td>
</tr>
<tr>
<td>31</td>
<td>0.4279E</td>
<td>0.6599E</td>
</tr>
<tr>
<td>33</td>
<td>0.4404E</td>
<td>0.6490E</td>
</tr>
<tr>
<td>35</td>
<td>0.4534E</td>
<td>0.6346E</td>
</tr>
<tr>
<td>37</td>
<td>0.4671E</td>
<td>0.6076E</td>
</tr>
<tr>
<td>39</td>
<td>0.4819E</td>
<td>0.5672E</td>
</tr>
<tr>
<td>-41</td>
<td>0.4983E</td>
<td>0.5172E</td>
</tr>
<tr>
<td>-43</td>
<td>0.5167E</td>
<td>0.4576E</td>
</tr>
<tr>
<td>-45</td>
<td>0.5385E</td>
<td>0.3852E</td>
</tr>
<tr>
<td>-47</td>
<td>0.5659E</td>
<td>0.2930E</td>
</tr>
<tr>
<td>-49</td>
<td>0.6065E</td>
<td>0.1865E</td>
</tr>
<tr>
<td>-50</td>
<td>0.6429E</td>
<td>0.1136E</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>---</td>
<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>1</td>
<td>0.3431E</td>
<td>01</td>
</tr>
<tr>
<td>2</td>
<td>0.4045E</td>
<td>01</td>
</tr>
<tr>
<td>3</td>
<td>0.4356E</td>
<td>01</td>
</tr>
<tr>
<td>4</td>
<td>0.4589E</td>
<td>01</td>
</tr>
<tr>
<td>5</td>
<td>0.4785E</td>
<td>01</td>
</tr>
<tr>
<td>6</td>
<td>0.4960E</td>
<td>01</td>
</tr>
<tr>
<td>7</td>
<td>0.512E</td>
<td>01</td>
</tr>
<tr>
<td>8</td>
<td>0.5269E</td>
<td>01</td>
</tr>
<tr>
<td>9</td>
<td>0.5407E</td>
<td>01</td>
</tr>
<tr>
<td>10</td>
<td>0.5543E</td>
<td>01</td>
</tr>
<tr>
<td>11</td>
<td>0.5666E</td>
<td>01</td>
</tr>
<tr>
<td>12</td>
<td>0.5812E</td>
<td>01</td>
</tr>
<tr>
<td>13</td>
<td>0.5946E</td>
<td>01</td>
</tr>
<tr>
<td>14</td>
<td>0.6079E</td>
<td>01</td>
</tr>
<tr>
<td>15</td>
<td>0.6213E</td>
<td>01</td>
</tr>
<tr>
<td>16</td>
<td>0.6356E</td>
<td>01</td>
</tr>
<tr>
<td>17</td>
<td>0.6511E</td>
<td>01</td>
</tr>
<tr>
<td>18</td>
<td>0.6667E</td>
<td>01</td>
</tr>
<tr>
<td>19</td>
<td>0.6822E</td>
<td>01</td>
</tr>
<tr>
<td>20</td>
<td>0.7006E</td>
<td>01</td>
</tr>
<tr>
<td>21</td>
<td>0.7155E</td>
<td>01</td>
</tr>
<tr>
<td>22</td>
<td>0.7412E</td>
<td>01</td>
</tr>
<tr>
<td>23</td>
<td>0.7666E</td>
<td>01</td>
</tr>
<tr>
<td>24</td>
<td>0.7986E</td>
<td>01</td>
</tr>
<tr>
<td>25</td>
<td>0.8457E</td>
<td>01</td>
</tr>
<tr>
<td>26</td>
<td>0.8885E</td>
<td>01</td>
</tr>
<tr>
<td></td>
<td>I</td>
<td>T</td>
</tr>
<tr>
<td>---</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>1</td>
<td>0.5882E-01</td>
<td>0.0000E+00</td>
</tr>
<tr>
<td>3</td>
<td>0.6655E-01</td>
<td>0.1379E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.7037E-01</td>
<td>0.2176E+00</td>
</tr>
<tr>
<td>7</td>
<td>0.7325E-01</td>
<td>0.2787E+00</td>
</tr>
<tr>
<td>9</td>
<td>0.7565E-01</td>
<td>0.3286E+00</td>
</tr>
<tr>
<td>11</td>
<td>0.7777E-01</td>
<td>0.3701E+00</td>
</tr>
<tr>
<td>13</td>
<td>0.7971E-01</td>
<td>0.4006E+00</td>
</tr>
<tr>
<td>15</td>
<td>0.8154E-01</td>
<td>0.4254E+00</td>
</tr>
<tr>
<td>17</td>
<td>0.8323E-01</td>
<td>0.4635E+00</td>
</tr>
<tr>
<td>19</td>
<td>0.8488E-01</td>
<td>0.4586E+00</td>
</tr>
<tr>
<td>21</td>
<td>0.8659E-01</td>
<td>0.4662E+00</td>
</tr>
<tr>
<td>23</td>
<td>0.8811E-01</td>
<td>0.5055E+00</td>
</tr>
<tr>
<td>25</td>
<td>0.8971E-01</td>
<td>0.5097E+00</td>
</tr>
<tr>
<td>27</td>
<td>0.9133E-01</td>
<td>0.4981E+00</td>
</tr>
<tr>
<td>29</td>
<td>0.9258E-01</td>
<td>0.4796E+00</td>
</tr>
<tr>
<td>31</td>
<td>0.9466E-01</td>
<td>0.4760E+00</td>
</tr>
<tr>
<td>33</td>
<td>0.9639E-01</td>
<td>0.4434E+00</td>
</tr>
<tr>
<td>35</td>
<td>0.9837E-01</td>
<td>0.4581E+00</td>
</tr>
<tr>
<td>37</td>
<td>0.1063E+02</td>
<td>0.4215E+00</td>
</tr>
<tr>
<td>39</td>
<td>0.1024E+02</td>
<td>0.3945E+00</td>
</tr>
<tr>
<td>41</td>
<td>0.1046E+02</td>
<td>0.3631E+00</td>
</tr>
<tr>
<td>43</td>
<td>0.1072E+02</td>
<td>0.3209E+00</td>
</tr>
<tr>
<td>45</td>
<td>0.1101E+02</td>
<td>0.2702E+00</td>
</tr>
<tr>
<td>47</td>
<td>0.1139E+02</td>
<td>0.2091E+00</td>
</tr>
<tr>
<td>49</td>
<td>0.1194E+02</td>
<td>0.1311E+00</td>
</tr>
<tr>
<td>50</td>
<td>0.1245E+02</td>
<td>0.8008E-01</td>
</tr>
</tbody>
</table>
\[ t = 0.26 \]

<table>
<thead>
<tr>
<th>( I )</th>
<th>( T )</th>
<th>( W )</th>
<th>( I )</th>
<th>( T )</th>
<th>( W )</th>
<th>( I )</th>
<th>( T )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8411E 01</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.9575E 01</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.1012E 02</td>
<td>0.0000E 00</td>
</tr>
<tr>
<td>3</td>
<td>0.9251E 01</td>
<td>0.1204E 00</td>
<td>3</td>
<td>0.1050E 02</td>
<td>0.1148E 00</td>
<td>3</td>
<td>0.1117E 02</td>
<td>0.1124E 00</td>
</tr>
<tr>
<td>5</td>
<td>0.9739E 01</td>
<td>0.1907E 00</td>
<td>5</td>
<td>0.1057E 02</td>
<td>0.1816E 00</td>
<td>5</td>
<td>0.1155E 02</td>
<td>0.1779E 00</td>
</tr>
<tr>
<td>7</td>
<td>0.1037E 02</td>
<td>0.2443E 00</td>
<td>7</td>
<td>0.1132E 02</td>
<td>0.2329E 00</td>
<td>7</td>
<td>0.1191E 02</td>
<td>0.2281E 00</td>
</tr>
<tr>
<td>9</td>
<td>0.1035E 02</td>
<td>0.2877E 00</td>
<td>9</td>
<td>0.1162E 02</td>
<td>0.2745E 00</td>
<td>9</td>
<td>0.1221E 02</td>
<td>0.2684E 00</td>
</tr>
<tr>
<td>11</td>
<td>0.1060E 02</td>
<td>0.3204E 00</td>
<td>11</td>
<td>0.1188E 02</td>
<td>0.3085E 00</td>
<td>11</td>
<td>0.1248E 02</td>
<td>0.3014E 00</td>
</tr>
<tr>
<td>13</td>
<td>0.1082E 02</td>
<td>0.3315E 00</td>
<td>13</td>
<td>0.1212E 02</td>
<td>0.3400E 00</td>
<td>13</td>
<td>0.1271E 02</td>
<td>0.3422E 00</td>
</tr>
<tr>
<td>15</td>
<td>0.1102E 02</td>
<td>0.3873E 00</td>
<td>15</td>
<td>0.1233E 02</td>
<td>0.3748E 00</td>
<td>15</td>
<td>0.1293E 02</td>
<td>0.3660E 00</td>
</tr>
<tr>
<td>17</td>
<td>0.1121E 02</td>
<td>0.3903E 00</td>
<td>17</td>
<td>0.1254E 02</td>
<td>0.3989E 00</td>
<td>17</td>
<td>0.1314E 02</td>
<td>0.3730E 00</td>
</tr>
<tr>
<td>19</td>
<td>0.1140E 02</td>
<td>0.4019E 00</td>
<td>19</td>
<td>0.1272E 02</td>
<td>0.3982E 00</td>
<td>19</td>
<td>0.1335E 02</td>
<td>0.3883E 00</td>
</tr>
<tr>
<td>21</td>
<td>0.1158E 02</td>
<td>0.4337E 00</td>
<td>21</td>
<td>0.1253E 02</td>
<td>0.4014E 00</td>
<td>21</td>
<td>0.1355E 02</td>
<td>0.3898E 00</td>
</tr>
<tr>
<td>23</td>
<td>0.1177E 02</td>
<td>0.4420E 00</td>
<td>23</td>
<td>0.1312E 02</td>
<td>0.4053E 00</td>
<td>23</td>
<td>0.1375E 02</td>
<td>0.3910E 00</td>
</tr>
<tr>
<td>25</td>
<td>0.1156E 02</td>
<td>0.4222E 00</td>
<td>25</td>
<td>0.1331E 02</td>
<td>0.4043E 00</td>
<td>25</td>
<td>0.1353E 02</td>
<td>0.3908E 00</td>
</tr>
<tr>
<td>27</td>
<td>0.1214E 02</td>
<td>0.4130E 00</td>
<td>27</td>
<td>0.1350E 02</td>
<td>0.4117E 00</td>
<td>27</td>
<td>0.1414E 02</td>
<td>0.3972E 00</td>
</tr>
<tr>
<td>29</td>
<td>0.1233E 02</td>
<td>0.4061E 00</td>
<td>29</td>
<td>0.1371E 02</td>
<td>0.3987E 00</td>
<td>29</td>
<td>0.1433E 02</td>
<td>0.4031E 00</td>
</tr>
<tr>
<td>31</td>
<td>0.1253E 02</td>
<td>0.4008E 00</td>
<td>31</td>
<td>0.1350E 02</td>
<td>0.3965E 00</td>
<td>31</td>
<td>0.1454E 02</td>
<td>0.4073E 00</td>
</tr>
<tr>
<td>33</td>
<td>0.1273E 02</td>
<td>0.3952E 00</td>
<td>33</td>
<td>0.1411E 02</td>
<td>0.3954E 00</td>
<td>33</td>
<td>0.1476E 02</td>
<td>0.4035E 00</td>
</tr>
<tr>
<td>35</td>
<td>0.1293E 02</td>
<td>0.3828E 00</td>
<td>35</td>
<td>0.1433E 02</td>
<td>0.3865E 00</td>
<td>35</td>
<td>0.1458E 02</td>
<td>0.3839E 00</td>
</tr>
<tr>
<td>37</td>
<td>0.1316E 02</td>
<td>0.3769E 00</td>
<td>37</td>
<td>0.1456E 02</td>
<td>0.3558E 00</td>
<td>37</td>
<td>0.1522E 02</td>
<td>0.3409E 00</td>
</tr>
<tr>
<td>39</td>
<td>0.1339E 02</td>
<td>0.3501E 00</td>
<td>39</td>
<td>0.1481E 02</td>
<td>0.3287E 00</td>
<td>39</td>
<td>0.1547E 02</td>
<td>0.3076E 00</td>
</tr>
<tr>
<td>41</td>
<td>0.1364E 02</td>
<td>0.3192E 00</td>
<td>41</td>
<td>0.1557E 02</td>
<td>0.3010E 00</td>
<td>41</td>
<td>0.1573E 02</td>
<td>0.2787E 00</td>
</tr>
<tr>
<td>43</td>
<td>0.1393E 02</td>
<td>0.2822E 00</td>
<td>43</td>
<td>0.1537E 02</td>
<td>0.2693E 00</td>
<td>43</td>
<td>0.1624E 02</td>
<td>0.2457E 00</td>
</tr>
<tr>
<td>45</td>
<td>0.1426E 02</td>
<td>0.2380E 00</td>
<td>45</td>
<td>0.1572E 02</td>
<td>0.2283E 00</td>
<td>45</td>
<td>0.1639E 02</td>
<td>0.2079E 00</td>
</tr>
<tr>
<td>47</td>
<td>0.1468E 02</td>
<td>0.1843E 00</td>
<td>47</td>
<td>0.1615E 02</td>
<td>0.1767E 00</td>
<td>47</td>
<td>0.1684E 02</td>
<td>0.1727E 00</td>
</tr>
<tr>
<td>49</td>
<td>0.1530E 02</td>
<td>0.1155E 00</td>
<td>49</td>
<td>0.1680E 02</td>
<td>0.1112E 00</td>
<td>49</td>
<td>0.1750E 02</td>
<td>0.1088E 00</td>
</tr>
<tr>
<td>50</td>
<td>0.1588E 02</td>
<td>0.7051E-01</td>
<td>50</td>
<td>0.1749E 02</td>
<td>0.6802E-01</td>
<td>50</td>
<td>0.1811E 02</td>
<td>0.6656E-01</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
<td>I</td>
<td>T</td>
<td>W</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----</td>
<td>-------</td>
<td>-----</td>
<td>----</td>
<td>-------</td>
<td>-----</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.1142E 02</td>
<td>0.000CE CO</td>
<td>1</td>
<td>0.1151E 02</td>
<td>0.000CE CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1242E 02</td>
<td>0.108CE CO</td>
<td>3</td>
<td>0.1252E 02</td>
<td>0.106CE CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1293E 02</td>
<td>0.1704E CO</td>
<td>5</td>
<td>0.1343E 02</td>
<td>0.1678E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.1330E 02</td>
<td>0.2183E CO</td>
<td>7</td>
<td>0.1382E 02</td>
<td>0.2150E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.1361E 02</td>
<td>0.2573E CO</td>
<td>9</td>
<td>0.1413E 02</td>
<td>0.2526E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.1389E 02</td>
<td>0.2887E CO</td>
<td>11</td>
<td>0.1442E 02</td>
<td>0.2842E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.1414E 02</td>
<td>0.3061E CO</td>
<td>13</td>
<td>0.1468E 02</td>
<td>0.3130E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.1437E 02</td>
<td>0.3129E CO</td>
<td>15</td>
<td>0.1491E 02</td>
<td>0.3453E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>0.1459E 02</td>
<td>0.3594E CO</td>
<td>17</td>
<td>0.1513E 02</td>
<td>0.3429E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>0.1480E 02</td>
<td>0.3586E CO</td>
<td>19</td>
<td>0.1535E 02</td>
<td>0.3704E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>0.1511E 02</td>
<td>0.3524E CO</td>
<td>21</td>
<td>0.1556E 02</td>
<td>0.3743E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>0.1521E 02</td>
<td>0.3492E CO</td>
<td>23</td>
<td>0.1577E 02</td>
<td>0.3772E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.1541E 02</td>
<td>0.3504E CO</td>
<td>25</td>
<td>0.1597E 02</td>
<td>0.3794E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>0.1563E 02</td>
<td>0.3626E CO</td>
<td>27</td>
<td>0.1618E 02</td>
<td>0.3842E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>0.1584E 02</td>
<td>0.3638E CO</td>
<td>29</td>
<td>0.1640E 02</td>
<td>0.3889E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>0.1605E 02</td>
<td>0.3588E CO</td>
<td>31</td>
<td>0.1662E 02</td>
<td>0.3788E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>0.1628E 02</td>
<td>0.3404E CO</td>
<td>33</td>
<td>0.1685E 02</td>
<td>0.3536E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.1652E 02</td>
<td>0.3251E CO</td>
<td>35</td>
<td>0.1759E 02</td>
<td>0.3563E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>0.1677E 02</td>
<td>0.3202E CO</td>
<td>37</td>
<td>0.1734E 02</td>
<td>0.3453E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>0.1743E 02</td>
<td>0.2968E CO</td>
<td>39</td>
<td>0.1760E 02</td>
<td>0.3101E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>0.1773E 02</td>
<td>0.2711E CO</td>
<td>41</td>
<td>0.1788E 02</td>
<td>0.2813E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>0.1762E 02</td>
<td>0.2401E CO</td>
<td>43</td>
<td>0.1821E 02</td>
<td>0.2489E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>0.1799E 02</td>
<td>0.2192E CO</td>
<td>45</td>
<td>0.1858E 02</td>
<td>0.2129E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>0.1846E 02</td>
<td>0.1656E CO</td>
<td>47</td>
<td>0.1995E 02</td>
<td>0.1636E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>0.1914E 02</td>
<td>0.1040E CO</td>
<td>49</td>
<td>0.1975E 02</td>
<td>0.1024E CO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.1979E 02</td>
<td>0.6357E-01</td>
<td>50</td>
<td>0.2140E-01</td>
<td>0.6256E-01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \tau = 9.021 )</td>
<td>( \tau = 9.900 )</td>
<td>( \tau = 10.04 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I )</td>
<td>( T )</td>
<td>( W )</td>
<td>( I )</td>
<td>( T )</td>
<td>( W )</td>
<td>( I )</td>
<td>( T )</td>
<td>( W )</td>
</tr>
<tr>
<td>1</td>
<td>0.1364E 02</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.1521E 02</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.1546E 02</td>
<td>0.0000E 00</td>
</tr>
<tr>
<td>3</td>
<td>0.1471E 02</td>
<td>0.1001E 00</td>
<td>3</td>
<td>0.1632E 02</td>
<td>0.9613E-01</td>
<td>3</td>
<td>0.1658E 02</td>
<td>0.9548E-01</td>
</tr>
<tr>
<td>5</td>
<td>0.1526E 02</td>
<td>0.1590E 00</td>
<td>5</td>
<td>0.1690E 02</td>
<td>0.1524E CC</td>
<td>5</td>
<td>0.1715E 02</td>
<td>0.1514E 00</td>
</tr>
<tr>
<td>7</td>
<td>0.1566E 02</td>
<td>0.2039E 00</td>
<td>7</td>
<td>0.1732E 02</td>
<td>0.1954E CC</td>
<td>7</td>
<td>0.1758E 02</td>
<td>0.1941E 00</td>
</tr>
<tr>
<td>9</td>
<td>0.1600E 02</td>
<td>0.2406E 00</td>
<td>9</td>
<td>0.1767E 02</td>
<td>0.2298E 00</td>
<td>9</td>
<td>0.1794E 02</td>
<td>0.2284E 00</td>
</tr>
<tr>
<td>11</td>
<td>0.1630E 02</td>
<td>0.2712E 00</td>
<td>11</td>
<td>0.1798E 02</td>
<td>0.2572E 00</td>
<td>11</td>
<td>0.1825E 02</td>
<td>0.2573E 00</td>
</tr>
<tr>
<td>13</td>
<td>0.1657E 02</td>
<td>0.3007E 00</td>
<td>13</td>
<td>0.1826E 02</td>
<td>0.2747E 00</td>
<td>13</td>
<td>0.1854E 02</td>
<td>0.2853E 00</td>
</tr>
<tr>
<td>15</td>
<td>0.1681E 02</td>
<td>0.3214E 00</td>
<td>15</td>
<td>0.1852E 02</td>
<td>0.2986E 00</td>
<td>15</td>
<td>0.1880E 02</td>
<td>0.3142E 00</td>
</tr>
<tr>
<td>17</td>
<td>0.1705E 02</td>
<td>0.3391E 00</td>
<td>17</td>
<td>0.1877E 02</td>
<td>0.3112E 00</td>
<td>17</td>
<td>0.1905E 02</td>
<td>0.3141E 00</td>
</tr>
<tr>
<td>19</td>
<td>0.1728E 02</td>
<td>0.3508E 00</td>
<td>19</td>
<td>0.1900E 02</td>
<td>0.3194E 00</td>
<td>19</td>
<td>0.1928E 02</td>
<td>0.3312E 00</td>
</tr>
<tr>
<td>21</td>
<td>0.1750E 02</td>
<td>0.3644E 00</td>
<td>21</td>
<td>0.1923E 02</td>
<td>0.3310E 00</td>
<td>21</td>
<td>0.1951E 02</td>
<td>0.3517E 00</td>
</tr>
<tr>
<td>23</td>
<td>0.1772E 02</td>
<td>0.3698E 00</td>
<td>23</td>
<td>0.1946E 02</td>
<td>0.3393E 00</td>
<td>23</td>
<td>0.1974E 02</td>
<td>0.3563E 00</td>
</tr>
<tr>
<td>25</td>
<td>0.1754E 02</td>
<td>0.3694E 00</td>
<td>25</td>
<td>0.1969E 02</td>
<td>0.3373E 00</td>
<td>25</td>
<td>0.1957E 02</td>
<td>0.3545E 00</td>
</tr>
<tr>
<td>27</td>
<td>0.1816E 02</td>
<td>0.3682E 00</td>
<td>27</td>
<td>0.1952E 02</td>
<td>0.3363E 00</td>
<td>27</td>
<td>0.2021E 02</td>
<td>0.3495E 00</td>
</tr>
<tr>
<td>29</td>
<td>0.1839E 02</td>
<td>0.3685E 00</td>
<td>29</td>
<td>0.2016E 02</td>
<td>0.3327E 00</td>
<td>29</td>
<td>0.2044E 02</td>
<td>0.3440E 00</td>
</tr>
<tr>
<td>31</td>
<td>0.1862E 02</td>
<td>0.3530E 00</td>
<td>31</td>
<td>0.2040E 02</td>
<td>0.3298E 00</td>
<td>31</td>
<td>0.2069E 02</td>
<td>0.3454E 00</td>
</tr>
<tr>
<td>33</td>
<td>0.1886E 02</td>
<td>0.3456E 00</td>
<td>33</td>
<td>0.2065E 02</td>
<td>0.3195E 00</td>
<td>33</td>
<td>0.2094E 02</td>
<td>0.3441E 00</td>
</tr>
<tr>
<td>35</td>
<td>0.1911E 02</td>
<td>0.3467E 00</td>
<td>35</td>
<td>0.2091E 02</td>
<td>0.3087E 00</td>
<td>35</td>
<td>0.2120E 02</td>
<td>0.3329E 00</td>
</tr>
<tr>
<td>37</td>
<td>0.1937E 02</td>
<td>0.3069E 00</td>
<td>37</td>
<td>0.2119E 02</td>
<td>0.3026E 00</td>
<td>37</td>
<td>0.2148E 02</td>
<td>0.3135E 00</td>
</tr>
<tr>
<td>39</td>
<td>0.1965E 02</td>
<td>0.2895E 00</td>
<td>39</td>
<td>0.2148E 02</td>
<td>0.2772E 00</td>
<td>39</td>
<td>0.2177E 02</td>
<td>0.2785E 00</td>
</tr>
<tr>
<td>41</td>
<td>0.1995E 02</td>
<td>0.2658E 00</td>
<td>41</td>
<td>0.2179E 02</td>
<td>0.2554E 00</td>
<td>41</td>
<td>0.2208E 02</td>
<td>0.2540E 00</td>
</tr>
<tr>
<td>43</td>
<td>0.2029E 02</td>
<td>0.2360E 00</td>
<td>43</td>
<td>0.2214E 02</td>
<td>0.2264E 00</td>
<td>43</td>
<td>0.2243E 02</td>
<td>0.2250E 00</td>
</tr>
<tr>
<td>45</td>
<td>0.2068E 02</td>
<td>0.1999E 00</td>
<td>45</td>
<td>0.2255E 02</td>
<td>0.1914E 00</td>
<td>45</td>
<td>0.2284E 02</td>
<td>0.1903E 00</td>
</tr>
<tr>
<td>47</td>
<td>0.2117E 02</td>
<td>0.1551E 00</td>
<td>47</td>
<td>0.2366E 02</td>
<td>0.1489E 00</td>
<td>47</td>
<td>0.2336E 02</td>
<td>0.1479E 00</td>
</tr>
<tr>
<td>49</td>
<td>0.2151E 02</td>
<td>0.9731E-01</td>
<td>49</td>
<td>0.2382E 02</td>
<td>0.9379E-01</td>
<td>49</td>
<td>0.2413E 02</td>
<td>0.9318E-01</td>
</tr>
<tr>
<td>50</td>
<td>0.2259E 02</td>
<td>0.5940E-01</td>
<td>50</td>
<td>0.2454E 02</td>
<td>0.5730E-01</td>
<td>50</td>
<td>0.2484E 02</td>
<td>0.5694E-01</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
<td>I</td>
<td>T</td>
<td>W</td>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>-----</td>
<td>-----------------</td>
<td>-----------</td>
<td>-----</td>
<td>-----------------</td>
<td>-----------</td>
<td>-----</td>
<td>-----------------</td>
<td>-----------</td>
</tr>
<tr>
<td>1</td>
<td>0.1703E C2</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.1872E C2</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.2046E C2</td>
<td>0.0000E 00</td>
</tr>
<tr>
<td>2</td>
<td>0.1820E 02</td>
<td>0.9223E-01</td>
<td>2</td>
<td>0.1952E C2</td>
<td>0.8888E-01</td>
<td>2</td>
<td>0.2172E 02</td>
<td>0.8546E-01</td>
</tr>
<tr>
<td>3</td>
<td>0.1884E C2</td>
<td>0.1464E CC</td>
<td>3</td>
<td>0.2055E C2</td>
<td>0.1407E CC</td>
<td>3</td>
<td>0.2237E 02</td>
<td>0.1357E CC</td>
</tr>
<tr>
<td>4</td>
<td>0.1924E 02</td>
<td>0.1872E 00</td>
<td>4</td>
<td>0.2101E 02</td>
<td>0.1864E 00</td>
<td>4</td>
<td>0.2285E 02</td>
<td>0.1741E CC</td>
</tr>
<tr>
<td>5</td>
<td>0.1961E 02</td>
<td>0.2201E 00</td>
<td>5</td>
<td>0.2146E 02</td>
<td>0.2125E 00</td>
<td>5</td>
<td>0.2325E 02</td>
<td>0.2049E CC</td>
</tr>
<tr>
<td>6</td>
<td>0.1954E 02</td>
<td>0.2465E 00</td>
<td>6</td>
<td>0.2174E 02</td>
<td>0.2387E 00</td>
<td>6</td>
<td>0.2360E 02</td>
<td>0.2299E CC</td>
</tr>
<tr>
<td>7</td>
<td>0.2023E 02</td>
<td>0.2691E 00</td>
<td>7</td>
<td>0.2233E 02</td>
<td>0.2942E 00</td>
<td>7</td>
<td>0.2468E 02</td>
<td>0.2471E CC</td>
</tr>
<tr>
<td>8</td>
<td>0.2050E 02</td>
<td>0.2779E CC</td>
<td>8</td>
<td>0.2259E 02</td>
<td>0.2922E CC</td>
<td>8</td>
<td>0.2448E 02</td>
<td>0.2552E CC</td>
</tr>
<tr>
<td>9</td>
<td>0.2076E 02</td>
<td>0.2796E 00</td>
<td>9</td>
<td>0.2265E 02</td>
<td>0.3071E 00</td>
<td>9</td>
<td>0.2475E 02</td>
<td>0.2899E CC</td>
</tr>
<tr>
<td>10</td>
<td>0.2149E 02</td>
<td>0.3071E 00</td>
<td>10</td>
<td>0.2311E 02</td>
<td>0.3164E 00</td>
<td>10</td>
<td>0.251C 02</td>
<td>0.2971E CC</td>
</tr>
<tr>
<td>11</td>
<td>0.2125E 02</td>
<td>0.3167E CC</td>
<td>11</td>
<td>0.2335E 02</td>
<td>0.3147E CC</td>
<td>11</td>
<td>0.2552E 02</td>
<td>0.3035E CC</td>
</tr>
<tr>
<td>12</td>
<td>0.2145E 02</td>
<td>0.3227E CC</td>
<td>12</td>
<td>0.2360E 02</td>
<td>0.3034E CC</td>
<td>12</td>
<td>0.2578E 02</td>
<td>0.3061E CC</td>
</tr>
<tr>
<td>13</td>
<td>0.2172E 02</td>
<td>0.3256E CC</td>
<td>13</td>
<td>0.2384E 02</td>
<td>0.3174E CC</td>
<td>13</td>
<td>0.2648E 02</td>
<td>0.3036E CC</td>
</tr>
<tr>
<td>14</td>
<td>0.2156E 02</td>
<td>0.3490E CC</td>
<td>14</td>
<td>0.2410E 02</td>
<td>0.3212E CC</td>
<td>14</td>
<td>0.2672E 02</td>
<td>0.2974E CC</td>
</tr>
<tr>
<td>15</td>
<td>0.2212E 02</td>
<td>0.3273E CC</td>
<td>15</td>
<td>0.2436E 02</td>
<td>0.3110E CC</td>
<td>15</td>
<td>0.2659E 02</td>
<td>0.2899E CC</td>
</tr>
<tr>
<td>16</td>
<td>0.2247E 02</td>
<td>0.3223E CC</td>
<td>16</td>
<td>0.2463E 02</td>
<td>0.3111E CC</td>
<td>16</td>
<td>0.2665E 02</td>
<td>0.2764E CC</td>
</tr>
<tr>
<td>17</td>
<td>0.2272E 02</td>
<td>0.3119E CC</td>
<td>17</td>
<td>0.2492E 02</td>
<td>0.2978E CC</td>
<td>17</td>
<td>0.2710E 02</td>
<td>0.2614E CC</td>
</tr>
<tr>
<td>18</td>
<td>0.2299E 02</td>
<td>0.2914E CC</td>
<td>18</td>
<td>0.2521E 02</td>
<td>0.2843E CC</td>
<td>18</td>
<td>0.2752E 02</td>
<td>0.2382E CC</td>
</tr>
<tr>
<td>19</td>
<td>0.2328E 02</td>
<td>0.2832E CC</td>
<td>19</td>
<td>0.2553E 02</td>
<td>0.2659E CC</td>
<td>19</td>
<td>0.2888E 02</td>
<td>0.2382E CC</td>
</tr>
<tr>
<td>20</td>
<td>0.2359E 02</td>
<td>0.2707E CC</td>
<td>20</td>
<td>0.2586E 02</td>
<td>0.2358E CC</td>
<td>20</td>
<td>0.2787E 02</td>
<td>0.2260E CC</td>
</tr>
<tr>
<td>21</td>
<td>0.2392E 02</td>
<td>0.2439E CC</td>
<td>21</td>
<td>0.2624E 02</td>
<td>0.2091E CC</td>
<td>21</td>
<td>0.2827E 02</td>
<td>0.2240E CC</td>
</tr>
<tr>
<td>22</td>
<td>0.2428E 02</td>
<td>0.2173E CC</td>
<td>22</td>
<td>0.2668E 02</td>
<td>0.1773E CC</td>
<td>22</td>
<td>0.2872E 02</td>
<td>0.1716E CC</td>
</tr>
<tr>
<td>23</td>
<td>0.2471E 02</td>
<td>0.1838E CC</td>
<td>23</td>
<td>0.2723E 02</td>
<td>0.1379E CC</td>
<td>23</td>
<td>0.2928E 02</td>
<td>0.1332E CC</td>
</tr>
<tr>
<td>24</td>
<td>0.2524E 02</td>
<td>0.1426E CC</td>
<td>24</td>
<td>0.2806E 02</td>
<td>0.0867E-01</td>
<td>24</td>
<td>0.3014E 02</td>
<td>0.0375E-01</td>
</tr>
<tr>
<td>25</td>
<td>0.2604E 02</td>
<td>0.8964E-01</td>
<td>25</td>
<td>0.2832E 02</td>
<td>0.5297E-01</td>
<td>25</td>
<td>0.3053E 02</td>
<td>0.5112E-01</td>
</tr>
<tr>
<td>I</td>
<td>T</td>
<td>W</td>
<td>I</td>
<td>T</td>
<td>W</td>
<td>I</td>
<td>T</td>
<td>W</td>
</tr>
<tr>
<td>----</td>
<td>---------</td>
<td>----------</td>
<td>----</td>
<td>---------</td>
<td>----------</td>
<td>----</td>
<td>---------</td>
<td>----------</td>
</tr>
<tr>
<td>1</td>
<td>0.2222E 02</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.2422E 02</td>
<td>0.0000E 00</td>
<td>1</td>
<td>0.2630E 02</td>
<td>0.0000E 00</td>
</tr>
<tr>
<td>3</td>
<td>0.2351E 02</td>
<td>0.8288E-01</td>
<td>3</td>
<td>0.2556E 02</td>
<td>0.7995E-01</td>
<td>3</td>
<td>0.2768E 02</td>
<td>0.7776E-01</td>
</tr>
<tr>
<td>5</td>
<td>0.2419E 02</td>
<td>0.1318E 00</td>
<td>5</td>
<td>0.2627E 02</td>
<td>0.1270E 00</td>
<td>5</td>
<td>0.2840E 02</td>
<td>0.1246E 00</td>
</tr>
<tr>
<td>7</td>
<td>0.2469E 02</td>
<td>0.1689E 00</td>
<td>7</td>
<td>0.2678E 02</td>
<td>0.1631E 00</td>
<td>7</td>
<td>0.2894E 02</td>
<td>0.1611E 00</td>
</tr>
<tr>
<td>9</td>
<td>0.2510E 02</td>
<td>0.1988E 00</td>
<td>9</td>
<td>0.2721E 02</td>
<td>0.1920E 00</td>
<td>9</td>
<td>0.2938E 02</td>
<td>0.1907E 00</td>
</tr>
<tr>
<td>11</td>
<td>0.2546E 02</td>
<td>0.2237E 00</td>
<td>11</td>
<td>0.2758E 02</td>
<td>0.2148E 00</td>
<td>11</td>
<td>0.2977E 02</td>
<td>0.2155E 00</td>
</tr>
<tr>
<td>13</td>
<td>0.2579E 02</td>
<td>0.2449E 00</td>
<td>13</td>
<td>0.2762E 02</td>
<td>0.2280E 00</td>
<td>13</td>
<td>0.3013E 02</td>
<td>0.2343E 00</td>
</tr>
<tr>
<td>15</td>
<td>0.2610E 02</td>
<td>0.2672E 00</td>
<td>15</td>
<td>0.2823E 02</td>
<td>0.2431E 00</td>
<td>15</td>
<td>0.3046E 02</td>
<td>0.2546E 00</td>
</tr>
<tr>
<td>17</td>
<td>0.2638E 02</td>
<td>0.2799E 00</td>
<td>17</td>
<td>0.2853E 02</td>
<td>0.2509E 00</td>
<td>17</td>
<td>0.3077E 02</td>
<td>0.2651E 00</td>
</tr>
<tr>
<td>19</td>
<td>0.2666E 02</td>
<td>0.2889E 00</td>
<td>19</td>
<td>0.2881E 02</td>
<td>0.2554E 00</td>
<td>19</td>
<td>0.3106E 02</td>
<td>0.2782E 00</td>
</tr>
<tr>
<td>21</td>
<td>0.2653E 02</td>
<td>0.2958E 00</td>
<td>21</td>
<td>0.29C9E 02</td>
<td>0.2654E 00</td>
<td>21</td>
<td>0.3135E 02</td>
<td>0.2821E 00</td>
</tr>
<tr>
<td>23</td>
<td>0.2719E 02</td>
<td>0.3001E 00</td>
<td>23</td>
<td>0.2937E 02</td>
<td>0.2700E 00</td>
<td>23</td>
<td>0.3163E 02</td>
<td>0.2859E 00</td>
</tr>
<tr>
<td>25</td>
<td>0.2746E 02</td>
<td>0.3012E 00</td>
<td>25</td>
<td>0.2964E 02</td>
<td>0.2729E 00</td>
<td>25</td>
<td>0.3152E 02</td>
<td>0.2869E 00</td>
</tr>
<tr>
<td>27</td>
<td>0.2772E 02</td>
<td>0.2995E 00</td>
<td>27</td>
<td>0.2982E 02</td>
<td>0.2815E 00</td>
<td>27</td>
<td>0.322E 02</td>
<td>0.2884E 00</td>
</tr>
<tr>
<td>29</td>
<td>0.2759E 02</td>
<td>0.2997E 00</td>
<td>29</td>
<td>0.302E 02</td>
<td>0.2774E 00</td>
<td>29</td>
<td>0.3249E 02</td>
<td>0.2804E 00</td>
</tr>
<tr>
<td>31</td>
<td>0.2828E 02</td>
<td>0.3018E 00</td>
<td>31</td>
<td>0.3049E 02</td>
<td>0.2790E 00</td>
<td>31</td>
<td>0.3279E 02</td>
<td>0.2798E 00</td>
</tr>
<tr>
<td>33</td>
<td>0.2856E 02</td>
<td>0.2973E 00</td>
<td>33</td>
<td>0.3079E 02</td>
<td>0.2669E 00</td>
<td>33</td>
<td>0.3310E 02</td>
<td>0.2718E 00</td>
</tr>
<tr>
<td>35</td>
<td>0.2887E 02</td>
<td>0.2915E 00</td>
<td>35</td>
<td>0.3110E 02</td>
<td>0.2551E 00</td>
<td>35</td>
<td>0.3342E 02</td>
<td>0.2593E 00</td>
</tr>
<tr>
<td>37</td>
<td>0.2918E 02</td>
<td>0.2721E 00</td>
<td>37</td>
<td>0.3143E 02</td>
<td>0.2350E 00</td>
<td>37</td>
<td>0.3375E 02</td>
<td>0.2469E 00</td>
</tr>
<tr>
<td>39</td>
<td>0.2952E 02</td>
<td>0.2473E 00</td>
<td>39</td>
<td>0.3178E 02</td>
<td>0.2219E 00</td>
<td>39</td>
<td>0.3411E 02</td>
<td>0.2315E 00</td>
</tr>
<tr>
<td>41</td>
<td>0.2988E 02</td>
<td>0.2184E 00</td>
<td>41</td>
<td>0.3216E 02</td>
<td>0.2095E 00</td>
<td>41</td>
<td>0.3450E 02</td>
<td>0.2095E 00</td>
</tr>
<tr>
<td>43</td>
<td>0.3028E 02</td>
<td>0.1957E 00</td>
<td>43</td>
<td>0.3257E 02</td>
<td>0.1892E 00</td>
<td>43</td>
<td>0.3493E 02</td>
<td>0.1846E 00</td>
</tr>
<tr>
<td>45</td>
<td>0.3075E 02</td>
<td>0.1661E 00</td>
<td>45</td>
<td>0.3306E 02</td>
<td>0.1607E 00</td>
<td>45</td>
<td>0.3543E 02</td>
<td>0.1556E 00</td>
</tr>
<tr>
<td>47</td>
<td>0.3134E 02</td>
<td>0.1291E 00</td>
<td>47</td>
<td>0.3366E 02</td>
<td>0.1247E 00</td>
<td>47</td>
<td>0.3666E 02</td>
<td>0.1210E 00</td>
</tr>
<tr>
<td>49</td>
<td>0.3222E 02</td>
<td>0.9114E-01</td>
<td>49</td>
<td>0.3457E 02</td>
<td>0.7842E-01</td>
<td>49</td>
<td>0.3700E 02</td>
<td>0.7604E-01</td>
</tr>
<tr>
<td>50</td>
<td>0.3304E 02</td>
<td>0.4956E-01</td>
<td>50</td>
<td>0.3542E 02</td>
<td>0.4789E-01</td>
<td>50</td>
<td>0.3787E 02</td>
<td>0.4644E-01</td>
</tr>
<tr>
<td>I</td>
<td>T (°C)</td>
<td>W</td>
<td>I</td>
<td>T (°C)</td>
<td>W</td>
<td>I</td>
<td>T (°C)</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2839E-02</td>
<td>0.0000E+00</td>
<td>1</td>
<td>0.3061E-02</td>
<td>0.0000E+00</td>
<td>1</td>
<td>0.3285E-02</td>
<td>0.0000E+00</td>
</tr>
<tr>
<td>3</td>
<td>0.2918E-02</td>
<td>0.7551E-01</td>
<td>3</td>
<td>0.3207E-02</td>
<td>0.7358E-01</td>
<td>3</td>
<td>0.3435E-02</td>
<td>0.7134E-01</td>
</tr>
<tr>
<td>5</td>
<td>0.3055E-02</td>
<td>0.1207E+00</td>
<td>5</td>
<td>0.3284E-02</td>
<td>0.1174E+00</td>
<td>5</td>
<td>0.3514E-02</td>
<td>0.1137E+00</td>
</tr>
<tr>
<td>7</td>
<td>0.3110E-02</td>
<td>0.1557E+00</td>
<td>7</td>
<td>0.3341E-02</td>
<td>0.1513E+00</td>
<td>7</td>
<td>0.3573E-02</td>
<td>0.1464E+00</td>
</tr>
<tr>
<td>9</td>
<td>0.3156E-02</td>
<td>0.1840E+00</td>
<td>9</td>
<td>0.3389E-02</td>
<td>0.1786E+00</td>
<td>9</td>
<td>0.3622E-02</td>
<td>0.1727E+00</td>
</tr>
<tr>
<td>11</td>
<td>0.3137E-02</td>
<td>0.2069E+00</td>
<td>11</td>
<td>0.3431E-02</td>
<td>0.2012E+00</td>
<td>11</td>
<td>0.3605E-02</td>
<td>0.1943E+00</td>
</tr>
<tr>
<td>13</td>
<td>0.3233E-02</td>
<td>0.2253E+00</td>
<td>13</td>
<td>0.3469E-02</td>
<td>0.2209E+00</td>
<td>13</td>
<td>0.3706E-02</td>
<td>0.2115E+00</td>
</tr>
<tr>
<td>15</td>
<td>0.3268E-02</td>
<td>0.2394E+00</td>
<td>15</td>
<td>0.3504E-02</td>
<td>0.2378E+00</td>
<td>15</td>
<td>0.3714E-02</td>
<td>0.2256E+00</td>
</tr>
<tr>
<td>17</td>
<td>0.3300E-02</td>
<td>0.2530E+00</td>
<td>17</td>
<td>0.3537E-02</td>
<td>0.2514E+00</td>
<td>17</td>
<td>0.3775E-02</td>
<td>0.2348E+00</td>
</tr>
<tr>
<td>19</td>
<td>0.3330E-02</td>
<td>0.2654E+00</td>
<td>19</td>
<td>0.3569E-02</td>
<td>0.2628E+00</td>
<td>19</td>
<td>0.3808E-02</td>
<td>0.2483E+00</td>
</tr>
<tr>
<td>21</td>
<td>0.3359E-02</td>
<td>0.2754E+00</td>
<td>21</td>
<td>0.3592E-02</td>
<td>0.2693E+00</td>
<td>21</td>
<td>0.3839E-02</td>
<td>0.2458E+00</td>
</tr>
<tr>
<td>23</td>
<td>0.3389E-02</td>
<td>0.2674E+00</td>
<td>23</td>
<td>0.3629E-02</td>
<td>0.2746E+00</td>
<td>23</td>
<td>0.3870E-02</td>
<td>0.2449E+00</td>
</tr>
<tr>
<td>25</td>
<td>0.3418E-02</td>
<td>0.2708E+00</td>
<td>25</td>
<td>0.3659E-02</td>
<td>0.2765E+00</td>
<td>25</td>
<td>0.3913E-02</td>
<td>0.2479E+00</td>
</tr>
<tr>
<td>27</td>
<td>0.3447E-02</td>
<td>0.2679E+00</td>
<td>27</td>
<td>0.3650E-02</td>
<td>0.2736E+00</td>
<td>27</td>
<td>0.3922E-02</td>
<td>0.2488E+00</td>
</tr>
<tr>
<td>29</td>
<td>0.3477E-02</td>
<td>0.2664E+00</td>
<td>29</td>
<td>0.3721E-02</td>
<td>0.2710E+00</td>
<td>29</td>
<td>0.3984E-02</td>
<td>0.2453E+00</td>
</tr>
<tr>
<td>31</td>
<td>0.3508E-02</td>
<td>0.2586E+00</td>
<td>31</td>
<td>0.3752E-02</td>
<td>0.2633E+00</td>
<td>31</td>
<td>0.3956E-02</td>
<td>0.2445E+00</td>
</tr>
<tr>
<td>33</td>
<td>0.3539E-02</td>
<td>0.2517E+00</td>
<td>33</td>
<td>0.3785E-02</td>
<td>0.2554E+00</td>
<td>33</td>
<td>0.4030E-02</td>
<td>0.2372E+00</td>
</tr>
<tr>
<td>35</td>
<td>0.3572E-02</td>
<td>0.2414E+00</td>
<td>35</td>
<td>0.3818E-02</td>
<td>0.2477E+00</td>
<td>35</td>
<td>0.4065E-02</td>
<td>0.2280E+00</td>
</tr>
<tr>
<td>37</td>
<td>0.3607E-02</td>
<td>0.2266E+00</td>
<td>37</td>
<td>0.3854E-02</td>
<td>0.2343E+00</td>
<td>37</td>
<td>0.4101E-02</td>
<td>0.2137E+00</td>
</tr>
<tr>
<td>39</td>
<td>0.3644E-02</td>
<td>0.2144E+00</td>
<td>39</td>
<td>0.3892E-02</td>
<td>0.2175E+00</td>
<td>39</td>
<td>0.4145E-02</td>
<td>0.2028E+00</td>
</tr>
<tr>
<td>41</td>
<td>0.3684E-02</td>
<td>0.1981E+00</td>
<td>41</td>
<td>0.3933E-02</td>
<td>0.1967E+00</td>
<td>41</td>
<td>0.4183E-02</td>
<td>0.1876E+00</td>
</tr>
<tr>
<td>43</td>
<td>0.3729E-02</td>
<td>0.1773E+00</td>
<td>43</td>
<td>0.3979E-02</td>
<td>0.1729E+00</td>
<td>43</td>
<td>0.4230E-02</td>
<td>0.1674E+00</td>
</tr>
<tr>
<td>45</td>
<td>0.3780E-02</td>
<td>0.1507E+00</td>
<td>45</td>
<td>0.4032E-02</td>
<td>0.1465E+00</td>
<td>45</td>
<td>0.4284E-02</td>
<td>0.1420E+00</td>
</tr>
<tr>
<td>47</td>
<td>0.3845E-02</td>
<td>0.1173E+00</td>
<td>47</td>
<td>0.4059E-02</td>
<td>0.1139E+00</td>
<td>47</td>
<td>0.4353E-02</td>
<td>0.1106E+00</td>
</tr>
<tr>
<td>49</td>
<td>0.3941E-02</td>
<td>0.7377E-01</td>
<td>49</td>
<td>0.4158E-02</td>
<td>0.7171E-01</td>
<td>49</td>
<td>0.4455E-02</td>
<td>0.6960E-01</td>
</tr>
<tr>
<td>50</td>
<td>0.4031E-02</td>
<td>0.4505E-01</td>
<td>50</td>
<td>0.4251E-02</td>
<td>0.4381E-01</td>
<td>50</td>
<td>0.4550E-02</td>
<td>0.4252E-01</td>
</tr>
</tbody>
</table>
The time required for total stratification is

$$\tau = 0.626$$

For times greater than $\tau=0.63$,

$$\frac{dT}{d\tau} = 2 \tau^{4/9}$$

for $0.75 \leq Z \leq 0.20$ and where the symbols are defined in Chapter 6.
In this Appendix, the model which was used by Drake (16) to predict the core thermal gradient in a uniformly heated vertical cylinder is generalized to account for all vertical enclosures of arbitrary cross-sections.

Consider a vertical container of arbitrary geometry given in Figure L.1. Let \( a \) be the cross-sectional area, \( p \) be the perimeter of the segment where boundary layer flow occurs, and \( L \) be the height of the container. Assume that the "mixing region" is of depth \( x \) at some uniform temperature, \( T_L \). The rate of change of \( T_L \) with time can be calculated from an energy balance if the boundary flow rate and temperature are known and if the inflow from the boundary layer equals the outflow to the main core region, i.e., the depth of the "mixing" region does not change with time. Then, if an energy reference level of \( T_L \) is chosen, an analysis of the "mixing region" model shows:

\[
\begin{align*}
\text{Energy in} & = \rho c_p p \int w(T - T_L) \, dy \, dt \quad \text{L.1} \\
\text{Energy out} & = 0 \quad \text{(Fluid at } T_L) \quad \text{L.2} \\
\text{Energy change} & = \rho c_p a_1 \Delta x \, dT_L \quad \text{L.3}
\end{align*}
\]

so that an energy balance yields:

\[
\frac{dT_L}{dt} = \frac{p}{a_1 \Delta x} \int w(T - T_L) \, dy 
\quad \text{L.4}
\]

Now, assume that the rate of change of the linear core temperature gradient is due to the variation in \( T_L \) in the "mixing region". That is,

\[
\frac{d}{dt} \left( \frac{dT_L}{dz} \right) = \frac{1}{(L - \Delta x)} \frac{dT_L}{dt} \quad \text{L.5}
\]
Figure L.1
Generalized Cylinder
Then,
\[
\frac{d}{dt} \left( \frac{d}{dz} \right) = \frac{1}{(L - \Delta x)} \frac{P}{\Delta x} \int w(T - T_L) dy \]  
L.6

The enthalpy flow integral, \( \int w(T - T_L) dy \), can be calculated by the methods described in Chapter 5 and do not depend on the container cross-section.

Rewriting Equation L.6 as
\[
\frac{d}{dt} \left( \frac{d}{dz} \frac{T_0 - T_L}{(qL/k)} \right) = \frac{L}{(L - \Delta x)} \frac{D_{e}P}{a_1} \frac{L}{\Delta x} 
\]

or
\[
\int \left( \frac{w_{D}D_{e}}{a} \right) \frac{R_a^{1/5}}{(qL/k)} \frac{T - T_L}{R_a^{1/5}} \frac{L}{D_{e}} d(y/D_{e}) \]  
L.7

or
\[
\frac{d}{d\tau_e} \left( \frac{d}{dz} \right) = \frac{L}{(L - \Delta x)} \frac{pD_{e}}{a_1} \int v_T dY_e \]  
L.8

where
\[
\tau_e = \frac{at}{L \Delta x} R_a^{1/5} 
\]

\[
W_e = \frac{w_{D}D_{e}}{a} \frac{1}{R_a^{1/5}} \]  
L.9

\[
T = \frac{T - T_0}{(qL/k)} R_a^{1/5} \]  
L.10

\[
Y_e = \frac{y}{D_{e}} \]

and \( D_{e} \) is an equivalent diameter which is equal to
\[
D_{e} = \frac{4a_1}{p} \]  
L.11

Therefore, for a cylinder
\[
D_{e} = 4 \frac{(\pi/4)}{\pi D_0} = D \]  
L.11

Letting
\[
C_s = \frac{L}{(L - \Delta x)} \frac{L}{\Delta x} \]  
L.11
and using the definitions of $\tau$, $W$, $T$ given by Equation 5.3.35 and defining $Y = y/D$, Equation L.8 becomes

$$\frac{d}{d\tau} \left( \frac{dT}{dZ} \right) = 4C_s \int WdT dY \quad \text{L.12}$$

And from experimental data, Drake (16) found that

$$\frac{dT}{dZ} = 4\tau^{4/9} \quad \text{5.5.1}$$

Now, for a rectangular container such as that used in this study,

$$D_e = 4 \frac{HD}{2H} = 2D \quad \text{L.13}$$

where $H$ is the width of the heating panel. Then, using the definitions of $\tau$, $W$, $T$ given by Equation 5.3.35, it is easily seen that

$$\frac{d}{d\tau} \left( \frac{dT}{dZ} \right) = 2C_s \int WdT dY \quad \text{L.14}$$

That is, the rate of change of the core temperature gradient for a rectangular cavity is half that for a uniformly heated vertical cylinder. Therefore, since $WdT dY$ is independent of the geometrical shape (true, there is no interaction between boundary layers), for a rectangular container with uniformly heated vertical sidewalls

$$\frac{dT}{dZ} = 2\tau^{4/9}. \quad \text{L.15}$$
APPENDIX M

FINITE DIFFERENCE EQUATIONS FOR THE UNIFORMLY
HEATED VERTICAL PLATE BOUNDARY LAYER EQUATIONS

In this Appendix, the finite difference equations for
the case of a uniformly heated vertical plate in isothermal
and non-isothermal surroundings are developed. Then the
properties of these equations are examined. Finally, a
stability analysis of the finite difference equations is
presented.

M.1 Development of the Finite Difference Equations

M.1.1 The Grid System

Solution of partial differential equations using
finite difference equations involves the solution of a set
of finite difference analogs at given spatial points for a
given time. The spatial points, or grid points, are formed
by zoning the Y-Z space into IMAX by JMAX finite areas
each of height \( Z = Z_{\text{MAX}}/\text{IMAX} \) and width \( Y = Y_{\text{MAX}}/\text{JMAX} \) as
shown in Figure M.1.

In the differential problem, the plate is of infinite
height and the distance from the plate extends to infinity.
However, since a finite number of grid points must be used
in the difference analog, the height of the plate, \( Z_{\text{MAX}} \),
and the maximum distance from the plate, \( Y_{\text{MAX}} \), must be
limited. Selection of these distances are discussed
further in a later section.

In the following discussion, the functional dependence
of the variables with respect to time and position on the
grid is denoted in finite-difference notation as:

\[
T(Z,Y,\tau) = T(\text{i}, Z, \text{j}, Y, \sum_{\text{i}=1}^{\text{n}} \Delta \text{i}) = T_{\text{i, j}}^{n}
\]  

M.1.1
LEGEND
● - interior grid points
● - boundary points
● - temperature, $T_{i,j}$
● - velocity, $W_{i,j}$
● - velocity, $V_{i,j}$

FIGURE M.1 GRID SYSTEM FOR THE FINITE DIFFERENCE SCHEME
where \( i, j \) are the space indices, \( n \) is the time index, and \( \Delta t_i \) is the time increment of the \( i \)th iteration of the finite difference equations.

### M.1.2 The Finite Difference Equations

The finite difference scheme used in this work involves an implicit formulation of the momentum and energy equations in the \( Y \)-direction combined with an explicit formulation in the \( Z \)-direction. The momentum and energy equations are written in finite difference form as follows:

\[
\begin{align*}
\frac{w_{i+1,j}^{n+1} - w_{i,j}^n}{\Delta t} + \frac{w_{i,j}^n}{\Delta z} &+ \frac{v_{i,j}^n}{\Delta y} = \frac{w_{i+1,j+1}^{n+1} - w_{i,j}^n}{\Delta t} \\
\frac{T_{i+1,j}^{n+1} - T_{i,j}^n}{\Delta t} + \frac{T_{i,j}^n}{\Delta z} &+ \frac{v_{i,j}^n}{\Delta y} = \frac{T_{i+1,j+1}^{n+1} - T_{i,j}^n}{\Delta t} \\
\end{align*}
\]

Then, the finite difference equations are swept in the \( Z \)-direction using the following equations

\[
\begin{align*}
\frac{w_{i+1,j}^{n+1} - w_{i,j}^n}{\Delta t} &+ \frac{w_{i,j}^n}{\Delta z} &+ \frac{v_{i,j}^n}{\Delta y} \neq \frac{w_{i+1,j+1}^{n+1} - w_{i,j}^n}{\Delta t} \\
\frac{T_{i+1,j}^{n+1} - T_{i,j}^n}{\Delta t} &+ \frac{T_{i,j}^n}{\Delta z} &+ \frac{v_{i,j}^n}{\Delta y} \neq \frac{T_{i+1,j+1}^{n+1} - T_{i,j}^n}{\Delta t} \\
\end{align*}
\]
\[
\frac{T^{n+1}_{i,j} - T^n_{i,j}}{\Delta t} + \frac{W^n_{i,j} - T^{n+1}_{i-1,j}}{\Delta Z} + \frac{V^n_{i,j} - T^{n+1}_{i,j+1}}{\Delta Y} = T^{n+1}_{i,j} - T^n_{i,j} \\
= \frac{T^{n+1}_{i,j-1} - 2T^n_{i,j} + T^{n+1}_{i,j+1}}{\Delta Y^2} - \frac{W^n_{i,j} \partial T}{\partial Z}
\]
M.1.2d

where the prime denotes the function calculated using Equations M.1.2a and M.1.2b. Note that the first derivative is forward differenced in the Y-direction and backward differenced in the Z-direction. The continuity equation is written in a backward difference form as follows:

\[
\frac{W^{n+1}_{i,j} - W^n_{i,j}}{\Delta Z} + \frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta Y} = 0
\]
M.1.2e

Finally, the boundary conditions are imposed on the exterior grid points shown in Figure M.1 are:

\[
W^n_{i,0} = V^n_{i,0} = 0 \quad 0 \leq i \leq I, \text{ all } n
\]

\[
W^n_{0,j} = V^n_{0,j} = T^n_{0,j} = 0 \quad 0 \leq j \leq J, \text{ all } n
\]

M.1.2f

\[
W^n_{i,j} = V^n_{i,j} = T^n_{i,j} = 0 \quad 0 \leq i \leq I, \text{ all } n
\]

and

\[
W^0_{i,j} = V^0_{i,j} = T^0_{i,j} = 0 \quad 0 \leq i \leq I, 0 \leq j \leq J \text{ initially}
\]

Equations M.1.2, then, are the finite difference analogs of the partial differential equations except for the wall heat flux boundary condition

\[
\frac{\partial T}{\partial Y} = -\frac{1}{Ra^{1/4}}
\]
M.1.3

Specification of this boundary condition is deferred to Section M.1.3. Reasons for this deferment will become apparent in the following discussion.
Also, note that because of the backward differencing of the first derivative in the Z-direction, a boundary condition need not be specified at \( i = -1 \).

Equation M.1.2a can be rewritten as

\[
A_j w_{1,j-1}^{n+1} + B_j w_{1,j}^{n+1} + C_j w_{1,j+1}^{n+1} = D_j \quad M.1.4a
\]

where

\[
A_j = -\text{Pr}(\Delta t/\Delta Y^2)
\]

\[
B_j = 1.0 - v_{1,j}^n(\Delta t/\Delta Y) + 2.0 \text{Pr}(\Delta t/\Delta Y^2)
\]

\[
C_j = v_{1,j}^n(\Delta t/\Delta Y) - \text{Pr}(\Delta t/\Delta Y^2)
\]

\[
D_j = w_{1,j}^n - (w_{1,j}^n - w_{1,j-1}^n)(\Delta t/\Delta Z) + \text{Pr} T_{1,j}^n
\]

Furthermore, using Equation M.1.2f the general equation for the case when \( j = 1 \) is

\[
B_1 w_{1,1}^{n+1} + C_1 w_{1,2}^{n+1} = D_1 \quad M.1.4b
\]

Similarly, when \( j = J-1 \)

\[
A_{J-1} w_{1,J-2}^{n+1} + B_{J-1} w_{1,J-1}^{n+1} = D_{J-1} \quad M.1.4c
\]

Therefore, for the \( i \)th row, a system of equations is written as follows:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
B_1 & C_1 & & & & & w_{1,1}^{n+1} & & & & D_1 \\
A_2 & B_2 & C_2 & & & & w_{1,2}^{n+1} & & & & D_2 \\
& & A_3 & B_3 & C_3 & & w_{1,3}^{n+1} & & & & D_3 \\
& & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& & A_{J-3} & B_{J-3} & C_{J-3} & & w_{1,J-3}^{n+1} & & & & D_{J-3} \\
& & A_{J-2} & B_{J-2} & C_{J-2} & & w_{1,J-2}^{n+1} & & & & D_{J-2} \\
& & A_{J-1} & B_{J-1} & & & w_{1,J-1}^{n+1} & & & & D_{J-1}
\end{array}
\]
or, using standard matrix notation,

\[ ||A|| \cdot ||W|| = ||D|| \]  \hspace{1cm} M.1.5b

Equation M.1.5b can be readily solved using schemes for solving tridiagonal matrices such as noted by Nobel (38) and Peaceman (42). (A listing of the algorithm used in this work is given in Appendix E.)

However, the set of energy equations does not lend itself to such a solution for, in the case of \( j = 1 \),

\[ A'_0 T^{n+1}_{1,0} + B'_1 T^{n+1}_{1,1} + C'_1 T^{n+1}_{1,2} = D'_1 \]  \hspace{1cm} M.1.6

Therefore, in order to write the energy difference equations in the form of Equation M.1.5, values of \( T^{n+1}_{1,0} \) must be written using the heat flux boundary condition and temperature values at adjacent points. As will subsequently be shown in Section M.1.3, a general three point formula can be written such that

\( \left( c_0 T^{n+1}_{1,0} + c_1 T^{n+1}_{1,1} + c_2 T^{n+1}_{1,2} \right)/Y = -1/\text{Ra}^{1/4} \)  \hspace{1cm} M.1.7

Then, combining Equations M.1.6 and M.1.7, \( T^{n+1}_{1,0} \) can be eliminated. Now, for the ith row, a set of equations can be written which is similar in form to Equation M.1.5b, i.e.,

\[ ||B|| \cdot ||T|| = ||E|| \]  \hspace{1cm} M.1.8

Solution of the finite difference equations is effected by the following procedure. Equation M.1.8 is used to advance the temperature field from time level \( n \) to time level \( n+1 \) by sweeping in the Y-direction. Then, the temperature field is swept in the Z-direction using Equation M.1.2d. Then, the vertical velocity field is similarly advanced using Equations M.1.5b and M.1.2c. Finally, the horizontal velocity field is advanced to the time level \( n+1 \) using Equation M.1.2e. This iterative
technique is repeated until steadystate results are obtained.

The finite difference equations were solved using an IBM 360 computer. Calculation times were of the order of 0.2 minutes for the case of the isothermal surroundings problem. A listing of the program used is given in Appendix E.

M.1.3 Specification of the Heat Flux Boundary Condition

As seen in the previous section, values of $T_{i,0}^{n+1}$ must be approximated using the heat flux boundary condition and values of temperature at adjacent points. Several approximations can be made. A few of the more common possibilities are now considered.

1. Linear Approximation

In this case, a forward difference approximation of the first derivative is equated to the wall heat flux to yield the following two-point formula:

$$\frac{T_{i+1,0}^{n+1} - T_{i,0}^{n+1}}{\Delta Y} = -\frac{1}{Ra^{1/4}}$$  \hspace{1cm} M.1.9

11. Taylor's Series Approximation

A Taylor's series expansion for $T_{1,1}^{n+1}$ is written about the point $(1,0)$ so that

$$T_{1,1}^{n+1} = T_{1,0}^{n+1} + \Delta Y \left( \frac{\partial T_{1,0}^{n+1}}{\partial Y} \right) + \frac{\Delta Y^2}{2} \frac{\partial^2 T_{1,0}^{n+1}}{\partial Y^2}$$  \hspace{1cm} M.1.10a

Similarly, a Taylor series expansion for $T_{1,2}^{n+1}$ is written about the same point, $(1,0)$:

$$T_{1,2}^{n+1} = T_{1,0}^{n+1} + 2\Delta Y \left( \frac{\partial T_{1,0}^{n+1}}{\partial Y} \right) + \frac{4\Delta Y^2}{2} \frac{\partial^2 T_{1,0}^{n+1}}{\partial Y^2}$$  \hspace{1cm} M.1.10b
Combining Equations M.1.10a and M.1.10b and equating the first derivative to the heat flux, one obtains

\[ \frac{3}{2} T_{1,0}^{n+1} - 2 T_{1,1}^{n+1} + \frac{1}{2} T_{1,2}^{n+1} = \frac{\Delta Y}{\text{Ra}^{1/4}} \] \hspace{1cm} \text{M.1.10c}

which is the required Taylor's series three point formula.

iii. Generalized Polynomial Approximation

Assume a general three point formula of the following form:

\[ \frac{c_0 T_{1,0}^{n+1} + c_1 T_{1,1}^{n+1} + c_2 T_{1,2}^{n+1}}{\Delta Y} = - \frac{1}{\text{Ra}^{1/4}} \] \hspace{1cm} \text{M.1.11}

It is now required that Equation M.1.11 fit a 0th order, and a 1st order polynomial, i.e., the equation must fit curves of the form

\[ T_{1,j}^{n+1} = T_{1,0}^{n+1} \] \hspace{1cm} \text{M.1.12a}

and

\[ T_{1,j}^{n+1} = T_{1,0}^{n+1} + a Y \] \hspace{1cm} \text{M.1.12b}

Substituting these into Equation M.1.11 one obtains

\[ c_0 + c_1 + c_2 = 0 \] \hspace{1cm} \text{M.1.13a}

and

\[ \frac{(c_0 + c_1 + c_2) T_{1,0}^{n+1} + a Y c_1 + 2a Y c_2}{\Delta Y} = a \] \hspace{1cm} \text{M.1.13b}

Combining these two Equations, one obtains

\[ c_1 + 2c_2 = 1 \] \hspace{1cm} \text{M.1.13c}

Using Equations M.1.13a and M.1.13c, the constants \( c_0 \) and \( c_1 \) can be written in terms of \( c_2 \) so that

\[ c_0 = -1 + c_2 \]

\[ c_1 = 1 - 2c_2 \] \hspace{1cm} \text{M.1.14}
The generalized three point formula is then written as
\[ c_0 T_{1,0}^{n+1} + c_1 T_{1,1}^{n+1} + c_2 T_{1,2}^{n+1} = \frac{\Delta y}{H} \frac{1}{1/4} \]  \[ \text{M.1.11} \]

with the constants being defined by Equation M.1.14 and \( c_2 \) being an arbitrary constant. It is this constant which is used in this work.

Equation M.1.11 is equivalent to Equations M.1.9 and M.1.10c when the constant \( c_2 \) is defined as 0 and -0.5 respectively.

M.2 Properties of Finite Difference Equations

M.2.1 Uniformly Heated Vertical Plate in Isothermal Surroundings

In this section, stability and convergence of the finite difference analogs (Equations M.1.2) of the boundary layer equations (Chapter 5) will be considered. Finally, the effect of arbitrarily specifying the wall heat flux boundary condition using Equation M.1.7 will be discussed.

If one defines \( D_s \) as the exact solution of the partial differential equations, \( F_s \) as the exact solution of the finite difference equations, and \( N_s \) as the numerical solution of the finite difference equations, then, \( (D_s - F_s) \) is known as the truncation error, and \( (F_s - N_s) \) is the numerical error. To find the conditions under which \( F_s = D_s \) is a problem of convergence. To find the conditions under which the numerical error, \( (F_s - N_s) \), is small throughout the entire region of integration is a problem of stability. The problem of stability of the finite difference equations for the case of a uniformly heated vertical plate in isothermal surroundings will be considered first.
Stability of the Finite Difference Equations

A method for predicting the stability of finite difference equations which has had much favor in the literature (27, 40, 42, 45) is the so-called von Neumann stability analysis. This analysis was developed for partial differential equations with constant coefficients. However, it has achieved some usefulness in problems where variable coefficients are present, e.g., equations such as Equation M.1.2. For this case, the coefficients are assumed to be constant over a time step, $\Delta t$. Then, a Fourier expansion of a line of errors is made and the progress of the general term of the expansion is followed. Conditions under which the errors do not grow without bound are then established and these become the stability criteria of the finite difference scheme.

Let the exact solution of the difference Equation M.1.2b at some time level $n+1$ be $T_{i,j}^{n+1}$ at the grid point $(i,j)$. In the numerical solution, there will be some error, $\varepsilon_{i,j}^{n+1}$, associated with this temperature so that the calculated temperature is $T_{i,j}^{n+1} + \varepsilon_{i,j}^{n+1}$. Since the numerical solution must also satisfy Equation M.1.2b, substitution of the numerical solution into Equation M.1.2b and then subtracting the exact solution from it yields an expression for the growth of errors, viz.,

$$\frac{\varepsilon_{i,j}^{n+1} - \varepsilon_{i,j}^n}{\Delta t} + \frac{\varepsilon_{i,j}^n - \varepsilon_{i-1,j}^n}{\Delta z} + \frac{\varepsilon_{i,j}^n - \varepsilon_{i,j+1}^{n+1}}{\Delta y} = 0$$

Now, following the argument of O'Brien (40), concentrate on a single line of errors, $E(z)$, and make a harmonic decomposition of it so that

$$E(z) = \sum_{m} \sum_{n} A_{mn} e^{i\alpha_m z} e^{i\beta_n y}$$

M.2.2
This summation must reduce to the correct error value at each point. Because of superposition, one can consider a single term in Equation M.2.2, e.g., $e^{i\alpha Z} e^{i\beta Y}$. A solution of Equation M.2.1 which reduces to $e^{i\alpha Z} e^{i\beta Y}$ at $\tau = 0$ is $e^{\gamma t} e^{i\alpha Z} e^{i\beta Y}$. In order that the error not grow with time, it is sufficient that

$$|e^\gamma| \leq 1$$

Substitution of Equation M.2.3 into M.2.1 and simplification yields

$$\Gamma^* = \left| \frac{1 - W_{i,j}^n \frac{\Delta t}{\Delta Z} (1 - e^{i\alpha \Delta Z})}{1 + V_{i,j}^n \frac{\Delta t}{\Delta Y} (e^{i\beta \Delta Y} - 1) - \frac{2\Delta t}{\Delta Y^2} (\cos \beta \Delta Y - 1)} \right| \leq M.2.4$$

A similar analysis for Equation M.1.2d can be shown (see M.3) to yield

$$\Gamma = \frac{1 - \gamma^* (V_{1,j}^n \frac{\Delta t}{\Delta Y} (e^{i\beta \Delta Y} - 1) - \frac{2\Delta t}{\Delta Y^2} (\cos \beta \Delta Y - 1))}{1 + W_{1,j}^n \frac{\Delta t}{\Delta Z} (1 - e^{i\alpha \Delta Z})} \leq M.2.5$$

where $\gamma$ is the overall amplification factor for the finite difference scheme, Equations M.1.2b and M.1.2d. In the case of a uniformly heated vertical plate in isothermal surroundings, the horizontal velocity, $V_{1,j}^n$, is always negative and the vertical velocity component, $W_{1,j}^n$, is always positive. For this case, as shown later,

$$\Gamma < 0$$

Therefore, this scheme is unconditionally stable.

Similarly, it can be shown that the scheme for the momentum Equations M.1.2a and M.1.2c are also unconditionally stable for this set of conditions.

Table M.1 lists the limits of the various parameters which were tested to prove that the finite difference equations were indeed stable. From these tests, it was
concluded that the implicit alternating difference scheme (IAD) used in this work is unconditionally stable as predicted by the von Neumann analysis.

**TABLE M.1**

| RANGE OF PARAMETERS TESTED FOR STABILITY OF IAD SCHEME |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \Delta t \) | \( \Delta t/\Delta z \) | \( \Delta t/\Delta y \) | \( \Delta t/\Delta y^2 \) | \( P_t \) | \( R_A \) |
| 1.0              | 6.0             | 0.6             | 0.36            | 10              | 10^6            |
| 2.0              | 12.0            | 1.2             | 0.72            | 10              | 10^6            |
| 3.0              | 18.0            | 1.8             | 1.08            | 10              | 10^6            |
| 0.25             | 1.25            | 0.125           | 0.09            | 10              | 10^6            |
| 1.0              | 6.0             | 1.2             | 0.36            | 10              | 10^6            |
| 1.0              | 10.0            | 0.6             | 0.36            | 10              | 10^6            |
| 1.0              | 6.0             | 0.6             | 0.36            | 100             | 10^6            |
| 1.0              | 6.0             | 0.6             | 0.36            | 10              | 10^{10}         |
| 1.0              | 6.0             | 0.6             | 0.36            | 10              | 10^{12}         |

M.2.1.2 Convergence of the Finite Difference Equations

As seen previously, convergence is defined as the difference between the exact solution of the finite difference equations and the exact solution of the differential equations. This section considers the effect of the various parameters on the convergence properties of the finite difference equations.

Solution of the finite difference scheme (Equations M.1.2) is actually the numerical solution, \( N_s \). Comparison of these results to the analytical results, i.e., \((D_s - N_s)\), contains both the truncation error, \((D_s - F_s)\), and the numerical error, \((F_s - N_s)\), since

\[
D_s - N_s = (D_s - F_s) + (F_s - N_s)
\]  
M.2.7
Section M.2.1.1 showed that the finite difference scheme was stable under all conditions, hence, the numerical error did not increase without bound. This means that deviations of the numerical solution from the analytical can be attributed solely to truncation error.

The main parameters in the difference scheme are:

i. The time increment, $\Delta t$

The time increment was varied from 0.25 to 3.0. The steady state results agreed well for all of the tests. However, a slight oscillation in the transient results was detected for $\Delta t = 3.0$ (see Section M.3.1.4). Therefore, in order to ensure convergent results, the time increment should be kept less than 2.0.

ii. The number of grid points along the plate, IMAX

The effect of this parameter is seen in Figure M.2. The steady state temperature results are convergent for all IMAX. However, the maximum velocity changes with increasing IMAX up to IMAX = 10. No further change in velocity was observed for values of IMAX greater than this. However, since the deviation of $W$ was less than 5 percent for IMAX = 6, it was concluded that adequate steady state results could be obtained at this smaller value and that the number of calculations required would be minimized.

iii. The number of grid points perpendicular to the plate, JMAX

The effect of this parameter is seen in Figure M.3. The solutions for JMAX's of 30 and 15 both converged to the same answer. Therefore, it seems that the number of grid points in the Y-direction is not an important parameter in determining the accuracy of the solution. However, note that in both cases there is at least one grid point between the wall and the maximum velocity. This grid point is necessary if the maximum velocity and its position is to be determined accurately.
\[ Pr = 10 \]
\[ Ra = 10^6 \]
\[ Z = 1.00 \]

- Imax = 6
- Imax = 10

**Figure M.2** The effect of Imax on the boundary layer solution when \( T_\infty = T_0 \)
Figure MG3 The Effect of Jmax on the Boundary Layer Solution when $T_\infty = T_0$. 

Pr = 10
Rd = 10^6
Z = 1.00

- Jmax = 30
- Jmax = 15
iv. The distance, $Y_{MAX}$, at which the boundary conditions at $Y \to \infty$ are imposed

The effect of this parameter is shown in Figure M.4. The boundary conditions in Equation M.1.2f for $Y \to \infty$ were imposed at $Y_{MAX} = 25$ and $Y_{MAX} = 15$. The temperature solutions deviate markedly. The reason for this deviation is not to be attributed to truncation error but rather to the "porous wall" boundary condition (Equation 2.3.18) at $Y_{MAX} = 15$. The boundary condition at $Y \to \infty$ is equivalent to having a porous wall there so that the vertical velocity is zero while the horizontal velocity is constant, i.e., $W = 0$ and $\frac{\partial V}{\partial Y} = 0$. At $Y = 15$, a substantial vertical velocity component is still present. By placing a wall there, the viscous forces are increased, thereby reducing the flow in the region near the wall. However, since the temperature boundary layer thickness is substantially less than 15, it remains unaltered. Therefore, the boundary conditions for the surroundings must be placed sufficiently far from the plate so that the vertical component of velocity, $W$, is very small compared to the maximum.

In summary, convergence is obtained when the boundary conditions are imposed at a sufficient distance from the wall so that the viscous forces are not markedly increased in that region. It is this parameter, $Y_{MAX}$, and not $Y$, which determines convergence provided at least one of the grid points lies between the wall and the maximum vertical velocity. The value of $\Delta Z$ influences the maximum vertical velocity for $Z \lesssim 0.1$. However, the differences are small so that the value of $\Delta Z = 0.167$ which was chosen in order to minimize the computational effort was considered adequate to give convergent results.

The parameters chosen in the above tests were shown to give consistent results, i.e., varying the grid size or increasing $Y_{MAX}$ greater than 25 did not alter the resulting solution. The question remains as to whether
FIGURE M.4 THE EFFECT OF $Y_{\text{MAX}}$ ON THE BOUNDARY LAYER SOLUTION WHEN $T_{\infty} = T_0$
this solution is in fact identical to an analytical one. Figures M.5 to M.8 compare the computed solutions to the analytical solutions obtained by Chang et al. [8] for a variety of Prandtl numbers and Rayleigh numbers. It is seen that agreement is good for all cases. For the case of Prandtl number equal to 100, the results seem to be slightly lower for the computed solution than the similar solution. However, this is due to the fact that the momentum boundary layer thickness is much greater than YMAX = 25 where the porous wall boundary was applied.

Hence, the velocities were lowered due to the additionally imposed shear field.

The most serious discrepancy occurs in the horizontal velocity. Figures M.9 and M.10 show these for various Prandtl numbers and modified Rayleigh numbers. The reason for this discrepancy is two-fold. Firstly, the horizontal inertial terms are small for most of the calculations. Therefore, the velocity, V, can vary over a wide region without seriously affecting the solution. Therefore, it can be subject to relatively large errors while the momentum and energy equations are satisfied. Secondly, the analytical solution predicts that at Z = 0, V \rightarrow \infty .

This solution is not possible on the computer. Hence, the "leading edge" introduces an error into the horizontal velocity component. However, the results are sufficiently accurate to still permit some idea of the amount of entrainment involved for the case of a uniformly heated vertical plate in isothermal surroundings.

v. Transient convergence

In addition to steady state convergence, there is also the problem of transient solution convergence, i.e., how well does the transient numerical solution agree with the transient solution of the differential equations? Figures M.11 and M.12 show the transient numerical solutions of the wall temperature and maximum velocity respectively. For a
FIGURE M.5 COMPARISON OF COMPUTED VELOCITY AND TEMPERATURE PROFILES TO THE CLASSIC SIMILAR SOLUTION - \( \Phi(\eta) \), \( \psi = 106 \).
FIGURE M.6 COMPARISON OF THE COMPUTED VELOCITY AND TEMPERATURE PROFILES TO THE CLASSIC SIMILAR SOLUTION - PR = 10, RA = 10^6. 
FIGURE M.7 COMPARISON OF THE COMPUTED VELOCITY AND TEMPERATURE PROFILES TO THE CLASSIC SIMILAR SOLUTION

\( Pr = 100, Ra^+ = 10^6 \)
FIGURE M.8 COMPARISON OF THE COMPUTED VELOCITY AND TEMPERATURE PROFILES TO THE CLASSIC SIMILAR SOLUTION - PR = 10, RA⁺ = 10¹² -
FIGURE M.9  COMPARISON OF THE HORIZONTAL VELOCITY TO THE CALCULATED VALUE FROM THE CLASSIC SIMILAR SOLUTION -EFFECT OF PRANDTL NUMBER-
Figure M.10 COMPARISON OF THE HORIZONTAL VELOCITY TO THE CALCULATED VALUE FROM THE CLASSIC SIMILAR SOLUTION - EFFECT OF RAYLEIGH NUMBER -
FIGURE M.11
THE EFFECT OF THE TIME STEP ON THE TRANSIENT TEMPERATURE SOLUTION

Note: The solid line is the explicit solution

\[ \tau = \frac{a t}{L^2} \quad \text{Re}^{1/2} \]

- Pr=10
- Re=10^6
- Z = 1.00
- \( \nabla \) - Dt=0.25
- \( \bullet \) - Dt=0.50
- \( \circ \) - Dt=1.00
- \( \blacksquare \) - Dt=2.00
- \( \triangle \) - Dt=3.00

The dashed line represents conduction only.
FIGURE 8.12
THE EFFECT OF THE TIME STEP ON THE TRANSIENT VELOCITY SOLUTION

Explicit Sol'n

Pr = 10
Ra = 10^6
Z = 1.00

- ▼ - Dt = 0.25
- ○ - Dt = 0.50
- ● - Dt = 1.00
- ■ - Dt = 2.00
- △ - Dt = 3.00

MAXIMUM VERTICAL VELOCITY, W

\[ \tau = \frac{t a}{L^2} \left( \frac{Ra}{t} \right)^{1/2} \]
large range of time step increments, \( \Delta \tau \), the transient solutions agreed well with one another. However, two discrepancies are noticeable from the plots. The first is that the wall temperature is not equal to zero at \( \tau = 0 \) as is required by the initial conditions. Siegel (54) showed that the initial wall temperature transient for the case of an isothermal wall and a uniformly heated wall in isothermal surroundings followed the conduction equation for the particular boundary condition. For the case of a uniformly heated vertical plate in isothermal surroundings, Carslaw and Jaeger (7) show that the transient solution for conduction into a semi-infinite medium is:

\[
T = 2^{\pi^{1/2}} \Re^{1/4} \tau^{1/2}
\]

As seen from Figure M.11, the numerical solution is much too high. However, at \( 2 \leq \tau \leq 5 \), the numerical solution indicates that the wall temperature increases as the \( 1/2 \)-power of \( \tau \). Furthermore, the explicit solution using the same type of space formulation of the derivatives had even a greater variance. These discrepancies are all a result of the formulation of the wall heat flux boundary condition (see Section M.1.3). In calculating the explicit solution a linear approximation was used to estimate the boundary heat flux (Equation M.1.9 or Equation M.1.11 with \( c_2 = 0 \)). The implicit alternating direction solutions were solved using an empirical value of \( c_2 \) which gave the best steady state results (see Section M.2.3). If, however, a value of \( c_2 \) equal to zero were used in the IAD scheme, the transient solution followed that obtained using the explicit formulation of the equations. That is to say, the value of \( c_2 \) chosen markedly affects the initial wall temperature, and therefore, the transient solution. The reason for this is that the computational scheme, and in particular Equation 4.4.11, is incapable of predicting values of the temperature at distances less than \( \Delta Y \) from
the wall. Therefore, it artificially calculates a wall temperature which is necessary to conduct the wall heat away as if the first zero value of temperature were at ΔY. For \( \tau = 0 \), Equation M.1.11 predicts that a wall temperature of

\[
T_w = \Delta Y / (c_o \, Ra^{1/4}) \tag{M.2.9}
\]

is necessary in order that the wall heat flux could be removed when \( T_{1,1} \) and \( T_{1,2} \) are zero.

From Equation M.2.9 it is seen that at \( \tau = 0 \) the wall temperature approaches zero as \( Y \to 0 \). Therefore, in order for the finite difference scheme to be able to predict the correct transient solution, it is necessary either to

1. refine the grid spacing near the wall for the first few iterations;

2. allow \( c_2 \) to vary with time in some prescribed way which will allow \( T_w \to 0 \) as \( \tau \to 0 \);

3. develop a scheme for predicting the wall heat flux which is independent of the grid spacing. This type of modification was beyond the scope of this work.

It should be noted that the problem is not present for the case of a isothermal plate in isothermal surroundings. For this case, the wall temperature is always defined. However, errors do result in predicting the wall heat flux for the first few iterations. As seen from the work of Neuner (27) and Hellums (27), the transient solutions tend to be more accurate for the isothermal plate case since the wall temperature is known.

Even though quantitative conclusions cannot be made from the computed transients, several observations can be made. From Figure M.11, one can see that the conduction region for the transient numerical solution lasts for about \( \tau = 5 \) with the final steady state occurring at \( \tau = 9 \).
Therefore, the intermediate region lasts for about a
\( \Delta \tau \approx 4 \). On the other hand, if the solution had followed
the conduction curve, the conduction transient would have ended at \( \tau \approx 8 \). Since the only difference between the
numerical and analytical solutions is due to the initial
temperature conditions, it is reasonable to assume that
the intermediate region would be of the same duration as
in the numerical case. Therefore, steady state can be
assumed to occur at \( \tau \approx 12 \). Siegel (54) predicts that
steady state should occur at \( \tau \approx 19.6 \). Hellums (27)
found that Siegel predicted times which were also too long
for the case of natural convection from an isothermal
plate in isothermal surroundings. The discrepancy is
probably due in most part to the selection of the criteria
at which the numerical solution is assumed to have reached
steady state. In this work, agreement of the velocity and
temperature fields to within 1 percent between any two
time levels constituted steady state. Certainly, stricter
convergence criteria would have extended the intermediate
region thereby reducing the difference in times between
Siegel's and this work.

Finally, some word should be said about the inter-
mediate region. Siegel (54), Hellums (27) and Neuner (37)
all predict an "overshoot" in temperature and velocity,
i.e., the velocity and temperature reach values higher than
those at steady state. As seen from Figures M.11 and M.12
the explicit solution does predict such an "overshoot"
while the IAD scheme does not. The lack of "overshoot" is
due to the fact that the vertical grid size, \( \Delta Z \), was too
large. While a \( \Delta Z = 0.167 \) was found to give adequate agree-
ment between the numerical and analytical steady state
solution, it was found that a value of \( \Delta Z \) of 0.067 or
less was required before convergent transient solutions
were obtained. These convergent solutions displayed the
predicted "overshoot". However, since the transient
solution cannot be relied upon because of the above mentioned problems, the results of these tests were not presented.

In summary, convergent steady state solutions were obtained for the case of a uniformly heated alternating direction scheme. For convergence to occur, it was necessary to have the horizontal grid size sufficiently small so that at least one point lies between the maximum velocity and the wall. Also, the horizontal grid had to extend sufficiently so that the velocity at \( J_{MAX} \) is negligible. Steady state temperature and velocity profiles agreed well with the analytical solution of Chang (8) for all vertical grid sizes tested \( (0.167 \leq \Delta Z \leq 0.05) \). However, grid sizes of \( \Delta Z < 0.067 \) had to be used for convergence of the transient solution.

The numerical transient solution deviated from the predicted analytical conduction equation because of the inability of the computational scheme to predict the initial wall temperature adequately.

M.2.1.3 The Effect of the Wall Heat Flux Approximation on the Solution

In Section M.1. , it was pointed out that the wall heat flux could not be directly specified. The derivative of temperature had to be approximated by some formula incorporating the wall temperature and the temperatures at one or two adjacent points. For this work, a general three point formula (Equation M.1.11) was used to approximate the first derivative. The effect of the values of \( c_2 \) chosen on the solution of the boundary layer equations for a uniformly heated vertical plate in isothermal surroundings will be considered in this section.

Figure M.13 compares the solutions obtained for three different values of \( c_2 \). These values correspond to a
$Z = 1.00$
$Pr = 10$
$Re = 10^6$

- $c_2 = -0.167$
- $c_2 = 0.0$
- $c_2 = -0.5$

**FIGURE M.13** THE EFFECT OF $c_2$ ON THE BOUNDARY LAYER SOLUTION WHEN $T = T_0$
linear approximation of the temperature derivative, 
\( \partial T/\partial Y \), i.e., Equation M.1.9, a Taylor's series approxi-
mation, i.e., Equation M.1.10c, and an empirical approxi-
mation whose value of \( c_2 \) was calculated using the known
analytical solution. This latter method will be discussed
further below. The values of \( c_2 \) for these three methods
were 0.0, -0.5, and -0.167 respectively. As seen from
Figure M.13, best agreement for the temperature solution
was for the calculated value of \( c_2 \). The linear approxi-
mation tended to overestimate the wall temperature slightly,
while the Taylor's series approximation underestimated
the wall temperature. However, except for the wall temper-
ature, all of the calculated temperature profiles agreed
very well with the analytical solution. Similarly, the
velocity profiles agreed quite well except in the region
of the maximum velocity. Then the linear and empirical
approximations gave the best agreement while the Taylor's
series approximation deviated significantly.

The empirical values of \( c_2 \) were calculated from the
analytical solution of Chang et al. (8). The temperature
data from this work were fitted to Equation M.1.11, and
the resulting values of \( c_2 \) for various grid spacings, \( \Delta Y \),
were plotted in Figure M.14.

Several important features of Figure M.14 should be
noted. The value of \( c_2 \) is independent of the Prandtl
number up to values of \( \Delta Y/Ra_{\infty}^{1/20} \leq 1.5 \). At grid spacings
greater than 1.5, a Prandtl number dependency becomes
apparent—the greater the Prandtl number, the greater is
the rate of change of \( c_2 \) with grid spacing. This is due
to the fact that the temperature profiles are very similar
in shape near the wall for all Prandtl numbers. However,
the thermal boundary layer thickness decreases with Prandtl
number (see Chapter 5). The thinner boundary layer causes
the fit (which is valid at small \( \Delta Y \)'s) to begin deviating.
This deviation is of course more pronounced at thinner
FIGURE M.14
GENERALIZED PLOT OF $c_2$
FOR A UNIFORMLY HEATED VERTICAL PLATE IN ISOTHERMAL SURROUNDINGS

$Pr=100$, $10$, $1$
boundary layers, i.e., higher Prandtl numbers. Another interesting feature is that the value of $\Delta Y/\text{Ra}_z^{1/20} \approx 1.5$ is approximately where the peak of the velocity profiles occurs (see Chapter 5). Since the grid spacings must be smaller than this to ensure accuracy, a value of $c_2$ approximately equal to $-0.17$ is optimal for all cases of interest. Finally, note that the value of $c_2$ is dependent on $\text{Ra}_z^+$. Therefore, $c_2$ should vary with height up the plate.

An attempt to measure the error introduced by assuming $c_2$ constant rather than a function of height showed that the solution obtained differed negligibly if the constant value of $c_2$ was chosen at $Z = 1$.

In summary, the finite difference solution requires an arbitrary means of postulating the wall heat flux boundary condition. A generalized three point formula (Equation M.1.11) was chosen for this work. An empirical value of $c_2 = -0.167$ gave the best results for all cases of interest. However, if an analytical solution were not available from which $c_2$ could be calculated, a two point linear approximation ($c_2 = 0$) could be used to obtain adequate results. In order to use the value of $c_2 = -0.17$, the grid spacing must be such that at least one grid point lies between the wall and the point of maximum velocity. If the grid spacing is greater than this, Figure M.14 should be used for obtaining values of $c_2$. However, grid spacings greater than $\Delta Y/\text{Ra}_z^{1/20} \approx 1.5$ are not recommended.

M.2.2 Uniformly Heated Vertical Plate in Surroundings Whose Temperature Increases Linearly with Height

Natural convection from a uniformly heated vertical plate in surroundings where the temperature increases linearly with distance from the leading edge of the plate will now be considered. This section re-examines the
stability and convergence criteria in terms of this problem. The solution of the natural convection equations for the case is sufficiently different from the case when the surroundings are isothermal that the stability and convergence may be affected. For example, it can be shown (Appendix M) that while the IAD scheme used in this work is unconditionally stable when \( W > 0 \), there are stability restrictions when \( W < 0 \). A negative vertical velocity does occur for the case of a linear bulk temperature gradient (see Chapter 5).

Rather than solve the finite difference analogs of the boundary layer equations directly for various Prandtl numbers, Rayleigh numbers and bulk temperature gradients, a new set of dimensionless equations were written. The finite difference equations (Equation M.1.2) can be used to solve the boundary layer equations if the variables are defined as in Equation 5.3.12 and

\[
\frac{\partial \theta}{\partial z} = 4 \\
Ra^+ = 0.0625
\]

Then, the only remaining parameter is the Prandtl number. It is this scheme which was used in this work.

M.2.2.1 Stability of the Finite Difference Equations

The time step, \( \Delta \tau \), and the grid size were varied in order to determine their effect on the stability of the finite difference equations. The results of these tests are given in Table M.2. The results of these tests indicate that the maximum negative velocities are of the order

\[
W_{\text{max}} = -0.015 \\
V_{\text{max}} = -0.25
\]
### TABLE M.2
PARAMETERS INVOLVED IN FINITE DIFFERENCE COMPUTATION OF NATURAL CONVECTION
FROM A UNIFORMLY HEATED VERTICAL PLATE IN NON-ISOTHERMAL SURROUNDINGS

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>J</td>
<td>Z</td>
<td>Y</td>
<td>τ</td>
<td>Z</td>
<td>Y</td>
<td>( \frac{T}{Z} )</td>
<td>( \frac{Y}{Y^2} )</td>
<td>( \frac{2Pr}{Y^2} )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.1</td>
<td>1.67</td>
<td>0.5</td>
<td>0.06</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>1.0</td>
<td>1.67</td>
<td>0.5</td>
<td>0.6</td>
<td>2.0</td>
<td>8.0</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.25</td>
<td>1.67</td>
<td>0.5</td>
<td>0.15</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>30</td>
<td>0.15</td>
<td>1.67</td>
<td>1.0</td>
<td>0.09</td>
<td>0.15</td>
<td>0.3</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.4</td>
<td>1.67</td>
<td>0.5</td>
<td>0.24</td>
<td>0.8</td>
<td>3.2</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.31</td>
<td>1.67</td>
<td>0.5</td>
<td>0.186</td>
<td>0.62</td>
<td>2.48</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>30</td>
<td>0.4</td>
<td>1.67</td>
<td>1.0</td>
<td>0.24</td>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.15</td>
<td>1.67</td>
<td>0.5</td>
<td>0.09</td>
<td>0.30</td>
<td>1.2</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.15</td>
<td>1.67</td>
<td>0.5</td>
<td>0.09</td>
<td>0.3</td>
<td>1.2</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>15</td>
<td>0.1</td>
<td>1.67</td>
<td>0.5</td>
<td>0.06</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>
The von Neumann stability analysis (Appendix M) predicts that when $W < 0$, $V < 0$

$$\frac{|W| \Delta T}{\Delta Z} > 1$$

However, it can be shown that for all the cases in Table M.2 $|W| \frac{\Delta T}{\Delta Z} \ll 1$. The reason for this inconsistency between the theoretical prediction and experimental evidence is not apparent. In fact, the runs in Table M.2 do not reveal a consistent stability criteria. The only observation which can be made is that the maximum time interval is a function of Prandtl number. It was found that for Prandtl numbers less than 10, a time increment, $\Delta T_0$, less than 0.31 were necessary to ensure stability. For Prandtl number of 100, the time increment had to be reduced to 0.1 to maintain stability. As will be seen later, a time increment of 0.15 was used in the final tests for all runs except for the case of Prandtl number equal to 100 with good results.

M.2.2.2 Convergence of the Finite Difference Equations

In this section convergence of the finite difference scheme will be considered. As in Section M.2.1.2, solutions of Equations M.1.2 for a uniformly heated vertical plate in surroundings whose temperature increased linearly with height will be examined from two points of view. The effect of grid size and time increment on the solution will be tested. Also, the numerical solution will be compared to Equation 5.3.5, since this is the solution which must result at distances far from the leading edge of the plate.

The main parameters in the IAD scheme which may affect convergence are:
1. The time increment, $\Delta \tau_0$

As seen in Table M.2, the time increment was varied from 0.1 to 1.0. The effect of the time increment on the temperature and velocity solutions is shown in Figures M.15 and M.16. The solution agrees very well with the asymptotic solution (Equation 5.3.5) for both the velocity and temperature fields for $\Delta \tau_0$ less than 0.25. The velocity solution deviates markedly from the asymptotic solution for $\Delta \tau_0 = 0.5$, and becomes unstable for $\Delta \tau_0 = 1.0$. Indication that the stability and convergence of finite difference equations is limited by the momentum equation is given by the fact that greater deviation occurs in the velocity solution for a given time step size.

11. The number of grid points along the plate, $\text{IMAX}$, and the corresponding grid spacing, $\Delta Z_0$

Numerical solutions were obtained for $\text{IMAX} = 6$ and $\text{IMAX} = 10$ and corresponding values of $\Delta Z_0$ of 1.67 and 1.0. The solutions agreed well with each other and the analytical asymptotic solution. Therefore, unlike the case of a uniformly heated plate in isothermal surroundings, the solution for this case was convergent even at these large $\Delta Z_0$'s.

111. The number of grid points perpendicular to the plate, $\text{JMAX}$

The effect of the number of grid points perpendicular to the plate on the temperature and velocity profiles is shown in Figures M.17 and M.18 respectively. Thirty grid points were necessary to obtain convergent solutions which agree with the analytical solution. Fewer grid points reduced the amount of energy and fluid convected in the "reverse flow" region.
FIGURE M.15
THE EFFECT OF THE TIME STEP SIZE ON THE TEMPERATURE PROFILE

\[ Pr = 10 \]

- $\Delta \tau_0 = 0.1$
- $\Delta \tau_0 = 0.25$
- $\Delta \tau_0 = 0.5$
- unstable: $\Delta \tau_0 = 1.0$

analytical
FIGURE M.16
THE EFFECT OF THE TIME STEP SIZE ON THE VELOCITY PROFILE

analytical

Pr = 10

- $\Delta \tau_b = 0.1$
- $\Delta \tau_b = 0.25$
- $\Delta \tau_b = 0.5$
- unstable $\Delta \tau_b = 1.0$
iv. The distance, $Y_{\text{MAX}}$, at which the boundary conditions at $Y_0$ are imposed

As seen in Figures M.19 and M.20, the distance at which the boundary conditions are imposed is an important parameter. The solution of the finite difference equations with $Y_{\text{MAX}} = 15$ gave better results than that for $Y_{\text{MAX}} = 30$. In Section M.2.1.2, it was concluded that the boundary conditions must be applied at a distance sufficiently far from the wall so that the vertical velocity at that point is small. In this case, a value of $Y_{\text{MAX}} = 15$ is sufficiently far from the wall. Increasing the distance $Y_{\text{MAX}}$ while maintaining the number of horizontal grid points constant increases the value of $Y_0$. For the case tested, $Y_0$ was sufficiently large when $Y_{\text{MAX}} = 30$ so that it fell further from the wall than the maximum velocity. This resulted in reduced accuracy of the numerical solution. Therefore, $Y_0$ must be less than 0.7 in order to maintain the required accuracy.

v. Transient convergence

Figures M.21 and M.22 show the wall temperature and maximum velocity ($Y_0 = 1$) as a function of time, $\tau_0$. Agreement is good for time steps of 0.1 and 0.15 for both variables. As was mentioned in Section M.2.1.2, the convergent numerical solution of the transient differs from that which would be obtained analytically because of the wall heat flux boundary condition approximation. However, since the $Y_0$ is much smaller than for the case of a uniformly heated vertical plate in isothermal surroundings, agreement between the initial wall temperature transient and the transient heat conduction solution is much better.

Following the same arguments as in Section M.2.1.2, one can see that the time scale of the initial transient is about $\tau = 0.6$. The intermediate transient lasts for about $\tau_0 = 4.5$. Therefore, the steady state temperature profiles occur at a $\tau_0 = 5$. The velocity profiles seem to
FIGURE M.17
THE EFFECT OF JMAX
ON THE TEMPERATURE SOLUTION

\[ \Delta \tau_0 = 0.15 \]

- JMAX = 30
- JMAX = 15

analytical
FIGURE M.18
THE EFFECT OF JMAX ON
THE VELOCITY SOLUTION

$\Delta \tau_0 = 0.15$

- JMAX = 30
- JMAX = 15

analytical
take a slightly longer time to achieve steady state (Appendix I), \( \tau_0 \approx 6.5 \). Whether this is due to the oscillatory instability of the physical system as reported by Cheese\-wright (8) or is purely an artifice of the computational system is not clear. No doubt further analysis of the finite difference equations, and in particular their stability characteristics, is required before confidence can be attached to the transient results.

In summary, the finite difference scheme in Equation M.1.2 could be modified to solve the problem of natural convection from a uniformly heated vertical plate in surroundings with a linear vertical temperature gradient. Stability and convergence could be obtained if the time-step size were kept below 0.15. As in the case of a uniformly heated plate in isothermal surroundings, the horizontal grid size had to be kept small enough so that at least one point fell between the wall and maximum velocity. Unlike the previous case, the vertical grid size could be quite large ( \( Z_0 \approx 1.67 \) ) and still obtain convergent results.

M.2.2.3 The Effect of the Wall Heat Flux Specification on the Solution

In Section M.1.2, the need for an arbitrary specification of the wall heat flux condition was discussed. In Section M.2.1.3, best agreement between the numerical and analytical solutions for the case of a uniformly heated vertical plate in isothermal surroundings was obtained when a generalized three point formula using a coefficient \( c_2 \) determined from Figure M.14 was used. To ensure such agreement for this case, another set of values of \( c_2 \) must be derived since the solutions for the two cases are markedly different. This new set of values of
FIGURE M.19
THE EFFECT OF YMAX ON THE TEMPERATURE SOLUTION

$\Delta T_0 = 0.15$

- $YMAX = 15$
- $YMAX = 30$

analytical

$T_0$

$Y_0$
FIGURE M.20
THE EFFECT OF $Y_{MAX}$ ON THE VELOCITY SOLUTION

$\Delta \tau_0 = 0.15$

- $Y_{MAX} = 15$
- $Y_{MAX} = 30$

$\dot{w}_n$ vs. $Y_n$
The effect of the time step size on the transient temperature solution.

- $\Delta \tau = 0.15$
- $\Delta \tau = 0.10$

Pr = 10
YMAX = 15
JMAX = 30
ZMAX = 10
IMAX = 6

FIGURE M.21
THE EFFECT OF THE TIME STEP SIZE ON THE TRANSIENT TEMPERATURE SOLUTION
The effect of the time step size on the transient velocity solution.

- $\Delta \tau_b = 0.15$
- $\Delta \tau_b = 0.10$

FIGURE M.22

Pr = 10
YMAX = 15  JMAX = 30
ZMAX = 10  IMAX = 6
c₂ to be used are given in Figure M.23.

Unlike the case of the uniformly heated plate in isothermal surroundings, the values of c₂ change markedly with the grid spacing over all ranges. At small values of ΔY₀, the values of c₂ are comparable for both cases. This is, of course, to be expected since the temperature profiles are similar very near the wall.

Also, since the results are independent of Prandtl number and Rayleigh number, Figure M.23 is valid for all cases of interest. Assuming that the conclusions reached in Section M.2.1.3 apply here, it is recommended that the grid spacing be such that at least one point lies between the wall and maximum velocity to ensure the accuracy of the solution. Therefore, for the case of a uniformly heated vertical plate in non-isothermal surroundings,

$$\Delta Y_0 \leq 0.9$$

and values of c₂ be chosen using Figure M.23.

M.3 Stability of the Finite Difference Equations for the Layer Problem

The stability of the finite difference approximations to the boundary layer equations is considered for the case of a uniformly heated plate in isothermal and non-isothermal surroundings. In particular, the same difference formulation in space is compared for explicit, semi-implicit (this term is defined below), and implicit alternating direction schemes.

Briefly, the boundary layer equations of interest are:

$$\frac{\partial T}{\partial t} + W \frac{\partial T}{\partial Z} + V \frac{\partial T}{\partial Y} = \frac{\partial^2 T}{\partial Y^2} - \frac{W T_{\infty}}{\partial Z}$$  \hspace{1cm} \text{M.3.1a}

$$\frac{\partial W}{\partial t} + W \frac{\partial W}{\partial Z} + V \frac{\partial W}{\partial Y} = \text{Pr} T + \text{Pr} \frac{\partial^2 W}{\partial Y^2}$$  \hspace{1cm} \text{M.3.1b}
FIGURE M.23
GENERALIZED PLOT OF $c_2$
FOR A UNIFORMLY HEATED
VERTICAL PLATE IN
SURROUNDINGS WITH A
LINEAR TEMPERATURE
GRADIENT
Defining a new set of parameters:

\[ \delta_t f = f_{i,j}^{n+1} - f_{i,j}^n \]
\[ \delta_z f = f_{i,j}^n - f_{i-1,j}^n \]
\[ \delta_y f = f_{i,j+1}^n - f_{i,j}^n \]
\[ \delta_{yy} f = f_{i,j-1}^n - 2f_{i,j}^n + f_{i,j+1}^n \]

where \( f \) is the dependent variable, one can write Equation M.1 into a set of generalized finite difference equations as follows:

\[ \frac{\delta_t T}{\Delta t} + \phi_1 \frac{W_{i,j}^n}{\Delta Z} \delta_z T^n + \phi_2 \frac{W_{i,j}^n}{\Delta Z} \delta_z T^{n+1} + \phi_3 \frac{V_{i,j}^n}{\Delta Y} \delta_y T^{n+1} \]
\[ + \phi_4 \frac{V_{i,j}^n}{\Delta Y} \delta_y T^n = \delta_y \frac{T^{n+1}}{\Delta Y^2} + \phi_4 \frac{\delta_y T^n}{\Delta Y^2} - \frac{W_{i,j}^n}{\Delta Z} \]

\[ \frac{\delta_t W}{\Delta t} + \phi_1 \frac{W_{i,j}^n}{\Delta Z} \delta_z W^n + \phi_2 \frac{W_{i,j}^n}{\Delta Z} \delta_z W^{n+1} + \phi_3 \frac{V_{i,j}^n}{\Delta Y} \delta_y W^{n+1} \]
\[ + \phi_4 \frac{V_{i,j}^n}{\Delta Y} \delta_y W^n = Pr \frac{T^{n+1}}{\Delta Y} + \phi_3 Pr \frac{\delta_y T^n}{\Delta Y^2} + \phi_4 Pr \frac{\delta_y T^n}{\Delta Y^2} \]

\( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are constants equal to 1 or 0 depending upon the type of time scheme used. If \( \phi_1 = \phi_4 = 1 \), and \( \phi_2 = \phi_3 = 0 \) then the difference equations are explicit in time. On the other hand, if \( \phi_1 = \phi_4 = 0 \), and \( \phi_2 = \phi_3 = 1 \) the equations are implicit in time. Two other combinations are possible. One is when \( \phi_1 = \phi_3 = 1, \phi_2 = \phi_4 = 0 \). This makes the equations implicit in the Y-direction and explicit in the Z-direction. It is this scheme which will be defined as the semi-implicit formulation. The remaining combination is \( \phi_1 = \phi_3 = 0, \phi_2 = \phi_4 = 1 \). Then, the equations are explicit in the Y-direction and implicit in the Z-direction. This scheme will be known as the semi-explicit scheme.
The stability of these various finite difference schemes will be considered using the so-called von Neumann analysis. In this type of analysis, the coefficients $w_{1,j}^n$ and $v_{1,j}^n$ are assumed constant over a time step, $t$. Then, a Fourier expansion of a line of errors is made and the progress of the general term of the expansion is followed.

This analysis will be illustrated in detail using the energy finite difference equation, Equation M.3.3a. Let the exact solution of the difference equation (Equation M.3.3a) be $T_{1,j}^{n+1}$ at time $t$ and grid point $(1,j)$. The numerical solution contains a certain error, $e_{1,j}^{n+1}$, due to the accumulation of "round off" errors. Therefore, the numerical solution at $t$ and $(1,j)$ is $T_{1,j}^{n+1} + e_{1,j}^{n+1}$. This solution must satisfy Equation M.3.3a also. So that, substituting the numerical solution into Equation M.3.3a, one obtains

$$\delta_t \left( T + e \right) + \frac{W_{1,j}^{n+1}}{\Delta t} \left( \delta_z (T + e)^n + \delta_z (T + e)^{n+1} \right) + \frac{V_{1,j}^n}{\Delta y}$$

$$\left( \delta_y (T + e)^{n+1} + \delta_y (T + e)^n \right) = \frac{1}{\Delta y^2} \left( \delta_y (T + e)^{n+1} \right)$$

$$+ \delta_y (T + e)^n = W_{1,j}^n \left( \frac{\partial T_n}{\partial Z} \right)$$

Subtracting Equation M.3.3a from Equation M.3.3b, one finds that

$$\delta_t \frac{\delta_z}{\Delta t} + \delta_1 \frac{W_{1,j}^{n+1}}{\Delta x} \delta_z e^n + \delta_2 \frac{W_{1,j}^{n+1}}{\Delta x} \delta_z e^{n+1} + \delta_3 \frac{V_{1,j}^n}{\Delta y} \delta_y e^{n+1}$$

$$+ \delta_4 \frac{V_{1,j}^{n+1}}{\Delta y} \delta_y e^n = \delta_3 \delta_y e^{n+1/\Delta y^2} + \delta_4 \delta_y e^{n/\Delta y^2}$$

Now, following the argument of O'Brien (33a, one considers the numerical solution of Equation M.3.3a. At each line, a group of errors is introduced which propagates as in Equation M.3.4b. Concentrating on a single line of
errors, $E(z)$, and making a harmonic decomposition of it, one obtains

$$E(z) = \sum_{m}^{X} A_{m} e^{i\alpha Z} e^{i\beta Y} \tag{M.3.5}$$

This summation must reduce to the correct error value at each grid point. Because of the superposition principle, only a single error term, $e^{i\alpha Z} e^{i\beta Y}$, need be considered. $\alpha$ and $\beta$ are any terms in the sequence of frequencies $\{a_m\}, \{\beta_n\}$ respectively. Now, the solution of Equation M.3.4b which reduces to $e^{i\alpha Z} e^{i\beta Y}$ at $t = 0$ is $e^{i\lambda t} e^{i\alpha Z} e^{i\beta Y}$. In order that the initial error not grow at $t$ increases, it is necessary and sufficient that

$$\left| \frac{d}{dt} \right| e^{i\lambda t} \leq 1 \tag{M.3.6}$$

Substitution of the error term into Equation M.3.4b, one obtains upon simplification:

$$\int = \frac{1 - \phi_1 \frac{\Delta t}{\Delta Z} \left(1 - e^{-i\alpha \Delta Z}\right) - \phi_2 \frac{\Delta t}{\Delta Y} e^{i\beta \Delta Y} - 1}{1 + \phi_2 \frac{\Delta t}{\Delta Z} e^{i\alpha \Delta Z} + \phi_3 \frac{\Delta t}{\Delta Y} e^{i\beta \Delta Y} - 1}$$

$$+ \phi_4 \frac{2\Delta t}{\Delta Y} \cos \beta \Delta Y - 1$$

$$= \phi_3 \frac{2\Delta t}{\Delta Y} \cos \beta \Delta Y - 1 \tag{M.3.7}$$

Similarly, if $\epsilon^{n+1}_{1,j}$ is the error due to "round off" error in the momentum equation (Equation M.3.3b), the error propagates in the momentum equation as

$$\frac{\delta_x \epsilon^{n+1}}{\Delta t} + \phi_1 \frac{w^n_{1,1}}{\Delta Z} \delta_z \epsilon^{n} + \phi_2 \frac{w^n_{1,1}}{\Delta Z} \delta_z \epsilon^{n+1} + \phi_3 \frac{v^n_{1,1}}{\Delta Y}$$

$$\delta_y \epsilon^{n+1} + \phi_4 \frac{v^n_{1,1}}{\Delta Y} \delta_y \epsilon^{n} = \phi_3 \frac{\delta_y v^n_{1,1}}{\Delta Y^2} + \phi_4 \frac{\delta_y v^n_{1,1}}{\Delta Y^2} \tag{M.3.8}$$

Now, using the same arguments as for the energy equation, a solution of M.3.7 is
so that defining
\[ f' = f e^{iy't} \]
substitution of the solution into Equation M.8 yields:
\[ f' = \frac{1 - \phi_1 W_1^n \Delta t (1 - e^{-i \alpha \Delta Z}) - \phi_4 V_1^n \Delta t (e^{i \beta Y} - 1)}{1 + \phi_2 W_1^n \Delta t (1 - e^{-i \alpha \Delta Z}) - \phi_3 V_1^n \Delta t (e^{i \beta Y} - 1)} \]
\[ + \frac{\phi_4 \text{Pr} \frac{t}{Y^2} (\cos \beta Y - 1)}{1 + \phi_3 \text{Pr} \frac{t}{Y^2} (\cos \beta Y - 1)} \]

M.3.9

The criteria for stability of the equations are that
\[ f < 1 \]
\[ |f'| < 1 \]

This criteria will now be applied to the various finite difference schemes.

M.3.1 The Explicit Formulation, \( \phi_1 = \phi_4 = 1, \phi_2 = \phi_3 = 0 \)

For this case,
\[ f = \left| 1 - \frac{t}{Z W_1^n} (1 - e^{-i \alpha Z}) - \frac{t}{Y V_1^n} (e^{i \beta Y} - 1) \right| \]
\[ + \frac{2 \frac{t}{Y^2} (\cos \beta Y - 1)}{1 + \phi_3 \text{Pr} \frac{t}{Y^2} (\cos \beta Y - 1)} \]

M.3.10

The stability of the scheme depends upon the size and sign of the various parameters in Equation M.3.10. By noting that
- $1 \leq e^{-1\alpha Z} \leq 1$
- $1 \leq e^{1\beta Y} \leq 1$ \hspace{1cm} M.3.12
- $1 \leq \cos(\beta \Delta Y) \leq 1$

Limiting values of Equation M.3.10 can be calculated. These stability limitations are listed in Table M.3.

Similarly, stability criteria for the momentum equation can be found. For the explicit formulation,

$$f' = 1 - \frac{\Delta t}{\Delta z}w_{i,j}^{n}(1 - e^{-1\alpha \Delta Z}) - v_{1,j}^{n} \frac{\Delta t}{\Delta Y}(e^{1\beta \Delta Y} - 1)$$
$$+ Pr \frac{2\Delta t}{\Delta Y^2} (\cos \beta \Delta Y - 1) \leq 1$$ \hspace{1cm} M.3.13

The limits of the various parameters which will ensure stability of the momentum equation are also listed in Table M.3.

It should be noted that this scheme is stable only for

$$w_{i,j}^{n} > 0$$ \hspace{1cm} M.3.14

Also, when $v_{1,j}^{n} > 0$, the energy equation limits the maximum grid size and the time increment which can be used. However, when $v_{1,j}^{n} > 0$, the momentum equation is the one which limits the parameter values for stability.

**M.3.2 Implicit Formulation, $\phi_1 = \phi_4 = 0, \phi_2 = \phi_3 = 1$**

For this case,

$$f = \left| 1 - \frac{\Delta t}{\Delta z}w_{i,j}^{n}(1 - e^{-1\alpha \Delta Z}) + v_{1,j}^{n} \frac{\Delta t}{\Delta Y}(e^{1\beta \Delta Y} - 1) - \frac{2\Delta t}{\Delta Y^2} (\cos \beta \Delta Y - 1) \right| \leq 1$$ \hspace{1cm} M.3.15
### TABLE M.3

**STABILITY CRITERIA FOR THE EXPLICIT FORM OF THE ENERGY EQUATION AND MOMENTUM EQUATION**

<table>
<thead>
<tr>
<th>Energy Equation</th>
<th>Momentum Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{1,j}^{n} &gt; 0 ) ( v_{1,j}^{n} &lt; 0 )</td>
<td>( w_{1,j}^{n} &lt; 0 ) ( v_{1,j}^{n} &gt; 0 )</td>
</tr>
<tr>
<td>( \Delta t \frac{\Delta w}{\Delta z} + \Delta t \frac{\Delta v}{\Delta y} + 2 \Delta t \frac{\Delta w}{\Delta y^2} \leq 1 )</td>
<td>( 1 + 2 \frac{\Delta t}{\Delta y} v &gt; 2 \frac{\Delta t}{\Delta z} w + 2 \frac{\Delta t}{\Delta y} ) and ( 1 + 2 \frac{\Delta t}{\Delta y} v \leq 2 \frac{\Delta t}{\Delta y} + 4 \frac{\Delta t}{\Delta y^2} ) and ( \frac{\Delta t}{\Delta y} v + 2 \frac{\Delta t}{\Delta y^2} \leq 1 )</td>
</tr>
<tr>
<td>( \Delta t \frac{\Delta w}{\Delta z} + 2 \rho \frac{\Delta t}{\Delta y} \frac{\Delta v}{\Delta y} \leq 1 )</td>
<td>( 1 + 2 \frac{\Delta t}{\Delta y} v &gt; 2 \frac{\Delta t}{\Delta z} w + 2 \frac{\Delta t}{\Delta y} ) and ( 1 + 2 \frac{\Delta t}{\Delta y} v \leq 2 \frac{\Delta t}{\Delta y} + 4 \frac{\Delta t}{\Delta y^2} ) and ( \frac{\Delta t}{\Delta y} v + 2 \frac{\Delta t}{\Delta y^2} \leq 1 )</td>
</tr>
</tbody>
</table>

**NOTE:** \( W \equiv |w_{1,j}^{n}| \)

\( V \equiv |v_{1,j}^{n}| \)
Table M.4 lists the conditions under which the implicit formulation is stable. When \( W_{i,j}^n \leq 0 \), as in the case of the explicit formulation, the finite difference equations are unstable. When \( W_{i,j}^n > 0 \), then the equations are stable. In the case of \( V_{i,j}^n > 0 \), there is a limitation on the grid size and time increment. The most interesting feature to note is that when one scheme is unstable the other is stable. Also, the grid size limits are the same for both cases. Therefore, if one is interested in solving a problem where

\[ W_{i,j}^n > 0 \text{ and } V_{i,j}^n > 0 \]

then a combination of the explicit and implicit schemes might be profitable.

**M.3.3 Semi-implicit Formulation, \( \phi_1 = \phi_3 = 1, \phi_2 = \phi_4 = 0 \)**

For this case,

\[
\begin{align*}
\left\{ \cdot \right\} &= \left| \frac{1 - \frac{\Delta t}{\Delta z} W_{i,j}^n (1-e^{-i\alpha \Delta Z})}{1 + \frac{\Delta t}{\Delta y}(e^{i\beta \Delta Y} - 1) - 2\text{Pr} \frac{\Delta t}{\Delta y^2} (\cos \beta \Delta Y - 1)} \right| \leq 1 \\
\left\{ \cdot \right\} &= \left| \frac{1 - \frac{\Delta t}{\Delta z} V_{i,j}^n (1-e^{-i\alpha \Delta Z})}{1 + \frac{\Delta t}{\Delta y}(e^{i\beta \Delta Y} - 1) - 2\text{Pr} \frac{\Delta t}{\Delta y^2} (\cos \beta \Delta Y - 1)} \right| \leq 1
\end{align*}
\]

Table M.5 gives the stability criteria for this scheme. Stability is only possible for this scheme if the vertical velocity, \( W_{i,j}^n \), is positive.
**TABLE M.5**

STABILITY CRITERIA FOR THE SEMI-IMPLICIT FORM OF
THE ENERGY AND MOMENTUM EQUATIONS

<table>
<thead>
<tr>
<th>$w_{1,j}^n &gt; 0$, $v_{1,j}^n &gt; 0$</th>
<th>$w_{1,j}^n &lt; 0$, $v_{1,j}^n &gt; 0$</th>
<th>$w_{1,j}^n &lt; 0$, $v_{1,j}^n &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} \leq \frac{\Delta t}{\Delta z} \leq 1$</td>
<td>$\frac{1}{2} \leq \frac{\Delta t}{\Delta z} \leq 1$</td>
<td>$\frac{1}{2} \leq \frac{\Delta t}{\Delta z} \leq 1$</td>
</tr>
<tr>
<td>If $\frac{2\Delta t}{\Delta y} \geq 1 + \frac{4\Delta t}{\Delta y^2}$</td>
<td>Unstable</td>
<td>Unstable</td>
</tr>
<tr>
<td>Then $\frac{W}{\Delta z} \leq \frac{V}{\Delta y} - \frac{2}{\Delta y^2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| $\frac{1}{2} \leq \frac{\Delta t}{\Delta z} \leq 1$ | $\frac{1}{2} \leq \frac{\Delta t}{\Delta z} \leq 1$ | $\frac{1}{2} \leq \frac{\Delta t}{\Delta z} \leq 1$ |
| If $\frac{2\Delta t}{\Delta y} \geq 1 + \frac{4Pr\Delta t}{\Delta y^2}$ | Unstable | Unstable |
| then $\frac{W}{\Delta z} \leq \frac{V}{\Delta y} - \frac{2Pr}{\Delta y^2}$ | |

Note: $W \equiv |w_{1,j}^n|$
$V \equiv |v_{1,j}^n|$
M.3.4 Semi-explicit Formulation, $\phi_1 = \phi_3 = 0$, $\phi_2 = \phi_4 = 1$

The stability criteria for this scheme are:

$$
\left| 1 - v^n \frac{\Delta t}{\Delta Y} (e^{i\beta \Delta Y} - 1) + \frac{2\Delta t}{\Delta Y} (\cos \beta \Delta Y - 1) \right| \leq 1 \quad M.3.19
$$

$$
\left| 1 - v^n \frac{\Delta t}{\Delta Y} (e^{i\beta \Delta Y} - 1) + \frac{2\Pr \Delta t}{\Delta Y^2} (\cos \beta \Delta Y - 1) \right| \leq 1 \quad M.3.20
$$

Table M.6 gives the stability criteria for this scheme.

M.3.5 Uniformly Heated Plate Convection

The present work considered natural convection from a uniformly heated vertical plate for both isothermal and non-isothermal surroundings. This section deals with the effectiveness of the various schemes presented above in relation to the problem at hand. Table M.2 lists typical values of the parameters which were found to comprise the stability limitations above. Using the values in Table M.7, maximum values of the time increment can be calculated for the various schemes. These are listed in Table M.8. From Table M.8, it can be seen that only the explicit formulation permits calculation of the natural convection flows for a uniformly heated plate in surroundings with a linear temperature gradient. However, results have shown that the total number of iterations required are approximately 250. This requires a relatively large amount of computer time. This is compared to 50 iterations for the case of the explicit formulation of the problem of a uniformly heated plate in isothermal surroundings. The semi-implicit formulation requires only 25 iterations.
TABLE M.6

STABILITY CRITERIA FOR THE SEMI-EXPLICIT FORM OF
THE ENERGY AND MOMENTUM EQUATIONS

<table>
<thead>
<tr>
<th>Energy Equation</th>
<th>Momentum Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>If (\frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} \leq 1), Stable [\frac{\hat{u}}{\hat{u}} - 2\frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}}]</td>
<td>If (\frac{4}{\sqrt{2}} + 2\frac{\hat{u}}{\hat{u}} \leq 1), Stable [\frac{\hat{u}}{\hat{u}} - 2\frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}}]</td>
</tr>
<tr>
<td>(\frac{2\hat{u}}{\hat{u}} + \frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}})</td>
<td>(\frac{2\hat{u}}{\hat{u}} + \frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}})</td>
</tr>
<tr>
<td>Energy Equation</td>
<td>Momentum Equation</td>
</tr>
<tr>
<td>If (\frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} \leq 1), Stable [\frac{\hat{u}}{\hat{u}} - 2\frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}}]</td>
<td>If (\frac{4}{\sqrt{2}} + 2\frac{\hat{u}}{\hat{u}} \leq 1), Stable [\frac{\hat{u}}{\hat{u}} - 2\frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}}]</td>
</tr>
<tr>
<td>(\frac{2\hat{u}}{\hat{u}} + \frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}})</td>
<td>(\frac{2\hat{u}}{\hat{u}} + \frac{\hat{u}}{\hat{u}} \leq \frac{\hat{u}}{\hat{u}})</td>
</tr>
<tr>
<td>Note: (W =</td>
<td>\hat{u}</td>
</tr>
</tbody>
</table>

\[\hat{u} \geq 0 \quad \hat{u} \leq 0 \quad \hat{u} > 0 \quad \hat{u} < 0 \quad \hat{u} > 0 \quad \hat{u} < 0 \quad \hat{u} > 0 \quad \hat{u} < 0\]
### TABLE M.7

VALUES OF PARAMETERS IN FINITE DIFFERENCE EQUATIONS

<table>
<thead>
<tr>
<th></th>
<th>$W_{\text{max}}$</th>
<th>$V_{\text{max}}$</th>
<th>$W_{\text{min}}$</th>
<th>$V_{\text{min}}$</th>
<th>$P_r$</th>
<th>$Z$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isothermal Surroundings</td>
<td>0.2</td>
<td>0</td>
<td>0.0</td>
<td>-1.9</td>
<td>10</td>
<td>0.17</td>
<td>2.5</td>
</tr>
<tr>
<td>Non-isothermal Surroundings</td>
<td>0.32</td>
<td>0</td>
<td>-0.14</td>
<td>-0.24</td>
<td>10</td>
<td>1.67</td>
<td>0.5</td>
</tr>
</tbody>
</table>

### TABLE M.8

MAXIMUM TIME INCREMENT POSSIBLE FOR STABILITY

<table>
<thead>
<tr>
<th>SCHEME</th>
<th>Isothermal</th>
<th>Non-Isothermal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit Formulation</td>
<td>0.19</td>
<td>0.019</td>
</tr>
<tr>
<td>Implicit Formulation</td>
<td>stable</td>
<td>unstable</td>
</tr>
<tr>
<td>Semi-implicit Formulation</td>
<td>0.84</td>
<td>unstable</td>
</tr>
<tr>
<td>Semi-explicit Formulation</td>
<td>0.25</td>
<td>unstable</td>
</tr>
</tbody>
</table>
for this latter case. Therefore, these formulation, while they may be feasible for the case of a uniformly heated plate in isothermal surroundings, are not feasible when the plate is placed in non-isothermal surroundings. In order to overcome these severe time restrictions, another formulation must be considered. One scheme which has been used with success in the past is the Implicit Alternating Direction formulation developed by Peaceman and Rachford (35). This method will now be evaluated.

M.3.5.1 Implicit Alternating Direction Formulation

In this method, the finite difference equations are advanced from a time level n to a time level n+1 in two steps. The first step uses the semi-implicit formulation to advance the temperature to the n+1th level. The resulting temperature field is \( T_{i,j}^{n+1} \). Then, using these calculated values in place of \( T_{i,j}^{n} \), the semi-explicit method is used to reiterate the energy equation through the same time step. Then, the resulting field is \( T_{i,j}^{n+1} \).

That is, the first step is

\[
\begin{align*}
T_{i,j}^{n+1} &= T_{i,j}^{n} + \frac{T_{i,j}^{n} - T_{i-1,j}^{n}}{\Delta t} + W_{i,j} \frac{T_{i,j}^{n} - T_{i-1,j}^{n}}{\Delta Z} + V_{i,j} \frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta Y} \\
&= \frac{T_{i,j}^{n+1} - 2T_{i,j}^{n} + T_{i,j+1}^{n+1}}{\Delta Y} - W_{i,j} \frac{\partial T}{\partial Z} \quad \text{M.3.21}
\end{align*}
\]

Then, the second step is

\[
\begin{align*}
T_{i,j}^{n+1} &= T_{i,j}^{n} + \frac{T_{i,j}^{n} - T_{i,j-1}^{n}}{\Delta t} + W_{i,j} \frac{T_{i,j}^{n} - T_{i,j-1}^{n}}{\Delta Z} + V_{i,j} \frac{T_{i,j+1}^{n+1} - T_{i,j}^{n+1}}{\Delta Y} \\
&= \frac{T_{i,j}^{n+1} - 2T_{i,j}^{n} + T_{i,j+1}^{n+1}}{\Delta Y} - W_{i,j} \frac{\partial T}{\partial Z} \quad \text{M.3.22}
\end{align*}
\]
Defining the amplification factor for the first step as \( f^n \) and for the second step as \( f \), it can be shown that by a von Neumann analysis

\[
\left| f^n \right| = \left| \frac{1 - W_{1,j}^n \frac{\Delta t}{\Delta Z}(1 - e^{-i\alpha \Delta Z})}{1 + V_{1,j}^n \frac{\Delta t}{\Delta Y}(e^{i\beta \Delta Y} - 1) - 2\frac{\Delta t}{\Delta Y}(\cos \beta \Delta Y - 1)} \right| 
\]

\[
\left| f \right| = \left| \frac{1 - W_{1,j}^n \frac{\Delta t}{\Delta Z}(e^{i\beta \Delta Y} - 1) - 2\frac{\Delta t}{\Delta Y}(\cos \beta \Delta Y - 1)}{1 + W_{1,j}^n \frac{\Delta t}{\Delta Z}(1 - e^{-i\alpha \Delta Z})} \right| 
\]

For this two step procedure, the amplification error must be such that \( \left| f^n \right| \leq 1 \) for stability. Table M.9 lists the criteria necessary for stability for the various possible conditions.

Similarly, it can be shown that for the momentum equation the amplification factors for the two step process are

\[
\left| f^0 \right| = \left| \frac{1 - W_{1,j}^n \frac{\Delta t}{\Delta Z}(1 - e^{-i\alpha \Delta Z})}{1 + V_{1,j}^n \frac{\Delta t}{\Delta Y}(e^{i\beta \Delta Y} - 1) - 2\frac{\Delta t}{\Delta Y}(\cos \beta \Delta Y - 1)} \right| 
\]

\[
\left| f^1 \right| = \left| \frac{1 - W_{1,j}^n \frac{\Delta t}{\Delta Y}(e^{i\beta \Delta Y} - 1) - 2\frac{\Delta t}{\Delta Y}(\cos \beta \Delta Y - 1)}{1 + W_{1,j}^n \frac{\Delta t}{\Delta Z}(1 - e^{-i\alpha \Delta Z})} \right| 
\]

and that the criteria for stability is

\[
\left| f \right| \leq 1
\]
# TABLE M.9

**STABILITY CRITERIA FOR THE IAD FORM OF THE ENERGY AND MOMENTUM EQUATIONS**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Energy Equation</th>
<th>Momentum Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{1,j} &gt; 0 ) ( v_{1,j} &lt; 0 ) ( w_{1,j} &gt; 0 ) ( v_{1,j} &gt; 0 ) ( w_{1,j} &lt; 0 ) ( v_{1,j} &lt; 0 ) ( w_{1,j} &lt; 0 ) ( v_{1,j} &gt; 0 )</td>
<td>[ \frac{2}{\Delta y} &gt; \nu ] Stable ( \frac{\nu + \frac{\partial}{\partial y} (\frac{\partial w}{\partial y})}{\Delta y} \geq 1 + \frac{\Delta t}{\Delta y^2} ]</td>
<td>[ \frac{2}{\Delta y} &gt; \nu ] Stable ( \frac{\nu + \frac{\partial}{\partial y} (\frac{\partial w}{\partial y})}{\Delta y} \geq 1 + \frac{\Delta t}{\Delta y^2} ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{\nu + \frac{\partial}{\partial y} (\frac{\partial w}{\partial y})}{\Delta y} \geq 1 ]</td>
<td>[ \frac{\nu + \frac{\partial}{\partial y} (\frac{\partial w}{\partial y})}{\Delta y} \geq 1 ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{\nu}{\Delta y} \geq \nu + \frac{\partial}{\partial y} ]</td>
<td>[ \frac{\nu}{\Delta y} \geq \nu + \frac{\partial}{\partial y} ]</td>
</tr>
<tr>
<td></td>
<td>[ \frac{\nu}{\Delta y} \leq \frac{\nu}{\Delta y} \leq \frac{\nu}{\Delta y} \leq \frac{\nu}{\Delta y} ]</td>
<td>[ \frac{\nu}{\Delta y} \leq \frac{\nu}{\Delta y} \leq \frac{\nu}{\Delta y} \leq \frac{\nu}{\Delta y} ]</td>
</tr>
</tbody>
</table>

**NOTE:**

\( w = |w_{1,j}| \)

\( v = |v_{1,j}| \)
APPENDIX N
DATA REDUCTION PROCEDURE

In this Appendix, a step-wise procedure is outlined for the reduction of the data which was used in this program. A sample calculation is not provided for two reasons. Firstly, references to appendices which contain sample calculations for various parts of the reduction procedure have already been presented. Redundant calculations would serve no further purpose. Secondly, it is believed that the step-wise procedures outlined below are sufficient to trace the data reduction procedure so that the magnitude of errors and the reliability of the data can be adequately judged.

N.1 Thermocouple Location

It was necessary to locate the position of the thermocouples since accurate placement was not possible because of the wire fraility. This was achieved by using photographic techniques. The following steps were part of the procedure:

1. Pictures of the thermocouple positions were taken.
   The picture magnification was known. These pictures (usually about 3 to a set) were made into a composite photograph.

2. The position of each thermocouple was established by calculating its position relative to a known reference point on the container. Optical effects were taken into account for each case.

3. At least one other picture was used to confirm the thermocouple placement and to check the accuracy of the technique. It was found that the position differed by less than 1/16 in. in the horizontal direction and 1/8 in. in the vertical direction.

4. These positions were used for all of the tests at a given height. One check was made after four tests had been run as to the additional error which may be added due to the movement of the thermocouples between tests. The
difference from the initial values was found to be insignificant.

N.2 Temperature Reduction
1. Readings of time and chart reading were taken from the test charts.

2. These were recorded on computer cards. Data recorded included the test number, time, thermocouple number, and chart reading.

3. A computer program was used to reduce the data to temperature vs. time readings for each thermocouple. The output gave the thermocouple position, temperature and time.

4. This data was plotted for each thermocouple (Fig. 5.1)

5. A cross-plot was then made at selected times giving the temperature as a function of position for a given time. (Figures 5.2-5.4)

6. Heat losses for the test were calculated as shown in Appendix F. From this, the modified Rayleigh Number was calculated. The mean fluid temperature was used for determining the fluid properties at each time.

7. Using the modified Rayleigh number obtained in step 6, dimensionless temperatures and positions were calculated. These were plotted in Figure 6.9.

N.3 Velocity Reduction
1. A transparent reproduction of a 160x220 mm. graph was made using the "Xerox" process.

2. This transparency was then placed over the streak photograph in such a way that the top of the graph coincided with the liquid surface and one vertical side with the container wall. The upper left hand corner was designated as (0,0).

3. The coordinates of the top of the streak and the bottom of the streak were recorded. This was done for approximately 500 streaks per picture.
4. These coordinates were then transferred to a computer program along with the picture magnification and exposure time where the velocities and coordinates were determined. The results of this analysis is given in Appendix J for two times. Errors in velocity were judged to be about ± 30 per cent.

5. The vertical velocities were grouped into heights of 0.1, e.g., all measurements between 0.90 ≤ z/L ≤ 0.80 were deemed to be at a height of z/L = 0.85. These were plotted as a function of width. An example is given in Figure N.1. A curve representing a "best fit" was then drawn.

6. The results of these curves were then cross-plotted as a function of height for increments of y/D = 0.1.

7. From the equation

\[ \psi = -\frac{1}{L} \frac{\partial V}{\partial (z/L)} \]  

it is seen that the stream function at a given y/D can be obtained by integrating the area under the curves plotted in step 6.

8. The streamfunction, \( \psi \), as a function of plate height was then plotted for each y/D.

9. Contour maps of the streamlines were then determined from the graphs of step 8. For example, if a contour of \( \psi = 0.1 \) were required, the (y/D, z/L) coordinates from each graph of step 8 would be plotted. These points were then joined to form the contours as shown in Figures 4.9 and 4.10.

10. The maximum streamlines in Figures 6.10, 6.11 were obtained by picking the maximum value of the contour plot at any height.
Test W-2-H1
120 sec.

$z/L = 0.885$

FIGURE N.1  HORIZONTAL VELOCITY VS. WIDTH
APPENDIX O

BIOGRAPHICAL NOTE

The author, Edward Stephen Matulevicius, was born in Montreal, Quebec, on September 4, 1942. He received his elementary and high school education in Verdun, Quebec, attending St. Thomas More Elementary School and St. Willibrord High School. He was graduated from St. Willibrord High School in June, 1959.

He entered McGill University in Montreal in September, 1959, and was awarded a B. Eng. degree from the Department of Chemical Engineering in June, 1964. Following this, he entered the Department of Chemical Engineering at Massachusetts Institute of Technology in September, 1964. He was awarded a S.M. degree in Chemical Engineering in February, 1966, after which he continued in the Graduate School in the doctoral program of the Department of Chemical Engineering at M.I.T. While attending the Graduate School, the author was a half-time research assistant, a full-time research assistant, and a teaching assistant.

The author is a member of the honorary societies Tau Beta Pi, Phi Epsilon Alpha, and Sigma Xi. He is also a member of the professional societies American Institute of Chemical Engineers, and the Canadian Society for Chemical Engineering.

The author has accepted full-time employment with the Esso Research and Engineering Company in Linden, New Jersey.