Three-index symmetric matter representations of SU(2) in F-theory from non-Tate form Weierstrass models

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ABSTRACT: We give an explicit construction of a class of F-theory models with matter in the three-index symmetric (**4**) representation of SU(2). This matter is realized at codimension two loci in the F-theory base where the divisor carrying the gauge group is singular; the associated Weierstrass model does not have the form associated with a generic SU(2) Tate model. For 6D theories, the matter is localized at a triple point singularity of arithmetic genus $g = 3$ in the curve supporting the SU(2) group. This is the first explicit realization of matter in F-theory in a representation corresponding to a genus contribution greater than one. The construction is realized by "unHiggsing" a model with a U(1) gauge factor under which there is matter with charge $q = 3$. The resulting SU(2) models can be further unHiggsed to realize non-Abelian $G_2 \times$ SU(2) models with more conventional matter content or SU(2)^3 models with trifundamental matter. The U(1) models used as the basis for this construction do not seem to have a Weierstrass realization in the general form found by Morrison-Park, suggesting that a generalization of that form may be needed to incorporate models with arbitrary matter representations and gauge groups localized on singular divisors.

Contents

1 Introduction

F-theory [1–3] provides a very general string-theoretic approach to constructing low-energy theories of supergravity coupled to gauge fields and matter. In particular, F-theory extends the approach of type IIB string theory to include non-perturbative seven-brane configurations that produce a rich variety of structures for low-energy physics. F-theory uses the axiodilaton of the IIB theory to encode an elliptic fibration over the compactification space.

A beautiful mathematical correspondence originally elucidated by Kodaira [4] relates singularities in the elliptic fibration over (complex) codimension one subspaces (divisors) in the compactification space to Dynkin diagrams, encoding the physical non-Abelian gauge content of the theory in geometric structure. This correspondence is well-understood, and has been used to study low-energy theories with exceptional gauge groups (E_6, E_7, E_8) and non-simply laced groups (Sp(N), F_4, G_2) in addition to the usual groups such as SU(N) that have standard realizations on D-branes in perturbative string theory. A similar correspondence holds between codimension two singularities in elliptic fibrations and the representation content of matter in F-theory models, but this correspondence is at present only partially understood despite much recent work in the F-theory community on the

explicit resolution of codimension two singularities [5–12]. In this paper we explore some explicit examples of F-theory models with novel matter content as a step towards a more general understanding of the codimension two generalization of the Kodaira story.

Some hints towards a general structure underlying the proposed correspondence between codimension two singularities in elliptic fibrations and representation theory of semi-simple Lie groups were given in [7, 13]. For any representation $\mathbf{R}$ of a Lie group G, there is a number g_R given by

$$g_R = \frac{\lambda}{12} \left(2\lambda C_{\mathbf{R}} + B_{\mathbf{R}} - A_{\mathbf{R}}\right), \tag{1.1}$$

where $A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}$ are numerical coefficients associated with the representation $\mathbf{R}$ through

$$\mathrm{Tr}\,_{\mathbf{R}} F^2 = A_{\mathbf{R}} \mathrm{Tr}\, F^2 \tag{1.2}$$

$$\mathrm{Tr}\,_{\mathbf{R}} F^4 = B_{\mathbf{R}} \mathrm{Tr}\, F^4 + C_{\mathbf{R}} (\mathrm{Tr}\, F^2)^2, \tag{1.3}$$

and λ is a group-dependent constant, with $\lambda = 1$ for $\mathrm{SU}(N)$. Here Tr refers to the trace in the fundamental representation, while $\mathrm{Tr}_{\mathbf{R}}$ corresponds to the trace in the representation $\mathbf{R}$. By manipulation of the anomaly cancellation formulae of 6D supergravity, it was suggested in [13] that $g_{\mathbf{R}}$ should have a natural geometric interpretation as a genus contribution to the divisor (curve) supporting the gauge group. Previous analyses of specific cases have supported this hypothesis. For $\mathrm{SU}(N)$, k-index antisymmetric representations all have $g_{\mathbf{R}} = 0$, and these are precisely the representations that can be realized on a smooth genus 0 curve in a 6D F-theory model. The adjoint and (two-index) symmetric matter representations of $\mathrm{SU}(N)$ both have $g_{\mathbf{R}} = 1$. In 6D models where G is realized on a smooth curve of genus g, there are g matter fields in the adjoint representation. We expect that for all representations with $g_{\mathbf{R}} > 0$ other than the adjoint, $g_{\mathbf{R}}$ represents the *arithmetic genus* contribution from a singularity p on the divisor C that supports the group G, where p supports matter in the representation $\mathbf{R}$.

As discussed in general terms in [7, 14], the two-index symmetric representation of $\mathrm{SU}(N)$ is expected to be realized on ordinary double point singularities of the singular curve C carrying the group. Recently, two explicit constructions of classes of models containing matter in the two-index symmetric representation (**6**) of $\mathrm{SU}(3)$ were given [15, 16]. Direct construction of Weierstrass models with $g_{\mathbf{R}} > 0$ matter representations other than the adjoint appears to be quite subtle, as the algebraic structure of e.g. $\mathrm{SU}(N)$ models with such matter requires an intricate cancellation in the vanishing of the discriminant to high order on C that relies on the singular nature of C and the consequent non-UFD (Universal Factorization Domain) structure of the ring of functions on C. Such models thus cannot be realized as Weierstrass forms from generic constructions in the standard Tate approach used in e.g. [6, 17], or using a naive power series analysis using generic factorization properties of functions in C as in [7]. Lacking a general theory of Weierstrass forms for models with such exotic matter representations, explicit constructions of symmetric matter representations have so far used indirect approaches. In [15], the symmetric representation of $\mathrm{SU}(3)$ was constructed by identifying models with Abelian groups $\mathrm{U}(1)\times\mathrm{U}(1)$ and appropriate charges that lift to the symmetric representation of $\mathrm{SU}(3)$ after unHiggsing. This is the general approach we use in this paper. In [16], the symmetric representation of $\mathrm{SU}(3)$ was identified

by Higgsing a theory with a larger (SU(6)) group so that the symmetric matter naturally appeared after the Higgsing. This gives a complementary perspective on the construction of such models that we also incorporate into the analysis of this paper. A more direct approach to constructing Weierstrass models for these kinds of situations where the ring of functions on the singular divisor C is not a UFD will be presented elsewhere [18].

In this paper we focus on the three-index symmetric (**4**) representation of SU(2), associated with the Young diagram ☐☐☐. We realize this representation by unHiggsing Abelian models constructed in [19] with U(1) gauge group and matter of charge $q = 3$. For SU(2), there is no quartic Casimir, so the group coefficient $B_{\mathbf{R}}$ vanishes, and we have $A_{\mathbf{4}} = 10$, $C_{\mathbf{4}} = 41$ for the **4** representation. These coefficients are readily verified by using a field strength F proportional to the generator T_3, which takes the form $\mathrm{diag}(1/2, -1/2)$ in the fundamental representation and $\mathrm{diag}(3/2, 1/2, -1/2, -3/2)$ in the three-index symmetric representation **4**. It follows from (1.1) that the genus contribution from a full hypermultiplet in the **4** representation of SU(2) is $g_{\mathbf{4}} = 6$. Because this representation is self-conjugate (pseudoreal), we can have matter in a half-hypermultiplet, giving a genus contribution $\frac{1}{2}g_{\mathbf{4}} = 3$. From the point of view of 6D anomaly cancellation, the contribution of a half-hypermultiplet in the **4** representation combined with 7 hypermultiplets in the fundamental **2** representation are *anomaly equivalent* [7, 20] to the contribution of 3 hypermultiplets in the adjoint **3** representation along with 7 uncharged hypermultiplets. We thus expect that we may find half-hypermultiplets of the **4** representation of SU(2) at arithmetic genus 3 singularities in a curve C supporting the gauge group in a general complex surface base B. We see that this works out as expected in the explicit constructions we present here based on unHiggsing the U(1) models in [19]. As in the previous explicit constructions of symmetric (**6**) matter representations of SU(3), the models that we find have a non-Tate realization of the gauge group SU(2) in the Weierstrass model. This matches with the general expectations of the analysis of [16] and seems to be related to another curious feature of the construction shown here, which is that the involved U(1) model of [19] does not have the general form considered in [21]. We discuss these connections further in the conclusions section at the end of the paper.

The structure of this paper is as follows. In section 2 we review the U(1) models of [19] with charge $q = 3$ matter. In section 3, we unHiggs these U(1) models to SU(2) models with matter in the **4** representation. In section 4, we consider further unHiggsing to non-Abelian gauge groups with other matter content, and section 5 contains some concluding remarks.

2 Abelian F-theory models with matter of charge $q = 3$

In this section, we review a construction of a family of F-theory compactifications with gauge group $G = \mathrm{U}(1)$ and matter with U(1) charges $q = 1, 2, 3$. These compactifications were first studied in [19], to which we refer for further details. In section 2.1, we briefly recall the construction of the elliptically fibered Calabi-Yau manifolds, denoted by X, specifying these compactifications. We then summarize the matter spectrum of the resulting effective theories in section 2.2. We conclude this discussion in section 2.3 by presenting explicit models with base $B = \mathbb{P}^2$.

2.1 Geometry of the elliptic fibration

We consider elliptically fibered Calabi-Yau manifolds $\pi : X \to B$ with base manifold B. The elliptic fiber $\mathcal{E} = \pi^{-1}(p)$ over a generic point $p \in B$ is given by the Calabi-Yau hypersurface in the del Pezzo surface dP_1, which is the blow-up of $\mathbb{P}^2$ at a point; this space is also known as the first Hirzebruch surface $\mathbb{F}_1$. F-theory compactifications on such Calabi-Yau manifolds X were first analyzed in detail in [19], whose notation and conventions we follow. In summary, the resulting low-energy effective theories have $G = \mathrm{U}(1)$ gauge group and charged matter with $\mathrm{U}(1)$ charges $q = 1, 2, 3$.

The Calabi-Yau manifold X is constructed as the hypersurface

$$p := s_1 u^3 e^2 + s_2 u^2 v e^2 + s_3 u v^2 e^2 + s_4 v^3 e^2 + s_5 u^2 w e + s_6 u v w e + s_7 v^2 w e + s_8 u w^2 + s_9 v w^2 = 0 \,, \tag{2.1}$$

in the ambient space of a dP_1 fibration over B. Here the coefficients s_i are sections of line bundles on the base B, to be specified momentarily, and the variables $[u\!:\!v\!:\!w\!:\!e]$ are the homogeneous coordinates on dP_1, which is the ambient space of the generic elliptic fiber $\mathcal{E}$; the weights of the coordinates are $(1, 1, 1, 0)$ and $(0, 0, 1, 1)$ with respect to two $\mathbb{C}^*$ actions on dP_1. The blow down map from dP_1 to $\mathbb{P}^2$ is given by $[u\!:\!v\!:\!w\!:\!e] \mapsto [ue\!:\!ve\!:\!w]$ so that e vanishes on the exceptional divisor E of dP_1. The del Pezzo surface dP_1 is toric; it is described by a reflexive polyhedron that we depict, along with its dual polyhedron, in figure 1.

The Calabi-Yau condition for X implies that the hypersurface constraint (2.1) has to be a well-defined section of the anti-canonical bundle of the ambient space given by the dP_1 fibration over B. This requires that the coordinates $[u\!:\!v\!:\!w\!:\!e]$ and the coefficients s_i are sections of the following line bundles:

Section	Line bundle	Section	Line bundle
u	$\mathcal{O}(H - E + \mathcal{S}_9 + K_B)$	s_1	$\mathcal{O}(-3K_B - \mathcal{S}_7 - \mathcal{S}_9)$
v	$\mathcal{O}(H - E + \mathcal{S}_9 - \mathcal{S}_7)$	s_2	$\mathcal{O}(-2K_B - \mathcal{S}_9)$
w	$\mathcal{O}(H)$	s_3	$\mathcal{O}(-K_B + \mathcal{S}_7 - \mathcal{S}_9)$
e_1	$\mathcal{O}(E)$	s_4	$\mathcal{O}(2\mathcal{S}_7 - \mathcal{S}_9)$
		s_5	$\mathcal{O}(-2K_B - \mathcal{S}_7)$
		s_6	$\mathcal{O}(-K_B)$
		s_7	$\mathcal{O}(\mathcal{S}_7)$
		s_8	$\mathcal{O}(-K_B + \mathcal{S}_9 - \mathcal{S}_7)$
		s_9	$\mathcal{O}(\mathcal{S}_9)$

$$\tag{2.2}$$

Here we denote the line bundle associated to a divisor D by $\mathcal{O}(D)$, $-K_B$ is the anti-canonical divisor of B and the classes H, E are the pullback of the hyperplane on $\mathbb{P}^2$ and the exceptional divisor on the dP_1-fiber, respectively. We note that the two divisor classes $\mathcal{S}_7$ and $\mathcal{S}_9$, which are the classes of the coefficients s_7 and s_9, are free discrete parameters determining the topology of X. When $\mathcal{S}_7 = \mathcal{S}_9 = -K_B$, the dP_1 fibration over the base B is trivial and the s_i are all sections of the line bundle $\mathcal{O}(-K_B)$. Other values of $\mathcal{S}_7$ and $\mathcal{S}_9$ parametrize a two-parameter family of twisted dP_1 bundles over B.
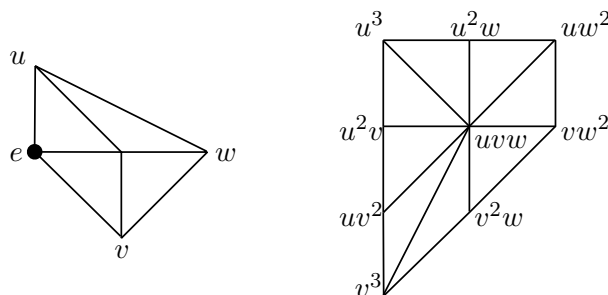
Figure 1. Polyhedron for dP_1 and its dual with corresponding monomials (in the patch $e = 1$). The toric zero section $\hat{c}_0$ is indicated by the dot.

The Weierstrass model of (2.1) and a Tate form for it are readily computed for example using Nagell's algorithm [19]. As the explicit expressions for the Weierstrass coefficients f, g, the discriminant $\Delta = 4f^3 + 27g^2$ as well as the Tate coefficients are rather lengthy, we relegate them to (A.1) and (A.3) in appendix A. The computation of Δ reveals that X generically does not exhibit any codimension one singularities, which implies the absence of a non-Abelian gauge group in the F-theory effective theory.[1]

The elliptic fibration of X has two sections, one of which being the zero section $\hat{c}_0$ and the second one, denoted by $\hat{c}_1$, generating its rank one Mordell-Weil group (MW-group) of rational sections. Consequently, the gauge group G of F-theory on X is

$$G = \mathrm{U}(1). \tag{2.3}$$

More explicitly, the two sections of X are given by the intersection of $e = 0$ with (2.1), which we choose as the zero section $\hat{c}_0$, and by the second point of intersection of the line $t_P := s_8 u + s_9 v = 0$ with X, besides $e = 0$ where the intersection is tangent. Thus, the MW-group of X is not realized torically, i.e. the extra section is not simply the intersection of a toric divisor of dP_1 with the elliptic curve. In terms of the homogeneous coordinates on the dP_1-fiber, the sections read

$$\hat{c}_0 = X \cap \{e = 0\}: \quad [-s_9 : s_8 : 1 : 0], \tag{2.4}$$
$$\hat{c}_1 = X \cap \{t_P = 0\}: \quad [-s_9 : s_8 : s_1 s_9^3 - s_2 s_9^2 s_8 + s_3 s_9 s_8^2 - s_4 s_8^3 : s_7 s_8^2 - s_6 s_9 s_8 + s_5 s_9^2].$$

The Weierstrass coordinates of the section $\hat{c}_1$ are given in (A.2) in appendix A, while $\hat{c}_0$ maps to the zero section in Weierstrass form. The Shioda map of the section $\hat{c}_1$ is computed to be [19]

$$\sigma(\hat{c}_1) = C_1 - C_0 + 3K_B + \mathcal{S}_7 - 2\mathcal{S}_9, \tag{2.5}$$

where C_1, C_0 denote the divisor classes of the rational sections $\hat{c}_1$ and $\hat{c}_0$. The Kaluza-Klein reduction of the M-theory three-form C_3 along the $(1,1)$-form associated to $\sigma(\hat{c}_1)$ yields the U(1) gauge field in the effective theory [3, 24]. The (negative of the) height pairing is

$$b_{11} = 2(-3K_B + 2\mathcal{S}_9 - \mathcal{S}_7), \tag{2.6}$$

which enters a Green-Schwarz counterterm in the F-theory effective action [21, 24].

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[1]We do not consider the non-Abelian gauge groups that would be imposed by choosing bases B with non-Higgsable clusters [22, 23]. However, the analysis can be extended straightforwardly.

We emphasize here that the locus in B where the coordinates (2.4) of the two sections agree is given by

$$z_1 := s_7 s_8^2 - s_6 s_8 s_9 + s_5 s_9^2 = 0 \,. \tag{2.7}$$

At points where $z_1 = 0$, a rescaling under the second $\mathbb{C}^*$ makes the two sections in (2.4) equivalent. Note that z_1 is precisely the z-coordinate of $\hat{c}_1$ in Weierstrass form, cf. (A.2). Thus, the homology class of the divisor in B along which $\hat{c}_0 \cong \hat{c}_1$ is $[z_1] = -2K_B + 2\mathcal{S}_9 - \mathcal{S}_7$ as follows from (2.2).

Furthermore, we observe that the Calabi-Yau constraint (2.1) is invariant under the $\mathbb{Z}_2$-symmetry $u \leftrightarrow v$ given that we also exchange $s_1 \leftrightarrow s_4$, $s_2 \leftrightarrow s_3$, $s_5 \leftrightarrow s_7$ and $s_8 \leftrightarrow s_9$. According to (2.2), this amounts to exchanging

$$\mathcal{S}_7 \mapsto \mathcal{S}_7' := -2K_B - \mathcal{S}_7 \,, \qquad \mathcal{S}_9 \mapsto \mathcal{S}_9' := -K_B + \mathcal{S}_9 - \mathcal{S}_7 \,. \tag{2.8}$$

This symmetry relates Calabi-Yau manifolds X with the same base B, but different values for $\mathcal{S}_7$ and $\mathcal{S}_9$. Indeed, we can check that the key geometric properties of X are invariant under the symmetry $u \leftrightarrow v$. In particular, this implies that the effective theories of F-theory on X that are related by (2.8) have to be identical.

Relation to $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$-elliptic fibrations. Before delving into the analysis of codimension two singularities of X, we elaborate on the relation to elliptic fibrations with generic elliptic fiber in $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$ considered in [21]. We will see that elliptic fibrations with generic elliptic fiber in dP_1 that satisfy the additional condition $[s_8] = 0$ or $[s_9] = 0$ are equivalent to those with elliptic fiber in $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$. Indeed, we first note that a general elliptic fibration X described by (2.1) has to have non-vanishing and general coefficients s_i. This necessitates that all divisor classes in (2.2) are effective, i.e. $[s_i] \geq 0$. Second, we see that a model with constant s_8 (or s_9) allows performing the variable transformation $u = u' - v s_9 / s_8$ $(v = v' - u s_8 / s_9)$ so that we effectively achieve $s_9 \equiv 0$ $(s_8 \equiv 0)$.[2] As is clear from the dual polyhedron in figure 1, removing s_9 (s_8) amounts to blowing up dP_1 at $u = e = 0$ $(v = e = 0)$, i.e. adding the vertex with coordinates $(-2, 1)$ (or $(-1, -1)$) to the polyhedron of dP_1. The resulting polyhedron is precisely the one of $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$ and the Calabi-Yau constraint (2.1) can be readily written in the form of [21], as claimed. We will also see this equivalence on the level of the matter spectrum in section 2.2. Note however that, as we discuss in further detail in later sections, in the generic case where $s_8, s_9 \neq 0$, this class of U(1) models cannot be written in the Morrison-Park form from [21].

More extremely, we can relax the effectiveness constraint $[s_8] \geq 0$ or $[s_9] \geq 0$ completely. In both cases, the model defined by (2.1) still defines a sensible elliptically fibered Calabi-Yau manifold. However, there will be a codimension one singularity of Kodaira type I_2 at $s_9 = 0$ or $s_8 = 0$, respectively, as analyzed in [19, 25]. It can be resolved globally by the blow-ups in dP_1 at $v = e = 0$ or $u = e = 0$, respectively, resulting again in the new ambient space $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$. Thus, we see that the elliptic fibrations with their generic elliptic fibers in $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$ can be thought of as arising from the Calabi-Yau manifold X via the specialization $s_8 = 0$ or $s_9 = 0$, respectively, in (2.1).

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[2]The symmetry $u \leftrightarrow v$ exchanges $s_8 \to s_9$ and the two case of constant s_8 or s_9.

Rep	Multiplicity	Locus
$\mathbf{1}_3$	$x_{\mathbf{1}_3} = \mathcal{S}_9 \cdot (-K_B + \mathcal{S}_9 - \mathcal{S}_7)$	$V(I_{(3)}) := \{s_8 = s_9 = 0\}$
$\mathbf{1}_2$	$x_{\mathbf{1}_2} = 6K_B^2 - K_B \cdot (4\mathcal{S}_9 - 5\mathcal{S}_7)$ $+ \mathcal{S}_7^2 + 2\mathcal{S}_7\mathcal{S}_9 - 2\mathcal{S}_9^2$	$V(I_{(2)}) := \{s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3$ $= s_7 s_8^2 + s_5 s_9^2 - s_6 s_8 s_9 = 0\} \backslash\ V(I_{(3)})$
$\mathbf{1}_1$	$x_{\mathbf{1}_1} = 12K_B^2 - K_B \cdot (8\mathcal{S}_7 - \mathcal{S}_9)$ $- 4\mathcal{S}_7^2 + \mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_9^2$	$V(I_{(1)}) := \{(2.9)\} \backslash\ (V(I_{(2)}) \cup V(I_{(3)}))$

Table 1. Charged matter under U(1) and codimension two fibers of X.

2.2 The matter spectrum

The matter spectrum of the F-theory compactification on X is derived by analyzing the singularities of the elliptic fibration that arise over codimension two loci in the base. Since the Calabi-Yau manifold X has a non-trivial MW-group generated by $\hat{c}_1$, it automatically has Kodaira fibers of type I_2 at the codimension two loci in B along which

$$y_1 = fz_1^4 + 3x_1^2 = 0 \tag{2.9}$$

is satisfied [21]. Here f and g enter the Weierstrass form of X and $[y_1 : x_1 : z_1]$ are the Weierstrass coordinates of $\hat{c}_1$ given in (A.1) and (A.2), respectively. The matter located at (2.9) is automatically charged under the U(1) gauge field corresponding to $\hat{c}_1$.

The locus (2.9) is reducible with three irreducible components, as e.g. shown by a primary decomposition (see [26, 27] for more details on the necessary technical tools), corresponding to three different matter representations. The full matter spectrum derived in [19] is given in table 1, which includes the U(1)-charges, the multiplicities $x_{\mathbf{R}}$ of 6D charged hyper multiplets in the representation $\mathbf{R}$ and the codimension two loci supporting the respective fibers. Here, we use the notation $V(I)$ for the vanishing set of an ideal I.

The matter spectrum of X is completed by the number of neutral hyper multiplets H_{neut}. It has been computed in [19] to be

$$H_{\text{neutral}} = 13 + 11K_B^2 + K_B \cdot (3\mathcal{S}_7 + 4\mathcal{S}_9) + 3\mathcal{S}_7^2 - 2\mathcal{S}_7 \cdot \mathcal{S}_9 + 2\mathcal{S}_9^2. \tag{2.10}$$

Employing this, together with the charged spectrum in table 1, anomaly-freedom of the 6D U(1) SUGRA theory is readily checked, following the general prescription of [28, 29]. We note that the matter spectra in table 1 and in (2.10) are invariant under the $\mathbb{Z}_2$-symmetry (2.8) of X.

We stress that one main difference of the matter spectrum in table 1 and the one of $\text{Bl}_1 \mathbb{P}^2(1,1,2)$-elliptic fibrations studied in [21] is the presence of matter fields with $q = 3$. In turn, it is expected that models without these matter fields should be already described by the models in [21]. Indeed, employing the discussion at the end of the previous section, Calabi-Yau manifolds X with $x_{\mathbf{1}_3} = 0$, which requires either $[s_8] = 0$ or $[s_9] = 0$, are geometrically completely equivalent to $\text{Bl}_1 \mathbb{P}^2(1,1,2)$-elliptic fibrations and so are the effective theories, as expected.

2.3 Models over $B = \mathbb{P}^2$

We conclude the discussion of F-theory compactified on the Calabi-Yau manifold X by considering the concrete examples with base $B = \mathbb{P}^2$. In this case we have $-K_B = \mathcal{O}_{\mathbb{P}^2}(3)$ and $\mathcal{S}_7$ and $\mathcal{S}_9$ can be associated with non-negative integers since the second homology of $\mathbb{P}^2$ is one-dimensional and generated by the hyperplane H_B of $\mathbb{P}^2$. We can then solve the conditions imposed by effectiveness of the divisor classes $[s_i] \geq 0$, $i = 1, \ldots, 9$, given in (2.2), as in [26]. This yields the allowed region for the pair $(\mathcal{S}_7, \mathcal{S}_9)$ shown in figure 2. We immediately notice that this region is precisely given by the toric polytope of dP_1 rescaled by 3, which is precisely the anti-canonical class of $\mathbb{P}^2$ in units of H_B.

Next we determine the matter spectrum of X for the concrete base $\mathbb{P}^2$ employing table 1. We recall the $\mathbb{Z}_2$-symmetry (2.8) relating Calabi-Yau manifolds X with different values for $(\mathcal{S}_7, \mathcal{S}_9)$. In the allowed region in figure 2, this symmetry exchanges points on the lines $\mathcal{S}_9 = x$ and $\mathcal{S}_9 = \mathcal{S}_7 - 3 + x$ for $x = 0, \ldots, 6$. As the effective theories of F-theory on X are related accordingly, as discussed before, and as $\mathcal{S}_7 = 3$ is the fixed line under (2.8), we only have to list models and corresponding spectra for $\mathcal{S}_7 \leq 3$. We obtain the following list for the degrees of the sections s_i entering the Calabi-Yau constraint (2.1) and of the matter multiplicities $x_{\mathbf{R}}$:

$(\mathcal{S}_7, \mathcal{S}_9)$	$[s_1]$	$[s_2]$	$[s_3]$	$[s_4]$	$[s_5]$	$[s_6]$	$[s_8]$	$(x_{\mathbf{1}_3}, x_{\mathbf{1}_2}, x_{\mathbf{1}_1})$
$(0,0)$	9	6	3	0	6	3	3	$(0, 54, 108)$
$(1,0)$	8	6	4	2	5	3	2	$(0, 40, 128)$
$(2,0)$	7	6	5	4	4	3	1	$(0, 28, 140)$
$(3,0)$	6	6	6	6	3	3	0	$(0, 18, 144)$
$(1,1)$	7	5	3	1	5	3	3	$(3, 52, 125)$
$(2,1)$	6	5	4	3	4	3	2	$(2, 42, 138)$
$(3,1)$	5	5	5	5	3	3	1	$(1, 34, 143)$
$(1,2)$	6	4	2	0	5	3	4	$(8, 60, 120)$
$(2,2)$	5	4	3	2	4	3	3	$(6, 52, 134)$
$(3,2)$	4	4	4	4	3	3	2	$(4, 46, 140)$
$(2,3)$	4	3	2	1	4	3	4	$(12, 58, 128)$
$(3,3)$	3	3	3	3	3	3	3	$(9, 54, 135)$
$(2,4)$	3	2	1	0	4	3	5	$(20, 60, 120)$
$(3,4)$	2	2	2	2	3	3	4	$(16, 58, 128)$
$(3,5)$	1	1	1	4	3	3	5	$(25, 58, 119)$
$(3,6)$	0	0	0	0	3	3	6	$(36, 54, 108)$

$$(2.11)$$

The spectrum of the remaining theories in the allowed region in figure 2 can be obtained by application of the $\mathbb{Z}_2$-symmetry (2.8). We note that all the spectra in (2.11) are different. In particular, the number of matter fields with charge $q = 2$ is always larger than zero, which will be important for the unHiggsing of X discussed next.

We conclude by noting that the four models with $x_{\mathbf{1}_3} = 0$ are precisely four of the possible seven $\mathrm{Bl}_1\mathbb{P}^2(1,1,2)$-elliptic fibrations that can be constructed on $B = \mathbb{P}^2$ and
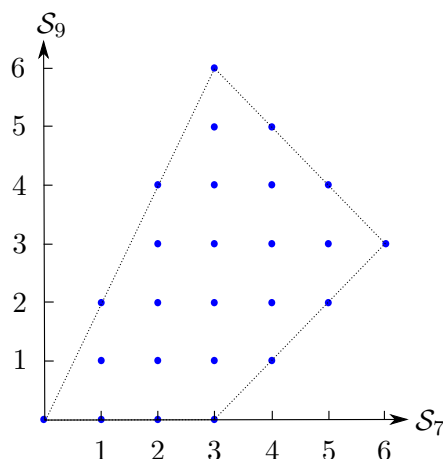
Figure 2. Allowed region for the pair $(\mathcal{S}_7, \mathcal{S}_9)$ specifying X for $B = \mathbb{P}^2$.

without an I_2 singularity at codimension one, i.e. an extra SU(2) gauge group factor. The role of the parameter b in [21, 30] is played by $b \equiv s_5$, which assumes values from $[b] = 3, \ldots, 6$ in the allowed region. In order to obtain the remaining three models with $[b] = [s_5] = 0, 1, 2$, we have to relax effectiveness of the class $[s_8]$. The three missing models are then given at $(\mathcal{S}_7, \mathcal{S}_9) = (4, 0), (5, 0), (6, 0)$.

3 Matter in the three-index symmetric representation 4 of SU(2)

We begin this section by briefly recalling the general geometrical procedure that corresponds to an unHiggsing of a U(1) to a non-Abelian gauge symmetry in F-theory. We will focus on unHiggsings that preserve the rank of the gauge group. General discussions of rank-preserving unHiggsings of U(1)'s in F-theory can be found in [15, 19, 21, 30, 31].

An F-theory compactification with a U(1)^m gauge symmetry is specified by a Calabi-Yau manifold $X_{n+1}^{(m)}$ with a MW-group of rank m. The Abelian gauge symmetry of the theory is unHiggsed to a non-Abelian one by performing a geometric transition from $X_{n+1}^{(m)}$ to a new Calabi-Yau manifold $X_{n+1}^{(0)}$ with a trivial MW-group; the manifold $X_{n+1}^{(0)}$ is obtained by tuning the complex structure of $X_{n+1}^{(m)}$ so that all its rational sections are placed on top of each other. Typically, this process induces codimension one singularities of the elliptic fibration of $X_{n+1}^{(0)}$ that produce a non-Abelian gauge group in the final "unHiggsed" theory. This can be thought of as a transition that takes "horizontal" divisors in the Calabi-Yau manifold associated with sections into "vertical" divisors associated with resolved Kodaira singularities over divisors in the base. For example, it is shown in [21, 30] that a model with a single U(1) gauge group can be unHiggsed to a model with SU(2) or larger non-Abelian gauge group[3] and the general unHiggsings of two or more U(1)'s are studied in [15]. Concrete unHiggsings of toric models with up to three U(1)'s and of general U(1)×U(1) F-theory compactification are discussed in [19] and [15].

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[3]In some cases, particularly when there are additional non-Abelian factors present before the unHiggsing, the unHiggsed model can develop problematic singularities.
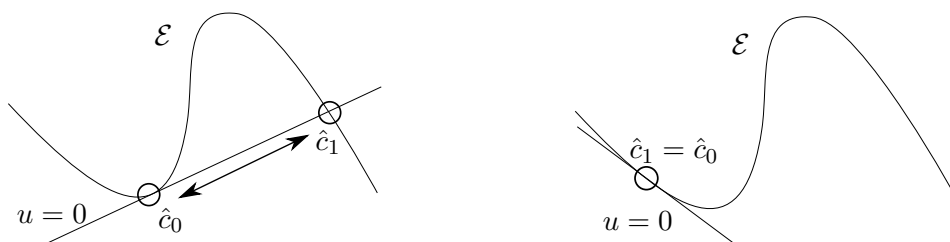
Figure 3. UnHiggsing by tuning the complex structure of X, shown on the left, so that $\hat{c}_0 = \hat{c}_1$ in the generic elliptic fiber $\mathcal{E}$ of X as shown on the right.

In this section, we analyze the unHiggsing of the Abelian F-theory model defined by the Calabi-Yau manifold X in (2.1) that has one U(1). We thus identify $X^{(1)} \equiv X$. This model unHiggses to a non-Abelian theory with $G = $ SU(2) gauge group, similar to the models in [21, 30]. The corresponding geometrical tuning of X to a manifold $X^{(0)} \equiv X^{\text{SU}(2)}$ with trivial MW-group but I_2 singularities at codimension one is discussed in section 3.1. Then, we show that the structure of codimension two singularities in X that is responsible for the presence of matter fields with U(1)-charge $q = 3$ in F-theory yields a novel singularity structure in the unHiggsed geometry $X^{\text{SU}(2)}$: the I_2 singularities corresponding to the SU(2) gauge group occur on a singular divisor $t = 0$ with a triple point singularity. Most notably, it seems that the triple point singularity can not be deformed without affecting the I_2 singularity of the elliptic fibration of $X^{\text{SU}(2)}$. This interplay between singularities of the divisor $t = 0$ and the singularity of the elliptic fibration yields a new non-Tate Weierstrass model with I_2 singularities at codimension one. Furthermore, as demonstrated in section 3.3, F-theory on $X^{\text{SU}(2)}$ yields the first explicit realization of SU(2) gauge theories with the three-index symmetric representation, which is located precisely at the triple point singularity of the SU(2) divisor $t = 0$. We support this observation by matching the effective theories before and after the Higgsing in section 3.4. We conclude our discussion by explicitly constructing all elliptic fibrations $X^{\text{SU}(2)}$ with base $B = \mathbb{P}^2$.

3.1 UnHiggsing U(1) → SU(2) in geometry

We begin by recalling that the elliptically fibered Calabi-Yau manifold X given in (2.1) has two rational sections $\hat{c}_0$ and $\hat{c}_1$ with fiber coordinates (2.4). The unHiggsing of the U(1) gauge symmetry of F-theory on X is performed by tuning its complex structure so that the two rational sections $\hat{c}_0$ and $\hat{c}_1$ of the elliptic fibration become identical, i.e. $\hat{c}_0 \equiv \hat{c}_1$, as shown in figure 3. As discussed before in (2.7), these two sections coincide precisely if $z_1 \equiv 0$, where z_1 is the z-coordinate of the section $\hat{c}_1$ in Weierstrass form. Thus, the relevant tuning of the complex structure of X is given by

$$z_1 = s_7 s_8^2 - s_6 s_8 s_9 + s_5 s_9^2 \to 0 \,. \tag{3.1}$$

We denote the resulting tuned Calabi-Yau manifold by $X^{\text{SU}(2)}$ for reasons that become clear below.

There are a number of remarks in order. First, we emphasize that we have to forbid the special solution $s_8 = s_9 \equiv 0$ to (3.1). This is clear from table 1 because there is matter

with $q = 3$ located at this locus in X. This implies that imposing $s_8 = s_9 \equiv 0$ globally by tuning the complex structure of X would render the resulting elliptic fibration of $X^{\mathrm{SU}(2)}$ singular everywhere, which does not define a good F-theory model. In fact, we consider solutions to (3.1) with general s_8, s_9 in order to preserve in the unHiggsing to $X^{\mathrm{SU}(2)}$ the geometric structure in X giving rise to matter with charge $q = 3$.

Second, the tuning (3.1) induces a codimension one singularity of Kodaira type I_2. This is immediately clear from table 1 and can be checked formally for example by using the Weierstrass form, see appendix A. Indeed, the locus $V(I_{(2)})$ in table 1, which supports codimension two I_2 singularities corresponding to matter with charge $q = 2$, is promoted to codimension one in B if we perform the tuning $z_1 \to 0$. The locus of I_2 singularities is then given by

$$t := s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3 = 0\,, \tag{3.2}$$

whose class is $[t] = [s_1] + 3[s_9] = -3K_B + 2\mathcal{S}_9 - \mathcal{S}_7$ according to (2.2).

Thus, we see that the gauge group G of F-theory on $X^{\mathrm{SU}(2)}$, which has a trivial MW-group of rational sections, is given by

$$G = \mathrm{SU}(2)\,. \tag{3.3}$$

The U(1) gauge group of X has been unHiggsed in a rank-preserving way to SU(2).

Third, we point out that generically, if all s_i in z_1 are non-trivial and general polynomials, the tuning (3.1) sets a non-trivial polynomial on B to zero. A general solution to this relation can be identified when the base is smooth (which we assume) and the corresponding ring of sections can be treated as a UFD, for example when the base is $B = \mathbb{P}^2$, where the sections are simply homogeneous polynomials of various degrees in the homogeneous coordinates. In this case, for example, every factor in s_9 must be a factor of either s_7 or s_8. We assume that s_8 and s_9 have no common factors since they could be factored out of z_1, and as mentioned above the solution $s_8 = s_9 \equiv 0$ does not give a good F-theory model. The general solution to (3.1) with relatively prime s_8 and s_9 is then given by (cf. [32])

$$s_5 = s_8 \sigma_5\,, \qquad s_6 = s_8 \sigma_7 + s_9 \sigma_5\,, \qquad s_7 = s_9 \sigma_7\,. \tag{3.4}$$

Here σ_5 and σ_7 are arbitrary sections of $\mathcal{O}(-K_B - \mathcal{S}_9)$ and $\mathcal{O}(\mathcal{S}_7 - \mathcal{S}_9)$, as follows from (2.2). Clearly, a necessary condition for the existence of this solution is effectiveness of $[s_5] - [s_8] = -K_B - \mathcal{S}_9$ and $\mathcal{S}_7 - \mathcal{S}_9$ for the sections σ_5 and σ_7 to exist, respectively.

The constraint (3.1) can also be solved simply by setting

$$s_5 = s_6 = s_7 \equiv 0\,. \tag{3.5}$$

Note that this is a special case of (3.4), where $\sigma_5 = \sigma_7 = 0$, and does not require effectiveness of $-K_B - \mathcal{S}_9$ or $\mathcal{S}_7 - \mathcal{S}_9$. We emphasize that this tuning is clearly always possible on any base B. The charged matter spectrum of F-theory on $X^{\mathrm{SU}(2)}$ obtained by this tuning agrees with that obtained by the tuning (3.4). This follows from consistency with the Higgsing back to X together with the fact, which we checked in an explicit computation, that the additional tuning $\sigma_5 = \sigma_7 \equiv 0$ does not change the singularities of $X^{\mathrm{SU}(2)}$. Thus, we will

for the remainder of this work consider the solution (3.5). Finally, we note that simple tunings achieving $z_1 \to 0$ are possible if s_8 or s_9 are constants, *i.e.* in the absence of matter with U(1)-charge $q = 3$, cf. table 1; for example, if s_8 is constant, we can always solve (3.1) by $s_7 = \frac{1}{s_8^2}(s_6 s_8 s_9 - s_5 s_9^2)$.

Let us further elaborate on the geometry of $X^{\mathrm{SU}(2)}$. First, we emphasize that the divisor $t = 0$ defined in (3.2) has triple point singularities at the locus of points defined by $s_8 = s_9 = 0$; i.e., three of its branches cross at the common locus $s_8 = s_9 = 0$. Focusing on complex two-dimensional bases B, $t = 0$ defines a Riemann surface with arithmetic genus g computed as

$$g = 1 + \frac{1}{2}[t] \cdot ([t] + K_B) = p_g + \frac{1}{2} \sum_p m_p(m_p - 1) \,. \tag{3.6}$$

Here the first equality follows from adjunction whereas in the second equality we split the arithmetic genus into the geometric genus p_g and contributions from all singular points p of $t = 0$ with multiplicity m_p, see e.g. [7]. Each triple point singularity of $t = 0$ has multiplicity $m_p = 3$ and contributes 3 to the arithmetic genus g of t as it can be deformed into three ordinary double point singularities, each of which contributes one to g. We will discuss the physical interpretation of the triple point singularity in section 3.3, where we show that each triple point singularity supports a half-hypermultiplet of matter in the three-index symmetric **4** representation of SU(2).

We conclude by noting that the geometric genus p_g of the curve $t = 0$ is greater or equal to one for effective classes of s_8 and s_9. This follows from the genus formula (3.6) as

$$
\begin{aligned}
p_g &= 1 + \frac{1}{2}(-2K_B + [s_8] + [s_9]) \cdot (-K_B + [s_8] + [s_9]) - 3[s_8][s_9] \\
&\geq 1 + \frac{1}{2}3[s_9] \cdot (-K_B + [s_8] + [s_9]) - 3[s_8] \cdot [s_9] = 1 + \frac{1}{2}3[s_9] \cdot (-K_B + [s_9]) - \frac{3}{2}[s_8] \cdot [s_9] \\
&\geq 1 + \frac{1}{2}3[s_9] \cdot [s_8] - \frac{3}{2}[s_8] \cdot [s_9] = 1 \,,
\end{aligned}
\tag{3.7}
$$

where we used, employing (2.2), that $[t] = -2K_B + [s_8] + [s_9]$ in the first equality, then $-2K_B \geq 2[s_9] - [s_8]$ following from $[s_1] \geq 0$ in the first inequality and $-K_B + [s_9] \geq [s_8]$ as follows from $[s_7] \geq 0$ in the last inequality. Field theoretically, this is relevant since we expect the geometric genus to give rise to p_g nonlocal adjoint matter fields. At least one adjoint matter field is required to Higgs the SU(2) gauge theory specified by $X^{\mathrm{SU}(2)}$ back to the original U(1) theory, so if the triple point singularities do not support localized adjoint matter then it is clear that the geometric genus of t must be positive. In addition, we emphasize that $g \geq 1$ is equivalent to $[z_1] \geq 0$ as we have the relation

$$[t] = [z_1] - K_B \,, \tag{3.8}$$

which follows from (2.7) and (2.2). This implies that $[t]$ is always effective as we have $-K_B \geq 0$ and $[z_1] \geq 0$, which is necessary for the existence of a non-trivial section z_1 allowing for the deformation of the model $X^{\mathrm{SU}(2)}$ back to X. The p_g adjoint Higgs VEV's can be thought of as corresponding to the deformations in $z_1 \neq 0$.

3.2 Novel matter structure from non-Tate Weierstrass forms

The Weierstrass model of the unHiggsed theory $X^{\mathrm{SU}(2)}$ is obtained using the tuning (3.5) in the general Weierstrass model of X given in (A.1). The resulting SU(2) model is specified by the Weierstrass coefficients

$$f = \frac{1}{3}\left(-\left(s_3^2 - 3s_2 s_4\right)s_8^2 + (s_2 s_3 - 9s_1 s_4)s_9 s_8 - \left(s_2^2 - 3s_1 s_3\right)s_9^2\right), \qquad (3.9)$$

$$g = \frac{1}{27}\left(-2(s_3^3 - 9s_2 s_3 s_4 + 27 s_1 s_4^2)s_8^3 - 6(s_2 s_3^2 + 3s_2^2 s_4 - 9s_1 s_3 s_4)s_8^2 s_9\right.$$
$$\left. + 6s_3(2s_2^2 - 3s_1 s_3)s_8 s_9^2 - 2s_2^3 s_9^3\right) + \left(s_1 s_4 - \frac{1}{3}s_2 s_3\right)T.$$

Here we have replaced the variable t defined in (3.2) for the moment by the formal variable T. While the formal expansion of f and g is thus ambiguous, it is clear that f is not naturally written in a form containing terms proportional to T as there are no cubic terms in s_8, s_9, and this form of g is a fairly natural way of combining terms with a term linear in T. Alternative presentations of g lead to equivalent conclusions but with different algebra. From (3.9), we readily compute the discriminant $\Delta = 4f^3 + 27g^2$. We emphasize that for T being an abstract variable, we do not obtain a vanishing of Δ. However, we see that

$$(4f^3 + 27g^2)\big|_{T=0} \sim s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3, \qquad (3.10)$$

which agrees precisely with t given in (3.2). Thus, for the special choice $T \equiv t$ we obtain a vanishing of Δ to first order. In fact, if we set $T \equiv t$ we see that Δ vanishes also to second order at $t = 0$ due to additional cancellations. We then obtain

$$\Delta = t^2 \Delta', \qquad \Delta' = 4s_1 s_3^3 + 4s_2^3 s_4 - 18 s_1 s_2 s_3 s_4 + 27 s_1^2 s_4^2 - s_2^2 s_3^2. \qquad (3.11)$$

Here, the remainder Δ' of the discriminant is in the class $[\Delta'] = -6K_B + 2S_7 - 4S_9$ so that $[\Delta] = [\Delta'] + 2[t] = -12K_B$.

In summary, we see that the singularity structure of the elliptic fibration defined by the Weierstrass model with (3.9) crucially depends on the particular form of $t = 0$ with triple point singularities at $s_8 = s_9 = 0$. In particular, the forms in (3.9) do not have the structure needed for an SU(2) singularity through Tate's algorithm [6, 33], and do not have the form expected for an SU(2) on a smooth divisor $t = 0$, because the induced ring of local functions is not a universal factorization domain [7]. Thus, we refer to the model (3.9) and models of similar type more generally as *non-Tate form* Weierstrass models. Explicitly, we observe that the Tate coefficients

$$a_1 = a_3 = 0, \qquad a_2 = -s_3 s_8 - s_2 s_9, \qquad a_4 = s_2 s_4 s_8^2 + (s_2 s_3 - 3s_1 s_4)s_8 s_9 + s_1 s_3 s_9^2$$
$$a_6 = -s_1 s_4^2 s_8^3 + (2s_1 s_3 - s_2^2)s_4 s_8^2 s_9 + (2s_2 s_4 - s_3^2)s_1 s_8 s_9^2 - s_1^2 s_4 s_9^3 \qquad (3.12)$$

for (3.9) that naively follow from (A.3) by the tuning (3.5) do not exhibit the vanishing orders in Tate's algorithm for the realization of an SU(2) gauge group [6, 33].

We conclude by noting that (3.9) assumes the normal form of a Weierstrass model with I_2 singularities as dictated by Tate's algorithm if s_8 or s_9 are constants, i.e. $t = 0$ is smooth. For example, if $s_9 = \text{const.}$ we can shift the variables so that $t \equiv s_1$ and (3.9) assumes the form of a Weierstrass model with I_2 singularities in [7].

SU(2)-rep	Multiplicity	Fiber	Locus
4	$x_{\mathbf{4}} = \frac{1}{2}\mathcal{S}_9 \cdot (-K_B + \mathcal{S}_9 - \mathcal{S}_7)$	I_0^{*ns}	$V_{\text{Sing}} = \{s_8 = s_9 = 0\}$
3	$x_{\mathbf{3}} = \frac{1}{2}[t] \cdot ([t] + K_B) + 1 - 6x_{\mathbf{4}}$	I_2	$V_{\text{SU}(2)} = \{t = 0\}$
2	$x_{\mathbf{2}} = 2(3K_B^2 - K_B \cdot (2\mathcal{S}_7 - \mathcal{S}_9)$ $\quad -\mathcal{S}_7^2 + \mathcal{S}_7 \cdot \mathcal{S}_9 - \mathcal{S}_9^2) + 2x_{\mathbf{4}}$	I_3	$V(\mathfrak{p}_1) \cup V_{\text{Sing}}$

Table 2. Matter spectrum of $X^{\text{SU}(2)}$. Shown is the multiplicity of full hypermultiplets in a 6D SUGRA theory. We note that there is only a half-hypermultiplet in the $\mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}$ at each ordinary triple point $s_8 = s_9 = 0$ of $t = 0$.

3.3 The non-Abelian matter spectrum

We are now in a position to determine the matter spectrum of F-theory on the Calabi-Yau manifold $X^{\text{SU}(2)}$. For the reader only interested in the results of this analysis, we summarize the matter content in table 2.

We begin with the matter content localized at codimension one. As noted before, the SU(2) gauge algebra is supported on a Riemann surface $t = 0$ of higher (arithmetic) genus g, which is computed via (3.6). As $t = 0$ has a number of $[s_8] \cdot [s_9]$ ordinary triple point singularities, each of which contribute 3 to g, we obtain the topological genus p_g

$$p_g = g - 3[s_8] \cdot [s_9], \tag{3.13}$$

which is explicitly given in the first line of (3.7). In a 6D compactification, the topological genus p_g gives rise to p_g hypermultiplets in the adjoint representation $\mathbf{3}$ of the SU(2) gauge group on $t = 0$ [34]. Employing (2.2), this gives the multiplicity $x_{\mathbf{3}}$ in the second row in table 2.

Next let us consider the matter contribution of the triple point singularities at the loci $s_8 = s_9 = 0$. One way to attain a triple point singularity on a divisor supporting an SU(2) is to take a Tate model for an SU(2) on a smooth divisor $\tilde{t}$, and then to deform the divisor to get a triple point singularity. In this scenario, the triple point can be viewed as a limit of three double point singularities. Furthermore, each double point is reached in a limit of a family of smooth surfaces; reasoning following [16], each such double point must be associated with an adjoint representation since there is no intermediate opportunity for a matter transition through a superconformal fixed point, and for similar reasons the triple point in the Tate construction must then represent three adjoint matter multiplets. For the non-Tate model found here, however, the arithmetic genus three singularity may give a matter content with a half-hypermultiplet in the triple symmetric $\mathbf{4}$ representation. To distinguish these possibilities, further analysis is needed. In the following section we argue that by matching the matter content with the Higgsed U(1) theory, the only consistent possibility is that each triple point carries a half-hypermultiplet in the $\mathbf{4}$ representation. This gives the multiplicity $x_{\mathbf{4}}$ in the first row in table 2.

Another approach, in principle, to determining the matter content at the intersection point is to explicitly resolve the singularity of the Calabi-Yau manifold over the triple

intersection point. This is an interesting direction for study, which we leave the details of for future work. We make several comments, however. First, the local analysis will determine the representation of SU(2)×SU(2)×SU(2) realized at the intersection of three independent divisors. This will either give three bifundamental type representations, corresponding to the possibility of three adjoints for the SU(2) on the connected divisor, or a trifundamental[4] representation $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$, which would break up into a $\mathbf{4}$ and two fundamental $\mathbf{2}$'s when the divisor is connected and we embed SU(2) $\subset$ SU(2) × SU(2) × SU(2). (Actually, we would get a half trifundamental, as this representation is self-conjugate). Note that while for a larger group like SU(3), the precise matter content, such as the presence of an adjoint *vs.* a symmetric + antisymmetric, depends on how the divisor connects to itself, i.e. on whether the local representation on each branch is fundamental or antifundamental, that distinction is irrelevant for SU(2) where the fundamental representation is self-conjugate. In any case, this analysis suggests that when the triple point gives a triple-symmetric $\mathbf{4}$ representation there will also be two fundamental $\mathbf{2}$ representations present.

The Kodaira singularity at the triple points is of type I_0^*. Since this is a codimension two singularity, the split/non-split distinction and monodromy structure is not relevant in the same way as it is for codimension one singularities, where it would determine whether the gauge group would be G_2 or SO(8). For six-dimensional theories, this singularity arises at a point, so there is no question of monodromy, and the Dynkin diagram associated with the singularity is a D_4. Locally, the matter structure associated with the codimension two singularity is determined by the embedding of the three single nodes associated with the A_1 SU(2) factors on the branches of the I_2 locus into the D_4. This can be done in an essentially unique way that respects the permutation symmetry on the A_1 factors by embedding the three A_1 factors as the three outer nodes of the Dynkin diagram D_4. The central node then represents a matter state that is charged under all three SU(2) factors, and thus associated with the trifundamental representation $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$, which yields the $\mathbf{4} + \mathbf{2} + \mathbf{2}$ representation upon the embedding of SU(2) $\subset$ SU(2) × SU(2) × SU(2) by identifying the three SU(2) factors as discussed above. This gives strong evidence from the group theory point of view that indeed the local D_4 structure at the triple point must be associated with the $\mathbf{4}$ representation of the SU(2) on the I_2 locus. A more explicit resolution of this singularity is left to future work. Note that for 4D F-theory models, the codimension two D_4 singularity arises over a curve in the base threefold. While there may be nontrivial monodromy around this curve, this simply corresponds to the identification of the different SU(2) factors on the branches of the I_2 locus that enter the triple point. Since these branches are already identified globally, this does not modify the above conclusion that the resulting matter content should include the $\mathbf{4}$ representation of the SU(2).

Finally, we use the Weierstrass model (3.9) to find the codimension two singularities of $X^{\mathrm{SU}(2)}$ at the intersection $t = \Delta' = 0$ with Δ' given in (3.11). The computation of the primary decomposition of the ideal $I := \{t, \Delta'\}$ yields two prime ideals, which we denote by $\mathfrak{p}_1$ and $\mathfrak{p}_2$. As these ideals are generated by 14 and six polynomials, respectively,

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[4]The possibility of a trifundamental representation arising at a triple point of an I_2 locus was also discussed in [35].

we do not present their explicit forms here. Consequently, the variety $V(I)$ is reducible with irreducible components $V(\mathfrak{p}_1)$ and $V(\mathfrak{p}_2)$ that can be shown, employing the resultant technique as in [26], to have multiplicities 1 and 2 inside $V(I)$, respectively. Thus, we find the homology relation

$$[V(I)] = [V(\mathfrak{p}_1)] + 2[V(\mathfrak{p}_2)]. \tag{3.14}$$

The individual homology classes are computed as explained in [27] to be

$$[V(\mathfrak{p}_1)] = 2(3K_B^2 - K_B \cdot (2\mathcal{S}_7 - \mathcal{S}_9) - \mathcal{S}_7^2 + \mathcal{S}_7 \cdot \mathcal{S}_9 - \mathcal{S}_9^2),$$
$$[V(\mathfrak{p}_2)] = 6K_B^2 - K_B \cdot (\mathcal{S}_9 - 2\mathcal{S}_7) + 3\mathcal{S}_7 \cdot \mathcal{S}_9 - 3\mathcal{S}_9^2. \tag{3.15}$$

Next, we determine the singularity type of $X^{\mathrm{SU}(2)}$ along these two irreducible components. By reducing the Weierstrass coefficients f, g and the discriminant Δ given in (3.9) and (3.11) as well as the Tate coefficients (3.12) modulo the ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, respectively, we find Kodaira singularities of type I_3 and III, respectively. Thus, the locus $V(\mathfrak{p}_1)$ supports a number of $[V(\mathfrak{p}_1)]$ matter fields in the fundamental representation $\mathbf{2}$ of SU(2), as shown in the last line of table 2, while no matter fields are located on $V(\mathfrak{p}_2)$ since the type III fiber is just a degenerated I_2 fiber with no additional $\mathbb{P}^1$ harboring matter states. In a compactification on a threefold $X^{\mathrm{SU}(2)}$ to 6D, the found matter fields form a full hypermultiplet. The multiplicity of matter fields in the $\mathbf{2}$ representation is given in the last line of table 2, where we have added $[s_8] \cdot [s_9]$ fundamentals contributed by the ordinary triple point singularities of $t = 0$, matching the analysis of the local trifundamental representation mentioned above.

We conclude by noting that the anomaly coefficient b of the 6D SUGRA theory given by F-theory on the threefold $X^{\mathrm{SU}(2)}$ is given by the class of t, i.e. it reads

$$b^{\mathrm{SU}(2)} = [t] = -2K_B + [s_8] + [s_9] = -3K_B + 2\mathcal{S}_9 - \mathcal{S}_7. \tag{3.16}$$

Employing this coefficient, the spectrum in table 2, $a = K_B$, and the anomaly coefficients $(A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}) = (1, 0, \frac{1}{2}), (4, 0, 8), (10, 0, 41)$ for the SU(2)-representations $\mathbf{R} = \mathbf{2}, \mathbf{3}, \mathbf{4}$, respectively, we readily check that the two 6D gauge and mixed gauge-gravity anomalies are cancelled. For the anomaly cancellation to work, following the genus analysis, it is necessary that there is only a half-hypermultiplet in the representation $\mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}$ at each triple point singularity of $t = 0$, as indicated in table 2. Note furthermore, as mentioned earlier, that there is an anomaly equivalence

$$\frac{1}{2}\mathbf{4} + 7 \times \mathbf{2} \leftrightarrow 3 \times \mathbf{3} + 7 \times \mathbf{1}. \tag{3.17}$$

This shows that with the number of matter fields in the fundamental identified above, it is not possible to satisfy the anomaly conditions when the triple intersection point supports three adjoints and any positive number of fundamental representations. This provides an alternative argument using only anomaly conditions and counting of known singularity types that the matter content at the triple points is $\frac{1}{2} \times \mathbf{4} + \mathbf{2}$ as identified above. For more details on the relevant anomaly cancellation conditions in the context of F-theory, see e.g. the review [36]. Finally, note that as found in [16], we expect that the total number of

fields that must be brought together to explicitly undergo a transition like (3.17) will bring the theory to a superconformal transition point, where a tensor branch is also available. A more explicit treatment of such transitions will be presented elsewhere.

3.4 Matching effective theories through the Higgs transition

Next, we match the effective field theory of F-theory on $X^{\mathrm{SU}(2)}$ with the Abelian model obtained by F-theory on X. We show that the two theories are related under a Higgsing by matter in the adjoint representation. As mentioned above, this corresponds to the extremal transition $X^{\mathrm{SU}(2)} \to X$ induced by switching on the deformation parameter z_1 defined in (3.1).

We begin by matching the charged matter spectrum of the non-Abelian model in table 2 with the one of the Abelian model in table 1 through the adjoint Higgsing. First, we note the following branching of SU(2) representations under the breaking $\mathrm{SU}(2) \to \mathrm{U}(1)$:

$$\mathbf{4} \to \mathbf{1}_3 \oplus \mathbf{1}_{-3} \oplus \mathbf{1}_1 \oplus \mathbf{1}_{-1} \,, \qquad \mathbf{3} \to \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_0 \,, \qquad \mathbf{2} \to \mathbf{1}_1 \oplus \mathbf{1}_{-1} \,. \qquad (3.18)$$

Here we have computed U(1)-charges using the generator $2\sigma_3$, where σ_3 is the third Pauli matrix of SU(2). Next, we use the fact that a hypermultiplet with charge q is composed of states with charge q and $-q$ to eliminate negative charges. Finally, employing that two hypermultiplets with charges $q = \pm 2$, respectively, from the adjoint representation are eaten up in the Higgsing by the massive W-bosons of the broken SU(2) vector multiplet, we obtain an Abelian theory with the following numbers $x_{\mathbf{1}_q}$ of hypermultiplets with charges $q = 1, 2, 3$:

$$x_{\mathbf{1}_3} = 2x_{\mathbf{4}} \,, \qquad x_{\mathbf{1}_2} = 2(x_{\mathbf{1}_3} - 1) \,, \qquad x_{\mathbf{1}_1} = 2(x_{\mathbf{1}_4} + x_{\mathbf{1}_1}) \,. \qquad (3.19)$$

Comparing with the matter spectrum in table 1, using table 2, we see that we precisely reproduce the effective theory of F-theory on X. Furthermore, we note that the anomaly coefficient b in (2.6) of the Abelian theory is $2b^{\mathrm{SU}(2)}$ with $b^{\mathrm{SU}(2)}$ given in (3.16) as expected. This in particular implies an anomaly free theory in 6D. This precise matching between the spectra gives a rigorous argument for the presence of $\mathbf{4}$ matter at the triple point singularities, matching with the results of the arguments given in the previous section; this is the only matter content that would give a consistent U(1) theory after Higgsing.

Next, we note that the number of complex structure moduli increases in the Higgsing, corresponding geometrically to the deformations $X^{\mathrm{SU}(2)} \to X$. The new complex structure moduli are naturally associated with the deformation parameters in z_1. We expect therefore that the number of independent parameters that deform z_1 away from the locus $z_1 = 0$ will match the number of Higgs VEVs, i.e. neutral hypermultiplets in the $\mathbf{3}$ representation.[5] As there are $x_{\mathbf{3}} = p_g$ matter fields in the $\mathbf{3}$ representation, each of which has one neutral component, we expect p_g new moduli and deformation parameters in z_1. To be concrete, for the concrete base $B = \mathbb{P}^2$ we can compute the change in the number of complex structure moduli by a counting of monomials in appropriate classes. First, we compute the number

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[5]Note that there is no D-term condition in an adjoint Higgsing.

$x_{\mathbf{3}}$ of adjoint fields in the representation $\mathbf{3}$ and Higgs VEVs according to table 2 as

$$x_{\mathbf{3}} = 28 - \frac{15}{2}\mathcal{S}_7 + \frac{1}{2}\mathcal{S}_7^2 + 6\mathcal{S}_9 + \mathcal{S}_7 \cdot \mathcal{S}_9 - \mathcal{S}_9^2 \,, \qquad (3.20)$$

where we have used that $K_B = \mathcal{O}_{\mathbf{P}^2}(-3)$. Explicitly computing the number of deformation parameters in z_1, assuming the generic form (3.4) for the solution to $z_1 = 0$, we can parameterize the deformations by replacing σ_5, σ_7 by generic s_5, s_6, s_7. The number of independent monomials in a degree d divisor class is $m[d] = (d+1)(d+2)/2$, allowing us to confirm that the number of independent degrees of freedom that deform $z_1 \neq 0$ is

$$m[s_5] + m[s_6] + m[s_7] - m[\sigma_5] - m[\sigma_7] = x_{\mathbf{3}} \,. \qquad (3.21)$$

In principle, it should also be possible to check whether the number of independent Weierstrass moduli in both the SU(2) and U(1) models involved match precisely with the number of neutral scalar fields expected from the gravitational anomaly cancellation condition $H - V = 273 - 29T$. While the computation just performed demonstrates that the difference between these numbers is correctly captured by the deformation parameters in z_1, there is some redundancy in our parameterization of these models through the s_i's; removing this redundancy and identifying the proper number of independent degrees of freedom in the Weierstrass model would a useful check to determine whether the models presented here are the most general forms for the given spectra, or only represent a subset of the possibilities.

3.5   Models over $B = \mathbb{P}^2$

We conclude the discussion of F-theory compactified on the Calabi-Yau manifold $X^{\mathrm{SU}(2)}$ with the concrete models obtained for $B = \mathbb{P}^2$.

We begin by considering the generic class of SU(2) models on $\mathbb{P}^2$. When the SU(2) is realized on a smooth divisor of degree d, the genus of the corresponding curve is $g = (d-1)(d-2)/2$. This is the number of matter fields in the adjoint ($\mathbf{3}$) representation. From explicit construction or anomaly cancellation, it is straightforward to determine that the number of fundamental ($\mathbf{2}$) matter fields is $x_{\mathbf{2}} = 16 + 6d^2 - 16g$. This parameterizes the full spectrum of F-theory constructions on $\mathbb{P}^2$ with an SU(2) gauge group realized on a smooth divisor. Using the anomaly equivalence (3.17), we expect that we can exchange 3 adjoints and seven uncharged moduli in any of these models for a half-hypermultiplet in the $\mathbf{4}$ representation and seven fundamentals. For example, when $d = 8$, we have a genus 21 curve, and the generic matter content consists of 21 adjoints and 64 fundamentals. We would expect anomaly-equivalent models with $21 - 3x$ adjoints, x half-hypermultiplets in the $\mathbf{4}$ representation, and $64 + 7x$ hypermultiplets in the fundamental representation. These classes of models (for general d) comprise all models that are consistent from the low-energy 6D supergravity point of view, and that have no tensor multiplets, an SU(2) gauge group, and matter in only the $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ representations. We might hope to identify in F-theory using the approach described here all such models that have at least one adjoint representation that can be Higgsed to give a U(1) theory with charges up to $q = 3$.

Next we recall that the Calabi-Yau manifold $X^{\mathrm{SU}(2)}$ is defined by (2.1) with tuned complex structure so that $s_5 \equiv s_6 \equiv s_7 \equiv 0$. Thus, the model exists as long as all other sections s_i exist, i.e. are associated to effective divisor classes. By explicitly solving the effectiveness conditions implied by this, we again obtain the allowed region in figure 2. For every Abelian model X there exists a corresponding model $X^{\mathrm{SU}(2)}$ and vice versa. For each of these 16 inequivalent models (recall the $\mathbb{Z}_2$-symmetry (2.8)) we readily compute all divisor classes $[s_i]$, the class of the SU(2)-divisor $t = 0$ as well as the charged matter spectrum in table 2. We obtain:

$(\mathcal{S}_7, \mathcal{S}_9)$	$[s_1]$	$[s_2]$	$[s_3]$	$[s_4]$	$[s_5]$	$[s_6]$	$[s_8]$	$[t]$	$(x_{\mathbf{4}}, x_{\mathbf{3}}, x_{\mathbf{2}})$
$(0,0)$	9	6	3	0	6	3	3	9	$(0, 28, 54)$
$(1,0)$	8	6	4	2	5	3	2	8	$(0, 21, 64)$
$(2,0)$	7	6	5	4	4	3	1	7	$(0, 15, 70)$
$(3,0)$	6	6	6	6	3	3	0	6	$(0, 10, 72)$
$(1,1)$	7	5	3	1	5	3	3	10	$(\frac{3}{2}, 27, 61)$
$(2,1)$	6	5	4	3	4	3	2	9	$(1, 22, 68)$
$(3,1)$	5	5	5	5	3	3	1	8	$(\frac{1}{2}, 18, 71)$
$(1,2)$	6	4	2	0	5	3	4	12	$(4, 31, 56)$
$(2,2)$	5	4	3	2	4	3	3	11	$(3, 27, 64)$
$(3,2)$	4	4	4	4	3	3	2	16	$(2, 24, 68)$
$(2,3)$	4	3	2	1	4	3	4	13	$(6, 30, 58)$
$(3,3)$	3	3	3	3	3	3	3	12	$(\frac{9}{2}, 28, 63)$
$(2,4)$	3	2	1	0	4	3	5	15	$(10, 31, 50)$
$(3,4)$	2	2	2	2	3	3	4	14	$(8, 30, 56)$
$(3,5)$	1	1	1	4	3	3	5	16	$(\frac{25}{2}, 30, 47)$
$(3,6)$	0	0	0	0	3	3	6	18	$(18, 28, 36)$

(3.22)

There are some remarks in order. First, we note that in the absence of triple point singularities of $t = 0$, its minimal degree is 6. However, in that case the model $X^{\mathrm{SU}(2)}$ is completely equivalent to the elliptic fibrations by quartics in $\mathrm{Bl}_1 \mathbb{P}^2(1, 1, 2)$ of Morrison, Park [21], as mentioned before. Thus, there have to exist models with $[t] = 3, 4, 5$. As discussed before at the end of section 3.1, these can be obtained from X if we relax the effectiveness condition on $[s_8]$. Indeed, we can then lower the degree of $[t] = [s_1]$ to 3, as expected.

Second, in the case with ordinary triple point singularities on t, we observe that our list (3.22) does not produce all models that seem geometrically possible. For example, a model with $[t] = 5$ has an arithmetic genus of $g = 6$ which seems to allow for one ordinary triple point singularity while still exhibiting a geometric genus $p_g = 3$, i.e. adjoints for a Higgsing to an Abelian theory. Similar models with a different number of ordinary triple points than in (3.22) seem to be constructable also for higher degree curves $t = 0$. Naively it would seem that we can simply choose, for example $[s_1] = [s_2] = [s_3] = [s_4] = 2$ and

$[s_8] = [s_9] = 1$ in the Weierstrass form (3.9). While this set of choices are not compatible with effectiveness of all divisor classes in (2.2), this would seem to define a well-defined Weierstrass model with the SU(2) structure of interest realized on a quintic curve with a single triple point at the intersection $s_8 = s_9 = 0$. The issue, however, is that since f is of degree 12 and g of degree 18 in homogeneous coordinates $[x : y : z]$, this leads to a problematic (6, 12) singularity when $z \to 0$. The compatibility of the divisor classes with (2.2) avoids this problem. It would be interesting to understand whether the absence of these models is a mere artifact of how the Weierstrass form (3.9) is constructed, or whether this is an indication of a fundamental limitation in the spectrum of models available from F-theory, or even in 6D supergravity consistent with quantum gravity constraints. A systematic mathematical classification of Weierstrass models of elliptic fibrations with I_2 singularities over singular divisors would help to answer this question.

4 Further unHiggsing to larger non-Abelian groups

In this section we discuss the possibility to further unHiggs the non-Abelian model defined by F-theory on $X^{\mathrm{SU}(2)}$. Here, we are motivated by the search for a resulting non-Abelian theory that has a standard matter spectrum consisting only of fundamentals, anti-fundamentals and adjoints. In this case, the geometric realization of the corresponding elliptic fibration should follow the standard rules of Tate's algorithm. Starting with these standard Tate Weierstrass models the inverse process of the unHiggsing described here can then be understood as a deformation (re-Higgsing) of these Weierstrass models to a non-Tate Weierstrass model. Systematizing this deformation procedure outlined below may shed light on the general construction of non-Tate Weierstrass forms with novel matter structures in F-theory. For a recent application of this idea, we refer the reader to [16].

Here, we discuss two unHiggsing, one to models with $G_2 \times \mathrm{SU}(2)$ gauge group and standard matter content given by adjoints and (bi-)fundamentals and one to models with $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge group and with a matter content that includes trifundamental matter.

4.1 UnHiggsing SU(2) with the 4 representation to $\mathrm{SU}(2) \times G_2$

One possible unHiggsing of F-theory on $X^{\mathrm{SU}(2)}$ yields a theory with $G_2 \times \mathrm{SU}(2)$ gauge group on two different divisors and with a standard matter spectrum consisting of adjoints, fundamentals and bifundamentals. The unHiggsing is achieved by imposing

$$s_8 \equiv a s_9 \tag{4.1}$$

for an appropriate section $a \in \mathcal{O}(-K_B - \mathcal{S}_7)$, which can exist if $-K_B - \mathcal{S}_7$ is an effective class (if $[s_9] \geq [s_8]$, we can impose the inverse relation $s_9 = b s_8$ for appropriate b.).

With this tuning, the SU(2) divisor $t = 0$ defined in (3.2) degenerates as

$$t = s_9^3(s_4 a^3 - s_3 a^2 + s_2 a - s_1), \tag{4.2}$$

so that its triple point singularities disappear at the cost of an overall factor of s_9^3. Indeed, the Weierstrass model (3.9) reduces to the form

$$f = \left(-\frac{1}{3}\tilde{s}_2^2 + \tilde{s}_3\tilde{s}_1\right)s_9^2, \qquad g = \left(-\frac{2}{27}\tilde{s}_2^3 + \frac{1}{3}\tilde{s}_2\tilde{s}_3\tilde{s}_1 - s_4\tilde{s}_1^2\right)s_9^3,$$

$$\Delta = -16\tilde{s}_1^2 s_9^6 \Delta', \qquad \Delta' = -\tilde{s}_2^2\tilde{s}_3^2 + 4\tilde{s}_1\tilde{s}_3^3 + 4\tilde{s}_2^3\tilde{s}_4 - 18\tilde{s}_1\tilde{s}_2\tilde{s}_3\tilde{s}_4 + 27\tilde{s}_1^2\tilde{s}_4^2, \quad (4.3)$$

where we made the definitions

$$\tilde{s}_1 = s_1 + a s_2 - a^2 s_3 + a^3 s_4, \qquad \tilde{s}_2 = s_2 - 2a s_3 + 3a^2 s_4, \qquad \tilde{s}_3 = s_3 - 3a s_4. \quad (4.4)$$

The Weierstrass form (4.3) reveals the presence of singularities of Kodaira types I_2 at $\tilde{s}_1 = 0$ and I_0^* at $s_9 = 0$, respectively. We readily observe that (4.3) is of the normal form of a Weierstrass model with I_2 singularity following from Tate's algorithm or the analysis in [7]. Using the orders of vanishing of the Tate coefficients (A.3) in the limit (4.1), which are $(\infty, 1, \infty, 2, 3)$, or by computing the irreducible monodromy cover [20], we see that the singularity at $s_9 = 0$ is non-split, i.e. of type $I_0^{*\mathrm{ns}}$ yielding a G_2 gauge symmetry [6]. Thus, F-theory on $X^{\mathrm{SU}(2)}$ with the tuning (4.1) has the gauge group

$$G = \mathrm{SU}(2) \times G_2. \quad (4.5)$$

Note that the Weierstrass form (4.3), like (3.9), are acceptable for choices of s_1, s_9 that violate the effectiveness conditions (2.2). However, if in addition also (4.4) is to be satisfied, i.e. if the model shall be deformable back to $X^{\mathrm{SU}(2)}$, such models suffer from the same issue discussed earlier and have problems with bad singularities at infinity. For example, there should be no problem in tuning, for example, a G_2 on a line $[s_9] = 1$ and an SU(2) on a conic $[s_1] = 2$. This, however, would imply that $[s_8] = -2$, i.e., that (4.4) breaks down. As we see below, in this case there is insufficient matter to carry out the Higgsing that is needed to deform the model to return to the SU(2) models where (3.9) is valid, explaining the absence of a corresponding SU(2) model.

The matter content of the F-theory effective field theory can be derived from the Weierstrass model (4.3). As we will discuss, due to the presence of the G_2 gauge group, matter representations arise both at codimension one, i.e. are non-local, as well as at codimension two loci where the singularities of the elliptic fibration enhance. Before presenting the details of this analysis, we summarize the derived matter spectrum in table 3. We emphasize again that the spectrum only contains fundamental and adjoint representations, which can be attributed to the smoothness of both gauge divisors $\tilde{s}_1 = 0$ and $s_9 = 0$ as well as the standard form of the Weierstrass model. We note that there is an additional Kodaira singularity of type III at the codimension two locus $\tilde{s}_1 = \tilde{s}_2 = 0$ that does not give rise to matter fields.

Cancellation of 6D anomalies can be checked using the group theory coefficients $(A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}) = \left(1, 0, \frac{1}{4}\right), \left(4, 0, \frac{5}{2}\right), \left(1, 0\frac{1}{2}\right), (4, 0, 8)$ for the G_2-representations $\mathbf{R} = \mathbf{7}, \mathbf{14}$ and the SU(2)-representations $\mathbf{R} = \mathbf{2}, \mathbf{3}$, respectively, given for example in [28]. The coefficients $b^{\mathrm{SU}(2)} = [s_1] = -3K_B - \mathcal{S}_7 - \mathcal{S}_9$ and $b^{G_2} = \mathcal{S}_9$ enter the 6D GS-counterterms.

Rep	Multiplicity	Fiber	Locus
$(\mathbf{2},\mathbf{7})$	$x_{(\mathbf{2},\mathbf{7})} = \frac{1}{2}\mathcal{S}_9 \cdot (-3K_B - \mathcal{S}_7 - \mathcal{S}_9)$	I_2^*	$V_{\text{bf}} = \{\tilde{s}_1 = s_9 = 0\}$
$(\mathbf{1},\mathbf{7})$	$x_{(\mathbf{1},\mathbf{7})} = \mathcal{S}_9 \cdot (-2K_B + \mathcal{S}_7 - \mathcal{S}_9)$	$-$	non-local
$(\mathbf{2},\mathbf{1})$	$x_{(\mathbf{2},\mathbf{1})} = \frac{1}{2}(-4K_B + 4\mathcal{S}_7 - 3\mathcal{S}_9)(-3K_B - \mathcal{S}_7 - \mathcal{S}_9)$	I_3	$V_2 = \{\tilde{s}_1 = 4\tilde{s}_2\tilde{s}_4 - \tilde{s}_3^2 = 0\}$ $\cup V_{\text{bf}}$
$(\mathbf{3},\mathbf{1})$	$x_{(\mathbf{3},\mathbf{1})} = 1 + \frac{1}{2}(-3K_B - \mathcal{S}_7 - \mathcal{S}_9) \cdot (-2K_B - \mathcal{S}_7 - \mathcal{S}_9)$	I_2	$V_{\text{SU}(2)} = \{\tilde{s}_1 = 0\}$
$(\mathbf{1},\mathbf{14})$	$x_{(\mathbf{1},\mathbf{14})} = 1 + \frac{1}{2}\mathcal{S}_9 \cdot (\mathcal{S}_9 + K_B)$	I_0^*	$V_{\text{G}_2} = \{s_9 = 0\}$

Table 3. Matter spectrum of F-theory on $X^{\text{SU}(2)}$ with the tuning (4.1) to a model with gauge group $G_2 \times \text{SU}(2)$. Shown are the multiplicities of full hypermultiplets in a 6D SUGRA theory.

Next, we explain the derivation of the matter spectrum given in table 3. We begin with the non-local matter. As both the G_2 and the SU(2) divisors are smooth, there are $g = 1 + \frac{1}{2}\mathcal{S}_9 \cdot (\mathcal{S}_9 + K_B)$ adjoint matter fields in the $\mathbf{14}$ representation of G_2 and $g_{\text{SU}(2)} = 1 + \frac{1}{2}(-3K_B - \mathcal{S}_7 - \mathcal{S}_9) \cdot (-2K_B - \mathcal{S}_7 - \mathcal{S}_9)$ adjoints in the $\mathbf{3}$ representation of SU(2), respectively. This yields the last two lines of table 3.

For G_2, the fundamental representation $\mathbf{7}$ is in general non-local, as already discussed in [20]. The multiplicity of this representation is given[6] by the difference $g' - g$. Here g is the genus of the G_2-divisor $s_9 = 0$ and g' is the genus of the threefold cover[7] of the curve $s_9 = 0$ with branch points p given by the codimension two enhancement points $s_9 = \Delta' = 0$ with Δ' given in (4.3) [20]. Using the Riemann-Hurwitz formula for the genus g' of a ramified covering of a genus g Riemann surface [37],

$$g' = \frac{1}{2}\left(2 + N(2g - 2) + \sum_p (e_p - 1)\right), \qquad (4.6)$$

where N is the degree of the covering, p are its branch points and e_p denotes the ramification index at p, we obtain using $N = 3$ and $e_p = 2$ at all p:

$$g' - g = (-2K_B + \mathcal{S}_7 - \mathcal{S}_9) \cdot \mathcal{S}_9. \qquad (4.7)$$

This follows as there are $\mathcal{S}_9 \cdot [\Delta'] = 2\mathcal{S}_9 \cdot (-3K_B + \mathcal{S}_7 - 2\mathcal{S}_9)$ identical branch points p and since $g = 1 + \frac{1}{2}\mathcal{S}_9 \cdot (\mathcal{S}_9 + K_B)$. We note that (4.7) is precisely the multiplicity in the second line of table 3.

The enhancement points $V_{\text{bf}} = \{s_9 = \tilde{s}_1 = 0\}$ support bifundamental matter. The $(\mathbf{2},\mathbf{7})$ representation is self-conjugate, and thus allows for half-hypermultiplets; indeed, as encountered in the context of non-Higgsable clusters [22], each such point supports a half-hypermultiplet in this representation. The number of bifundamentals is thus given by $\mathcal{S}_9 \cdot (-3K_B - \mathcal{S}_7 - \mathcal{S}_9)$ yielding the first line in table 3. In addition to supporting bifundamentals, at the intersection points $s_9 = \tilde{s}_1 = 0$ there must also be one additional $(\mathbf{2},\mathbf{1})$ representation at $s_0 = \tilde{s}_1 = 0$. As in the analysis of [20, 22], this can be seen by

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[6]Thanks to D. Morrison for discussions on this point.

[7]This is expected as the gauge group G_2 arises by acting with the outer automorphism $\mathbb{Z}_3$ on the Dynkin diagram of SO(8).

analyzing the matter structure through the monodromy cover; the G_2 can be enhanced to an SO(7), under which the $\mathbf{7} + \mathbf{1}$ of G_2 combine to a spinor $\mathbf{8}$ representation. Taking into account the SU(2) fundamentals at the points $V_2 = \{\tilde{s}_1 = 4\tilde{s}_2\tilde{s}_4 - \tilde{s}_3^2 = 0\}$ of I_3 fibers, we obtain, using (2.2), the third line of table 3.

Deformations of $I_2 \times I_0^{*\mathrm{ns}}$ Weierstrass models. Finally, we reverse our perspective and apply the above results to describe how to deform an elliptic fibration with standard I_2 and $I_0^{*\mathrm{ns}}$ singularities, i.e. an F-theory geometry with SU(2) $\times G_2$ gauge symmetry, to a "Higgsed" elliptic fibration with only an I_2 singularity, i.e. an SU(2) gauge group, but codimension two singularities giving rise to the discussed matter in the three-index symmetric tensor representation. The idea is to start with the tuned geometry specified by the Weierstrass model (4.3) and to view the original model defined by $X^{\mathrm{SU}(2)}$ as a deformation thereof. To this end, we introduce the deformation parameter

$$\epsilon := s_8 - as_9 \tag{4.8}$$

describing the deviation from the tuning (4.1). The class of s_8, expressed in terms of the classes of the SU(2) and G_2 divisors $\tilde{s}_1$ and s_9, respectively, reads

$$[\epsilon] = 2[s_9] + [\tilde{s}_1] + 2K_B \,, \tag{4.9}$$

which imposes a minimal degree of $\tilde{s}_1$ and s_9 for the deformations ϵ to exist. In addition, this implies that the degree of ϵ is completely fixed if the degrees of $\tilde{s}_1$ and s_9 are given. Employing the parametrization of the Weierstrass model (4.3) in terms of the sections $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{s}_3$, s_4 and s_9 as well as the definition of ϵ, we express the deformed Weierstrass model (3.9) as

$$f = \left(-\frac{1}{3}\tilde{s}_2^2 + \tilde{s}_3\tilde{s}_1\right)s_9^2 + \left(\frac{1}{3}\tilde{s}_2\tilde{s}_3 - 3\tilde{s}_1\tilde{s}_4\right)s_9\epsilon + \left(\tilde{s}_2\tilde{s}_4 - \frac{1}{3}\tilde{s}_3^2\right)\epsilon^2 \,, \tag{4.10}$$

$$g = \left(-\frac{2}{27}\tilde{s}_2^3 + \frac{1}{3}\tilde{s}_2\tilde{s}_3\tilde{s}_1 - \tilde{s}_4\tilde{s}_1^2\right)s_9^3 + \left(\tilde{s}_1\left(\tilde{s}_2\tilde{s}_4 - \frac{2}{3}\tilde{s}_3^2\right) + \frac{1}{9}\tilde{s}_2^2\tilde{s}_3\right)s_9^2\epsilon$$

$$+ \left(\frac{1}{9}\tilde{s}_2\tilde{s}_3^2 - \frac{2}{3}\tilde{s}_2^2\tilde{s}_4 + \tilde{s}_1\tilde{s}_3\tilde{s}_4\right)s_9\epsilon^2 + \left(\frac{1}{3}\tilde{s}_2\tilde{s}_3\tilde{s}_4 - \frac{2}{27}\tilde{s}_3^3 - \tilde{s}_1\tilde{s}_4^2\right)\epsilon^3 \,.$$

We readily check that (4.10) reduces to (4.3) in the limit $\epsilon \to 0$. Its I_2 singularity is located, in the employed parametrization, at

$$t = -\tilde{s}_1 s_9^3 + \tilde{s}_2 s_9^2\epsilon - \tilde{s}_3 s_9\epsilon^2 + \tilde{s}_4\epsilon^3 = 0 \tag{4.11}$$

with ordinary triple point singularities at $s_9 = \epsilon = 0$.

In field theory, the above deformation corresponds to a Higgsing of the SU(2)$\times G_2$ theory. Indeed, we see that the spectrum in table 3 exactly reproduces the SU(2) spectrum in table 2 as

$$x_{\mathbf{4}} = x_{(\mathbf{2},\mathbf{7})} + 2(x_{(\mathbf{1},\mathbf{14})} - 1)\,, \quad x_{\mathbf{3}} = x_{(\mathbf{3},\mathbf{1})} + 2x_{(\mathbf{2},\mathbf{7})} + x_{(\mathbf{1},\mathbf{7})} + x_{(\mathbf{1},\mathbf{14})} - 1\,,$$

$$x_{\mathbf{2}} = 2x_{(\mathbf{1},\mathbf{7})} + x_{(\mathbf{2},\mathbf{7})} + x_{(\mathbf{2},\mathbf{1})}\,, \tag{4.12}$$

where the -2 and -1 in the multiplicities take into account the fields eaten up by the massive gauge bosons. This corresponds to the group theoretical breaking

$$\mathrm{SU}(2) \times G_2 \supset \mathrm{SU}(2)^3 \longrightarrow \mathrm{SU}(2)\,, \tag{4.13}$$

where we first embed the regular subgroup $\mathrm{SU}(2)^2$ into G_2 and then break to $\mathrm{SU}(2)$. The relevant representations branch as

$$\begin{aligned}
(\mathbf{1}, \mathbf{14}) &\cong (\mathbf{1}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{4}) \longrightarrow \mathbf{3} \oplus 3 \cdot \mathbf{1} \oplus 2 \cdot \mathbf{4}\,, \\
(\mathbf{2}, \mathbf{7}) &\cong (\mathbf{2}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}) \longrightarrow \mathbf{4} \oplus \mathbf{2} \oplus 2 \cdot (\mathbf{3} \oplus \mathbf{1})\,, \\
(\mathbf{1}, \mathbf{7}) &\cong (\mathbf{1}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}) \longrightarrow \mathbf{3} \oplus 2 \cdot \mathbf{2}\,, \\
(\mathbf{3}, \mathbf{1}) &\cong (\mathbf{3}, \mathbf{1}, \mathbf{1}) \to \mathbf{3}\,, \qquad (\mathbf{2}, \mathbf{1}) \cong (\mathbf{2}, \mathbf{1}, \mathbf{1}) \to \mathbf{2}\,.
\end{aligned} \tag{4.14}$$

Here, we denote by $\cong$ the presentation of $\mathrm{SU}(2) \times G_2$ irreducible representations as (reducible) representations of its subgroup $\mathrm{SU}(2)^3$. The embedding of the final $\mathrm{SU}(2)$ gauge group into $\mathrm{SU}(2)^3$ is such that representations of the middle $\mathrm{SU}(2)$ go to multiple copies of singlets and the tensor product of the representations of the two outer $\mathrm{SU}(2)$'s is formed, i.e. $(\mathbf{R}, \mathbf{R}', \mathbf{R}'') \to \dim(\mathbf{R}') \cdot (\mathbf{R} \otimes \mathbf{R}'')$.

The Higgs fields leading to the particular branching (4.14) transform in the $\mathrm{SU}(2) \times G_2$-representation $(\mathbf{2}, \mathbf{7})$. There are 17 vector multiplets before and three after Higgsing. The 14 vector multiplets that get massive in the Higgsing transform according to the first line in (4.14) as one $\mathbf{3}$, three singlets $\mathbf{1}$ and two $\mathbf{4}$'s of the final $\mathrm{SU}(2)$. They eat up hypermultiplets in the broken $(\mathbf{2}, \mathbf{7})$ in the corresponding representations in the second line of (4.14). Thus, for this Higgsing to be possible there have to be four half-hypermultiplets in the real representation $(\mathbf{2}, \mathbf{7})$.[8] The Higgs VEVs have to be turned on along the singlet components in the second line of (4.14). As just mentioned, three $\mathrm{SU}(2)$-singlet hypermultiplets are eaten up by the massive vector multiplets. Thus, also three complex Higgs VEVs have to be fixed by supersymmetry. It would be interesting to understand this condition explicitly on the level of D-term constraints in the 6D effective SUGRA theory, which should describe the full moduli space of the resulting Higgsed theory being parametrized by all singlets in the breaking (4.14) with three fields fixed by D-flatness. Note that in the case mentioned above, for example, where on $\mathbb{P}^2$ we can tune a G_2 factor on a line, $[s_9] = 1$, and an $\mathrm{SU}(2)$ on a conic, $[s_1] = 2$, there are only two half-hypermultiplets in the $(\mathbf{2}, \mathbf{7})$ representation, explaining the inability to Higgs the model in this and other such cases, and correlating with the absence of an appropriate $\mathrm{SU}(2)$ model violating the effectiveness constraints from (2.2).

4.2 UnHiggsing $\mathrm{SU}(2)$ with 4 to $\mathrm{SU}(2)^3$ with trifundamentals

We conclude with a brief discussion of a different unHiggsing of the $\mathrm{SU}(2)$ model defined by F-theory on $X^{\mathrm{SU}(2)}$ leading to a theory with three $\mathrm{SU}(2)$ gauge algebras on three different divisors and with a matter spectrum which necessarily has to contain a trifundamental representation besides the standard adjoint, fundamental and bifundamental representation.

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[8]The number of half-hypers in the $(\mathbf{2}, \mathbf{7})$ is given, according to table 3, by $\mathcal{S}_9 \cdot (-3K_B - \mathcal{S}_7 - \mathcal{S}_9) \geq \mathcal{S}_9 \cdot (-2K_B - \mathcal{S}_9)$ for $[s_8] \geq [s_9]$. E.g. for $B = \mathbb{P}^2$ all models of the form (4.3) have at least 5 half-hypers.

The unHiggsing is preformed by imposing that the divisor $t = 0$ defined in (3.2) factorizes as

$$t = s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3 \overset{!}{=} \prod_{i=1}^{3} (a_i s_8 + b_i s_9) \,. \tag{4.15}$$

This imposes the obvious constraints of the form

$$s_4 = a_1 a_2 a_3 \,, \quad s_3 = -a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_3 b_1 \,, \quad s_2 = a_1 b_2 b_3 + a_2 b_1 b_3 + a_3 b_1 b_2 \,, \quad s_1 = -b_1 b_2 b_3 \,. \tag{4.16}$$

We note that under this tuning, the Weierstrass model (3.9) that is obtained by the special solution (3.5) develops six singularities of Kodaira type I_2. This is attributed to the fact that the simple solution overspecializes the complex structure of $X^{\mathrm{SU}(2)}$, leading to spurious singularities.

A more general Weierstrass form is obtained over a UFD using the tuning (3.4). In this case, imposing the conditions (4.16) introduces three singularities of Kodaira type I_2 along the three divisors

$$t_i := a_i s_8 + b_i s_9 = 0 \,, \qquad i = 1, 2, 3 \,. \tag{4.17}$$

The resulting Weierstrass model is algebraically very complex. Instead of presenting it here, we just mention its key properties. A careful analysis of its codimension two singularities reveals that the resulting model has matter in the fundamental representations w.r.t. all three SU(2) factors as well as in all possible bifundamental representations of two SU(2)'s. Most notably, at the codimension two locus $s_8 = s_9 = 0$ the three SU(2) divisors $t_i = 0$ intersect. Employing the fact that the Weierstrass model is not of the standard I_2 form following from Tate's algorithm, it can be argued that there is trifundamental matter located at these points. This is also required by the Higgsing back to the original SU(2) model specified by $X^{\mathrm{SU}(2)}$.

We will return to analyzing SU(2)^3 models with trifundamental matter and their (un-)Higgsings in future work [18].

5 Conclusions

In this paper we have presented an explicit construction of a class of Weierstrass models that realize matter in the three-index symmetric (**4**) representation of SU(2). For 6D F-theory models, this matter is localized at triple point singularities in the curve C carrying the gauge group. Such singularities have a contribution $g_a = 3$ to the arithmetic genus of C, matching with the formula (1.1) and the conjectured interpretation of this formula in [13]. To our knowledge, this represents the first explicit realization in the F-theory literature of any matter representation with a genus contribution $g > 1$ through a Weierstrass model.

In the Weierstrass models studied here the gauge group lives on a curve of the form $t = A\xi^3 + B\xi^2\eta + C\xi\eta^2 + D\eta^3$, where the triple point singularities are found at the locus of points satisfying $\xi = \eta = 0$. This is closely parallel to the framework of [15, 16], where two-index symmetric matter was found to live on curves of the form $t = A\xi^2 + B\xi\eta + C\eta^2$. Here, as in those papers, the vanishing of the discriminant Δ to order N for an I_N singularity

depends on the singular structure of t, and the Weierstrass model does not take the simple form that follows when one starts from the general Tate model for an I_N singularity on a general divisor t and transforms to Weierstrass form. This matches with the analysis of [16], in which transitions between theories with different matter content were studied. It was found there that for 6D theories, a transition between two models with distinct matter representations and a given gauge group occurs when the model passes through a superconformal fixed point. Indeed, by continuity it seems impossible to change matter representations without such a transition when the gauge group is kept fixed. Thus, for example, tuning a Tate type model with an $SU(N)$ gauge group on a smooth curve C and then taking a singular limit of C cannot change the matter content, so the full genus contribution must still come from adjoint matter in any model where the Weierstrass model comes from the generic Tate I_N form. This explains the necessity for the remarkable algebraic structure involved in the realizations of the symmetric matter representations found in this and previous works.

Another remarkable feature of the analysis here is that the Weierstrass form of the U(1) models of [19] that we have used does not seem to fit in the general classification given in [21]. In that paper a general argument was given suggesting that any F-theory model with an Abelian factor should have a Weierstrass description of the form

$$y^2 = x^3 + \left(c_1 c_3 - b^2 c_0 - \frac{1}{3}c_2^2\right)x + \left(c_0 c_3^2 - \frac{1}{3}c_1 c_2 c_3 + \frac{2}{27}c_2^3 - \frac{2}{3}b^2 c_0 c_2 + \frac{1}{4}b^2 c_1^2\right) . \quad (5.1)$$

The Weierstrass models for U(1) theories with charge $q = 3$ matter we consider here, do not, however, seem to take this form [19]. In fact, we would have a problem if they did. It was argued in [21, 30] that in any U(1) model of the form (5.1), taking $b \to 0$ gives an unHiggsing to an $SU(2)$ model. The resulting $SU(2)$ model, however is always in the form that follows by starting with a generic Tate I_2 construction, with the $SU(2)$ realized on the divisor $\{c_3 = 0\}$, and transforming to Weierstrass form. It seems then from the discussion above and the analysis of [16] that any such $SU(2)$ can only have $g_R > 0$ matter coming from adjoint representations and cannot include exotic matter such as three-index symmetric matter representations. Thus, the existence of these constructions seems to suggest that there must be a more general class of U(1) models than those constructed in [21]. We can understand this further by considering that in [21] the form (5.1) arose from a situation where the extra section had an explicit description through

$$[x, y, z] = \left[c_3^2 - \frac{2}{3}b^2 c_2, -c_3^3 + b^2 c_2 c_3 - \frac{1}{2}b^4 c_1, b\right] . \quad (5.2)$$

Comparing to the expressions for the section $[x_1, y_1, z_1]$ in appendix A, we find that in our case there is a similar description, where identifying $b \equiv z_1 = s_7 s_8^2 - s_6 s_8 s_9 + s_5 s_9^2$ the section can be described in the form

$$[x, y, z] = \left[c_3^2 - \frac{2}{3}bc_2, -c_3^3 + bc_2 c_3 - \frac{1}{2}b^2 c_1, b\right] . \quad (5.3)$$

Understanding better how to construct more general classes of U(1) models with higher charges that allow unHiggsing to non-Abelian $SU(2)$ models with exotic matter representations may shed light on the general construction of Weierstrass models where gauge groups

are realized on singular divisors. A natural starting point, for example, is the complete intersection U(1)^3 model in [38]

This paper has presented a novel and specific example of a rather remarkable geometric and algebraic structure that can arise in F-theory, adding to the small set of explicit classes of Weierstrass models known that realize exotic matter representations. There are many ways in which it would be interesting to expand on these developments, both in terms of this and other specific realizations and in terms of more general theoretical structures.

For the specific class of representations studied here, namely the **4** of SU(2), it would be interesting to analyze the dual heterotic models in cases with a smooth heterotic dual, as was done for the two-index symmetric representation of SU(3) in [16]. Also following the lines of [16], it seems that analogous constructions to those found here can be realized explicitly through exotic matter transitions in a further unHiggsed non-Abelian theory; results on this will be presented elsewhere [18].[9]

In principle, the methods used here could be used to construct larger exotic SU(N) representations. To follow the same logic as that presented here for higher-dimensional representations of SU(2), for example, we would need to identify models with U(1) gauge fields and matter fields transforming under representations of charge $q > 3$. More generally, it would be desirable to address the general challenge of classifying the algebraic structures that can be used in the Weierstrass model to construct general gauge groups over singular divisors, and to bring together algebraic, geometric, and field theory understandings of these more exotic matter representations along with their Higgsings and unHiggsings to theories with Abelian or higher-rank non-Abelian gauge theories. This seems like a rich arena for exploration, with highly intricate and nontrivial structure in the Weierstrass models encoding these features, and we anticipate that further study of these questions will lead to additional novel results and increased understanding. Finally, getting a systematic handle on the types of codimension two singularities that can be realized in Weierstrass models for elliptically fibered Calabi-Yau manifolds would be an important step towards completing the systematic classification of such geometries [22, 39–43].

Acknowledgments

A   Representation in Tate and Weierstrass form

Here we present the explicit expressions for the Weierstrass model of the dP_1-elliptic fibration X in (2.1). We refer the reader to [19] for more details.

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[9]We thank Nikhil Raghuram for discussions related to this issue.

We apply Nagell's algorithm to the cubic (2.1) with respect to the point $\hat{c}_0 \cap \mathcal{E}$ to obtain a birational map to its WSF. We determine the functions f, g of this WSF to be given by

$$f = \frac{1}{48} \left(24 \left(2 \left(s_2 s_4 s_8^2 + s_1 \left(s_7^2 - 3 s_4 s_9 \right) s_8 + s_5 \left(s_4 s_5 + s_2 s_7 \right) s_9 + s_3 \left(s_5 s_7 s_8 + s_9 \left(s_2 s_8 + s_1 s_9 \right) \right) \right) \right. \right.$$
$$\left. \left. - s_6 \left(s_4 s_5 s_8 + s_2 s_7 s_8 + \left(s_3 s_5 + s_1 s_7 \right) s_9 \right) \right) - \left(s_6^2 - 4 \left(s_5 s_7 + s_3 s_8 + s_2 s_9 \right) \right)^2 \right)$$

$$g = \frac{1}{864} \left(\left(s_6^2 - 4 \left(s_5 s_7 + s_3 s_8 + s_2 s_9 \right) \right)^3 - 36 \left(2 \left(s_2 s_4 s_8^2 + s_1 \left(s_7^2 - 3 s_4 s_9 \right) s_8 + s_5 \left(s_4 s_5 + s_2 s_7 \right) s_9 \right. \right. \right.$$
$$\left. + s_3 \left(s_5 s_7 s_8 + s_9 \left(s_2 s_8 + s_1 s_9 \right) \right) \right) - s_6 \left(s_4 s_5 s_8 + s_2 s_7 s_8 + \left(s_3 s_5 + s_1 s_7 \right) s_9 \right) \right) \left(s_6^2 - 4 s_5 s_7 - 4 s_3 s_8 \right.$$
$$\left. - 4 s_2 s_9 \right) + 216 \left(\left(s_2^2 - 4 s_1 s_3 \right) s_7^2 s_8^2 + s_4^2 \left(s_5^2 - 4 s_1 s_8 \right) s_8^2 - 2 s_7 \left(s_2 \left(s_3 s_5 + s_1 s_7 \right) - 2 s_1 s_3 s_6 \right) s_9 s_8 \right.$$
$$+ \left(\left(s_3 s_5 - s_1 s_7 \right)^2 - 4 s_1 s_3^2 s_8 \right) s_9^2 + 2 s_4 \left(-2 s_1^2 s_9^3 + 2 \left(s_1 s_5 s_6 - s_2 \left(s_5^2 - 2 s_1 s_8 \right) \right) s_9^2 \right.$$
$$\left. \left. \left. - s_8 \left(2 s_8 s_2^2 - 2 s_5 s_6 s_2 + 2 s_1 s_6^2 + s_1 s_5 s_7 + s_3 \left(s_5^2 - 4 s_1 s_8 \right) \right) s_9 + \left(2 s_1 s_6 - s_2 s_5 \right) s_7 s_8^2 \right) \right) \right) \qquad (A.1)$$

We observe that there is no factorization of the discriminant Δ following from f and g indicating the absence of codimension one singularities and a non-Abelian gauge group.

Furthermore, we plug the coordinates of the rational section (2.4) into this map to obtain its coordinates in WSF,

$$z_1 = s_7 s_8^2 - s_6 s_8 s_9 + s_5 s_9^2, \qquad (A.2)$$

$$x_1 = \frac{1}{12} \left(12 s_1^2 s_9^6 + 4 \left(2 s_2 \left(s_5^2 - 3 s_1 s_8 \right) - 3 s_1 s_5 s_6 \right) s_9^5 + \left(\left(s_6^2 - 4 s_5 s_7 \right) s_5^2 + 12 \left(s_2^2 + 2 s_1 s_3 \right) s_8^2 - 4 \left(4 s_3 s_5^2 \right. \right. \right.$$
$$\left. + s_2 s_6 s_5 - 3 s_1 \left(s_6^2 + 2 s_5 s_7 \right) \right) s_8 \right) s_9^4 - 2 s_8 \left(-4 \left(s_6 s_7 + 3 s_4 s_8 \right) s_5^2 + \left(s_6^3 - 10 s_3 s_8 s_6 + 4 s_2 s_7 s_8 \right) s_5 \right.$$
$$\left. + 2 s_8 \left(9 s_1 s_6 s_7 + 6 s_1 s_4 s_8 + s_2 \left(s_6^2 + 6 s_3 s_8 \right) \right) \right) s_9^3 + s_8^2 \left(s_6^4 - 2 s_5 s_7 s_6^2 - 8 s_5^2 s_7^2 + 12 \left(s_3^2 + 2 s_2 s_4 \right) s_8^2 \right.$$
$$\left. - 4 \left(9 s_4 s_5 s_6 - s_7 \left(5 s_2 s_6 + 6 s_1 s_7 \right) + s_3 \left(s_6^2 + 2 s_5 s_7 \right) \right) s_8 \right) s_9^2 - 2 s_8^3 \left(12 s_3 s_4 s_8^2 + 2 \left(s_7 \left(s_3 s_6 + 4 s_2 s_7 \right) \right. \right.$$
$$\left. \left. - 3 s_4 \left(s_6^2 + 2 s_5 s_7 \right) \right) s_8 + s_6 s_7 \left(s_6^2 - 4 s_5 s_7 \right) \right) s_9 + s_8^4 \left(\left(s_6^2 - 4 s_5 s_7 \right) s_7^2 + 4 \left(2 s_3 s_7 - 3 s_4 s_6 \right) s_8 s_7 + 12 s_4^2 s_8^2 \right) \right),$$

$$y_1 = \frac{1}{2} \left(2 s_1^3 s_9^9 + s_1 \left(2 s_2 \left(s_5^2 - 3 s_1 s_8 \right) - 3 s_1 s_5 s_6 \right) s_9^8 + \left(\left(s_3 s_5^2 - s_2 s_6 s_5 + s_1 \left(s_6^2 - s_5 s_7 \right) \right) s_5^2 \right. \right.$$
$$\left. + 6 s_1 \left(s_2^2 + s_1 s_3 \right) s_8^2 + \left(-2 s_2^2 s_5^2 + 2 s_1 s_2 s_6 s_5 + s_1 \left(3 s_1 \left(s_6^2 + 2 s_5 s_7 \right) - 4 s_3 s_5^2 \right) \right) s_8 \right) s_9^7$$
$$- s_8 \left(2 \left(s_2^3 + 6 s_1 s_3 s_2 + 3 s_1^2 s_4 \right) s_8^2 - \left(s_5 s_6 s_2^2 + \left(6 s_3 s_5^2 - 4 s_1 \left(s_6^2 + 2 s_5 s_7 \right) \right) s_2 + s_1 \left(6 s_4 s_5^2 + 2 s_3 s_6 s_5 \right. \right. \right.$$
$$\left. - 9 s_1 s_6 s_7 \right) \right) s_8 + s_5 \left(3 s_4 s_5^3 + 2 s_3 s_6 s_5^2 - 3 s_2 s_7 s_5^2 - 2 s_2 s_6^2 s_5 + s_1 s_6 s_7 s_5 + 2 s_1 s_6^3 \right) \right) s_9^6 + s_8^2 \left(s_1 s_6^4 \right.$$
$$- s_2 s_5 s_6^3 + s_3 s_5^2 s_6^2 + 7 s_1 s_5 s_7 s_6^2 + 9 s_4 s_5^3 s_6 - 8 s_2 s_5^2 s_7 s_6 + s_1 s_5^2 s_7^2 + 6 \left(s_3 \left(s_2^2 + s_1 s_3 \right) + 2 s_1 s_2 s_4 \right) s_8^2$$
$$- s_3 s_5^3 s_7 + \left(s_2^2 s_6^2 - 4 s_3^2 s_5^2 - 8 s_2 s_4 s_5^2 - 6 s_1 s_4 s_6 s_5 + 6 s_1^2 s_7^2 + 2 s_2 \left(s_2 s_5 + 7 s_1 s_6 \right) s_7 + s_3 \left(2 s_1 \left(s_6^2 + 2 s_5 s_7 \right) \right. \right.$$
$$\left. \left. - 6 s_2 s_5 s_6 \right) \right) s_8 \right) s_9^5 - s_8^3 \left(s_8 \left(6 s_2 s_8 - 5 s_5 s_6 \right) s_3^2 - 5 s_6 s_7 \left(s_5^2 - 2 s_1 s_8 \right) s_3 + 5 s_7 \left(s_6 s_8 s_2^2 + 2 s_1 s_7 s_8 s_2 \right. \right.$$
$$\left. - s_2 s_5 \left(s_6^2 + s_5 s_7 \right) + s_1 s_6 \left(s_6^2 + 2 s_5 s_7 \right) \right) + s_4 \left(5 \left(2 s_6^2 + s_5 s_7 \right) s_5^2 - 10 \left(s_3 s_5 + s_2 s_6 \right) s_8 s_5 \right.$$
$$\left. + 6 \left(s_2^2 + 2 s_1 s_3 \right) s_8^2 \right) \right) s_9^4 + s_8^4 \left(2 \left(s_3^3 + 6 s_2 s_4 s_3 + 3 s_1 s_4^2 \right) s_8^2 - \left(6 s_4^2 s_5^2 + s_3^2 s_6^2 - 4 \left(s_2^2 + 2 s_1 s_3 \right) s_7^2 \right. \right.$$
$$\left. + 2 s_3 \left(s_3 s_5 - 3 s_2 s_6 \right) s_7 + 2 s_4 \left(s_2 s_6^2 + 7 s_3 s_5 s_6 - 3 s_1 s_7 s_6 + 2 s_2 s_5 s_7 \right) \right) s_8 + 5 \left(s_4 s_5 s_6 \left(s_6^2 + 2 s_5 s_7 \right) \right.$$
$$\left. \left. + s_7 \left(s_7 \left(2 s_1 s_6^2 - s_2 s_5 s_6 + s_1 s_5 s_7 \right) - s_3 s_5 \left(s_6^2 + s_5 s_7 \right) \right) \right) \right) s_9^3 - s_8^5 \left(3 s_8 \left(2 s_2 s_8 - 3 s_5 s_6 \right) s_4^2 + \left(s_6^4 \right. \right.$$
$$\left. + \left(7 s_5 s_7 - 4 s_3 s_8 \right) s_6^2 + 2 s_2 s_7 s_8 s_6 + s_5^2 s_7^2 - 8 s_3 s_5 s_7 s_8 + 6 s_8 \left(s_8 s_3^2 + s_1 s_7^2 \right) \right) s_4 + s_7 \left(s_6 s_8 s_3^2 - \left(s_6^3 \right. \right.$$
$$\left. \left. + 8 s_5 s_7 s_6 - 6 s_2 s_7 s_8 \right) s_3 + s_7 \left(9 s_1 s_6 s_7 + s_2 \left(s_6^2 - s_5 s_7 \right) \right) \right) \right) s_9^2 + s_8^6 \left(3 s_8 \left(-s_6^2 - 2 s_5 s_7 + 2 s_3 s_8 \right) s_4^2 \right.$$
$$+ s_7 \left(2 s_6^3 + s_5 s_7 s_6 - 2 s_3 s_8 s_6 + 4 s_2 s_7 s_8 \right) s_4 + s_7^2 \left(2 s_8 s_3^2 - 2 s_6^2 s_3 - 3 s_5 s_7 s_3 + 3 s_1 s_7^2 + 2 s_2 s_6 s_7 \right) \right) s_9$$
$$\left. + s_8^7 \left(-2 s_8^2 s_4^3 + 3 s_6 s_7 s_8 s_4^2 + s_7^2 \left(-s_6^2 + s_5 s_7 - 2 s_3 s_8 \right) s_4 + s_7^3 \left(s_3 s_6 - s_2 s_7 \right) \right) \right).$$

The Weierstrass form (A.1) can be obtained from a Tate model with the following Tate coefficients [19]:

$$a_1 = s_6, \quad a_2 = -s_5 s_7 - s_3 s_8 - s_2 s_9, \quad a_3 = -(s_4 s_5 + s_2 s_7) s_8 - (s_3 s_5 + s_1 s_7) s_9, \tag{A.3}$$

$$a_4 = s_1 s_3 s_9^2 + (s_2 (s_5 s_7 + s_3 s_8) + s_4 (s_5^2 - 3 s_1 s_8)) s_9 + s_8 (s_1 s_7^2 + s_3 s_5 s_7 + s_2 s_4 s_8),$$

$$a_6 = -s_1^2 s_4 s_9^3 - (s_2 s_4 (s_5^2 - 2 s_1 s_8) + s_1 (s_3 (s_5 s_7 + s_3 s_8) - s_4 s_5 s_6)) s_9^2 - s_8 (s_4 s_8 s_2^2 + (s_1 s_7^2 - s_4 s_5 s_6) s_2$$
$$+ s_1 s_4 (s_6^2 + s_5 s_7) + s_3 ((s_2 s_5 - s_1 s_6) s_7 + s_4 (s_5^2 - 2 s_1 s_8))) s_9 - s_8^2 (s_2 s_4 s_5 s_7 + s_1 (s_8 s_4^2 + s_3 s_7^2 - s_4 s_6 s_7)).$$

References

[1] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022] [INSPIRE].

[2] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114] [INSPIRE].

[3] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2.*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161] [INSPIRE].

[4] K. Kodaira, *On compact analytic surfaces: Ii*, *Ann. Math.* **77** (1963) 563.

[5] S.H. Katz and C. Vafa, *Matter from geometry*, *Nucl. Phys.* **B 497** (1997) 146 [hep-th/9606086] [INSPIRE].

[6] M. Bershadsky, K.A. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys.* **B 481** (1996) 215 [hep-th/9605200] [INSPIRE].

[7] D.R. Morrison and W. Taylor, *Matter and singularities*, *JHEP* **01** (2012) 022 [arXiv:1106.3563] [INSPIRE].

[8] M. Esole and S.-T. Yau, *Small resolutions of* SU(5)-*models in F-theory*, *Adv. Theor. Math. Phys.* **17** (2013) 1195 [arXiv:1107.0733] [INSPIRE].

[9] H. Hayashi, C. Lawrie, D.R. Morrison and S. Schäfer-Nameki, *Box Graphs and Singular Fibers*, *JHEP* **05** (2014) 048 [arXiv:1402.2653] [INSPIRE].

[10] M. Esole, S.-H. Shao and S.-T. Yau, *Singularities and Gauge Theory Phases*, *Adv. Theor. Math. Phys.* **19** (2015) 1183 [arXiv:1402.6331] [INSPIRE].

[11] A.P. Braun and S. Schäfer-Nameki, *Box Graphs and Resolutions I*, *Nucl. Phys.* **B 905** (2016) 447 [arXiv:1407.3520] [INSPIRE].

[12] A. Grassi, J. Halverson and J.L. Shaneson, *Matter From Geometry Without Resolution*, *JHEP* **10** (2013) 205 [arXiv:1306.1832] [INSPIRE].

[13] V. Kumar, D.S. Park and W. Taylor, 6*D supergravity without tensor multiplets*, *JHEP* **04** (2011) 080 [arXiv:1011.0726] [INSPIRE].

[14] V. Sadov, *Generalized Green-Schwarz mechanism in F-theory*, *Phys. Lett.* **B 388** (1996) 45 [hep-th/9606008] [INSPIRE].

[15] M. Cvetič, D. Klevers, H. Piragua and W. Taylor, *General* U(1) × U(1) *F-theory compactifications and beyond: geometry of unHiggsings and novel matter structure*, *JHEP* **11** (2015) 204 [arXiv:1507.05954] [INSPIRE].

[16] L.B. Anderson, J. Gray, N. Raghuram and W. Taylor, *Matter in transition*, *JHEP* **04** (2016) 080 [arXiv:1512.05791] [INSPIRE].

[17] S. Katz, D.R. Morrison, S. Schäfer-Nameki and J. Sully, *Tate's algorithm and F-theory*, *JHEP* **08** (2011) 094 [arXiv:1106.3854] [INSPIRE].

[18] D. Klevers, D.R. Morrison, N. Raghuram and W. Taylor, to appear.

[19] D. Klevers, D.K. Mayorga Pena, P.-K. Oehlmann, H. Piragua and J. Reuter, *F-Theory on all Toric Hypersurface Fibrations and its Higgs Branches*, *JHEP* **01** (2015) 142 [arXiv:1408.4808] [INSPIRE].

[20] A. Grassi and D.R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*, *Commun. Num. Theor. Phys.* **6** (2012) 51 [arXiv:1109.0042] [INSPIRE].

[21] D.R. Morrison and D.S. Park, *F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds*, *JHEP* **10** (2012) 128 [arXiv:1208.2695] [INSPIRE].

[22] D.R. Morrison and W. Taylor, *Classifying bases for* 6D *F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072 [arXiv:1201.1943] [INSPIRE].

[23] D.R. Morrison and W. Taylor, *Non-Higgsable clusters for* 4D *F-theory models*, *JHEP* **05** (2015) 080 [arXiv:1412.6112] [INSPIRE].

[24] D.S. Park, *Anomaly Equations and Intersection Theory*, *JHEP* **01** (2012) 093 [arXiv:1111.2351] [INSPIRE].

[25] L.B. Anderson, I. García-Etxebarria, T.W. Grimm and J. Keitel, *Physics of F-theory compactifications without section*, *JHEP* **12** (2014) 156 [arXiv:1406.5180] [INSPIRE].

[26] M. Cvetič, D. Klevers and H. Piragua, *F-Theory Compactifications with Multiple* U(1)*-Factors: Constructing Elliptic Fibrations with Rational Sections*, *JHEP* **06** (2013) 067 [arXiv:1303.6970] [INSPIRE].

[27] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral Four-Dimensional F-theory Compactifications With* SU(5) *and Multiple U(1)-Factors*, *JHEP* **04** (2014) 010 [arXiv:1306.3987] [INSPIRE].

[28] J. Erler, *Anomaly cancellation in six-dimensions*, *J. Math. Phys.* **35** (1994) 1819 [hep-th/9304104] [INSPIRE].

[29] D.S. Park and W. Taylor, *Constraints on* 6D *Supergravity Theories with Abelian Gauge Symmetry*, *JHEP* **01** (2012) 141 [arXiv:1110.5916] [INSPIRE].

[30] D.R. Morrison and W. Taylor, *Sections, multisections and* U(1) *fields in F-theory*, arXiv:1404.1527 [INSPIRE].

[31] T.W. Grimm, A. Kapfer and D. Klevers, *The Arithmetic of Elliptic Fibrations in Gauge Theories on a Circle*, *JHEP* **06** (2016) 112 [arXiv:1510.04281] [INSPIRE].

[32] C. Lawrie and D. Sacco, *Tate's algorithm for F-theory GUTs with two* U(1)*s*, *JHEP* **03** (2015) 055 [arXiv:1412.4125] [INSPIRE].

[33] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, in *Lecture Notes in Mathematics. Vol. 476: Modular functions of one variable IV*, Springer, Berlin Germany (1975), pg. 33.

[34] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys.* **B 471** (1996) 195 [hep-th/9603150] [INSPIRE].

[35] L. Bhardwaj, M. Del Zotto, J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *F-theory and the Classification of Little Strings*, *Phys. Rev.* **D 93** (2016) 086002 [arXiv:1511.05565] [INSPIRE].

[36] W. Taylor, *TASI Lectures on Supergravity and String Vacua in Various Dimensions*, arXiv:1104.2051 [INSPIRE].

[37] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, New York U.S.A. (2014).

[38] M. Cvetič, D. Klevers, H. Piragua and P. Song, *Elliptic fibrations with rank three Mordell-Weil group: F-theory with* $U(1) \times U(1) \times U(1)$ *gauge symmetry*, *JHEP* **03** (2014) 021 [arXiv:1310.0463] [INSPIRE].

[39] D.R. Morrison and W. Taylor, *Toric bases for* 6*D F-theory models*, *Fortsch. Phys.* **60** (2012) 1187 [arXiv:1204.0283] [INSPIRE].

[40] W. Taylor, *On the Hodge structure of elliptically fibered Calabi-Yau threefolds*, *JHEP* **08** (2012) 032 [arXiv:1205.0952] [INSPIRE].

[41] G. Martini and W. Taylor, 6*D F-theory models and elliptically fibered Calabi-Yau threefolds over semi-toric base surfaces*, *JHEP* **06** (2015) 061 [arXiv:1404.6300] [INSPIRE].

[42] S.B. Johnson and W. Taylor, *Calabi-Yau threefolds with large* $h^{2,1}$, *JHEP* **10** (2014) 23 [arXiv:1406.0514] [INSPIRE].

[43] W. Taylor and Y.-N. Wang, *Non-toric Bases for Elliptic Calabi-Yau Threefolds and* 6*D F-theory Vacua*, arXiv:1504.07689 [INSPIRE].