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Compactness results for pseudo-holomorphic curves in symplectic cobordisms

by

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Abstract. We prove a compactness theorem for pseudo-holomorphic curves with totally real boundary condition in 4-dimensional symplectic cobordisms, using Taubes' theory of positive cohomology assignments. We define a relative intersection number for surfaces in a smooth manifold X^4 with boundary on a prescribed submanifold $Y^2 \subset X$. Using this relative intersection number, we extend Taubes' theory to treat pseudo-holomorphic curves in an almost complex manifold X^4, J with boundary on a fixed totally real submanifold $Y \subset X$.

Thesis Supervisor: Tomasz S. Mrowka
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Contents

CHAPTER 1

Introduction

A symplectic manifold (X, ω) with boundary $\partial X = M_+ \amalg M_-$ is called a *symplectic cobordism* if both $M_\pm$ have contact type in X. By definition, this means that there are vector fields $V_\pm$ defined near and transverse to $M_\pm$ which expand the symplectic form: $\mathcal{L}_{V_\pm}\omega = \omega$. We require V_+ to point outwards along M_+ and we call M_+ the ω-convex end of X. Similarly, V_- is required to point inwards along M_-, and we call M_- the ω-concave end of X. We will say that X is a symplectic cobordism *from M_- to M_+* in order to distinguish its concave end from its convex end.

The most basic example is that of a trivial symplectic cobordism from a contact manifold M, ξ to itself. In this case, we endow $X = [0,1] \times M$ with the symplectic form $\omega = d(e^t\alpha)$, where α is any contact form for the contact structure on M. Other examples of symplectic cobordisms come from classical phase spaces, complex geometry, and from Weinstein's procedure for performing contact surgery using symplectic handlebodies [**W**].

The existence of a symplectic cobordism from one contact manifold M_-, ξ_- to another M_+, ξ_+ does not guarantee the existence of a symplectic cobordism from M_+, ξ_+ to M_-, ξ_-. Thus, unlike the classical notion of cobordism, symplectic cobordism does not define an equivalence relation on the class of contact manifolds. It only defines a partial ordering on this class. Yet it remains an interesting problem to determine what a symplectic cobordism tells us about the contact manifolds that bound it.

Recall, for example, that a contact 3-manifold M, ξ is *overtwisted* if there exists an embedded disk $Y \subset M$ such that ∂Y is Legendrian for ξ, and Y is everywhere transverse to ξ except at one singular point $e \in Y$ (which must be elliptic). If M, ξ is not overtwisted, then it is called *tight*. In 1989, Eliashberg [**E1**] showed that the overtwisted structures on a closed 3-manifold M are essentially classified by homotopy classes of 2-plane distributions on M. Therefore the problem of classifying contact structures on 3-manifolds amounts to classifying the tight structures. In general, however, it is difficult to determine when a given contact structure is tight. The next result of Eliashberg [**E2**] was thus significant for it showed that many known contact manifolds were tight.

THEOREM 1 (Eliashberg). *A symplectically fillable contact manifold is tight.*

A contact manifold is (strongly) *symplectically fillable* if there exists a symplectic manifold X, ω such that $\partial X = M$ and M is ω-convex in X. In other words, X is a symplectic cobordism from the empty manifold to M. If we think of the empty contact manifold as being tight, then a natural question to ask in light of Eliashberg's result is the following:

(1) *Let M be a contact 3-manifold. If there exists a symplectic cobordism from a tight contact manifold to M, then must M be tight?*

Following Gromov [**Gr**], Eliashberg [**E2**] and Hofer [**H**], one strategy for answering (1) would be to study pseudo-holomorphic curves in the symplectic cobordism in question. To understand this strategy, let us briefly recall Hofer's analysis [**H**] of pseudo-holomorphic curves in the symplectization of a contact manifold M.

Let α be a contact form for the contact structure ξ on M, and let v_α be the associated Reeb vector field, defined by

$$\iota_{v_\alpha} d\alpha = 0 \quad , \quad \iota_{v_\alpha} \alpha = 1 \, .$$

If J is a complex structure on ξ that is compatible with $d\alpha|_\xi$, then J extends to an almost complex structure on the symplectization $\mathbb{R} \times M$ by setting

$$J(\partial_t) = v_\alpha \quad , \quad J(v_\alpha) = \partial_t \, . \tag{2}$$

Now suppose that M is an overtwisted contact 3-manifold, and let $Y \subset M$ be an overtwisted disk with elliptic singularity e. By a result known as Bishop's theorem (see [**BG**]), we find a family of J-holomorphic disks $f_n : D \to \mathbb{R} \times M$ that "fill" Y. That is, for each n, $f_n(\partial D)$ is an embedded loop in Y and winds precisely once around e.

Hofer made two observations about the Bishop family of disks. First, by the Maximum Principle, each loop $f_n(\partial D)$ runs transverse to the characteristic foliation induced by ξ on Y; indeed, $f_n(\partial D)$ is transverse to ξ itself. Second, by the Implicit Function theorem, the family of unparametrized disks is open: if $f : (D, \partial D) \to (\mathbb{R} \times M, Y)$ is J-holomorphic, then there is a local family $\{f_\nu\}$ of these disks such that $f_{\nu=0}$ equals f. Combining these observations with a description of the bubbles that might form off the Bishop family of disks, Hofer proved the existence of at least one closed orbit of the Reeb vector field on M. Thus he proved an extension of the Weinstein conjecture for overtwisted contact manifolds (see [**H**] for more details).

Now, returning to (1), let X, ω be a symplectic cobordism from M_- to M_+, and suppose that M_+ is overtwisted. By Moser's theorem, both $M_\pm$ have collar neighborhoods in X that are symplectomorphic to neighborhoods in their symplectizations. Let $\mathcal{J}_{\mathrm{rel}}(X, \partial X)$ be the set of ω-compatible almost complex structures on X that, in a neighborhood of $M_\pm$, are compatible with $d\alpha_\pm$ on $\xi_\pm$ and satisfy (2). If $Y \subset M_+$ is an overtwisted disk and $J \in \mathcal{J}_{\mathrm{rel}}(X, \partial X)$, then Bishop's theorem still applies and we find as before a family of filling disks

$$f_n : (D, \partial D) \to (X, Y) \, .$$

As in Hofer's case, each $f_n(\partial D)$ is transverse to the characteristic foliation on Y, and the family $\{f_n\}$ is open. If the Bishop family were also closed, then the disks could be continued until the boundary of some first disk, say $f_{n_0}(\partial D)$, met and was tangent to ∂Y. But ∂Y is Lagrangian, so this would contradict the fact that $f_{n_0}(\partial D)$ is transverse to the characteristic foliation on Y.

The Bishop disks should be regarded as genus 0 J-holomorphic curves in X with boundary on the totally real submanifold $Y \backslash \{e\} \subset M_+$. In general, in the interest of answering (1) and of finding relationships between the convex and concave ends of a symplectic cobordism (of arbitrary dimension), we would like to consider higher genus curves in X with a totally real boundary condition in $M_+ \cup M_-$. However, to effectively use these curves to understand the topology of X, M_+ or M_-, we need a compactification of the associated moduli space.

This motivates the main theorem of this dissertation.

THEOREM 2. *Let X^4, ω be a symplectic cobordism with compatible relative almost complex structure J, and let $Y \subset X$ be a compact, totally real submanifold. Suppose $f_n : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$ is a sequence of J-holomorphic curves with uniformly bounded Hofer energy:*

$$\mathcal{E}(f_n) \leq C \qquad \textit{for all } n. \tag{3}$$

Then there exists a punctured Riemann surface Σ' and a J-holomorphic map $f' : (\Sigma', \partial\Sigma') \to (\widetilde{X}, Y)$ that has finitely many singular points, and there is a subsequence of $\{f_n\}$ such that $f_n(\Sigma)$ converges smoothly to $f'(\Sigma')$ in $\widetilde{X}$, uniformly in compact regions of $\widetilde{X}$.

Theorem 2 serves two purposes: it supplies a compactness result for J-holomorphic curves in a *non-compact* symplectic manifold with cylindrical ends; and it supplies a compactness result for curves with boundary on a prescribed totally real submanifold. We should point out that pseudo-holomorphic curves with Lagrangian and totally real boundary conditions have been extensively studied by Oh, *et al.*; see [**Oh**], [**KOh**]. To the author's knowledge, however, no compactness theorem for curves in non-compact symplectic manifolds has hitherto appeared in the literature. Hofer [**H**] suggested that compactness for curves in $\mathbb{R} \times M$ could be proved by following Parker & Wolfson's proof [**PW**] of Gromov compactness in closed symplectic manifolds. The latter uses an argument dating back to Sacks & Uhlenbeck [**SU**] which we have been unable to adapt for symplectic cobordisms and symplectic manifolds with cylindrical ends. In the Appendix, we give some indication of why the Sacks-Uhlenbeck argument is difficult to implement in the simplest case of a symplectization (or trivial cobordism) $\mathbb{R} \times M$.

We have ultimately resorted to an entirely different method to prove Theorem 2. This method is based on a regularity theorem proved by Taubes [**T**] in order to prove equivalence of the Seiberg-Witten and Gromov-Witten invariants for a symplectic 4-manifold X. The regularity theorem, or "recognition principle" as we call it, enables us to recognize the 2-currents that come from integration over the support of a J-holomorphic curve in X: they are precisely the ones which carry a generalized local intersection number, or *cohomology assignment*, which evaluates *positively* on local J-holomorphic disks in X. Because these currents can be defined locally, and because a positive cohomology assignment is really a local object, this method easily gives rise to a proof of the compactness theorem in the neighborhood of any point in *any* symplectic manifold, closed, compact, or otherwise. We use Aronszajn's unique continuation principle [**A**] to paste the resulting limit curves together where they coincide.

The proof of Theorem 2 can now be outlined as follows. For each J-holomorphic curve in the sequence, we get a current by integrating over the support of the curve. The mass of this current is bounded in terms of the Hofer energy of the curve. Thus, from a sequence of curves with uniformly bounded Hofer energy, we get a sequence of currents with uniformly bounded mass. We find a subsequence that converges to a limit current $\mathcal{F}$, with $\operatorname{spt}(\mathcal{F}) = S$, and S carries a positive cohomology assignment. Now apply the recognition principle to conclude that S is a J-holomorphic curve in X, and its positive cohomology assignment comes from local intersection with S.

The bulk of this dissertation is occupied with the proof of the recognition principle which is needed to prove Theorem 2. In Chapters 2 and 3, we supply some background on monotonicity theorems and basic measure theory (including currents) that is needed in the later chapters. In Chapter 4, we set the stage for the main compactness theorem, by reviewing the definitions of a symplectic cobordism and its completion, and defining the Hofer energy for pseudo-holomorphic curves in the completion. In some cases of interest, such as curves with Lagrangian boundary condition, or the Bishop family of disks, there is a natural bound on the Hofer energy in terms of the relative homology class represented by these curves. This motivates the compactness theorem, which appears in Chapter 4 as Theorem 13. We outline why the hypotheses needed to apply the recognition principle, namely the finite Hausdorff measure condition, are satisfied by the limit current S.

In Chapter 5, we give an exposition of Taubes' positive cohomology assignments and regularity theorem. In Chapter 6, we define a local, relative intersection number for 2-manifolds in X^4 with boundary on a prescribed codimension 2 submanifold. Applied to half-disks with boundary on a totally real submanifold $Y \subset X$, this relative intersection number allows us to extend Taubes' theory to handle relative 2-currents with boundary supported in Y. Finally, in Chapter 7, we supply a proof of Lemma 41 (see [**T**, Lemma5.5]), which is needed to construct local families of pseudo-holomorphic disks. The existence of these disks is crucial in showing that S with its positive cohomology assignment can locally be expressed as the graph of a pseudo-holomorphic function. By the reflection principle, Lemma 41 also generates for us local families of half-disks with boundary on a totally real submanifold, which we need to prove regularity at the boundary of S.

We have yet to examine the singular points of the limit curve, to show they are finite in number, and to show that the limit curve has finite topology. These will be done in a subsequent paper.

CHAPTER 2

Monotonicity

THEOREM 3 (Wirtinger inequality). *Let X^{2n}, ω be a symplectic manifold with compatible almost complex structure J, and let Σ be a J-holomorphic curve in X.*

i) If a real, 2-dimensional submanifold $\Sigma' \subset X$ is homologous to Σ, then

$$\mathrm{Area}(\Sigma') \geq \mathrm{Area}(\Sigma)$$

and equality holds if and only if Σ' is J-holomorphic.

ii) Let $Y^n \subset X$ be a totally real submanifold. If a real, 2-dimensional submanifold $\Sigma' \subset X$ is homologous to Σ relative to Y, then

$$\mathrm{Area}(\Sigma') \geq \mathrm{Area}(\Sigma)$$

and equality holds if and only if Σ' is J-holomorphic.

PROOF. The proof is a consequence of the Cauchy-Schwarz inequality (see [**AL**, p.100]). Let $\Sigma' \subset X$ be any oriented submanifold, and let $x \in \Sigma'$. If e, f is an oriented orthonormal basis for $T_x\Sigma'$, then

$$0 \leq \omega(e,f) = g(Je,f) \leq \|Je\| \cdot \|f\| \tag{4}$$

where $g = \omega(\cdot, J\cdot)$ is the compatible hermitian metric. Equality holds in (4) if and only if $T_x\Sigma'$ is a complex vector space.

Now (4) implies that $0 \leq \omega|_{\Sigma'} \leq \mathrm{dvol}_{g,\Sigma'}$. Therefore, if Σ' is homologous to Σ, then

$$\mathrm{Area}(\Sigma) = \int_\Sigma \omega = \int_{\Sigma'} \omega \leq \mathrm{Area}(\Sigma')$$

and equality holds if and only if Σ' is J-holomorphic.

For the boundary case, the same argument works. We only need to observe that the equation $\int_\Sigma \omega = \int_{\Sigma'} \omega$ holds because Y is Lagrangian and $[\Sigma - \Sigma'] = 0$ in $H_2(X, Y)$. □

THEOREM 4 (Isoperimetric inequality). *Let X^{2n}, J be an almost complex manifold with at most cylindrical ends, and let h be a compatible hermitian metric which equals the product metric on the ends of X.*

i) There exist constants ϵ_0 and $C > 0$ depending only on X, J and h so that, if Σ is a J-holomorphic curve in X with $\mathrm{diam}(\Sigma) \leq \epsilon_0$, then any subdomain $\Omega \subset \Sigma$ whose boundary is homeomorphic to a circle satisfies

$$\mathrm{Area}_h(\Omega) \leq C\, length_h^2(\partial\Omega) \ .$$

ii) Let $Y^n \subset X^{2n}$ be a compact, totally real submanifold. There exist constants $\epsilon_0, C > 0$ depending only on X, J, h and Y so that if Σ is a

J-holomorphic curve in X with $\mathrm{diam}(\Sigma) \leq \epsilon_0$, *then any subdomain $\Omega \subset \Sigma$ whose boundary is homeomorphic to a circle satisfes*

$$\mathrm{Area}_h(\Omega) \leq C \cdot \mathrm{length}_h^2(\partial\Omega \backslash \partial_Y \Omega),$$

where $\partial_Y \Omega = \partial\Omega \cap Y$.

PROOF. Fix $x \in X$ and let ω_0 be the constant 2-form on $T_x X$ defined by

$$\omega_0(v, J_x v) = h_x(v, v) \qquad , v \in T_x X.$$

Setting $\omega = (\exp^{-1})^* \omega_0$, we obtain a local symplectic form that tames J on a neighborhood U of x.

Next, define a hermitian metric g on U by setting

$$g(u,v) = \frac{1}{2}[\omega(u, Jv) + \omega(v, Ju)] \qquad \text{for } y \in U \text{ and } u, v \in T_y X. \tag{5}$$

Both g and h are uniformly equivalent to a Euclidean metric on a smaller set $V \subset U$. Since X is either compact or has at most cylindrical ends, we can assume that V is a ball of radius ϵ_0, independent of x.

By our assumptions on f and by (5), Ω minimizes g-area for its boundary curve γ, and lies in a neighborhood V as constructed above. We can make an arbitrarily small perturbation of Ω so that γ is a smooth Jordan curve. The solution Γ of the Euclidean Plateau problem in V with boundary γ is a smooth minimal disc (see [**Law**]), and

$$\mathrm{Area}_{euc}(\Gamma) \leq C \cdot \mathrm{length}_{euc}^2(\gamma)$$

by the classical isoperimetric inequality. But g is uniformly equivalent to the Euclidean metric, so we have

$$\mathrm{Area}_g(\Omega) \leq \mathrm{Area}_g(\Gamma) \leq C\mathrm{Area}_{euc}(\Gamma) \leq C\mathrm{length}_{euc}^2(\gamma) \leq C\mathrm{length}_g^2(\gamma)\,.$$

This proves (i), because g and h are uniformly equivalent.

To prove (ii), let $y \in Y$ and let e, f be an oriented orthonormal basis for $T_y Y$. Since Y is totally real, the vectors e, Je, f, Jf form an oriented basis for $T_y X$, and via the exponential map they define coordinates x^1, y^1, x^2, y^2 on a neighborhood of y in X. With respect to these coordinates, Y is the set where $y^1 = y^2 = 0$. Therefore Y is Lagrangian with respect to the local symplectic form $\omega = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$ in U.

Note that ω and J are compatible at y, since

$$\omega_y(\cdot, J_y \cdot) = (dx^1)^2 + (dy^1)^2 + (dx^2)^2 + (dy^2)^2.$$

It follows that ω tames J on a neighborhood U of y. Define a hermitian metric g in U by (5) above. Then g and h are uniformly equivalent to a euclidean metric in a smaller neighborhood V of y. By compactness of Y, we can assume that V is a ball of radius ϵ_0 independent of y.

If we let γ denote the oriented 1-cycle which is the boundary of Ω, then γ decomposes as a sum $\gamma = \alpha + \beta$, where $\alpha = \partial_Y \Omega$ and $\beta = \overline{\partial\Omega \backslash \partial_Y \Omega}$. Let $\beta_1, \ldots, \beta_l$ denote the oriented components of $\partial\Omega \backslash \partial_Y \Omega$, so that $\beta = \beta_1 + \cdots + \beta_l$. For each i, let p_i, q_i be the endpoints of the arc β_i, so that $\partial\beta_i = q_i - p_i$. There is an arc $\tilde{\beta}_i$ contained in Y of length

$$\mathrm{length}(\tilde{\beta}_i) \leq C_1 \cdot \mathrm{length}(\beta_i)$$

such that $\partial\tilde{\beta}_i = q_i - p_i$ as well. Since Y is compact, the constant C_1 may be taken to be independent of y.

Now let Γ be the solution to the Plateau problem in V with boundary $\beta - \tilde{\beta} = \beta_1 - \tilde{\beta}_1 + \cdots \beta_l - \tilde{\beta}_l$. Since V is contractible, Γ is homologous relative Y to Ω, and by Lemma 3 (ii), $\mathrm{Area}_g(\Omega) \le C\,\mathrm{Area}_g(\Gamma)$. We conclude as before that

$$\mathrm{Area}_g(\Omega) \le C \cdot \mathrm{length}^2(\beta - \tilde{\beta}) \le C(1 + C_1)^2 \cdot \mathrm{length}^2(\beta).$$

Since g and h are uniformly equivalent in V, this completes the proof. □

COROLLARY 5 (Monotonicity). *Let X^{2n}, J be an almost complex manifold with compatible hermitian metric h.*

> *i) There exist constants $c, r_0 > 0$ depending only on X, J, h such that the following is true. Let $S \subset X$ be a compact, J-holomorphic curve, and let $B = B_r(x)$ be a ball of radius $r < r_0$ centered at $x \in S$ such that ∂S is contained in the complement of B. Then*

$$Area(S \cap B_r(x)) \ge cr^2\,. \tag{6}$$

> *ii) Let $Y^n \subset X$ be a compact, totally real submanifold. There exist constants c, r_0 depending on X, J, h, Y so that (6) holds for every compact, J-holomorphic curve S and every ball $B = B_r(x)$ of radius $r < r_0$ centered at $x \in S$ which satisfies $\partial S \subset (X \backslash B) \cup Y$.*

PROOF. To prove (i), define a function $A(r) = \mathrm{Area}(S \cap B_r(x))$ for $0 \le r < r_0$. For almost every $r < r_0$, $\Omega_r = S \cap B_r(x)$ is a smooth subsurface of S with smooth boundary. Thus $L(r) = \mathrm{length}(\partial\Omega_r)$ is defined and equals $A'(r)$ for almost every r. By the isoperimetric inequality, $\sqrt{A(r)} \le C \cdot L(r) = C \cdot A'(r)$. Integrating $A'/\sqrt{A}$ from 0 to δ, we obtain (6).

To prove (ii), take $L(r) = \mathrm{length}(\partial\Omega_r \backslash \partial_Y \Omega_r)$, which equals $A'(r)$, and follow the arguments of case (i). □

CHAPTER 3

Currents

In this chapter, we supply some background on currents that will prove useful in Chapters 4–6. Throughout this chapter, X will denote a smooth manifold, possibly non-compact, M will denote a smooth submanifold of X, and U an open subset of X.

1. Basic defintions.

A current is simply a linear functional on the space of smooth, compactly supported forms in X. To make this precise, let $\Omega_c^k(U)$ denote the vector space of all C^∞ compactly supported k-forms μ with $\mathrm{spt}(\mu) \subset U$. We topologize this vector space in the following manner (see [**F**, §4.1.1] or [**deR**, §9] for more details). Given a compact subset $K \subset U$, let $\Omega_K(U) \subset \Omega_c(U)$ denote the space of forms μ with $\mathrm{spt}(\mu) \subset K$. For each $j = 0, 1, 2, \ldots$ and each $\mu \in \Omega_c(U)$, let

$$||\mu||_j = \max_{|\alpha| \leq j} \max_U |D^\alpha \mu| .$$

This sequence of norms defines a topology on $\Omega_K(U)$, and the topology we take on $\Omega_c(U)$ is the largest for which the inclusion maps $\Omega_K(U) \to \Omega_c(U)$ are all continuous.

This topology is complete, and a sequence $\mu_n \in \Omega_c(U)$ converges to μ if and only if

$$||\mu_n - \mu||_j \to 0 \quad \text{as } n \to +\infty$$

for each j, and there is a compact subset $K \subset U$ that contains every $\mathrm{spt}(\mu_n)$.

Definition. Let $\mathcal{D}^k(U)$ denote the topological dual of $\Omega_c^k(U)$. An element $\mathcal{F} \in \mathcal{D}^k(U)$ is called a *current* on U.

Examples.

1. The example of most interest to us is the current coming from a k-chain γ in X. In this case, $\mathcal{F}_\gamma \in \mathcal{D}^k(U)$ is defined by
$$\mathcal{F}_\gamma(\mu) = \int_\gamma \mu \qquad , \text{ for } \mu \in \Omega_c^k(U) \ .$$
2. Let η be any $(n-k)$-form on X, where $n = \dim X$. Then $\mathcal{F}_\eta$ defined by
$$\mathcal{F}_\eta(\mu) = \int_X \eta \wedge \mu \qquad , \text{ for } \mu \in \Omega_c^k(U) ,$$
is a k-current in U.
3. Let $x \in U$. The delta distribution δ_x is a 0-current in U, with
$$\delta_x(f) = f(x) \qquad , \text{ for all } f \in C_c^\infty(U) \ .$$

Boundary and Pushforward of a Current. The *boundary* of a current $\mathcal{F} \in \mathcal{D}^k(U)$ is the $(k-1)$-current given by the formula

$$\partial\mathcal{F}(\mu) = \mathcal{F}(d\mu) \qquad , \text{ for } \mu \in \Omega_c^{k-1}(U) \ .$$

Let X, X' be smooth manifolds and let $U \subset X$, $U' \subset X'$ be open subsets. Given a proper map $\varphi : U \to U'$ and a current $\mathcal{F} \in \mathcal{D}^k(U)$, the *pushforward* of $\mathcal{F}$ by φ is the current in U' defined by the formula

$$\varphi_*\mathcal{F}(\eta) = \mathcal{F}(\varphi^*\eta) \qquad , \text{ for } \eta \in \Omega_c^k(X') \ .$$

It is straightforward to check that these notions coincide with the notions of boundary and pushforward for chains, that is, $\partial\mathcal{F}_\gamma = \mathcal{F}_{\partial\gamma}$ and $\varphi_*\mathcal{F}_\gamma = \mathcal{F}_{\varphi_*\gamma}$.

Mass of a Current. Assume now that X carries a Riemannian metric. Then we can define a norm on $\Omega_c^*(U)$, called the *comass* norm by some and the C^0 norm by others, by setting

$$\mathbb{M}(\mu) = \sup_U \|\mu\| \ .$$

The dual norm on $\mathcal{D}^*(U)$ is called the mass norm, and

$$\mathbb{M}(\mathcal{F}) = \sup_{\mathbb{M}(\mu)\leq 1} \mathcal{F}(\mu)$$

is called the *mass* of the current $\mathcal{F}$.

Recall that we have fixed, once and for all, the C^∞ topology on $\Omega_c^*(U)$; this is the topology with respect to which currents were defined. The comass norm also defines a topology on $\Omega_c^*(U)$, which coincides with the C^0 topology, and is weaker than the C^∞ topology.

On $\mathcal{D}^*(U)$ we will take the *weak* topology generated by sets of the form

$$\{\mathcal{F} : \varepsilon_1 < \mathcal{F}(\mu) < \varepsilon_2\}$$

with $\mu \in \Omega_c^*(U)$, and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. This is the weakest topology for which the linear maps

$$\begin{array}{rccc} \hat{\mu}: & \mathcal{D}^*(U) & \longrightarrow & \mathbb{R} \\ & \mathcal{F} & \longmapsto & \mathcal{F}(\mu) \end{array} ,$$

for $\mu \in \Omega_c^*(U)$, are all continuous. (Thus it is weaker than the mass norm topology.) Equivalently, the weak topology is characterized by the fact that a map $\Psi : Y \to \mathcal{D}^*(U)$ is continuous if and only if $\hat{\mu} \circ \Psi$ is continuous for all μ.

THEOREM 6 (Alaoglu's Theorem). *The unit ball*

$$B_{\mathbb{M}} = \{F \in \mathcal{D}^*(U) : \mathbb{M}(\mathcal{F}) \leq 1\}$$

is compact in the weak topology on $\mathcal{D}^*(U)$.

Relative Currents. Let $Y \subset X$ be a closed submanifold, and let ι denote the inclusion. Generalizing the notion of a local relative chain $\gamma \in C_k(U, U \cap Y)$, we say $\mathcal{F} \in \mathcal{D}^k(U)$ is a *relative current* if there exists $\mathcal{G} \in \mathcal{D}^{k-1}(U \cap Y)$ so that $\partial\mathcal{F} = \iota_*\mathcal{G}$. We will denote the subspace of all relative currents in $\mathcal{D}^k(U)$ by $\mathcal{D}^k(U, Y)$.

LEMMA 7. *Let $Y \subset X$ be a closed submanifold, let $U \subset X$ be open, and let $\mathcal{F} \in \mathcal{D}^k(U)$. The following conditions are equivalent:*

i) $\mathcal{F} \in \mathcal{D}^k(U, U \cap Y)$.

ii) $\partial\mathcal{F}(\mu) = 0$ *whenever* $\mu \in \Omega_c^k(U)$ *and* $\mu|_Y$ *is exact.*

PROOF. Suppose $\mathcal{F} \in \mathcal{D}^k(U, U \cap Y)$. Let $\mu \in \Omega_c^k(U)$ with $\mu|_Y = d\alpha$ for some $\alpha \in \Omega_c^{k-1}(U \cap Y)$. Because Y is a closed submanifold, we can find a $(k-1)$–form $\tilde{\alpha}$ supported in a tubular neighborhood of Y, whose restriction to Y equals α. Then

$$\partial\mathcal{F}(\mu) = \mathcal{G}(\iota^*\mu) = \mathcal{G}(d\alpha) = \mathcal{G}(\iota^* d\tilde{\alpha}) = \partial\mathcal{F}(d\tilde{\alpha}) = 0\,.$$

Conversely, suppose that $\mathcal{F}$ satisfies condition *(ii)*. Define a current $\mathcal{G} \in \mathcal{D}^{k-1}(U \cap Y)$ as follows. Given $\alpha \in \Omega_c^{k-1}(U \cap Y)$, choose $\tilde{\alpha} \in \Omega_c^{k-1}(U)$ as before so that $\tilde{\alpha}|_Y = \alpha$. Then put $\mathcal{G}(\alpha) = \mathcal{F}(d\tilde{\alpha})$. It is easy to verify that $\mathcal{G}$ is well-defined, and that $\iota_*\mathcal{G} = \partial\mathcal{F}$. □

By the way, $\Omega_c^k(U, Y)$ will denote the subspace of $\Omega_c^k(U)$ consisting of all k–forms μ such that $\mu|_Y$ is exact.

COROLLARY 8. *$\mathcal{D}^k(U, Y)$ is closed in $\mathcal{D}^k(U)$ with respect to the weak topology.*

2. Other facts from measure theory.

Hausdorff measure. Let (X, g) be a metric space. To define Hausdorff measure in X, first we define preliminary measures μ_δ on X, for $\delta > 0$.

Given $S \subset X$, let $\mathcal{B}$ be a countable collection of balls covering S, where each ball B is assumed to have radius

$$\text{radius}(B) < \delta.$$

Given $k \in \mathbb{N}$, associate to $\mathcal{B}$ the quantity

$$\sum_{B \in \mathcal{B}} [\text{radius}(B)]^k,$$

and then set

$$\mu_\delta(S) = \inf_{\mathcal{B}} \left\{ \sum_{B \in \mathcal{B}} [\text{radius}(B)]^k \right\},$$

where the infimum is taken over all collections $\mathcal{B}$ of the sort described above.

If $\delta_1 < \delta_2$, then $\mu_{\delta_1}(S) \geq \mu_{\delta_2}(S)$. By definition, the *k-dimensional Hausdorff measure* of S is the quantity

$$\mathfrak{H}^k(S) = \sup_{\delta \to 0} \mu_\delta(S).$$

We note that if $\mathfrak{H}^{k_0}(S) < +\infty$, then $\mathfrak{H}^k(S) = 0$ for all $k > k_0$. The 0-dimensional Hausdorff measure $\mathfrak{H}^0$ is simply the counting measure of S.

We list here some additional measure theoretic facts (see [**F**]) that will prove useful in Chapters 5 and 7.

THEOREM 9 (Federer, 2.10.27). *Let X, Y be metric spaces, and let $X \times Y$ carry the product metric. Suppose that X is boundedly compact. Then for any subset $A \subset X \times Y$ and any integer $k, m \geq 0$ we have*

$$\int_X \mathfrak{H}^k(\{y \mid (x, y) \in A\})\, d\mathfrak{H}^m x \leq \frac{\alpha(k)\alpha(m)}{\alpha(k+m)} \cdot \mathfrak{H}^{k+m}(A),$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$.

CHAPTER 4

Compactness in Symplectic Cobordisms

Throughout this chapter, X^{2n}, ω denotes a symplectic cobordism from its concave end M_-, ξ_- to its convex end M_+, ξ_+. Recall that there exists an outward/inward pointing, conformally symplectic vector field $V_\pm$, defined in a neighborhood of $M_\pm$ such that $\alpha_\pm = \iota_{V_\pm}\omega|_{M_\pm}$ is a contact form for the contact structure on $M_\pm$. Using Moser's method, we find a collar neighborhood of M_+ (resp. M_-) in X that is symplectomorphic to $(1-\varepsilon, 1] \times M_+$ (resp. $[0, \varepsilon) \times M_-$) together with the symplectic form $d(e^t\alpha_+)$ (resp. $d(e^t\alpha_-)$).

With these identifications we form the *completion* $\widetilde{X}$ of X by gluing $[1, +\infty) \times M_+$ to X along its convex end, and then gluing $(-\infty, 0] \times M_-$ to X along its concave end. We will refer to $[1, +\infty) \times M_+$ as the positive or convex end of $\widetilde{X}$, and $(-\infty, 0] \times M_-$ as the negative or concave end of $\widetilde{X}$. The symplectic form on X extends naturally to a symplectic form on $\widetilde{X}$, which we also denote ω, by setting $d(e^t\alpha_\pm)$ on the two ends.

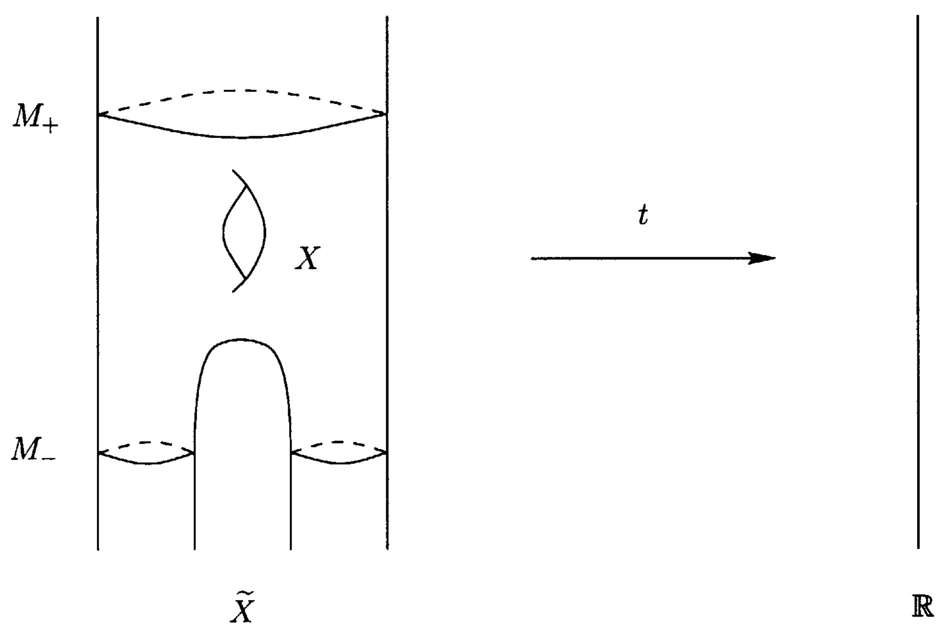

FIGURE 4-1. A symplectic cobordism.

Next let $v_{\alpha_\pm} \in \Gamma(TM_\pm)$ denote the Reeb vector field associated to $\alpha_\pm$. An almost complex structure J on X, compatible with ω, is called a *compatible relative almost complex structure* if in the neighborhood of $M_\pm$, it satisfies:

$$J(\partial_t) = v_{\alpha_\pm} \qquad , \qquad J(v_{\alpha_\pm}) = -\partial_t$$

$$J \text{ is compatible with } d\alpha_\pm \text{ on } \xi_\pm\,.$$

In this case, J extends to an almost complex structure on $\widetilde{X}$, which we also denote J. The set of all compatible relative almost complex structures is denoted $\mathcal{J}_{\text{rel}}(X, \partial X)$.

For fixed $J \in \mathcal{J}_{\text{rel}}(X, \partial X)$, our goal is to study J-holomorphic curves in $\widetilde{X}$ with boundary on a prescribed compact, totally real submanifold $Y \subset \widetilde{X}$.

Once we have fixed a choice of $J \in \mathcal{J}_{\text{rel}}(X, \partial X)$, we might choose to work with the metric $\omega(\cdot, J\cdot)$ on $\widetilde{X}$, but this metric is not complete and its injectivity radius goes to zero as $t \to -\infty$ along the negative, or concave, end of $\widetilde{X}$. Therefore, following Hofer [**H**], let us choose a metric g which equals $\omega(\cdot, J\cdot)$ in X, and equals the product metric $dt \wedge \alpha_\pm + d\alpha_\pm(\cdot, J\cdot)$ on the ends of $\widetilde{X}$.

1. Hofer energy.

It will be convenient to extend the time function $(t, p) \mapsto t$, which at present is defined only on the ends of $\widetilde{X}$, to a function $t : \widetilde{X} \to \mathbb{R}$ in such a way that $t^{-1}[0,1]$ precisely equals our original manifold X. (Compare Figure 4-1.) Now define a collection of functions

$$\mathcal{C} = \{\varphi : \mathbb{R} \to [\tfrac{1}{2}, \tfrac{7}{2}] \mid \varphi' \geq 0 \text{ and } \varphi(t) = e^t \text{ on } (0,1)\}.$$

Each $\varphi \in \mathcal{C}$ pulls back via t to a function $\widetilde{X} \to [\frac{1}{2}, \frac{7}{2}]$, and satisfies $d(\varphi\alpha_\pm) = d(e^t\alpha_\pm) = \omega$ on $(0, \varepsilon) \times M_- \cup (1-\varepsilon, 1) \times M_+$. Therefore, setting

$$\omega_\varphi = \begin{cases} \omega \text{ on } X \\ d(\varphi\alpha_+) \text{ on } (1-\varepsilon, 1) \times M_+ \\ d(\varphi\alpha_-) \text{ on } (0, \varepsilon) \times M_- \end{cases},$$

we obtain a well-defined 2-form on $\widetilde{X}$.

For a Riemann surface Σ and a J-holomorphic mapping $f : \Sigma \to \widetilde{X}$, we set

$$\mathcal{E}_\varphi(f) = \int_\Sigma f^*\omega_\varphi$$

for each $\varphi \in \mathcal{C}$, and define the *Hofer energy* of f to be

$$\mathcal{E}(f) = \sup_{\varphi \in \mathcal{C}} \mathcal{E}_\varphi(f) = \sup_{\varphi \in \mathcal{C}} \int_\Sigma f^*(\omega_\varphi).$$

For each relatively compact, open subset $U \subset \widetilde{X}$, we can associate to f and Σ the 2-current $\mathcal{F} \in \mathcal{D}^2(U)$ whose value on a compactly supported 2-form $\mu \in \Omega^2_c(U)$ equals

$$\mathcal{F}(\mu) = \int_{f(\Sigma)} \mu.$$

If $f(\partial\Sigma)$ lies on a compact submanifold $Y \subset \widetilde{X}$, then $\partial\mathcal{F} = \iota_*\mathcal{G}$ where $\mathcal{G} \in \mathcal{D}_1(U \cap Y)$ is the 1-current defined by $f(\partial\Sigma) \subset Y$. So in fact $\mathcal{F} \in \mathcal{D}_2(U, U \cap Y)$ is a relative current. In the next lemma, we see how the mass of $\mathcal{F}$ is related to the Hofer energy of f.

LEMMA 10. $\mathbb{M}(\mathcal{F}) \leq \mathcal{E}(f) \cdot \{3 + \operatorname{diam}(t(U))\}$.

PROOF. Let $a = \inf_U t$ and $b = \sup_U t$, so that $\operatorname{diam}(t(U)) = b - a$. In the first case, suppose that $a < b \leq 0$. Take a sequence of functions $\varphi_k \in \mathcal{C}$ that uniformly approximate the function

$$\varphi(t) = \begin{cases} \frac{1}{2} & \text{if } t \leq a \\ \frac{1}{2} + \frac{1}{2(b-a)} \cdot (t-a) & \text{if } a \leq t \leq b \\ 1 & \text{if } b \leq t \leq 0 \\ e^t & \text{if } 0 \leq t \leq 1 \\ e & \text{if } t \geq 1 \end{cases} .$$

Since $\mathcal{E}(f) \geq \mathcal{E}_{\varphi_k}(f)$ for all k, it follows that

$$\begin{aligned}
\mathcal{E}(f) &\geq \int_\Sigma \varphi'(t \circ f) \,|\, d(t \circ f)|^2 + \varphi(t \circ f) \,|\, \pi df|^2 \\
&\geq \int_{(t \circ f)^{-1}[a,b]} \frac{1}{2(b-a)} \,|\, d(t \circ f)|^2 + \frac{1}{2}|\, \pi df|^2 \\
&\geq \frac{1}{b-a} \min(1, b-a) \cdot \frac{1}{2} \int_{f^{-1}(U)} |\, df|^2 \\
&\geq \frac{1}{\max(1, b-a)} \cdot E(f; f^{-1}(U)) \,. \qquad (7)
\end{aligned}$$

Here, the energy $E(f; f^{-1}(U)) = \frac{1}{2} \int_{f^{-1}(U)} |\, df|^2$ is computed using the metric g.

For any compactly supported 2-form μ with $\operatorname{spt}(\mu) \subset U$ and $\|\mu\| \leq 1$, we have

$$\begin{aligned}
\mathcal{F}(\mu) &= \int_\Sigma \mu(f_x, f_y)\, dx\, dy \\
&= \int_{f^{-1}(U)} \mu(f_x, f_y)\, dx\, dy \\
&\leq \int_{f^{-1}(U)} \|\mu\| \frac{1}{2} |\, df|^2 dx\, dy \\
&\leq \int_{f^{-1}(U)} \frac{1}{2} |\, df|^2\, dx\, dy \\
&= E(f; f^{-1}(U)) \,.
\end{aligned}$$

Therefore, by (7),

$$\mathbb{M}(\mathcal{F}) \leq \mathcal{E}(f) \cdot \max\{1, \operatorname{diam}(t(U))\} \,. \qquad (8)$$

In a similar manner we can show that (8) holds when $1 \leq a < b$. If $0 \leq a < b \leq 1$, then

$$\mathcal{E}_\varphi(f) \geq \int_{f^{-1}(U)} f^*\omega_\varphi = \int_{f^{-1}(U)} f^*\omega = \frac{1}{2} \int_{f^{-1}(U)} |df|^2 = E(f; f^{-1}(U))$$

for all $\varphi \in \mathcal{C}$, because f is J-holomorphic. Therefore $\mathcal{E}(f) \geq E(f; f^{-1}(U))$ and we conclude as in the first case that (8) holds.

In the general case, let $U_1 = U \cap t^{-1}(-\infty, 0)$, $U_2 = U \cap t^{-1}(0,1)$ and $U_3 = U \cap t^{-1}(1, +\infty)$. For each $i = 1, 2, 3$, define a current $\mathcal{F}_i$ in U_i by

$$\mathcal{F}_i(\mu) = \int_{f(\Sigma)} \mu \qquad , \quad \mu \in \Omega^2_c(U_i)\,.$$

We have shown that (8) holds for each $\mathcal{F}_i$ in U_i. Hence

$$\mathbb{M}(\mathcal{F}) \leq \mathbb{M}(\mathcal{F}_1) + \mathbb{M}(\mathcal{F}_2) + \mathbb{M}(\mathcal{F}_3) \leq \mathcal{E}(f) \cdot \{3 + \operatorname{diam}(t(U))\}\ ,$$

and we are done. □

Note that in the proof we have shown

$$\operatorname{Area}(f(\Sigma) \cap U) \leq \mathcal{E}(f) \cdot \{1 + \operatorname{diam}(t(U))\}\ . \tag{9}$$

If Y is totally real, then by the monotonicity formula (6), there are constants r_0 and $c > 0$ such that

$$\operatorname{Area}(f(\Sigma) \cap B_r(x)) \geq cr^2 \qquad \text{for } r \leq r_0\,,\ x \in f(\Sigma)\,,$$

provided $\partial f(\Sigma)$ is contained in Y or in the complement of $B_r(x)$. Setting

$$c(U) = \frac{1}{c} \cdot \{1 + \operatorname{diam}(t(U))\}\,, \tag{10}$$

we obtain the next result.

Lemma 11. *Let $Y \subset \widetilde{X}$ be a compact, totally real submanifold and let $U \subset \widetilde{X}$ be open and relatively compact. There exist positive constants c and r_0 depending only on U so that for any pseudo-holomorphic map $f : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$, the following is true: if $r < r_0$ and $\mathcal{B}$ is a collection of disjoint balls of radius r centered at points in $f(\Sigma) \cap U$, then the number of elements in $\mathcal{B}$ is at most*

$$\frac{\mathcal{E}(f) \cdot c}{r^2}\ .$$

2. Uniform energy bound.

In some cases, knowing the relative homology class represented by $f : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$ gives us a bound on the Hofer energy of f.

For example, if $Y \subset X$ is Lagrangian, then $\omega_\varphi|_Y = \omega|_Y = 0$ for all $\varphi \in \mathcal{C}$. If $f : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$ is a J-holomorphic curve in the class $A \in H_2(X, Y)$, then

$$\int_{f(\Sigma)} \omega_\varphi \ = \ < \omega_\varphi, A > \ = \ < \omega, A >$$

for all φ. Thus $\mathcal{E}(f) \ \leq \ < \omega, A >$.

In case $n = 2$, then we are interested in the situation where Y is a compact surface, with or without boundary, in $\partial X = M_+ \amalg M_-$. By the contact condition, the points at which Y is tangent to $\xi_\pm$ are generically isolated and finite in number. Therefore, apart from these isolated points of complex tangency, Y is totally real in $\widetilde{X}$.

There is a function h on Y such that

$$\omega|_Y = h \cdot \mathrm{dvol}_Y,$$

and we define the ω*-volume* of Y to be the non-negative quantity

$$\mathrm{vol}_\omega(Y) = \int_Y |\omega| = \int_Y |h| \cdot \mathrm{dvol}_Y .$$

If Y is Lagrangian for ω, then $\mathrm{vol}_\omega(Y) = 0$.

LEMMA 12. *Let $Y \subset M_+ \cup M_-$ be compact and let $A \in H_2(\widetilde{X}, Y)$. There exists a constant $C > 0$ depending only on A such that $\mathcal{E}(f) \le C$ for all J-holomorphic curves $f : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$ representing A.*

PROOF. Since $J \in \mathcal{J}_{\mathrm{rel}}(X, \partial X)$, we have

$$-dd^J(e^t) = -d(e^t dt \circ J) = d(e^t \alpha_+) = \omega = \omega_\varphi \qquad , \ \varphi \in \mathcal{C}$$

on the set $t^{-1}(1-\varepsilon, 1] \approx (1-\varepsilon, 1] \times M_+$. Likewise, $-dd^J(e^t) = \omega_\varphi$ in $[0, \varepsilon) \times M_-$ for all $\varphi \in \mathcal{C}$. It follows that $\omega_\varphi + dd^J(e^t)$ is identically zero on $M_+ \cup M_-$, and so on Y as well.

If $f : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$ is a J-holomorphic curve in the class A, then for any $\varphi \in \mathcal{C}$ we have

$$\begin{aligned} \mathcal{E}_\varphi(f) &= \int_{f(\Sigma)} (\omega_\varphi + dd^J(e^t)) - \int_{f(\partial\Sigma)} d^J(e^t) \\ &= < \omega_\varphi + dd^J(e^t), A > + e \int_{f(\partial_+\Sigma)} \alpha_+ - \int_{f(\partial_-\Sigma)} \alpha_- \\ &\le \ < \omega + dd^J(e^t), A > + \mathrm{vol}_\omega(Y) \ . \end{aligned}$$

As the last quantity is independent of $\varphi \in \mathcal{C}$, this proves the lemma. □

3. Compactness.

THEOREM 13. *Let X^4, ω be a symplectic cobordism with compatible relative almost complex structure J, and let $Y \subset X$ be a compact, totally real submanifold. Suppose $f_n : (\Sigma, \partial\Sigma) \to (\widetilde{X}, Y)$ is a sequence of J-holomorphic curves with uniformly bounded Hofer energy:*

$$\mathcal{E}(f_n) \le C \qquad \textit{for all } n. \tag{11}$$

Then there exists a punctured Riemann surface Σ' and a J-holomorphic map $f' : (\Sigma', \partial\Sigma') \to (\widetilde{X}, Y)$ that has finitely many singular points, and there is a subsequence of $\{f_n\}$ such that $f_n(\Sigma)$ converges smoothly to $f'(\Sigma')$ in $\widetilde{X}$, uniformly in compact regions of $\widetilde{X}$.

PROOF. Let $U \subset \widetilde{X}$ be open and relatively compact. For each n, let $\mathcal{F}_n \in \mathcal{D}_2(U, U \cap Y)$ be the relative current in U defined by integration over $\mathcal{F}_n(\Sigma)$. By Lemma 10, we have the uniform mass bound $\mathbb{M}(\mathcal{F}_n) \le C \cdot \{1 + \mathrm{diam}(t(U))\}$ for all n. So by Alaoglu's theorem, there is a subsequence $\{\mathcal{F}_n\}$ that converges weakly to a current $\mathcal{F} \in \mathcal{D}_2(U, Y)$. Let S denote the support of $\mathcal{F}$ in U, with $\partial S \subset Y \cap U$. By Lemmas 14 and 15 below, S and ∂S have finite 2- and 1-dimensional Hausdorff measure, respectively. Therefore, by the recognition principle (Chap. 5, Theorem 17 and Chap. 6, Theorem 33), S is the image of a (possibly disconnected) pseudo-holomorphic curve in U.

Thus, for each relatively compact, open subset $U \subset \widetilde{X}$, we have a relative current $\mathcal{F}_U$ in U with support $\mathrm{spt}(F_U) = S_U$, and we have a Riemann surface Σ_U, a domain $\Omega_U \subset \Sigma_U$, and a J-holomorphic map $f_U : \Omega_U \to \widetilde{X}$ such that $S_U = \mathrm{Image}(f_U)$. Since $\widetilde{X}$ can be covered by such open sets of this sort, we obtain, by Aronszajn's unique continuation principle [**A**], a J-holomorphic curve $S \subset \widetilde{X}$ with the property that $S \cap U$ equals S_U for any relatively compact, open $U \subset \widetilde{X}$.

By [**T**], the singular points of S are isolated and finite in number in any compact region of $\widetilde{X}$. In fact, a refinement of Taubes' analysis will show that S has finitely many singular points and has finite topology. (Additional details will be given in a forthcoming paper.) Therefore we are done. □

LEMMA 14 (Taubes). *Let $U \subset X$ be open and let* $\mathrm{diam}(t(U))$ *be finite. Then the support S of $\mathcal{F}$ in U has finite 2-dimensional Hausdorff measure.*

PROOF. For each $N >> 0$, set $r_N = 16^{-N}$. Given n and N, let $\mathcal{B}_{n,N}$ be a maximal set of disjoint balls of radius r_N centered at points in $f_n(D) \cap U$. Label these points as $x_1^{n,N}, \dots x_l^{n,N}$. By Lemma 11 and the estimate (11), the number l is at most

$$\frac{c_1}{(r_N)^2} = c_1 \cdot 16^{2N} \ ,$$

where c_1 is some integer constant independent of n and N. By repeating the point $x_1^{n,N}$ if necessary, we will assume that $l = c_1 \cdot 16^{2N}$.

Note that the balls B_{2r_N} of twice the radius cover $f_n(D) \cap U$. Let $W_{n,N}$ denote the union of the balls of radius $4r_N$. This set is a neighborhood of $f_n(D) \cap U$ in U.

Now fix N and i, and let $n \to +\infty$. By a diagonal argument we may assume, up to taking a subsequence, that $x_i^{n,N}$ converges to some point $x_i^N \in \overline{U}$. Let W_N denote the union of the closed, radius $4r_N$ balls centered at these points, that is,

$$W_N = \bigcup_{i=1}^{l} B_{4r_N}(x_i^N) \, .$$

We claim that $W_{N+1} \subset W_N$. To see why, note that y lies in W_{N+1} if and only if

$$\mathrm{dist}(y, x_i^{N+1}) \leq \frac{1}{4} \cdot 16^{-N}$$

for one of the points x_i^{N+1}. Given arbitrary $\varepsilon > 0$, we have

$$\mathrm{dist}(x_i^{N+1}, x_i^{n,N+1}) < \varepsilon \quad \text{for all sufficiently large}\, n.$$

There is some j so that

$$\mathrm{dist}(x_i^{n,N+1}, x_j^{n,N}) \leq 2 \cdot 16^{-N} \, ,$$

and moreover,

$$\mathrm{dist}(x_j^{n,N}, x_j^N) < \varepsilon \quad \text{for all sufficiently large } n.$$

Thus the distance from y to the set $\{x_1^N, \dots, x_l^N\}$ is at most $9/4 \cdot 16^{-N} + 2\varepsilon$. As ε was arbitrary, this proves the claim.

Now by construction, the set

$$W = \bigcap_N W_N$$

has finite 2-dimensional Hausdorff measure. For given $\delta > 0$, the balls $B_{4r_N}(x_i^N)$, $i = 1, \dots, l$, cover W and can be made to have radius $< \delta$. Therefore

$$\mu_\delta(W) \leq l \cdot (4r_N)^2 = \frac{c_1}{(r_N)^2} \cdot 16r_N^2 = 16c_1,$$

and $\mathfrak{H}^2(W) = \sup_\delta \mu_\delta(W) \leq 16c_1 < +\infty$.

Finally, we claim that $S = \mathrm{spt}(\mathcal{F})$ is contained in W, and so has finite 2-dimensional Hausdorff measure. It suffices to prove that $\mathcal{F}(\alpha) = 0$ for all compactly supported forms α with $\mathrm{spt}(\alpha) \subset U \backslash W$. If α is such a form, then $\mathrm{spt}(\alpha) \subset U \backslash W_N$ for some large N. Consequently, $\mathrm{spt}(\alpha) \subset U \backslash W_{n,N}$ and $\mathcal{F}_n(\alpha) = 0$ for all $n >> 0$. Since $\mathcal{F}$ is the weak limit of the sequence $\{\mathcal{F}_n\}$, it follows that $\mathcal{F}(\alpha) = 0$, as required. □

The next lemma relies on the fact that Y is both compact and totally real in X.

LEMMA 15. *Let $U \subset X$ be an open subset such that* $\mathrm{diam}(t(U))$ *is finite. Then the support $\partial S = S \cap Y$ of $\partial \mathcal{F}$ has finite 1-dimensional Hausdorff measure.*

CHAPTER 5

A Recognition Principle

In this chapter we describe Taubes' method [**T**] for determining when a closed subset S of X, J is a J-holomorphic curve. To get a basic idea of how this method works, consider the Euclidean situation where we must determine when a subvariety of $\mathbb{C}^N$ is a holomorphic curve.

THEOREM 16 (**Euclidean Recognition Principle**). *A smooth, oriented, real 2-dimensional submanifold $S \subset \mathbb{C}^N$ is complex analytic if and only if the local intersection of S with every complex hyperplane is positive.*

PROOF. First, it is easy to show that the tangent space to S at any point x is a complex line in C^N – for otherwise we can find a complex hyperplane that intersects S negatively. Next, take a point $x \in S$, set $L = T_xS$, and take a complex hyperplane L' such that $\mathbb{C}^N = L \oplus L'$. In a neighborhood of x, we can represent S as the graph of a smooth function $\Phi : L \to L'$,

$$S = \{z + \Phi(z) | z \in L\}.$$

If $y \in S$ is another point in this neighborhood, then T_yS, L, L' are all complex subspaces of $\mathbb{C}^N$. This can only be if Φ is holomorphic so this completes the proof. □

Let us extract the following ideas from the Euclidean case:

(1) If $S \subset \mathbb{C}^4$ is complex analytic, then S should intersect every local holomorphic disk (along any direction) positively.
(2) Exhibit S locally as the graph of a holomorphic function $\Phi : D \to \mathbb{C}$ (in the case above, D is a neighborhood of 0 in T_xS). That Φ is holomorphic follows from positivity of the intersections with S.

The proof of Taubes' recognition principle is based on these ideas, so to outline that result we need to do three things. First, we must construct local (families of) pseudoholomorphic disks lying along arbitrary directions – we will do this in Chapter 7. Second, we must find an appropriate function Φ whose graph is S – this will be done using the family of parallel disks constructed in Corollary 42. Finally, since we only have, a priori, a closed subset $S \subset X$ (with finite 2-dimensional Hausdorff measure), we must decide what it means for S to have a local intersection number, and for S to intersect local pseudo-holomorphic disks positively. We can settle this matter now with the following definitions.

DEFINITION. *Let U be an open subset of X, and let $S \subset U$. An S-*admissible disk *in U is a map $\sigma : D \to X$ defined on an open disk $D \subset \mathbb{C}$, with image $\sigma(D) \subset U$, which extends to a continuous map $\overline{D} \to X$ such that $\sigma(\partial D)$ is contained in the complement of $\overline{S}$.*

DEFINITION. *Let U be an open subset of X, and let $S \subset U$ be closed. A* cohomology assignment *for S in U is a map*

$$I : \{S\text{-}admissible\ disks\ in\ U\} \longrightarrow \mathbb{Z}$$

that satisfies the following criteria:

(1) *If $\sigma : D \to U$ is S-admissible and* $\mathrm{Image}(\sigma, D) \subset U \setminus S$*, then* $I(\sigma, D) = 0$.
(2) *If $\sigma_0, \sigma_1 : D \to U$ are admissible and homotopic via an S-admissible homotopy in U, then $I(\sigma_0) = I(\sigma_1)$. (An S-admissible homotopy in U is a homotopy $\tau : [0,1] \times D \to U$ which extends continuously to $[0,1] \times \overline{D} \to U$ in such a way that $\tau([0,1] \times \partial D) \subset U \setminus S$.)*
(3) *If $\sigma : D \to U$ is admissible and $\tau : D' \to D$ is a proper map of degree k, then $I(\sigma \circ \tau, D') = k \cdot I(\sigma, D)$.*
(4) *If $\sigma : D \to U$ is admissible and $\sigma^{-1}(S)$ is contained in a disjoint union of disks $\bigcup_i D_i \subset D$, then $I(\sigma, D) = \sum_i I(\sigma, D_i)$.*

Finally, if $I(\sigma) > 0$ for all embedded, pseudo-holomorphic, S-admissible disks $\sigma : D \to U$ satisfying $\sigma^{-1}(S) \neq \emptyset$, then we call I a *positive* cohomology assignment for S in U.

Thus a positive cohomology assignment has all the best characteristics of an intersection pairing with a (possibly multicovered) pseudoholomorphic curve.

If Σ is a Riemann surface, $f : \Sigma \to X$ is a J-holomorphic map and $U \subset X$ is open, then a natural positive cohomology assignment for $S = f(\Sigma) \cap U$ in U comes from intersection with S: any S-admissible disk $\sigma : D \to U$ can be perturbed admissibly so as to become transverse to f, and then we define $I_f(\sigma, D)$ to be the signed sum of points in the finite set $\{(w, z) \in D \times \Omega \mid \sigma(w) = f(z)\}$.

THEOREM 17 (**Symplectic Recognition Principle**). *Let X^4, ω be a symplectic manifold with a compatible almost complex structure J. Let $S \subset X$ be a closed subset with finite 2-dimensional Hausdorff measure. If S carries a (global) positive cohomology assignment I, then there is a Riemann surface Σ, a domain $\Omega \subset \Sigma$, and a J-holomorphic map $f : \Omega \to X$ with finitely many singular points such that $S = f(\Omega)$ and $I = I_f$.*

THEOREM 18 (**Local Statement**). *Let J, ω be an almost complex structure and compatible symplectic form on $\mathbb{C}^2$, both standard at the origin. Let $U \subset \mathbb{C}^2$ be open, and let $S \subset U$ be a connected, closed subset with $\mathfrak{H}^2(S) < +\infty$. If S carries a positive cohomology assignment I in U, then there is a Riemann surface Σ, a domain $\Omega \subset \Sigma$, and a J-holomorphic map $f : \Omega \to U$ with finitely many singular points such that $S = f(\Omega)$ and $I = I_f$.*

To prove the recognition principle from the local statement, note that if I is a positive cohomology assignment for S in X and $V \subset X$ is open, then I restricts to a positive cohomology assignment for $S \cap V$ in V. (Any $(S \cap V)$-admissible disk in V is S-admissible in X.) Denote this restriction by I_V.

Now let g be the metric $\omega(\cdot, J\cdot)$ on X. At a point $x \in X$, $J|_x$ allows us to identify the tangent space T_xX with $(\mathbb{C}^2, J_0)$. Use the exponential map to identify a neighborhood U of 0 in T_xX with a neighborhood V of x in X, and then pull each of J, g and ω back to U.

If $x \in S$, then by the local version of the recognition principle, we find a Riemann surface Σ, a domain $\Omega \subset \Sigma$, and a $(\exp^* J)$-holomorphic map $f : \Omega \to \mathbb{C}^2$ so that

$f(\Omega) \subset U$, $f(\Omega) = \exp^{-1}(S)$, and $I_f = \exp^{-1} I$. Let us denote the pushforward $\exp \circ f$ by f as well. Then clearly $S \cap V = f(\Omega)$ and I_V equals I_f.

Thus in the neighborhood of any point $x \in S$, we can express the pair (S, I) as $(f(\Omega), I_f)$ for some J-holomorphic map $f : \Omega \to X$. By Aronszajn's unique continuation principle, any two such expressions agree on overlaps. Therefore, away from its boundary, S is a J-holomorphic subvariety of X, with finitely many singular points in each compact region in X.

1. Behavior at a Point.

We focus our attention on a single point in S, which we may as well take to be $\mathbf{0}$. The space of J_0-hermitian coordinates centered at $\mathbf{0}$ is parameterized by $U(2)$, and the space of J_0-complex lines through $\mathbf{0}$ is a copy of $\mathbb{C}P^1$. There is a natural bundle map

$$\pi : U(2) \longrightarrow \mathbb{C}P^1 \approx U(2)/(U(1) \times U(1))$$

which associates, to a choice of hermitian coordinates $\{z^0, z^1\}$, the "vertical" line $z^0 = 0$.

Let $\gamma \to \mathbb{C}P^1$ denote the tautological line bundle, and let $E \to U(2)$ denote the pullback of γ via π. We think of E as the bundle of "vertical lines", and its orthogonal complement $E^\perp$ in $U(2) \times \mathbb{C}^2$ as the bundle of "horizontal lines". By Corollaries 42 and 43, there exist $R > 0$ and bundle maps θ, ξ so that the following diagram commutes:

$$\begin{tikzcd}
 & & D_R(E) \arrow[r, hook, "(0,\mathrm{id})"] \arrow[dl] \arrow[d, "\pi"] & D_R(E^\perp) \oplus D_R(E) \arrow[r, "\theta"] & U(2) \times \mathbb{C}^2 \arrow[d, "\pi"] \\
 & U(2) \arrow[dl] \arrow[d, "\pi"] & D_R(\gamma) \arrow[rr, "\xi"] \arrow[dl] & & \mathbb{C}P^1 \times \mathbb{C}^2 \\
(\mathbf{0}, J_0) & \mathbb{C}P^1 \arrow[l] & & &
\end{tikzcd}$$

Here, $D_R(\gamma)$, $D_R(E)$ and $D_R(E^\perp)$ denote the respective radius R disk bundles. The maps to $(\mathbf{0}, J_0)$ on the far left are of course trivial and are included to remind us that we are centered at the point $\mathbf{0} \in \mathbb{C}^2$, and that $U(2)$ and $\mathbb{C}P^1$ are specified by the complex structure J_0.

By construction, the restriction of θ to any vertical fiber $\omega \times D_R(E|_{\{z^0,z^1\}})$, $\{z^0, z^1\} \in U(2)$, has J-holomorphic image in $\{z^0, z^1\} \times \mathbb{C}$. Likewise, the restriction of ξ to any fiber in $D_R(\gamma)$ has J-holomorphic image. For any $\kappa \in \mathbb{C}P^1$, the fiber over κ is denoted $D_R(\kappa)$.

Lemma 19. *Suppose $U \subset \mathbb{C}^2$ is an open neighborhood of $\mathbf{0}$, $S \subset U$ is closed, and $\mathfrak{H}^2(S) < +\infty$. Then the set*

$$\Upsilon = \{\kappa \in \mathbb{C}P^1 \mid \xi(D_R(\kappa)) \cap S \text{ is finite}\}$$

has full measure in $\mathbb{C}P^1$.

Proof. Pick arbitrary coordinates $\{z^0, z^1\}$ in $\mathbb{C}^2$, and let κ_0 be the vertical line $z_0 = 0$. There is a neighborhood of κ_0 in $\mathbb{C}P^1$ over which $D_R(\gamma)$ is identified with $D_\delta \times D_R$, for some $\delta > 0$. Any two disks intersect each other discretely, so the set $A = \xi^{-1}(S)$ has finite 2-dimensional Hausdorff measure in $D_\delta \times D_R$. Applying Lemma 9, we have, for $k \geq 0$,

$$\int_{D_\delta} \mathfrak{H}^k\left((b \times D_R) \cap A\right) d\mathfrak{H}^2 b \leq \frac{\alpha(k)\alpha(2)}{\alpha(k+2)} \cdot \mathfrak{H}^{k+2}(A)$$

When $k > 0$, the right hand side of this inequality is zero. When $k = 0$, we see that $\mathfrak{H}^0\left((b \times D_R) \cap \xi^{-1}(S)\right) < +\infty$, and hence that $\xi(b \times D_R) \cap S$ is finite, for almost every direction $b \in D_\delta$.

Since $\{z^0, z^1\}$ and $\kappa_0 = \{z^0 = 0\}$ were arbitrary, this proves the lemma. □

LEMMA 20. *If, in addition to the hypotheses of Lemma 19, we assume that S has a positive cohomology assignment I in U, then $\Upsilon \subset \mathbb{C}P^1$ is open.*

PROOF. Fix an element $\kappa_0 \in \Upsilon$. We will show that for all κ sufficiently close to κ_0 in $\mathbb{C}P^1$, the intersection $\xi(D_R(\kappa)) \cap S$ contains at most $m = I(\xi, D_R(\kappa_0))$ points.

To do so, let $\{z^0, z^1\}$ be hermitian coordinates in $\mathbb{C}^2$ for which κ_0 is the vertical line $z^0 = 0$. For each $b \in \mathbb{C}$, let κ_b denote the complex line $z^0 - bz^1 = 0$. The coordinate choice gives a local trivialization of $D_R(\gamma)$ near κ_0, so ξ is regarded as a map $\mathbb{C} \times D_R \to \mathbb{C}^2$.

For each $b \in \mathbb{C}$, define $\xi_b : D_R \to \mathbb{C}^2$ by $\xi_b(z) = \xi(b, z)$. Each $\xi_b^{-1}(S)$ is a closed subset of D_R and, by assumption, $\xi_0^{-1}(S) = \{0\}$. Thus, given $r < R$, there exists $\epsilon > 0$ such that $\xi_b^{-1}(S) \subset D_r$ for all $b \in D_\epsilon \subset \mathbb{C}$. Moreover, since S carries a positive cohomology assignment, $\xi_b^{-1}(S)$ has at most m components.

Suppose there is some b, $|b| < \epsilon$ such that $\xi_b^{-1}(S)$ contains $m+1$ points. Using Lemma 45, we will perturb $\xi_b(D_R)$ and obtain a contradiction. First, we need the following technical results.

Claim. Assume that $\xi_b^{-1}(S) \subset D_R$ has m components and contains $m+1$ points. Let $d > 0$ denote the diameter of the largest component of $\xi_b^{-1}(S)$. There exists a positive number $r < R$ so that the following hold:

(1) There is a ball $D_r(p) \subset D_R$ such that $\xi_b^{-1}(S) \cap D_r(p) \subset D_{\frac{r}{4}}(p)$.
(2) There are $m+1$ points $\zeta_1, \ldots, \zeta_{m+1} \in \xi_b^{-1}(S) \cap D_r(p)$ such that $|\zeta_i - \zeta_j| \geq \frac{d}{2(m+1)}$ for $i \neq j$.
(3) There is a bound $8d \leq r \leq 8^{m+1}d$.

Note that d can be made arbitrarily small by taking $|b|$ sufficiently small.

To prove the claim, let $\Gamma = \xi_b^{-1}(S)$ and let Γ_0 be the largest component of Γ, with $d = \operatorname{diam}(\Gamma_0)$. There exist points p, $p' \in \Gamma_0$ such that $|p - p'| = \frac{d}{2}$. For each $l = 1, 2, \ldots, m+1$ there is a point $\zeta_l \in \Gamma_0$ such that

$$|p - \zeta_l| = l \cdot \frac{d}{2(m+1)} \tag{12}$$

(Otherwise, p and p' lie in different components of Γ_0.)

We have $\Gamma_0 \subset B_{2d}(p)$ and if $\Gamma \cap B_{8d}(p) \subset B_{2d}(p)$ then we are done. If not, then there are other components of Γ that meet $B_{8d}(p)$. Let Γ_1 be the union of these components. Then $\Gamma_1 \subset B_{2\cdot 8d}(p)$ and if $\Gamma \cap B_{64d} \subset B_{16d}$, we are done.

Otherwise, continue in this fashion to obtain a sequence Γ_j of unions of components of Γ such that $\Gamma_j \cap B_{8^j d}(p) \neq \emptyset$ and $\Gamma_j \subset B_{2\cdot 8^j d}$ for each j. This process must stop after m repetitions because Γ only has m components. At the final stage, we have

$$\Gamma \cap B_{8^{m+1}d}(p) \subset B_{2\cdot 8^m d}(p).$$

Taking $r = 8^{m+1}d$, this proves the claim.

Now translate the origin in $\mathbb{C}$ to p. The restriction of ξ_b to $D_r(p)$ takes the form $u : D_r \to \mathbb{C}$, and the map

$$\begin{array}{ccc} D_r & \longrightarrow & \mathbb{C}^2 \\ z & \mapsto & (u(z), z) \end{array}$$

has J-holomorphic image. By taking ϵ sufficiently small and $|b| < \epsilon$, we can ensure that d and r are small enough for the hypotheses of Lemma 45 to be satisfied. Therefore we get a 1-parameter family of admissible disks $\sigma_{\mathrm{a}} : D_r \to \mathbb{C}^2$, $\mathrm{a} \in D_\delta$, and each disk $\sigma_{\mathrm{a}}(D_r)$ is J-holomorphic and passes through the $m+1$ points $(u(\zeta_i), \zeta_i)$. At $\mathrm{a} = 0$, we have $\sigma_0 = (u, \mathrm{id}|_{D_r})$, so the homotopy invariance of I implies that $I(\sigma_{\mathrm{a}}, D_r) = I(\sigma_0, D_r) = m$ for all $\mathrm{a} \in D_\delta$.

By Lemma 45 (iii), $\sigma_{\mathrm{a}}(D_r)$ and $\sigma_{\mathrm{a}'}(D_r)$ intersect discretely when $\mathrm{a} \neq \mathrm{a}'$. Therefore $\sigma^{-1}(S)$ has finite 2-dimensional Hausdorff measure in $D_\delta \times D_r$ and, by Lemma 9, there are values of a arbitrarily close to 0 for which the intersection $\sigma_{\mathrm{a}}(D_r) \cap S$ contains the $m+1$ points $(u(\zeta_i), \zeta_i))$, $i = 1, \ldots, m+1$. Thus we have a contradiction, and the lemma is proved. □

If $\kappa \in \Upsilon$, then $\xi(D_r(\kappa)) \cap S = \{\mathbf{0}\}$ for all sufficiently small $r \leq R$. We set

(13) $$\nu_0(\kappa) = I(\xi, D_r(\kappa)) \text{ for any sufficiently small } r.$$

This quantity is well-defined because of property (4) of a cohomology assignment. Now define the *multiplicity* of $\mathbf{0}$ in S to be

(14) $$\nu_0 = \inf_{\kappa \in \Upsilon} \nu_0(\kappa).$$

Note that if $\xi\left((D_r(\kappa)) \cap S = \{\mathbf{0}\}\right.$, and $\kappa' \in \Upsilon$ is sufficiently close to κ, then

$$\xi\left(D_r(\kappa')\right) \cap S \subset \xi(D_{\frac{r}{2}}(\kappa')).$$

Consequently $\xi : D_r(\kappa') \to U$ is S-admissible and, by property (2) of cohomology assignments, $I(\xi, D_r(\kappa')) = I(\xi, D_r(\kappa))$. Since $\nu_0(\kappa') \leq I(\xi, D_r(\kappa'))$, we have proved:

(15) $$\nu_0(\kappa') \leq \nu_0(\kappa) \text{ for all } \kappa' \text{ sufficiently close to } \kappa.$$

If there is a neighborhood N of κ such that $\nu_0(\kappa') = \nu_0(\kappa)$ for all $\kappa' \in N$, then we call κ a *stable direction* through $\mathbf{0}$. For example, the infimum in (14) is taken over a collection of positive integers, so is achieved for some $\kappa \in \Upsilon$. This direction κ is stable. The set of all stable directions through $\mathbf{0}$ will be denoted Υ_s.

LEMMA 21. *Υ_s is open and dense in $\mathbb{C}P^1$.*

PROOF. It follows from the definition that Υ_s is open in Υ. If $\kappa \in \Upsilon$ and N is a small neighborhood of κ, then by (15), $\nu_0(\kappa') \leq \nu_0(\kappa)$ for all $\kappa' \in N$. Hence $\{\nu_0(\kappa') | \kappa' \in N\}$ is a finite set of positive numbers and there exists some $\kappa'' \in N$ with $\nu_0(\kappa'') = \inf_{\kappa' \in N} \nu_0(\kappa')$. This κ'' is stable, so we have shown that Υ_s is dense in Υ.

Since Υ is open and dense in $\mathbb{C}P^1$, we are done. □

LEMMA 22. *Let κ be a stable direction through $\mathbf{0}$, and let $r < R$ be small enough that $\xi(D_r(\kappa)) \cap S = \{\mathbf{0}\}$. If $\kappa' \in \Upsilon$ is sufficiently close to κ, then $\xi\,(D_r(\kappa')) \cap S = \{\mathbf{0}\}$ as well.*

PROOF. Take $r < R$ so that $\xi\,(D_r(\kappa)) \cap S = \{\mathbf{0}\}$ and $\nu_0(\kappa) = I(\xi, D_r(\kappa))$. In proving (15), we found a neighborhood N of κ such that

$$\nu_0(\kappa') \leq I(\xi, D_r(\kappa') = I(\xi, D_r(\kappa)) = \nu_0(\kappa)$$

for all $\kappa' \in N$. Since κ is stable, we have (up to taking a smaller neighborhood) $\nu_0(\kappa') = \nu_0(\kappa)$ for all $\kappa' \in N$. Hence $\nu_0(\kappa') = I(\xi, D_r(\kappa'))$ for $\kappa' \in N$ and, by the definition of I and the definition of $\nu_0(\kappa')$, $\xi\,(D_r(\kappa')) \cap S = \{\mathbf{0}\}$. $\square$

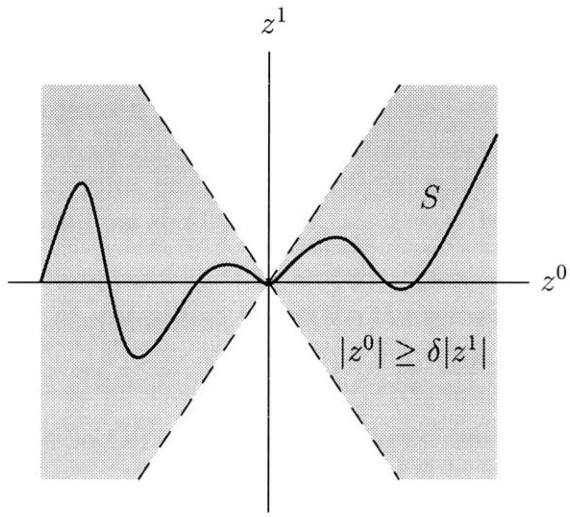

FIGURE 5-1

COROLLARY 23. *Let $\sigma : D \to U$ be an embedded, J-holomorphic, S-admissible disk in U. Suppose that $\sigma(D)$ intersects S only at $\mathbf{0}$. Then $I(\sigma, D) \geq \nu_0$.*

PROOF. We can assume that $\sigma^{-1}(\mathbf{0}) = 0$. Let κ_0 be the tangent line to $\sigma(D)$ at $\mathbf{0}$ and choose coordinates $\{z^0, z^1\}$ so that $\kappa_0 = \{z^0 = 0\}$.

In the first case, assume that κ_0 is stable. By Lemma 22, there exists $\delta > 0$ such that $\xi^{-1}(S) \cap \{z^0 - bz^1 = 0\} = \{0\}$ when $|b| \leq \delta$. In other words, $\xi^{-1}(S)$ is contained in the region where $|z^0| \geq \delta|z^1|$ (see Figure 5-1). Thus if $D_r \subset D$ is any sufficiently small disk containing 0, then $\sigma(D_r)$ is contained in the region where $|z^0| \leq \frac{\delta}{2}|z'|$, and $\sigma|_{D_r} : D_r \to U$ is homotopic to $\xi : D_r(\kappa_0) \to U$ through S-admissible disks. By properties (2) and (4) of cohomology assignments, $I(\sigma, D) = I(\sigma, D_r) = I(\xi, D_r(\kappa_0))$. Now take r small enough that $I(\xi, D_r(\kappa_0)) = \nu_0(\kappa_0)$. Then $I(\sigma, D) = \nu_0(\kappa_0) \geq \nu_0$.

In the second case, κ_0 is not stable but can be approximated by stable directions, by Lemma 21. Express σ near 0 as a graph over κ_0. Thus we have $D_r(\kappa_0) \approx D_r \subset D$ and $u : D_r(\kappa_0) \to \mathbb{C}$ so that $\sigma(z) = (u(z), z)$ for all $z \in D_r(\kappa_0)$. Let σ_κ be the perturbation of σ given by Lemma 44. By taking κ arbitrarily close to κ_0, we can assume that $\sigma_\kappa : D_R \to \mathbb{C}^2$ is S-admissible and admissibly homotopic to $\sigma|_{D_r}$ (see the proof of Lemma 20), and that κ is a stable direction. Therefore $I(\sigma, D) = I(\sigma, D_r) =$

$I(\sigma_\kappa, D_r)$. But recall that the image of σ_κ is J-holomorphic, and I is positive. It follows that $\nu_0 \leq I(\sigma_\kappa, D_r) = I(\sigma, D)$, as needed. $\square$

2. Local behavior.

We are working in a neighborhood U of $\mathbf{0}$ in $\mathbb{C}^2$, and with an almost complex structure J in U that is compatible with the metric g.

The discussion of the previous section is valid at each point of U: for each $x \in U$, J_x is a complex structure on $\mathbb{C}^2$ centered at x, and the almost complex structure J equals J_x at x. The set of J_x-hermitian coordinates centered at x is a copy of $U(2)$ over x, and the space of J_x-complex lines through x is a copy of $\mathbb{C}P^1$ over x.

Thus we obtain over U a principal $U(2)$-bundle P, a $\mathbb{C}P^1$-bundle Q, and a bundle map

$$\begin{tikzcd} P \arrow[rr, "\pi"] \arrow[dr] & & Q \arrow[dl] \\ & (U,J) & \end{tikzcd}$$

which, fibrewise, maps a choice of J_x hermitian coordinates centered at x to the corresponding "vertical" line through X. The line bundles $\gamma \to \mathbb{C}P^1$ and $E, E^\perp \to U(2)$ of the previous chapter also generalize to line bundles $\gamma \to Q$, $E = \pi^*\gamma \to P$ and $E^\perp \to P$. For example, the fiber of E over a point $(x, \{z^0, z^1\}) \in P$ is the J_x complex line $z^0 = 0$.

Corollaries 42 and 43 give us bundle maps Θ, Ξ and a commutative diagram

(16)
$$\begin{tikzcd} & & D_R(E) \arrow[r, hook, "(0,\mathrm{id})"] \arrow[dl] \arrow[d, "\pi"] & D_R(E^\perp) \oplus D_R(E) \arrow[r, "\Theta"] & P \times \mathbb{C}^2 \arrow[d, "\pi"] \\ & P \arrow[dl] \arrow[d, "\pi"] & D_R(\gamma) \arrow[rr, "\Xi"] \arrow[dl] & & Q \times \mathbb{C}^2 \\ (U,J) & Q \arrow[l] & & & \end{tikzcd}$$

with the property that the image under $\Theta|_{(x,\{z^0,z^1\})}$ of each vertical disk $w \times D_R(E|_{(x,\{z^0,z^1\})})$, centered at the point $x \in U$ with coordinates $\{z^0, z^1\}$, is J-holomorphic in $\mathbb{C}^2$. (By the commutativity of the diagram, the image under Ξ of each fiber in $D_R(\gamma)$ is also J-holomorphic.)

Now introduce the closed subset $S \subset U$ with finite 2-dimensional Hausdorff measure, and positive cohomology assignment I.

We summarize the results of the previous section with the following

PROPOSITION 24. *For each $x \in S$, the following are true*

i) The set $\Upsilon|_x = \{\kappa \in Q_x \mid \Xi|_{(x,\kappa)}(D_R(\kappa)) \cap S$ is finite $\}$ is open and has full measure in $Q_x \approx \mathbb{C}P^1$.

ii) Given $\kappa \in \Upsilon|_x$, let $\nu_x(\kappa) = I\left(\Xi|_{(x,\kappa)}, D_r(\kappa)\right)$ for any sufficiently small r. The multiplicity of x in S defined as $\nu_x = \inf \nu_x(\kappa)$ is a positive integer.

iii) The set of stable directions through x,

$$\Upsilon_s|_x = \{\kappa \in \Upsilon|_x : \nu_x(\kappa') = \nu_x(\kappa) \text{ for all nearby directions } \kappa'\}$$

is open and dense in Q_x.

iv) If κ is a stable direction through x and $\Xi(D_r(\kappa)) \cap S = \{x\}$, then $\Xi(D_r(\kappa')) \cap S = \{x\}$ for all $\kappa' \in \Upsilon|_x$ sufficiently close to κ.

v) If $\sigma : D \to U$ is an embedded, J-holomorphic, S-admissible disk and $\sigma(D) \cap S = \{x\}$, then $I(\sigma, D) \geq \nu_x$.

A point $x \in S$ is called *regular* if $\nu_y > \nu_x$ for all $y \in S$ lying in some neighborhood of x. Otherwise, x is called *singular*. Note that singular points have $\nu_x \geq 2$.

LEMMA 25. *If x is any point in S, then $\nu_y \leq \nu_x$ for all y in some neighborhood of x.*

PROOF. Let κ_0 be a stable direction through x so such that $\nu_x(\kappa_0) = \nu_x$, and let $\{z^0, z^1\}$ be hermitian coordinates centered at x such that $\kappa_0 = \{z^0 = 0\}$. By Lemma 22, there exists $\delta > 0$ such that the closed set $\Xi_x^{-1}(S)$ is contained in the region $\{|z^0| \geq \delta |z^1|\}$. Moreover, there is a small $r > 0$ so that $I(\Xi_x, D_r(\kappa_0)) = \nu_x(\kappa_0) = \nu_x$. If $y \in S$ is sufficiently close to x, then Corollary 42 provides an S-admissible disk $\sigma = \theta_w : D_r \to U$ (for some $w \in D_r$) such that $y \in \sigma(D_r) \cap S$, and also an admissible homotopy between σ and $\Xi_x|_{D_r(\kappa_0)}$. By Lemma 44, we can perturb σ through S-admissible disks to an S-admissible disk $\sigma_\kappa : D_r \to U$ which is tangent at y to a stable direction $\kappa \in \Upsilon|_y$. By construction, the image of σ_κ is J-holomorphic, and also I is assumed to be positive. Thus if we choose $r' < r$ so that $\sigma_\kappa(D_{r'}) \cap S = \{y\}$, we find

$$\nu_y \leq \nu_y(\kappa) = I(\sigma_\kappa, D_{r'}) \leq I(\sigma_\kappa, D_r) = I(\Xi_x, D_r(\kappa_0)) = \nu_x.$$

□

In summary, if $x \in S$ is a regular point, then there is a neigborhood N_x of x such that $\nu_y = \nu_x$ for all $y \in S \cap N_x$.

LEMMA 26. *The set S_{reg} of regular points is open and dense in S.*

PROOF. The preceding discussion shows that S_{reg} is open.

If $x \in S$ and N_x is an open neighborhood of x, then the function defined on N_x by $y \mapsto \nu_y$ is positive and integer valued. Therefore its minimum is attained, and any point at which the minimum is attained is regular. □

3. The regular points of S.

Continuing our proof of Theorem 18, recall that $U \subset \mathbb{C}^2$ is open, J and g are compatible almost complex structure and metric on U, $S \subset U$ is closed with finite 2-dimensional Hausdorff measure, and I is a positive cohomology assignment on S.

In this section, we show that the open, dense set S_{reg} consisting of the regular points of S has the structure of an open J-holomorphic submanifold of U. Following the strategy outlined in Chapter 5, we exhibit S in the neighborhood of a regular point x as the graph of a Lipschitz function $\Phi : D \to \mathbb{C}$. Thus S has the structure of a Lipschitz submanifold near x. We then show that Φ is holomorphic at 0. The proof of Theorem 31 follows when we allow x to vary through S_{reg}.

Fix a point $x \in S_{reg}$ and let κ_0 be a stable direction through x such that $\nu_x = \nu_x(\kappa_0)$. Let $\{z^0, z^1\}$ be hermitian coordinates on $\mathbb{C}^2$ centered at x such that $\kappa_0 = \{z^0 = 0\}$. In these coordinates, S is the graph of a function $\Phi : D_r(\kappa_0^\perp) \to \mathcal{D}_r(\kappa_0)$ which we define as follows.

Let $\theta_w : D_R \to U$ be the family of disks parameterized by $w \in D_R(\kappa_0^\perp)$, given to us by Lemma 42. Recall also the map Θ in (16) which, at the point $(x, \{z^0, z^1\}) \in P$, coincides with θ:

$$\Theta_{(x,\{z^0,z^1\})}(w, z) = \theta_w(z) \text{ for all } (w, z) \in D_R(\kappa_0^\perp) \times D_R(\kappa_0)$$

Since $\theta_0(D_R) \cap S$ is finite and $\theta_0(0) = x$, we have $\theta_0(D_r) \cap S = \{x\}$ for all sufficiently small r.

LEMMA 27. *For all $(\tilde{x}, \{\tilde{z}^0, \tilde{z}^1\})$ in a neighborhood of $(x, \{z^0, z^1\})$ in P, and all w in a neighborhood D_r of 0, the intersection*

$$\Theta_{(\tilde{x},\{\tilde{z}^0,\tilde{z}^1\})}(w \times D_R) \cap S \tag{17}$$

contains precisely one point.

PROOF. Duplicating the proof of Lemma 20, we find that for all $(\tilde{x}, \{\tilde{z}^0, \tilde{z}^1\})$ sufficiently close to $(x, \{z^0, z^1\})$ and all w sufficiently close to 0, the intersection (17) is finite and contains at most $\nu_0 = I(\theta_0, D_r)$ points. Let k denote the number of points.

Since x is regular, we can assume that each point y in (17) has multiplicity $\nu_y = \nu_x$. The image of $\Theta_{(\tilde{x},\{\tilde{z}^0,\tilde{z}^1\})}$ on each vertical fiber $w \times D_R$ is J-holomorphic, so by Proposition 24 (v),

$$I\left(\Theta_{(\tilde{x},\{\tilde{z}^0,\tilde{z}^1\})}, w \times D_R\right) \geq k \cdot \nu_0. \tag{18}$$

By the homotopy invariance of I, we also have

$$I\left(\Theta_{(\tilde{x},\{\tilde{z}^0,\tilde{z}^1\})}, w \times D_R\right) = I\left(\Theta_{(\tilde{x},\{z^0,z^1\})}, 0 \times D_r\right) = \nu_0 \ ,$$

and the latter is compatible with (18) only if $k = 1$. This proves the lemma. □

Consequently, for each $w \in D_r$ there is a unique point $\Phi(w) \in D_r(\kappa_0)$ so that $\theta(w \times D_r) \cap S = \theta(w, \Phi(w))$. This defines the function $\Phi : D_r(\kappa_0^\perp) \to D_r(\kappa_0)$.

Near x, S can be identified with its preimage under the diffeomorphism θ:

$$\theta^{-1}(S) = \{(w, \Phi(w)) \mid w \in D_r\}. \tag{19}$$

This is none other than the graph of Φ. Since $\theta^{-1}(S)$ is closed, Φ is at least continuous.

LEMMA 28. *Let $x \in S_{reg}$ and define $\Phi : D_r \to D_r$ so that* (19) *holds. There is a constant $k > 0$ such that*

$$|\Phi(w) - \Phi(w')| < k \cdot |w - w'| \textit{ for all } w, w' \in D_r.$$

In other words, Φ is a Lipschitz function on D_r.

PROOF. Identify S near $\mathbf{0}$ with $\theta^{-1}(S) = \{(w, \Phi(w)) \mid w \in D_r\}$ and use the identifications of (16) to apply Proposition 24 (iv). For each point $x = (w, \Phi(w))$, we find coordinates $\{\tilde{z}^0, \tilde{z}^1\}$ centered at x and $\delta_x > 0$ so that $\theta^{-1}(S)$ is contained in the region $\{|\tilde{z}^0| \geq \delta_x |\tilde{z}^1|\}$. Taking r smaller if necessary, we may take $\delta > 0$ to be independent of x and $\{\tilde{z}^0, \tilde{z}^1\}$ to be arbitrarily close (in norm) to $\{z^0, z^1\}$. Then any other point $x' = (w', \Phi(w'))$ must satisfy

$$\left|z^0(x') - z^0(x)\right| \leq \frac{\delta}{2}\left|z^1(x') - z^1(x)\right|,$$

or equivalently

$$|w' - w| \leq \frac{\delta}{2}\left|\Phi(w') - \Phi(w)\right|.$$

$\square$

Allowing x to vary through S_{reg}, we obtain the next result.

COROLLARY 29. *There is an open, dense subset of S that has the structure of a Lipschitz submanifold of U.*

LEMMA 30. *Let $x \in S_{reg}$ and define $\Phi : D_r \to D_r$ so that* (19) *holds. Then $\Phi : D_r \to D_r$ is holomorphic at 0.*

PROOF. Since $\Phi(0) = 0$, we need to prove that the function

$$\partial\Phi(w) = \frac{\Phi(w)}{w}, \quad w \in D_r \setminus \{0\}$$

extends continuously over 0.

We know by Lemma 28 that $\partial\Phi$ is bounded on $D_r \setminus 0$. Therefore if $\partial\Phi$ fails to be continuous at 0, we can find two sequences w_i and w_i' converging to 0 such that $\lim \partial\Phi(w_i) \neq \lim \partial\Phi(w_i')$. For each i, set $\lambda_i = \partial\Phi(w_i)$ and $\lambda_i' = \partial\Phi(w_i')$. We can assume that $\lim|\lambda_i| \leq \lim|\lambda_i'|$, and that

$$2\epsilon > |\lambda_i - \lambda_j'| > \epsilon$$

for some $\epsilon > 0$, for all i and j.

Fix some $j \gg i \gg 0$ so that $|w_j'| \ll |w_i|$ and, for ease of notation, set

$$w = w_i \quad , \quad w' = w_j'$$

$$\lambda = \lambda_i = \frac{\Phi(w)}{w} \quad , \quad \lambda' = \lambda_j' = \frac{\Phi(w')}{w'}.$$

By construction, the quadratic

$$f(z) = \frac{\lambda - \lambda'}{w - w'} z^2 + \frac{\lambda' w - \lambda w'}{w - w'} z$$

satisfies: $f(0) = 0$, $f(w) = \Phi(w)$, $f(w') = \Phi(w')$. Therefore the map

$$\begin{array}{rccl} (20) \qquad & \mathbf{q}: & \longrightarrow & U \\ & z & \mapsto & (z, f(z)) \end{array}$$

passes through the three points $(0,0)$, $(w, \Phi(w))$, $(w', \Phi(w'))$ in S. To make this map S-admissible, recall that S (or $\theta^{-1}(S)$) lies in the region $\{|z^1| \leq k|z^0|\}$, where k is the Lipschitz constant of Φ. If ρ is chosen so that

$$|f(z)| > 2k|z| \text{ for all } z \in \partial D_\rho, \tag{21}$$

then both (20) and small perturbations of (20) will be admissible disks in U. Let us take

$$\rho = 100|w| \cdot \frac{k + |\lambda'|}{\epsilon}.$$

Then it is straightforward to check that (21) is fulfilled for this choice of ρ.

Note that k, $|\lambda'|$ and ε are independent of $|w|$, so by taking i very large (so that $|w| = |w_i|$ is very small), we can assume that ρ is less than the constant R_0 of Lemma 41. We can also assume that $|f(z)| \leq 44(k + |\lambda'|)\rho$ is less than the constant C_1 of Lemma 41. Therefore when $a_0,\ a_1,\ a_2 \in \mathbb{C}$ are sufficiently small, Lemma 41 gives us a function ψ, which depends smoothly on a_0, a_1, a_2, and $z \in D_\rho$, so that the perturbed map

$$\sigma(z) = \big(z, f(z) + a_2 z^2 + a_1 z + a_0 + \psi(z)\big) \tag{22}$$

has J-holomorphic image in $\mathbb{C}^2$. If $|w|$ and $|w'|$ are sufficiently small, then by Lemma 41 and the inverse function theorem, there exist unique $a_0,\ a_1,\ a_2$ satisfying the simultaneous equations

$$\begin{aligned} a_2 w^2 + a_1 w + a_0 + \psi(w) &= 0 \\ a_2\,(w')^2 + a_1 w' + a_0 + \psi(w') &= 0 \\ a_0 + \psi(0) &= 0. \end{aligned}$$

Hence σ passes through the three points $(0,0)$, $\theta(w, \Phi(w))$, $\theta(w', \Phi(w'))$ in S.

Now translate the origin to $p = \frac{w}{2}$ and perturb σ using Lemma 45 with $m = 3$, $\zeta_1 = 0$, $\zeta_2 = w$ and $\zeta_3 = w'$. The hypotheses of Lemma 45 are satisfied when $|w|$ is sufficiently small, for then

$$\sum_{i=1}^{3} \prod_{j \neq i} |\zeta_i - \zeta_j| = |w||w'| + |w||w - w'| + |w'||w' - w| > C\rho^{3-\frac{1}{3}}$$

$$\prod_{j=1}^{3} |\zeta_j - p| = \left|\frac{w}{2}\right|^2 \cdot \left|w' - \frac{w}{2}\right| > C \cdot \rho^{3+\frac{2}{3}},$$

where C is the constant of Lemma 45. We therefore obtain a 1-parameter family of J-holomorphic S-admissible disks $\sigma_{\mathrm{a}}(D_\rho) \subset U$, $\mathrm{a} \in D_\rho$, such that $\sigma_{\mathrm{a}}(D_\rho)$ and $\sigma_{\mathrm{a}'}(D_\rho)$ intersect discretely when $\mathrm{a} \neq \mathrm{a}'$. Applying Lemma 9, we find a full measure set of $\mathrm{a} \in D_\rho$ such that $\sigma_{\mathrm{a}}(D_\rho)$ and S intersect discretely.

Fix one such $\mathrm{a} \in D_\rho$. By construction, $\sigma_{\mathrm{a}}(D_\rho) \cap S$ contains at least the three points $\theta(0,0)$, $\theta(w, \Phi(w))$, $\theta(w', \Phi(w'))$, so $I(\sigma_{\mathrm{a}}, D_\rho) \geq 3\nu_x$ by Proposition 24 (v) and the assumed regularity of $x \in S$. On the other hand, σ_{a} is homotopic through admissible disks to $\sigma_0 = \sigma$, and σ is admissibly homotopic to the quadratic map $\mathbf{q}(z) = (z, f(z))$. It is a simple matter to check that $\mathbf{q}$ is admissibly homotopic to the doubly covered vertical disk

$$z \mapsto \theta\left(0, \tfrac{\lambda - \lambda'}{w - w'} z^2\right), \quad z \in D_\rho.$$

Therefore, by properties 2 and 3 of cohomology assignments,

$$I(\sigma_{\mathrm{a}}, D_\rho) = I(\sigma, D_\rho) = I(\mathbf{q}, D_\rho) = 2 \cdot I(\theta_0, D_\rho) = 2\nu_x.$$

We thus have $2\nu_x = I(\sigma_{\mathrm{a}}, D_\rho) \geq 3\nu_x$, which is impossible as $\nu_x > 0$. Therefore our initial assumption on the behavior of $\partial\Phi$ at 0 is false, and we conclude that Φ is holomorphic at 0. $\square$

An immediate consequence of Lemma 30 is the following.

THEOREM 31. *If $S \subset U$ is closed, $\mathfrak{H}^2(S) < +\infty$ and I is a positive cohomology assignment on S in U, then the set of regular points of S is an embedded, J-holomorphic submanifold of U.*

CHAPTER 6

The Boundary Case

In standard transversality and intersection theory, there is no well-defined intersection number between two maps $f : M \to X$, $g : N \to X$ when one of M and N has non-empty boundary. Under homotopy, points of intersection can slide off the boundary, so the usual count of points (with or without sign) does not yield a well-defined number. For this reason a cohomology assignment on $S \subset X$, which one should think of as a generalized local intersection pairing with S, is defined only on the class of S-admissible disks in X. The restriction $\sigma(\partial D) \subset X \setminus S$ guarantees that intersection points do not slide off via the boundary ∂D.

If X^4 is an oriented manifold and $Y^2 \subset X$ is a 2-dimensional orientable submanifold then it is possible to define an intersection pairing on the class of smooth maps

$$\{f : M^2 \to X \mid M \text{ is a surface and } f(\partial M) \subset Y\}.$$

Let $f : (M, \partial M) \to (X, Y)$ and $g : (N, \partial N) \to (X, Y)$ be two such maps and suppose the following are true:

i) f and g are transverse in X;

ii) $f|_{\partial M}$ and $g|_{\partial N}$ are transverse in Y;

iii) The set $\{(p, q) \in M \times N \mid f(p) = g(q)\}$ is finite and consists of points (p, q) such that $p \in \operatorname{Int} M$, $q \in \operatorname{Int} N$, or $p \in \partial M$, $q \in \partial N$.

When these conditions are satisfied, we say that f and g are *relatively transverse* and we define the *relative intersection pairing* of f and g to be the signed sum

$$I(f, g) = \sum_{\substack{(p,q) \in \operatorname{Int} M \times \operatorname{Int} N \\ f(p) = g(q)}} \varepsilon_{p,q} + \frac{1}{2} \sum_{\substack{(p,q) \in \partial M \times \partial N \\ f(p) = g(q)}} \varepsilon_{p,q}, \tag{23}$$

where $\varepsilon_{p,q} = +1$ if $T_{f(p)}X$ and $f_* T_p M \oplus g_* T_q N$ are equal as oriented vector spaces, and $\varepsilon_{p,q} = -1$ if the orientations are opposite.

Note that the sign $\varepsilon_{p,q}$ at a point $(p, q) \in \partial M \times \partial N$ is not determined by comparing $f_* T_p(\partial M) \oplus g_* T_q(\partial N)$ to $T_{f(p)}Y$. In fact we have not specified an orientation on Y.

THEOREM 32. *Let $f_0, f_1 : (M, \partial M) \to (X, Y)$ and $g : (N, \partial N) \to (X, Y)$ be smooth maps of surfaces and suppose that f_0 and f_1 are relatively transverse to g. If $F(t, p) = f_t(p)$ is a homotopy between f_0 and f_1 such that $f_t(\partial M) \subset Y$ for all t, then $I(f_0, g) = I(f_1, g)$.*

Such a homotopy will be called a *relative homotopy.* Now if $f : (M, \partial M) \to (X, Y)$ and $g : (N, \partial N) \to (X, Y)$ are any two maps (possibly not relatively transverse), then we can find maps $f' : (M, \partial M) \to (X, Y)$ and $g' : (N, \partial N) \to (X, Y)$ relatively homotopic to f and g respectively, so that f' and g' are relatively transverse. Then $I(f', g')$ is defined as above, and we set $I(f, g) = I(f', g')$. By Theorem 32, this relative intersection number is well-defined.

PROOF OF THEOREM 32. We can assume that $F^{-1}(Y) = I \times \partial M$ and $g^{-1}(Y) = \partial N$, that is,

(24) the normal vector to $I \times \partial M$ in $I \times M$ (resp.to ∂N in N) is mapped into the normal bundle of Y in X.

Because f_t is required to map ∂M into Y for all t, we can not generically assume that F is transverse to g along $I \times \partial M$. However, by Sard's theorem we can perturb F so that

(25) $F^\circ = F|_{I\times \operatorname{Int} M}$ and $g^\circ = g|_{\operatorname{Int} N}$ are transverse in X, and

(26) $F^+ = F|_{I\times\partial M}$ and $g^+ = g|_{\partial N}$ are transverse in Y.

Therefore the set $W = \{(t,p,q) \mid F(t,p) = g(q)\}$ equals the union $W^\circ \cup W^+$, where

$$W^\circ = \{(t,p,q) \in I \times \operatorname{Int} M \times \operatorname{Int} N \mid F^\circ(t,p) = g^\circ(q)\}$$

is a 1-manifold with boundary and possibly some ends in $I \times \operatorname{Int} M \times \operatorname{Int} N$, and

$$W^+ = \{(t,p,q) \in I \times \partial M \times \partial N \mid F^+(t,p) = g^+(q)\}$$

is a 1-manifold with boundary in $I \times \partial M \times \partial N$.

If W° has any open ends, these will converge to $I \times \partial M \times \partial N$, so to points in W^+. We will presently show that after an arbitrarily small perturbation of F, there can be at most finitely many points where this can happen, and that $\overline{W^\circ}$ is a smooth manifold with boundary on $(\partial I \times \operatorname{Int} M \times \operatorname{Int} N) \cup W^+$. The perturbation will leave F a relative homotopy and preserve (24), (25), (26) above. Moreover, an interior intersection point of weight ± 1 converges to the boundary (in the sense that W° meets W^+) only if it combines with a boundary intersection point of weight $\mp\frac{1}{2}$ to become a single boundary intersection point with weight $\pm\frac{1}{2}$. Thus the total intersection number (23) remains invariant as time t progresses.

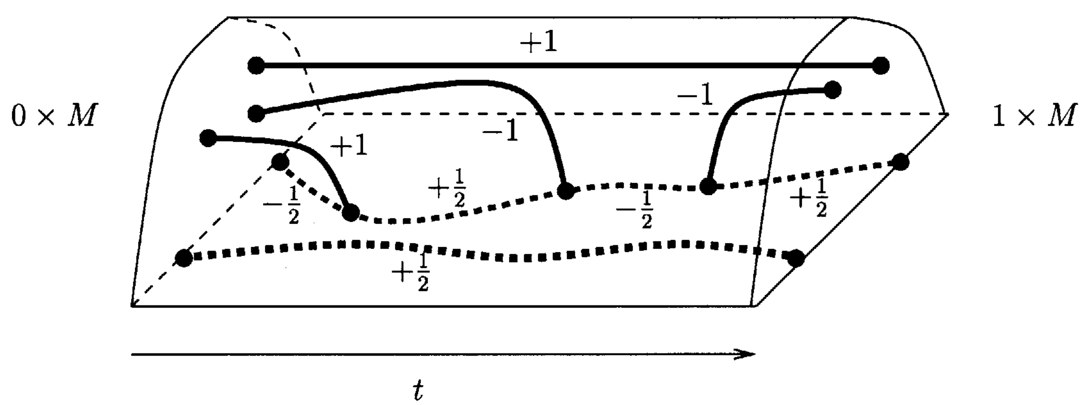

FIGURE 6-1. Behavior of intersection points in $I \times M$. Solid lines represent W° while dotted lines represent W^+.

Let $D^+ = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, \quad y \geq 0\}$ denote the upper half disk in $\mathbb{R}^2$, with boundary $\partial^+ D^+ = \{(x,y) \in D^+ \mid y = 0\}$. By our assumptions (24), (25) and

(26), we can find coordinates x_1, y_1, x_2, y_2 on X near Y so that Y is given by the local equations $y_1 = y_2 = 0$, and so that F and g have the following local expressions:

$$\begin{aligned} F : I \times D^+ &\longrightarrow \mathbb{R}^4_{x_1,y_1,x_2,y_2} \\ (t,x,y) &\longmapsto (x, y, \alpha(t,x,y), \beta(t,x,y)) \\ g : D^+ &\longrightarrow \mathbb{R}^4 \\ (x,y) &\longmapsto (x,y,0,0) \end{aligned}$$

where $\beta(t,x,0) = 0$ for all t, x. Therefore W° is the set where $y > 0, \alpha = \beta = 0$, and W^+ is the set where $y = 0, \alpha = 0$.

Since F^+ is transverse to g^+ in Y, it follows that F fails to be transverse to g at the points $(t,x,0)$ where the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial t} \\ 0 & 0 & \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial t} \end{bmatrix}_{y=0} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial t} \\ 0 & 0 & 0 & \frac{\partial \beta}{\partial y} & 0 \end{bmatrix}$$

does not have full rank. This occurs precisely when $\frac{\partial \beta}{\partial y}(t,x,0) = 0$.

Suppose that $\frac{\partial \beta}{\partial y} = 0$ at the point $(0,0,0)$. By making an arbitrarily small perturbation of α in the x direction, we can assume that $\frac{\partial \alpha}{\partial x}(0,0,0) \neq 0$. Then make a change of coordinates so that $\alpha(t,x,y) = x$. By Taylor's theorem, there are functions $f, g, h : I \times D^+ \to \mathbb{R}$ so that

$$\beta = y \cdot (tf + xg + yh) \tag{27}$$

in a neighborhood of $(0,0,0)$. By making another arbitrarily small perturbation we can assume that f, g and h are not zero at $(0,0,0)$.

Thus in the vicinity of $(0,0,0)$, we have $W^+ = \{(t,x,y) \mid x = y = 0\}$ and $W^\circ = \{(t,x,y) \mid y > 0 \text{ and } y \cdot (tf + yh) = 0\}$. As $f(0,0,0)$ and $h(0,0,0)$ are both nonzero, it follows immediately that $\overline{W^\circ}$ is a smooth manifold with boundary on W^+, and that $(0,0,0)$ is isolated as a point of intersection of $\overline{W^\circ}$ and W^+.

It remains to determine the weights ± 1, or $\pm\frac{1}{2}$, associated to the points of W°, or W^+, located near $(0,0,0)$. The sign of the weights equals the sign of the determinant

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} \\ 0 & 0 & \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} \end{vmatrix} \tag{28}$$

which, for $\alpha = x$, equals the sign of $\frac{\partial \beta}{\partial y}$. By (27), $\beta_y(t,0,0) = tf(t,0,0)$ for any t. Hence the weight at $(t,0,0) \in W^+$ equals $\frac{1}{2} \cdot \operatorname{sign}(tf)$. On the other hand, at the point $(t,0,y) \in W^\circ$, we have $tf + yh = 0$, and $\operatorname{sign}(t) = -\operatorname{sign}(h) \cdot \operatorname{sign}(f)$. By (27), $\beta_y(t,0,y) = y \cdot \left(t\frac{\partial f}{\partial y} + h + y + \frac{\partial h}{\partial y}\right)$, so when $y > 0$ and t are sufficiently small, we have

$$\operatorname{sign}(\beta_y) = \operatorname{sign}(yh) = \operatorname{sign}(h) = -\operatorname{sign}(tf).$$

Therefore the weight at $(t,0,y) \in W^\circ$ equals $-\operatorname{sign}(tf)$. The four possibilities are depicted in Figure 6-2.

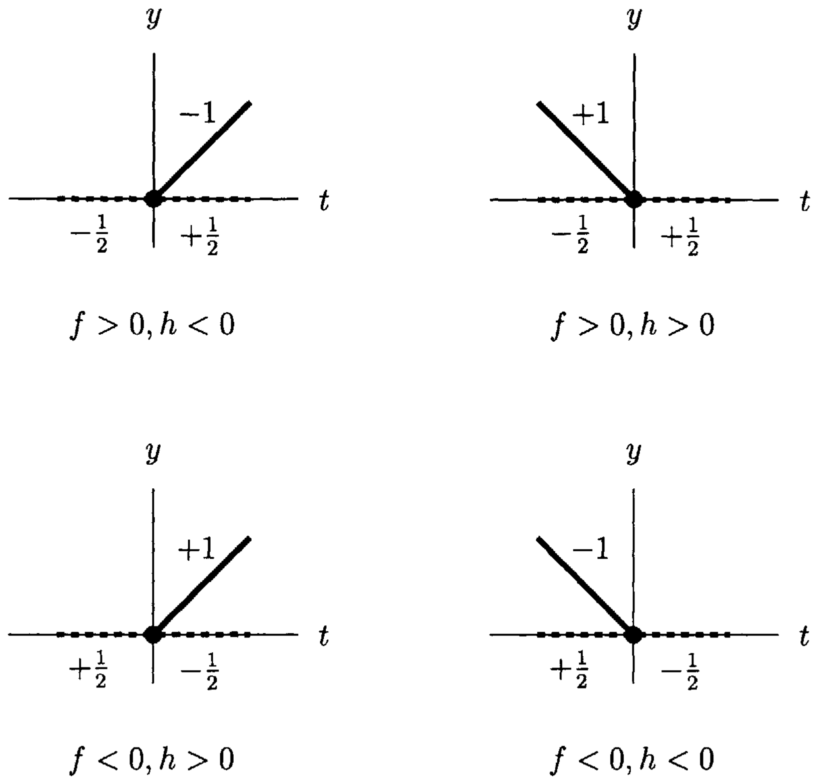

FIGURE 6-2. W° (solid) meets W^+ (dotted) transversally at the origin. When $\operatorname{sign}(t) = -\operatorname{sign}(hf)$, then $y > 0$.

In any case, the sum of the weights is constant with respect to time t.

We have shown that, after an arbitrarily small perturbation, F is transverse to g on the complement of a finite set of points in $I \times D^+$. The points where F and g are not transverse are the boundary points of $\overline{W^\circ}$, where W° and W^+ meet transversally. The weights carried by points in $W = W^\circ \cup W^+$ change with t according to the four pictures in Figure 6-2. Therefore the intersection number $I(f_t, g)$ is invariant with respect to t in a neighborhood of the points $\overline{W^\circ} \cap W^+$. Outside of $\overline{W^\circ} \cap W^+$, W is a 1-manifold with boundary, so the usual argument (see [**GP**, Chap. 3]) shows that $I(f_t, g)$ is everywhere invariant under time. We conclude that $I(f_0, g) = I(f_1, g)$. $\square$

We want to prove the following local recognition principle with totally real boundary conditions.

THEOREM 33. *Let J, ω be an almost complex structure and compatible symplectic form on $\mathbb{C}^2$, and let $Y \subset \mathbb{C}^2$ be a real 2-dimensional plane that is everywhere totally real with respect to J. Let $U \subset \mathbb{C}^2$ be open and let $S \subset U$ be a connected closed subset with boundary $\partial S \subset Y$, such that $\mathfrak{H}^2(S) < +\infty$ and $\mathfrak{H}^1(\partial S) < +\infty$. If S carries a positive relative cohomology assignment I in $(U, Y \cap U)$, then there is a Riemann surface Σ with boundary, a domain $\Omega \subset \Sigma$ with boundary $\partial^+\Omega = \Omega \cap \partial\Sigma$, and a J-holomorphic map $f : (\Omega, \partial^+\Omega) \to (U, Y)$ such that $S = f(\Omega)$, $\partial S = f(\partial^+\Omega)$, and $I = I_f$.*

LEMMA 34. *Let $Y \subset \mathbb{C}^2$ be a J_0-totally real 2-plane, and let $U \subset \mathbb{C}^2$ be an open neighborhood of* $\mathbf{0}$. *If $(S, \partial S) \subset (U, Y)$ is closed, and $\mathfrak{H}^1(\partial S) < +\infty$, then the set*

$$\Lambda = \{\lambda \in S^1 \mid \xi_\lambda(\partial^+ D_R^+) \cap \partial S \text{ is finite}\}$$

has full measure in S^1.

LEMMA 35. *If in addition to the hypotheses of Lemma 34 we assume that $(S, \partial S)$ has a positive relative cohomology assignment I in (U, Y), then $\xi(D_R^+(\lambda)) \cap S$ is finite for each $\lambda \in \Lambda$.*

COROLLARY 36. *Let $Y \subset \mathbb{C}^2$ be a totally real 2-plane, let $U \subset \mathbb{C}^2$ be open, and let $(S, \partial S) \subset (U, Y)$ be closed with $\mathfrak{H}^2(S) < +\infty$ and $\mathfrak{H}^1(\partial S) < +\infty$. Then $\Lambda \subset S^1$ is open.*

If $\lambda \in \Lambda$, then $\xi(D_r^+(\lambda)) \cap S = \{\mathbf{0}\}$ for all sufficiently small r. Therefore we set

$$\nu_0(\lambda) = 2 \cdot I(\xi, D_r^+(\lambda))$$

for any sufficiently small r. If $\lambda' \in \Lambda$ is sufficiently close to λ, then

$$\xi(D_r^+(\lambda')) \cap S \subset \xi(D_{r/2}^+(\lambda')).$$

Hence $\xi : (D_r^+(\lambda'), \partial^+ D_r^+(\lambda')) \to (U, Y)$ is S-admissible and, by homotopy invariance, $I(\xi, D_r^+(\lambda')) = I(\xi, D_r^+(\lambda))$. It follows that $\nu_0(\lambda') \leq \nu_0(\lambda)$ for all λ' sufficiently close to λ in Λ. If in fact $\nu_0(\lambda') = \nu_0(\lambda)$ for all λ' in some neighborhood of λ, then we call λ a stable (real) direction through $\mathbf{0}$. The set of stable directions in Λ will be denoted Λ_s.

Next define the multiplicity of $\mathbf{0}$ in S to be

$$\nu_0 = \inf_{\lambda \in \Lambda} \nu_0(\lambda)$$

and note that the infimum is taken over a set of positive integers. Therefore ν_0 is a positive integer and equals $\nu_0(\lambda)$ for some $\lambda \in \Lambda$. The latter direction λ must, by definition, be stable.

LEMMA 37. *The set Λ_s of stable directions in Y through* $\mathbf{0}$ *is an open subset of* S^1.

CHAPTER 7

Constructing Families of Pseudo-holomorphic Disks

Let $U \subset \mathbb{C}^2$ be an open subset containing the origin, and let J_0 denote the standard complex structure on $\mathbb{C}^2$. Fix a metric g on U. Let J be a compatible almost complex structure on U that is standard at the origin, and let ω be a tame symplectic structure on U that is standard at the origin. In terms of complex coordinates $z^0 = x^1+ix^2, z^1 = x^3 + ix^4$ on U, we have $\omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$, and

$$J_0 = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix} \quad , \quad j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

At a point $\mathbf{z} = (z^0, z^1)$ in $\mathbb{C}^2$, write J as

$$J_{\mathbf{z}} = \begin{bmatrix} A_{\mathbf{z}} & B_{\mathbf{z}} \\ C_{\mathbf{z}} & D_{\mathbf{z}} \end{bmatrix},$$

where A, B, C, D are real 2x2 matrices. Since $J = J_0$ at the origin, each of $\|A - j\|, \|D - j\|, \|B\|, \|C\|$ is small when $\|\mathbf{z}\|$ is small.

Next let D_R denote the open disk of radius R in $\mathbb{C}$. Given a smooth function $u : D_R \to \mathbb{C}$ and a complex polynomial $f : D_R \to \mathbb{C}$, our plan is to find a smooth function $\psi : D_R \to \mathbb{C}$ so that the map

$$\begin{aligned} \mathbf{q} : \ D_R &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto (u(z) + f(z) + \psi(z), z) \end{aligned}$$

has J-holomorphic image in $\mathbb{C}^2$. Note that this is weaker than asking for the map q itself to be J-holomorphic (in the sense that $Jd\mathbf{q} = d\mathbf{q}j_0$).

In terms of the complex Gaussian coordinates, we have $\mathbf{q} = (q, id_{\mathbb{C}})$, with $q = u + f + \psi$. Then the condition that $\mathbf{q}$ have J-holomorphic image is equivalent to the equation

$$B + Adq - dqD - dqCdq \equiv 0 \tag{29}$$

which, since $J^2 = -I$, is equivalent to

$$\bar\partial \psi = -\bar\partial u + \frac{1}{2} j\{B + (A - j)dq - dq(D - j) - dqCdq\}. \tag{30}$$

Let Q_ψ be the (0,1)-form appearing on the right hand side of equation (30). To solve (30), we look for ψ as a fixed point of a functional

$$\text{“}F(\psi) = \bar\partial^{-1}(Q_\psi)\text{”}$$

defined on a suitable Banach space.

1. Hölder spaces and estimates for $\bar{\partial}$.

Let $\mathcal{K}^{k,\alpha}(D_R)$ be the Banach space of functions $\psi : \mathbb{C} \to \mathbb{C}$ of class $C^{k,\alpha}$ such that the restriction of ψ to $\mathbb{C}\backslash D_R$ is holomorphic, and such that $|\psi| \to 0$ as $|z| \to +\infty$. (Note then that $\psi|_{\partial D_R} \in \text{Span}\{e^{in\theta}\}_{n<0}$.)

For any $\psi \in \mathcal{K}^{k,\alpha}(D_R)$, the holomorphic map $\psi|_{\mathbb{C}\backslash D_R}$ is uniquely determined by $\psi|_{\partial D_R}$. Therefore the following Hölder norm on D_R also defines a norm on $\mathcal{K}^{k,\alpha}(D_R)$:

$$\|\psi\| = \|\psi\|_{k,\alpha} = \|\psi\|_{k,\alpha;D_R} = \sum_{j=0}^{k} R^j \|D^j\psi\|_{0;D_R} + R^{k+\alpha}[D^k\psi]_{\alpha;D_R} .$$

Here, $\|\cdot\|_{0;D_R}$ denotes the C^0 norm on D_R,

$$\|D^j\psi\|_{0;D_R} = \sup_{|\beta|=j} \|D^\beta\psi\|_{0;D_R} ,$$

and $[\cdot]_\alpha$ the Hölder norm, given by

$$[\phi]_{\alpha;D_R} = \sup_{x,y\in D_R} \frac{|\phi(x) - \phi(y)|}{|x-y|^\alpha} .$$

Some relevant background on Hölder spaces may be found in [**GT**, Chap.4]. We will need the following easily verifiable facts:

(1) If $k' + \alpha' > k + \alpha$, then $\mathcal{K}^{k',\alpha'}(\overline{D_R}) \subset \mathcal{K}^{k,\alpha}(\overline{D_R})$.
(2) If $\varphi \in \mathcal{K}^{k,\alpha}$, $\psi \in \mathcal{K}^{k',\alpha'}$ and $k' + \alpha' > k + \alpha$, then $\varphi\psi \in \mathcal{K}^{k,\alpha}$.
(3) If $\psi \in \mathcal{K}^{k+1,\alpha}(D_R)$, then $\|D\psi\|_{k,\alpha} \le \frac{1}{R}\|\psi\|_{k+1,\alpha}$.

LEMMA 38. *Let Ω be an open, proper subset in $\mathbb{R}^2$. Given a bounded function $\varphi \in C^\alpha(\Omega)$, set*

$$\psi(z) = \frac{1}{2\pi i}\int_\Omega \frac{1}{z-w}\,\varphi(w)dw\,d\bar{w} . \tag{31}$$

Then $\psi \in C^1(\Omega)$, and $\frac{\partial\psi}{\partial\bar{z}} = \varphi$ in Ω. Moreover, for any $z \in \Omega$,

$$\partial_z\psi(z) = \frac{1}{2\pi i}\int_{\Omega_0} \frac{1}{(z-w)^2}\,(\varphi(z) - \varphi(w))dw\,d\bar{w} . \tag{32}$$

Here Ω_0 is any domain containing Ω and φ is extended to vanish outside Ω.

The same methods used in [**GT**, Chap.4] to solve the Dirichlet problem for Poisson's equation yield the following estimates for $\bar{\partial}$.

LEMMA 39. *Let $B_1 = B_R(0) \subset B_2 = B_{2R}(0)$ be concentric balls in $\mathbb{C}$. Let $\varphi \in C^\alpha(\bar{B}_2)$, $0 < \alpha < 1$, and define ψ as in (31). Then $\psi \in C^{1,\alpha}(\bar{B}_1)$ and*

$$\|D\psi\|_{0;\bar{B}_1} + R^\alpha[D\psi]_{\alpha;\bar{B}_1} \le C \cdot \{\|\varphi\|_{0;\bar{B}_2} + R^\alpha[\varphi]_{\alpha;\bar{B}_2}\} ,$$

for some constant $C = C(\alpha)$.

COROLLARY 40. *Let $\varphi \in C^\alpha_0(D_R)$. For $z \in \mathbb{C}$, set*

$$\psi(z) = \frac{1}{2\pi i}\int_{D_R} \frac{1}{z-w}\,\varphi(w)dw\,d\bar{w} .$$

Then $\psi \in \mathcal{K}^{1,\alpha}(D_R)$ and

$$\|\psi\|_{1,\alpha;D_R} \le C(\alpha)R \cdot \|\varphi\|_{0,\alpha;D_{2R}} = C(\alpha)R \cdot \|\varphi\|_{0,\alpha;D_R} . \tag{33}$$

PROOF. Extend φ to be 0 outside of D_R. Then ψ is holomorphic in $\mathbb{C}\backslash D_R$. Also, $|\psi| \to 0$ as $|z| \to +\infty$. Therefore $\psi \in \mathcal{K}^{1,\alpha}(D_R)$, and (33) follows from Lemma 39. □

2. The main lemma.

We are now prepared to find a solution to equation (30) on D_R. To do so, let us fix a cut-off function $\chi : \mathbb{C} \to [0,1]$ with the property that $\chi(z) = 1$ for $|z| \leq 1$, and $\chi(z) = 0$ for $|z| \geq 3/2$. For any $R > 0$, define χ_R to be the function

$$\chi_R(z) = \chi(\frac{z}{R}) \quad , z \in \mathbb{C}.$$

In Lemma 41, we use χ_R to define a cut-off equation on D_{2R} (see (35) below) whose restriction to the smaller disk D_R is equation (30). We find a solution to this cut-off equation by a fixed point argument in the Banach space $\mathcal{K}^{2,\alpha}(D_{2R})$, and then restrict this solution to D_R to solve (30).

LEMMA 41. *Let $U \subset \mathbb{C}^2$ be open, let g be a metric on U, and let J be a compatible almost complex structure that equals J_0 at the origin. There exist constants $R_0, C_0, C_1 > 0$ depending only on g, J such that the following are true.*

i) Let $R < R_0$. For each smooth map $u : D_{2R} \to \mathbb{C}$ with $\|u\|_{2,\alpha;D_{2R}} < R/C_2$, and each degree m complex polynomial $f = a_m z^m + \cdots + a_0$ with $\|f\|_{2,\alpha;D_{2R}} < R$, there exists a unique $\psi \in \mathcal{K}^{2,\alpha}(D_{2R})$ with

$$\|\psi\|_{2,\alpha;D_{2R}} \leq C_0 \cdot (\|u\|_{2,\alpha;D_{2R}} + R\|f\|_{2,\alpha;D_{2R}} + R^2) \tag{34}$$

that solves the equation

$$\bar\partial\,\psi = \chi_R \cdot \left\{ -\bar\partial\, u + \frac{j}{2}\left[B|_{\mathbf{q}_\psi} + (A-j)|_{\mathbf{q}_\psi} \cdot dq_\psi - dq_\psi \cdot (D-j)|_{\mathbf{q}_\psi} - dq_\psi \cdot C|_{\mathbf{q}_\psi} \cdot dq_\psi\right] \right\}, \tag{35}$$

$$\mathbf{q}_\psi = (q_\psi, id) = (u + f + \psi, id)\,.$$

In particular, the restriction of $\mathbf{q}_\psi$ to D_R has J-holomorphic image in $\mathbb{C}^2$.

ii) The function ψ is smooth.

iii) The function ψ is also C^∞ with respect to the coefficients $a_0, \ldots, a_m$ of f. There exist constants c_k depending only on k, J and g so that

$$\|\frac{\partial\psi}{\partial a_k}\| \leq c_k \cdot R^{k+1}.$$

iv) Suppose that the map

$$\begin{array}{rcl} D_R & \longrightarrow & \mathbb{C}^2 \\ z & \longmapsto & (u(z), z) \end{array}$$

has J-holomorphic image. Then for any complex polynomial $f = a_m z^m + \cdots + a_0$, the function ψ obtained in (i) satisfies:

$$\|\psi\|_{2,\alpha} \leq C_0 R \cdot \|f\|_{2,\alpha;D_{2R}}\,.$$

PROOF. For each $\psi \in \mathcal{K}^{2,\alpha}(D_{2R})$, let $\mathbf{q}_\psi = (q_\psi, id) = (u + f + \psi, id)$ and define

$$Q_\psi = -\,\bar\partial\, u + \frac{j}{2}\{B|_{\mathbf{q}_\psi} + (A-j)|_{\mathbf{q}_\psi} \cdot dq_\psi - dq_\psi \cdot (D-j)|_{\mathbf{q}_\psi} - dq_\psi \cdot C|_{\mathbf{q}_\psi} \cdot dq_\psi\}\,. \tag{36}$$

The cut-off form $\chi_R \cdot Q_\psi$ lies in $C_0^{1,\alpha}(D_{2R})$, so by Corollary 40,

$$F(\psi)(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{dw}{z-w}\, \chi_R \cdot Q_\psi \tag{37}$$

is a well-defined element of $\mathcal{K}^{2,\alpha}(D_{2R})$. This defines a functional $F : \mathcal{K}^{2,\alpha}(D_{2R}) \to \mathcal{K}^{2,\alpha}(D_{2R})$, and any fixed point ψ^* of F must solve (35) on D_{2R}. Since $\chi_R \equiv 1$ on D_R, ψ^* solves equation (30) on D_R. Consequently the image of $\mathbf{q}_\psi|_{D_R}$ is J-holomorphic in $\mathbb{C}^2$. We will look for a fixed point of F on a neighborhood of 0 in $\mathcal{K}^{2,\alpha}(D_{2R})$.

Suppose there is a positive number ε so that $\|D_\psi F\|_{2,\alpha} \le \frac{1}{2}$ when $\|\psi\|_{2,\alpha} \le \varepsilon$. Then

$$\|F(\psi) - F(\varphi)\| \le \frac{1}{2}\|\psi - \varphi\| \quad \text{for } \psi, \varphi \in B_\varepsilon(0),$$

and

$$\|F(\psi)\| \le \frac{1}{2}\|\psi\| + \|F(0)\| \le \frac{\varepsilon}{2} + \|F(0)\| \quad \text{for } \psi \in B_\varepsilon(0).$$

If $\|F(0)\| \le \frac{\varepsilon}{2}$, then F maps $B_\varepsilon(0)$ to itself and is a contraction there. So F has a unique fixed point $\psi^* \in B_\varepsilon(0)$, which is what we want. Furthermore, $\|\psi^*\| \le 2\|F(0)\|$. In the steps that follow, we will make the necessary estimates to find a suitable value for ε.

Step 1. There is a constant C depending only on J, χ and α so that if $R < 1$ and $\|\mathbf{q}\|_{1,\alpha;D_{2R}} \le 50R$, then the following hold:

$$\|(J - J_0) \circ \mathbf{q}\|_{1,\alpha;D_{2R}} \le C \cdot \|\mathbf{q}\|_{1,\alpha;D_{2R}} \tag{38}$$

$$\|D(J - J_0) \circ \mathbf{q}\|_{1,\alpha;D_{2R}} \le C\,. \tag{39}$$

The proof follows from the fact that J equals J_0 at the origin, and from the multiplicative property of Hölder norms.

Step 2. If $R < 1$ and $\|\mathbf{q}\|_{2,\alpha;D_{2R}} \le 50R$, then

$$\|D_\psi F\|_{1,\alpha;D_{2R}} \le C_0 \cdot (R + \|\mathbf{q}\|_{2,\alpha;D_{2R}}) \tag{40}$$

for some constant $C_0 = C_0(J, \chi, \alpha)$.

Proof. Differentiating (36) and using Step 1, we obtain

$$\begin{aligned}
\|D_\psi Q\|_{1,\alpha} \;\le\;& \|D(J - J_0) \circ \mathbf{q}\|_{1,\alpha} \cdot (1 + 2\|dq\|_{1,\alpha} + \|dq\|_{1,\alpha}^2) \\
&+ 2\|(J - J_0) \circ \mathbf{q}\|_{1,\alpha} \cdot \|\frac{\delta}{\delta\psi} dq\|_{1,\alpha} \\
&+ 2\|(J - J_0) \circ \mathbf{q}\|_{1,\alpha} \cdot \|\frac{\delta}{\delta\psi} dq\|_{1,\alpha} \cdot \|dq\|_{1,\alpha} \\
\le\;& C \cdot (1 + \|dq\|_{1,\alpha})^2 + 2C\|\mathbf{q}\|_{1,\alpha} \cdot \|\frac{\delta}{\delta\psi} dq\|_{1,\alpha} \cdot (1 + \|dq\|_{1,\alpha})\,.
\end{aligned}$$

Now $\|\frac{\delta q}{\delta\psi}\|_{1,\alpha} = 1$, so that $\|\frac{\delta}{\delta\psi} dq\|_{1,\alpha} \le \frac{1}{R}$. It follows from Corollary 40 and (37) that if $\|\mathbf{q}\|_{2,\alpha} \le 50R$, then

$$\begin{aligned}
\|D_\psi F\|_{2,\alpha;D_{2R}} \;\le\;& C(\alpha)R \cdot \|\chi_R \cdot D_\psi Q\|_{1,\alpha;D_{2R}} \;\le\; C(\alpha,\chi)R \cdot \|D_\psi Q\|_{1,\alpha;D_{2R}} \\
\le\;& C \cdot (R + \|q\|_{2,\alpha})(1 + \|dq\|_{1,\alpha}) + 100CR \cdot (1 + \|dq\|_{1,\alpha}) \\
\le\;& 151C \cdot (R + \|q\|_{2,\alpha})\,.
\end{aligned}$$

Step 3. If $R < 1$ and $\|\mathbf{q}_0\|_{2,\alpha;D_{2R}} \leq 50R$, then at $\psi = 0$,

$$\|F(0)\|_{2,\alpha;D_{2R}} \leq C_1\{\|u\|_{2,\alpha} + (\|u\|_{2,\alpha} + \|f\|_{2,\alpha} + R)^2\}, \tag{41}$$

for some constant $C_1 = C_1(J, \chi, \alpha)$.

Proof. When $\psi = 0$, then $\mathbf{q}_0 = (q_0, id_{\mathbb{C}}) = (u + f, id_{\mathbb{C}})$. Using (38) and (36), observe that

$$\|Q_0\|_{1,\alpha;D_{2R}} \leq \|du\|_{1,\alpha;D_{2R}} + C\,\|\mathbf{q}_0\|_{1,\alpha;D_{2R}} \cdot \{1 + 2\|dq_0\|_{1,\alpha;D_{2R}} + \|dq_0\|^2_{1,\alpha;D_{2R}}\}.$$

As $\|\mathbf{q}_0\|_{2,\alpha;D_{2R}} \leq 50R$, it follows that

$$\begin{aligned}\|F(0)\|_{2,\alpha;D_{2R}} &\leq C(\alpha,\chi)\{R\|du\|_{1,\alpha} + 50CR^2 \cdot (1 + \|dq_0\|_{1,\alpha})^2\} \\ &\leq C(\alpha,\chi)\{\|u\|_{2,\alpha} + 50C \cdot (R + \|q_0\|_{2,\alpha})^2\} \\ &\leq C_1\{\|u\|_{2,\alpha} + (R + \|u\|_{2,\alpha} + \|f\|_{2,\alpha})^2\}.\end{aligned}$$

Step 4. Here, we use the results of Steps 1-3 to find a fixed point of F on an ε-neighborhood of 0 in $\mathcal{K}^{2,\alpha}(D_{2R})$.

Let $R_0 = 1/60C_0(1 + C_1)$, where C_0 and C_1 are the constants of Steps 2 and 3, respectively. Let $R < R_0$ and suppose that $\|u\|_{2,\alpha;D_{2R}} < R/C_1$, while $\|f\|_{2,\alpha;D_{2R}} < R$. Define $F : \mathcal{K}^{2,\alpha}(D_{2R}) \to \mathcal{K}^{2,\alpha}(D_{2R})$ as in (37).

Next take $\varepsilon = 8C_1(\|u\|_{2,\alpha} + R\|f\|_{2,\alpha} + R^2)$ and note that by our assumptions on $\|u\|, \|f\|$ and R, we have $\varepsilon < 24R$. Thus if $\|\psi\|_{2,\alpha} \leq \varepsilon$, then $\|\mathbf{q}_\psi\|_{2,\alpha} < 29R < 50R$, and so by Step 2,

$$\begin{aligned}\|D_\psi F\|_{2,\alpha;D_{2R}} &\leq C_0 \cdot (R + \|\mathbf{q}\|_{2,\alpha;D_{2R}}) \\ &\leq 30C_0 \cdot R \\ &\leq \frac{1}{2}.\end{aligned}$$

Moreover by Step 3, $\|F(0)\|_{2,\alpha;D_{2R}} \leq C_1\{\|u\|_{2,\alpha} + (\|u\|_{2,\alpha} + \|f\|_{2,\alpha} + R)^2\} < 4C_1(\|u\|_{2,\alpha} + R\|f\|_{2,\alpha} + R^2) = \frac{1}{2}\varepsilon$.

We can therefore conclude that F is a contraction on $B_\varepsilon(0) \subset \mathcal{K}^{k,\alpha}(D_{2R})$. Let ψ^* be the unique fixed point of $F|_{B_\varepsilon(0)}$. Then

$$\|\psi^*\|_{2,\alpha} = \|F(\psi^*)\|_{2,\alpha} \leq \frac{1}{2}\|\psi^* - 0\|_{2,\alpha} + \|F(0)\|_{2,\alpha}$$

and hence $\|\psi^*\|_{2,\alpha} \leq 2\|F(0)\|_{2,\alpha;D_{2R}}$. Since $\bar\partial\,\psi^* = \chi_R \cdot Q_{\psi^*}$, it follows that the restriction of $\mathbf{q}_{\psi^*}$ to B_R has J-holomorphic image. The smoothness of ψ^* follows from a standard bootstrap argument.

To prove *(iii)*, consider $\mathbf{q}, Q, F$ as functions of $\mathbf{a} = (a_m, \ldots, a_1, a_0) \in \mathbb{C}^m$ and $\psi \in \mathcal{K}^{2,\alpha}(D_{2R})$. Thus,

$$q(\mathbf{a}, \psi)(z) = (u(z) + a_m z^m + \cdots + a_1 z + a_0 + \psi(z), z),$$

$$Q(\mathbf{a}, \psi) = -\bar\partial\,u + \frac{j}{2}\{B|_{\mathbf{q}_{\mathbf{a},\psi}} + (A-j)|_{\mathbf{q}_{\mathbf{a},\psi}} \cdot dq - dq \cdot (D-j)|_{\mathbf{q}_{\mathbf{a},\psi}} - dq \cdot C|_{\mathbf{q}_{\mathbf{a},\psi}} \cdot dq\}, \tag{42}$$

$$F(\mathbf{a}, \psi) = \bar\partial^{-1}(\chi_R \cdot Q(\mathbf{a}, \psi)).$$

From the equation $\psi^*(\mathbf{a}) = F(\mathbf{a}, \psi^*(\mathbf{a}))$, we find that ψ^* is differentiable with respect to a_k and

$$\frac{\delta\psi^*}{\delta a_k}(\mathbf{a}) = [I - \frac{\delta F}{\delta\psi}(\mathbf{a}, \psi^*_{\mathbf{a}})]^{-1} \cdot \frac{\delta F}{\delta a_k}(\mathbf{a}, \psi^*_{\mathbf{a}}) \ . \tag{43}$$

Since $\|\frac{\delta F}{\delta\psi}\|_{2,\alpha} = \|D_\psi F\|_{2,\alpha} \leq \frac{1}{2}$ for $\|\psi\|_{2,\alpha} \leq \varepsilon$, it follows that

$$\|\frac{\delta\psi^*}{\delta a_k}\|_{2,\alpha} \leq 2\|\frac{\delta F}{\delta a_k}(\mathbf{a}, \psi^*_{\mathbf{a}})\|_{2,\alpha} \leq 2C(\alpha)R\,\|\frac{\delta Q}{\delta a_k}(\mathbf{a}, \psi^*_{\mathbf{a}})\|_{1,\alpha} . \tag{44}$$

On the other hand, $\delta\mathbf{q}/\delta a_k = p_k$, where $p_k : \mathbb{C} \to \mathbb{C}^2$ is the function $z \mapsto (z^k, 0)$, satisfying $\|p_k\|_{1,\alpha;D_{2R}} \leq c_k \cdot R^k$. Differentiating (42) with respect to a_k, we therefore have

$$\begin{aligned}
\|\frac{\delta Q}{\delta a_k}\|_{1,\alpha} \ &\leq \ \|D(J - J_0) \circ \mathbf{q}\| \cdot \|\frac{\delta\mathbf{q}}{\delta a_k}\| \cdot (1 + 2\|dq\| + \|dq\|^2) \\
&\quad + 2\|(J - J_0) \circ \mathbf{q}\| \cdot \|\frac{\delta}{\delta a_k}(dq)\| \cdot (1 + \|dq\|) \\
&\leq \ c_k\, R^k \cdot (1 + \|dq\|)^2 + c_k\, \|\mathbf{q}\| \cdot R^{k-1} \cdot (1 + \|dq\|) \\
&\leq \ (51 + 50) \cdot 51\, c_k R^k
\end{aligned}$$

Combined with (44), this proves *(iii)*.

Finally, to prove *(iv)*, write Q as a function of f, ψ :

$$Q_f(\psi) = Q(\mathbf{a}, \psi) .$$

To say that $(u(z), z)$ is J-holomorphic is to say that $Q_0(0) = 0$. In this case, there is a constant C_2 such that

$$\|Q_f(0)\| \leq C_2 \cdot \|f\|$$

for all complex polynomials f of sufficiently small norm. But $\|\psi^*_f\|' \leq 2\|F_f(0)\|' \leq 2C(\alpha)R \cdot \|Q_f(0)\|'$, so this does the trick. $\square$

3. Families of disks.

Recall that on the open set $U \subset \mathbb{C}^2$, we have a metric g and a compatible almost complex structure J which equals J_0 at the origin. Fix complex coordinates $z^0 = x^1 + ix^2$, $z^1 = x^3 + ix^4$ in $\mathbb{C}^2$.

For example, if we take $u = 0$ in Lemma 41, then by (34) we have

$$\sup_{D_{2R}} |\psi^*| \leq \|\psi^*\|_{2,\alpha;D_{2R}} \leq C_0 \cdot (R\|f\| + R^2) .$$

It follows that $\sup_{D_{2R'}} |\psi^*| \leq C' \cdot R'$ for all $R' \leq R$, and hence $\psi^*(0) = 0$.

COROLLARY 42. *There exist constants $R_0 > 0$ and C depending only on J and g such that, for $R < R_0$, there is a diffeomorphism $\theta : D_R \times D_R \to U \subset \mathbb{C}^2$ with the following properties:*

i) For each $w \in D_R$, the map $\theta_w = \theta(w, \cdot) : D_R \to U$ has J-holomorphic image.

ii) For each $w \in D_R$, $\theta(w, 0) = (w, 0)$.

iii) For each w, $\mathrm{dist}(\theta(w, z), (w, z)) \leq C\, R \cdot |z|$.

iv) There is a constant $C_k = C(J, g, k)$ so that $|D^k(\theta_w)| \leq C_k \cdot R$ for each $w \in D_R$.

v) The image of θ_0 is tangent to the vertical line $\{z^0 = 0\}$ at $\mathbf{0}$.

PROOF. Let R_0 be the constant of Lemma 41, and let $R < R_0$. Given $w \in D_R$, take $u = 0$ and $f = w$ in Lemma 41. Then we obtain a smooth function $\psi_w : D_{2R} \to \mathbb{C}$ for which the map

$$\begin{array}{rccl} \mathbf{q}_w : & D_R & \longrightarrow & \mathbb{C}^2 \\ & z & \longmapsto & (w + \psi_w(z), z) \end{array}$$

has J-holomorphic image.

Set $\theta(w, z) = (w + \psi_w(z), z)$. By Lemma 41 (3), θ is smooth in w and in z, and

$$D\theta = \begin{pmatrix} I + \frac{\delta\psi}{\delta w} & \frac{\delta\psi}{\delta z} \\ 0 & I \end{pmatrix}.$$

Moreover, there is a constant c_0 so that $\|\delta\psi/\delta w\| \le c_0 \cdot R$ for all $R < R_0$. Thus when R_0 is sufficiently small, $D\theta$ is invertible, and by the inverse function theorem, θ is a diffeomorphism onto its image.

By construction, each θ_w has J-holomorphic image. For any $R' \le R$, (34) yields

$$\sup_{D_{2R'}} |\psi_w| \le \|\psi_w\|_{2,\alpha;D_{2R'}} \le C_0 \cdot (R'|w| + (R')^2) \le 2C_0 R R'.$$

Thus $\psi_w(0) = 0$ and $\theta(w, 0) = (w, 0)$. A similar argument will prove *(v)*.

Another consequence of (34) is that

$$R \cdot \sup_{D_{2R}} |D\psi_w| \le \|\psi_w\|_{2,\alpha;D_{2R}} \le 2C_0 R^2.$$

Hence $\mathrm{dist}(\theta(w,z),(w,z)) \le C|\psi_w(z)| \le C \cdot \sup_{D_{2R}} |D\psi_w| \cdot |z| \le CR \cdot |z|$. This proves *(iii)*.

From the proof of Lemma 41, recall that $\bar\partial\psi = \frac{i}{2}\{B + (A - j)dq - dq(D - j) - dq\,C\,dq\}$ and that $\|\psi_w\|$ is sufficiently small that we have $\|q\| < 50R$. In our case, $q = (q, \mathrm{id}) = (w + \psi_w, \mathrm{id})$. By Corollary 40 and (38) we have, for each $k > 0$,

$$\begin{aligned} \|\psi\|_{k,\alpha} &\le CR \cdot \|B + (A - j)\cdot d\psi - d\psi \cdot (D - j) - d\psi \cdot C \cdot d\psi\|_{k-1,\alpha} \\ &\le CR \cdot \|\psi\|_{k-1,\alpha} \cdot \left(1 + 2\|d\psi\|_{k-1,\alpha} + \|d\psi\|^2_{k-1,\alpha}\right) \\ &\le C \cdot \|\psi\|_{k-1,\alpha} \cdot (R + \|\psi\|_{k,\alpha}) \cdot (1 + \|d\psi\|_{k-1,\alpha}) \\ &\le 51C \cdot \|\psi\|_{k-1,\alpha}\,(R + \|\psi\|_{k,\alpha})\,\| \\ &\le 51^2 CR \cdot \|\psi\|_{k-1,\alpha}. \end{aligned}$$

This proves *(iv)*. □

Let κ denote the "vertical" line $z^0 = 0$, and $\kappa^\perp$ the "horizontal" line $z^1 = 0$. Letting $D_R(\kappa)$, resp. $D_R(\kappa^\perp)$, denote the disk of radius R in κ, resp. $\kappa^\perp$, we interpret θ as a map $D_R(\kappa) \times D_R(\kappa^\perp) \to \mathbb{C}^2$.

The above construction works regardless of our original choice of hermitian coordinates in $\mathbb{C}^2$. The set of such coordinates is parametrized by $U(2)$, and there is a natural map $\pi : U(2) \to \mathbb{C}P^1 \approx U(2)/U(1) \times U(1)$ which sends a choice of coordinates to the corresponding vertical line. Let $\gamma \to \mathbb{C}P^1$ denote the tautological line bundle and let $E \to U(2)$ denote the pullback of γ via π. We refer to E as the "bundle of horizontal lines". Applying Corollary 42 for each choice of hermitian coordinates, we obtain a bundle map

$$
\begin{tikzcd}
D_R(E) \times D_R(E^{\perp}) \arrow[rr] \arrow[dr] & & U(2) \times \mathbb{C}^2 \arrow[dl] \\
& U(2) &
\end{tikzcd}
$$

which, fiberwise, has all the properties *(i)–(v)* of Corollary 42.

COROLLARY 43. *Let $D_R(\gamma)$ be the radius R disk bundle in γ. There exists a constant $R_0 > 0$ depending only on J and g so that for each $R < R_0$, there is a smooth map $\xi : D_R(\gamma) \to U$ with the following properties.*

i) ξ maps the zero section to $\mathbf{0}$ and embeds the complement of the zero section.

ii) For each $\kappa \in \mathbb{C}P^1$, $\xi\,(D_R(\kappa))$ is J-holomorphic in $\mathbb{C}^2$ and is tangent to κ at $\mathbf{0}$.

iii) If $\{z^0, z^1\}$ are hermitian coordinates in $\mathbb{C}^2$ and κ_0 is the vertical line $z^0 = 0$, then $\xi|_{D_R(\kappa_0)} : D_R(\kappa_0) \to \mathbb{C}^2$ coincides with the map $\theta_0 : D_R(\kappa_0) \to \mathbb{C}^2$ obtained in Corollary 42.

PROOF. Fix $\kappa_0 \in \mathbb{C}P^1$ and choose a lift $\{z^0, z^1\} \in U(2)$. The map $\theta_0 : D_R(\kappa_0) \to \mathbb{C}^2$ which, in coordinates $\{z^0, z^1\}$, is defined by

$$\theta_0(z) = (\psi(z), z)\,, \quad z \in D_R,$$

is tangent to κ_0 at $\mathbf{0} = \theta_0(0)$, by Corollary 42, *(v)*. The ψ in question is the unique element of $\mathcal{K}^{2,\alpha}(D_{2R})$ with norm bounded by (34) satisfying the equation

$$\bar{\partial}\,\psi = \chi_R \cdot \frac{j}{2}\left\{B + (A - j)\cdot d\psi - d\psi\cdot(D - j) - d\psi\cdot C\cdot d\psi\right\}. \tag{45}$$

If $\{\tilde{z}^0, \tilde{z}^1\} \in U(2)$ also maps to κ_0 under π, then there exist $\lambda_0, \lambda_1 \in U(1)$ so that $(\tilde{z}^0(x), \tilde{z}^1(x)) = (\lambda_0 z^0(x), \lambda_1 z^1(x))$ for all $x \in \mathbb{C}^2$. (In other words, $\{\tilde{z}^0, \tilde{z}^1\}$ differs from $\{z^0, z^1\}$ by the diagonal matrix with entries λ_0, λ_1.) Let

$$\tilde{\theta}_0(\tilde{z}) = \left(\tilde{\psi}(\tilde{z}), \tilde{z}\right), \quad \tilde{z} \in D_R,$$

be the map obtained by applying Corollary 42 in the coordinates $\{\tilde{z}^0, \tilde{z}^1\}$. If we make the substitution $\tilde{z} = \lambda_1 z$ and switch back to the coordinates $\{z^0, z^1\}$, then we find

$$\tilde{\theta}_0(z) = \left(\frac{1}{\lambda_0}\tilde{\psi}(\lambda_1 z), z\right),$$

and the function $z \mapsto \frac{1}{\lambda_0}\tilde{\psi}(\lambda_1 z)$ also satisfies (45). Moreover, this function is an element of $\mathcal{K}^{2,\alpha}(D_{2R})$ whose norm $\left\|\frac{1}{\lambda_0}\tilde{\psi}(\lambda_1 z)\right\| \leq \|\tilde{\psi}\|$ is bounded by (34). (It suffices to check that the non-negative Fourier coefficients of the restriction to ∂D_{2R} are all zero, which is true because $\tilde{\psi} \in \mathcal{K}^{2,\alpha}(D_{2R})$.) By uniqueness, it follows that $\psi(z) = \frac{1}{\lambda_0}\tilde{\psi}(\lambda_1 z)$ for all $z \in D_R$.

Thus $\tilde{\theta}_0$ and θ_0 define the same map $D_R(\kappa_0) \to \mathbb{C}^2$, and we can define $\xi : D_R(\kappa_0) \to \mathbb{C}$ independently of the choice of lift for κ_0. $\square$

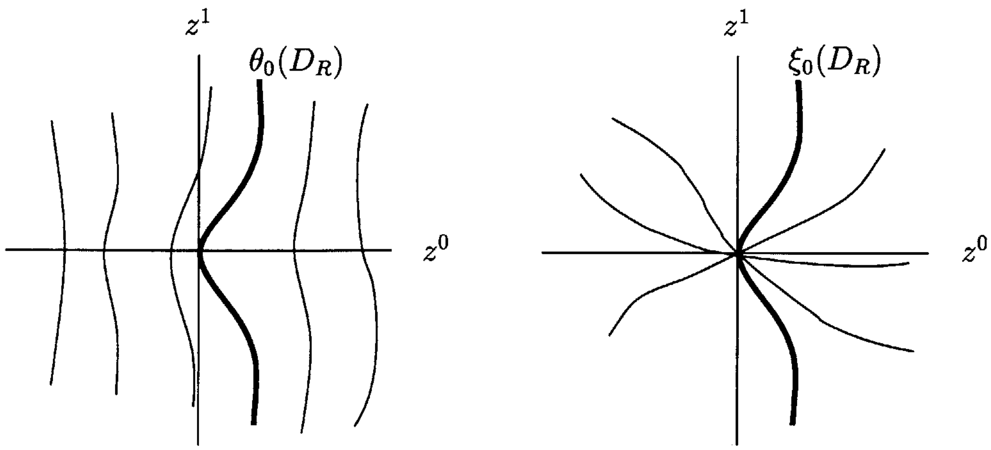

FIGURE 7-1. $\theta_w(D_R)$ versus $\xi_b(D_R)$.

LEMMA 44. *There is a constant $R_0 > 0$ so that the following are true. Fix hermitian coordinates in $\mathbb{C}^2$ and let*

$$\begin{aligned} \sigma : D_R &\longrightarrow U \\ z &\longmapsto (u(z), z) \end{aligned}$$

be a map defined on D_R for some $R < R_0$, whose image is J-holomorphic. Let κ_0 denote the tangent line to $\sigma(D_R)$ at $\sigma(0)$. For all κ sufficiently close to κ_0, there is a perturbation $\sigma_\kappa : D_R \to U$ of σ such that

i) $\sigma_\kappa(0) = \sigma_0(0)$,
ii) σ_κ is tangent to κ at 0,
iii) $\sigma_\kappa(D_R)$ is a J-holomorphic subvariety of U.

PROOF. We can assume that $u(0) = 0$, and then pick hermitian coordinates $\{z^0, z^1\}$ so that $\kappa_0 = \{z^0 = 0\}$. For $b \in \mathbb{C}$, we identify κ_b near κ_0 with the line $z^0 - bz^1 = 0$. The perturbation will take the form

$$(u(z) + a_1 z + a_0 + \psi_{a_0,a_1}(z), z)\,, \tag{46}$$

for suitable choices of $a_0, a_1 \in \mathbb{C}$.

If a_0, a_1 are sufficiently small, then by Lemma 41 we can find ψ_{a_0,a_1} so that the image of (46) is J-holomorphic. This map passes through $\mathbf{0}$ at $z = 0$ provided

$$a_0 + \psi_{a_0,a_1}(0) = 0. \tag{47}$$

By Lemma 41 (*iii*), $\left\|\frac{\delta\psi}{\delta a_0}\right\| < c_0 R_0$, so if R_0 is sufficiently small, the map $a_0 \mapsto a_0 + \psi_{a_0,a_1}(0)$ is invertible, and takes the value 0 at $a_0 = 0$. Therefore the constraint (47) defines a_0 as a smooth function of a_1.

Finally, consider the function $b(a_1) = a_1 + \psi'(0)$, with derivative

$$\frac{db}{da_1} = 1 + \frac{\delta\psi'}{\delta a_1}(0).$$

Again, we have $\left\|\frac{\delta\psi'}{\delta a_1}(0)\right\| < c_1 R_0 < 1$ provided R_0 is sufficiently small. In this case, there is a neighborhood of $a_1 = 0$ that is mapped diffeomorphically by b onto a neighborhood of $b(0) = 0$.

In summary, we have $\delta > 0$ and smooth functions $a_1(b)$, $a_0(a_1) = a_0(a_1(b))$ for $|b| < \delta$, so that each of the maps σ_{κ_b} defined by (46) satisfies $\sigma_{\kappa_b}(0) = \mathbf{0}$ and $\sigma'_{\kappa_b}(0) = \big(a_1(b) + \psi'_{a_0,a_1}(0), 1\big) = (b, 1)$. This completes the proof. $\square$

LEMMA 45. *There are constants $R_0, \delta, C > 0$ depending only on J and g so that the following hold.*

i) Let $R < R_0$ and let $u : D_R \to \mathbb{C}$ be a smooth map such that $(u, \mathrm{id}\,|_{D_R}) : D_R \to \mathbb{C}^2$ has J-holomorphic image. If $\zeta_1, \ldots, \zeta_m$ are distinct points in D_R such that

$$\sum_{i=1}^{m} \prod_{j \neq i} |\zeta_i - \zeta_j| > C R^{m-\frac{1}{3}}, \tag{48}$$

then there is a 1-parameter family of maps

$$\sigma_{\mathrm{a}} : D_R \longrightarrow \mathbb{C}^2, \quad \mathrm{a} \in \mathbb{C}, |\mathrm{a}| < \delta$$

so that each disk $\sigma_{\mathrm{a}}(D_R)$ is J-holomorphic and passes through $(u(\zeta_j), \zeta_j)$ for each $j = 1, \ldots, m$. Furthermore, $\sigma_0 = (u, \mathrm{id}\,|_{D_R})$.

ii) Each σ_{a} has the form

$$\sigma_{\mathrm{a}}(z) = \left(u(z) + \mathrm{a} \prod_{j=1}^{m} (z - z_j) + \psi(z), z \right), \quad z \in D_R,$$

where $\underline{z} = (z_1, \ldots, z_m)$ depends smoothly on $\underline{\zeta} = (\zeta_1, \ldots, \zeta_m)$, and

$$|\underline{z} - \underline{\zeta}| < C_m R^{\frac{5}{3}}$$

for some constant $C_m = C_m(J, g)$.

iii) If, in addition, $\prod_{j=1}^{m} |\zeta_j| > C R^{m+\frac{2}{3}}$, then σ_{a} intersects $\sigma_{\mathrm{a}'}$ discretely for all $\mathrm{a}' \neq \mathrm{a}$.

PROOF. When $\mathrm{a} \in \mathbb{C}$ and $z_j \in \mathbb{C}$ are sufficiently small, we obtain ψ depending smoothly on a and z_j so that the map

$$\mathbf{q} = \left(u + \mathrm{a} \prod_{j=1}^{m} (z - z_j) + \psi, \mathrm{id} \right)$$

has J-holomorphic image. The condition that $\mathbf{q}$ pass through the points $(u(\zeta_i), \zeta_i)$ amounts to having

$$f(\zeta_i) + \psi_f(\zeta_i) = \mathrm{a} \prod_{j=1}^{m} (\zeta_i - z_j) + \psi_f(\zeta_i) = 0 \tag{49}$$

for each $i = 1, \ldots, m$. Fixing a in (49), we now solve for $\underline{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m$ in terms of $\underline{\zeta} = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$. For each i, set

$$g_i(z_1, \ldots, z_m) = \mathrm{a} \prod_{j=1}^{m} (\zeta_i - z_j) \in \mathbb{C},$$

$$h_i(z_1, \ldots, z_m) = \psi_{\mathrm{a} \prod (z - z_j)}(\zeta_i) \in \mathbb{C}.$$

Then $g_i(\zeta_1, \dots, \zeta_m) = 0$, and Taylor's formula with remainder yields

$$\begin{aligned}
f(\zeta_i) + \psi_f(\zeta_i) &= g_i(\underline{z}) + h_i(\underline{z}) \\
&= g_i(\underline{\zeta}) + \sum_k \frac{\partial g_i}{\partial z_k}(\underline{\zeta}) \cdot (z_k - \zeta_k) + h_i(\underline{\zeta}) + \mathcal{R}_i(\underline{\zeta}, \underline{z} - \underline{\zeta}) \\
&= \sum_k \frac{\partial g_i}{\partial z_k}(\underline{\zeta}) \cdot \mu_k + h_i(\underline{\zeta}) + \mathcal{R}_i(\underline{\zeta}, \underline{\mu}),
\end{aligned}$$

where we have set $\mu_k = z_k - \zeta_k$ and $\underline{\mu} = (\mu_1, \dots, \mu_m)$, and where

$$\mathcal{R}(\underline{\zeta}, \underline{z}) = \int_0^1 (1-t) \sum_{k,l} \frac{\partial^2 g}{\partial z_k \partial z_l}(\underline{\zeta} + t\underline{\mu}) \cdot \mu_k \cdot \mu_l \, dt + \int_0^1 \sum_k \frac{\partial h}{\partial z_k}(\underline{\zeta} + t\underline{\mu}) \cdot \mu_k \, dt$$

is the remainder term. To solve equation (49) for $z_1, \dots, z_m$ it is thus sufficient to find a fixed point $\underline{\mu}$ of the function

$$G : \mu \longmapsto -[Dg(\underline{\zeta})]^{-1} \cdot \left(h(\underline{\zeta}) + \mathcal{R}(\underline{\zeta}, \underline{\mu})\right),$$

where $[Dg(\underline{\zeta})]$ denotes the matrix $\left[\frac{\partial g_i}{\partial z_k}(\underline{\zeta})\right]$.

By lemma 41 we have the following estimates:

$$\begin{aligned}
|h(\underline{\zeta})| &\le \|\psi_{\mathrm{a}\prod(z-\zeta_j)}\|_{2,\alpha} \le C \cdot R \cdot |\mathrm{a}| \cdot \prod_{j=1}^m \|z - \zeta_j\| \le c_m \cdot R^{m+1} \cdot |\mathrm{a}|, \\
|\mathcal{R}(\underline{\zeta}, \underline{\mu})| &\le C_m \left(R^{m-2} \cdot |\mathrm{a}|\, |\underline{\mu}|^2 + R^m |\mathrm{a}|\, |\underline{\mu}|\right), \\
\left|\frac{\partial \mathcal{R}}{\partial \underline{\mu}}\right| &\le C_m \left(R^{m-3} \cdot |\mathrm{a}|\, |\underline{\mu}|^2 + R^{m-2} |\mathrm{a}|\, |\underline{\mu}| + R^m |\mathrm{a}|\right).
\end{aligned}$$

Moreover, the matrix $[Dg(\zeta)]$ is diagonal and its (i,i)-th entry equals

$$\frac{\partial g_i}{\partial z_i} = -\mathrm{a} \prod_{j \ne i} (\zeta_i - \zeta_j).$$

Thus, by (48), $\|Dg(\underline{\zeta})\| > C|\mathrm{a}|R^{m-\frac{2}{3}}$, and

$$\begin{aligned}
|D_{\underline{\mu}} G| &\le C_m \left(R^{\frac{2}{3}-3} |\underline{\mu}|^2 + R^{\frac{2}{3}-2} |\underline{\mu}| + R^{\frac{2}{3}}\right), \\
|G(0)| &\le C_m R^{\frac{5}{3}}
\end{aligned}$$

for some constant C_m depending only on m.

Let $\varepsilon = 2C_m R^{\frac{5}{3}}$. If $\underline{\mu} < \varepsilon$, then $|D_{\underline{\mu}} G| < C(R + R^{\frac{1}{3}} + R^{\frac{2}{3}})$ for some big constant C. If R_0 is sufficiently small, it follows that $|D_{\underline{\mu}} G| < \frac{1}{2}$. By definition, we also have $|G(0)| \le \frac{\varepsilon}{2}$. Therefore, as pointed out in the proof of Lemma 41, we can conclude that G has a unique fixed point $\underline{\mu}$ of length $|\underline{\mu}| < 2C_m R^{\frac{5}{3}}$. Taking $(z_1, \dots, z_m) = (\zeta_1 + \mu_1, \dots, \zeta_m + \mu_m)$, we obtain the map $\sigma_{\mathrm{a}} = \left(u + \mathrm{a}\prod_{j=1}^m (z - z_j) + \psi_{\mathrm{a}\prod(z-z_j)}, \mathrm{id}\right)$ passing through each of the points $(u(\zeta_i), \zeta_i)$. By uniqueness, $\sigma_0 = (u, \mathrm{id}\,|_{D_R})$.

Writing $\mathrm{a}\prod(z - z_j) = a_m z^m + \cdots + a_0$, we have

$$\left.\frac{\delta\sigma_{\mathrm{a}}}{\delta \mathrm{a}}\right|_{\mathrm{a}=0} = \prod_{j=1}^{m}(z - z_j) + \sum_{k=0}^{m} \frac{\delta\psi}{\delta a_k} \cdot \frac{\partial a_k}{\partial \mathrm{a}}.$$

Combined with Lemma 41 (iii), this gives us

$$\begin{aligned}\left\| \left.\frac{\delta\sigma_{\mathrm{a}}}{\delta \mathrm{a}}\right|_{\mathrm{a}=0} (0) \right\| &\geq \prod_{j=1}^{m} |z_j| - \sum_{k=0}^{m} c_k R^{k+1} \cdot |\underline{z}|^{m-k} \\ &\geq \prod_{j=1}^{m} |\zeta_j| - C \sum_{k=0}^{m-1} |\underline{\zeta}|^k |\underline{\mu}|^{m-k} - C \cdot R^{m+1} \\ &\geq \prod_{j=1}^{m} |\zeta_j| - C \cdot R^{m+\frac{2}{3}}.\end{aligned}$$

If $\prod_{j=1}^{m} |\zeta_j| > C \cdot R^{m+\frac{2}{3}}$, then the map $\mathrm{a} \mapsto \sigma_{\mathrm{a}}(0)$ is a local diffeomorphism of a neighborhood of $\mathrm{a} = 0$ onto its image. Thus, when a and a' are sufficiently small and $\mathrm{a} \neq \mathrm{a}'$, the range of σ_{a} does not equal the range of $\sigma_{\mathrm{a}'}$. By Aronszajn's unique continuation principle [**A**], the images of σ_{a} and $\sigma_{\mathrm{a}'}$ therefore intersect discretely. $\square$

4. Families of half-disks on a totally real plane.

The goal of this section is to produce families of pseudo-holomorphic half-disks with boundary on a totally real 2-plane $Y \subset \mathbb{C}^2$. Working in a neighborhood U of $\mathbf{0}$, we can change the metric near Y to assume that Y is orthogonal to JY. Then any orthogonal basis for Y is also a unitary basis for $\mathbb{C}^2 = Y \oplus JY$ and so defines hermitian coordinates $z^1 = x^1 + iy^1$, $z^2 = x^2 + iy^2$. With respect to these coordinates, $Y = \{y^1 = y^2 = 0\}$ is a copy of the standard $\mathbb{R} \oplus \mathbb{R} \subset \mathbb{C}^2$. Then using the reflection principle we will obtain the results of this section as variations on the results of Section 7.

Consider anew $\mathbb{C}^2$ with standard complex coordinates $z^1 = x^1 + iy^1$, $z^2 = x^2 + iy^2$, and let Y be the real 2-plane $\{(x^1, y^1, x^2, y^2) \mid y^1 = y^2 = 0\}$. On a neighborhood U of the origin, we are given an almost complex structure J and a compatible metric g, and both are standard at $\mathbf{0}$. A *half-disk with boundary on* Y is a smooth map $\mathbf{q} : (D^+, \partial^+ D^+) \to (\mathbb{C}^2, Y)$ defined on the upper half-disk

$$D^+ = \{z = x + iy \in \mathbb{C} \mid x^2 + y^2 < 1, y \geq 0\},$$

which maps the boundary $\partial^+ D^+ = \{x + iy \in D \mid y = 0\}$ into Y. If $\mathbf{q}(z) = \big(q(z), z\big)$ for all $z \in D^+$, then $q : D^+ \to \mathbb{C}$ must map $\partial^+ D^+$ into $\mathbb{R}$. Therefore q extends by reflection to a smooth map $D \to \mathbb{C}$:

$$q(\overline{z}) = \overline{q(z)} \text{ for all } z \in D.$$

Now let $D_R^+ \subset \mathbb{C}$ be the upper half-disk of radius R, let $u : D_R^+ \to \mathbb{C}$ be a smooth function mapping $\partial^+ D_R^+$ into $\mathbb{R}$, and let $f(z) = a_m z^m + \cdots + a_0$ be a real polynomial. Our task is to find a smooth function $\psi : D_R^+ \to \mathbb{C}$ so that the image of

$$\mathbf{q} = (u + f + \psi, \mathrm{id})$$

is J-holomorphic in U and so that the boundary $\mathbf{q}(\partial^+ D_R^+)$ lies on Y. (We will call $\mathbf{q}$ a J-holomorphic half-disk with boundary on Y.)

Let $\mathbb{C}^+$ denote the upper half plane $\{x+iy \in \mathbb{C} \mid y \geq 0\}$. Our search for ψ leads us to define the Banach space $\mathcal{K}^{2,\alpha}(D_R^+)$ consisting of functions $\psi : (\mathbb{C}^+, \mathbb{R}) \to (\mathbb{C}, \mathbb{R})$ of class $C^{2,\alpha}$ on $\mathrm{Int}(\mathbb{C}^+)$ such that $|\psi(z)| \to 0$ as $|z| \to +\infty$, and such that the restriction of ψ to $\mathrm{Int}(\mathbb{C}^+ \setminus D_R^+)$ is holomorphic.

If $\psi \in \mathcal{K}^{2,\alpha}(D_R^+)$, then ψ maps $\mathbb{R} \subset \mathbb{C}^+$ to $\mathbb{R}$, so the reflection of ψ is a map $\mathbb{C} \to \mathbb{C}$ of class $C^{2,\alpha}$ whose restriction to $\mathbb{C} \setminus D_R$ is holomorphic. Thus $\mathcal{K}^{2,\alpha}(D_R^+)$ can be identified with the closed subspace of $\mathcal{K}^{2,\alpha}(D_R)$ consisting of functions that are symmetric with respect to the real axis, that is, $\psi(\bar{z}) = \overline{\psi(z)}$ for all $z \in \mathbb{C}$.

LEMMA 46. *Let $U \subset \mathbb{C}^2$ be open, and let J and g be a compatible almost complex structure and metric on U, both standard at the origin. Let $\chi : \mathbb{C} \to [0,1]$ be a smooth, radially symmetric cut-off function satisfying $\chi(z) = 1$ for $|z| \leq 1$ and $\chi(z) = 0$ for $|z| \geq \frac{3}{2}$.*

There exist constants $R_0, C_0, C_1 > 0$ depending only on g, J and χ such that the following hold.

i) Let $R < R_0$, let $u : (D_{2R}^+, \partial^+ D_{2R}^+) \to (\mathbb{C}, \mathbb{R})$ be a smooth function and let $f(z) = a_m z^m + \cdots + a_0$ be a degreem polynomial with real coefficients. Suppose that $\|u\|_{2,\alpha;D_{2R}^+} < \frac{R}{C_2}$ and $\|f\|_{2,\alpha;D_{2R}^+} < R$. Then there is a unique $\psi \in \mathcal{K}^{2,\alpha}(D_{2R}^+)$ satisfying the estimate

$$\|\psi\|_{2,\alpha;D_{2R}^+} \leq C_0 \cdot \left(\|u\|_{2,\alpha;D_{2R}^+} + R\|f\|_{2,\alpha;D_{2R}^+} + R^2\right) \tag{50}$$

which solves the following equation in $\mathrm{Int}(\mathbb{C}^+)$:

(51)

$$\bar{\partial}\,\psi = \chi_R \cdot \left\{ -\,\bar{\partial}\,u + \frac{j}{2}\left[B|_{\mathbf{q}_\psi} + (A-j)|_{\mathbf{q}_\psi} \cdot dq_\psi - dq_\psi \cdot (D-j)|_{\mathbf{q}_\psi} - dq_\psi \cdot C|_{\mathbf{q}_\psi} \cdot dq_\psi\right] \right\},$$

$$\mathbf{q}_\psi = (q_\psi, id) = (u + f + \psi, id).$$

ii) The function ψ is smooth in $\mathrm{Int}(\mathbb{C}^+)$.

iii) The restriction $\mathbf{q}_\psi|_{D_R^+} : D_R^+ \to \mathbb{C}^2$ is a J-holomorphic half-disk with boundary on the standard totally real 2-plane $\mathbb{R} \oplus \mathbb{R} \subset \mathbb{C}^2$.

iv) The function ψ is C^∞ with respect to the coefficients $a_0, \ldots, a_m$ of f. Moreover there exist constants $c_k = c_k(g, J, \chi)$ such that

$$\left\| \frac{\partial \psi}{\partial a_k} \right\| \leq c_k \cdot R^{k+1}.$$

v) If the map $(u, \mathrm{id}) : D_R^+ \to \mathbb{C}^2$ has J-holomorphic image, then for any real polynomial $f = a_m z^m + \cdots + a_0$, the function ψ obtained in (i) satisfies:

$$\|\psi\|_{2,\alpha;D_{2R}^+} \leq C_0 R \cdot \|f\|_{2,\alpha;D_{2R}^+}.$$

PROOF. The function u on D_{2R}^+ extends to a complex-valued function on D_{2R} satisfying $u(\bar{z}) = \overline{u(z)}$ and the real polynomial f naturally satisfies $f(\bar{z}) = \overline{f(z)}$. Identify $\mathcal{K}^{2,\alpha}(D_{2R}^+)$ with the closed subspace

$$\{\psi \in \mathcal{K}^{2,\alpha}(D_{2R}) \mid \psi(\bar{z}) = \overline{\psi(z)}\}$$

of $\mathcal{K}^{2,\alpha}(D_{2R})$, and consider the restriction of the functional F defined by (37) to $\mathcal{K}^{2,\alpha}(D_{2R}^+)$. The proof and estimates of Lemma 41 can be duplicated to show that F has a unique fixed point $\psi^* \in \mathcal{K}^{2,\alpha}(D_{2R}^+)$, if only we can show that F maps $\mathcal{K}^{2,\alpha}(D_{2R}^+)$ to itself. But the latter is straightforward: for any $\psi \in \mathcal{K}^{2,\alpha}(D_{2R}^+)$, there is a function η_ψ satisfying $\eta_\psi(\overline{z}) = \overline{\eta_\psi(z)}$ on $\mathbb{C}$ such that the 1-form Q_ψ defined in (36) satisfies $dz \wedge Q_\psi(z) = \eta_\psi(z) dz \wedge d\bar{z}$. Therefore

$$\begin{aligned}
\overline{F_\psi(z)} &= \overline{\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\chi_R \cdot \eta_\psi(w)}{z - w}\, dw\, d\bar{w}} \\
&= \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{\chi_R \cdot \eta_\psi(\bar{w})}{\bar{z} - \bar{w}}\, d\bar{w}\, dw \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\chi_R \cdot \eta_\psi(w)}{\bar{z} - w}\, dw\, d\bar{w} \\
&= F_\psi(\bar{z}),
\end{aligned}$$

which is to say that $F_\psi \in \mathcal{K}^{2,\alpha}(D_{2R}^+)$. $\square$

We want to apply Lemma 46 to the following situation. Let $U \subset \mathbb{C}^2$ be open and fix an almost complex structure J in U. Let g be any metric on U compatible with J. A real 2-plane $Y \subset \mathbb{C}^2$ is *totally real* in U if at each point $p \in Y \cap U$ we have $T_pY \cap J_pT_pY \cong Y \cap J_pY = \mathbf{0}$. In $\mathbb{C}^2$, Y is totally real if and only if it is never a J_p-complex line, for $p \in Y$. Since $\mathbb{C}^2$ equals the direct sum $Y \oplus J_pY$, any basis $\mathbf{v}_1, \mathbf{v}_2$ for Y is also a complex basis for $\mathbb{C}^2$. By altering the metric g in a neighborhood of Y in U, we can assume that for all p, Y is orthogonal to J_pY with respect to g_p. Hence any oriented orthonormal basis for Y is also a unitary basis for $(\mathbb{C}^2, J_p, g_p)$ and defines hermitian coordinates $z^1 = x^1 + iy^1, z^2 = x^2 + iy^2$ centered at p with respect to which $Y = \{y^1 = y^2 = 0\}$. We are therefore in a position to apply Lemma 46, as we did Lemma 41 in Section 7, to construct families of J-holomorphic half-disks with boundary on Y, tangent to prescribed directions, or passing through prescribed points. Because we have altered the metric g near Y, the constants of Lemma 46 will now also depend on Y, but in no way on u or f.

COROLLARY 47. *Let Y be a totally real subspace of $\mathbb{C}^2$ and let $z^1 = x^1 + iy^1, z^2 = x^2 + iy^2$ be complex coordinates in $\mathbb{C}^2$ defined by an oriented orthonormal basis $\mathbf{v}_1, \mathbf{v}_2$ for Y. There exist positive constants R_0, C depending only on J, g and Y so that, for each $R < R_0$, there is a diffeomorphism $\theta : D_R \times D_R \to U$ with the following properties:*

i) For each $w \in D_R$, the image of $\theta_w = \theta(w, \cdot) : D_R \to U$ is a J-holomorphic disk passing through $\theta_w(0) = (w, 0)$.
ii) For each real *$w \in D_R$, the restriction of θ_w to each of D_R^+ and D_R^- is a J-holomorphic half-disk with boundary on Y, that is, $\theta_w(\partial^+ D_R^+), \theta_w(\partial^- D_R^-) \subset Y$ are oriented curves passing through $(w, 0)$.*
iii) For $w = 0$, the oriented curve $\theta_0(\partial^+ D_R^+)$ in Y is tangent at $\mathbf{0}$ to the vector $\mathbf{v}_2$, and the half-disk $\theta_0(D_R^+)$ is tangent at $\mathbf{0}$ to the complex line $\{z^2 = 0\}$.
iv) For each $w \in D_R$, $\mathrm{dist}(\theta(w, z), (w, z)) \leq C\, R \cdot |z|$.

v) There is a constant $C_k = C(J, g, k, Y)$ such that $|D^k(\theta_w)| \leq C_k \cdot R$ for all $w \in D_R$.

PROOF. Use Lemmas 41 and 46, and follow the proof of Corollary 42, making use of the uniqueness property of ψ. □

The set of oriented orthonormal frames in Y is parametrized by the group $O(2)$, which double covers the space $S^1 = SO(2)$ of oriented directions in Y. Each choice of vertical direction $\mathbf{v}_2 \in S^1$ determines a J-holomorphic half-disk with boundary on Y, according to the following variation on Corollary 43.

COROLLARY 48. *Let Y be a totally real subspace of $\mathbb{C}^2$. There exists a positive constant $R_0 > 0$ depending only on J, g and Y, and for each $R < R_0$ there exists a smooth map $\xi : S^1 \times D_R^+ \to U$ such that the following hold:*

i) ξ maps $S^1 \times \partial^+ D_R^+$ into Y, maps $S^1 \times 0$ to $\mathbf{0}$, and restricts to an embedding $S^1 \times \mathrm{Int}(D_R^+) \hookrightarrow U \subset \mathbb{C}^2$.
ii) Given $\mathbf{v} \in S^1$, let $z^1 = x^1 + iy^1, z^2 = x + iy^2$ be complex coordinates for which $Y = \{y^1 = y^2 = 0\}$ and $\mathbf{v}$ spans the line $\{x^1 = y^1 = y^2 = 0\}$. Then with respect to these coordinates, $\xi(\mathbf{v}, z) = \xi(-\mathbf{v}, -z)$ for all real $z \in \partial^+ D_R^+$.
iii) For each $\mathbf{v} \in S^1$, $\xi_{\mathbf{v}}(D_R^+) = \xi(\mathbf{v} \times D_R^+)$ is a J-holomorphic half-disk with boundary on Y; moreover the oriented boundary $\xi_{\mathbf{v}}(\partial^+ D_R^+)$ is tangent to the oriented vector $\mathbf{v}$ at $\mathbf{0}$.
iv) Let $\mathbf{v}_1, \mathbf{v}_2$ be an oriented orthonormal basis for Y, and let $z^1 = x^1 + iy^1, z^2 = x + iy^2$ be the associated complex coordinates for $\mathbb{C}^2$. Then $\xi_{\mathbf{v}_2} : D_R^+ \to U$ coincides with the map $\theta_0|_{D_R^+} : D_R^+ \to U$ obtained in Corollary 47.

COROLLARY 49. *Let $Y \subset \mathbb{C}^2$ be a totally real plane and let $z^1 = x^1 + iy^1, z^2 = x + iy^2$ be complex coordinates in $\mathbb{C}^2$ defined by an oriented orthonormal basis $\mathbf{v}_1, \mathbf{v}_2$ for Y. There is a positive constant $R_0 = R_0(J, g, Y)$ such that the following are true. Let $R < R_0$ and let*

$$\begin{aligned} \sigma : D_R &\longrightarrow U \subset \mathbb{C}^2 \\ z &\longmapsto \big(u(z), z\big) \end{aligned}$$

be a J-holomorphic half-disk with boundary $\sigma(\partial^+ D_R^+)$ contained in Y. Let σ^+ denote the restriction of σ to $\partial^+ D_R^+$ and suppose that the tangent vector to σ^+ at 0 is a positive multiple of $\mathbf{v} \in S^1$. Then for all $\mathbf{v}' \in S^1$ sufficiently close to $\mathbf{v}$, there is a J-holomorphic half-disk $\sigma_{\mathbf{v}'} : D_R^+ \to U$ such that

i) $\sigma_{\mathbf{v}'}(\partial^+ D_R^+)$ is contained in Y,
ii) $\sigma_{\mathbf{v}'}(0) = \sigma_{\mathbf{v}}(0)$,
iii) $\sigma_{\mathbf{v}'}^+$ is tangent to $\mathbf{v}'$ at 0.

LEMMA 50. *Let $U \subset \mathbb{C}^2$ be open and let J, g be a compatible almost complex structure and metric in U. Let $Y \subset \mathbb{C}^2$ be a totally real 2-plane for J. There exist constants $R_0, \delta, C > 0$ depending only on J, g and Y such that the following hold.*

i) Let $R < R_0$ and let $u : D_R^+ \to \mathbb{C}$ be a smooth map such that $(u, \mathrm{id}) : D_R^+ \to \mathbb{C}^2$ has J-holomorphic image and maps $\partial^+ D_R^+$ into

Y. Let $\zeta_1, \ldots, \zeta_m$ be distinct points in $\partial^+ D_R^+$ such that

(52) $$\sum_{i=1}^{m} \prod_{j \neq i} |\zeta_i - \zeta_j| > CR^{m-\frac{1}{3}}.$$

Then there is a real 1-parameter family of maps

$$\sigma_{\mathrm{a}} : D_R^+ \longrightarrow \mathbb{C}^2, \quad \mathrm{a} \in \mathbb{R}, |\mathrm{a}| < \delta$$

such that each $\sigma_{\mathrm{a}}(D_R^+)$ is a J-holomorphic half-disk with boundary on Y, passing through the m points $(u(\zeta_j), \zeta_j)$. Furthermore, $\sigma_0 = (u, \mathrm{id})$.

ii) Each σ_{a} has the form

$$\sigma_{\mathrm{a}}(z) = \left(u(z) + \mathrm{a} \prod_{j=1}^{m} (z - z_j) + \psi(z), z \right), \quad z \in D_R,$$

where $\underline{z} = (z_1, \ldots, z_m)$ depends smoothly on $\underline{\zeta} = (\zeta_1, \ldots, \zeta_m)$, and $|\underline{z} - \underline{\zeta}| < C_m R^{\frac{5}{3}}$ for some constant $C_m = C_m(J, g, Y)$.

iii) If in addition to the hypotheses, we have $\prod_{j=1}^{m} |\zeta_j| > CR^{m+\frac{2}{3}}$, then σ_{a} and $\sigma_{\mathrm{a}'}$ intersect discretely when $\mathrm{a}' \neq \mathrm{a}$.

APPENDIX A

The Sacks-Uhlenbeck argument

In this appendix we show how some obvious attempts to use a Sacks-Uhlenbeck type argument to prove compactness for J-holomorphic curves in the symplectization of a contact manifold M, ξ might fail. For a basic description of the Sacks-Uhlenbeck method, we refer the reader to [**P**] and [**PW**].

Fix an almost complex structure J on $\mathbb{R} \times M$ such that J is compatible with $d\alpha$ on $\xi = \operatorname{Ker} \alpha$ amd $J(\partial_t) = v_\alpha$, the Reeb vector field associated to α. If $u : \Sigma \to \mathbb{R} \times M$ is a J-holomorphic map of a punctured Riemann surface Σ into the symplectization of M, then the Hofer energy of u equals

$$\mathcal{E}(u) = \sup_{\mathcal{C}} \int_\Sigma u^* d(\varphi\alpha) \ ,$$

where $\mathcal{C}$ denotes the set of functions $\varphi : \mathbb{R} \to [\frac{1}{2}, 1]$ such that $\varphi' \geq 0$.

LEMMA 51 (sup estimate for $\mathcal{E}$). *There exist $\hbar, C > 0$ so that if $u : B_r \to \mathbb{R} \times M$ is J-holomorphic and*

$$\sup_\varphi \int_{B_r} u^* d(\varphi\alpha) < \hbar \ ,$$

then

$$\sup_{B_{r/2}} |\nabla u| < \frac{C}{r} \ .$$

PROOF. By scaling, it suffices to prove the result for $r = 1$. In this case, if the result fails to hold, then for any $\hbar, C > 0$, we can find a J-holomorphic map $u : B_r \to \mathbb{R} \times M$ for which $\sup_\varphi \int_{B_r} u^* d(\varphi\alpha) < \hbar$ and $\sup_{B_{r/2}} |\nabla u| \geq C$.

We will take a sequence $\hbar_k \to 0$, $C_k = 1/\hbar_k^2 \to +\infty$ and $u_k : B_r \to \mathbb{R} \times M$ such that

$$\mathcal{E}(u_k; B_r) = \sup_\varphi \int_{B_r} u_k^* d(\varphi\alpha) < \hbar_k \quad \text{and} \quad \sup_{B_{r/2}} |\nabla u_k| \geq C_k \, .$$

The trick is to choose $\hbar_k$ and C_k wisely.

Take points $z_k \in B_{r/2}$ such that $|\nabla u_k(z_k)| = \sup_{B_{r/2}} |\nabla u_k|$. Applying Lemma 26 of [**H**] with $X = B_r$, $\phi = |\nabla u_k|$, $x = z_k$, and $\varepsilon = \hbar_k$, we get $x' = z_k' \in B_{r/2}$ and $\varepsilon_k' > 0$ such that

a) $\varepsilon_k' \leq \hbar_k$ and $|\nabla u_k(z_k')| \cdot \varepsilon_k' \geq |\nabla u_k(z_k)| \cdot \hbar_k$
b) $|z_k - z_k'| \leq 2\hbar_k$
c) $2|\nabla u_k(z_k')| \geq |\nabla u_k(z)|$ for all $z \in B_r$ satisfying $|z - z_k'| \leq \varepsilon_k'$.

Now we can assume that $\hbar_k < r/16$ for all k, or at any rate for all sufficiently large k. Therefore we have

a) $\varepsilon_k' \leq \hbar_k$ and $|\nabla u_k(z_k')| \cdot \varepsilon_k' \geq \frac{1}{\hbar_k}$
b) $z_k' \in B_{2\hbar_k}(z_k) \subset B_{5r/8}$

c) $|\nabla u_k(z)| \leq 2|\nabla u_k(z'_k)|$ for all $z \in B_{\varepsilon'_k}(z'_k) \subset B_{r/16}(z'_k) \subset \mathrm{Int}(B_{3r/4})$.

For convenience, relabel ε'_k as ε_k and z'_k as z_k. Set $R_k = |\nabla u_k(z_k)|$ and $H_k = H(u_k(z_k))$. Recalling that $\varepsilon_k R_k \to +\infty$ as $k \to \infty$, we define, for $z \in B_{\varepsilon_k R_k}(0)$,

$$v_k(z) = T_{H_k} \circ u_k(z_k + \frac{z}{R_k}) \ ,$$

where T_H is translation by $-H$ in $\mathbb{R} \times M$. Thus we have a sequence of J-holomorphic maps defined on increasing balls in $\mathbb{C}$ which satisfy:

1) $\sup_\varphi \int_{B_{\varepsilon_k R_k}} v_k^* d(\varphi\alpha) \leq \mathcal{E}(u_k; B_r) < \hbar_k \to 0$
2) $|\nabla v_k| \leq 2$ on $B_{\varepsilon_k R_k}$, and $|\nabla v_k(0)| = 1$
3) For each k, $v_k(0)$ lies in the compact set $H^{-1}(0)$.

Thus we get a subsequence $\{v_k\}$ that converges in C^1_{loc} to a J-holomorphic map $v_\infty : \mathbb{C} \to \mathbb{R} \times M$. Moreover,

$$\mathcal{E}(v_\infty) = 0 \qquad |v_\infty(0)| = 1 \qquad |v_\infty| \leq 2 \text{ on } \mathbb{C} \ .$$

The first property implies that v_∞ is constant, but this contradicts the second. With this contradiction in hand, we have proved the lemma. □

The next result is a version of Lemma 51 for curves with boundary. Given a subset $A \subset \mathbb{C}$, we let $A^+ = \{z \in A | \mathrm{Im} z \geq 0\}$. So for example B_r^+ is the closed upper half disc of radius r in $\mathbb{C}$. On the boundary $\partial^+ B_r^+ = \{z \in B_r^+ | \mathrm{Im} z = 0\}$, we require that a map $u : B_r^+ \to \mathbb{R} \times M$ take values in $Y \subset 0 \times M$.

LEMMA 52. *There exist $\hbar, C > 0$ such that if $u : B_r^+ \to \mathbb{R} \times M$ is J-holomorphic with $u(\partial^+ B_r^+) \subset Y$, and if*

$$\sup_\varphi \int_{B_r^+} u^* d(\varphi\alpha) < \hbar \ ,$$

then

$$\sup_{B_{r/2}^+} |\nabla u| < \frac{C}{r} \ .$$

PROOF. Again, by rescaling, we can assume that $r = 1$. If the result fails to hold, then we take a sequence $\hbar_k \to 0$ with $\hbar_k < r/16$ for all k, and $C_k = 1/\hbar_k^2 \to +\infty$. For each k, there is a J-holomorphic map $u_k : (B_r^+, \partial^+ B_r^+) \to (\mathbb{R} \times M, Y)$ with

$$\sup_\varphi \int_{B_r^+} u_k^* d(\varphi\alpha) < \hbar_k \quad \text{and} \quad \sup_{B_{r/2}^+} |\nabla u_k| \geq C_k \ .$$

Again, by Lemma 26 of [**H**], we get $\varepsilon_k > 0$ and $z_k \in B_r^+$ such that

a) $\varepsilon_k \leq \hbar_k$ and $|\nabla u_k(z_k)| \cdot \varepsilon_k \geq \frac{1}{\hbar_k}$
b) $z_k \in B_{2\hbar_k}(z_k) \subset B_{r/8}(z_k) \subset B_{5r/8}^+$
c) $|\nabla u_k(z)| \leq 2|\nabla u_k(z_k)|$ for all $z \in B_{\varepsilon_k}(z_k) \subset B_{r/16} \subset (\mathrm{Int} B_{3r/4})^+$.

As before, we set $R_k = |\nabla u_k(z_k)|$ and $H_k = H(u_k(z_k))$, and observe that $R_k \cdot \varepsilon_k \to +\infty$ as $k \to \infty$. Up to taking a subsequence, we can assume that z_k converges to $z_\infty \in B$.

There are two cases to consider: either $R_k \cdot \mathrm{dist}(z_k, \partial^+ B) \to +\infty$, or $R_k \cdot \mathrm{dist}(z_k, \partial^+ B) \to \rho < +\infty$ (where $\mathrm{dist}(z_k, \partial^+ B)$ is simply equal to $\mathrm{Im} z_k$.)

In the first case, we renormalize as in the proof of Lemma 51 to get a sequence of maps converging in C^1_{loc} to a map $v : \mathbb{C} \to \mathbb{R} \times M$. As in Lemma 51, this leads to a contradiction.

In the second case, set $\rho_k = R_k \cdot \mathrm{Im} z_k$ and define $v_k : B^+_{\varepsilon_k R_k}(i\rho_k) \to \mathbb{R} \times M$ by the formula

$$v_k(z) = u_k(\frac{z}{R_k} + \mathrm{Re} z_k) \ .$$

Then we have a sequence of J-holomorphic maps defined on increasing domains in $\mathbb{C}^+$ satisfying:

1) $\sup_\varphi \int_{B^+_{\varepsilon_k R_k}(i\rho_k)} v_k^* d(\varphi\alpha) \leq \mathcal{E}(u_k; B_r^+) < \hbar_k \to 0$
2) $|\nabla v_k| \leq 2$ on $B^+_{\varepsilon_k R_k}(i\rho_k)$, and $|\nabla v_k(i\rho_k)| = 1$
3) $v_k(z) \in Y$ when $z \in B^+_{\varepsilon_k R_k}(i\rho_k)$ and $\mathrm{Im} z = 0$.

We get a subsequence $\{v_k\}$ converging in C^1_{loc} to a J-holomorphic map $v_\infty : \mathbb{C}^+ \to \mathbb{R} \times M$ with the properties

$$v_\infty(z) \in Y \text{ for all } z \in \mathbb{R} \subset \mathbb{C}^+ \qquad \mathcal{E}(v_\infty) = 0 \qquad |v_\infty(i\rho)| = 1 \,.$$

The last two properties are contradictory, so this proves the lemma. □

To complete the Sacks-Uhlenbeck argument, we would like to proceed in the following manner (see [**P**], [**PW**]).

1. The Covering argument

Consider a sequence $u_k : (D, \partial D) \to (\mathbb{R} \times M, Y)$ or $u_k : (\Sigma, \partial\Sigma) \to (\mathbb{R} \times M, Y)$ of J-holomorphic maps with uniformly bounded Hofer-energy,

$$\mathcal{E}(u_k; D) \leq E_0 \text{ for all } k \,.$$

For the moment, fix a positive radius r and cover D by r-balls so that the $r/2$-balls also cover, and so that any point of D lies in at most 10 balls. Let $\hbar$ be the minimum of the $\hbar$'s occurring in Lemmas 51 and 52. Now suppose the following were true:

If $D_1, D_2 \subset \Sigma$ are disjoint open sets, then

(53) $\mathcal{E}(u; D_1 \cup D_2) = \mathcal{E}(u; D_1) + \mathcal{E}(u; D_2)$ for any qualifying $u : \Sigma \to \mathbb{R} \times M$.

Then for any k, there are at most $l = 10E_0/\hbar$ "bad" balls on which u_k has energy $\geq \hbar$. As $k \to \infty$, the centers of these bad balls converge (up to taking a subsequence) to $x_1, \ldots, x_l \in D$. Setting $\Omega(r) = \cup_i B_r(x_i)$, we have, thanks to Lemmas 51 and 52 above, a subsequence $\{u_k\}$ that converges in C^1 on $D \backslash \Omega(r)$.

(To see how this works, take $K > 0$ so that $|x_i(k) - x_i| < r/4$ for all i when $k \geq K$. If $x \in D\backslash\Omega(r)$, then for all i and all $k \geq K$, we have $|x - x_i(k)| > r/2$. Hence x lies in a ball $B_r(y_k)$ that is good for u_k, that is, $\mathcal{E}(u_k; B_r(y_k)) < \hbar$. It follows that $\sup_{B_{r/2}(y_k)} |\nabla u_k| \leq C/r$ and that $|\nabla u_k(x)| \leq C/r$, independently of k. This is true for any $x \in D\backslash\Omega(r)$, so we find

$$\sup_{D\backslash\Omega(r)} |\nabla u_k| \leq C/r \text{ for all } k \ .$$

Thus we get a subsequence that converges on $D\backslash\Omega(r)$ to a map $D\backslash\Omega(r) \to \mathbb{R} \times M$.)

Now let $r \to 0$. We can assume $x_1(r), \ldots, x_l(r)$ converge to $\tilde{x}_1, \ldots, \tilde{x}_l$ in D as $r \to 0$. For each r, we have a subsequence $\{u_k\}$ that converges in C^1 on $D\backslash\Omega(r)$. By taking a diagonal subsequence, we obtain a sequence $\{u_k\}$ that converges in C^1_{loc} to a J-holomorphic map $u_\infty : D\backslash\{\tilde{x}_1, \ldots, \tilde{x}_l\} \to \mathbb{R} \times M$.

Now to finish, we need the following Removable Singularities Theorem, which was proved on page 3 of [**HWZ**].

THEOREM 53. *Let $u : \bar{D}\backslash 0 \to \mathbb{R} \times M$ be a J-holomorphic map with finite energy $\mathcal{E}(u; D)$. Either the image of u lies in a compact region, in which case the singularity is removable; or, u has a positive or a negative end, i.e. near the puncture, u is asymptotic to a cylinder over a periodic orbit.*

Unfortunately, Hofer's energy is not additive in the sense of (53). For let $x : I \to M$ be any trajectory of the Reeb vector field X_α, so that $\dot{x}(t) = X_\alpha(x(t))$ for $t \in I$. Given $a < b$, define $u : (a, b) \times I \to \mathbb{R} \times M$ by $u(s, t) = (s, x(t))$. This map u is J-holomorphic, and for any $\varphi \in \mathcal{C}$,

$$\int_{(a,b)\times I} u^* d(\varphi\alpha) = \varphi(b) - \varphi(a) \ .$$

Therefore $\mathcal{E}(u) = \sup_\varphi(\varphi(b) - \varphi(a)) = \frac{1}{2}$. But this is true for any a, b we like, so $\mathcal{E}$ can't be additive.

2. φ-energy.

The Sacks-Uhlenbeck argument with $\mathcal{E}$ substituted for the usual energy did not work because $\mathcal{E}$ is not additive with respect to domains. Note however that for any fixed $\varphi \in \mathcal{C}$, the "φ-energy" (or "$\mathcal{E}_\varphi$-energy")

$$\mathcal{E}_\varphi(u; \Omega) = \int_\Omega u^* \omega_\varphi$$

is additive: $\mathcal{E}_\varphi(u; D_1 \cup D_2) = \mathcal{E}_\varphi(u; D_1) + \mathcal{E}_\varphi(u; D_2)$ whenever $D_1 \cap D_2 = \phi$.

Now one might try to substitute $\mathcal{E}_\varphi$ for $\mathcal{E}$ in the covering argument of the previous section. To be precise, if $\{u_k\}$ is a family of J-holomorphic curves with $\mathcal{E}(u_k) < E_0$ for all k, then for any fixed $\varphi \in \mathcal{C}$, we have a uniform bound

$$\mathcal{E}_\varphi(u_k) \leq \mathcal{E}(u_k) < E_0 \qquad \text{for all } k \ .$$

Since $\mathcal{E}_\varphi$ is additive on domains, the entire discussion of the previous section seems to go through without difficulty, setting us on our happy way to completing the proof of the compactness theorem.

Alas, the $\mathcal{E}_\varphi$-energy is not invariant under translations in $\mathbb{R} \times M$, and consequently does not yield a sup estimate such as those in Lemmas 51 and 52. Without the sup estimate, we can not begin to appeal to the Arzelá-Ascoli Theorem to get C^1 convergence.

To see what goes wrong, consider again the renormalization step in the proof of Lemma 51 or 52. We are assuming fixed some $\varphi \in \mathcal{C}$, and we have $\hbar_k \to 0$, $C_k \to +\infty$ and $u_k : D_r \to \mathbb{R} \times M$ so that

$$\int_{D_r} u_k^* \omega_\varphi \leq \hbar_k \qquad \text{and} \qquad \sup_{D_{r/2}} |\nabla u_k| \geq C_k \ .$$

From [**H**, Lemma 26] we get $\varepsilon_k \to 0$ and $z_k \in D_{r/2}$ so that

$$\varepsilon_k R_k \to +\infty \ , \qquad \text{where} \qquad R_k = |\nabla u_k(z_k)| \ ,$$

$$\text{and} \qquad |\nabla u_k(z)| \le 2R_k \qquad \text{for all } z \in B_{\varepsilon_k}(z_k) \ .$$

Now we renormalize by setting

$$v_k(z) = u_k(z/R_k + z_k) \qquad \text{for } z \in B_{\varepsilon_k R_k}(0) \subset \mathbb{C} \ .$$

The resulting sequence of maps satisfy:

$$|\nabla v_k(0)| = 1 \qquad |\nabla v_k(z)| \le 2 \qquad \text{for } z \in B_{\varepsilon_k R_k}(0) \ . \tag{54}$$

Also, for each k, we have

$$\int_{B_{\varepsilon_k R_k}} v_k^* \omega_\varphi < \hbar_k \ ,$$

and the latter quantity goes to zero.

We can assume by taking a subsequence if necessary that $z_k \to z_\infty$ in $D_{r/2}$. If $u_k(z_k) = v_k(0)$ stays in a bounded region, then there is a subsequence that converges. Together with the estimate (54), this implies that $\{v_k\}$ has a subsequence converging in C^1_{loc} on $\mathbb{C}$ to a map $v_\infty : \mathbb{C} \to \mathbb{R} \times M$.

On the other hand, if $H \circ u_k(z_k) \to -\infty$ then the " bubble is going off the neck". To catch it, we translate the images in the target: set $H_k = H \circ u_k(z_k)$ and let T_{H_k} denote translation down by H_k in $\mathbb{R} \times M$. That is, if $p \in \mathbb{R} \times M$, then $H(T_{H_k}(p)) = H(p) - H_k$. Now define instead

$$v_k(z) = T_{H_k} \circ u_k(z/R_k + z_k) \qquad \text{for} \qquad z \in B_{\varepsilon_k R_k} \ .$$

Then $H(v_k(0)) = H(T_{H_k} \circ u_k(z_k)) = H_k - H_k = 0$, and we still have the estimate (54), because we are using the translationally invariant cylindrical metric on $\mathbb{R} \times M$.

Therefore we get a subsequence $\{v_k\}$ that converges in C^1_{loc} to a map $v_\infty : \mathbb{C} \to \mathbb{R} \times M$ with $H(v_\infty(0)) = 0$ and $|\nabla v_\infty(0)| = 1$, that is, v_∞ is not constant.

As for the estimate on $\mathcal{E}_\varphi(u) = \int_D u^* d(\varphi\alpha)$, note that this quantity is *not* invariant under the translation T_{H_k}. The $\mathcal{E}_\varphi$-energy can increase under translation, and we could wind up with $\mathcal{E}_\varphi(v_k) > \hbar_k$.

For a trivial example, let A be any constant with $1/2 \le A < 1$, and let $\varphi \equiv A$ on $(-\infty, \tau_0]$, $\varphi \equiv 1$ on $[0, +\infty)$ and $\varphi' > 0$ on $(\tau_0, 0)$. Suppose that $u^* d\alpha = 0$. If $H \circ u(D_\varepsilon(z)) \subset (-\infty, \tau_0)$, then clearly $\mathcal{E}_\varphi(u) = 0$. But the translate $T_{H \circ u(z)} \circ u$ meets the region $H^{-1}(\tau_0, 0)$, so it has positive $\mathcal{E}_\varphi$-energy.

Here is a slightly more interesting example. Suppose we have a trajectory $x : \mathbb{R} \to M$ of the Reeb vector field, i.e. $\dot{x} = X_\alpha|_x$, that is defined for all time $t \in \mathbb{R}$. For example, if M has a closed characteristic parametrized by $x : [0, T] \to M$, then extend x periodically to be defined on the whole of $\mathbb{R}$.

Take the standard complex structure on $\mathbb{R} \times \mathbb{R}$, which in coordinates (s,t) is written $j(\partial_s) = \partial_t$. Given the trajectory x, define a map $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times M$ by

$$u(s,t) = (-e^{-s} \cos t, x(e^{-s} \sin t)) \ .$$

Recall that on $\mathbb{R} \times M$, J is chosen compatible with $d\alpha$ on ξ, and $J\partial_\tau = X_\alpha$, $JX_\alpha = -\partial_\tau$. Now since

$$u_s = e^{-s}(\cos t)\, \partial_\tau - e^{-s}(\sin t)\, X_\alpha$$
$$u_t = e^{-s}(\sin t)\, \partial_\tau + e^{-s}(\cos t)\, X_\alpha \,,$$

it is clear that $Ju_s = u_t$, and u is J-holomorphic.

Note that $|\nabla u| = \sqrt{2} e^{-s}$, so that if $s << 0$, then $|\nabla u|$ is big. We next use this observation to prove that the sup estimate fails for many choices of $\varphi \in \mathcal{C}$.

LEMMA 54. *Let $\varphi \in \mathcal{C}$ and let* $\lim_{\tau \to -\infty} \varphi(\tau) = A$, $A \in [1/2, 1]$. *Suppose that*

$$\lim_{\tau \to -\infty} (\varphi(\tau) - A) \cdot |\tau| = 0 \ .$$

Then for any $r < \pi/2$, the J-holomorphic map $u : \mathbb{R}^2 \to \mathbb{R} \times M$ given by

$$u(s,t) = (-e^{-s}\cos t, x(e^{-s}\sin t))$$

satisfies

$$\mathcal{E}_\varphi(u; D_r(s)) \to 0 \quad as \ \ s \to -\infty$$

$$\sup_{D_{r/2}(s)} |\nabla u|^2 \to +\infty \quad as \ \ s \to -\infty \ .$$

PROOF. Since u is always parallel to ∂_τ and X_α,

$$\int_{D_r(s)} u^* d(\varphi\alpha) = \int_{D_r(s)} \varphi'(H \circ u)\frac{1}{2}|du|^2 ds \wedge dt \ .$$

In our case, $H \circ u = -e^{-s}\cos t$ and we can bound the integral on the right by

$$\int_{-r}^{r} \int_{s-r}^{s+r} \varphi'(-e^{-s}\cos t) e^{-2s} ds dt \ .$$

Make the substitutions $\ell = \cos t$ and $u = -e^{-s}\ell$ in the interior integral to obtain

$$\int_{s-r}^{s+r} \varphi'(-e^{-s}\cos t)e^{-2s} ds \ = \ \frac{1}{\ell^2}\{\varphi(u_1)u_1 - \varphi(u_2)u_2 + \int_{u_1}^{u_2} \varphi(u)du\} \ . \tag{55}$$

Since φ is everywhere increasing and $\varphi(u_1) \geq A$, (55) is bounded by

$$\frac{1}{\ell^2}\{A - \varphi(u_2)\} u_1 \ . \tag{56}$$

Noting that $u_1 = -e^{-(s-r)}\ell$ and $u_2 = -e^{-(s+r)}\ell = e^{-2r}u_1$ and that $0 < \cos r \leq \cos t \leq 1$, we see that

$$(56) \quad \leq \quad \frac{e^{2r}}{\cos^2 r}\{\varphi(-e^{-s-r}\cos r) - A\} e^{-s-r} \ .$$

Hence

$$\mathcal{E}_\varphi(u; D_r(s)) \ \leq \ Cr\, e^{2r}\{\varphi(-e^{-s-r}\cos r) - A\} e^{-s-r}\cos r \ \ ,$$

and the constant C depends only on r. As $s \to -\infty$, then also $-e^{-s-r}\cos r \to -\infty$, so the right hand expression goes to 0.

Finally, $\sup_{D_{r/2}(s)} |\nabla u|^2 = 2e^{-2(s-\frac{r}{2})}$, which blows up as $s \to -\infty$. □

COROLLARY 55. *Let φ satisfy the conditions of Lemma 54. Then for any sequences $\hbar_k \to 0$ and $C_k \to +\infty$, we can find a sequence of J-holomorphic maps $u_k : D_r \to \mathbb{R} \times M$ so that*

$$\mathcal{E}_\varphi(u_k; D_r) < \hbar_k \qquad and \qquad \sup_{D_{r/2}} |\nabla u_k|^2 > C_k \ .$$

Next we show definitively that the sup estimate fails for any $\varphi \in \mathcal{C}$.

LEMMA 56. *Let $\varphi \in \mathcal{C}$. There are constants $E, C > 0$ and a sequence of J-holomorphic maps $u_k : D_1 \to \mathbb{R} \times M$ satisfying*

$$\mathcal{E}_\varphi(u_k; D_1) < \frac{C}{k} \qquad \textit{and} \qquad \sup_{D_{1/2}} |du_k|^2 > Ck^2$$

for all k.

PROOF. Fix any J-holomorphic map $u : D_1 \to \mathbb{R} \times M$ with $u^* d\alpha = 0$. We can assume that $H \circ u < 0$ on D. Set $E = \frac{1}{2} \int_D |du|^2 ds \wedge dt$.

Given k, consider $v(z) = u(z^k)$, $z \in D$. We have $E(v) = \frac{1}{2} \int_D |dv|^2 = kE$. If $\sup_{D_{r/2}} |du|^2$ occurs at $r_0 e^{i\theta_0} \in D_{r/2}$, then it is easy to check that

$$\sup_{D_{r/2}} |dv|^2 \geq k^2 r_0^{2k-2} \sup_{D_{r/2}} |du|^2 \ . \tag{57}$$

As $\varphi \in \mathcal{C}$, we know that $\lim_{\tau \to -\infty} \varphi'(\tau) = 0$. Hence for any k, there exists τ_k so that $0 \leq \varphi' < 1/k^2$ on $(-\infty, \tau_k)$. Set $v_k(z) = T_{-\tau_k} \circ v(z) = T_{-\tau_k} \circ u(z^k)$. Then $H \circ v_k < \tau_k$ on D, so the image of v_k is contained in $H^{-1}(-\infty, \tau_k)$. Therefore,

$$\mathcal{E}_\varphi(v_k; D) = \int_D v_k^*(\varphi' d\tau \wedge \alpha + \varphi \, d\alpha) = \int_D \varphi'(H \circ v_k) \frac{1}{2} |dv_k|^2 < \frac{1}{k^2} E(v) = \frac{1}{k} E \ .$$

By (57), we also have $\sup_{D_{r/2}} |dv_k|^2 = \sup_{D_{r/2}} |dv|^2 \geq Ck^2 \sup_{D_{r/2}} |du|^2$. So we are done. $\square$

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