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CORRECTION TO
“COMBINATORICS AND GEOMETRY OF POWER IDEALS”:
TWO COUNTEREXAMPLES FOR POWER IDEALS
OF HYPERPLANE ARRANGEMENTS

FEDERICO ARDILA AND ALEXANDER POSTNIKOV

ABSTRACT. We disprove Holtz and Ron’s conjecture that the power ideal $C_{\mathcal{A},-2}$ of a hyperplane arrangement \mathcal{A} (also called the internal zonotopal space) is generated by \mathcal{A} -monomials. We also show that, in contrast with the case $k \geq -2$, the Hilbert series of $C_{\mathcal{A},k}$ is not determined by the matroid of \mathcal{A} for $k \leq -6$.

Remark. This note is a corrigendum to our article [1], and we follow the notation of that paper.

1. INTRODUCTION

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in a vector space V ; say $H_i = \{x \mid l_i(x) = 0\}$ for some linear functions $l_i \in V^*$. Call a product of (possibly repeated) l_i ’s an \mathcal{A} -*monomial* in the symmetric algebra $\mathbb{C}[V^*]$. Let $\text{Lines}(\mathcal{A})$ be the set of lines of intersection of the hyperplanes in \mathcal{A} . For each $h \in V$ with $h \neq 0$, let $\rho_{\mathcal{A}}(h)$ be the number of hyperplanes in \mathcal{A} not containing h . Let $\rho = \rho(\mathcal{A}) = \min_{h \in V}(\rho_{\mathcal{A}}(h))$. For all integers $k \geq -(\rho + 1)$, consider the *power ideals*

$$I_{\mathcal{A},k} := \left\langle h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \in V, h \neq 0 \right\rangle, \quad I'_{\mathcal{A},k} := \left\langle h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \in \text{Lines}(\mathcal{A}) \right\rangle$$

in the symmetric algebra $\mathbb{C}[V]$. It is convenient to regard the polynomials in $I_{\mathcal{A},k}$ as differential operators, and to consider the space of solutions to the resulting system of differential equations:

$$C_{\mathcal{A},k} = I_{\mathcal{A},k}^{\perp} := \left\{ f(x) \in \mathbb{C}[V^*] \mid h \left(\frac{\partial}{\partial x} \right)^{\rho_{\mathcal{A}}(h)+k+1} f(x) = 0 \text{ for all } h \neq 0 \right\}$$

which is known as the *inverse system* of $I_{\mathcal{A},k}$. Define $C'_{\mathcal{A},k}$ similarly. These objects arise naturally in numerical analysis, algebra, geometry, and combinatorics. For references, see [1, 3].

One important question is to compute the Hilbert series of these spaces of polynomials, graded by degree, as a function of combinatorial invariants of \mathcal{A} . Frequently, the answer is expressed in terms of the Tutte polynomial of \mathcal{A} . This has been done

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successfully in many cases. One strategy used independently by different authors has been to prove the following:

- (i) There is a spanning set of \mathcal{A} -monomials for $C_{\mathcal{A},k}$.
- (ii) There is an exact sequence $0 \rightarrow C_{\mathcal{A}\setminus H,k}(-1) \rightarrow C_{\mathcal{A},k} \rightarrow C_{\mathcal{A}/H,k} \rightarrow 0$ of graded vector spaces.
- (iii) Therefore, the Hilbert series of $C_{\mathcal{A},k}$ is an evaluation of the Tutte polynomial of \mathcal{A} .

Here $\mathcal{A}\setminus H$ and \mathcal{A}/H are the deletion and contraction of H , respectively.

For $k \geq -1$, this method works very nicely. Dahmen and Micchelli [2] were the first ones to do this for $C'_{\mathcal{A},-1}$. Postnikov, Shapiro, and Shapiro [5] did it for $C_{\mathcal{A},0}$, while Holtz and Ron [3] did it for $C'_{\mathcal{A},0}$. In [1] we did it for $C_{\mathcal{A},k}$ for all $k \geq -1$, and showed that $C'_{\mathcal{A},0} = C_{\mathcal{A},0}$ and $C'_{\mathcal{A},-1} = C_{\mathcal{A},-1}$.

For $k \leq -3$ this approach does not work in full generality. In [1] we showed that (i) is false in general for $C_{\mathcal{A},k}$, and left (ii) and (iii) open, suggesting the problem of measuring $C_{\mathcal{A},k}$. For $k \leq -6$, (ii) and (iii) are false, as we will show in Propositions 4 and 5, respectively. In fact, we will see that the Hilbert series of $C_{\mathcal{A},k}$ is not even determined by the matroid of \mathcal{A} .

The intermediate cases are interesting and subtle, and deserve further study; notably the case $k = -2$, which Holtz and Ron call the *internal zonotopal space*. In [3] they proved (ii) and (iii) and conjectured (i) for $C'_{\mathcal{A},-2}$. In [1, Proposition 4.5.3] – a restatement of Holtz and Ron’s Conjecture 6.1 in [3] – we put forward an incorrect proof of this conjecture; the last sentence of our argument is false. In fact their conjecture is false, as we will see in Proposition 2.

2. THE CASE $k = -2$: INTERNAL ZONOTOPAL SPACES

Before showing why Holtz and Ron’s conjecture is false, let us point out that the remaining statements about $C_{\mathcal{A},-2}$ that we made in [1] are true. The easiest way to derive them is to prove that $C_{\mathcal{A},-2} = C'_{\mathcal{A},-2}$, and simply note that Holtz and Ron already proved those statements for $C'_{\mathcal{A},-2}$:

Lemma 1. *We have $C_{\mathcal{A},k} = C'_{\mathcal{A},k}$ for any k with $-(\rho + 1) \leq k \leq 0$.*

Proof. By [1, Theorem 4.17] we have $I_{\mathcal{A},0} = I'_{\mathcal{A},0}$, so it suffices to show that $I_{\mathcal{A},j} = I'_{\mathcal{A},j}$ implies that $I_{\mathcal{A},j-1} = I'_{\mathcal{A},j-1}$ as long as these ideals are defined. If $I_{\mathcal{A},j} = I'_{\mathcal{A},j}$, then for any $h \in V \setminus \{0\}$ we have $h^{\rho_{\mathcal{A}}(h)+j+1} = \sum f_i h_i^{\rho_{\mathcal{A}}(h_i)+j+1}$ for some polynomials f_i , where the h_i ’s are the lines of the arrangement. As long as the exponents are positive, taking partial derivatives in the direction of h gives $h^{\rho_{\mathcal{A}}(h)+j} = \sum g_i h_i^{\rho_{\mathcal{A}}(h_i)+j}$ for some polynomials g_i . □

The following result shows that (i) does not hold for $C_{\mathcal{A},-2}$.

Proposition 2. [3, Conjecture 6.1] *is false: The “internal zonotopal space” $C_{\mathcal{A},-2}$ is not necessarily spanned by \mathcal{A} -monomials.*

Proof. Let \mathcal{H} be the hyperplane arrangement in \mathbb{C}^4 determined by the linear forms $y_1, y_2, y_3, y_1 - y_4, y_2 - y_4, y_3 - y_4$. We have

$$I'_{\mathcal{H},-2} = \langle x_1^1, x_2^1, x_3^1, (\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_4 + x_4)^2 \rangle = \langle x_1, x_2, x_3, x_4^2 \rangle$$

as $\epsilon_1, \epsilon_2, \epsilon_3$ range over $\{0, 1\}$. The other generators of $I_{\mathcal{H}, -2}$ are of degree at least 3, and are therefore in $I'_{\mathcal{H}, -2}$ already, so

$$I_{\mathcal{H}, -2} = \langle x_1, x_2, x_3, x_4^2 \rangle, \quad C_{\mathcal{H}, -2} = \text{span}(1, y_4).$$

Therefore $C_{\mathcal{H}, -2}$ is not spanned by \mathcal{H} -monomials. □

As Holtz and Ron pointed out, if [3, Conjecture 6.1] had been true, it would have implied [3, Conjecture 1.8], an interesting spline-theoretic interpretation of $C_{\mathcal{A}, -2}$ when \mathcal{A} is unimodular. The arrangement above is unimodular, but it does not provide a counterexample to [3, Conjecture 1.8]. In fact, Matthias Lenz [4] has recently put forward a proof of this weaker conjecture.

3. THE CASE $k \leq -6$

In this section we show that when $k \leq -6$, the Hilbert series of $C_{\mathcal{A}, k}$ is not a function of the Tutte polynomial of \mathcal{A} . In fact, it is not even determined by the matroid of \mathcal{A} . Recall that $\rho = \rho(\mathcal{A}) := \min_{h \in V} (\rho_{\mathcal{A}}(h))$. Say $h \in V$ is *large* if it is on the maximum number of hyperplanes, so $\rho_{\mathcal{A}}(h) = \rho$.

Lemma 3. *The degree 1 component of $C_{\mathcal{A}, -\rho}$ is*

$$(C_{\mathcal{A}, -\rho})_1 = (\text{span}\{h \in V : h \text{ is large}\})^\perp$$

in V^* .

Proof. An element f of $C_{\mathcal{A}, -\rho}$ needs to satisfy the differential equations $h(\partial/\partial x)^{\rho_{\mathcal{A}}(h) - \rho + 1} f(x) = 0$ for all non-zero vectors $h \in V$. If f is linear, then this condition is trivial unless h is large; and in that case it says that $f \perp h$. □

Proposition 4. *For $k \leq -6$, the Hilbert series of $C_{\mathcal{A}, k}$ is not determined by the matroid of \mathcal{A} .*

Proof. First assume $k = -2m$. Let L_1, L_2, L_3 be three different lines through 0 in \mathbb{C}^3 and consider an arrangement \mathcal{A} of $3m$ (hyper)planes consisting of m generically chosen planes H_{i1}, \dots, H_{im} passing through L_i for $i = 1, 2, 3$. Then $\rho = 2m$ and the only large lines are L_1, L_2 , and L_3 . Therefore $\dim(C_{\mathcal{A}, -2m})_1$ equals 1 if L_1, L_2, L_3 are coplanar, and 0 otherwise. However, the matroid of \mathcal{A} does not know whether L_1, L_2, L_3 are coplanar.

More precisely, consider two versions \mathcal{A}_1 and \mathcal{A}_2 of the above construction; in \mathcal{A}_1 the lines L_1, L_2, L_3 are coplanar, and in \mathcal{A}_2 they are not. Notice that \mathcal{A}_1 and \mathcal{A}_2 have the same matroid: the rank 3 matroid whose non-bases are the triples $\{H_{ia}, H_{ib}, H_{ic}\}$ for $1 \leq i \leq 3$ and $1 \leq a < b < c \leq m$. However, $\dim(C_{\mathcal{A}_1, -2m})_1 \neq \dim(C_{\mathcal{A}_2, -2m})_1$.

The case $k = -2m - 1$ is similar. It suffices to add a generic plane to the previous arrangements. □

Proposition 5. *For $k \leq -6$, the sequence of graded vector spaces*

$$0 \rightarrow C_{\mathcal{A} \setminus H, k}(-1) \rightarrow C_{\mathcal{A}, k} \rightarrow C_{\mathcal{A}/H, k} \rightarrow 0$$

of [1, Proposition 4.4.1] is not necessarily exact, even if H is neither a loop nor a coloop.

Proof. We will not need to recall the maps that define this sequence; we will simply show an example where right exactness is impossible because $\dim(C_{\mathcal{A},k})_1 = 0$ and $\dim(C_{\mathcal{A}/H,k})_1 = 1$. We do this in the case $k = -2m$; the other one is similar.

Consider the arrangement $\mathcal{A} = \mathcal{A}_2$ of the proof of Proposition 4 and the plane $H = H_{11}$. We have $\dim(C_{\mathcal{A},-2m})_1 = 0$. In the contraction \mathcal{A}/H , the planes H_{12}, \dots, H_{1m} become the same line L_1 in H , while the other $2m$ planes of \mathcal{A} become generic lines in H . Therefore $\rho(\mathcal{A}\setminus H) = 2m$ and $(C_{\mathcal{A}/H,-2m})_1 = L_1^\perp$ in H^* , which is one-dimensional. \square

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DEPARTMENT OF MATHEMATICS, SAN FRANCISCO STATE UNIVERSITY, 1600 HOLLOWAY AVENUE, SAN FRANCISCO, CALIFORNIA 94110

E-mail address: federico@sfsu.edu

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MASSACHUSETTS 02139

E-mail address: apost@math.mit.edu