Announcements

- Turn in problem set 3
- Pick up problem set 3 solutions, problem set 4, lecture notes, slides

Quantum Harmonic Oscillator

- Quadrature-measurement statistics and phase space
- Characteristics functions and the Wigner distribution
- Positive operator-valued measurement of $\hat{a}$
Quadrature-Measurement Statistics: Summary

<table>
<thead>
<tr>
<th>State</th>
<th>$\langle \hat{a}(t) \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>n\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\alpha\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\beta; \mu, \nu\rangle$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>$\langle \Delta \hat{a}_1^2(t) \rangle$</th>
<th>$\langle \Delta \hat{a}_2^2(t) \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>n\rangle$</td>
<td>$(2n + 1)/4$</td>
</tr>
<tr>
<td>$</td>
<td>\alpha\rangle$</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$</td>
<td>\beta; \mu, \nu\rangle$</td>
<td>$</td>
</tr>
</tbody>
</table>

Quadrature-Measurement Statistics of $|n\rangle$ and $|\alpha\rangle$

- Number State

\[ a_1(t) = \text{Re}[ae^{-j\omega t}] \]

- Coherent State

\[ a_1(t) = \text{Re}[ae^{-j\omega t}] \]
Quadrature-Measurement Statistics of $|\beta; \mu, \nu\rangle$

- Amplitude-Squeezed State
  \[ a_1(t) = \text{Re}[ae^{-j\omega t}] \]

- Phase-Squeezed State
  \[ a_1(t) = \text{Re}[ae^{-j\omega t}] \]

Classical Random Variable Review

- Probability Density Function: \( p_x(X) \)

- Characteristic Function: \( M_x(jv) \)

- Fourier Relationship
  \[
  M_x(jv) \equiv \langle e^{jvx} \rangle = \int_{-\infty}^{\infty} dX \, e^{jvX} \, p_x(X)
  \]
  \[
  p_x(X) = \int_{-\infty}^{\infty} \frac{dv}{2\pi} \, e^{-jvX} \, M_x(jv)
  \]
Quadrature-Measurement Statistics

- Characteristic Function for the Quadrature Measurement

\[ M_{a_1}(j\nu) \equiv \langle e^{j\nu \hat{a}_1(t)} \rangle = \langle e^{j\nu [\hat{a}_1 \cos(\omega t) + \hat{a}_2 \sin(\omega t)]} \rangle \]

\[ \hat{a}(t) = \hat{a} e^{-j\omega t} \rightarrow M_{a_1}(j\nu) = \chi_W(\zeta^*, \zeta) \big|_{\zeta = j\nu/2} \]

- Wigner Characteristic Function

\[ \chi_W(\zeta^*, \zeta) \equiv \langle e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger} \rangle = \langle e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} \rangle e^{-|\zeta|^2/2} \]

Quadrature-Measurement Statistics of \( |n\rangle \)

- Wigner Characteristic Function of the Number State

\[ \chi_W(\zeta^*, \zeta) = \langle n | e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} | n \rangle e^{-|\zeta|^2/2} = L_n(|\zeta|^2)e^{-|\zeta|^2/2} \]

- Quadrature-Measurement Probability Density Function

\[ p_{a_1}(\alpha_1) = \int_{-\infty}^{\infty} \frac{dv}{2\pi} L_n(v^2/4)e^{-v^2/8}e^{-jv\alpha_1} \]

\[ = \sqrt{\frac{2}{\pi}} \frac{e^{-2\alpha_1^2}}{2^n n!} [H_n(\sqrt{2}\alpha_1)]^2 \]
Quadrature-Measurement Statistics of $|n\rangle$

- Example: $|n\rangle$, with $n = 10$

\[
\mathcal{P}_{a_1}(t)(\alpha_1)
\]

Wigner Distribution in $(\alpha_1, \alpha_2)$ Space

- Inverse Transform of the Wigner Characteristic Function

\[
W(\alpha, \alpha^*) = \int \frac{d^2\zeta}{\pi^2} \chi W(\zeta^*, \zeta) e^{\zeta^* \alpha - \zeta \alpha^*}
\]

- Vacuum-State Wigner Distribution:

\[
W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha|^2} = \text{2-D Gaussian pdf}
\]

- One-Photon Wigner Distribution:

\[
W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha|^2} (4|\alpha|^2 - 1) \neq \text{valid 2-D pdf}
\]
# Measuring the $\hat{a}$ Operator

- **Definition:** Measurement of the $\hat{a}$ Operator
  - yields an outcome that is a complex number $\alpha = \alpha_1 + j\alpha_2$
  - joint probability density for getting this outcome is
    \[
    p(\alpha) = \frac{|\langle \alpha | \psi \rangle|^2}{\pi}
    \]
- **Consistency Checks:**
  \[
  p(\alpha) \geq 0
  \]
  \[
  \int d^2\alpha p(\alpha) = \langle \psi | \left( \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| \right) |\psi\rangle = 1
  \]

## Measuring the $\hat{a}$ Operator: Summary

<table>
<thead>
<tr>
<th>State</th>
<th>$\langle \alpha \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>n\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\beta\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\beta; \mu, \nu\rangle$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>$\langle \Delta \alpha_1^2 \rangle$</th>
<th>$\langle \Delta \alpha_2^2 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>n\rangle$</td>
<td>$(n + 1)/2$</td>
</tr>
<tr>
<td>$</td>
<td>\beta\rangle$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$</td>
<td>\beta; \mu, \nu\rangle$</td>
<td>$(</td>
</tr>
</tbody>
</table>
Coming Attractions: Lectures 8 and 9

- Lecture 8:
  Quantum Harmonic Oscillator
  - Reconciling the $\hat{a}$ measurement with the notion of observables
  Single-Mode Photodetection
  - Direct Detection — semiclassical versus quantum
  - Homodyne Detection — semiclassical versus quantum

- Lecture 9:
  Single-Mode Photodetection
  - Heterodyne Detection — semiclassical versus quantum
  - Realizing the $\hat{a}$ measurement