Models for
Pricing and Inventory Management
of Seasonal Products

by

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Submitted to the Sloan School of Management
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ABSTRACT

Seasonal products, such as fashion apparel, are characterized by high demand uncertainty, supply inflexibility and a limited market life. These factors result in frequent mismatches between supply and demand for these products, leading to costly stockouts and forced markdowns. The ordering and pricing decisions made by a retailer prior to, and during, the sales season are of crucial importance in addressing this problem. In this dissertation, we develop models to help retail managers make these decisions. In addition, we identify and study certain structural properties of seasonal product pricing models. Our analytical and computational analyses provide a number of insights about the nature of optimal pricing strategies for seasonal products. The main contributions of our research are summarized below.

We propose and investigate four intuitive conjectures about the structure of a single store pricing model. We show through a series of counter-intuitive examples that these conjectures are not true in general. We derive a set of sufficient conditions under which the conjectures can be guaranteed to hold for certain cases and present examples of some common demand functions and distributions that satisfy these conditions. This work provides some interesting economic insights on optimal retail pricing behavior under different demand and inventory conditions. We also develop a heuristic and an upper bounding procedure for the single store model based on the solution of a related deterministic model.

We develop a method for incorporating observed sales data to update the demand distribution in the pricing model. Our methodology allows for a number of non-stationary features in the demand distribution, such as changes in product price and in customers' value for the product over time. We present computational test results that demonstrate the effectiveness of our methodology and that provide interesting economic insights into pricing behavior under conditions of demand over- and under-estimation.

We formulate models for multi-store pricing and inventory management that address different distribution practices in retailing, including the use of a distribution center, pre-season shipment of the product to individual stores, and inter-store transfers. We develop and test a set of heuristics and an upper bounding procedure for these models.

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INTRODUCTION

Retailers who carry seasonal merchandise, such as air conditioners or summer wear, often have to place orders for these items much in advance of the sale season. Once the initial order is placed, they may have limited, if any, flexibility in placing reorders. In addition, such merchandise may only be in demand at the stores during the season. Since the demand for the merchandise is not perfectly predictable at the order time, retailers may be left with an excess or shortage of merchandise as the sale season progresses. These mismatches between supply and demand can be costly, and can be mitigated in part through price markdowns, possible inter-store transfer of merchandise, possible reorders, and other decisions made by the retailer. Retail merchandise managers face a challenging task as they attempt to make pricing and inventory management decisions under these uncertain demand and constrained supply conditions.

The increased application of information technology in retail store operations has had a dramatic impact on the amount of sales and inventory information available to managers and the speed with which it becomes available. As a consequence, there is a significant opportunity today for the development of computer-based models that will utilize the available data to help retail managers address the pricing and inventory management issues described above.

This dissertation focuses on the development of such models. In particular, our objective is to develop models to help retail managers make pricing, markdown, ordering and related decisions in the best possible manner given the observed and projected demand and supply conditions. Our focus is on developing and analyzing pricing and inventory management models and on developing optimal (i.e., revenue/profit maximizing) strategies to aid in managerial decision making. A second objective of our work is to identify and study certain structural properties of such models in order to gain qualitative and analytical insights into the nature of optimal pricing and inventory decisions.
Past research on pricing and inventory management of seasonal products is limited in scope. Some marketing scientists have focused on developing simple models of the firm in order to study the qualitative nature of optimal medium-to-long term pricing and inventory policies, but these are of little relevance in the short-term, inventory constrained context of seasonal products. Researchers in stochastic inventory theory have developed a large body of knowledge on optimal inventory policies under diverse demand and supply conditions, but they have for the most part ignored the pricing issue. Some researchers have recently developed models of dynamic pricing and inventory management for seasonal products, but their work ignores a number of important aspects of the retail environment that we are interested in studying. These aspects include the potential for learning about demand as sales are observed, the ability to reorder during the season and pricing and inventory management in multi-store environments. In addition, we examine a number of fundamental properties of the seasonal product pricing problem that have escaped attention in the past.

Our research can be viewed as lying at the interface of marketing and operations management. The marketing issues that we consider include pricing, markdowns and demand learning, and the operational issues include the ordering of seasonal merchandise from suppliers and the distribution of this merchandise to the member stores.

We approach the season product pricing and inventory management problem from two directions in this dissertation. We study a basic model for the problem in order to analyze certain fundamental and intuitive properties and identify conditions under which these properties can be guaranteed to hold. We also develop extensions of the basic models to incorporate certain important aspects of retail environments, such as multiple stores and demand learning. Our research is summarized below.

**Basic single store problem**
In Chapter 3, we focus on basic, single store versions of the seasonal product pricing and inventory management problem, similar to the models of seasonal product pricing developed by management scientists in recent years. We study this problem in the following ways.
• **Structural properties:** We analyze basic models of optimal seasonal product pricing that include limited inventory, no reordering, a single store, and either deterministic or stochastic demand. We study the following structural properties for these models:
  - The optimal price is non-decreasing in the level of demand (the term ‘level of demand’ is defined in Chapter 3)
  - The optimal price is non-increasing in the level of inventory
  - The marginal revenue is non-increasing in the level of inventory
  - When the level of demand is decreasing over time, the retailer does not benefit from reserving some stock of the product for future periods.

Through a series of counter-intuitive examples, we show how all these properties are not true in general. We derive sufficient conditions that guarantee the first three properties, and provide examples of demand models for which these conditions are satisfied. The fourth property entails the study of an extended model that allows the retailer to set a *sales limit* in each period in addition to the price. We show how this flexibility in reserving stock for later periods can benefit the retailer even under conditions where the value for the product decreases over time. We prove, however, that under a non-increasing price constraint, or under continuous price revisions, the retailer will not have an incentive to limit sales in any period.

• **Heuristic and Upper Bound:** We analyze and solve a deterministic model of the seasonal product pricing problem, and use this to develop a heuristic for the stochastic version of the model. We present results of computational tests that indicate that this heuristic performs very well in practice and often significantly outperforms fixed-price heuristics proposed recently by other researchers. The heuristic provides some interesting insights into the nature of optimal pricing strategies, and also serves as a fundamental building block for heuristics for more computationally demanding problems, such as the ones discussed in Chapters 4 and 5. We also prove that the deterministic problem, derived by replacing all the demand random variables in the stochastic problem by their expected values, provides an upper bound to the stochastic problem.
Optimal pricing with demand learning
In Chapter 4, we develop a method for incorporating data on observed sales to update the demand distribution, and embed this method within an optimal pricing model. In contrast to most existing approaches, our methodology allows for a number of non-stationary features in the demand distribution, such as changes in product price and in customers' value for the product. We also develop an extension to the heuristic discussed above to cut down on the computational requirements for the resulting model, and present computational test results that demonstrate the effectiveness of our methodology and the performance of the heuristic. These tests also provide some interesting insights into pricing behavior under various conditions.

Multi-store models
In Chapter 5, we formulate three models for multi-store pricing and inventory management. Since it is not feasible to solve these models in an optimal fashion, we develop a number of heuristics and some upper bounds for these models. We also present computational results for some of these heuristics.

In Chapter 2, we present a review of related research. A technical appendix and a list of references are included after Chapter 5.
CHAPTER 2

LITERATURE REVIEW

Our work is related to the research literature in the following areas:

- Dynamic pricing
- Integrated production-marketing problems
- Dynamic pricing of seasonal products
- Economic and empirical research on fashion goods pricing
- Inventory models with demand learning

In this chapter, we provide a brief discussion of the relevant research from these areas and how it relates to our work.

Before discussing the literature in the above areas, we would like to draw a distinction between what we term as decision support models and economic models. (Economic models are often also called theoretical models - see, for instance, Murthy, 1983). These comments will allow us later to highlight the difference between some of the models we have developed (in Chapters 4 and 5) and certain models in the literature. The two classes of models referred to above can be contrasted along the following dimensions:

- *Modeling objective:* The goal of a decision support model is to provide a practical quantitative analysis tool to help decision makers address specific problems. In contrast, economic models primarily aim to generate general qualitative insights about issues such as the effects of different market factors or firms' strategies on firm performance, and the nature of optimal decision policies.

- *Modeling detail:* Decision support models are designed to incorporate enough detail about a specific problem context so as to provide an adequate representation of the decision makers' environment and thereby lead to valid recommendations. On the other hand,
economic models often make a number of simplifying assumptions in order to isolate and study a few issues of interest.

- **Parameter estimation**: A decision support model is designed so that its parameters may be accurately estimated using actual data or, on occasion, managerial judgment. In economic models, by contrast, the parameters may be aggregate theoretical quantities that may be difficult to estimate accurately. These models often seek to study the model's behavior over a general range of values for the model parameters.

- **Model solution**: Decision support models rarely lend themselves to analytic, closed form solutions. It is often computationally impractical, if not infeasible, to solve them in an optimal fashion, and heuristics and simulation methods are then employed to provide timely, practical solutions. In contrast, economic models are typically designed so as to be analytically tractable. This is necessary to allow the researcher to derive general conclusions without needing to use specific parameter values in the model.

The distinctions between these two classes of models is not rigid: economic models can on occasion serve as foundations for the development of decision support models, and, conversely, simplified versions of decision support models may at times lend themselves to fruitful economic analysis. Typically, however, a modeler would have either the first or the second type of model in mind when building a model. A number of the models that exist in the literature, and that will be discussed below, are economic models that are not practical for providing specific recommendations in specific business settings. We have employed both approaches in this dissertation - in studying certain structural properties of the seasonal product pricing problem, we have used an economic modeling approach; however, in addressing certain important practical issues, we have used a decision support modeling approach.

We now turn to a discussion of the literature in the areas listed above.

### 2.1 Dynamic Pricing

Marketing scientists have been studying the issue of the optimal dynamic pricing of products for a long time. A bulk of their research has focused on the manufacturer's problem of pricing a
durable product, and our comments here will be limited to this research since the proposed research can itself be viewed as focusing on durable as opposed to repeat-purchase products. Rao (1984), Rao (1993), Gijsbrechts (1993), and Kalish (1988) provide some recent reviews of research on dynamic pricing.

Issues studied

This line of research has investigated the nature of optimal dynamic pricing policies over the lifetime of a durable product under different conditions internal and external to the manufacturer. Researchers have mostly employed continuous-time optimal control techniques to formulate and analyze economic models of dynamic pricing. Some of the models use a two- or three-period dynamic programming structure. These models typically assume demand to be a deterministic function of price. Some of the dynamic issues that these models have attempted to study are the following:


- **Diffusion** (Kalish, 1985, Eliashberg & Jeuland, 1986, Rao & Bass, 1990, Horsky, 1990): This includes two dynamic factors - the word-of-mouth effect and the saturation effect. The word-of-mouth effect causes product awareness to increase with increased sales, leading to increased demand. The saturation effect causes demand to decrease as more products are sold due to a gradual saturation of the market.

- **Technological obsolescence**: This effect leads to a decrease in the value of the product over time due to the introduction of technologically superior products. A high rate of technological innovation in an industry may cause the value of products to drop rapidly with time, as is often the case with many hi-tech products today.

- **Externality effect** (Dhebar & Oren, 1985): This refers to the increase in the value of the product caused by the increase in the product's customer base, as is the case for many communications products.
Other dynamic issues considered include inventory holding costs and discount rates. Specific models that have been formulated and analyzed have included different subsets of the above issues.

Pricing strategies

Three general types of pricing strategies are characterized under this research. A myopic pricing strategy is defined as the static optimal pricing solution that results from the firm optimizing price in each period without consideration of the dynamic effects. A price skimming strategy is a dynamic pricing strategy where the initial price is above the myopic price, while a penetration strategy is a dynamic pricing strategy where the initial price is below the myopic price.

Competitive environment

Most of these models assume a monopolistic environment, thereby ignoring competitive issues. Some models have adopted a duopolistic (Moorthy, 1988) or oligopolistic (Rao and Bass, 1985, Choi et al., 1990, Dockner and Jorgenson, 1988), framework and have proceeded to analyze competitive strategies using game theoretic techniques. Eliashberg and Jeuland (1986) studied the issue of competitive entry by assuming an initially monopolistic market followed by the entry of a competitor in the second period.

Customer expectations

In certain dynamic pricing contexts, it may be important to consider the issue of customer expectations. Customers may alter their purchase decisions if they expect the price of a product to move up or down in the future. Some models have been developed to investigate this issue. A few (e.g., Besanko and Winston, 1990) assume full customer expectations, where customers are assumed to be fully informed of the future pricing strategy, and model the firm-customer interaction through game theory. The assumption of full customer expectation is obviously quite strong, and some attempts have been made to instead incorporate partial expectations. Holak et al. (1987) found that consumers incorporate expectations of future prices in their purchase behavior (in the context of innovative consumer durables, such as PCs), but only in a simple manner that is consistent with partial expectations.
Conclusions derived

The models developed have been analyzed to study general conditions under which price skimming or penetration pricing are optimal. The conclusions derived have been primarily of a qualitative nature, and, for illustrative purposes, we list two of these below:

- Kalish (1983) shows that in a monopolistic setting with learning curve effects and no other dynamic factors, the optimal price will decrease over time and the average price would be lower than that for a myopic monopolist.
- Besanko and Winston (1990) show that the optimal price path for a monopolist is lower when there are fully informed, rational consumers than when there are myopic consumers. One conclusion they draw is that a firm that wrongly assumes consumers to be myopic when in fact they are in fact fully informed and rational may see significantly less profits than if it rightly assumed full expectations from the consumers.

Limitations

The dynamic pricing models in marketing science are economic models that serve a useful purpose in providing general, qualitative insights about the interplay between a firm's pricing decision and different factors in the firm's internal and market environment over the medium to long term. However, they are unsuitable in a decision support context since they are based on a number of simplifying assumptions. In addition, these pricing models do not address certain key aspects of the retail environment, such as the stochastic nature of demand (and the associated potential for demand learning), the inflexibility in supply, and the coordination of pricing and logistical decisions across the multi-store network.

Some of these unexplored issues have been highlighted by researchers reviewing the literature on dynamic pricing. One of the conclusions by Gijsbrechts (1993) is that a fruitful area for new work would be the development of pricing systems that help managers decide on appropriate pricing schemes in their particular situation. Kalish (1988) mentions that the research area relating to
dynamic pricing under uncertain demand with demand learning is, 'though important, still in its infancy'.

Our goal has, in part, been to fill some of the above gaps in the pricing research literature by using the retailing of seasonal products as a specific problem setting.

2.2 Joint Ordering-Pricing Models

An informative review of ordering-pricing models can be found in the survey of joint production-marketing models by Eliashberg and Steinberg (1991). In our review below, we only mention papers that have studied some of the structural properties of interest to us in our analysis of the basic single store pricing problem in Chapter 3. To our knowledge, only two of these properties have been studied by past researchers - the concavity of the optimal value function, and the non-increasing nature of price as a function of initial inventory.

Thowsen (1975) considered an demand model with additive uncertainty of the form
\[ b(p) = D(p) + X, \]
where \( b(p) \) is a random variable representing demand at price \( p \) in any given period, \( D(p) \) is a deterministic function of price and \( X \) is a random variable with \( E[X] = 0 \).

The author derived a list of sufficient conditions under which the optimal price is non-increasing in inventory and the optimal value function is concave for such a demand model. One situation where these conditions are satisfied is when \( D(p) \) is linear and \( X \) is in the class PF\(_2\). The PF\(_2\) class contains many common distributions, such as the exponential, uniform, normal, and truncated normal.

Polatoglu (1991) considered a single-period version of our seasonal product pricing model and showed that the optimal expected revenue function was unimodal in the initial level of inventory when the demand model had one of the following forms:
\[ b(p) = D(p) + X, \]
where \( D(p) \) is a linear function of price and \( X \) is a uniform random variable with \( E[X] = 0 \).
\( \delta(p) = D(p) \zeta \), where \( D(p) \) is a linear function of price and \( \zeta \) is an exponential random variable.

This result is related to our study of the concavity of the optimal revenue function in Chapter 3, since unimodality is a relaxation of the concavity property stated for the optimal expected revenue function in that conjecture. Zabel (1970) also proved the same result for the second demand model given above.

Thomas (1973) studied a multiperiod joint ordering-pricing model with linear ordering costs plus a fixed order placement cost in each period. Demand was taken to be a random variable with a distribution that depended on price and which satisfied the stochastic monotonicity property described in Condition 3.2.1.3 in Chapter 3. The author noted that an "(s,S,p)" ordering policy was optimal for this problem for a range of numerically tested problems, and suggested that such a policy may be optimal for this problem under 'fairly general conditions'. Using our counterexample to Conjecture 3 of Section 3.2.2, it can be shown that even for the deterministic demand case such a result would not hold without the imposition of more conditions.

As we discuss later, Bitran and Mondschein (1993) and Gallego and Van Ryzin (1995) studied continuous time versions of the seasonal product pricing problem. They showed that under certain assumptions, price is non-increasing in inventory and the optimal revenue function is concave for the continuous time model with demand (for each price level) being given by a Poisson point process.

Lodish (1980) provided a framework for analyzing dynamic pricing and ordering problems that is essentially a very generic version of the stochastic dynamic programming formulations we consider. The author's focus was also on seasonal products, and the paper describes an interesting application of the model to broadcast spot pricing. Some of the demand parameters in the model were derived through subjective management estimates. The author did not present any analysis of the model to identify properties of optimal solutions, nor did he present any heuristics to make the model computationally tractable in general situations.
Limitations

The two structural properties mentioned earlier have been analyzed in the literature only under fairly restrictive conditions. For instance, for the additive demand model (used by Polotaglu and Thowsen), the variance of demand is always independent of its expected value, while for the multiplicative demand model (used by Polotaglu), this variance is always proportional to the square of the expected value. Our analysis of these properties employs a very general form of a demand model, and we derive sufficient conditions that guarantee these properties for some cases of the seasonal product pricing problem. Finally, we believe that the counterexamples we provide to each of the properties are also new contributions in this area that, through their counterintuitive nature, provide some interesting insights into the problem.

2.3 Dynamic Pricing of Seasonal Products

A few researchers (Bitran and Mondschein 1993, 1995 and Gallego and Van Ryzin 1994a) have recently attempted to model the dynamic pricing problem for seasonal products under stochastic demand. These models assume that the merchandise can only be ordered once before the start of the sales season, with no reordering capability during the season. Other key features of these models are the following:

- **Demand:** Gallego and Van Ryzin model demand as a Poisson process where the arrival (more appropriately, 'purchase') rate at time t depends on the price at time t. Bitran and Mondschein model price in a two-phased manner. They consider a Poisson process based on store arrival distribution, which is independent of the product price, and a reservation price distribution for the product that determines the fraction of arrivals that will purchase the product at any given price. This demand model appears to be a more behaviorally appealing representation of the store purchase process. The authors show, however, that this model is actually equivalent to one with a price-dependent Poisson purchase process. Gallego and Van Ryzin (1994a) study an extension of their model to the case of a compound Poisson purchase arrival process, where arriving customers can purchase in larger than unit quantities. Most of the structural results in these papers are based on the assumption of a
homogeneous Poisson arrival process. However, the dynamic programming model in Bitran and Mondschein (1993, 1995) can be solved for non-homogeneous Poisson arrivals as well.

- **Optimal control:** A stochastic dynamic programming framework is used in all these papers. Bitran and Mondschein (1993, 1995) study both continuous and discrete time formulations for their model, while Gallego and Van Ryzin (1994a) assume a continuous time framework. The continuous time approach facilitates the derivation of some interesting theoretical results, while the discrete time approach is more realistic from an application standpoint, since in any actual retail environment we expect that time would need to be modeled in rather large-sized discrete steps.

- **Single products:** Both Bitran and Mondschein (1993, 1995) and Gallego and Van Ryzin (1994a) formulate models that consider pricing for a single product.

- **Single stores:** Again, both the Bitran and Mondschein (1993, 1995) and Gallego and Van Ryzin (1994a) models address the pricing problem from a single-store standpoint. These models would be applicable to multi-store contexts if the pricing decisions were unrelated across stores. However, as we will discuss in the following chapter, this is often not the case. Bitran and Mondschein (1993) actually do formulate a pricing and inter-store transfer problem for a multi-store network. However, they assume that prices can be set independently across stores, and their model cannot be solved adequately due to computational problems.

- **Other issues:** Gallego and Van Ryzin (1994a) consider some other issues such as holding costs and discount rates. They also study such issues as overbooking & cancellations and backlogging that may be appropriate in certain service contexts such as airline and hotel reservations, but are not very relevant to the retail store merchandise setting that we have focused on.

Two other papers in this line of research are Gallego and Van Ryzin (1994b) and Feng and Gallego (1994). In the former, the authors attempt to extend the analysis in Gallego and Van Ryzin (1994a) to a multi-product case where all the products share a common set of resources (raw materials) and the demand for a product is related to its own price as well as the prices of the other products. A deterministic version of the resulting model is solved under the assumption that
the Poisson arrival processes for all products are homogeneous, and this solution is used to construct heuristics for the original problem. Feng and Gallego (1994) study the problem of determining the single point in time over the planning horizon at which the price of a product should be modified (raised or lowered). They present a continuous-time optimal stochastic control formulation for the problem with demand being modeled in the Poisson arrival process manner described above, and the initial and revised prices are taken as fixed. While the authors present some general insights into the nature of the optimal policy, their model is very restricted from a practical standpoint for the kinds of situations that are the focus of our study.

2.4 Economic and Empirical Research

Lazear (1986) developed a simple two-period model for the pricing of a perishable commodity under uncertain demand. The author was able to derive a number of interesting conclusions through a fairly simple retail model. A key conclusion in this paper is that high initial uncertainty about demand will result in a more steep pricing policy (higher price in first period followed by a higher expected markdown for second period) than low demand uncertainty.

Lazear also showed how the information gained by the retailer through observing demand in one period would result in a revised demand distribution for the next period that would then determine the optimal price for this second period. This issue of demand learning was studied empirically by Sass (1988) using data on pricing behavior from residential housing markets. Among Sass' conclusions was the following: The better informed the retailer is ex ante, the tighter the prior on consumers' reservation prices and the less the learning that will occur over time. This will cause prices to be more rigid over time.

Pashigan (1988) provided an extension to Lazear's model and performed some empirical analysis to infer that the large variation in clothing prices seen in recent years was influenced by the increase in fashion variety (with the consequent increase in demand uncertainty for any one fashion product). Further empirical investigations of this model were done by Pashigan and Bowen (1991).
While the above research provides some important insights into the nature of dynamic pricing policies for fashion products and some other goods, it is not aimed at providing specific procedures or modeling aids to retail planners to support them in making pricing or other decisions.

2.5 Models of Demand Learning

The issue of dynamic demand learning has been studied by many researchers in inventory control contexts. A number of them involve the application of a Bayesian approach to incorporate demand learning in a periodic review stochastic inventory model. We describe this Bayesian learning approach - in the inventory control context - below and then briefly discuss some specific work in this area. We then cite some addition work on modeling demand learning that is not based on the Bayesian approach.

The periodic review stochastic inventory problem with demand learning is typically formulated as a dynamic program. Demand in any period is assumed to be a random variable with a distribution function that has an unknown parameter, say $\alpha$. The planner is expected to specify a prior distribution on $\alpha$. After each time period, the observed demand is used to revise the prior into a posterior distribution through the application of Bayes' rule. The prior is required to be conjugate to the demand distribution, thus allowing the posterior to be calculated in a simple manner. A parameter that summarizes the posterior distribution is included in the state space of the dynamic program so that the updated demand information is passed on from one period to the next. A second variable in the state space is the current inventory level.

Some researchers (Scarf, 1959, 1960, Azoury, 1979, 1985) have identified conditions under which the Bayesian model can be reformulated as a dynamic program with a single state variable that incorporates information both on the current inventory level as well as on the observed demand from past periods.

Murray and Silver (1966) present an inventory model for style goods where the number of potential buyers in each period is known. Each of these buyers purchases the product with
probability $\rho$, and this probability is unknown at the start of the season. The authors assume a beta prior on $\rho$, which is revised over time as demand is observed. The price of the product is assumed to be fixed throughout the season. The paper contains an interesting discussion of a ‘state aggregation’ procedure to help decrease the computational requirements of the resulting model.

Popovic (1987) presents an inventory control model with demand learning where the demand distribution is non-stationary. He uses a Bayesian approach with demand modeled as a Poisson distribution whose arrival rate is unknown and changing over time. The arrival rate $\lambda_t$ at time $t$ is modeled as $\lambda_t = \lambda(k+1)t^k$, for some known positive integer $k$ and some unknown $\lambda$. By assuming a gamma prior distribution on $\lambda$, the author shows how the Bayesian approach could be applied to the problem.

An alternative Bayesian learning model has been proposed by Chang and Fyffe (1971). The authors consider demand over a season consisting of $T$ periods. Aggregate demand for the season is represented through a random variable, $D$, which is assumed to be normally distributed with an unknown mean $\mu$ and variance $\nu$. The demand in any period $t$ is assumed to be of the form $d_t = D s_t + X_t$, where $\nu_t$ is the estimated proportion of the aggregate season demand that falls in period $t$ (so that $\sum_{t=1}^{T} s_t = 1$), and $X_t$ is a noise term that is independent of $D$ and the other $X_t$'s and is distributed normally with zero mean and variance $\nu_t$. Both $s_t$ and $\nu_t$ are constants that may be computed using historical data. As demand is observed, the distribution of $D$ is updated through an application of Bayes' rule, leading to revised forecasts for sales in each time period through the above equation. This model of demand has been used by Crowston, Hausman and Kampe (1973) in a multistage production planning context.

Iyer and Eppen (1995a, 1995b) describe a methodology in which demand in each period is assumed to be based on one of a set of 'pure demand processes'. The actual underlying demand process is unknown to the user, and a (discrete) prior distribution is defined over the set of demand processes. This prior is updated after each demand observation using Bayes rule. The
demand processes are required to satisfy certain conditions that allow the demand learning process to be embedded in a dynamic optimization model in an analytically convenient fashion. Among the demand processes that meet these conditions are the normal, negative binomial and Poisson.

Hausman (1969) shows that under certain circumstances, the ratios of successive demand forecasts can, as a first approximation, be treated as independent random variables distributed according to the lognormal distribution. Based on his observations, some researchers have studied the problem of demand forecasting for style goods by making a markovian assumption on demand under which the demand $D_t$ in period $t$ is related to past demand only through the demand $D_{t-1}$ in the last period. Hausman and Peterson (1972) use this approach in studying a multiproduct problem.

All the models discussed above suffer from one key limitation that prevents us from utilizing their approaches in our setting - they assume that there is no price change affecting demand over time. Incorporating this non-stationary price behavior in a Bayesian learning framework is a key contribution of our research in Chapter 4 to the literature on demand learning.

Smith et al (1994) present a two-stage sales forecasting methodology that models demand as a function of price and a combination of other marketing and environmental factors. The parameters of the model are first estimated through regression analysis using historical data. In stage 2, the key parameters of this model are updated through a discounted least squares procedure.
CHAPTER 3

THE SINGLE STORE PROBLEM: STRUCTURAL PROPERTIES AND MODEL SOLUTION

3.1 Introduction

As discussed in Chapter 1, pricing policies for seasonal products are often governed by the objective of maximizing revenues from a fixed stock of product within a limited sale period. We define the seasonal product pricing problem to be that of determining the dynamic optimal pricing policy for retail products in contexts that include the two factors identified above: a fixed stock of product and a limited sale period. Examples of applicable contexts abound: fashion apparel, seasonal sports and home equipment, vacation cruises, and airline, hotel and car reservations. In this chapter, we formulate the seasonal product pricing problem analytically, discuss some of its structural properties, and present a heuristic and upper bound for the problem.

In certain seasonal product contexts, the retailer may be in a position to place one or more reorders for the product during the season. While our focus in this paper is mainly on situations where no reorders are possible (which is often the case with seasonal products), we will comment in Section 3.3.4 on the extension of our work to contexts where reordering is possible.

Another factor that is often present in such seasonal product environments is a high level of demand uncertainty. Goods that have a high fashion content have highly unpredictable demand, and even for non-fashion seasonal items such as air conditioners, it is often not feasible to predict demand accurately at the beginning of the season. The product’s price therefore needs to be adjusted dynamically to incorporate new demand information and to balance supply with demand over the sale period. We will consider both deterministic and stochastic demand conditions in our analysis of the problem.
Some researchers (Gallego and Van Ryzin, 1994, Bitran and Mondschein, 1995) have recently developed models of the seasonal product pricing problem and devised solution approaches for these models. In this chapter, we add to this emerging body of research by developing a heuristic and upper bounding scheme for the problem, and extending the pricing model to allow for a single reorder to be placed during the season. In addition, we attempt to provide some general insights into the nature of this pricing problem through a series of structural conjectures, counterexamples and sufficient conditions, touching upon issues that have either been ignored by past researchers or been studied under very restrictive conditions.

We provide an overview of this chapter in the remainder of this section.

The seasonal product pricing problem will be formulated in formal analytical terms in Section 3.2.1. It may be described as follows: A retailer has a certain initial level of inventory of a product in stock at the beginning of a season, which consists of a certain fixed number of periods. Demand in each period is either a deterministic function of price (the deterministic demand case) or a random variable whose distribution depends on price (the probabilistic demand case). It may also depend on the period in question - that is, the demand distribution (or demand function) at any fixed price \( p \) may be non-stationary. The retailer initially decides on the price to charge for the product in period 1. After each period, the retailer checks the current inventory level and determines the price to be charged in the next period. The pricing problem is to determine a pricing strategy that maximizes the expected revenues over the sales season. We do not consider inventory holding costs or discount rates in this problem. These additional factors could be included without affecting the key results in this chapter, and we have chosen to leave them out in order to keep the exposition simple. We assume that demand is a smooth and decreasing function of price. (Throughout this chapter, we will use the term 'smooth' to denote a function that has a continuous second derivative with respect to the variable being considered). While demand being decreasing in price has a clear meaning in the deterministic demand context, we will use a stochastic dominance-based extension of this condition for the probabilistic case, and this will be defined in Section 3.2.1.
In framing the pricing problem in the above fashion, we have chosen to focus on certain key factors in the retailer’s environment while ignoring other issues. We have chosen a fairly basic version of the problem for our analysis since it allows us to study the interactions of a few fundamental elements of the problem - such as the price, the level of inventory, and the level of demand - and to derive a number of interesting analytical and economic insights about them.

Although the literature has examined both continuous-time and discrete-time models, we have focused our study on the discrete time case for the following key reason: the continuous-time model, while representing an interesting idealized case, provides pricing strategies that are not practical, since price revisions are allowed at all points in time. In addition, properties that hold in the continuous time case do not directly transfer over to the more realistic discrete time case. For instance, the sales limit issue we consider in Conjecture 4 (discussed below) will not arise in a continuous time model for in that context a retailer would never benefit from imposing such a limit.

We will investigate the following conjectures related to the above problem. Conjectures 1-3 below are stated with reference to the following quantities in any period $t$: the inventory level at the beginning of period $t$, the optimal price for period $t$, the level of demand in period $t$, and the marginal revenue in period $t$. (The level of demand is meant to capture factors that determine the demand for the product, such as the size of the market, the availability of competitive products, and the attractiveness of the product. By an increase in the level of demand, we mean that the demand at each price point has shifted upward. This concept will be defined more rigorously in Section 3.2.1. The marginal revenue at any given inventory level $I$ is defined as the incremental expected revenue that the retailer will receive from having an additional unit of inventory.)

Conjecture 1: The optimal price is non-increasing in the level of inventory.

Conjecture 2: The optimal price is non-decreasing in the level of demand.

Conjecture 3: The marginal revenue is non-increasing in the level of beginning inventory.

(This is equivalent to stating that the optimal revenue function is concave in the initial level of inventory.)
Conjecture 4: The retailer does not benefit from reserving some stock of the product for future periods (and thereby limiting sales in early periods) if the level of demand is decreasing over time.

As we discuss in Section 3.2.1, these conjectures are quite intuitive. However, we will show through a series of counterexamples in Section 3.2.2 that they are not true in general. Thus, we need to impose further conditions on the problem in order to guarantee these results. We derive a set of sufficient conditions in Section 3.2.3 and present examples of commonly used demand functions and distributions that satisfy these conditions for certain cases of the seasonal product pricing problems. We also highlight certain open analytical issues for further research.

Some researchers (e.g., Thowsen, 1975, Bitran and Mondschein, 1993, Gallego and Van Ryzin, 1995) have studied the properties in Conjectures 1 and 3 in the past for certain special cases of the seasonal product pricing problem we study in this chapter. Each of these special cases arise from assuming a specific form for the price-dependent demand distribution. In contrast, in Section 3.2 we study these conjectures under very general demand conditions. The main contributions of Section 3.2 are the identification of certain structural properties of the seasonal product pricing problem, the demonstration through counterexamples that these properties do not hold in general, and the derivation of conditions under which the properties can in fact be guaranteed to hold. We also attempt to provide some interesting economic insights on optimal retail pricing behavior under different demand and inventory conditions.

We begin Section 3.3 by showing how the seasonal product pricing problem under deterministic demand can be formulated as a non-linear program and that, under Condition 3.2.3.1, this is a concave maximization problem that can be solved efficiently through a binary search procedure. We also prove that the deterministic demand problem that is based on using the expected demand at each price provides an upper bound to the underlying probabilistic demand problem. Finally, we propose a new heuristic for the seasonal product pricing problem under stochastic demand that is based on solving this expected value problem repeatedly. The heuristic provides a simple, easy-to-implement pricing scheme to retail managers, and acts as a building block for heuristics
that we will develop for the more complicated pricing problems considered in Chapters 4 and 5. In Section 3.3.3, we describe computational tests which indicate that the heuristic and the upper bound perform very well in practice. The results also suggest that the optimal fixed price heuristic proposed by Gallego and Van Ryzin (1994) can perform poorly under conditions of non-stationary demand.

3.2 Structural Properties

3.2.1 ANALYTICAL FORMULATIONS

In this section, we formulate the seasonal product pricing problem and the conjectures mentioned in the introduction in analytical terms. We assume that the retailer has T periods in which to sell the product, and that the salvage value for the product at the end of T periods is zero. The results presented in this chapter remain valid for the case of a linear salvage value function also.

3.2.1.1 DEMAND

In the deterministic demand case, demand at price p in period t is a deterministic quantity, denoted by \( D_t(p) \). The following conditions are assumed regarding \( D_t(p) \):

**Condition 3.2.1.1:** \( D_t(\cdot) \) is a decreasing function of p for all t, i.e.,

\[ p_1 > p_2 \text{ implies } D_t(p_1) < D_t(p_2) \text{ for all } p_1, p_2 > 0 \]

**Condition 3.2.1.2:** \( D_t(p) \) is a smooth function of p for all t.

Note that Condition 3.2.1.1 implies that \( D_t(\cdot) \) is invertible.

In the probabilistic demand case, demand at price p in period t is a random variable, denoted by \( \mathcal{S}_t(p) \). We assume that \( \mathcal{S}_t(p) \) is a continuous random variable for all t, and denote its distribution and density functions by \( F_t(\cdot|p) \) and \( f_t(\cdot|p) \) respectively. For both the deterministic and probabilistic demand cases, we assume that demand is continuous and that the inventory comes in continuous

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amounts. With the exception of Theorem 3.2.3.5, the results of this chapter can, with appropriate reformulations of the conditions discussed, be extended to the case of probabilistic demand with discrete demand distributions and discrete inventory levels.

The following conditions will be assumed regarding $\delta_t(p)$. Throughout this chapter, we will use the term stochastic dominance to refer to first order stochastic dominance.

**Condition 3.2.1.3:** For any $t$, and any prices $p_1 > p_2 > 0$, $\delta_t(p_1)$ is stochastically dominated by $\delta_t(p_2)$, i.e., $F_t(x|p_1) \geq F_t(x|p_2)$ for all $x \geq 0$

**Condition 3.2.1.4:** $F_t(x|p)$ is a smooth function of $x$ and $p$ for all $t$

Condition 3.2.1.3 is a way of translating the inverse demand-price relationship of the deterministic case (Condition 3.2.1.1) to the probabilistic case. It may be described in words as follows: *If, in any period, the retailer raises the price, then the probability that the demand in that period will exceed some given level $X$ will ‘decrease’, and this holds true for all levels $X$. (We actually mean ‘not increase’ instead of ‘decrease’ in the preceding statement. Throughout this chapter, we will use the terms ‘decreasing’ for ‘non-increasing’, ‘increasing’ for ‘non-decreasing’, etc., when we are focusing on intuitive explanations, and will indicate this abuse of language by using quotation marks. The formal definitions, conjectures and theorems in the chapter, however, will be stated using the correct terminology.)*

Conditions 3.2.1.2 and 3.2.1.4 are assumed purely for analytical convenience, and are common assumptions in economic models of demand.

Note that the above conditions imply the following:

$$\frac{\partial D_t(p)}{\partial p} \leq 0 \quad \text{for all } t, p \quad \text{(for the deterministic case)}$$

$$\frac{\partial F_t(x|p)}{\partial p} \geq 0 \quad \text{for all } x > 0, t, p \quad \text{(for the probabilistic case)}$$
3.2.1.2 PRICING MODEL

The seasonal product pricing problem for the probabilistic demand case is a stochastic dynamic program as described below. We model the time index $t$ as increasing with time (so that $t = 1$ is the first period and $t = T$ is the last) and define the following functions:

$V_t(I) = \text{The maximum expected revenue from period } t \text{ onwards, given that the retailer has } I \text{ units of product in stock at the beginning of period } t.$

$W_t(I, p) = \text{The maximum expected revenue from period } t \text{ onwards, given that the retailer has } I \text{ units of product in stock at the beginning of period } t \text{ and charges a price } p \text{ in period } t.$

---

**Seasonal Product Pricing Problem**

*Backward recursion equation*

$V_t(I) = \max_{p, \delta_t(p), I} W_t(I, p), \text{ where}$

$W_t(I, p) = E[p \min\{\delta_t(p), I\} + V_{t+1}(I - \min\{\delta_t(p), I\})]$

*Boundary conditions*

$V_T(I) = 0 \text{ for all } I$

$V_t(0) = 0 \text{ for all } t$

---

One special case of this model, mentioned in Bitran and Mondschein (1993, 1995), is where $\delta_t(p)$ is a Poisson random variable. A continuous time version of the Poisson demand case has also been analyzed by Gallego and Van Ryzin (1994) and Bitran and Mondschein (1994). We will consider this Poisson demand model later when we present examples of demand distributions.
The seasonal product pricing problem under deterministic demand can be expressed as a special case of the above dynamic program. We replace the random variable \( \beta_t(p) \) by \( D_t(p) \) and the backward recursion equation involving \( W_t(I,p) \) now reads as follows:

\[
W_t(I,p) = p \min\{D_t(p), I\} + V_{t+1}(I- \min\{D_t(p), I\})
\]

### 3.2.1.3 MODEL WITH SALES LIMIT

We describe below an extension to the dynamic programming model formulated above that considers the issue of a sales limit as expressed in Conjecture 4. In this model, the retailer is allowed to limit the sales in any period to an amount less than the inventory available at the beginning of that period. The retailer therefore has to make two decisions at the beginning of each period: what price to charge in that period, and what limit to impose on sales in that period.

We define the two functions \( V_t(I) \) and \( W_t(I,p,S) \) as follows:

\[
V_t(I) = \begin{cases} 
\text{The maximum expected revenue from period } t \text{ onwards, given that the retailer has } I \text{ units of product in stock at the beginning of period } t. 
\end{cases}
\]

\[
W_t(I,p,S) = \begin{cases} 
\text{The maximum expected revenue from period } t \text{ onwards, given that the retailer has } I \text{ units of product in stock at the beginning of period } t, \text{ charges a price } p \text{ in period } t, \text{ and imposes a sales limit of } S \text{ on period } t. \text{ It is assumed that } S \leq I.
\end{cases}
\]

---

**Seasonal Product Pricing Problem with Sales Limit**

**Backward recursion equation**

\[
V_t(I) = \max_{p \geq 0, S \leq 0} W_t(I,p,S), \quad \text{where}
\]

\[
W_t(I,p,S) = \mathbb{E}[p \min\{\beta_t(p), S\} + V_{t+1}(I- \min\{\beta_t(p), S\})]
\]

**Boundary conditions**

\[
V_T(I) = 0 \quad \text{for all } I
\]

\[
V_t(0) = 0 \quad \text{for all } t
\]
3.2.1.4 LEVEL OF DEMAND

We describe below how the notion of a level of demand is expressed in our analytical framework. Consider first the deterministic demand case. We model the demand function $D_t(p)$ as a function of a non-negative parameter, $\alpha$, which we call the demand parameter. When relevant, we will therefore write the demand function as $D_t(p|\alpha)$. We now wish to define the following property: *The level of demand at each price is 'increasing' in $\alpha$.* This is done by assuming the following:

**Condition 3.2.1.5:** $\alpha_1 > \alpha_2 > 0$ implies that $D_t(p|\alpha_1) \geq D_t(p|\alpha_2)$ for all $t, p$.

Thus, a higher value for $\alpha$ implies a higher (strictly, at least as high) level of demand at each price $p$. For instance, suppose the demand function is linear:

$$D_t(p) = A_t - b_t p, \quad p \leq A_t/b_t$$

It is straightforward to observe that a higher value for $A_t$ should lead to a higher level of demand at each price, and so the above condition is satisfied if we take $\alpha = A_t$. For analytical convenience, we will assume the following condition:

**Condition 3.2.1.6:** $D_t(p|\alpha)$ is a smooth function of $\alpha$ for all $t, p$.

The extension of this idea to the probabilistic case is analogous to the extension of Condition 3.2.1.1 to Condition 3.2.1.3. We again assume that the random variable $\mathcal{A}_t(p)$ is associated with a demand parameter $\alpha$. Where relevant, we will denote this random variable by $\mathcal{A}_t(p|\alpha)$. Analogous to Conditions 3.2.1.5 and 3.2.1.6, we will assume the following:

**Condition 3.2.1.7:** $\alpha_1 > \alpha_2 > 0$ implies that $\mathcal{A}_t(p|\alpha_1)$ stochastically dominates $\mathcal{A}_t(p|\alpha_2)$ for all $t, p$, i.e., $F[x|p, \alpha_1] \leq F[x|p, \alpha_2]$ for all $x \geq 0$.

**Condition 3.2.1.8:** $F_t(x|p, \alpha)$ and $f_t(x|p, \alpha)$ are smooth functions of $x$, $p$ and $\alpha$ for all $t$. 

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The use of $\alpha$ to parametrically capture the notion of the *level of demand* will be further illustrated in the examples that follow below.

### 3.2.1.5 CHANGE IN LEVEL OF DEMAND OVER TIME

We model increasing or decreasing trends in demand over time in the following manner. Consider the deterministic demand case where we have $T$ demand functions $\{D_t(p): t = 1, \ldots, T\}$ over the $T$ periods of the season. To model decreasing demand levels over time, we will assume the following:

**Condition 3.2.1.9:**

$$D_t(p) \geq D_{t+1}(p) \quad \text{for all } t, p.$$  

Hence, in words, demand is decreasing over time if, at all price levels $p$, the demand $D_t(p)$ is decreasing in $t$. If the demand functions $D_t(p)$ were all of the form:

$$D_t(p) = D_t(p|\alpha_t) \text{ for all } t,$$

where $\alpha_t$ is a demand parameter, then, given Condition 3.2.1.5, we would have Condition 3.2.1.9 being equivalent to the following condition:

$$\alpha_{t+1} \leq \alpha_t \text{ for all } t.$$  

This approach can be extended to the probabilistic demand case in a straightforward manner, yielding the condition below:

**Condition 3.2.1.10:** $D_t(p)$ stochastically dominates $D_{t+1}(p)$ for all $t, p$, i.e., $F_t[x|p] \leq F_{t+1}[x|p]$ for all $x \geq 0$, and all $t, p$.

As for the deterministic case, if the demand random variables $D_t(p)$ were all of the form:

$$D_t(p) = D_t(p|\alpha_t) \text{ for all } t,$$

where $\alpha_t$ is a demand parameter, then, given Condition 3.2.1.7, we would have Condition 3.2.1.10 being equivalent to the following condition:

$$\alpha_{t+1} \leq \alpha_t \text{ for all } t.$$  

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Increasing trends in demand could be modeled in a similar manner.

3.2.1.6 EXAMPLES
We provide below a number of examples of demand models (demand functions \( D(p) \) and random variables \( \delta(p) \)) that we will use in this chapter to illustrate the application of the sufficient conditions to be derived in Section 3.2.3.

**Deterministic case**

3.2.1.6.1 Linear: \( D(p) = A - bp \) where \( A > 0, b > 0 \) and \( p \geq A/b \)

The parameter \( A \) may represent the size of the customer population, while \( b \) may represent their price sensitivity to the product. Either \( A \) or \( 1/b \) could be taken as the demand parameter \( \alpha \) – in the former case, an increase in \( \alpha \) may signify an increase in the overall customer population, and in the latter case, it may signify a decrease in customers' price sensitivity to the product.

3.2.1.6.2 Weibull: \( D(p) = Ae^{-bp} \) where \( A > 0, b > 0 \) and \( a > 0 \)

Here too, \( A \) may represent the size of the customer population, while \( b \) may represent their price sensitivity to the product. Again, either \( A \) or \( 1/b \) could be taken as the demand parameter \( \alpha \). When \( a = 1 \), we get the exponential demand function.

We term this demand function as a Weibull function since it can be derived by assuming that the customer population has a Weibull distribution of reservation prices. A customer is said to have a reservation price of \( r \) if he or she will purchase the product for any price \( \leq r \). The cumulative distribution function for a Weibull is given by: \( F(r) = 1 - e^{-br} \). This allows us to derive the Weibull-based demand function as follows:

\[
D(p) = (\text{Total population of customers}) \times (\text{Fraction of customers who will purchase at price } p) = A(1-F(r)) = Ae^{-br}.
\]
Both the linear and Weibull demand functions satisfy Conditions 3.2.1.1, 3.2.1.2, and they both satisfy Conditions 3.2.1.5 and 3.2.1.6 for \( \alpha = A \) and for \( \alpha = 1/b \).

**Probabilistic case**

3.2.1.6.3 **Multiplicative Noise:** \( \delta(p) = D(p)\xi \), where \( D(p) \) is a deterministic demand function that satisfies Conditions 3.2.1.1 and 3.2.1.2, and \( \xi \) is a continuous random variable

This model satisfies Conditions 3.2.1.3 and 3.2.1.4. A special case of the multiplicative model is the ‘linear-exponential’ model where \( D(p) \) is linear and \( \xi \) follows an exponential distribution:

\[
\delta(p) = (A - bp)\xi,
\]

where \( \xi \) is an exponential random variable, and \( A > 0, b > 0, p \geq A/b \).

This model was studied by Zabel (1970) and Polatoglu (1991). In this case, also, Conditions 3.2.1.7 and 3.2.1.8 are satisfied when the demand parameter is taken to be \( A \) or \( 1/b \).

3.2.1.6.4 **Exponential-Exponential:** \( P[\delta(p) \geq x] = A e^{-bx}, \) where \( 1/b_p = e^{-bx} \), and \( A > 0, b > 0 \)

In this model, \( \delta(p) \) is an exponential random variable (scaled by a certain constant \( A \)) with a mean \( 1/b_p \) that is an exponential function of price. \( \delta(p) \) satisfies Conditions 3.2.1.3 and 3.2.1.4. We can again consider either one of \( A \) and \( 1/b \) as the demand parameter for this model, and in both cases, Conditions 3.2.1.7 and 3.2.1.8 are satisfied.

3.2.1.6.5 **Poisson-Weibull:** \( \delta(p) = \) Poisson r.v. with arrival rate \( \lambda(p) \), where

\[
\lambda(p) = A e^{-bp^*}, \quad \text{where} \quad A > 0, b > 0, a > 0.
\]

In this model, \( \delta(p) \) is a Poisson random variable with an arrival rate \( \lambda(p) \) that is a Weibull function of price. This demand model was employed by Bitran and Mondschein (1993, 1995). The case of Poisson distribution with an exponential arrival rate function (derived by setting \( a = 1 \)) was also employed by Gallego and Van Ryzin
(1995) in a continuous time context. This model satisfies Conditions 3.2.1.3 and 3.2.1.4. It also satisfies Conditions 3.2.1.7 and 3.2.1.8 for both \( \alpha = A \) and \( \alpha = 1/b \).

As discussed in Section 3.2.2, the multiplicative model is very restrictive in form. However, due in part to the analytical convenience that this model and the additive demand model offer, researchers in the past have almost exclusively focused on these demand models when analyzing structural properties of the type considered in Conjectures 1 and 3 in this chapter. The models in examples 3.2.1.5 and 3.2.1.6 do not fit into either of these forms, and they will be used to demonstrate the broader applicability of the conditions we derive in Section 3.2.3.

3.2.1.7 CONJECTURES
We now state the conjectures mentioned in Section 3.1 in formal analytical terms. We also provide some intuitive justification for these conjectures.

Consider the seasonal product pricing problem defined in Section 3.2.1.2, and suppose we are at the beginning of any period \( t \). We define \( p_d \) to be the optimal price in period \( t \) when the inventory at the beginning of period \( t \) is \( I \), and \( p_{wa} \) to be the optimal price in period \( t \) when the demand parameter in period \( t \) is \( \alpha \) (for a fixed level of period \( t \) inventory). We will assume in this chapter that this optimal price \( p_d \) (or \( p_{wa} \)) is unique. We do so to keep the exposition simple - the results in this chapter can be generalized to the case where the optimal price is not unique.

The first conjecture relates the optimal price in period \( t \) to the inventory level at the beginning of that period. It is intuitive to expect that as one increases the initial inventory level from low to high levels, the optimal initial price 'decreases' - as the retailer wants to induce the market to purchase a larger quantity - and then plateaus at some level beyond which it is not beneficial to the retailer to lower the price further. This leads to our first conjecture:

Conjecture 1: Consider the seasonal product pricing problem defined in Section 3.2.1.2 from any period \( t \) onwards. Suppose for each period \( r = t, \ldots, T \), the demand random
variable $s_t(p)$ (or, in the deterministic demand case, the demand function $D_t(p)$) satisfies Conditions 3.2.1.3 and 3.2.1.4 (Conditions 3.2.1.1 and 3.2.1.2 in the deterministic demand case). Then the optimal price $p_t$ is non-increasing in $I$.

In Conjecture 2, we wish to describe the influence of the level of demand in period $t$ on the price in that period. Suppose that the retailer revises his or her estimate of the level of demand in period $t$ upwards. Hence, at each price level, the demand in period $t$ has increased. One would intuitively expect that this should lead to an 'increase' in the optimal initial price, as expressed below:

**Conjecture 2:** Consider the seasonal product pricing problem defined in Section 3.2.1.2 from any period $t$ onwards. Suppose the demand random variable $s_t(p|\alpha)$ (or, in the deterministic demand case, the demand function $D_t(p|\alpha)$) satisfies Conditions 3.2.1.3, 3.2.1.4, 3.2.1.7 and 3.2.1.8, and for each period $r = t+1, \ldots, T$, the demand random variable $s_r(p)$ (or, in the deterministic demand case, the demand function $D_r(p)$) satisfies Conditions 3.2.1.3 and 3.2.1.4 (Conditions 3.2.1.1 and 3.2.1.2 in the deterministic demand case). Then the optimal price $p_{t\alpha}$ is non-decreasing in $\alpha$.

The marginal revenue function in period $t$ is given by $dV_t(I)/dI$. This will always be non-negative, since the retailer can choose to ignore the additional unit of inventory (also recall that we are not considering ordering costs or inventory holding costs). One may also, however, expect this marginal value to be 'decreasing' with the inventory level $I$ - as $I$ increases, the retailer would need to lower the initial price in order to sell the additional unit of product, and this would lead to lower incremental revenues from the additional unit. This leads to Conjecture 3:

**Conjecture 3:** Consider the seasonal product pricing problem defined in Section 3.2.1.2 from any period $t$ onwards. Suppose for each period $r = t, \ldots, T$, the demand random variable $s_r(p)$ (or, in the deterministic demand case, the demand function $D_r(p)$)
satisfies Conditions 3.2.1.3 and 3.2.1.4 (Conditions 3.2.1.1 and 3.2.1.2 in the deterministic demand case). Then the marginal revenue function $dV_d(I)/dI$ is non-increasing in $I$.

Finally, consider Conjecture 4. It is quite intuitive to expect that in certain cases, a retailer may benefit by reserving some of the inventory for future periods, as allowed by the model with sales limit. For instance, in the airline industry, booking limits are imposed on advance purchase fares in order to keep some inventory of seats reserved for later arrivals. In this illustrative context, the product is a seat on a particular flight, the initial inventory is the total number of seats on the flight, and the sales season is the time period in which the flight's seats are made available for sales, ending on the day the flight departs. These later arrivals typically constitute the 'high-end' segment of business travelers, while early arrivals typically constitute the 'low-end' segment of leisure travelers. Since the 'high end' travelers are willing to pay more for the 'product' (a seat on a certain flight), and since some of the inventory still needs to be sold at a lower fare to the low-end segment (as there is not enough demand by the high-end segment to sell all the seats on the aircraft), an airline benefits from making the product available at lower fares early in the season while imposing a booking limit to reserve a certain number of seats for sale in future periods. Similar 'yield management' practices are also prevalent in the hotel and car rental industries.

As illustrated by the airline example, the retailer may benefit from imposing a sales limit in the early part of the sale period for products that can be sold at higher prices later in the sale period. In contexts where the product's value decreases over the course of the season, however, one would not expect the retailer to benefit from reserving product for future periods in this manner. This decrease may come about both as a result of the product losing value to customers over time (as the useful life of the product decreases over the course of the season), as well as by the exit of high-end customers (who are willing to pay higher prices) and the entry of low-end customers (who are willing to pay only lower prices). Examples of such contexts are seasonal apparel and sportswear. This leads to the following conjecture:
Conjecture 4: Consider the probabilistic case of the seasonal product pricing problem with sales limit defined in Section 3.2.1.3. Suppose that for all \( r = 1, \ldots, T \), the demand random variables \( \delta_r(p) \) satisfy Conditions 3.2.1.3 and 3.2.1.10. Then \( W(t, p, S) \leq W(t, p, I) \) for all \( I \geq 0, P > 0 \) and \( S \leq I \). Hence, the retailer does not benefit from imposing a sales limit in period \( t \).

Note that unlike Conjectures 1-3, Conjecture 4 is only of interest in the probabilistic demand case. It is easy to see that a sales limit will never be useful in a deterministic demand context.

3.2.2 COUNTEREXAMPLES

In this subsection, we will discuss examples that contradict Conjectures 1-4. For Conjectures 1, 2 and 3, our examples are based on single period situations - that is, there is only one sale period and a single price set by the retailer. Also, for Conjectures 2 and 3, our examples are based on deterministic demand conditions. It is interesting that we can get counter-intuitive results even in the simple context of a single period and deterministic demand. By confounding our intuition, the results of this section provide us with fresh insights into the nature of the seasonal product pricing problem and motivate the search for conditions under which the conjectures may be guaranteed to hold true. This will lead us to the next section, where such conditions will be identified.

3.2.2.1 COUNTEREXAMPLE TO CONJECTURE 1

Consider the following seasonal product pricing problem. There is a single sale period \( (T = 1) \), and the demand is uncertain, with the demand random variable \( \delta(p) \) having a discrete distribution for all \( p \). The probability density function for \( \delta(p) \) is defined as follows.

For any price \( p \), the random variable \( \delta(p) \) has zero mass at all points except possibly at the points 0, 2 and 3. \( P[\delta(p) = 0] \) is given by the curve in Figure 3.2.2.1a
This curve is flat over the intervals \([0, 1] \), \([1.005, 1.4] \) and \([1.45, \infty) \). In the two intervals \((1, 1.005) \) and \((1.4, 1.45) \), \(P[\delta(p) = 0] \) is defined such that the curve is smooth and increasing in \(p \) over \([0, \infty) \). (We do not define the curve analytically in these two intervals since the particular values that \(P[\delta(p) = 0] \) takes in these intervals is not of consequence to our analysis.)

\[
P[\delta(p) = 2] \quad \text{and} \quad P[\delta(p) = 3] \quad \text{are given by:}
\]

\[
P[\delta(p) = 2] = \begin{cases} 
0.5 - P[\delta(p) = 0] & \text{for } p \in [0, 1.4] \\
0 & \text{for } p \in (1.4, \infty)
\end{cases}
\]

\[
P[\delta(p) = 3] = \begin{cases} 
0.5 & \text{for } p \in [0, 1.4] \\
1 - P[\delta(p) = 0] & \text{for } p \in (1.4, \infty)
\end{cases}
\]

\(P[\delta(p) = 2] \) and \(P[\delta(p) = 3] \) have been defined so that the sum of \(P[\delta(p) = 0], P[\delta(p) = 2], P[\delta(p) = 3] \) is 1 for all \(p \), as desired.

We note the following additional characteristics of this family of probability mass functions:

- At each price \(p \), the demand distribution is discrete, and \(P[\delta(p) = n] \) is a smooth function of \(p \) for all \(n \). Thus, \(\delta(p) \) satisfies (the discrete distribution analog of) Condition 3.2.1.3.
• For $p > q > 0$, $\beta(p)$ is stochastically dominated by $\beta(q)$. Thus, $\beta(p)$ satisfies (the discrete distribution analog of) Condition 3.2.1.4.

The following series of lemmas prove that the optimal price at inventory level 2 is less than the optimal price at inventory level 3. Thus, we would have a case where the optimal price increases when the initial inventory is increased, contradicting Conjecture 1.

**Lemma 3.2.2.1:** For all inventory levels $I$, the optimal price $p_I$ for the above problem lies in the region $[1, 1.005) \cup [1.4, 1.45)$.

**Lemma 3.2.2.2:** The optimal price $p_2$ for the above problem when the inventory level is 2 lies in the interval $[1, 1.005)$.

**Lemma 3.2.2.3:** The optimal price $p_3$ for the above problem when the inventory level is 3 lies in the interval $[1.4, 1.45)$.

Lemmas 3.2.2.2 and 3.2.2.3 establish the desired result.

Figure 3.2.2.1b shows the expected value function $E[\beta(p)]$ for this problem. This curve has the same ‘flat-drop’ form as the demand function considered later in the counterexample to Conjecture 3.

![Figure 3.2.2.1b: Expected value function](image)

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The above illustration employed a probabilistic demand context. For the seasonal product pricing problem with deterministic demand, Conjecture 1 is true - Conditions 3.2.1.1 and 3.2.1.2 suffice in guaranteeing that the optimal initial price, in the general multiperiod case, is a non-increasing function of the initial inventory level. This result is formally stated in the next section, and proved in the Appendix.

3.2.2.2 COUNTEREXAMPLE TO CONJECTURE 2

This example is based on a single period, deterministic demand problem. The demand function $D(p|\alpha)$ at two values $\alpha_1$ and $\alpha_2$ of the demand parameter $\alpha$, with $\alpha_1 > \alpha_2 > 0$, are as shown in Figure 3.2.2.2.

![Figure 3.2.2.2: Demand functions](image)
The demand functions and the price $p_1$ are chosen such that $p_1D(p_1|\alpha_1) > p_2D(p_2|\alpha_1)$. This condition will easily be satisfied by choosing a high-enough value for the point $D(p_1|\alpha_1)$. Note that $D(p|\alpha_1) = D(p|\alpha_2)$ for all $p \geq p_2$, and that $D(p|\alpha_1) > D(p|\alpha_2)$ for $p < p_2$, and so Condition 3.2.1.5 is satisfied for $\alpha_1$ and $\alpha_2$. For $\alpha_1 < \alpha < \alpha_2$, $D(p|\alpha)$ is defined in any manner that satisfies Conditions 3.2.1.1, 3.2.1.2, 3.2.1.5 and 3.2.1.6. It is easy to see that it will be feasible to do so, and we therefore do not furnish a specific analytical form for $D(p|\alpha)$ over this interval. The initial inventory $I$ is taken to be larger than $D(p_1|\alpha_1)$.

Lemma 3.2.2.4 shows that the optimal price decreases as $\alpha$ increases from $\alpha_2$ to $\alpha_1$. Here, $p_\alpha$ stands for the optimal price when the demand parameter is $\alpha$.

**Lemma 3.2.2.4:** For the pricing problem described above, $p_{\alpha_1} < p_{\alpha_2}$.

We offer the following explanation for why Conjecture 2 is not true in general. Consider the unconstrained optimal pricing problem:

$$\max_{p \geq 0} pD(p|\alpha)$$

where we are maximizing revenues over a single period without an inventory constraint. The optimal solution $\bar{p}_\alpha$ can be shown, by the first order optimality condition, to satisfy

$$\bar{p}_\alpha = -\frac{D(\bar{p}_\alpha|\alpha)}{\left(\frac{\partial D(\bar{p}_\alpha|\alpha)}{\partial p}\right)} \quad (3.2.2.1)$$

We observe that the optimal price depends both on the level of demand, $D(p|\alpha)$, and the rate of change of demand with price, $\left(\frac{\partial D(p|\alpha)}{\partial p}\right)$ (which we loosely term as the price sensitivity). An increase in the demand parameter will effect not only the level of demand, but potentially also the
price sensitivity of demand at various points on the demand curve. The overall effect of the increase in the demand parameter \( \alpha \) on \( \bar{p}_a \) will depend on the change in both of these quantities over all price levels. Note that the example assumes there was enough inventory to meet demand at the relevant prices, and so it is not necessary to assume a constrained inventory situation to derive this behavior.

The above situation could occur in practice when a retailer experiences a surge in demand caused by the entry of a number of predominantly low-end, price sensitive customers into the market. This may be caused by recent advertising or promotional activity by the retailer that generates a disproportionately greater response from the low-end customers. This may cause an overall increase in demand at each price level, but may still make it attractive for the retailer to lower the price since he or she may be able to sell a much higher amount at a lower price than was possible earlier.

3.2.2.3 COUNTEREXAMPLE TO CONJECTURE 3
This example is also based on a single period, deterministic demand problem. The demand function \( D(p) \), shown by the curve in Figure 3.2.2.3, is defined as:

\[
D(p) = \begin{cases} 
4 & \text{if } p \leq 60 \\
2 & \text{if } 65 \leq p \leq 100
\end{cases}
\]

In the intervals \((60, 65)\) and \((100, \infty)\), \(D(p)\) is defined in a way that allows the curve to be smooth and decreasing in \(p\) in the range \((0, \infty)\). We do not define the curve analytically in these two intervals since the particular values that \(D(p)\) takes in these intervals is not of consequence to our analysis. In addition, we require that \(D(p)\) decrease very dramatically as \(p\) goes beyond \$100, thus ensuring that the optimal price is \$100 when the inventory level is 2.
Figure 3.2.2.3: Demand function

The optimal prices at initial inventory levels of $I = 2, 3, \text{ and } 4$ are given by:

\[ p_I = \begin{cases} \$100 & \text{at } I = 2 \\ \$100 & \text{at } I = 3 \\ \$60 & \text{at } I = 4 \end{cases} \]

This yields $V(I) = \begin{cases} \$200 & \text{at } I = 2 \\ \$200 & \text{at } I = 3 \\ \$240 & \text{at } I = 4 \end{cases}$.

Hence $V(3) - V(2) = 0 < 40 = V(4) - V(3)$,

and so the rate at which $V(I)$ increases has increased in going from 2 to 3, contradicting Conjecture 3.

This counterexample hinges on the sudden drop in demand after the price level of $\$60$ followed by the flat demand between $\$65$ and $\$100$. This demand pattern makes the optimal price move in a very irregular fashion as a function of the initial inventory. This in turn causes $V(I)$ to increase as a function of $I$ in an irregular manner itself, leading to the non-concave behavior shown above. Note that though we have studied the values of $V(I)$ at the integral points $I = 2, 3 \text{ and } 4$, the example is applicable to a context where $I$ is a continuous variable.
Can the above situation arise in practice? One context where this might happen is when there are, say, two distinct segments of customers in the market - a high-end segment, willing to pay a high price (say, ≥ $100), and a low-end segment, willing to pay only a much lower price (say, ≈ $60) for the product. Under tight inventory conditions, the retailer will then focus on the high-end segment and charge a high price (≥ $100) for the product. When the inventory level exceeds the level that the high-end segment can buy, the retailer may still desist from lowering the price to clear the inventory if the size of the additional inventory is small, since the price would need to be lowered substantially in order to attract the other customers (from the low-end segment), and so the additional inventory will not be of any additional value to the retailer. When the inventory level is much higher, though, the retailer will be able to make higher revenues by lowering the price to attract the low-end segment and achieving much higher sales. At this point, the additional inventory will again be of positive value to the retailer. Note that we are implicitly assuming that the retailer cannot price discriminate by charging different prices to the two different customer segments. A counter-intuitive outcome of this behavior is that the retailer would be willing to pay the supplier a higher per-unit price when the supplier provides a high level of additional stock (over and above a small initial order level) than when the supplier provides lower levels of additional stock.

3.2.2.4 COUNTEREXAMPLE TO CONJECTURE 4

Finally, we consider the sales-limit issue introduced in Conjecture 4. Consider a two period problem where the demand in each period is given as follows:

In period 1,

\[
\begin{align*}
P[D(p) = 0] &= 1 \quad \text{for all } p \in (50, \infty) \\
P[D(p) = 5] &= 1 \quad \text{for all } p \in (25, 50) \\
P[D(p) = 30] &= .5 \quad \text{for all } p \in [0, 25] \\
P[D(p) = 50] &= .5 \quad \text{for all } p \in [0, 25].
\end{align*}
\]

In period 2,

\[
\begin{align*}
P[D(p) = 0] &= 1 \quad \text{for all } p \in (40, \infty)
\end{align*}
\]
\begin{align*}
P[D(p) = 5] &= 1 \quad \text{for all } p \text{ in } (20, 40] \\
P[D(p) = 20] &= 1 \quad \text{for all } p \text{ in } [0, 20]
\end{align*}

Figure 3.2.2.4 shows the demand model for periods 1 and 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{demand-price-relationship}
\caption{Demand-price relationship in periods 1 and 2.}
\end{figure}

It is straightforward to check that the demand model represented above satisfies Conditions 3.2.1.3 and 3.2.1.10. In particular, the level of demand decreases from period 1 to period 2. Suppose we start with 50 units of initial inventory. The optimal pricing strategy under the no-sales limit case would then be to charge $25 in period 1. Then, if the period 1 demand was 50, the inventory would be exhausted and the period 1 revenue would be $1,250. If the period 1 demand was 30, 20 units would be passed on to period 2 and the period 1 revenue would be $750. In that case, the optimal period 2 price would be $20 and the period 2 revenue would be $400. Hence the optimal expected revenue under no sales limit is \(0.5(\1,150 + \1,250) = \1,200\).

In the sales limit case, the optimal period 1 strategy would be to reserve 5 units of stock for period 2 and again charge $25. In this case, the period 1 revenues would be $1,125 under the high demand scenario and $750 under the low demand scenario. In the high demand scenario, period 1 sales would be limited to 45, and 5 units would be passed on to period 2. These would then be priced at $40 in period 2, fetching revenues of $200. In the low demand scenario, 20 units would be passed on to period 2, and these would be priced at $20 in that period, fetching revenues of $400. Hence the optimal expected revenues under a sales limit would be
.5(1,325 + 1,150) = $1,237.50. This is larger than the same figure for the model with no sales limit.

The sales limit was useful in the above examples because it enabled the retailer to 'price discriminate': the retailer was able to reserve some inventory to be sold at a higher price to a price-insensitive segment in period 2 and sell the rest at a lower price to a price sensitive segment in period 1. An alternative use of the sales limit, not illustrated in the above example, may be the following. At the beginning of each period, the retailer has to commit to a price for that period. This price is based on the available level of inventory and the retailer's expectation of current and future demand. If an unexpectedly large number of customers arrive in the current period to purchase the product, the retailer will be caught by surprise, and may then regret having sold so much product at the period 1 price instead of keeping some for selling at a higher price over the subsequent periods. A sales limit can therefore provide the retailer with the capability to insure against lost future revenues under unexpectedly high early demand conditions.

In certain retail environments, the retailer cannot raise the price over time - that is, the retailer can mark down prices but cannot mark them up. In such cases, the sales limit is not necessary, and we will prove this in the next section. Another context where a sales limit is not necessary is where the retailer can revise the price in a continuous manner, and this is also further discussed in the next section.

The sales limit may play a very critical role in certain situations - notably, one where the retailer does not know how customers value the product beforehand but learns of this over the course of the season. In this situation, the retailer may benefit substantially from insuring against an underestimation in demand by capping the level of sales allowed in the initial period. By capping sales, the retailer will be in a position to sell the remaining stock at a revised price in case demand has been significantly underestimated initially. The price revision can be made using the demand learning methodology we will present in the next section.
3.2.3 SUFFICIENT CONDITIONS

The counterexamples in Section 3.2.2 have demonstrated that the conjectures in Section 3.2.1 are not true in general. We turn now to the task of identifying conditions under which they can be guaranteed to hold.

We first derive a series of sufficient conditions under which Conjectures 1-3 can be guaranteed. This is done in three stages - we first consider the case of deterministic demand, then the case of uncertain demand in a single period context, and finally, the most general case of uncertain demand in a multi-period context. We employ this incremental approach in order to highlight the different conditions required under each of these contexts, and because we have limited results at this stage for the most general case of uncertain demand in a multi-period context. In each case, we provide examples of demand models (from among those discussed in Section 3.2.1.6) that satisfy the identified conditions. (We will not provide formal proofs for these claims in this thesis). Finally, we prove that in a common retail context - where the price is not allowed to increase over time - Conjecture 4 can be also guaranteed to hold. We also comment on another context - where the price can be revised on a continuous basis - in which Conjecture 4 can be guaranteed to hold.

The proofs for the results discussed in this section are provided in the Appendix.

As in Section 3.2.2, we will use the term $p_d$ to represent the optimal price in period $t$ when the inventory at the start of period $t$ is $I$, and the term $p_w$ to represent the optimal price in period $t$ when the demand parameter for period $t$ is $\alpha$ (for a fixed level of inventory at the beginning of period $t$).

3.2.3.1 DETERMINISTIC DEMAND

The counterexample to Conjecture 1 in the previous section was based on a probabilistic demand context. Our first result states that Conjecture 1 is true for the single period deterministic demand problem.
Theorem 3.2.3.1: Consider a special case of the seasonal product pricing problem defined in Section 3.2.1.2 where there is only one time period and demand is deterministic. Suppose $D_t(p)$ satisfies Conditions 3.2.1.1 and 3.2.1.2 for all $p$. Then the optimal price $p_{it}$ is non-increasing in $I$.

The extension of this result to the multi-period case, however, requires that the value function $V_t(I)$ be concave in $I$. This links Conjecture 1 with Conjecture 3, and so we turn first to defining sufficient conditions for Conjecture 3 to be true.

Theorem 3.2.3.2: Consider the seasonal product pricing problem under deterministic demand. Suppose, for all $t$, the demand function $D_t(.)$ satisfies Conditions 3.2.1.1 and 3.2.1.2. Also, suppose it satisfies the following condition:

$$2(D_t'(p))^2 - D_t(p)D_t''(p) \geq 0 \text{ for all } p > 0 \quad (3.2.3.1)$$

Then the value function $V_t(I)$ is concave in $I$ for all time periods $t$.

Corollary 3.2.3.1: Under the conditions defined in Theorem 3.2.3.2, the optimal price $p_{it}$ is non-increasing in $I$ for all $t$.

We have used the following notation above: $D'(p) = \frac{\partial D(p)}{\partial p}$, and $D''(p) = \frac{\partial^2 D(p)}{\partial p^2}$. Condition 3.2.3.1 will hold if $D_t(.)$ is concave (though this is not necessary for Condition 3.2.3.1), since then $D_t''(p) < 0$ for all $p$. Another way to state Condition 3.2.3.1 is by requiring that the term $[p + D(p)/D'(p)]$ be increasing in $p$.

The linear demand function (Example 3.2.1.6.1) satisfies Condition 3.2.3.1 for $A > 0$, $b > 0$ and $p \geq A/b$. The Weibull demand function (Example 3.2.1.6.2) satisfies Condition 3.2.3.1 for $A > 0$, $b > 0$ and $a \geq 1$.  

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We now turn our attention to Conjecture 2. We write the demand function for period \( t \) as \( D_t(\cdot|\alpha) \). Theorem 3.2.3.3 specifies conditions under which Conjecture 2 holds for this deterministic demand case.

**Theorem 3.2.3.3:** Consider the seasonal product pricing problem under deterministic demand. For any period \( t \), let the inventory level at the beginning of period \( t \) be fixed, and let \( p_{ta} \) be the optimal period \( t \) price when the demand function for period \( t \) is \( D_t(\cdot|\alpha) \). Suppose the demand functions \( D_t(\cdot|\alpha), D_{r+1}(\cdot|\alpha) \) (\( r = t+1,..T \)) satisfy Conditions 3.2.1.1, 3.2.1.2 and 3.2.3.1. Also, suppose \( D_t(\cdot|\alpha) \) satisfies Conditions 3.2.1.5, 3.2.1.6 and the following condition:

\[
D_t(p|\alpha) \left( \frac{\partial^2 D_t(p|\alpha)}{\partial \alpha \partial p} \right) - \left( \frac{\partial D_t(p|\alpha)}{\partial p} \right) \left( \frac{\partial D_t(p|\alpha)}{\partial \alpha} \right) \geq 0 \text{ for all } p, \alpha > 0
\]

(3.2.3.2)

Then \( p_{ta} \) is non-decreasing in \( \alpha \).

A different version of Conjecture 2 could be one where the retailer revises the demand information simultaneously for all the periods, i.e., the demand functions \( D_r(\cdot) \) (\( r = t,..T \)) are all functions of the same demand parameter \( \alpha \), and the change in \( \alpha \) affects them all simultaneously. This version of the conjecture is not considered in this thesis.

The linear demand function (Example 3.2.1.6.1) satisfies Condition 3.2.3.2 for \( A > 0, b > 0 \) and \( p \geq A/b \), for the case of \( \alpha = A \) as well as for \( \alpha = 1/b \). The Weibull demand function (Example 3.2.1.6.2) satisfies Condition 3.2.3.2 for \( A > 0, b > 0 \) and \( a \geq 1 \), for the case of \( \alpha = A \) as well as for \( \alpha = 1/b \).

It is worthwhile to pause at this stage and gain some intuition about Conditions 3.2.3.1 and 3.2.3.2. Since the demand function \( D_t(\cdot) \) is invertible, we can write the revenue function \( pD_t(p) \) as a function of the demand \( d = D_t(p) \) as follows:

\[
R_t(d) = D_t^{-1}(d).d
\]
One can show that Condition 3.2.3.1 is equivalent to requiring that $R_0(d)$ be concave in $d$. This is formally shown in the proof of Theorem 3.2.3.2 in the Appendix. A concave revenue function $R_0(d)$ is unimodal in demand, and this unimodality prevents the kind of behavior we observed in the value function $V(I)$ in Counterexample 3.

Condition 3.2.3.2 can be connected to Equation 3.2.2.1. It is easy to see that Condition 3.2.3.2 is equivalent to requiring that the derivative (with respect to $\alpha$) of the right hand side of the equation be non-negative.

### 3.2.3.2 UNCERTAIN DEMAND, SINGLE PERIOD

As demonstrated by the first counterexample in the previous section, Conjecture 1 is not true in the case of uncertain demand. Theorem 3.2.3.4 provides the additional conditions required to guarantee this result.

**Theorem 3.2.3.4:** Consider the single period seasonal product pricing problem under probabilistic demand. Suppose $\mathcal{A}(p)$ satisfies Conditions 3.2.1.3 and 3.2.1.4 for all $p$, and, additionally, it satisfies the following condition for all $p$ and $I$:

\[
(1 - F_1(I|p)) \int_0^1 F'_1(x|p) dx - F'_1(I|p) \int_0^I (1 - F_1(x|p)) dx \leq 0 \tag{3.2.3.3}
\]

Then the optimal price $p_{\Pi}$ is non-increasing in $I$.

We have used the following notation above: $F'(x|p) = \frac{\partial F(x|p)}{\partial p}$, and $F''(x|p) = \frac{\partial^2 F(x|p)}{\partial p^2}$.

Example of models for which Condition 3.2.3.3 is satisfied are:

- Multiplicative demand model (Example 3.2.1.6.3) when $\gamma$ follows a uniform or exponential distribution
- Exponential-exponential model (Example 3.2.1.6.4)
Poisson-Weibull model (Example 3.2.1.6.5)

Except for the case of the Poisson-Weibull demand model, we have analytical proofs that the demand models discussed satisfy the relevant conditions in this section as claimed. For the Poisson-Weibull case, we have checked the relevant conditions by running a set of numerical tests to investigate this issue. We used a grid of points ranging over different values of \( p, I, \) and the demand model parameters \( A, a \) and \( b \) (with all these variables non-negative, and with \( a \geq 1 \)). All the points checked satisfied the relevant conditions.

Some comments about the nature of Condition 3.2.3.3 are in order. First consider the following lemma.

**Lemma 3.2.3.1:** For any period \( t \),
\[
E[\min\{I, \delta_t(p)\}] = \int_0^1 (1 - F_t(x|p))dx.
\]

(We suppress the time period subscript 1 in the discussion below.)

Lemma 3.2.3.1 shows that for the single period model mentioned in Theorem 3.2.3.4:

Expected sales
\[
= \int_0^1 (1 - F(x|p))dx
\]

Also,

Probability of having a shortage = Probability that demand exceeds inventory
\[
= 1 - F(I|p)
\]

As price is increased, both of the above quantities will decrease. Condition 3.2.3.3 can be shown to be equivalent to requiring that

\[
\frac{\partial}{\partial p} \left( \frac{\int_0^1 (1 - F(x|p))dx}{(1 - F(I|p))} \right) \geq 0,
\]

which implies that *as price increases, the consequent decrease in expected sales is at a slower rate than the decrease in the probability of having a shortage.*

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The additional condition required for Conjecture 2 to hold is similar to Condition 3.2.3.3, as described below.

**Theorem 3.2.3.5:**  Consider the single-period seasonal product pricing problem under probabilistic demand. For a fixed level \( l \) of initial inventory, suppose \( \mathcal{A}(p) \) satisfies Conditions 3.2.1.1, 3.2.1.2, 3.2.1.5, and 3.2.1.6 for all \( p, \alpha \) and, additionally, also satisfies the following condition for all \( p, \alpha \):

\[
- \left( \int_0^l (1 - F_1(x|p, \alpha)) \, dx \right) \int_0^l \left( \frac{\partial F_1(x|p, \alpha)}{\partial \alpha} \right) \, dx \geq 0
\]

(3.2.3.4)

Then \( p_{1a} \) is non-decreasing in \( \alpha \).

Example of models for which Condition 3.2.3.4 is satisfied are:

- Exponential-exponential model (Example 3.2.1.6.4)
- Poisson-Weibull model (Example 3.2.1.6.5)

For the Poisson-Weibull demand model, we tested Condition 3.2.3.4 using \( \alpha = 1/b \). For the other model cited above, both \( \alpha = A \) and \( \alpha = 1/b \) were tested and the demand models were found to meet Condition 3.2.3.4 in both cases.

Condition 3.2.3.4 is similar to Condition 3.2.3.2 for the deterministic demand case, with the expected demand \( \int_0^l (1 - F_1(x|p, \alpha)) \, dx \) replacing the demand \( D_1(p|\alpha) \).

Conjecture 3 requires a fairly complicated condition defined in the following theorem.

**Theorem 3.2.3.6:**  Consider the single period seasonal product pricing problem under probabilistic demand. Suppose \( \mathcal{A}(p) \) satisfies Conditions 3.2.1.3 and 3.2.1.4 for all \( p \), and, additionally, it satisfies the following condition for all \( p \) and \( l \):

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\[ -\left( \left( 1 - F_1(l|x|p_1) \right)^\frac{1}{0} F'_1(x|p_1)dx - F'_1(l|x|p_1) \right) \left( 1 - F_1(x|p_1) \right)dx \right)^2 + \left[ 2 \left( \int_0^1 F'_1(x|p_1)dx \right)^2 + \int_0^1 F''_1(x|p_1)dx \int_0^1 \left( 1 - F_1(x|p_1) \right)dx \int_0^1 \left( 1 - F_1(x|p_1) \right)dx f_1(l|x|p_1) \right] \geq 0 \]

Then \( V_1(l) \) is concave in \( l \).

An example of a demand model for which Condition 3.2.3.5 is satisfied is the exponential-exponential model (Example 3.2.1.6.4)

3.2.3.3 UNCERTAIN DEMAND, MULTIPLE PERIODS

We now examine the most general case of the season product pricing problem - with uncertain demand and multiple periods. In the discussion below, we show how Theorems 3.2.3.4 and 3.2.3.5 can be extended to the multiple period context. Both these results will assume that the value function \( V_t(l) \) is concave, which is what Conjecture 3 asserts. (It is fairly easy to construct examples where the conjectures are violated when the value function \( V_t(l) \) is not concave in \( l \), even though \( V_t(l) \) may be increasing in \( l \).) We do not, at present, have a satisfactory set of sufficient conditions that guarantee Conjecture 3 in this multi-period context. However, as we will discuss later, we have strong computational evidence that \( V_t(l) \) is in fact concave (for all \( t \)) when the random variable \( \sigma(p) \) is of the Poisson-Weibull form.

Theorem 3.2.3.7: Consider the seasonal product pricing problem under probabilistic demand. Consider any period \( t \). Suppose \( V_{t+1}(l) \) is concave in \( l \). Further, suppose \( \alpha(p) \) satisfies Conditions 3.2.1.3, 3.2.1.4, and 3.2.3.3, and, in addition, satisfies the following condition for all \( p \):

\[
\frac{\frac{\partial^2 F_t(J|p)}{\partial p^2} - \frac{\partial F_t(J|p)}{\partial p}}{\frac{\partial F_t(J|p)}{\partial p}} \text{ is decreasing in } J \text{ for all } J \quad (3.2.3.6)
\]

Then \( p_t \) is non-increasing in \( l \).
Example of models for which Condition 3.2.3.6 is satisfied are:

- Multiplicative demand model (Example 3.2.1.6.3) when \( \lambda \) follows an exponential distribution
- Exponential-exponential model (Example 3.2.1.6.4)
- Poisson-Weibull model (Example 3.2.1.6.5)

**Theorem 3.2.3.8:** Consider the seasonal product pricing problem under probabilistic demand. Consider any period \( t \) and a fixed level \( I \) of inventory at the beginning of period \( t \). Suppose \( V_{t+1}(J) \) is concave in \( J \). Assume further that \( \mathcal{A}(p) \) satisfies Conditions 3.2.1.3, 3.2.1.4, 3.2.1.7, 3.2.1.8 and 3.2.3.4 for all \( p, \alpha \) and, in addition, satisfies the following condition for all \( p, \alpha \):

\[
\int_0^I \left( \frac{\partial^2 F(x|p,\alpha)}{\partial \alpha \partial p} \right) dx \quad \text{is increasing in } J \text{ for all } J. \quad (3.2.3.7)
\]

Then \( p_{t\alpha} \) is non-decreasing in \( \alpha \).

Example of models for which Condition 3.2.3.7 is satisfied are:

- Exponential-exponential model (Example 3.2.1.6.4)
- Poisson-Weibull model (Example 3.2.1.6.5)

Note that Conditions 3.2.3.6 and 3.2.3.7 are similar in the following sense. The term in Condition 3.2.3.6 can be written as

\[
\frac{\partial}{\partial J} \left( \int_0^I \frac{\partial F(x|p)}{\partial p} dx \right),
\]

while the term in Condition 3.2.3.7 can be written as

\[
\frac{\partial}{\partial \alpha} \left( \int_0^I \frac{\partial F(x|p,\alpha)}{\partial \alpha} dx \right).
\]

We have check for the concavity of \( V_t(I) \) for the seasonal product pricing problem under the Poisson-Weibull demand model by making a number of runs of the stochastic dynamic program.
for this problem. (The backward recursion solution approach for this dynamic program has been described in Bitran and Mondschein, 1993.) These runs have involved a range of settings for the initial inventory, the number of time periods, and the demand model parameters $A$, $b$ and $a$. In all of these computational tests, the function $V_t(I)$ has been found to be concave in $I$ for all period $t = 1, \ldots, T$. We therefore believe there is numerical evidence that Conjecture 3 is indeed true for the multi-period, Poisson-Weibull demand model. As we have shown above, Conjecture 3 implies Conjecture 1 and 2, and thus the computational evidence points towards these being true as well for this demand model.

3.2.3.4 SALES LIMIT
We now turn to Conjecture 4. We show below that the retailer will in fact not benefit from limiting sales in the early periods if he or she follows a non-increasing price policy - that is, if price is not allowed to increase over time. This situation is commonly observed in many retail settings, such as department stores, and may be motivated, for example, by concern about customers' negative reaction to price increases. We also comment briefly on another context in which a sales limit is not necessary - when the retailer is allowed to adjust the price on a continuous basis.

Retailers may often undertake promotional discounting to promote sales or awareness of a product, or to stimulate higher store traffic. In such cases, prices are temporarily reduced, and then raised to their original levels after the promotional period. Our discussion focuses on permanent markups or markdowns, and so the increase in price caused by the elimination of the temporary, promotional price, is not considered here.

Consider the dynamic programming model of the seasonal product pricing problem under sales limit, formulated in Section 3.2.1.3. We modify the model below to impose a non-increasing price constraint in every period. This is done by adding an additional variable, price, to the state space of the dynamic program. The value function is now defined as:
\[ V_t(I, P) = \text{The maximum expected revenue from period } t \text{ onwards, given that the retailer has } I \text{ units of product in stock at the beginning of period } t \text{ and the price charged in period } t-1 \text{ was } P. \]

The dynamic program can then be formally defined as:

---

**Seasonal Product Pricing Problem with Sales Limit and Non-Increasing Prices**

**Backward recursion equation**

\[
V_t(I, P) = \max_{P_{t+1} \geq 0, \sum_{S \geq 0} W_t(I, p, S), \text{ where}} \ E[p \ min\{D_t(p), S\} + V_{t+1}(I - \ min\{D_t(p), S\}, p)]
\]

**Boundary conditions**

\[
V_T(I, P) = 0 \quad \text{for all } I
\]

\[
V_t(0, P) = 0 \quad \text{for all } t
\]

---

Suppose we are in period \( t \) with the price in the last period being \( P \). Then the maximum amount of per-unit revenues that we could make, given the non-increasing price constraint, should be \( P \). This is formally proved by the following lemma.

**Lemma 3.2.3.2:**

\[
V_t(I, P) \leq V_T(S, P) + (I - S) P \quad \text{for all } t = 1, \ldots, T, \ I \geq 0, \ P > 0 \text{ and } S \leq I.
\]

This result suggests that, under a non-increasing price constraint, the retailer should not lose an opportunity to sell at the current price, since the per-unit future value of the product can never be higher than the current price. Thus, the retailer should not impose an artificial sales limit \( (S < I) \) to the current period sales. This is the essence of the proof for the claim in Theorem 3.2.3.8. This theorem shows that Conjecture 4 is true in cases where prices are not allowed to increase over time.
Theorem 3.2.3.8: \( W_t(l,p,S) \leq W_t(l,p,l) \) for all \( t=1, T, l \geq 0, P > 0 \) and \( S \leq l \).

A second situation in which a sales limit would not be necessary is one where the retailer can adjust the price on a continuous basis. In such cases, the price could always be adjusted upwards to an appropriate level whenever an unexpectedly high level of sales occurred over any period of time. In fact, the optimal price path for such a continuous time model involves a jump in price immediately after each sale (for examples of price paths, see the papers by Bitran and Mondshein, or Gallego and Van Ryzin). Thus, sales-limit extensions of the continuous time models of Bitran and Mondshein (1994, 1995) and Gallego and Van Ryzin (1994) are not necessary. We have assumed here that at any price \( p \) the probability of observing more than one unit of demand in any period of time of length \( \Delta t \) approaches zero as \( \Delta t \) approaches zero. In situations where this was not the case (for instance, where demand at any price \( p \) is a non-homogeneous Poisson point process), a sales limit may still be necessary. An example of this would be group bookings in the airline or hotel industries.

### 3.3 Heuristic And Upper Bound

In this section, we present a heuristic and an upper bound for the seasonal product pricing problem formulated in Section 3.2.1.2. Although it is computationally feasible to solve this problem in an optimal manner (through the standard backward recursion technique for dynamic programs), the heuristic and bound are still of interest to us because of two reasons. Firstly, the good performance of the heuristic, as evidenced in the computational tests performed, sheds some insight into the nature of optimal pricing policies for seasonal products. Secondly, the heuristic and bound are essential building blocks for the development of solution approaches for the more computationally demanding extensions of the seasonal product pricing problem, such as the problem with demand learning discussed in Chapter 4 and the multi-store problems discussed in Chapter 5.

Both the heuristic and bound are based on the expected demand approximation of the probabilistic seasonal product pricing problem. This approximation involves the replacement of the demand random variable (at any price) with its expected value, yielding a deterministic
seasonal product pricing problem. We discuss the solution of the deterministic problem below, and then describe how it leads to the heuristic and upper bound.

### 3.3.1 SOLUTION OF MODEL UNDER DETERMINISTIC DEMAND

In Section 3.2.1, we showed that the deterministic demand problem can be formulated as a special case of the dynamic programming model for the probabilistic demand problem. This deterministic demand problem can also be formulated as a non-linear program, as shown below:

**Seasonal Product Pricing Problem under Deterministic Demand**

**Formulation 1**

\[
\begin{align*}
\text{max} & \quad \sum_{t=1}^{T} p_t S_t \\
\text{s.t} & \quad S_t \leq D_t(p_t) \quad \forall \ t \\
& \quad \sum_{t=1}^{T} S_t \leq I \\
& \quad p_t \geq 0 \quad \forall \ t
\end{align*}
\]

Here, \( p_t \) is the price in period \( t \) (and is a decision variable for all \( t \)), \( D_t(\cdot) \) is the demand function in period \( t \), \( I \) is the initial inventory level, and \( S_t \) is the level of sales achieved in period \( t \).

At first glance, this formulation seems to differ from the dynamic programming formulation, since, in addition to setting the prices \( p_t \), it allows the retailer to determine the 'sales quota' \( S_t \) for period \( t \) (albeit under constrained conditions). However, as stated in Lemma 3.1 below, there must exist an optimal solution \( \{\bar{p}_t, \bar{S}_t\} \) to the above problem in which \( \bar{S}_t = D_t(\bar{p}_t) \) for all \( t \).
Lemma 3.3.1.1: There exists an optimal solution \( \{\bar{p}_t, \bar{S}_t\} \) to the non-linear programming problem above in which \( \bar{S}_t = D_t(\bar{p}_t) \) for all \( t \).

This result implies that one can view this non-linear program as providing us only the optimal prices \( p_t \), with \( S_t \) being automatically determined by the demand at these prices, which is similar to the dynamic programming model. We can re-express the non-linear program as follows:

---

**Seasonal Product Pricing Problem under Deterministic Demand**

**Formulation 2**

\[
\max \sum_{t=1}^{T} p_t D_t(p_t) \\
\sum_{t=1}^{T} D_t(p_t) \leq I \\
p_t \geq 0 \quad \forall \ t
\]

---

We will now re-express this in terms of the demand variables \( D_t(p_t) \). We assume here that the demand functions \( D_t(.) \) are invertible. Let \( d_t = D_t(p_t) \), so that \( D_t^{-1}(d_t) = p_t \). The above non-linear program can now be expressed as:

---

**Seasonal Product Pricing Problem under Deterministic Demand**

**Formulation 3**

\[
\max \sum_{t=1}^{T} d_t D_t^{-1}(d_t) \\
\sum_{t=1}^{T} d_t \leq I \quad (\lambda_t) \\
d_t \leq D_t(0) \quad \forall \ t \quad (\pi_t)
\]
The constraints \( d_t \leq D_t(0) \) have replaced the constraints \( p_t \geq 0 \). We have shown the dual multipliers for each constraint on the right hand side, and these will be referred to in the discussion that follows.

The above non-linear program has a concave objective function if Condition 3.2.3.1 is satisfied, as stated in the lemma below.

**Lemma 3.3.1.2:** Suppose the demand functions \( D_t(\cdot) \) satisfy Condition 3.2.3.1, i.e., that for all \( t \),

\[
2(D_t'(p))^2 - D_t(p)D_t''(p) \geq 0 \text{ for all } p > 0.
\]

Then the function \( F(d_1, \ldots, d_T) = \sum_{t=1}^{T} d_t D_t'(d_t) \) is concave in the \( d_t \)'s.

Since the constraints in the above non-linear program are linear, and the objective function is concave under Condition 3.2.3.1, it follows that under this condition the Karush-Kuhn-Tucker optimality conditions are both necessary and sufficient for the above problem. These conditions are given by:

\[
d_t \left( \frac{\partial D_t'(d_t)}{\partial d_t} + D_t'(d_t) - \lambda - \pi_t \right) = 0 \]

\[
\left( \sum_{t=1}^{T} d_t - 1 \right) \lambda = 0
\]

\[
(d_t - D_t(0)) \pi_t = 0
\]

\[
\sum_{t=1}^{T} d_t \leq 1
\]

\[
d_t \leq D_t(0)
\]

Lemma 3.3.1.3 states that the dual variables \( \pi_t \) must all be zero.

**Lemma 3.3.1.3:** Suppose \( \{\bar{d}_t, \bar{\pi}_t, \bar{\lambda}\} \) is a solution to the KKT conditions for the above non-linear program. Then we must have \( \bar{\pi}_t = 0 \) for all \( t \).

This result allows us to re-write the KKT conditions as follows:
\[
\frac{\partial D_i^t(d_i)}{\partial d_i} + D_i^t(d_i) - \lambda = 0 \\
\left( \sum_{t=1}^{T} d_t - I \right) \lambda = 0 \\
\sum_{t=1}^{T} d_t \leq I \\
d_t \leq D_t(0)
\]

To solve for the optimal solution \(\{\overline{d}_t, \overline{\lambda}\}\), we need to consider two cases:

Case 1: \(\overline{\lambda} = 0\)

Then the solution \(\{\overline{d}_t\}\) is derived by solving the equations

\[
\frac{\partial D_i^t(\overline{d}_i)}{\partial \overline{d}_i} + D_i^t(\overline{d}_i) = 0 \text{ for all } t.
\]

These constitute \(T\) equations in \(T\) unknowns.

Case 2: \(\overline{\lambda} > 0\)

Then the KKT conditions above imply that \(\sum_{t=1}^{T} \overline{d}_t = I\) \text{ and }

\[
\overline{d}_t \cdot \frac{\partial D_i^t(\overline{d}_i)}{\partial \overline{d}_i} + D_i^t(\overline{d}_i) - \overline{\lambda} = 0.
\]

These constitute \(T+1\) equations in \(T+1\) unknowns.

Theorem 3.3.1.1 shows how the equations in Cases 1 and 2 can be solved for \(\{\overline{d}_t\}\) in a straightforward manner.

**Theorem 3.3.1.1:** The solution to Case 1 above can be determined through a binary search on \(d_t\). The solution to Case 2 above can be determined by performing a simple line search on \(\lambda\).

### 3.3.2 HEURISTIC SOLUTION PROCEDURE

Our heuristic solution procedure is based on solving the expected price approximation to the seasonal product pricing problem on a rolling horizon basis. At the beginning of any period \(t\), the seasonal product pricing problem from period \(t\) onwards is solved heuristically by formulating and
solving the expected price approximation to the problem as described above. Then demand in period t is observed, the available inventory is recalculated, and the period t+1 problem is formulated and solved. The process repeats itself in this manner from t = 1 to t = T.

In Section 3.3.4, we present results from computational tests which suggest that this heuristic performs very well in practice.

3.3.3 UPPER BOUND
Theorem 3.3.3.1 states that the solution to the expected price approximation to the seasonal product pricing problem is an upper bound for the problem under Condition 3.2.3.1.

**Theorem 3.3.3.1:** Let $V_1(l)$ be the optimal value of the (probabilistic) seasonal product pricing problem where the expected price function in each time period, $\lambda_t(p) = E[D_t(p)]$, satisfies Condition 3.2.3.1 for each $p$ and $t$. Let $V_t(l)$ be the solution of the expected price approximation to the problem. Then $V_1(l) \leq V_t(l)$.

The performance of this bound on a set of test problems is reported in the next section.

3.3.4 COMPUTATIONAL TEST RESULTS
We report below on some computational test results that suggest that both the heuristic and the upper bound described above perform quite strongly. The results also indicate that the performance of the optimal fixed price heuristic suggested by Gallego and Van Ryzin (1994a) can be poor under conditions of non-stationary demand.

To examine these issues, we developed a test case with the following characteristics. There were four time periods. The demand model in each period was based on the two-phased approach described by Bitran and Mondschein (1993, 1995) and discussed in Chapter 2. We used a Poisson store arrival distribution combined with a Weibull reservation price distribution. The store arrival rate was set at 50 for each time period. The parameters for the Weibull reservation
price distribution were taken so as to get a decreasing mean reservation price over the four time periods. These parameters are given in Table 3.3.4.1, along with the associated mean of the reservation price distribution for each period.

<table>
<thead>
<tr>
<th>Period</th>
<th>Mean Res. Pr.</th>
<th>α</th>
<th>δ</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>62.73</td>
<td>5</td>
<td>-75</td>
<td>1/150</td>
</tr>
<tr>
<td>2</td>
<td>48.95</td>
<td>5</td>
<td>-75</td>
<td>1/135</td>
</tr>
<tr>
<td>3</td>
<td>26.00</td>
<td>5</td>
<td>-75</td>
<td>1/110</td>
</tr>
<tr>
<td>4</td>
<td>26.00</td>
<td>5</td>
<td>-75</td>
<td>1/110</td>
</tr>
</tbody>
</table>

Table 3.3.4.1: Parameters and means for Weibull reservation price distributions

Figure 3.3.4.1 graphs the reservation price distributions for all the periods.

Figure 3.3.4.1: Weibull reservation price distributions for time periods 1-4

Eight runs were made using this test case, with the initial inventory varying from 5 units to 150 units. For each run, we calculated the optimal pricing policy, the upper bound (based on the expected price approximation) and the optimal fixed price. We also implemented the expected price heuristic described in Section 3.3.2. The results of these tests are shown in Table 3.3.4.2.
Table 3.3.4.2: Computational test results for expected price heuristic, upper bound, and Gallego-Van Ryzin's optimal fixed price heuristic

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>95.4%</td>
<td>98.7%</td>
<td>95.5%</td>
</tr>
<tr>
<td>10</td>
<td>96.6%</td>
<td>99.5%</td>
<td>94.8%</td>
</tr>
<tr>
<td>25</td>
<td>97.8%</td>
<td>99.6%</td>
<td>94.4%</td>
</tr>
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<tr>
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<td>97.7%</td>
<td>99.9%</td>
<td>91.9%</td>
</tr>
<tr>
<td>100</td>
<td>98.3%</td>
<td>99.6%</td>
<td>91.7%</td>
</tr>
<tr>
<td>125</td>
<td>99.9%</td>
<td>99.7%</td>
<td>91.2%</td>
</tr>
<tr>
<td>150</td>
<td>100.0%</td>
<td>99.9%</td>
<td>91.4%</td>
</tr>
</tbody>
</table>

Each row in Table 3.3.4.2 presents the results of a single test problem. The first column indicates the initial inventory level for the test problem. The second column shows the ratio of the optimal solution value to the upper bound. The third column shows the ratio of the expected revenues from the heuristic procedure to the optimal solution, and the fourth column shows the ratio of the expected revenues from the optimal fixed price policy against the optimal solution.

The results in the table suggest that the upper bound and the heuristic solution approach both perform quite well. In contrast, the fixed price heuristic performs quite poorly, especially as the inventory level starts rising. We observe that all the ratios in the table appear to converge as the inventory level increases. This is no coincidence, for it can be shown in a straightforward manner that the upper bound and heuristic both converge in value to the optimal solution value as the level of initial inventory goes to infinity. This fact was also observed by Gallego and Van Ryzin (1994a), who proved some of these asymptotic properties for the continuous time version of the seasonal product pricing problem.

3.4 Conclusions And Future Research Issues

In this chapter, we have proposed and analyzed a number of conjectures regarding the structure of the seasonal product pricing problem. We have discussed the intuitive nature of the conjectures, and have demonstrated through a series of counterexamples that in fact
these conjectures are not in general true for the problem. Further, we have developed a number of sufficient conditions that guarantee the conjectures in many situations. These conditions are satisfied not only by certain commonly used demand models (such as the multiplicative uncertainty model under an exponential distribution), but also by other demand models that do not fit into their restrictive structure. Finally, we have developed a heuristic solution approach and an upper bound for the seasonal product pricing problem and have presented computational test results that suggest that these perform well even under non-stationary demand conditions.

A number of issues arising out of our analysis could be investigated further. These include the derivation of sufficient conditions that guarantee Conjecture 3 for the probabilistic, multi-period problem, and the analysis of other demand models with respect to the sufficient conditions that have been derived. One may also seek to determine if alternative sufficient conditions may be found (to those derived in this chapter) that may facilitate the analysis of larger classes of demand models.

In certain seasonal product contexts, retailers can reorder the product (usually only once or twice) during the season. This leads to an extension of the seasonal product pricing model that will incorporate the reordering decision. While a number of researchers have developed and analyzed joint ordering-pricing models in the past, they have used demand models that are specific in form (such as the additive and multiplicative uncertainty demand models described in Section 2.2). Examples of past work include Kunreuther and Richard (1969, 1971), Kunreuther and Schrage (1973) and Thomas (1970) for deterministic demand models, and Karlin and Carr (1962), Ernst (1970), Zabel (1972), Thomas (1974), Thowsen (1975), and Polatuglu (1991) for probabilistic demand models. A review of this area can be found in the survey of joint production-marketing models by Eliashberg and Steinberg (1991).

Some of the results in this chapter can be used to demonstrate certain structural properties for the joint ordering-pricing problem. For instance, consider a two-period seasonal product pricing model with a single reorder decision at the beginning of period 2, zero
lead time, and a linear ordering cost. Under the conditions specified in Theorem 3.2.3.6, it can be shown that the optimal period 2 ordering and pricing policy will be of the following (S,p) form: If the inventory I at the end of period 1 is less than S, order S - I and charge price p in period 2. If I ≥ S, order nothing (and set the same price as you would for the seasonal product pricing model without reordering). Other joint ordering-pricing situations could also be studied.

Finally, one may pose and analyze other structural properties of interest relating to the pricing problem considered in this chapter, or to extensions of this problem such as the one discussed above. These may, for instance, examine the relationship between the optimal price and the variance in the demand distribution (at each price p) in any period, or that between the optimal initial order quantity and the level of demand.
CHAPTER 4

OPTIMAL PRICING WITH DEMAND LEARNING

4.1 Introduction

A major source of the retailer’s demand uncertainty is often the lack of information about how attractive the product will be to customers. This is particularly the case for products such as fashion goods, where styles change every season. Sales observed during the course of the season can in such cases provide valuable information about market conditions. This sales-driven demand learning allows retailers to refine their demand forecasts and alter prices accordingly over time. In some situations where Quick Response strategies have helped to reduce production lead times substantially, retailers may also be in a position to use early sales information to place a second, and final, order for the product. For instance, two leading retailers, The Limited and Benetton, test market products in a few representative outlets in their chains early in the season and use the resulting demand information to determine the right quantities to order for the rest of the season (Hammond, 1992).

While this lack of information about the product’s attractiveness to customers often constitutes a major source of uncertainty, it is certainly not the only significant source - other unpredictable factors, such as the weather, and store traffic, also contribute to demand uncertainty. We therefore assume in this chapter that there are two sources of demand uncertainty for the retailer - one, related to the lack of information on product attractiveness, that is resolvable through sales observations, and the other that is not resolvable in this manner.

The above discussion suggests that a realistic model of dynamic pricing of seasonal products needs to incorporate a mechanism for demand learning. Existing dynamic pricing models (such as those developed by Gallego & Van Ryzin and Bitran & Mondschein) ignore the possible correlation in demand over time, and thereby ignore the potential for demand learning. In this
paper, we describe a modeling approach for using observed sales data to update demand information over time, and show how this can be embedded in an optimal dynamic pricing model. Our technique utilizes the Bayesian approach commonly employed in dynamic learning models such as inventory models that incorporate demand learning. It is distinguished from existing approaches, however, by its ability to address important sources of non-stationarity in the demand distribution, such as price changes and changes in customers' values for the product over time.

In addition to a description of the methodology, we present results from some preliminary computational tests based on representative data. These results indicate that the methodology is effective in estimating demand under a range of conditions. They also suggest that the incorporation of demand learning can lead to swift price corrections early in the season, and that this can substantially improve revenues in the seasonal product pricing context. In addition, the computational results provide insights into the nature of optimal dynamic pricing strategies in situations of over- or under-estimation of demand and in situations where prices are required to be non-increasing over time.

The remainder of this section discusses the demand model on which our demand learning methodology is based and the optimal pricing model within which we have embedded it, and comments on the other sections in this chapter.

**Demand Model**

We use the demand model developed by Bitran and Mondschein (1993, 1995), and discussed in the previous chapter. This model combines a Poisson store arrival process with random sampling from a reservation price distribution. As shown by Bitran and Mondschein (1995), for any given price \( p \) and any period \( t \), the resulting distribution of purchases in period \( t \) at price \( p \) is itself a Poisson distribution. (This result will be stated in more technical terms later in this chapter).

This model incorporates two sources of uncertainty - the number of store arrivals in any period (which is a Poisson process) and the "maximum willingness to pay" of the arriving customers (which are random samples from the reservation price distribution). In this chapter, we extend the
demand model by introducing a third source of uncertainty related to the retailer’s lack of information about how attractive the product is to the customer population. We assume that there is some parameter of the reservation price distribution that is unknown to the retailer at the beginning of the season. This parameter is revised at the end of each period based on observed sales in that period. Lack of knowledge about this parameter leads to the resolvable component of demand uncertainty, while the prior two factors lead to the unresolvable component.

**Optimal Pricing Model**

We utilize and extend the discrete-time seasonal product pricing model developed by Bitran and Mondschein (1993), and described in the previous chapter. There are \( T \) discrete periods in the season, and the retailer has a fixed stock of the product at the beginning of the season. Price revisions are allowed only at the beginning of each period. Product value at the end of the planning horizon is taken to be zero. (The model and methodology to be discussed can be modified in a straightforward manner to allow for a salvage value function.) The retailer's objective is to maximize the expected revenues over the planning horizon. Product cost is assumed to be a sunk cost, and is therefore not incorporated in the model. The resulting optimal pricing problem is formulated as a stochastic dynamic program, with the stages corresponding to the different time periods and the state in each period being the inventory level of the product at the start of the period.

To incorporate demand learning in the above model, we include two additional variables in the state space that allow us to transfer demand information from one period to the next. The resulting increase in the state space of the dynamic program can in certain cases effect the computational time in a significant manner, and we have developed a heuristic solution approach to the dynamic program that runs very efficiently. This heuristic has performed very well in the computational tests performed by us, as reported later in this chapter.

The rest of this chapter is organized as follows. Section 4.2 describes our demand learning methodology under various non-stationary demand conditions. Section 4.3 defines the basic optimal pricing model and shows how the demand learning technique can be embedded in the
model. Section 4.4 presents computational results and discusses the conclusions and insights arising from our research on demand learning. Finally, in Section 4.5, we summarize the key contributions made by our research and present further research issues arising from it.

4.2. Demand Learning Methodology

In this section, we describe our demand learning methodology in the context of the demand model discussed in Section 4.1. We assume that the store arrival rate in each period is known, and that the reservation price distributions are all functions of an unknown parameter $\delta$. For instance, these distributions may be exponential, with cumulative distribution functions given by $F_r(r) = F(r) = 1 - e^{-r}$, where the mean $1/\delta$ is unknown.

Our goal is to utilize observed sales data to update the value of this parameter $\delta$ from one period to the next. Our methodology is an extension of the standard Bayesian learning technique. The Bayesian approach applies to a context where we get observations from the same distribution over time and use these to update our information (represented in the form of a prior distribution) on some unknown parameter of that distribution. We have found it necessary to seek an extension to this approach since in our case the demand distribution (from which the demand data is observed) changes from one period to the next, due, primarily, to changes in price.

The Bayesian approach utilizes both the planner's initial estimate of demand and the observed sales data to revise the demand forecast. This is in contrast to alternative statistical estimation approaches such as MLE (maximal likelihood estimation), where only the observed sales data is utilized. As the volume of sales data grows, however, the planner's initial estimate has a progressively smaller impact on the demand estimate calculated using the Bayesian approach, and the Bayesian estimate becomes very similar to the maximum likelihood estimate (Hines and Montgomery, 1980, pg. 579). We have chosen to utilize the Bayesian approach since we believe that in actual applications it would be important to capture both the planners' initial judgment as well as the initial sales data in arriving at an accurate and robust estimate of demand during the early part of the season.
The discussion below is organized as follows. We first describe the Bayesian approach to demand learning under the assumption that the price, the store arrival rate, and the length of time periods stay constant. We then peel away the stationarity assumptions one by one. We describe our extension of the Bayesian approach to the case where price changes from one time period to the next. We show next how non-stationary store arrivals can be incorporated in our approach. We then describe a way of incorporating changes in the reservation price distribution over time within our model. At various points in the discussion, we illustrate our approach by using the example of the exponential reservation price distribution mentioned above. We provide examples of more general reservation price distributions to which our approach can be applied at the end of this section.

We assume for now that:

- The store arrival rate $A_t$ is constant ($= A$) across all time periods
- The reservation price distribution $F_t(\cdot|\delta)$ is constant ($= F(\cdot|\delta)$) across all time periods.

As discussed earlier (and stated later in Lemma 4.3.1), the purchase distribution at price $p$ in time period $t$ can be shown to be a Poisson distribution with the arrival rate $\lambda_t(p|\delta)$. The above assumptions imply that the purchase arrival rate $\lambda_t(p|\delta)=A_t(1-F_t(p|\delta))$ may be written simply as $\lambda(p|\delta)$ for all time periods. Note that $\lambda(p|\delta)=A(1-F(p|\delta))$. We have explicitly shown the dependence of $F$, $F_t$, $\lambda$, $\lambda_t$ on $\delta$. Due to its dependence on the unknown parameter $\delta$, the purchase arrival rate $\lambda(p|\delta)$ (for all prices $p$) is not known to the planner a priori. Instead, utilizing the Bayesian approach, we assume that the planner initially has a family of prior distributions on $\lambda(p)$, one for each price $p$, which will be updated at the end of every period ($t=1, 2, \ldots T$) through the incorporation of observed demand data. $\lambda(p)$ is therefore a random variable with a certain distribution that gets periodically revised. Note that the prior distribution is not on $\delta$, the unknown parameter, but on $\lambda(p)$. The priors are illustrated in Figure 4.2.1.
We assume that the following condition holds:

**Condition 4.2.1:** For any given value $\lambda$ of $\lambda(p)$, there is a unique value of $\delta$ such that $\lambda = \lambda(p|\delta)$, i.e., $\lambda = A(I-F(p|\delta))$ has a unique solution in $\delta$.

This condition is equivalent to assuming that the function $F(p|\delta)$ is invertible in $\delta$ for each price $p$. Note that the exponential distribution (with $\delta = 1/$mean) satisfies Condition 4.2.1.

### 4.2.1 CASE 1: CONSTANT PRICE

If the price $p$ were held constant over time, this Bayesian updating of the prior distribution would follow by a straightforward application of Bayes’ rule, as described below:

Suppose we enter period $t$ with a prior distribution $f_p(\lambda)$ on $\lambda(p)$, and we observe a demand of $n$ units in this period at price $p$. The posterior distribution on $\lambda$, $f_p(\lambda|n)$, can then be calculated by using Bayes’ rule as follows:

$$f_p(\lambda|n) = \frac{P[\text{Demand} = n|\lambda] \cdot f_p(\lambda)}{\int_0^\infty P[\text{Demand} = n|x] \cdot f_p(x)dx} = \frac{e^{-\lambda} \frac{n^\lambda}{n!} f_p(\lambda)}{\int_0^\infty e^{-x} \frac{x^n}{n!} f_p(x)dx}$$

The last equation follows from the fact that the demand distribution is Poisson, as mentioned in Section 4.1 and discussed further in Section 4.3. The revised distribution $f_p(\cdot|n)$ will now
represent the prior distribution for demand in period t+1, and the updating process will repeat itself.

For the updating process to be computationally feasible, one would need to devise a way of efficiently computing the posterior distribution. This can be achieved by starting the process with a prior that belongs to the family of conjugate distributions for the Poisson. One such conjugate for the Poisson is the two-parameter gamma distribution. The gamma includes a range of distribution forms over [0, ∞), and its density function is given by:

\[ g(x) = \frac{b^*}{\Gamma(a)} x^{a-1} e^{-bx}, \] where \( a \geq 1 \) and \( b > 0 \) are the parameters of the distribution.

When the prior \( f_p(.) \) is a gamma distribution with parameters \( (a_p, b_p) \), the posterior can be shown (DeGroot, pg. 323) to be given by:

\[ f_p(\lambda|n) = \frac{(b_p + 1)^{n} \times \Gamma}{\Gamma(a_p + n)} \times x^{\lambda - 1} e^{-(b_p + 1)x}, \] which is gamma with parameters \( a_p + n, b_p + 1 \).

Hence the calculation of the posterior distribution is reduced to simple parameter updates for \( a_p \) and \( b_p \).

4.2.2 CASE 2: VARYING PRICES

Our discussion above has assumed that the price remains fixed from one period to the next. We now need to include a mechanism that will allow us to use observed demand data at price \( p \) to learn about demand at any other price \( p^* \).

As in the above discussion, we will require that, for each price \( p \), the prior distribution on \( \lambda(p) \) is gamma with parameters \( (a_p, b_p) \). In order to use demand observations at a certain price \( p \) to update the prior distribution for the purchase rate at another price \( p^* \), we will connect together the parameters \( (a_p, b_p) \) of the gamma priors on \( \lambda(p) \) across different prices \( p \). This will facilitate the updating of the parameters \( (a_{p^*}, b_{p^*}) \) whenever we update the parameters \( (a_p, b_p) \). Since there are two parameters \( (a_p, b_p) \) for each price \( p \), we need two equations to link them together across \( p \).
We will derive these two equations by defining two desirable properties that should be satisfied by these parameters.

The first property is the expected purchase rate $E[\lambda(p)]$ (under the prior distribution for $\lambda(p)$) should have the same form across $p$ as the true underlying value of $\lambda(p)$, i.e.,

$$E[\lambda(p)] = A(1 - F(p|\delta)),$$

for some $\delta$ that is independent of $p$

Since the mean of a gamma $(a_p, b_p)$ distribution is given by $a/b$, this implies that

$$a_p/b_p = A(1 - F(p|\delta)),$$

for some $\delta$ that is independent of $p$ \hspace{1cm} (P1)

The second property is that the coefficient of variation of the gamma $(a_p, b_p)$ distributions should be independent of $p$, i.e.,

$$1/a_p = S$$

for some $S$ that is independent of $p$ \hspace{1cm} (P2)

The intuition behind property (P1) should be fairly evident to the reader upon some reflection. Property (P2) is based on the following motivation: for any price level $p$, the coefficient of variation of the gamma $(a_p, b_p)$ distribution reflects the level of uncertainty of the planner with respect to the arrival rate $\lambda(p)$. Property (P2) states that this level of uncertainty should not vary with price.

Alternative versions for this property could also be used. For instance, if the planner's level of uncertainty about demand was much greater for some 'medium' range of prices, but much lower for some extreme (low or high) range of prices, property P2 could instead be defined as

$$1/a_p = S/ (p-m)^2$$

for some constants $S$, $m$.

Here, $m$ would be a medium price level that would remain unchanged, while the value of $S$ would be revised over time as described in this section.

These equations allow us to link together the $(a_p, b_p)$ parameters across $p$. If we know the values of $a_p$, $b_p$ for some price $p$, we can calculate the values of $a_{p^*}$, $b_{p^*}$ for some other price $p^*$ using
equations (P1) and (P2) as follows. First, we use (P1) and (P2) with price $p$ to calculate the values for $\delta$ and $S$. Then we use these values of $\delta$ and $S$, along with price $p^*$, to calculate ($a_{p^*}$, $b_{p^*}$). Note that equation (P1) will yield a unique value for $\delta$ because of Condition 4.2.1.

To illustrate, suppose the reservation price distribution was exponential, as described above. Properties (P1) and (P2) can be written in this case as:

$$a_{p}/b_{p} = Ae^{-\delta p} \quad \text{(P1 - Exp)}$$

$$1/a_{p} = S \quad \text{(P2 - Exp)}$$

Suppose, at any time, we had the parameters ($a_{p}$, $b_{p}$) and wanted to determine the parameters ($a_{p^*}$, $b_{p^*}$). First, we would calculate $\delta$ and $S$ from the above equations, as follows:

$$\delta = -(1/p) \log(a_{p}/Ab_{p})$$

$$S = 1/a_{p}$$

We would then use equations (P1-Exp) and (P2-Exp) to calculate ($a_{p^*}$, $b_{p^*}$) using these values of $S$ and $\delta$, as follows:

$$a_{p^*} = 1/S, \quad b_{p^*} = 1/S Ae^{-\delta p^*}$$

We observe that, given properties (P1) and (P2), information about the prior distributions for all price levels $p$ is completely contained in the pair of parameters ($\delta, S$). We therefore term these the 'linking' parameters, and it is these parameters that will get revised from one time period to the next. They would also need to be initialized by the model user, and these initial values would reflect the initial state of knowledge of the planner. Thus, in the exponential reservation price distribution case, $\delta_0$ (the initial value of $\delta$) would reflect the best estimate of the inverse of the distribution mean, and $S_0$ (the initial value of $S$) would reflect the initial degree of uncertainty associated with this estimate. They would then get revised over time in the following manner:

Suppose we are in period $t$, and the current values of the linking parameters are $\delta$ and $S$. Suppose the price in period $t$ is $p$, and we observe a demand of $n$ units in period $t$. We would first calculate the parameters ($a_{p}$, $b_{p}$) for the prior on $\lambda(p)$ based on equations (P1) and (P2) using the values $\delta$ and $S$ for the linking parameters. Next, this prior distribution would be updated in the standard Bayesian manner on observing the demand. This would yield a posterior distribution on $\lambda(p)$ with parameters ($a_{p} + n$, $b_{p} + 1$). The revised values for $\delta$ and $S$ would then be recalculated from equations (P1) and (P2) using these updated values for ($a_{p}$, $b_{p}$). This process is outlined in Figure 4.2.2.
4.2.3 CASE 3: NON-STATIONARY STORE ARRIVAL DISTRIBUTION

We now consider the case where store arrival rates are not constant across time periods. The purchase rate now depends on the period under consideration in addition to the price, and so we denote it as $\lambda_t(p|\delta)$. Note that $\lambda_t(p|\delta) = A_t(1-F(p|\delta))$, where $A_t$ is the store arrival rate in period $t$. The gamma priors also depends on the time period, and so we denote the parameters of the gamma prior on $\lambda_t(p)$ as $(a_{pt}, b_{pt})$ for each price $p$ and time period $t$. As before, we will assume that the planner knows the value of $A_t$ for each time period $t$, but that the planner still does not know the value of $\lambda_t(p)$ since $\delta$ is not known.

A property of the gamma distribution that is useful in this case is the following:

**Lemma 4.2.1:** If $X$ is a random variable with a gamma $(a, b)$ distribution, and $Y = kX$ for some constant $k$, then $Y$ has a gamma $(a, bk)$ distribution.

Lemma 4.2.1 states a fairly standard result about the gamma distribution, and so we do not provide a proof for it here. We seek to modify the approach described in the previous case to allow for different values of $A_t$'s across time periods. The change we make will allow us to
translate a family of prior distributions on \( \{ \lambda_t(p) \}_p \) for period \( t \) to a family of prior distributions on \( \{ \lambda_{t+1}(p) \}_p \). We explain the nature of the change required below.

Suppose we have an updated gamma \((a_{pt}, b_{pt})\) prior on \( \lambda_t(p) \) at the end of period \( t \), and this yields the updated values of \( \delta \) and \( S \) from equations (P1) and (P2). In the previous case, a gamma \((a_{pt}, b_{pt})\) prior on \( \lambda_t(p) \) translated into a similar gamma \((a_{pt}, b_{pt})\) prior on \( \lambda_{t+1}(p) \) since the two random variables \( \lambda_t(p) \) and \( \lambda_{t+1}(p) \) had the same prior distribution. In the present case, however, \( \lambda_{t+1}(p) = (A_{t+1}/A_t)\lambda_t(p) \). Hence, Lemma 2.1 tells us that a gamma \((a_{pt}, b_{pt})\) distribution on \( \lambda_t(p) \) converts to a gamma \((a_{pt}, (A_t/A_{t+1})b_{pt})\) distribution on \( \lambda_{t+1}(p) \). Thus, the parameters of the prior distribution for \( \lambda_{t+1}(p) \) satisfy the equations:

\[
\frac{a_{pt+1}}{b_{pt+1}} = \frac{A_{t+1}}{A_t} \frac{a_{pt}}{b_{pt}} = \frac{A_{t+1}}{A_t} \frac{A_t(1 - F(p|\delta))}{A_t(1 - F(p|\delta))} = A_{t+1}(1 - F(p|\delta)) \]

and \(1/a_{pt+1} = S\)

These equations yield

\[
a_{pt+1} = 1/S \quad \text{and} \quad b_{pt+1} = \frac{1}{SA_{t+1}(1 - F(p|\delta))}
\]

### 4.2.4 CASE 4: NON-STATIONARY RESERVATION PRICE DISTRIBUTION

We are now ready to address the final stationarity assumption made earlier - that the reservation price distribution is stationary (i.e., constant over time). This distribution may change over time, for instance, because customers value the product less over time, or because the high reservation price customers leave the market early in the season. The former may be true in the case of seasonal apparel such as coats, and the latter in the case of new books or fashionwear. To address this problem, we would need to know how the reservation price distribution changes over time. We assume that this change occurs in the following manner:
Condition 4.2.2: There exist constants $R_2, \ldots, R_T$ such that $f_t(r) = f_t(R_tr)$ for all $r$ and $t > 1$, where $f_t(\cdot)$ is the density function for the reservation price distribution in period $t$.

Thus, under Condition 4.2.2, the reservation price distribution in any period $t > 1$ could be considered a 're-scaled' version of the distribution in period 1. We assume that reservation prices stay constant within any period, a reasonable assumption when the period lengths are of size, say, one day or a week. We also assume that the planner knows the trend in reservation prices as represented by the parameters $(R_2, \ldots, R_T)$. These may have been derived, for instance, through an examination of historical sales records for similar products.

We now describe how our demand learning methodology, as described for Case 2, can be adapted to the above situation. We describe this modification for the case of the exponential reservation price distribution below. We use Case 2 only for convenience of exposition, and our approach is equally applicable to the setting in Case 3.

The variable $\delta$ that is kept track of now represents the updated value of the parameter for the exponential reservation price distribution corresponding only to period 1. It is appropriately scaled in each period to derive the corresponding parameter for the reservation price distribution for that period. Suppose we are at the beginning of period $t$. Suppose the current values of the linking parameters are $\delta$ and $S$, and that the price in period $t$ is $p$. We describe below how we calculate the revised values of $\delta$ and $S$ at the end of period $t$, given a demand observation of $n$ in period $t$. First, the values $(a_p,b_p)$ of the gamma prior on $\lambda_t(p)$ need to be calculated. Given the relationship in Condition 4.2.2 between the reservation price distributions in periods 1 and $t$, the prior distribution on $\lambda_t(p)$ is the same as the prior distribution (at the beginning of period $t$) on $\lambda_1(p/R_t)$. The parameters for this distribution can be calculated using equations (P1-Exp) and (P2-Exp) with $p/R_t$ instead of $p$, to yield:

$$a_p = 1/S$$

and

$$b_p = \frac{1}{SA e^{-\delta p/R_t}}.$$
Next, these parameters are revised in the usual Bayesian fashion upon observing a demand level \( n \), giving us the updated values \((a_p + n, b_p + 1)\). Now we need to calculate the revised values of \( \delta \) and \( S \) in terms of these updated values. As in the equations above, we get:

\[
\begin{align*}
    a_p + n &= 1/S \\
    b_p + 1 &= \frac{1}{S \cdot A \cdot e^{-\frac{S}{R_i}}} \\
\end{align*}
\]

which yields:

\[
\begin{align*}
    \delta &= -\frac{R_p}{p} \log \left( \frac{a_p + n}{A(b_p + 1)} \right), \\
    S &= 1/(a_p + n)
\end{align*}
\]

4.2.5 ALTERNATIVE RESERVATION PRICE DISTRIBUTIONS

We have been illustrating our methodology through an exponential reservation price distribution. Here, we show how some alternative distributions could be used in its place.

- **Weibull, with unknown location parameter**: The Weibull is a three parameter family of distributions that allows us to model a fairly wide variety of unimodal reservation price distributions on \([\delta, \infty)\) for any \( \delta \in \mathbb{R} \). Its density and distribution functions are given by:

\[
\begin{align*}
    f(p) &= \alpha \beta^\alpha (p - \delta)^{\alpha-1} e^{-\beta(p - \delta)^\alpha}, \quad p \geq \delta \\
    F(p) &= 1 - e^{-\beta(p - \delta)^\alpha}, \quad p \geq \delta
\end{align*}
\]

where \( \delta \in \mathbb{R} \) is the location parameter, \( \beta > 0 \) is the scale parameter, and \( \alpha > 0 \) is the shape parameter.

The exponential distribution is a special case of the Weibull, derived by setting \( \alpha = 1 \) and \( \delta = 0 \).

In this case, the location parameter may be taken as unknown while the shape and the scale of the distribution are taken as known to the planner. This means that the parameters \( \beta \) and \( \alpha \) will remain fixed at some prespecified levels, while the location parameter \( \delta \) will be revised over the course of time. This can alternatively be viewed as revising the mean of the
distribution over time while maintaining the same shape and scale, since the mean of the
Weibull distribution is of the form $\delta + M(\alpha, \beta)$, where $M$ is a function of $\alpha$ and $\beta$. This
distribution can be shown to satisfy Conditions 4.2.1 and 4.2.2 as well.

In this case, we may write $\lambda(p)$ as:

$$\lambda(p) = Ae^{-\beta(p - \delta)^\alpha} \quad \text{for} \quad p \geq \delta$$

Property (P1) becomes:

$$a_p/b_p = A e^{-\beta\alpha(p - \delta)^\alpha} \quad , \quad p \geq \delta$$

As before, this equation, along with (P2), provides straightforward solutions for $\delta$ and $S$ in
terms of $(a_p, b_p)$, and vice versa.

We have assumed above that $p \geq \delta$. By assuming that the location parameter $\delta \leq 0$, we can
circumvent the case where we may have $p < \delta$. Even if $\delta$ were allowed to be positive, it can
be shown in a straightforward manner that the price will never be less than $\delta$ in the optimal
pricing model that we will discuss in the next section. Also, a comment is in order about the
case where $\delta < 0$, since the fact that this leads to 'negative' values for the reservation price
distribution may appear counter-intuitive. Actually, a Weibull distribution with density
function $f(.)$ for which the location parameter $\delta$ is negative can be replaced in the optimal
pricing model by a distribution with the same density function $f(x)$ for $x \geq 0$, and with a
probability $\int_0^\delta f(x)dx$ of being equal to zero. Since the price will never be negative, these two
distributions will be effectively equivalent in the demand behavior they model. This new
distribution can be viewed as corresponding to a population of customers with a segment that
has zero value for the product and the rest having values distributed as $f(x)$ for $x > 0$.

We have used this case of a Weibull reservation price distribution with an unknown location
parameter in our computational tests, discussed in Section 4.5.
• **Weibull, with unknown scale parameter:** An alternative to the above would be to use just the two parameter family of Weibull distributions (with $\delta = 0$) and assume that $\beta$ was the unknown parameter. In this case, we have

$$f(p) = \alpha \beta^p p^{a-1} e^{-\beta p^a}, \quad p \geq 0 \quad \text{and} \quad F(p) = 1 - e^{-\beta p^a}, \quad p \geq 0$$

In this case too Conditions 2.1 and 2.3 are satisfied. Property (P1) can now be expressed as:

$$a_p/b_p = A e^{-\beta p^a}$$

This equation, along with (P2), provides straightforward solutions for $\beta$ and $S$ in terms of $(a_p, b_p)$, and vice versa. We have done some additional computational tests with this case.

The two cases of the Weibull distribution discussed above are illustrated in Figure 4.2.5.

![Weibull distributions with unknown location and spread parameters](image)

**Figure 4.2.5: Weibull distributions with unknown location and spread parameters**

• **Gamma distribution with unknown mean:** A third alternative would be to use the gamma distribution. We can express the density function $f$ and the distribution function $F$ of a gamma distribution with mean $\mu$ and variance $\sigma^2$ as:

$$f(p) = \left( \frac{\mu^2}{\sigma^2} \right) \frac{\mu^2}{\sigma^2} p^{\mu-1} e^{-\frac{\mu}{\sigma^2} p}$$

$$F(p) = \int_0^p \left( \frac{\mu^2}{\sigma^2} \right) \frac{\mu^2}{\sigma^2} x^{\mu-1} e^{-\frac{\mu}{\sigma^2} x} dx$$
This distribution could be used in the following manner: It could be assumed that the planner knows how spread out the reservation price distribution is across the population of customers, though he or she may not know where this distribution is located. This means that the variance of the distribution would be fixed at some prespecified level \( \sigma^2 \), and the mean \( \mu \) would be unknown. Using the gamma instead of the exponential would entail making certain straightforward modifications in our demand learning process. Property (P1) will become:

\[
a_p/b_p = \alpha \int_0^\infty \left( \frac{\mu^2}{\sigma^2} \right)^{\alpha - 1} e^{-\mu^2/\sigma^2} \, \mu \, dr \\
\text{for some} \, \alpha > 0, \, \sigma^2 > 0 \quad \text{(P1-Gamma)}
\]

The one complicating issue that arises now is in the calculation of \( \mu \) in terms of \( p \), \( a_p \) and \( b_p \). The above equation does not allow us to solve for \( \mu \) analytically. This problem may be addressed in the following manner. Let us denote by \( I(p, \mu) \) the integral

\[
\int_0^\infty \left( \frac{\mu^2}{\sigma^2} \right)^{\alpha - 1} e^{-\mu^2/\sigma^2} \, \mu \, dr.
\]

The values of \( I(p, \mu) \) could be precalculated for a range of different values of \( p \) and \( \mu \) (both \( p \) and \( \mu \) would need to be discretized), and this matrix of values could then be used to find the appropriate value of \( \mu \) for given values of \( \alpha_p, \beta_p, p \) and \( A \) to solve the above equation. It can be shown that \( I(p, \mu) \) is increasing in \( \mu \), and so a simple binary search process could be employed in searching for the right value of \( \mu \).

### 4.3 Optimal Pricing With Demand Learning

In this section, we describe how the demand learning technique described in Section 4.2 can be embedded in the discrete time seasonal product pricing model developed by Bitran and Mondshein (1993). We begin with a discussion of the model developed by these authors. (We term this the basic pricing model to distinguish it from the extended model with demand learning that we present in this chapter). We then show how it can be extended to incorporate the demand learning mechanism, and how the resulting model may be solved computationally. We also describe a heuristic solution approach for the model with demand learning that allows for substantial run-time savings.
4.3.1 BASIC PRICING MODEL

This model was presented in general terms in Section 3.2.1.2. Below, we present the same model for the case where the demand distribution is based on the demand model presented by Bitran and Mondschein.

Notation

\( \eta_t \) = Number of customer arrivals in period \( t \) (a Poisson random variable)

\( A_t \) = Customers' arrival rate in period \( t \)

\( \lambda_t(p) \) = Number of purchases in period \( t \) at price \( p \) (a random variable)

\( I_0 \) = Total inventory at beginning of planning horizon

\( T \) = Number of time periods in planning horizon

\( f_t(\cdot) \) = probability density function for the reservation price distribution in time period \( t \)

\( F_t(\cdot) \) = cumulative distribution function for the reservation price distribution in time period \( t \)

\( V_t(I) \) = Maximum expected revenue from period \( t \) onwards when the initial inventory is \( I \).

The indices for the time periods increase with time, i.e., the sequence of time periods is given by \( 1,2,\ldots,T \).

---

**Basic Pricing Model**

**Backward recursion**

\[ V_t(I) = \max_{p \in \mathbb{R}} E[p \min\{\lambda_t(p), I\} + V_{t-1}(I- \min\{\lambda_t(p), I\})] \]

\[ = \max_{p \in \mathbb{R}} \sum_{d=0}^{I-1} \left( pd + V_{t-1}(I-d) \right) + P(\lambda_t(p) \geq 1)pI \]

**Boundary conditions**
\[ V_I(t) = 0 \quad \text{for all } I \]
\[ V_t(0) = 0 \quad \text{for all } t \]

The distribution for \( \delta_t(p) \) is given by:

\[ P(\delta_t(p) = d) = \sum_{n=0}^{\infty} P(\gamma_t = n)(1 - F_t(p))^d F_t(p)^{n-d} \quad (4.3.1) \]

The following result is discussed in Bitran and Mondschein (1993):

**Lemma 4.3.1:** \( \mathcal{A}(p) \) is a Poisson process with arrival rate rate \( \lambda_t(p) = A_t(1 - F_t(p)) \).

**Proof:** Follows by performing some straightforward algebraic manipulations on Equation 4.3.1.

Note that in the above model we are ignoring inventory holding costs and the time value of money - these entail straightforward modifications and do not affect the results in this chapter, and so, for expository simplicity, have not been modeled. In addition, we are assuming that there is no shortage cost aside from the lost opportunity to generate more revenues through additional sales.

**Model solution**

The dynamic program can be solved backwards in time. For each stage \( t \) and state variable \( I \), we need to solve a unidimensional non-linear optimization problem where the decision variable is the price \( p \). This basic model is not computationally demanding.

### 4.3.2 INCORPORATION OF DEMAND LEARNING

The demand learning technique developed in Section 4.3 requires that the parameters \((\delta, S)\) be transferred from one period to the next. We therefore need to include these parameters in the state space of the dynamic program. As mentioned in Section 4.3, these parameters are initialized by the model user, and they are then updated and transferred from one period to the next in the dynamic program. We provide below a formal description of how the demand learning process is
incorporated within the dynamic programming model. To simplify the description, we assume that there are only two periods in the model, that the store arrival rate and reservation price distribution are stationary, and that the reservation price distribution is exponential with an unknown mean. The extensions to the more general cases of nonstationary store arrivals, nonstationary reservation price distributions and alternative reservation price distribution forms, are straightforward.

---

**Extension of Pricing Model with Demand Learning**

**Initialization**

The user specifies the following parameters:

\[
\delta_0 = \text{Initial value of the parameter } \delta \\
S_0 = \text{Initial value of the parameter } S \\
I_0 = \text{Initial level of inventory} \\
A = \text{Store arrival rate}
\]

**State space:**

The state space at the beginning of period \(t = 1,2\) is given by \((I, \delta, S)\), where \(I\) is the level of inventory, and \((\delta, S)\) the updated values of the parameters, at the beginning of period \(t\) (or, equivalently, at the end of period \(t-1\)).

**Solution process**

The backward recursion approach is employed to solve the dynamic program. We therefore begin from period 2 (the last period), and calculate the value function for each state of the system, going then to period 1 and doing the same for the starting state \((I_0, \delta_0, S_0)\).

**Period 2**

In period 2, we solve the following optimization problem:
\[ V_2(I, \delta, S) = \max_{p \geq 0} E[p \min\{I, \mathcal{X}(p)\}\|\delta, S] \]

for each possible state \((I, \delta, S)\). For a given state \((I, \delta, S)\), this problem is solved by performing a line search on \(p\). We describe below how the expectation in the above equation is computed for a given state \((I, \delta, S)\) and a given value of \(p\).

First, the values for \(a_p\) and \(b_p\) are calculated using equations (P1 - Exp) and (P2 - Exp):

\[
a_p = \frac{1}{S} \quad \quad \text{and} \quad \quad b_p = \frac{1}{S} A e^{-\delta p}
\]

Now

\[
E[p \min\{I, \mathcal{X}(p)\}\|\delta, S] = \sum_{n=1}^{I-1} \frac{p^n}{n!} P[\mathcal{X}(p) = n|\delta, S] + p P[\mathcal{X}(p) \geq I|\delta, S] \\
= \sum_{n=1}^{I-1} \frac{p^n}{n!} P[\mathcal{X}(p) = n|\delta, S] + p \left( 1 - \sum_{n=1}^{I-1} P[\mathcal{X}(p) = n|\delta, S] \right)
\]

The above term would be calculable in a straightforward manner once we knew \(P[\mathcal{X}(p) = n|\delta, S]\) for each \(n\) from 1 to \(I-1\). For a given \(n\), this probability can be calculated as follows:

\[
P[\mathcal{X}(p) = n|\delta, S] = \int_0^\infty P[\mathcal{X}(p) = n|\lambda] \text{gamma}(\lambda|a_p, b_p) \lambda^d \text{d}\lambda \\
= \int_0^\infty \frac{\lambda^n e^{-\lambda} \lambda^{a_p-1} e^{-b_p \lambda}}{\Gamma(a_p)} \frac{b_p^{a_p}}{\Gamma(a_p)} \lambda^{b_p-1} \text{d}\lambda \\
= \frac{b_p^{a_p}}{n! \Gamma(a_p)} \int_0^\infty \lambda^{n} e^{-\lambda} \lambda^{b_p-1} \text{d}\lambda \\
= \frac{b_p^{a_p}}{n! \Gamma(a_p)} \frac{\Gamma(n+a_p)}{(b_p+1)^{n+a_p}}
\]

**Period 1**

In period 1, we need to solve the following optimization problem:
\[ V_1(I_0, \delta_0, S) \]
\[ = \max_{p, \delta_0} E \left[ p \min \{I_0, \delta(p)\} + VR_2 \left( I_0 - \min \{I_0, \delta(p)\}, \delta_0, S \right) \right] \]

\(\delta_0, S\) here are the updated values of \(\delta\) and \(S\) that are to be passed to the next period, and they are functions of \(\delta_0, S_0, n\) and \(p\). Here, \(n\) is the number of purchases observed in period 1. We show how these are calculated below.

The above problem is solved, again, by performing a line search on \(p\). We describe below how the expectation in the above equation is computed for a given value of \(p\).

As before, the values for \(a_p\) and \(b_p\) are calculated using equations (P1 - Exp) and (P2 - Exp):

\[ a_p = 1/ S_0 \quad \text{and} \quad b_p = 1/ S_0 Ae^{-\delta_0 p} \quad (4.31) \]

Now

\[ E \left[ p \min \{I_0, \delta(p)\} + VR_2 \left( I_0 - \min \{I_0, \delta(p)\}, \delta_0, S \right) \right] \]
\[ = \sum_{n=1}^{1-1} \left( pn + V_2(I_0 - n, \delta, S) \right) P\left[ I_0(p) = n \delta_0, S_0 \right] + p I_0 P\left[ \delta(p) \geq I_0, \delta_0, S_0 \right] \]
\[ = \sum_{n=1}^{1-1} \left( pn + V_2(I_0 - n, \delta, S) \right) P\left[ I_0(p) = n \delta_0, S_0 \right] + p I_0 \left( 1 - \sum_{n=1}^{1-1} P\left[ I_0(p) = n \delta_0, S_0 \right] \right) \]

Here, \((\tilde{\delta}, \tilde{S})\) are the revised values of \(\tilde{\delta}\) and \(\tilde{S}\) got by observing demand \(n\) at price \(p\). The above term could be computed in a straightforward manner once we have calculated \((\tilde{\delta}, \tilde{S})\), \(V_2(I_0-n, \tilde{\delta}, \tilde{S})\) and \(P[I_0(p) = n \delta_0, S]\) for each \(n\) from 1 to 1-1. We show below how, for a given value of \(n\), the quantities \((\tilde{\delta}, \tilde{S})\) and \(P[I_0(p) = n \delta_0, S]\) are calculated. The function value \(V_2(I_0-n, \tilde{\delta}, \tilde{S})\) is then known from the period 2 calculations done at the previous step of the algorithm.
\( P[\lambda(p) = n|\delta, S] \) is calculated just as in the previous step, with the values of the parameters \( a_p \) and \( b_p \) now coming from Equation (4.3.1). We are now left with the task of computing \((\bar{\delta}, \bar{S})\). This observed demand first results in a revision of the parameter values of the gamma prior on \( \lambda(p) \) from \((a_p,b_p)\) to \((a_p+n, b_p+1)\). This updating process is based on a direct application of Bayes' rule, as described in the previous section. Let us denote these revised parameter values by \((\bar{a}_p, \bar{b}_p)\), so that \( \bar{a}_p = a_p + n \) and \( \bar{b}_p = b_p + 1 \). The parameters \((\bar{\delta}, \bar{S})\) are derived by solving equations (P1) and (P2) with \((\bar{a}_p, \bar{b}_p)\) in place of \((a_p,b_p)\). This yields:

\[
\bar{\delta} = -\frac{1}{p} \log \left( \frac{\bar{a}_p}{\bar{b}_p} \right) \quad \text{and} \quad \bar{S} = \frac{1}{\bar{a}_p},
\]

that is, \( \bar{\delta} = -\frac{1}{p} \log \left( \frac{a_p + n}{Ab_p + 1} \right) \quad \text{and} \quad \bar{S} = \frac{1}{a_p + n} \)

For this demand learning methodology to be effective, we expect to observe the following trends:

- \( S \) should decrease over time, since it represents the amount of uncertainty that is left in the planner's knowledge about the demand function
- \( \delta \) should converge to its true underlying value over time (on average).

These expectations are supported by the results of the computational tests we have performed, as discussed in the next section.

**Model Solution**

The variables \( \delta \) and \( S \) need to be discretized since they are in the state space of the dynamic program. The variable \( \delta \) can be assumed to lie in some interval of reasonable length around the initial estimate \( \delta_0 \). \( S \) can be assumed to lie in some interval \([0, W]\) where \( W \) is an upper bound on \( S \). Once these intervals are identified, the variables can be discretized appropriately to balance solution accuracy with computational efficiency.
The increase in state space caused by the introduction of $\delta$ and $V$ has a significant effect on the computation time for solving the program. Depending on other model characteristics, the dynamic program may or may not have a satisfactory solution time.

One way to reduce the computational time requirement is by adapting to the above problem the expected value heuristic described in Section 3.3.2. Using this approach, at the beginning of period $t$, we solve the associated deterministic demand problem from period $t$ onwards and use the optimal price $p_t$ for period $t$ determined by solving this problem. Then we observe the demand $D_t(p)$, revise the state variables $(I, \delta, S)$ appropriately, and repeat the process in the next period. This heuristic has performed quite well in our computational tests, and we report on its performance in the next section.

### 4.4 Computational test results

In this section, we present summary results from some of the computational tests we have performed and draw some conclusions and insights from them. The test results provide evidence about the capability of our demand learning methodology to converge to the correct underlying reservation price distribution and about the strong performance of the heuristic solution approach. These results also yield some interesting insights into pricing behavior, sales patterns, and the supply-demand imbalance under different conditions.

We begin by describing our computational testing plan.

#### 4.4.1 TEST PLAN

Our objective in performing the computational tests was to study the following issues:

1. How well does the demand learning technique perform? Does $\delta$ converge to the correct underlying value, and does $S$ converge to zero?
2. What is the impact of over- or under-estimation in demand on the retailer's pricing behavior, on optimal revenues, and on the balance between supply and demand over the season?

3. Under situations of demand over- and under-estimation, what is the impact of demand learning on the retailer's pricing behavior, on optimal revenues, and on the balance between supply and demand over the season?

4. What is the impact of a non-increasing price constraint on the retailer's pricing behavior? How does the demand learning approach compare with the no demand learning approach when a non-increasing price constraint is imposed?

5. How well does the heuristic solution approach (described in Section 4) perform in relation to the more computationally demanding optimal solution approach?

To examine these issues, we developed a pair of base case problems with the following characteristics. There were four time periods, and the initial inventory was set at 50. The store arrival rate was set at 500 for each time period. The values of the parameters for the Weibull reservation price distribution were taken so as to get a mean reservation price of $100 along with a reasonable spread around this mean value. These parameter values were 2.0, .007 and -30 for $\alpha$, $\beta$ and $\delta$ respectively. The base case consisted of two versions - in the overestimation version, $\delta$ was overestimated by 30 (i.e., $\delta_0$ was set at 0), which resulted in an overestimation of the mean reservation price by $30. In the underestimation version, this mean was underestimated by $30 by setting $\delta_0 = -60$. The initial value for the coefficient of variation, $S$, was taken to be 0.5 in both cases. The parameters $\delta$ and $S$ were discretized by using 7 and 5 point grids for them respectively.

In addition to the pair of base case problems themselves, several modifications of this pair - involving changes in one or more variables - were utilized to examine the impact of those changes on the solution. There problems were solved separately using the following different approaches.

1. NO DL: The basic optimal pricing model (with no demand learning)

2. DL: The optimal pricing model with demand learning
3. **DL-HEUR:** The optimal pricing model with demand learning, using the heuristic solution technique.

4. **PERF INFO:** The basic optimal pricing model under perfect information, i.e., with the correct initial value of $\delta$.

**PERF INFO** and NO DL are based on the basic pricing model of Bitran and Mondschein (1993). They differ in that **PERF INFO** is based on perfect demand information, and NO DL is not. DL and DL-HEUR are both based on the demand learning model we have described in this chapter. They differ only in solution technique, as described above. The **PERF INFO** run was done in order to provide a baseline against which to compare the results from the other models - the solution under the **PERF INFO** model represented the ideal pricing behavior (and level of revenues) that would be evidenced if the reservation price distribution had been correctly estimated.

To analyze the solutions from these approaches for each test problem, we simulated the price policies recommended by each approach under the true demand conditions (i.e., using the true value of $\delta$). The simulations allowed us to calculate the expected values under these pricing strategies of a number of quantities of interest, such as the overall revenues, the price level and sales in each time period, and the values of the parameters $S$ and $\delta$ at the end of each time period.

Both the basic pricing model and the model with demand learning, in the forms they were described in the previous section, assume that price can be set at any value - i.e., prices are not restricted to some discrete set of allowable levels. This may constitute a small departure from realism, for retailers would typically have a list of feasible price levels that they would select from instead of allowing price to be set at any (continuous) level. We have performed a number of runs using discrete price sets (for instance, the set \{\$20, \$25, \$30, \ldots, \$95, \$100\}). These results are similar to those described in this section under continuous prices, and we therefore have not separately discussed the discrete-price context here. See Gallego and Van Ryzin (1994) and Bitran and Mondschein (1993, 1995) for additional discussions on using continuous versus discrete price levels in the seasonal product pricing context.
We now turn to a discussion of the results of the computational tests.

### 4.4.2 CONVERGENCE OF PARAMETERS

Tables 4.4.1 and 4.4.2 show the values of the parameters $S$ and $\delta$ at the end of time periods $t = 1$, 2 and 3 under different initial inventory levels. (Note that we have not shown the values of $\delta$ directly but instead have shown the values of the mean reservation price, a more meaningful quantity). The rest of the data for these problems was taken from the base case with demand overestimation. The results for the base case with demand underestimation were similar in nature.

<table>
<thead>
<tr>
<th>End of time period</th>
<th>Inventory</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>106.5</td>
<td>104.5</td>
<td>103</td>
<td>102.5</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>103</td>
<td>102</td>
<td>101.5</td>
<td>101</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>102</td>
<td>101.5</td>
<td>101</td>
<td>101</td>
</tr>
</tbody>
</table>

Initial Value = $130$
True Value = $100$

**Table 4.4.1: Average value of mean ($\delta + 130$)**

<table>
<thead>
<tr>
<th>End of time period</th>
<th>Inventory</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0.22</td>
<td>0.17</td>
<td>0.11</td>
<td>0.10</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.09</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.05</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Initial Value = 0.5

**Table 4.4.2: Average value of sq. coeff. of variation ($S$)**
These results indicate that, over time, the parameter $S$ converges to zero and the mean reservation price (and the parameter $\delta$) converges to its true underlying value. The convergence is faster when there are more sales observations in each time period (which happens when the initial inventory is higher), which is intuitive. These results indicate that the demand learning procedure does an effective job of recovering the true underlying reservation price distribution. This suggestion is corroborated by a number of additional computational tests we have done using different problem data.

### 4.4.3 PERFORMANCE OF MODELS UNDER DIFFERENT CONDITIONS

Table 4.4.3 shows the 'optimality gaps' between the optimal expected revenues under PERF INFO and the expected revenues under the approaches NO DL, DL and DL-HEUR under different initial inventory levels. The optimality gap (or % suboptimality) for NO DL is defined as the following quantity:

\[
100 \cdot \frac{\text{Optimal exp. revs. (under PERFINFO)} - \text{Exp. revs. (Under NO DL)}}{\text{Optimal exp. revs. (under PERFINFO)}}
\]

and it is defined similarly for DL and DL-HEUR.

Table 4.4.4 shows the same data under different levels of demand misestimation. It is striking to observe how poorly the NO DL method performs in many cases. In contrast, the DL model performs consistently well, even when the level of demand misestimation is very high. This suggests that demand learning can lead to a substantial improvement in revenue. We also note that DL-HEUR performs almost as well as DL in most cases. This suggests that a viable heuristic solution option exists for situations where the DL model may be too computationally demanding.

% Suboptimality of Revenues (Compared to PERF INFO)

<table>
<thead>
<tr>
<th>$I_0 =$</th>
<th>Overestimation (Mean $= $130 / d = 0)</th>
<th>Underestimation (Mean $= $70 / d = 30)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>No DL</td>
<td>19.9</td>
<td>20.2</td>
</tr>
<tr>
<td>DL</td>
<td>19</td>
<td>24</td>
</tr>
<tr>
<td>DL-Heur</td>
<td>2.9</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Table 4.4.3: Different Levels of Initial Inventory
% Suboptimality of Revenues (Compared to PERF INFO)

<table>
<thead>
<tr>
<th>Initial Mean ($\delta_0 + 130$)</th>
<th>Overestimation</th>
<th>Underestimation</th>
</tr>
</thead>
<tbody>
<tr>
<td>No DL</td>
<td>$$160$</td>
<td>$$130$</td>
</tr>
<tr>
<td>DL</td>
<td>52.5</td>
<td>20.2</td>
</tr>
<tr>
<td>DL - Heur</td>
<td>4.6</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Table 4.4.4: Different Levels of Initial Demand Misestimation

We note from Table 4.4.3 that, in the demand overestimation case, the basic model under imperfect information appears to perform progressively better with increasing levels of initial inventory, and that this behavior is reversed in the demand underestimation case. While we do not have strong intuition or analytical support for this observation, we can, however, determine the optimality gap between the basic model with imperfect demand information (the NO DL model) and the same model with perfect information (the PERF INFO model) as the level of inventory approaches infinity, and this result is formally stated below.

Lemma 4.4.1: Let $\delta_0$ be the initial estimate of the unknown parameter $\delta$ and let $\delta_a$ be its true underlying value. Let $V^{NO DL}(l)$ be the expected revenues derived from the price strategy recommended by NO DL, and $V^{PERF INFO}(l)$ be the expected revenues from the price strategy recommended by PERF INFO, when the initial inventory level is $l$. For each period $t$, let $p_t(\delta)$ be the price that maximizes the function $pA(1-F_l(p, \delta))$. Then
\[
\frac{V_{\text{NGDL}}^N(I)}{V_{\text{PERFINFO}}^N(I)} \to \frac{\sum_{t=1}^{T} p_t(\delta_0) A_t(1 - F_t(p_t(\delta_0)|\delta_s))}{\sum_{t=1}^{T} p_t(\delta_s) A_t(1 - F_t(p_t(\delta_s)|\delta_s))} \quad \text{as } I \to \infty
\]

**Proof:** As \( I \to \infty \), the dynamic program for the basic pricing problem separates into \( T \) independent pricing problems, one for each period \( t \), since the inventory constraint for the season is effectively removed. The optimal price in period \( t \) is then given by the price that maximizes the expected period \( t \) revenue, \( pA_t(1-F_t(p|\delta)) \). (For a more rigorous proof of this result, see the proof of Proposition 2 in Bitran and Mondschein, 1995.) This price is \( p_t(\delta_0) \) in the NO DL case, and \( p_t(\delta_s) \) in the PERF INFO case, and the result then follows.

It is straightforward to show that in the case where the reservation price distributions \( F_t \) are of the exponential type, the above ratio can be arbitrarily bad (i.e., can be arbitrarily close to zero).

### 4.4.4 Impact of Initial Level of Demand Uncertainty

Table 4.4.5 shows the 'optimality gaps' between the optimal expected revenues under PERF INFO and the expected revenues under the approaches NO DL, DL and DL-HEUR under different values of \( S_0 \). This data suggests that the DL and DL-HEUR models both perform fairly robustly with respect to \( S_0 \) as long as \( S_0 \) is not too small. This is a very desirable result since \( S_0 \) is probably the most unintuitive of all the parameters that needs to be specified by the user, and would therefore require the most judgment on the user's part. The poor performance by DL and DL-HEUR for the case where \( S_0 \) is very small is not surprising - a small \( S_0 \) implies that the user is very certain about the initial estimate \( \delta_0 \), and the Bayesian updating approach would then continue to give a large weight to this initial estimate instead of responding more to the observed sales, thus making the demand learning model behave very similar to the basic (no-demand learning) model.
% Suboptimality of Revenues (Compared to PERF INFO)

<table>
<thead>
<tr>
<th></th>
<th>Overestimation (Mean = $130, i.e., δ = 0)</th>
<th>Underestimation (Mean = $70, i.e., δ = -30)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01 0.3 0.5 1 2</td>
<td>0.01 0.3 0.5 1 2</td>
</tr>
<tr>
<td>No DL</td>
<td>20.2 20.2 20.2 20.2 20.2</td>
<td>6.8 6.8 6.8 6.8 6.8</td>
</tr>
<tr>
<td>DL</td>
<td>12.5 1.9 2.4 2.4 2.9</td>
<td>6.8 2.2 1.5 1.1 1</td>
</tr>
<tr>
<td>DL - Heur</td>
<td>13.9 2.6 2.3 2.3 2.2</td>
<td>5.2 3.4 3.2 3.4 4.5</td>
</tr>
</tbody>
</table>

Table 4.4.5: Different Levels of Initial Demand Uncertainty

4.4.5 EXPECTED PRICE AND SALES TRENDS

We now discuss the expected price and sales trends observed for the base case runs. This analysis sheds some valuable insights into dynamic pricing behavior under misestimated demand; and the effect of demand learning on this behavior.

Figure 4.4.1 shows the expected prices in periods 1-4 under PERF INFO, NO DL and DL for the base case, and Figure 4.4.2 shows the cumulative expected sales at the end of periods 1-4 for the same cases. The NO DL and DL runs were performed under the demand overestimation version of the base case.
Under perfect information, the ideal pricing strategy is to maintain a fairly level price and sales trend throughout the season (with a slight dip in the last period). When demand is overestimated, both NO DL and DL start the season with higher-than-ideal prices, as we would expect. The NO DL approach leads to gradual markdowns over time since the model finds an unexpectedly high
level of unsold inventory after each period. This occurs since it always expects demand at the price it sets to be higher than what occurs on average, since it has overestimated the reservation price distribution. As seen in Figure 4.4.2, the NO DL approach leads to slow initial sales, and while the successive markdowns do help to accelerate sales, they do not have an adequate enough impact because the model continues to expect higher sales than what will actually occur on the average. When the season ends, there is therefore a significant amount of unsold stock left.

Under the DL approach, the initial price is again higher (and sales again slower) than the ideal levels. However, the slow sales in period 1 lead this model to learn that it has overestimated demand, and it responds by dropping the price significantly in order to affect a price correction. This price is even lower than that from PERF INFO since in the DL case the period 1 sales have been very low, and so it needs to compensate for it by accelerating sales in the subsequent periods. Notice that DL does a very good job of balancing supply and demand by the end of the season by correcting its price path early in the season.

Figures 4.4.3 and 4.4.4 show the same output as Figures 4.4.1 and 4.4.2 for the demand underestimation version of the base case.

![Base Case with Underestimation, Expected Price Trends](image)

**Figure 4.4.3**
The ideal pricing policy under perfect information is exactly the same as previously seen, since this assumes perfect demand estimation (and is therefore not affected by the demand over or under estimation). When demand is underestimated, both NO DL and DL start the season with lower prices. The NO DL approach leads to gradual markups over time as the model finds an unexpectedly high level of sales in each period. This is so since it always expects demand at the price it sets to be lower than what occurs on average, since it has underestimated the reservation price distribution. As seen in Figure 4.4.2, the NO DL approach leads to higher-than-ideal levels of initial sales, and while the successive markups do help to slow down sales, they do not have an adequate enough impact because the model continues to expect lower sales than what will actually occur on the average. On the average, therefore, the inventory is exhausted much before the season ends, leading to a stockout. Under the DL approach, the initial price is again lower (and sales again higher) than the ideal levels. However, the high sales level in period 1 leads this model to learn that it has underestimated demand, and it responds by raising the price significantly in order to affect a price correction. This price is even higher than that from PERF INFO since in the DL case the period 1 sales have been very high, and so it is left with fewer units to sell in the
subsequent periods. DL again does a very good job of balancing supply and demand by the end of the season by correcting its price path early in the season.

The increase in prices recommended by the DL and NO DL models above may be infeasible in many retail contexts, for retail practice (based, say, on consumer expectations) may require that the price of the merchandise not be marked up within a season. While the above analysis may still provide useful insights about optimal pricing behavior under freer pricing conditions, it is also interesting to study the performance of different approaches under such a non-increasing price requirement.

4.4.6 IMPACT OF NON-INCREASING PRICE CONSTRAINT

In order to impose this requirement, the basic pricing model and the model with demand learning are extended in the following manner. The price variable is discretized (so that price can only assume one of a limited set of values). The state space for both models is augmented by a new variable that stands for the price determined for the previous period. This price variable acts as an upper bound on price in the current period. The dynamic programming formulation of this problem can be found in Bitran and Mondschein (1995). These authors also present a number of computational test results for this model. The formulation for the demand learning case can be derived in a simple manner as an extension to the one considered in this chapter, and so we do not provide a technical description here. The heuristic solution approach we described in Section 4 does not transfer over to this non-increasing price context.

Figure 4.4.5 shows the expected price trends from a series of runs of the different models with and without the non-increasing price constraint. These runs were made using the base case data presented earlier - in particular, the initial inventory level is 50, and the initial coefficient of variation S is 0.5 for the DL runs. Under price overestimation, the impact of the non-increasing constraint on the price trends under DL and NO DL is minimal, since the price tends to be revised downwards over time. Therefore, the DL and NO DL runs shown in Figure 4.4.5 correspond to
the base case of demand underestimation ($\delta_0 = -$60). All runs were done using a discrete set of prices based on $5 increments.

![Graph: Expected price trends with and without non-increasing price constraint](image)

**Figure 4.4.5: Expected price trends with and without non-increasing price constraint**

The following observations can be made about the price trends shown in the above figure:

- For each model (PERF INFO, NO DL, and DL), the imposition of the non-increasing price constraint leads to an increase in the initial (period 1) price. This is intuitive - since there will not be any recourse to revising the price upwards in the future, the retailer starts with a higher period 1 price in order to have more pricing flexibility (i.e., a greater range of feasible prices) in the future, and this flexibility is important because of the uncertainty in demand. (Note that, for the DL and NO DL models, this increase is *not* influenced by the fact that demand has been initially underestimated, since the first period price is determined by these models before any sales have been observed. We would therefore expect to see the same result even for the case of demand overestimation.)

- Since the DL and NO DL runs are based on demand underestimation, the initial prices under these runs are lower than the price under PERF INFO.
• For the non-increasing price case, the period 1 price under the DL model is higher than that under the NO DL model. In the DL case, the model recognizes that demand may have been significantly over or underestimated, and sets the initial price in a manner that will allow it to revise price appropriately as demand learning occurs over time. This initial price is therefore set at a higher value than for the NO DL model, since the DL model wants to gives itself more flexibility in the range of prices it can adopt in future periods in response to demand learning.

4.5. Conclusions And Future Research Issues

In this chapter, we have presented a methodology for dynamic demand learning in an environment where the demand distribution changes over time due to changes in price, customer arrival rate and customers’ values for the product. We have shown how this methodology can be incorporated in a model of dynamic pricing for seasonal products. In addition, we have performed a series of computational tests to evaluate our demand learning scheme and to study the performance of dynamic pricing strategies with and without demand learning. Some key conclusions and insights that we have derived from these computational tests are the following:

• Under the demand learning methodology presented in this chapter, the estimated reservation price distribution converges to the actual reservation price distribution and the variances of the priors converge to zero as more observations are gathered. The methodology is therefore successful in utilizing sales data to move from the planners’ initial estimate of customers’ values for the product to their true underlying values.

• When there is a significant misestimation of demand, the optimal pricing strategy under no demand learning will result in large revenue losses due to:
  - Overpricing and high levels of unsold stock (under demand overestimation), or
  - Underpricing and early stockouts (under demand underestimation)

• Under conditions of demand misestimation, dynamic demand learning can help to limit the revenue losses to relatively nominal levels.
• Under the demand learning approach, there is a swift price correction early in the season with not much learning thereafter. Thus, it is important to have good pricing flexibility early in the season.

• When a non-increasing price constraint is imposed, both the NO DL and DL approaches start with a higher initial price in order to maintain more pricing flexibility in future periods. However, the DL approach starts with a substantially higher initial price than the NO DL approach since it recognizes that, in addition to the residual uncertainty, demand may have been significantly over or underestimated initially.

While it has addressed a key limitation of existing models of seasonal product pricing, our research has also identified a number of methodological and empirical issues on which further work would be useful. These are discussed below.

• *Increased demand variability:* At any given price \( p \) and time period \( t \), we assume that the underlying demand distribution is Poisson. Since the variance of a Poisson distribution is equal to its mean, the use of this distribution strictly limits the variance of the underlying demand distribution. Alternative distribution forms such as the negative binomial and the normal allow for higher variances, and an extension of the demand learning approach to one of these cases may therefore be useful.

• *Test marketing, reorderings and channel arrangements:* In some situations, retailers may have the flexibility to place a second order for the product early in the sales season after gaining some limited demand information through observed sales at some stores or through consumer interviews. An appealing extension of the model in this chapter would be one that incorporates a reordering decision after the first period. Certain aspects of manufacturer-retailer contracts, such as a limit on the reorder quantity, end-of-season returns to the manufacturer, and backup agreements (such as those considered by Eppen and Iyer, 1995) may also be investigated through extensions of the model in this chapter. An analysis of such models may provide useful insights about the retailer's market testing, pricing and ordering strategy.

• *Multiparameter (or non-parametric) estimation:* Our demand learning method assumes that the retailer's lack of information can be captured via a reservation price distribution in which a
single parameter is unknown to the planner. Thus, for instance, in our computational tests, we have assumed that the planner knows the shape and scale of the (Weibull) reservation price distribution and does not know the location (or equivalently, the mean) of the distribution. A more powerful demand learning model would allow for even less of prior knowledge on the retailer's front by determining the true reservation price distribution with more limited prior information. Alternatively, one could use the "Weibull with unknown location parameter"-based context of our computational study and perform additional tests to evaluate the performance of the demand learning scheme in situations where the planner has misjudged the shape and scale of the reservation price distribution.

- **Non-stationary reservation price distribution**: Another useful extension may be in contexts where the reservation price distribution changes with time. Our approach assumes that this change follows a certain pattern which is known to the planner, and this may in certain contexts be a limiting assumption. An alternative technique might apply differential weights to the sales data from past periods to capture the trend in customers' values for the product in addition to revising the retailer's initial estimate of this distribution.

- **Empirical study**: The discussion in the last section has highlighted the importance of demand learning and of correcting prices swiftly in response to such learning. This analysis leads to a number of questions on industry practice that merit empirical investigation, such as the following: How effective are retail organizations in learning about demand early in the season and to what extent do they react through early price corrections? How does our methodology perform in comparison with the decisions made by merchandise managers at these organizations?
CHAPTER 5

MULTI-STORE MODELS

5.1 Introduction

In this chapter, we develop models that address pricing and inventory management problems faced by multi-store organizations. We formulate three multi-store models that reflect different distribution practices commonly found in the retail industry. (Our knowledge of these practices is based on our discussions with industry representatives during an investigation phase of our work) As in previous chapters, these models are formulated as dynamic programs. The state space for each of these models is very large, rendering the dynamic programs computationally infeasible. We describe three heuristic solution schemes and an upper bounding scheme for these models, and present computational test results for some of these schemes.

To our knowledge, Bitran and Mondschein (1993) are the only researchers thus far who have considered pricing decisions in a multi-store environment. These authors formulate a multiple store model where the merchandise is initially distributed across all the stores and is then transferred between stores on a periodic basis as desired. Each store sets its own price in each period. The resulting dynamic program is computationally intractable, and so the authors presented a heuristic approach and an upper bounding procedure for the problem.

There are two important shortcomings in Bitran and Mondschein's work. Firstly, their model is not appropriate for many retail contexts. In many retail organizations, such as regional department store chains, pricing is centralized - that is, the product is sold at the same price in all the company's stores, though this price, of course, may change over the course of the season. Also, in many contexts, merchandise is transferred to stores on a periodic instead of a one-time basis, and inter-store transfers are not allowed. A second shortcoming of this paper is that the
solution procedure presented for the heuristic becomes too unwieldy computationally even for problems of moderate size (e.g., 10-15 stores).

In this chapter, we address the first shortcoming by presenting three different models that address different distribution practices by the retailer. These models also assume that the pricing function is centralized - in each period, the same price is charged across all the stores. We address the model solution issue by presenting three heuristics and an alternative upper bounding procedure to the one presented by Bitran and Mondschein.

5.2 Models

We study the following multi-store models in this chapter. In all these models, the retailer is required to set the same price across all the stores in any given period.

**PRE:** In this *(predistribution)* model, the merchandise is pre-distributed to all the stores in the retail chain before the start of the sale season. No inter-store transfers are allowed at any time.

**PRETR:** This is the same as the model PRE, except that it allows for inter-store transfers (at a given unit transfer cost) on a periodic basis.

**POST:** In this *(postdistribution)* model, the merchandise is stored at a central warehouse, and is sent to the stores on a periodic basis (at a given unit distribution cost). Hence, in each period, the retailer has to determine both the price for the product as well as the amount of inventory to ship to each store from the central stocking facility.

In models PRETR and POST, the transfers (across stores, and from the central stocking facility) are assumed to take place instantaneously. Also, the unit inter-store transfer cost is assumed to be the same across all pairs of stores in model PRETR and the unit distribution cost is assumed to be the same across all stores in model POST. (As we describe later in this chapter, Heuristic 1 can be applied to model PRETR for the case of store-dependent transfer costs also)
5.2.1 PRE: PREDISTRIBUTION WITHOUT INTERSTORE TRANSFERS

Notation:

\( \delta_{st}(p) \) = Random variable representing demand at store \( s \) in period \( t \) at price \( p \)

\( C \) = Total initial inventory

\( I_{st} \) = Inventory level at store \( s \) at beginning of period \( t \)

\( VP_t(C) \) = Optimal expected revenues from period 1 onwards, given initial inventory of \( C \) units

\( VP_t(I_{1t}, \ldots I_{st}) \) = Optimal expected profits from period \( t \) onwards, given inventory level of \( I_{st} \) at store \( s \) at the beginning of period \( t \).

DP Recursion:

\[
VP_t(I_{1t}, \ldots, I_{st}) = \max_{p} \left\{ E_{\delta_{st}(p)} \left[ p \sum_{s=1}^{s} \min\{I_{st}, \delta_{st}(p)\} + VP_{t+1}(I_{1t+1}, \ldots, I_{st+1}) \right] \right\} \quad \text{for} \quad t = 2, \ldots, T
\]

\[
VP_t(C) = \max_{p, I_{1t} \rightarrow I_{st}} \left\{ E_{\delta_{st}(p)} \left[ p \sum_{s=1}^{s} \min\{I_{st}, \delta_{st}(p)\} + VP_{2}(I_{12}, \ldots, I_{s2}) \right] \right\}
\]

where the variables satisfy the following constraints:

\[
p, I_{st} \geq 0 \quad \text{for all } s
\]

\[
I_{st+1} = I_{st} - \min\{I_{st}, \delta_{st}(p)\} \quad \text{for all } s, t
\]

\[
\sum_{s=1}^{s} I_{st} = C
\]

5.2.2 PRETR: PREDISTRIBUTION WITH INTERSTORE TRANSFERS

Notation:

\( \delta_{st}(p) \) = Random variable representing demand at store \( s \) in period \( t \) at price \( p \)

\( C \) = Total initial inventory
\( J_{st} \) = Inventory level at store \( s \) at end of period \( t \)

\( I_{st} \) = Inventory level at store \( s \) at beginning of period \( t \) (after transfers have occurred)

\( v \) = Unit inter-store transfer cost

\( \text{VP}_t(J_{1t-1}, \ldots, J_{St-1}) = \text{Optimal expected profits from period } t \text{ onwards, given inventory level at store } s \)

\( \text{at the end of period } t-1 \text{ is } J_{st-1}. \)

**DP Recursion:**

\[
\text{VP}_t(J_{1t-1}, \ldots, J_{St-1}) = \max_{p, I_{1t-1} - I_{st}} \left\{ E_{\delta_1(p), \ldots, \delta_t(p)} \left[ p \sum_{s=1}^{S} \min\{I_{st}, \delta_{st}(p)\} + \text{VP}_{t+1}(J_{1t}, \ldots, J_{St}) \right] - \frac{1}{2} v \sum_{s=1}^{S} |I_{st} - J_{st-1}| \right\} \quad \text{for } t > 1
\]

\[
\text{VP}_1(C) = \max_{p, I_{11} - I_{s1}} \left\{ E_{\delta_1(p)} \left[ p \sum_{s=1}^{S} \min\{I_{s1}, \delta_{s1}(p)\} + \text{VP}_2(J_{11}, \ldots, J_{S1}) \right] \right\}
\]

where the variables satisfy the following constraints:

\( p, I_{st} \geq 0 \) \quad \text{for all } s, t

\( J_{st} = I_{st} - \min\{I_{st}, \delta_{st}(p)\} \) \quad \text{for all } s, t > 0

\[
\sum_{s=1}^{S} I_{st} = C
\]

\[
\sum_{s=1}^{S} I_{st} = \sum_{s=1}^{S} J_{st-1} \quad \text{for all } t > 1
\]

Note that \( \sum_{s=1}^{S} |I_{st} - J_{st-1}| \) calculates the total flow of inventory between stores *twice* - hence the use of term 1/2 in calculating the total inventory flow and transfer cost.

### 5.2.3 POST: POSTDISTRIBUTION

**Notation:**

\( \delta_{st}(p) \) = Random variable representing demand at store \( s \) in period \( t \) at price \( p \)

\( C \) = Total initial inventory

\( J_{st} \) = Inventory level at store \( s \) at end of period \( t \)
\[ I_s = \text{Inventory level at store } s \text{ at beginning of period } t \text{ (after transfer from central facility has occurred)} \]

\[ v = \text{Unit distribution cost from central facility to each store.} \]

\[ VP_t(J_{s1}, \ldots, J_{sT}) = \text{Optimal expected profits from period } t \text{ onwards, given inventory level at store } s \text{ at the end of period } t-1 \text{ is } J_{st-1}. \]

**DP Recursion:**

\[ VP_t(J_{s1}, \ldots, J_{sT}) = \max_{p, J_{s1}, \ldots, J_{sT}} \left\{ E_{a(p), \ldots, a(p)} \left[ p \sum_{s=1}^{S} \min(I_s, \beta_s(p)) + VP_{t+1}(J_{s1}, \ldots, J_{sT}) \right] - v \sum_{s=1}^{S} (I_s - J_{st-1}) \right\} \]

where the variables satisfy the following constraints:

\[ p \geq 0 \]

\[ J_s = I_s - \min \{ I_s, \beta_s(p) \} \quad \text{for all } s, t > 0 \]

\[ J_s = 0 \quad \text{for all } s \]

\[ \sum_{t=1}^{T} \sum_{s=1}^{S} (I_s - J_{st-1}) \leq C \quad (5.1) \]

\[ I_s \geq J_{st-1} \quad \text{for all } s, t \]

Note that \((I_s - J_{st-1})\), used in the objective function of the above maximization problem as well as in Equation 5.1, is the total amount of inventory distributed to store \(s\) at the beginning of period \(t\).

### 5.3 Heuristics

The state space for each of the above models is too large for even small-sized problems. We therefore need to develop ways to solve the above problems in an approximate manner. In this section, we present three heuristic schemes for solving the above models.

#### 5.3.1 HEURISTIC 1: TIME AGGREGATION

The first heuristic involves collecting together all the future periods into a single aggregate period and then solving the resulting model assuming that only a one-time pricing and (if applicable) distribution decision is to be made at the start of the planning horizon. This approximation
approach was used by Bitran and Mondschein for the PRETR model (with decentralized pricing). However, while their heuristic solution procedure is computationally intractable for even moderately sized problems, we present an efficient solution approach for our heuristic below. This solution strategy is based on recognizing that the optimization problem related to the heuristic has special structure. This heuristic is applicable to all three models considered in this chapter. We describe the heuristic below for the case of the PRETR model. The modifications required for the other models are straightforward, and we comment on these at the end of this section.

**PRETR model**

The time aggregation heuristic is applied on a rolling horizon basis. At the beginning of every period $t$, the remaining periods ($t, t+1, \ldots, T$) are aggregated into a single period and the resulting problem solved to yield the optimal "aggregate period" price and the optimal interstore transfer quantities. The actual quantities transferred between stores are then calculated in the following manner. First, the expected demand in period $t$ (at the optimal "aggregate period" price) is calculated at each store. This demand is scaled up by a pre-specified parameter which we term the *distribution parameter*. Then the net demand (scaled demand minus current inventory), if any, is calculated for each store. Merchandise is then transferred from stores with negative demand (excess inventory) to stores with positive demand (shortfall of inventory) up to a point where the net demand becomes zero at either all the stores with shortfall or all the store with excess. If there is not enough merchandise at the stores with excess inventory to satisfy the net demand at all the stores with a shortfall of inventory, then the excess inventory is allocated to the latter stores in a manner proportional to the demand at these stores.

We describe below how the time aggregation approach works for any period $t > 1$. We comment later on the modification required for period 1. Period 1 needs to be treated differently because the distribution decision in period 1 involves the initial allocation of the system inventory across all stores as opposed to a transfer of inventory between stores.
Suppose we are at the beginning of any period \( t > 1 \), with the inventory at store \( s \) being \( C_s \). We use many of the same parameters and variables defined in the previous section. In addition, we define the following variables:

\[
\beta_s(p) = \sum_{\tau=1}^{T} \beta_{s\tau}(p), \quad \text{and} \quad \lambda_s(p) = \sum_{\tau=1}^{T} \lambda_{s\tau}(p)
\]

\( \beta_s(p) \) is random variable with arrival rate \( \lambda_s(p) \).

By aggregating all the time periods \( \tau = t, t+1, \ldots, T \) together, we can approximate the multiple store model PRETR by the following optimization problem:

\[
\max_{p \geq 0} \left\{ \max_{(i_s,b_s)} \left\{ E_{A_s} \left[ \sum_{s=1}^{S} \min(A_s(p), I_s) \right] - \frac{V}{2} \sum_{s=1}^{S} |I_s - C_s| \right\} \right\} \quad \text{(Problem 5.3.1)}
\]

where the variables satisfy the following constraints:

\[
\sum_{s=1}^{S} I_s = \sum_{s=1}^{S} C_s \quad \text{and} \quad I_s \geq 0 \quad \forall s.
\]

Our approach to solve the above problem will be to search over all the possible price levels \( p \) and to solve the resulting inventory transfer problem, which we term \( \text{OPT}(p) \) - PRETR. (We assume here that the price is selected in each period from a discrete set of values.) Once we have done so for all price levels, the best overall solution would constitute the optimal solution to Problem 5.3.1. For this procedure to be practical, we must be able to solve \( \text{OPT}(p) \) - PRETR efficiently. We show how this can be done below.

\[
\text{OPT}(p) \text{- PRETR:} \quad \max_{(i_s,b_s)} \left\{ E \left[ p \sum_{s=1}^{S} \min(A_s(p), I_s) \right] - \frac{V}{2} \sum_{s=1}^{S} |I_s - C_s| \right\}
\]

\[
\quad \text{s.t.} \quad \sum_{s=1}^{S} I_s = \sum_{s=1}^{S} C_s \quad \text{and} \quad I_s \geq 0 \forall s
\]

\( \text{OPT}(p) \) - PRETR can be reformulated as a linear programming problem. Let \( M = \sum_{s=1}^{S} C_s \) represent the total amount of the product that is available in the system. We will replace the variables \( I_s \) in the above problem by new variables \( \{x_{ms}: m=1, \ldots, M \text{ and } s=1, \ldots, S\} \) defined as follows:
\[ x_{sm} = 0 \text{ or } 1 \quad \text{for all } s,m \quad (5.2) \]
\[ x_{sm} \leq x_{sm-1} \quad \text{for all } s, m=2,..,M \quad (5.3) \]

We ask the reader to imagine that at each store \( s \) there are \( M \) (single-unit) inventory slots. Then \( x_{sm} \) represents whether or not the \( m \)th inventory slot at store \( s \) has been filled at the beginning of the planning horizon, after the transfers have been made. We can thus replace \( I_s \) by \( \sum_{m=1}^{M} x_{sm} \) in the expectation below:

\[
E \left[ p \sum_{s=1}^{S} \min(D_s(p), I_s) \right] = E \left[ p \sum_{s=1}^{S} \min(D_s(p), \sum_{m=1}^{M} x_{sm}) \right] = p \sum_{s=1}^{S} \left( \sum_{j=1}^{\infty} P[D_s(p) = j] \min\left(j, \sum_{m=1}^{M} x_{sm}\right) \right) \\
= p \sum_{s=1}^{S} \left( \sum_{j=1}^{\infty} x_{sm} \sum_{j=1}^{\infty} P[D_s(p) \geq j] \right) \quad \text{A} \\
= p \sum_{s=1}^{S} \left( \sum_{j=1}^{\infty} P[D_s(p) \geq j] x_{sj} \right) \quad \text{B} \\
\]

We have used the following relation in going from term A to term B:

\[
\sum_{j=1}^{\infty} P[N_s(p) = j] \min(j, I) = \sum_{j=1}^{I} P[N_s(p) \geq j] 
\]

and term C has been derived from term B by using the definition of the variables \( x_{sm} \) from Equations (5.2) and (5.3).

By replacing the variables \( I_s \) with the variables \( x_{sm} \), and the expectation in \( \text{OPT}(p) - \text{PRETR} \) with term C, we can reformulate \( \text{OPT}(p) - \text{PRETR} \) as follows:
Problem OPT(p) - PRETR:

\[
\begin{align*}
\max & \quad p \sum_{s=1}^{S} \left( \sum_{j=1}^{M} P[\beta_j(p) \geq j] x_{sj} \right) - \nu \sum_{s=1}^{S} t_s \\
\text{s.t.} & \quad \sum_{s=1}^{S} \sum_{m=1}^{M} x_{sm} = M \quad (5.4) \\
& \quad t_s \geq \sum_{m=1}^{M} x_{sm} - C_s \quad \forall s \quad (5.5) \\
& \quad x_{sm}, t_s \geq 0 \quad \forall s, m \quad (5.7) \\
& \quad x_{sm} \leq x_{s m-1} \quad \forall s, m = 2, \ldots, M \quad (5.8) \\
& \quad x_{sm} = 0 \text{ or } 1 \quad \forall s, m \quad (5.9)
\end{align*}
\]

Note that we have introduced additional variables \( t_s \) that denote the 'net inflow' of inventory into store \( s \) through the inter-store transfers at the beginning of the planning horizon. Since \( P[\beta_j(p) \geq j] > P[\beta_j(p) \geq j+1] \) for all \( j \geq 0 \), it can be shown in a straightforward manner that the variables \( x_{sm} \) will automatically satisfy Constraints (5.8) and (5.9), and so these constraints can be eliminated from the above problem, leaving us with a linear programming formulation.

**Store-dependent transfer costs:** We have assumed above that the unit inter-store transfer cost is store-independent. Below, we describe the linear programming formulation for problem OPT(p) - PRETR for the case where the inter-store transfer costs are store-dependent. We define the following additional parameters and decision variables:

\( v_{ij} \) = unit transfer cost from store \( i \) to store \( j \).

\( y_{ij} \) = amount of product transferred from store \( i \) to store \( j \) at the beginning of the planning horizon.
Problem OPT(p) - PRETR - Store dependent transfer costs

\[
\max \sum_{s=1}^{S} \left( \sum_{j=1}^{M} p_j \cdot \delta_j(p) \geq j \right) x_{ij} - \left( \sum_{s=1}^{S} \sum_{r=1}^{S} v y_{sr} \right)
\]

s.t. \quad \sum_{s=1}^{S} \sum_{m=1}^{M} x_{sm} = M \quad (5.10)

\[
\sum_{r=1}^{S} y_{rs} \geq \sum_{m=1}^{M} x_{sm} - \left( C_r - \sum_{r=1}^{S} y_{sr} \right) \quad \forall s
\]

\[
x_{sm}, y_{sr} \geq 0 \quad \forall s, r, m \quad (5.12)
\]

\[
x_{sm} \leq x_{sm-1} \quad \forall s, m = 2, \ldots, M \quad (5.13)
\]

\[
x_{sm} = 0 \quad \text{or} \quad 1 \quad \forall s, m \quad (5.14)
\]

As before, it can be reasoned that Constraints (5.13) and (5.14) will be automatically satisfied, and thus they can be removed from the formulation, leaving us again with a linear program.

While the problem OPT(p) - PRETR may be solved as a linear program, we describe below a more direct solution method that is applicable for the original case of store-independent transfer costs.

**Greedy algorithm for OPT(p) - PRETR**

This algorithm is motivated by the following way of looking at the above optimization problem: There are a total of M units of inventory to be allocated to all the stores. We have to decide on the best allocation of this stock of inventory to all the stores, keeping in mind the expected revenue gained from having stock at a store and the cost incurred in transferring stock to a store. By considering the nature of the objective function for OPT(p) - PRETR, we can deduce the following:

For any inventory unit j = 1, ..., M, and store s = 1, ..., S, the following is true:
The \( j \)th unit of inventory that is allocated to store \( S \) brings with it an incremental expected revenue of \( pP[\delta_s(p) \geq j] \). This \( j \)th unit will also bring with it a transfer cost of \( v \) if it needs to be brought to store \( s \) from some other store, i.e., if \( j > C_s \).

Note that for each incremental unit \( j \) allocated to a store \( s \), the expected revenue for that unit is decreasing with \( j \), while the expected cost (which is 0 for the first \( C_s \) units and \( v \) after that) is non-decreasing with \( j \). Hence the incremental expected profit \( (= \text{expected revenue} - \text{cost}) \) is decreasing with \( j \). This observation allows us to utilize a greedy algorithm to perform the optimal allocation of the \( M \) units across the stores: We take the inventory units one by one and allocate the current unit to the current best store, where ‘best’ is determined by the incremental expected profit level for that store, updating our incremental expected profit level information for that store before we move to the next unit of inventory.

We provide a formal description of the algorithm below.

**Step 1. Initialize decision variables:**

Set \( I_s = 0 \) for all \( s = 1, \ldots, S \).

**Step 2. Set initial incremental expected profit levels:**

For each \( s = 1, \ldots, S \), set

\[
    r_s = \begin{cases} 
    pP[\delta_s(p) \geq I_s + 1], & \text{if } I_s < C_s \\
    pP[\delta_s(p) \geq I_s + 1] - v, & \text{if } I_s \geq C_s 
    \end{cases}
\]

**Step 3. Allocate next unit of inventory:**

Let \( s' = \arg\max_{s=1, \ldots, S} \{r_s\} \)

Set \( I_{s'} = I_{s'} + 1 \)

**Step 4: Reset incremental expected profit level for store \( s' \):**

\[
    r_{s'} = \begin{cases} 
    pP[\delta_{s'}(p) \geq I_{s'} + 1], & \text{if } I_{s'} < C_{s'} \\
    pP[\delta_{s'}(p) \geq I_{s'} + 1] - v, & \text{if } I_{s'} \geq C_{s'} 
    \end{cases}
\]

**Step 5. Iterate back or stop:**
If $\sum_{s=1}^{S} I_s = M$, stop. \{I_s\} is the optimal inventory allocation.

Else, go back to Step 3.

\[ \]  \[ \]

**Period 1 modifications**

In period 1, the distribution decision involves the initial allocation of inventory across all stores and there are no interstore transfers involved. We use the same general approach described above to solve the time aggregation problem associated with period 1. Suppose we have $M$ units of inventory in stock to be distributed across all the stores. Problem OPT(p) - PRETR now becomes:

**Problem OPT(p) - PRETR - Period 1:**

\[
\begin{align*}
\text{max} & \quad p \sum_{s=1}^{S} \left( \sum_{j=1}^{M} P[\beta_s(p) \geq j] x_{sj} \right) \\
\text{s.t.} & \quad \sum_{s=1}^{S} t_s = M \\
& \quad t_s \geq \sum_{m=1}^{M} x_{sm} \quad \forall s \\
& \quad x_{sm}, t_s \geq 0 \quad \forall s, m \\
& \quad x_{sm} \leq x_{sm-1} \quad \forall s, m = 2, \ldots, M \\
& \quad x_{sm} = 0 \text{ or } 1 \quad \forall s, m
\end{align*}
\]

This problem can again be shown to be a linear program. The greedy solution approach can be applied to solve OPT(p) - PRETR for period 1 with the following modifications:

**Step 2** Set initial *incremental expected profit* levels:

For each $s = 1, \ldots, S$, set

\[
 r_s = pP[\beta_s(p) \geq I_s + 1]
\]

**Step 4:** Reset *incremental expected profit* level for store $s'$:

\[
 r_{s'} = pP[\beta_s(p) \geq I_s + 1]
\]
Modifications for PRE, POST

The period 1 problem under the time aggregation approach for PRE and POST is the same as the one for PRETR.

For any period \( t > 1 \), there is only a pricing decision involved in PRE, and so the time aggregation method finds the optimal price in a straightforward manner. For POST, the problem OPT(p) for any period \( t > 1 \) is shown below. The inventory at the beginning of period \( t \) at each store \( s \) is \( C_s \) and at the DC is \( M - \sum_{s=1}^{S} C_s \).

Problem OPT(p) - POST:

\[
\begin{align*}
\max & \quad p \sum_{s=1}^{S} \left( \sum_{j=1}^{M} P[ \mathcal{E}_s(p) \geq j] x_{sj} \right) \\
\text{s.t.} & \quad \sum_{s=1}^{S} t_s = M - \sum_{s=1}^{S} C_s \\
& \quad t_s + C_s \geq \sum_{m=1}^{M} x_{sm} \quad \forall s \\
& \quad x_{sm}, t_s \geq 0 \quad \forall s, m \\
& \quad x_{sm} \leq x_{sm-1} \quad \forall s, m = 2, \ldots, M \\
& \quad x_{sm} = 0 \quad \text{or} \quad 1 \quad \forall s, m
\end{align*}
\]

This problem can again be shown to be a linear program. The greedy solution approach can be applied to solve OPT(p) - POST for period \( t \) with the following modifications:

Step 1. Initialize decision variables:

Set \( I_s = C_s \) for all \( s = 1, \ldots, S \).

Step 2. Set initial incremental expected profit levels:

For each \( s = 1, \ldots, S \), set
\[ r_s = pP[\delta_s(p) \geq I_s+1] - v \]

Step 4: Reset incremental expected profit level for store \( s' \):
\[ r_{s'} = pP[\delta_{s'}(p) \geq I_{s'}+1] - v \]

5.3.2 HEURISTIC 2: EXPECTED PRICE APPROXIMATION

Our second heuristic scheme involves a different way of approximating the multiple store problem. Unlike Heuristic 1, Heuristic 2 retains the multi-time period structure of the model, but it replaces each of the random variables \( \delta_n(p) \) in the model by its expected value \( E[\delta_n(p)] \), which we term \( \lambda_n(p) \), thereby leading to a deterministic dynamic optimization problem. For each of the models, this deterministic problem can be formulated as a non-linear program as described below.

As with Heuristic 1, this heuristic would also ideally be implemented on a rolling horizon basis. The non-linear programming problem associated with the heuristic differs in structure between period 1 and the other periods when applied on a rolling horizon basis, since individual stores may start with some non-zero level of inventory at the beginning of any period \( t > 1 \), while they start with zero inventory at the beginning of period 1. The period 1 problem is similar across all the three models PRE, PRETR and POST, and the solution to this problem is described below. The period \( t \) problem where \( t > 1 \) for POST and PRE is solved heuristically, and we describe this approach after the discussion of the period 1 problem. We do not present any solution for the PRETR model for the period \( t \) problem where \( t > 1 \).

In addition to its use within a heuristic solution scheme, the deterministic problem associated with the expected price approximation also helps to provide an upper bound for each model - the optimal solution to the period 1 problem discussed above is an upper bound for the actual problem for each of the models PRE, PRETR and POST. We prove this claim in the next section of this chapter.
PERIOD 1 PROBLEM
When demand is deterministic, and all stores start with zero inventory, there will be no interstore transfers. Also, in such cases the amount consumed at any given store \( s \) can be viewed as being delivered on a periodic distribution basis or on a one-time basis. It is therefore easy to see that the deterministic expected-value problem introduced above will be the same for all three models - PRE, PRETR and POST.

For PRE, PRETR and POST models:
\[
d_{st} = \text{amount of product transferred to store } s \text{ at beginning of period } t \\
I_{st} = \text{inventory level at store } s \text{ at beginning of period } t \text{ (after transfers)}
\]

\[
\max \sum_{t=1}^{T} \sum_{s=1}^{S} p_{st} S_{st} - v \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st}
\]

s.t.
\[
\sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \leq I
\]
\[
S_{st} \leq \lambda_{st}(p_{t,}) \quad \forall \ s, t
\]
\[
S_{st} \leq d_{st} \quad \forall \ s, t
\]
\[
p_{t,}, d_{st}, S_{st} \geq 0 \quad \forall \ s, t
\]

This model can be viewed as a generalization to the multi-store context of the single store deterministic model presented and analyzed in Section 3.3.1. Lemma 5.1 allows us to reformulate the above problem just as Lemma 3.3.1.1 allowed us to reformulate the single store problem.

**Lemma 5.1:** There exists an optimal solution \( \{\bar{p}_{t,}, \bar{S}_{st}, \bar{d}_{st}\} \) to the non-linear programming problem above in which \( \bar{S}_{st} = \bar{d}_{st} = \lambda_{st}(\bar{p}_{t,}) \) for all \( t, m \).

We can therefore reformulate the above problem as follows:
\[
\max \sum_{t=1}^{T} p_t \left( \sum_{s=1}^{S} \lambda_{at}(p_t) \right) - \nu \sum_{t=1}^{T} \left( \sum_{s=1}^{S} \lambda_{at}(p_t) \right)
\]
\[
s.t.
\sum_{t=1}^{T} \left( \sum_{s=1}^{S} \lambda_{at}(p_t) \right) \leq I \quad \forall \ t
\]
\[
p_t \geq 0 \quad \forall t
\]

Defining \( D_t(p_t) = \sum_{s=1}^{S} \lambda_{at}(p_t) \), we get:

\[
\max \sum_{t=1}^{T} p_t D_t(p_t) - \nu \sum_{t=1}^{T} D_t(p_t)
\]
\[
s.t.
\sum_{t=1}^{T} D_t(p_t) \leq I \quad \forall \ t
\]
\[
p_t \geq 0 \quad \forall t
\]

We assume that the \( \lambda_{at}(\cdot) \)'s satisfy the following condition:

**Condition 5.1:** \( \lambda_{at}(p) \) is a continuous and monotonically decreasing function of \( p \) for all \( s \) and \( t \).

It follows then that \( D_t(p) \) is a continuous and monotonically decreasing function of \( p \) also, and hence is invertible in \( p \). Let \( d_t = D_t(p_t) \), so that \( D_t^{-1}(d_t) = p_t \). The above problem can be expressed in terms of \( d_t \) and \( D_t^{-1} \) as:

\[
\max \sum_{t=1}^{T} d_t D_t^{-1}(d_t) - \nu \sum_{t=1}^{T} d_t
\]
\[
\sum_{t=1}^{T} d_t \leq I
\]
\[
d_t \leq D_t(0) \quad \forall \ t
\]

This is very similar to the final reformulation of the single store problem studied in Section 3.3.1.

The only difference is the addition of the linear term \( -\nu \sum_{t=1}^{T} d_t \) in the objective function for the above problem. As in Lemma 3.3.1.2, we can show that the objective function above is concave provided the following condition is satisfied:
Condition 5.2: $2(D_t(p))^2 - D_t(p)D_t''(p) \geq 0$ for all $p > 0$ and all $t$.

We can analyze and solve this problem in the same manner as done in Section 3.3.1, and so we do not repeat the discussion here. Writing Condition 5.2 in terms of $\lambda_{st}(p)$, we get

$$2\left(\sum_{s=1}^{S} \lambda_{st}'(p)\right)^2 - \left(\sum_{s=1}^{S} \lambda_{st}'(p)\right)\left(\sum_{s=1}^{S} \lambda_{st}''(p)\right) \geq 0 \quad \text{for all } p, t \text{ and } s, \quad \text{or}$$

Condition 5.2-\lambda: $\sum_{s=1}^{S} \left(2\left(\lambda_{st}'(p)\right)^2 - \lambda_{st}'(p)\lambda_{st}''(p)\right) + \sum_{s,s' \in [1, S]} \left(2\lambda_{st}'(p)\lambda_{st}'(p) - \lambda_{st}''(p)\lambda_{st}(p)\right) \geq 0$

for all $p$, $t$ and $s$.

Condition 5.2-\lambda will be satisfied if the functions $\lambda_{st}(p)$ are all concave in $p$ (though this is not a necessary condition). One result that will be useful for us in deriving the upper bound in the next section is the following:

Lemma 5.2: Let $V_t(I)$ be the optimal value for the expected price approximation problem from period $t$ onwards. Suppose that the expected demand functions $\lambda_{st}(p)$ satisfy conditions 5.1 and 5.2-\lambda. Then the function $V_t(I)$ is concave in $I$.

Lemma 5.2 is straightforward to prove. It involves recognizing that the expected price approximation problem (from any period $t$ onwards) is a concave maximization problem (when viewed in terms of the expected demand decision variables) with the parameter $I$ being the right hand side of a constraint in the problem.

Period $t > 1$ problem

For any period $t > 1$, the above model needs to be modified to take into account the inventory present at the stores at the beginning of the period. We describe here how this period $t$ problem may be solved in a heuristic manner for the POST and PRE models.
Suppose we are at the beginning of period \( t \) and the inventory level at each store \( s \) is \( C_s \). Then the period \( t \) problem can be written as:

\[
\begin{align*}
\text{max} & \quad \sum_{t=1}^{T} \sum_{s=1}^{S} p_t S_{st} - \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \\
\text{s.t.} & \quad \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \leq I \\
& \quad S_{st} \leq \lambda_{st}(p_t) \quad \forall \ s, \tau \\
& \quad S_{st} \leq d_{st} + C_{st} \quad \forall \ s, \tau \\
& \quad \sum_{t=1}^{T} C_{st} \leq C_s \quad \forall \ s \\
& \quad p_t, d_{st}, S_{st}, C_{st} \geq 0 \quad \forall \ s, \tau 
\end{align*}
\]

Here, \( C_{st} \) is the amount of store \( s \) inventory allocated to time period \( \tau \). One way to solve the above problem might be to dualize the inventory constraint \( \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \leq I \). This solution approach is complicated by the constraints \( \sum_{t=1}^{T} C_{st} \leq C_s \), since these don’t allow the problem to be decomposed across time periods after the constraint \( \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \leq I \) has been dualized. Our approach to solving this problem heuristically is based on pre-allocating the inventory \( C_s \) for each store \( s \) across all the time periods \( \tau = t \ldots T \) in some “reasonable” manner and then solving the resulting non-linear program.

The pre-allocation of the store inventories is done as follows. We fix some price level \( p \) (this may be some medium price level within the price range being considered), and calculate the expected demand \( \lambda_{st}(p) \) at price \( p \) for each store \( s \) and time period \( \tau = t \ldots T \). The initial inventory \( C_s \) at store \( s \) is then allocated across all the time periods \( \tau = t, \ldots T \) in the following manner:
\[ C_{rt} = \frac{\lambda_{rt}(p)}{\left( \sum_{\sigma=1}^{T} \lambda_{\sigma r}(p) \right)} C_s \]

One special case is where the \( \lambda_{rt}(\cdot) \)'s have the following form: \( \lambda_{rt}(p) = A_{rt} \lambda_r(p) \). In this case, we get

\[ C_{rt} = \frac{A_{rt}}{\left( \sum_{\sigma=1}^{T} A_{\sigma r} \right)} C_s \]

In this case, the quantities \( C_{rt} \) are independent of the price \( p \). If the demand distribution at each price and at each store was stationary over time, for instance, the inventory \( C_s \) will be allocated equally across all time periods \( \tau = t, \ldots, T \).

The optimization problem that now needs to be addressed is the following:

**Problem Primal:**

\[
\text{max} \quad \sum_{t=1}^{T} \sum_{s=1}^{S} p_t S_{st} - \nu \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st}
\]

s.t.

\[
\sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \leq 1
\]

\[ S_{st} \leq \lambda_{rt}(p_r) \quad \forall \ s, \tau \]

\[ S_{st} \leq d_{st} + C_{rt} \quad \forall \ s, \tau \]

\[ p_t, d_{st}, S_{st} \geq 0 \quad \forall \ s, \tau \]

We dualize the inventory constraint \( \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} \leq 1 \) by associating a lagrange multiplier \( \lambda \) with it.

This yields the following dual problem:
Problem Dual ($\lambda$):

$$\max \sum_{t=1}^{T} p_t \sum_{s=1}^{S} S_{st} - (v + \lambda) \sum_{t=1}^{T} \sum_{s=1}^{S} d_{st} + \lambda I$$

s.t.

$$S_{st} \leq \lambda_{st}(p_t) \quad \forall \ s, t$$

$$S_{st} \leq d_{st} + C_{st} \quad \forall \ s, t$$

$$p_t, d_{st}, S_{st} \geq 0 \quad \forall \ s, t$$

We now describe how this problem can be solved optimally. Notice first that for any fixed value of $\lambda$, this dual problem can be decomposed into $(T - t + 1)$ single-time period problems, with the period $\tau$ problem being the following:

Problem Dual ($\lambda, \tau$):

$$\max \ p_t \sum_{s=1}^{S} S_{st} - (v + \lambda) \sum_{s=1}^{S} d_{st} + \lambda I$$

s.t.

$$S_{st} \leq \lambda_{st}(p_t) \quad \forall \ s$$

$$S_{st} \leq d_{st} + C_{st} \quad \forall \ s$$

$$p_t, d_{st}, S_{st} \geq 0 \quad \forall \ s$$

It is easy to observe the following (and we therefore do not offer a formal proof of this lemma):

Lemma 5.3: Let $\{p_n, d_{st}, S_{st}\}$ be an optimal solution to Dual ($\lambda, \tau$). Then:

$$p_t < v + \lambda \Rightarrow d_{st} = 0, S_{st} = \min \{C_{st}, \lambda_{st}(p_{\tau})\} \text{ for all } s.$$  

$$p_t > v + \lambda \Rightarrow d_{st} = \max \{0, \lambda_{st}(p_{\tau}) - C_{st}\}, S_{st} = \lambda_{st}(p_{\tau}) \text{ for all } s.$$  

$$p_t = v + \lambda \Rightarrow \text{there exists a solution } \{p_n, d_{st}', S_{st}'\} \text{ with } d_{st}' = 0,$$

$$S_{st}' = \min \{C_{st}, \lambda_{st}(p_{\tau})\} \text{ for all } s.$$  

Lemma 5.3 allows us to solve problem Dual ($\lambda, \tau$) by performing a line search on the price variable $p_n$, since, for any given value of this variable, the optimal values of the other decision variables in Dual ($\lambda, \tau$) can be determined in a straightforward manner. We divide the range of
feasible price values into two intervals, \((0, v + \lambda]\) and \((v + \lambda, \infty)\), and find the optimal price in each of these intervals separately, as described below.

From Lemma 5.3, the problem Dual \((\lambda, \tau)\) can be reformulated in the following manner for the case where the price variable \(p_t\) lies in the interval \((0, v + \lambda]\):

**Problem Dual \((\lambda, \tau)\)-1**

\[
\max \quad p_t \sum_{s=1}^{S} \min\{C_{s\tau}, \lambda_{s\tau}(p_t)\} + \lambda I \\
\text{s.t.} \quad 0 \leq p_t \leq v + \lambda
\]

and it can be reformulated in the following manner for the case where the price variable \(p_t\) lies in the interval \([v + \lambda, \infty)\):

(At price level \(p_t = v + \lambda\), the objective function values for the two problems are the same, and hence we can to keep this point in the feasible regions of both problems.)

**Problem Dual \((\lambda, \tau)\)-2:**

\[
\max \quad p_t \sum_{s=1}^{S} \lambda_{s\tau}(p_t) - (v + \lambda) \sum_{s=1}^{S} \max\{0, \lambda_{s\tau}(p_t) - C_{s\tau}\} + \lambda I \\
\text{s.t.} \quad p_t \geq v + \lambda
\]

Since the only decision variable in each of the above problems is \(p_t\), the problems can be easily solved via line searches. In fact, one can show that the second problem above has a concave objective function, and so it can be solved even faster through a binary search procedure.

Thus, for any given value of \(\lambda\), the problem Dual \((\lambda)\) can be solved using the decomposition approach described above. Now consider the following lagrangean dual of problem Primal

**Problem Dual:**

\[
\min V(\text{Dual } (\lambda)) \\
\text{s.t. } \lambda \geq 0
\]
where $V(\text{Dual}(\lambda))$ is the optimal solution value to problem Dual ($\lambda$).

A standard result in duality theory is that the objective function of problem Dual is convex in $\lambda$ (Bazaraa, Sherali and Shetty, 1993, Theorem 6.3.1). Problem Dual is therefore easy to solve through a line search on $\lambda$. Let $\bar{\lambda}$ be the optimal solution for this problem, and let $p_t(\bar{\lambda})$ be the optimal price in period $t$ for problem Dual ($\bar{\lambda}$). It is easy to check that the objective function of problem Primal is concave, and all its constraints are linear with the exception of the set of constraints $\{s_{rt} \leq \lambda_{rt}(p_r)\}$. Under the assumption that the expected demand functions $\lambda_{rt}(p_r)$ are all concave, it follows from the Strong Duality Theorem (Bazaraa, Sherali and Shetty, 1993, Theorem 6.2.4) that $V(\text{Dual}) = V(\text{Primal})$, and in this case $p_t(\bar{\lambda})$ will be the optimal price for period $t$ for Primal. Our heuristic solution approach uses this value of $p_t(\bar{\lambda})$ as the price for period $t$ even when a duality gap may exist, i.e., even when the optimal solution to Dual may not directly yield the optimal solution to Primal.

In our computational tests, we have used a Weibull distribution-based form for the expected value functions $\lambda_{rt}(p_r)$. This does not necessarily lead to concavity for these functions. However, we have observed that in most test situations, the solution of problem Dual does lead directly to the solution of problem Primal.

### 5.3.3 HEURISTIC 3: STORE AGGREGATION

Our third heuristic is applicable for model PRE after the initial inventory allocation decision across stores has been made. The heuristic involves replacing the different demand processes across the stores with a single demand process. A single demand distribution is used to provide the demand levels at all the stores, and this helps in limiting the amount of information about the inventory in the system that needs to be passed from one period to the next, as we show below. One can view this heuristic as assuming that demand at all stores is perfectly linked to the value of some one underlying random variable. The heuristic is described in more detail below.
We use a single demand random variable \( \lambda_t(p) \) to provide an 'aggregate demand level' for each period in the model. This random variable is chosen to have a mean given by \( \lambda_t(p) = \sum_{s=1}^{S} \lambda_{st}(p) \), where \( p_t \) is the price charged in period \( t \) and \( \lambda_{st}(p_t) \) is the expected demand at store \( s \) in period \( t \), given the price \( p_t \). While we do not require \( \lambda_t(p) \) to have any specific distributional form (and hence leave this decision to the modeler), in situations where the random variables \( \lambda_{st}(p) \) all come from the same family of distributions (e.g., Poisson) it may be appropriate to take \( \lambda_t(p) \) from that family also.

Now suppose a price of \( p_1 \) is charged in period 1, and that the 'aggregate demand level' derived from the above aggregate demand random variable \( \lambda_t(p_1) \) is \( d_1 \). We calculate the corresponding demand level for each store \( s \) at price \( p_1 \) in the following manner

\[
D_{s1}(p_1) = \frac{\lambda_{s1}(p_1)}{\sum_{j=1}^{S} \lambda_{j1}(p_1)} \cdot d_1.
\]

(The demand level \( D_{s1}(p_1) \) many needed to be rounded to get an integral value.)

We assume that store \( s \) experiences a demand of \( D_{s1}(p_1) \) during period 1. The only information that needs to be passed to period 2 are the two numbers \( d_1 \) and \( p_1 \). From these numbers and the starting inventory levels \( \{C_{s0}\}_s \) at all the stores, the inventory levels \( \{C_{s1}\}_s \), at the beginning of period 2 can be derived as follows:

\[
C_{s1} = C_{s0} - \min \{C_{s0}, D_{s1}(p_1)\} \quad \text{for all } s.
\]

Now we repeat the above procedure for period 2. Suppose period 2's 'aggregate demand level' turns out to be \( d_2 \), and the price being considered is \( p_2 \). Then the demand at store \( s \) is assumed to be

\[
D_{s2}(p_1) = \frac{\lambda_{s2}(p_2)}{\sum_{j=1}^{S} \lambda_{j2}(p_2)} \cdot d_2,
\]

The information that needs to be passed to period 3 is \( (d_1, p_1, d_2, p_2) \). From these numbers, and the starting inventory levels \( \{C_{s0}\}_s \) at all the stores, the inventory level \( \{C_{s2}\}_s \), at the beginning of period 3 can be derived as follows:
\[ C_{s2} = C_{s0} - \min\{C_{s0}, D_{s1}(p_1) + D_{s2}(p_2)\} \]

We observe that the state space grows incrementally from one period to the next, with two variables added at each stage. In general, we will need to pass the data \((d_1, p_1, d_2, p_2, \ldots, d_t, p_t)\) from period \(t\) to period \(t+1\). This data would be used to calculate the inventory levels \(\{C_{s_t}\}\) at the beginning of period \(t+1\) as follows:

\[
C_{s_t} = C_{s0} - \min\left( C_{s0}, \sum_{i=1}^{t} D_{si}(p_i) \right) = C_{s0} - \min\left( C_{s0}, \sum_{i=1}^{t} \left( \frac{\lambda_{s_i}(p_2)}{\sum_{j=1}^{s} \lambda_{s_i}(p_j)} \right) d_i \right)
\]  \hspace{1cm} (5.15)

Given the growth in state space, this heuristic may be computationally tractable only for a few periods. For two special cases discussed below, however, the state space requirements can be reduced further. For the general case, we propose using this heuristic with only a few aggregate periods in the model. The heuristic is attractive from two standpoints:

- Unlike Heuristic 1, this method does not model the future only in terms of one period. Even in the general case described above, we expect that at least 3 or so periods could be modeled.
- Unlike Heuristic 2, this method does not eliminate the stochastic aspect of the problem completely.

We may therefore consider Heuristic 3 as a more balanced approach to approximating the multi-store problem, as illustrated below:
In certain cases that are discussed below, the state space requirements can be significantly reduced. Suppose that the demand process consists of a price-independent store arrival process combined with a random sampling from a reservation price distribution, as discussed in Chapter 4. Let $A_{st}$, $F_s(.)$ be the mean of the store arrival process and the reservation price distribution in time period $t$ at store $s$, respectively. Then

$$\lambda_{st}(p) = A_{st}(1-F_s(p)).$$

Equation (5.15) can then be written as

$$C_{st} = C_{s0} - \min \left( C_{s0}, \sum_{j=1}^{T} \sum_{i=1}^{T} d_{ij} A_{st} \left(1 - F_{s}(p_i)\right) \right)$$

Consider the following two special cases:

- **Similar reservation price distribution:** Suppose all stores had the same reservation price distribution in each time period $t$, i.e. $F_s(.) = F_t(.)$ for all $s$, $t$ (for some distribution function $F_i(.)$). Then Equation (5.16) becomes:

$$C_{st} = C_{s0} - \min \left( C_{s0}, \sum_{i=1}^{T} \frac{d_i A_{st}}{\sum_{j=1}^{T} A_{jj}} \right)$$
and so we would only need to pass the data \( d_1, d_2, \ldots, d_t \) from period \( t \) to period \( t+1 \), since this would suffice to calculate the quantity in the above equation. Thus we have cut down the state space requirements by half at each stage of the dynamic program.

- **Similar reservation price distribution and similar store arrival rate**: Suppose all stores had the same reservation price distribution and the same store arrival rate in each time period \( t \), i.e., \( F_s(.) = F_t(.) \) and \( A_s = A_t \) for all \( s, t \) (for some constant \( A_t \) and some distribution function \( F_t(.) \)). Then Equation (5.16) becomes

\[
C_{st} = C_{s0} - \min \left( C_{s0}, \frac{1}{t} \sum_{i=1}^{t} d_i \right)
\]

and so we would only need to pass the single data point \( \left( \sum_{i=1}^{t} d_i \right) \), since this would be suffice to calculate the quantity in the above equation. Hence we have reduced the state space requirements back to its original size - one.

An alternative way of using this heuristic scheme is to form clusters of stores. The demand random variables for all stores in a cluster are replaced by a single aggregate demand random variable. The aggregate demand is then decomposed into separate demand levels for each of the stores in the cluster in proportion to the expected demand levels at these stores at the price under consideration. Ideally, the stores that are in each cluster should have highly positively correlated demand - this may be facilitated, for instance, by forming clusters of neighboring stores. Used in this manner, the heuristic does not completely resolve the computational problem associated with the multiple store problem - rather, it presents a way of decreasing the computational burden in a reasonable manner by clustering stores together.

### 5.4 Upper bound

We present below an upper bound for the multi-store models PRE, PRETR and POST.
Our bound arises from the deterministic problem formulated in the expected value heuristic (Heuristic 2). The solution value for the period 1 problem associated with Heuristic 2 provides an upper bound for each of the three multi-store models. This is formally stated in Theorem 5.1.

Theorem 5.1: Let $V_1(I)$ be the optimal solution value for the PRE, PRETR or POST model with $I$ units of beginning inventory, and let $V_1(I)$ be the optimal solution for the associated expected price approximation. Then

$$V_1(I) \leq V_1(I)$$

5.5 Computational Test Results

In this section, we describe the results of the computational tests we have performed to test some of the multi-store model solution approaches developed in previous sections. We focus primarily on testing heuristics 1 and 2 and the upper bound on the POST model. The test results discussed below suggest that for the case where the demand distribution (at any given price) is Poisson, and there is a 'reasonable' amount of initial inventory, both heuristics 1 and 2 perform quite well with respect to the upper bound.

5.5.1 TEST PLAN

In our test results, we have utilized the demand model presented in Bitran and Mondschein (1993, 1995). In this model, each store experiences a Poisson store arrival process in each period, and the arriving customers are sampled from a reservation price distribution. An arriving customer will purchase the product if his or her reservation price is at least as high as the product's price. As shown by Bitran and Mondschein (1995), the combination of the Poisson store arrival process and the reservation price distribution- based customer purchase behavior leads to a Poisson purchase process with an arrival rate that depends on price.
We define a base case problem as follows. There are 4 time periods, and 5 stores. The initial inventory is 100 (leading to a inventory/store ratio of 20), and the Poisson store arrival rate is 500 for each period at each store. The reservation price distribution is similar across all stores and time periods (and is therefore stationary). It is given by a Weibull distribution of the form

\[ f(r) = ad^*(p-b)^{a-1}e^{-d^*(p-b)^a} \]

with parameters \( a = 2 \), \( b = 0 \) and \( d = 0.0067 \). The distribution multiplier is taken to be 1.25. (The distribution multiplier was defined in Section 5.3, within the discussion of Heuristic 1.)

Our computational test strategy was to implement heuristics 1 and 2, and the upper bound, on the base case as well as on variants of this test problem in order to study their performance under different conditions. In particular, we implemented them on the following problems (in addition to the base case):

(Note: In each case, the data for the problem, aside from the parameter mentioned, was from the base case)

- Inventory/store ratios of 5, 10 and 30 (Base case has inventory/store ratio of 20)
- 2, 10 and 25 stores (Base case has 5 stores)
- Distribution multipliers of 1.5 and 2.5 (Base case has distribution multiplier of 1.25)
- Inventory predistributed equally across all stores. (Base case had inventory at central location, to be distributed to stores over time)

In addition, we studied a set of problems where the reservation price distribution at each store is non-stationary. The data for this set of problems is described in Section 5.5.5.

In each of the tables below, the numbers correspond to the gap between the relevant heuristic solution value and the upper bound (measured as a percentage of the upper bound value).

### 5.5.2 VARYING INVENTORY/STORE RATIOS

Table 5.1 shows the results for this set of runs. There are two key conclusions suggested by these results:
- Heuristic 1 appears to perform better than Heuristic 2
- As the inventory/store ratio increases, the performance of Heuristics 1 and 2 improves, and/or the upper bound becomes tighter.

As illustrated in the second observation, since we do not know the actual optimal solution value, it is not possible in certain situations to separate the performance of the heuristics from the performance of the upper bounding procedure.

<table>
<thead>
<tr>
<th>Inventory/Store Ratio</th>
<th>5.0</th>
<th>10.0</th>
<th>20.0</th>
<th>30.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heuristic 1</td>
<td>5.6</td>
<td>4.1</td>
<td>2.9</td>
<td>2.5</td>
</tr>
<tr>
<td>Heuristic 2</td>
<td>10.1</td>
<td>6.7</td>
<td>5.7</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Table 5.1: Performance of Heuristics 1 and 2 under different inventory/store ratios

While Heuristic 1 appears to be superior to Heuristic 2 from the above results, it is important to note that these results are based on a test set where the reservation price distribution is stationary over time. Since Heuristic 1 aggregates all future time periods into a single period at each stage, we would expect that its performance would not be as good in situations where the reservation price distribution differed from one period to the next. This suspicion is corroborated by the results from the set of runs in Section 5.4.3.

5.5.3 VARYING NUMBER OF STORES

Table 5.2 presents the results for this set of runs. The actual multi-store problem POST gets increasingly complex computationally as the number of stores increases. It is therefore reassuring to see from the results in Table 5.2 that Heuristics 1 and 2, as well as the upper bound, appear to be very robust in their performance with respect to the number of stores. Additionally, the computational time for each heuristic and the upper bound is still very small (our implementations take less than 1 minute for each application of the heuristic/upper bound on a Pentium, 90 Mhz PC).
Table 5.2: Performance of Heuristics 1 and 2 under different number of stores

<table>
<thead>
<tr>
<th>No. of Stores</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heuristic 1</td>
<td>3.0</td>
<td>2.9</td>
<td>3.0</td>
<td>2.9</td>
</tr>
<tr>
<td>Heuristic 2</td>
<td>6.0</td>
<td>5.7</td>
<td>5.3</td>
<td>5.1</td>
</tr>
</tbody>
</table>

5.5.4 VARYING LEVELS OF INITIAL DISTRIBUTION

Table 5.3 shows the results for the runs where the distribution multiplier was set at 1.5 and 2.5, in addition to the base case run (dist. mult. = 1.25).

<table>
<thead>
<tr>
<th>Distribution Multiplier =</th>
<th>1.25</th>
<th>1.5</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heuristic 1</td>
<td>2.9</td>
<td>2.8</td>
<td>2.9</td>
</tr>
<tr>
<td>Heuristic 2</td>
<td>5.7</td>
<td>4.1</td>
<td>3.7</td>
</tr>
</tbody>
</table>

Table 5.3: Performance of Heuristics 1 and 2 under different values of distribution multiplier

The effect of a larger distribution multiplier is that more inventory is sent to the stores from the central stocking facility early in the season, as indicated by the level of period 1 shipments shown for the different cases in Table 5.4.

<table>
<thead>
<tr>
<th>Distribution Multiplier =</th>
<th>1.25</th>
<th>1.5</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heuristic 1</td>
<td>7</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>Heuristic 2</td>
<td>9</td>
<td>10</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 5.4: Period 1 shipments under different values of distribution multiplier

We note from Table 5.4 that the performance of both heuristics and the upper bound is very robust with respect to the tested range of values for the distribution.
5.5.5 VARYING DEGREE OF NON-STATIONARITY IN RESERVATION PRICE DISTRIBUTION

These runs are based on different problem specifications from those for the base case. In order to compare the solution for the multi-store problem with that for the single-store problem, we based these runs on the same data as that for the non-stationary problem considered in Section 3.3.4. The number of stores is 5 and the number of periods is 4. The initial inventory level is 250 (yielding an inventory/store ratio = 50), the arrival rate for each store in each period is 50, and the parameters of the reservation price distributions for the problems are as follows: \( a = 5 \) and \( b = -75 \) for all problems and all time periods. The values for \( d \) are given in Table 5.5 below (in each case, the value of \( d \) is similar across all stores).

<table>
<thead>
<tr>
<th>Problem</th>
<th>Time period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Very low</td>
<td>0.0067</td>
</tr>
<tr>
<td>Low</td>
<td>0.0067</td>
</tr>
<tr>
<td>Medium</td>
<td>0.0067</td>
</tr>
<tr>
<td>High</td>
<td>0.0067</td>
</tr>
<tr>
<td>Very high</td>
<td>0.0067</td>
</tr>
</tbody>
</table>

Table 5.5: Values of parameter \( d \) for periods 1-4

The values for the parameter \( d \) for the problems above have been chosen so as to yield a decreasing trend in the means of the reservation price distributions over time. The means for the associated reservation price distributions are shown in Table 5.6. By decreasing the mean more or less steeply over time, we derive different degrees of non-stationarity in the reservation price distribution. We have labeled these problems as above to reflect the relative degree of non-stationarity.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Time period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Very low</td>
<td>62</td>
</tr>
<tr>
<td>Low</td>
<td>62</td>
</tr>
<tr>
<td>Medium</td>
<td>62</td>
</tr>
<tr>
<td>High</td>
<td>62</td>
</tr>
<tr>
<td>Very high</td>
<td>62</td>
</tr>
</tbody>
</table>

Table 5.6: Mean reservation prices for periods 1-4
The results from the runs are given in Table 5.7.

<table>
<thead>
<tr>
<th>Degree of Non-Stationarity</th>
<th>Very Low</th>
<th>Low</th>
<th>Medium</th>
<th>High</th>
<th>Very High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heuristic 1</td>
<td>4.1</td>
<td>5.0</td>
<td>6.9</td>
<td>7.1</td>
<td>7.7</td>
</tr>
<tr>
<td>Heuristic 2</td>
<td>3.1</td>
<td>3.3</td>
<td>2.1</td>
<td>2.5</td>
<td>2.1</td>
</tr>
</tbody>
</table>

Table 5.7: Performance of Heuristics 1 and 2 under different degrees of non-stationarity

As we expected from our discussion in Section 5.5.2, the performance of Heuristic 1 is significantly degraded by the non-stationarity. However, Heuristic 2 performs very well in this case. Note that in this case we can ascertain that both Heuristic 2 and the upper bound are performing well, and that Heuristic 1 is not performing well, due to the differences in the solution-bound gaps derived for the two heuristics. We observe that while Heuristic 1's performance is poorer for greater degrees of non-stationarity, Heuristic 1's performance is not negatively impacted by this effect.

5.5.6 OTHER ISSUES

All the test problems discussed above have used similar reservation price distributions across all the stores. We tested heuristics 1 and 2 and the upper bound on a different test problem where the reservation price distribution differed across the stores. The results from these tests suggest that dissimilarity in reservation price distributions across stores does not influence the performance of the heuristics or the upper bound. This result is quite intuitive: both heuristics (as well as the upper bounding procedure) consider all the stores separately and therefore are able to appropriately consider differences in reservation price distributions across stores. This contrasts, for example, with the issue of dissimilar reservation price distributions across time - in that case, as discussed in Section 5.5.5, Heuristic 1's performance is degraded by this dissimilarity.
Finally, we ran the base case under the PREDIST model - where all the merchandise is pre-shipped to the stores in some optimal fashion. Again, we tested the performance of heuristics 1 and 2 and the upper bound for this model. We based the pre-shipment quantity on the solution of the period 1 deterministic problem in Heuristic 2. (Since the reservation price distributions are similar across all the stores in the base case, the initial inventory is divided equally among all the stores). Heuristic 1's solution value was $30,302 as opposed to its value of $30,544 for the base case under the POST model. Heuristic 2's solution value was $30,360 as opposed to $29675 for the base case under the POST model. These results suggest that the pre-shipment strategy does not prove sub-optimal for the retailer. This result is somewhat non-intuitive, for one would expect the retailer to benefit from having the flexibility (in the POST model) in moving the inventory only incrementally out of the central stocking site according to the manner in which demand unfolds at different stores. We believe that this result is due to the limited variability in demand that is inherent in the Poisson demand model, and we will revisit this issue in the next section as we lay out issues for further research.

5.6 Conclusions and Future Research Issues

We have formulated three models for the multi-store pricing and distribution problem in this chapter. All these models are computationally intractable for even small-sized problems, and this has motivated the need to develop heuristic solution schemes and an upper bounding scheme for these models. We have presented three heuristics and one upper bounding procedure in this chapter. The computational results presented in the last section indicate that the heuristics tested perform quite well under a range of conditions, and similarly for the upper bounding procedure. The tests also indicate that these performances deteriorate under the following conditions: when the 'relative' level of inventory (vis-a-vis the overall demand over time) is small, and when the reservation price distribution is non-stationary (this applies only to Heuristic 1). Some areas for future work are the following:
• **Greater demand variability:** The Bitran and Mondschein (1995) demand model used by us in our computational testing work is associated with a limited amount of demand variability due to its Poisson nature. An alternative would be to utilize a negative binomial distribution instead and to repeat the computational tests under varying levels of variance for this distribution. These results might provide different information about the performance of the heuristics and upper bound studied in the previous section, and they may also provide additional insights into optimal pricing and distribution behavior in multi-store environments.

• **Inter-store transfers:** We have not tested the PRETR model in the previous section. By implementing this model on a test problem set, we may be able to draw interesting insights into the nature of optimal inter-store transfer strategies.

• **Solution of Heuristic 2:** Our technique for solving the non-linear program associated with Heuristic 2 is itself a heuristic approach. It would be useful to find a provably optimal algorithm for this problem, or to show that one does not exist in general. If the case is the latter, it would then be of interest to examine the performance of the solution technique we described in Section 5.3 and to try to improve on this if possible.

• **Calculation of 'optimal value' for demand parameter under greater demand variability:** While the results in the previous section indicate that the heuristics and upper bound are fairly robust with respect to the setting of the demand parameter, this result may not be true when a demand model with greater demand variability is used, such as the one referred to in point 1 above. One possible form for the demand parameter in that case would be \( d = \alpha \cdot \sigma \), where \( \sigma \) is the standard deviation of demand and \( \alpha \) is a scale parameter.
APPENDIX

Proofs for Chapter 3

Proof of Lemma 3.2.2.1
First note that for all \( p \) in \([1.005, 1.4)\) and for all \( I \),
\[
V(I, p) = pE[\min\{\delta(p), I\}] < 1.4E[\min\{\delta(1.4), I\}] = V(I, 1.4).
\]
Similarly, for all \( p \) in \([0, 1)\) and all \( I \), \( V(I, p) < V(I, 1) \).
Also, for all \( p \) in \([1.45, \infty)\), and all \( I \), \( V(I, p) = 0 \) since \( P[\delta(p) = 0] = 1 \).
The above results imply that the optimal price lies in \([1, 1.005) \cup [1.4, 1.45)\) for all \( I \).

Proof of Lemma 3.2.2.2
\[
V(2, 1) = 1(1.P[\delta(1) = 1] + 2.P[\delta(1) > 1]) = 1(0 + 2(0.75)) = 1.5 \quad \text{(A1)}
\]
Also, for any \( p \) in \([1.4, 1.45)\), \( V(2, p) \leq p(2(0.5)) = p \leq 1.45 \quad \text{(A2)}
\]
Lemma 3.2.2.1 along with equations (A1) and (A2) imply that \( p_2 \) is in \([1, 1.005)\).

Proof of Lemma 3.2.2.3
\[
V(3, 1.4) = 1.4(1.P[\delta(1.4) = 1] + 2.[\delta(1.4) = 2] + 3.P[\delta(1.4) > 2])
= 1.4 (0 + 3 (.5)) = 2.1 \quad \text{(A3)}
\]
Also, for any \( p \) in \([1, 1.005)\),
\[
V(3, p) \leq p.E[\min\{\delta(1), 3\}] = p (2(.25)+3(.5)) = 2p \leq 2.01 < 2.1 \quad \text{(A4)}
\]
Lemma 3.2.2.1 along with equations (A4) and (A5) imply that \( p_3 \) is in \([1.4, 1.45)\).

Proof of Lemma 3.2.2.4
Define \( V(I, p|\alpha) = p.\min\{D(p|\alpha), I\} \)
Since \( D(p) \) is flat in the price range \([0, p_2]\), it follows that for all \( p < p_2 \), \( V(I, p|\alpha_2) < V(I, p_2|\alpha_2) \).
Hence,
\[
p_{\alpha_1} \geq p_2 \quad \text{ (A5)}
\]
Also, for all \( p \) in \([p_2, p_3]\),
\[ V(I, p|\alpha_1) \leq p_3. D(p|\alpha_1) \leq p_3. D(p_2) < p_1. D(p_1) = p_1. \min \{D(p_1), I\} = V(I, p_1|\alpha_1) \]

and for \( p \) in \([p_3, \infty)\), \( V(I, p_3|\alpha_1) = 0 \).

Hence, \( p_{\alpha_1} < p_2 \) \hspace{1cm} (A6)

(A5) and (A6) imply that \( p_{\alpha_1} < p_{\alpha_1} \).

\[ \square \]

**Proof of Theorem 3.2.3.1**

We prove this result in a more general setting than that addressed in the theorem. We assume that there exists a salvage value for the unsold product at the end of the period, with the salvage value function \( S(.) \) being concave. The original problem in the Theorem is a special case of this problem where the salvage value is zero (i.e., \( S(y) = 0 \) for all \( y \)). The more general formulation will be used later in deriving Corollary 3.2.3.1.

We define the value function \( V'(.) \) as:

\[
V'(I) = \max_{p \geq 0} W'(I, p), \text{ where } \\
W'(I, p) = p \min \{D(p), I\} + S(I - \min \{D(p), I\}).
\]

Let \( I, \bar{I} \) be any two inventory levels with \( \bar{I} > I \). We will first show that for the optimal price \( p_1 \) at inventory level \( I \), \( \min \{D(p_1), I\} = D(p_1) \). Suppose this were not true, i.e., suppose \( D(p_1) > I \). Let us increase \( p \) from \( p_1 \) to \( D^{-1}(I) \). Now

\[
W'(I, D^{-1}(I)) = D^{-1}(I) \min \{D(D^{-1}(I)), I\} + S(I - \min \{D(D^{-1}(I)), I\}) \\
= D^{-1}(I)I > p_1 I = V(I, p_1),
\]

which contradicts that fact that \( p_1 \) is an optimal price for the deterministic problem with inventory level \( I \). Hence we have the following result:

**Lemma A.1:** For the optimal price \( p_1 \) at inventory level \( I \),

\[ \min \{D(p_1), I\} = D(p_1) \] \hspace{1cm} (A7)

Since \( \bar{I} > I \), (A7) implies that
\[ \min \{ D(p), \bar{I} \} = D(p_1). \]  \hspace{1cm} (A8)

Since \( D(p) \) is decreasing in \( p \), (A7) and (A8) also imply that for all \( p > p_1 \)
\[ \min \{ D(p), I \} = D(p) \]  \hspace{1cm} (A9)
and \[ \min \{ D(p), \bar{I} \} = D(p). \]  \hspace{1cm} (A10)

Now consider any price \( p > p_1 \). We will show that \( W^*(\bar{I},p) < W^*(\bar{I},p_1) \). This will show that the optimal price \( p_1 \) at inventory level \( \bar{I} \) satisfies \( p_1 \leq p_1 \).

\[
W^*(\bar{I},p) - W^*(\bar{I},p_1) \\
= p \min \{ D(p), \bar{I} \} + S(\bar{I} - \min \{ D(p), \bar{I} \}) \\
- p_1 \min \{ D(p_1), \bar{I} \} - S(\bar{I} - \min \{ D(p_1), \bar{I} \}) \\
= p D(p) - p_1 D(p_1) + S(I - D(p)) - S(\bar{I} - D(p)) \hspace{1cm} \text{by (A8) and (A10)} \\
= p D(p) - p_1 D(p_1) + S(I - D(p) + \delta) - S(\bar{I} - D(p_1) + \delta) \hspace{1cm} \text{where} \ \delta = \bar{I} - I > 0 \\
\leq p D(p) - p_1 D(p_1) + S(I - D(p)) - S(I - D(p_1)) \hspace{1cm} \text{since} \ S(\cdot) \ \text{is concave} \\
= W^*(I,p) - W^*(I, p_1) \hspace{1cm} \text{by (A7) and (A9)} \\
< 0 \hspace{1cm} \text{by definition of} \ p_1.
\]

Hence \( W^*(\bar{I},p) < W^*(\bar{I},p_1) \), and so the optimal price \( p_1 \) at inventory level \( \bar{I} \) must satisfy \( p_1 \leq p_1 \).

\[ \square \]

**Proof of Theorem 3.2.3.2:**

We will first show that, for the problem with a concave salvage salvage value function that was discussed in the proof of Theorem 3.2.3.1, the value function \( V^*(I) \) is concave in \( I \) if the demand function \( D(p) \) satisfies Condition 3.2.3.1. This result will then be used to prove Theorem 3.2.3.2 by induction on \( t \).

We had shown in the proof of Theorem 3.2.3.1 that for the optimal solution \( p_1 \) at inventory level \( I \), \( D(p_1) \leq I \). Since \( D(p) \) is decreasing in \( p \), this also means that \( p_1 \geq D^{-1}(I) \). We can therefore write \( V^*(I) \) as follows:

\[ V^*(I) = \max_{p \leq D^{-1}(I)} p D(p) + S(I - D(p)). \]
Re-writing this in terms of the demand variable $d = D(p)$, we have

$$V'(I) = \max_{0 \leq d \leq I} d \, D^{-1}(d) + S(I - d).$$

We now show that this problem has a concave objective function. Then, since the feasible region for the problem is \(\{d: 0 \leq d \leq I\}\), it will follow that \(V'(I)\) is concave in \(I\). The objective function is the sum of the two functions \(d \, D^{-1}(d)\) and \(S(I - d)\). Since \(S(I-d)\) is concave in \(d\), so the objective function will be concave in \(d\) as long as \(d \, D^{-1}(d)\) is concave in \(d\). This is proved below.

Let \(F(d) = d \, D^{-1}(d)\)

Then \(\frac{\partial F(d)}{\partial d} = D^{-1}(d) + d \frac{\partial D^{-1}(d)}{\partial d}\), and so \(\frac{\partial^2 F(d)}{\partial d^2} = \frac{\partial^2 D^{-1}(d)}{\partial d^2} + 2 \frac{\partial D^{-1}(d)}{\partial d}\)

Now \(D^{-1}(d) = p\) and so

\[
\frac{\partial D^{-1}(d)}{\partial d} = \frac{\partial p}{\partial d} = \left(\frac{\partial d}{\partial p}\right)^{-1} = \left(\frac{\partial D(p)}{\partial p}\right)^{-1},
\]

and

\[
\frac{\partial^2 D^{-1}(d)}{\partial d^2} = \frac{\partial}{\partial d} \left(\frac{\partial D(p)}{\partial p}\right)^{-1} = \frac{\partial}{\partial d} \left(\frac{\partial D(p)}{\partial p}\right)^{-1} \frac{\partial D(p)}{\partial p} = -\frac{\partial^2 D(p)}{\partial p^2} \left(\frac{\partial D(p)}{\partial p}\right)^3.
\]

So \(\frac{\partial^2 D^{-1}(d)}{\partial d^2} + 2 \frac{\partial D^{-1}(d)}{\partial d} = 2 \left(\frac{\partial D(p)}{\partial p}\right)^2 - D(p) \frac{\partial^2 D(p)}{\partial p^2} \left(\frac{\partial D(p)}{\partial p}\right)^3\)

Since \(D(p)\) is decreasing in \(p\), the denominator in the last expression is negative. The numerator is non-negative since \(D(p)\) satisfies Condition 3.2.3.1. Hence \(\frac{\partial^2 F(d)}{\partial d^2} \leq 0\), and so \(F(d)\) is concave in \(d\). Hence, as reasoned above, \(V'(I)\) is concave in \(I\).

Now consider the multiperiod problem in the statement of Theorem 3.2.3.2. For \(t = T\), we have

\(V_T(I) = \max_{p \geq 0} p \min\{D_T(p), I\}\), which can be considered as the value function for a single period problem with zero salvage value, a special case of the single period problem considered above. \(V_T(I)\) is therefore concave in \(I\). Now suppose for some \(t\), \(V_{t-1}(I)\) is concave in \(I\). Consider
\[ V_t(I) = \max_{p \geq 0} p \min\{D_t(p), I\} + V_{t+1}(I - \min\{D_t(p), I\}) \]

This can now be considered as the value function for a single period problem with a concave salvage value function \( V_{t+1}(.) \). This allows us to conclude that \( V_t(.) \) is concave, and so, by induction, Theorem 3.2.3.2 is proved.

\[ \square \]

**Proof of Corollary 3.2.3.1**

\[ V_t(I) = \max_{p \geq 0} p \min\{D_t(p), I\} + V_{t+1}(I - \min\{D_t(p), I\}) \]

where, for notational convenience, we define \( V_{T+1}(.) \) as \( V_{T+1}(I) = 0 \) for all \( J \). By Theorem 3.2.3.2, \( V_{t+1}(.) \) is concave (this holds trivially for \( t = T \)). Hence, \( V_t(I) \) can be considered as the value function of a single-period problem with a concave salvage value function given by \( V_{t+1}(.) \). The result now follows from the proof of Theorem 3.2.3.1.

\[ \square \]

**Proof of Theorem 3.2.3.3**

As shown in the proof of Theorem 3.2.3.2, we can write \( V_t(I) \) as

\[ V_t(I, p|\alpha) = \max_{p \geq 0} p \ D_t(p|\alpha) + V_{t+1}(I - D_t(p|\alpha)) \]

(Note that we have explicitly shown the dependence of various functions on the demand parameter \( \alpha \)).

Let \( V_t(I, p|\alpha) = p \ D_t(p|\alpha) + V_{t+1}(I - D_t(p|\alpha)) \).

Then

\[
\frac{\partial V_t(I, p|\alpha)}{\partial p} = p \frac{\partial D_t(p|\alpha)}{\partial p} + D_t(p|\alpha) - V_{t+1}'(I - D(p|\alpha)) \frac{\partial D_t(p|\alpha)}{\partial p}
\]

\[
= \frac{\partial D_t(p|\alpha)}{\partial p} \left( p + \frac{D_t(p|\alpha)}{\partial D_t(p|\alpha)} - V_{t+1}'(I - D(p|\alpha)) \right)
\]

We have assumed above that \( \frac{\partial D_t(p|\alpha)}{\partial p} \neq 0 \). If this were not true, then we would have

\[
\frac{\partial V_t(I, p|\alpha)}{\partial p} = 0 \Rightarrow D_t(I, p|\alpha) = 0 \]

and then it is straightforward to show that the optimal price \( p_{\text{opt}} \) will not be unique.
Let \( G_t(I,p|\alpha) = p + \frac{D_t(p|\alpha)}{\partial D_t(p|\alpha)} - V'_t,I(I - D_t(p|\alpha)) \), so that

\[
\frac{\partial V_t(I,p|\alpha)}{\partial p} = \frac{\partial D_t(p|\alpha)}{\partial p} G_t(I,p|\alpha)
\]

Let \( \overline{p}_\alpha \) be the solution to the equation \( \frac{\partial V_t(I,p|\alpha)}{\partial p} = 0 \). Then \( G_t(I, \overline{p}_\alpha |\alpha) = 0 \). Also,

\[
\frac{\partial G_t(I,p|\alpha)}{\partial p} = 1 + \left( \frac{\partial D_t(p|\alpha)}{\partial p} \right)^2 - D_t(p|\alpha) \frac{\partial^2 D_t(p|\alpha)}{\partial p^2} + V''_{t,I}(I - D_t(p|\alpha)) \frac{\partial D_t(p|\alpha)}{\partial p}
\]

\[
= 2 \left( \frac{\partial D_t(p|\alpha)}{\partial p} \right)^2 - D_t(p|\alpha) \frac{\partial^2 D_t(p|\alpha)}{\partial p^2} + V''_{t,I}(I - D_t(p|\alpha)) \frac{\partial D_t(p|\alpha)}{\partial p}
\]

The first term on the right hand side is non-negative because of Condition 3.2.3.1. The second term is non-negative because \( V_{t+1} \) is non-positive as the function \( V_{t+1} \) is concave and \( \frac{\partial D_t(p|\alpha)}{\partial p} \) is non-positive (because of Condition 3.2.1.1). Hence \( \frac{\partial G_t(I,p|\alpha)}{\partial p} \) is non-negative, and so, since \( G_t(I, \overline{p}_\alpha |\alpha) = 0 \), it follows that for all \( p > \overline{p}_\alpha \), \( G_t(I, p |\alpha) \geq 0 \). Hence for all \( p > \overline{p}_\alpha \),

\[
\frac{\partial V_t(I,p|\alpha)}{\partial p} \leq 0, \text{ i.e., the function } V_t(I,p|\alpha) \text{ is non-increasing in } p \text{ for } p > \overline{p}_\alpha. \text{ This implies that}
\]

\[
p_{\alpha} = \max \{ D_t^{-1}(I|\alpha), \overline{p}_\alpha \}. \text{ We prove below that both}
\]

\( D_t^{-1}(I|\alpha) \) and \( \overline{p}_\alpha \) are non-decreasing in \( \alpha \). It will then follow that \( p_{\alpha} \) is non-decreasing in \( \alpha \) as well.

Let \( p^a = D_t^{-1}(I|\alpha) \), so that

\[
D_t(p^a|\alpha) = I.
\]

Differentiating with respect to \( \alpha \) gives:
\[
\frac{\partial D_t(p^a|\alpha)}{\partial \alpha} + \frac{\partial D_t(p^a|\alpha)}{\partial p} \frac{\partial p^a}{\partial \alpha} = 0, \quad \text{or} \quad \frac{\partial p^a}{\partial \alpha} = -\frac{\partial D_t(p^a|\alpha)}{\partial p} \frac{\partial \alpha}{\partial D_t(p^a|\alpha)}
\]

The numerator on the right hand side is non-negative and the denominator is non-positive, and so \(\frac{\partial p^a}{\partial \alpha}\) is non-negative, as desired.

Now consider \(\tilde{p}_a\). As mentioned earlier, we have \(G_t(I, \tilde{p}_a|\alpha) = 0\), which yields

\[
\tilde{p}_a = -\frac{D_t(\tilde{p}_a|\alpha)}{\partial D_t(\tilde{p}_a|\alpha)} + V_{t+1}'(I - D(\tilde{p}_a|\alpha))
\]

Differentiating this equation with respect to \(\alpha\), we get

\[
\frac{\partial \tilde{p}_a}{\partial \alpha} = \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha} \left(\frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial p} + \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha} \frac{\partial \tilde{p}_a}{\partial \alpha}\right) - D_t(\tilde{p}_a|\alpha) \left(\frac{\partial^2 D_t(\tilde{p}_a|\alpha)}{\partial \alpha^2} + \frac{\partial^2 D_t(\tilde{p}_a|\alpha)}{\partial \alpha \partial p} \frac{\partial \tilde{p}_a}{\partial \alpha}\right)
\]

\[
+ V_{t+1}''(I - D_t(\tilde{p}_a|\alpha)) \left(-\frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial p} \frac{\partial \tilde{p}_a}{\partial \alpha} - \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha} \frac{\partial \tilde{p}_a}{\partial \alpha}\right)
\]

Rearranging the terms yields

\[
\frac{\partial \tilde{p}_a}{\partial \alpha} = \frac{2\left(\frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial p}\right)^2 - D_t(\tilde{p}_a|\alpha) \frac{\partial^2 D_t(\tilde{p}_a|\alpha)}{\partial p^2}}{\left(\frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial p}\right)^2} + V_{t+1}''(I - D_t(\tilde{p}_a|\alpha)) \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial p}
\]

\[
= \left[D_t(\tilde{p}_a|\alpha) \left(\frac{\partial^2 D_t(\tilde{p}_a|\alpha)}{\partial \alpha \partial p} \frac{\partial \tilde{p}_a}{\partial \alpha} - \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha} \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha}\right)\right]
\]

\[
= \left[\frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial p}\right]^2 - V_{t+1}''(I - D_t(\tilde{p}_a|\alpha)) \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha}
\]

Term A is non-negative since \(D_t(p|\alpha)\) satisfies Condition 3.2.3.1. Term B is non-negative since

\[
V_{t+1}''(I - D_t(\tilde{p}_a|\alpha)) \text{ is non-positive (as } V_{t+1} \text{ is concave) and } \frac{\partial D_t(\tilde{p}_a|\alpha)}{\partial \alpha} \text{ is negative (from condition 3.1). Hence the term within the parentheses on the left hand side of the equation is non-}
\]

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negative. Term C is positive because of Condition 3.2.3.2, and term D is non-positive because V_{t+1} is concave and D_t(p|\alpha) satisfies Condition 3.2.1.5. Hence, the right hand side of the equation is non-negative. It therefore follows that \frac{\partial \bar{p}}{\partial \alpha} is non-negative. This completes our proof.

\[ \square \]

**Proof of Theorem 3.2.3.4**

This is proved within the proof for Theorem 3.2.3.7.

**Proof of Lemma 3.2.3.1**

\[
E[\min\{I, \delta(p)\}] = \int_0^1 f(x|p)x dx + \int_1^\infty f(x|p) dx
\]

\[
= I F(I|p) - \int_0^1 F(I|p) dx + I (1 - F(I|p))
\]

\[
= \int_0^1 (1 - F(x|p)) dx
\]

\[ \square \]

**Proof of Theorem 3.2.3.5**

This is proved within the proof for Theorem 3.2.3.8.

**Proof of Theorem 3.2.3.6**

(We have suppressed the time subscript 1 below.)

\[
V(I) = \max_{p \geq 0} V(I, p), \text{ where}
\]

\[
V(I, p) = E[p \min\{D(p), I\}]
\]

\[
= p \int_0^1 (1 - F(x|p)) dx \quad \text{(by Lemma 3.2.3.1)}
\]

So \[
\frac{\partial V(I, p)}{\partial p} = -p \int_0^1 F'(x|p) dx + \int_0^1 (1 - F(x|p)) dx \quad \text{(A11)}
\]

The first order optimality condition implies that \[
\frac{\partial V(I, p)}{\partial p} = 0 \quad \text{for} \quad p = p_1. \text{ Hence, by equation (A11),}
\]
\[-p_1 \int_0^1 F'(x|p_1)dx + \int_0^1 (1 - F(x|p_1))dx = 0 \quad \text{(A12)}\]

where we have again used the notation \( F'(x|p) = \frac{\partial F(I, p)}{\partial p} \).

This yields,

\[
p_1 = \frac{\int_0^1 (1 - F(x|p_1))dx}{\int_0^1 F'(x|p_1)dx} \quad \text{(A13)}
\]

Note that since \( F'(x|p) \geq 0 \) by Condition 3.2.1.4, and since \( F'(x|p) \) is continuous in \( x \) by Condition 3.2.1.3, \( \int_0^1 F'(x|p)dx = 0 \) implies that \( F'(x|p) = 0 \) for all \( x \) in \([0,1]\). Then, given the continuity of \( F(x|p) \) in \( x \) (by Condition 3.2.1.3) and the fact that \( F(0|p) = 0 \), we must have \( F(x|p) = 0 \) for all \( x \) in \([0,1]\). We assume that for any inventory level \( I \) of interest and for all \( p \), there exists some \( x \) in the range \([0,1]\) for which the density function \( f(x|p) \) is positive. Under this assumption, and Conditions 3.2.1.3 and 3.2.1.4, it is straightforward to then see that \( \int_0^1 F'(x|p)dx > 0 \). We will continue to make this assumption throughout our analysis in this chapter. This assumption is satisfied by the multiplicative demand model (Example 3.2.1.6.3) with exponential uncertainty, the exponential-exponential demand model (Example 3.2.1.6.4) as well as the Poisson-Weibull demand model (Example 3.2.1.6.5), for any positive value of the inventory level \( I \).

Differentiating the above equation with respect to \( I \), we get:

\[
p'_1 = \frac{\int_0^1 F'(x|p_1)dx \left[ 1 - F(I|p) \right] - \int_0^1 F'(x|p_1)p'_1 dx - \int_0^1 (1 - F(x|p_1))dx \left[ F'(I|p_1) + \int_0^1 F''(x|p_1)p'_1 dx \right]}{\left( \int_0^1 F'(x|p_1)dx \right)^2}
\]

where \( p'_1 = \frac{\partial p_1}{\partial I} \), \( F''(x|p) = \frac{\partial^2 F(I, p)}{\partial p^2} \).
which implies that

\[ p'_i = \frac{(1 - F(I|p_i)) \int_0^1 F'(x|p_i)dx - F'(I|p_i) \int_0^1 F(x|p_i)dx}{2 \left( \int_0^1 F'(x|p_i)dx \right)^2 + \int_0^1 F''(x|p_i)dx \int_0^1 (1 - F(x|p_i))dx} \]  

(A14)

Now, \( V(I) \) can also be written as

\[ V(I) = p_i \int_0^1 (1 - F(x|p_i))dx \]

Differentiating with respect to \( I \), we get

\[ V'(I) = p'_i \int_0^1 (1 - F(x|p_i))dx + p_i \left[ (1 - F(I|p_i)) - \int_0^1 F'(x|p_i)p'_i dx \right] \]

Using equation (A12), this becomes

\[ V'(I) = p_i (1 - F(I|p_i)) \]

Differentiating again with respect to \( I \), we get

\[ V''(I) = p'_i (1 - F(I|p_i)) + p_i (-F'(I|p_i)p'_i - f(I|p_i)) \]

\[ = \frac{p'_i \left( (1 - F(I|p_i)) \int_0^1 F'(x|p_i)dx - F'(I|p_i) \int_0^1 F(x|p_i)dx \right) - \int_0^1 (1 - F(x|p_i))dx \cdot f(I|p_i)}{\int_0^1 F'(x|p_i)dx} \]

(using equation A13)

\[ = \left[ (1 - F(I|p_i)) \int_0^1 F'(x|p_i)dx - F'(I|p_i) \int_0^1 F(x|p_i)dx \right] - \left[ 2 \left( \int_0^1 F'(x|p_i)dx \right) + \int_0^1 F''(x|p_i)dx \int_0^1 (1 - F(x|p_i))dx \right] (1 - F(I|p_i))dx f(I|p_i) \]

(using equation A14)

\[ F'(x|p_i) \geq 0 \text{ for all } x > 0 \text{ since } F(.)|p \text{ is non-decreasing in } p. \text{ So } \int_0^1 F'(x|p_i)dx \geq 0. \text{ Hence the denominator in the previous expression is non-negative. Also, by Condition 3.2.3.5, the numerator in this expression is non-positive. Hence } V''(I) \leq 0, \text{ and so } V(I) \text{ is concave in } I. \]
Proof of Theorem 3.2.3.7

We will prove below that \( p'_1 = \frac{\partial p_1}{\partial I} \) is non-positive, which will imply the result.

Now

\[
V_t(I) = \max_{p \geq 0} V_t(I, p), \text{ where } \\
V_t(I, p) = E[p \min(D_t(p), I) + V_{t-1}(I- \min(D_t(p), I))]
\]

\[
= p \int_0^I (1 - F_t(x|p)) dx + E[V_{t-1}(I- \min(D_t(p), I))]
\]

(by Lemma 3.2.3.1).

Here

\[
E[V_{t-1}(I- \min(D_t(p), I))] = \int_0^I f_t(x|p) V_{t-1}(1-x) dx
\]

\[
= V_{t-1}(1-x) F_t(x|p) \bigg|_0^I + \int_0^I F_t(x|p) V'_{t-1}(1-x) dx
\]

\[
= \int_0^I F_t(x|p) V'_{t-1}(1-x) dx \text{ since } V_{t-1}(0) = F_{t-1}(0|p) = 0.
\]

We have implicitly assumed here that at each price, demand is non-negative with probability one.

Hence

\[
V_t(I, p) = p \int_0^I (1 - F_t(x|p)) dx + \int_0^I F_t(x|p) V'_{t-1}(1-x) dx
\]

For notational convenience, we suppress the subscripts \( t \) and \( t+1 \) from here on.

\[
\frac{\partial V(I, p)}{\partial p} = -p \int_0^I F'(x|p) dx + \int_0^I (1 - F(x|p)) dx + \int_0^I F'(x|p) V'(1-x) dx \tag{A15}
\]

The first order optimality condition implies that \( \frac{\partial V(I, p)}{\partial p} = 0 \) for \( p = p_t \). Hence, by equation (A11),

\[
-p_t \int_0^I F'(x|p_t) dx + \int_0^I (1 - F(x|p_t)) dx + \int_0^I F'(x|p_t) V'(1-x) dx = 0 \tag{A16}
\]

which yields,
\[
\begin{align*}
\tag{A17}
\frac{1}{\int_0^1 F'(x)p_1\,dx} \int_0^1 (1 - F(x)p_1)\,dx + \frac{1}{\int_0^1 F'(x)\,dx} \int_0^1 F'(x)p_1V'(1 - x)\,dx
\end{align*}
\]

Differentiating the above equation with respect to $I$, we get:

\[
\begin{align*}
p'_1 &= \frac{\int_0^1 F'(x)p_1\,dx \left[ F'(x)p_1 - \int_0^1 F'(x)p_1^2\,dx \right] - \int_0^1 (1 - F(x)p_1)\,dx \left[ F'(x)p_1 + \int_0^1 F''(x)p_1\,dx \right]}{\left( \int_0^1 F'(x)p_1\,dx \right)^2} \\
&\quad + \frac{\int_0^1 F'(x)p_1\,dx \left[ F'(x)p_1V(0) + \int_0^1 F''(x)p_1V'(1 - x)\,dx + \int_0^1 F'(x)p_1V''(1 - x)\,dx \right]}{\left( \int_0^1 F'(x)p_1\,dx \right)^2} \\
&\quad - \frac{\int_0^1 F'(x)p_1\,dx \left[ F'(x)p_1 + \int_0^1 F''(x)p_1\,dx \right]}{\left( \int_0^1 F'(x)p_1\,dx \right)^2}
\end{align*}
\]

which implies that

\[
\begin{align*}
p'_1 &= \frac{A}{B}
\end{align*}
\]

\[
\begin{align*}
\tag{A18}
&= \frac{(1 - F(x)p_1)\int_0^1 F'(x)p_1\,dx - F'(x)p_1\int_0^1 (1 - F(x)p_1)\,dx}{\left( \int_0^1 F'(x)p_1\,dx \right)^2} \left\{ \frac{B}{C} \right\}
\end{align*}
\]
The second order condition for optimality implies that \( \frac{\partial^2 V(I, p)}{\partial p^2} \leq 0 \) for \( p = p_i \), and so, by equation (A15), we get

\[
-p_i \int_0^1 F''(x|p_i)dx - 2 \int_0^1 F'(x|p_i)dx + \int_0^1 F''(x|p_i)V'(1 - x)dx \leq 0
\]  
(A19)

Substituting for \( p_i \) from equation (A17), this becomes:

\[
-2 \left( \int_0^1 F'(x|p_i)dx \right)^2 - \int_0^1 F''(x|p_i)dx \int_0^1 (1 - F(x|p_i))dx - \int_0^1 F'(x|p_i)V'(1 - x)dx \int_0^1 F''(x|p_i)dx + \int_0^1 F'(x|p_i)dx \int_0^1 F''(x|p_i)V'(1 - x)dx \leq 0
\]

\[
\int_0^1 F'(x|p_i)dx \geq 0, \text{ as discussed above. Hence we get, from the last inequality:}
\]

\[
-2 \left( \int_0^1 F'(x|p_i)dx \right)^2 - \int_0^1 F''(x|p_i)dx \int_0^1 (1 - F(x|p_i))dx - \int_0^1 F'(x|p_i)V'(1 - x)dx \int_0^1 F''(x|p_i)dx + \int_0^1 F'(x|p_i)dx \int_0^1 F''(x|p_i)V'(1 - x)dx \leq 0
\]

This implies that numerator in term A above is non-negative. The numerator in term B is non-positive by Condition 3.2.3.3. Let us consider the numerator in term C. Since

\[
\int_0^1 F'(x|p)V'(1 - x)dx = V'(1 - x) \int_0^1 F'(x|p)dx + \int_0^1 \left( \int_0^1 F'(y|p)dy \right) V''(1 - x)dx
\]

\[
= V'(0) \int_0^1 F'(x|p)dx + \int_0^1 \left( \int_0^1 F'(y|p)dy \right) V''(1 - x)dx
\]

this numerator becomes

\[
\int_0^1 F'(x|p_i)dx \int_0^1 F'(x|p_i)V''(1 - x)dx - \int_0^1 \left( \int_0^1 F'(y|p)dy \right) V''(1 - x)dx F'(l|p_i) \]

(A20)

From Condition 3.2.3.6, we have, for all \( 0 \leq x \leq I \), and all prices \( p \),
\[
\frac{F'(x|p)}{\int_0^x F'(y|p) dy} \geq \frac{F'(1|p)}{\int_0^1 F'(y|p) dy}.
\]
This implies that
\[
\int_0^1 F'(y|p) dy F'(x|p) \geq F'(1|p) \int_0^x F'(y|p) dy
\]
Since \(V(.)\) is concave, \(V''(I-x) \leq 0\) for all \(0 \leq x \leq I\), and so
\[
\left(\int_0^1 F'(y|p) dy\right) F'(x|p) V''(I-x) \leq F'(1|p) \left(\int_0^x F'(y|p) dy\right) V''(I-x).
\]
Integrating both sides with respect to \(x\) over the interval \([0, I]\), we get
\[
\int_0^x F'(x|p_t) dx \int_0^1 F'(x|p_t) V''(I-x) dx \leq \int_0^x \left(\int_0^1 F'(y|p) dy\right) V''(I-x) dx F'(1|p_t)
\]
Hence, from term (A20) above, the numerator of term C is non-positive. Therefore, the right hand side of equation (A14) is non-positive, and so \(p_t\) must be non-positive also.

Note that in the single period case considered in Theorem 3.2.3.4, the analysis above still applies, except, since in this case the term C will vanish, we will not need Condition 3.2.3.6.

\(\square\)

**Proof of Theorem 3.2.3.8**

This theorem can be proved in the same manner as Theorem 3.2.3.7. We write
\[
V_t(I|\alpha) = \max_{p>0} V_t(I, p|\alpha), \text{ where}
\]
\[
V_t(I, p|\alpha) = E[p \min\{D_t(p|\alpha), I\} + V_2(I- \min\{D_t(p|\alpha), I\})].
\]

Here, I will remain fixed and \(\alpha\) will vary, so we denote the optimal solution for this period t problem with demand parameter \(\alpha\) as \(p_\alpha\). We show that \(p_\alpha' = \frac{\partial p_\alpha}{\partial \alpha}\) is non-negative, which will imply the result.

Equation (A17) is now expressed as
\[
\begin{align*}
\rho_a &= \frac{\int_0^1 (1 - F(x|\rho_a, \alpha))dx}{\int_0^1 F'(x|\rho_a, \alpha)dx} + \frac{\int_0^1 F'(x|\rho_a, \alpha)V'(1-x)dx}{\int_0^1 F'(x|\rho_a, \alpha)dx}
\end{align*}
\]

and we will need to differentiate this with respect to \( \alpha \) instead of \( I \). The proof are very analogous to the proof for Theorem 3.2.3.7, and so we do not include the other steps here. Also, the proof for Theorem 3.2.3.5 is contained within this proof for Theorem 3.2.3.8 just as the proof for Theorem 3.2.3.4 was contained within that for Theorem 3.2.3.7.

Proof of Lemma 3.2.3.2
We define

\[ V_t(I,P,p) = \begin{cases} 
\text{the optimal expected revenues from period } t \text{ onwards when the inventory at the beginning of period } t \text{ is } I, \text{ the period } t-1 \text{ price is } P \text{ and the period } t \text{ price is } p. \\
E[p \min\{D_t(p), I\}] + V_{t+1}(I-\min\{D_t(p), I\})
\end{cases} \]

\( V_t(I,P,p) \) is not defined for \( P > P \), but it is otherwise not dependent on \( P \). We now have

\[ V_T(I,P) = \max_{P \geq P} V_t(I,P,p) \]

We will prove the result by backward induction on \( t \). For \( t = T \), we have

\[ V_T(I,P,p) = E[p \min\{D_T(p), I\}] = p \int_0^1 (1 - F_T(x|p))dx \quad \text{(by Lemma 3.2.3.1)} \]

Here, we have ignored the sales limit issue since, in period \( T \), there will clearly be no incentive for the retailer to limit the sales below the available inventory. For any \( S \leq I \), we have

\[ V_T(I,P,p) - V_T(S,P,p) = p \int_S^1 (1 - F_T(x|p))dx \leq (I-S)P \text{ since } 1-F_T(x|p) \text{ lies in } [0,1] \text{ for all } x \geq 0. \]

Hence,

\[ \max_{P \geq P} (V_T(I,P,p) - V_T(S,P,p)) \leq (I-S)P, \text{ and so } V_T(I,P) \leq V_T(S,P) + (I-S)P \]

and thus the result holds for \( t = T \). Let us assume that it holds for \( t+1 \). For period \( t \),
\[ V_t(I,P) = \max_{p_p=0} V_t(I,P,p), \text{ where} \]
\[ V_t(I,P,p) = \max_{p_p=0} E[p \min\{D_t(p), C\}] + E[V_{t+1}(I- \min\{D_t(p), C\}, p)] \]
\[ = \max_{p_p=0} \int_0^C f_t(x|p) p(x|p) dx + pC \int_0^C f_t(x|p) V_{t+1}(I-x) dx + V_{t+1}(I-C) \int_0^C f_t(x|p) dx \]

Consider any \( S \leq I \), and let \( C_S \) and \( C_I \) be the optimal sales limits for period \( t \) when the price in period \( t \) is \( p \) and the initial inventory levels are \( S \) and \( I \) respectively. Two cases arise:

**Case 1:** \( C_I \leq C_S \). In this case,

\[ V_t(I,P,p) - V_t(S,P,p) = \]
\[ \int_0^{C_I} f_t(x|p) \left( V_{t+1}(I-x, p) - V_{t+1}(S-x, p) \right) dx \]
\[ + \int_{C_I}^{C_S} f_t(x|p) \left( p(C_I-x) + V_{t+1}(I-C_I, p) - V_{t+1}(S-x, p) \right) dx \]
\[ + \int_{C_S}^\infty f_t(x|p) \left( p(C_I-C_S) + V_{t+1}(I-C_I, p) - V_{t+1}(S-C_S, p) \right) dx \]

\[ \leq \]
\[ \int_0^{C_I} f_t(x|p) p(I-S) dx \]
\[ + \int_{C_I}^{C_S} f_t(x|p) p(C_I-x + I - C_I - S + x) dx \]
\[ + \int_{C_S}^\infty f_t(x|p) p(C_I-C_S + I - C_I - S + C_S) dx \]
\[ = p(I-S) \]

**Case 2:** \( C_I > C_S \). It can be similarly shown in this case that \( V_t(I,P,p) - V_t(S,P,p) \leq p(I-S) \)

Hence, \( \max_{p_p=0} \{V_t(I,P,p) - V_t(S,P,p)\} \leq (I-S)p \), and so \( V_t(I,P) \leq V_t(S,P) + (I-S)p \).

The result now follows by induction.
Proof of Theorem 3.2.3.9

For any \( S \leq 1 \), \( W(I, p, S) = \)
\[
\begin{align*}
& \int_0^S f_1(x|p)x \, dx + p \int_0^S f_1(x|p) \, dx + \int_0^S f_1(x|p) \, dx
\end{align*}
\]
\[
\begin{align*}
& \frac{S}{S} f_1(x|p) v_{t,i}(I-x) \, dx + v_{t,i}(I-S) \int_0^S f_1(x|p) \, dx
\end{align*}
\]

Hence \( W(I, p, 1) - W(I, p, S) = \)
\[
\begin{align*}
& \int_0^S f_1(x|p) \left( p(x-S) + v_{t,i}(I-x, p) - v_{t,i}(I-S, p) \right) \, dx
\end{align*}
\]
\[
\begin{align*}
& + \int_0^S f_1(x|p) \left( p(I-S) - v_{t,i}(I-S, p) \right) \, dx
\end{align*}
\]
\[
\begin{align*}
& \geq \int_0^S f_1(x|p) \left( p(x-S) + p(I-x-I+S) \right) \, dx
\end{align*}
\]
\[
\begin{align*}
& + \int_0^S f_1(x|p) \left( p(I-S) - p(I-S) \right) \, dx
\end{align*}
\]
(Using Lemma 3.2.3.2)
\[
= 0
\]

Hence the result.

Proof of Lemma 3.3.1.1:
If \( \bar{S}_i \) were less than \( D_i(\bar{p}_i) \) for some \( t \) in a given solution, one could increase price \( \bar{p}_i \) while keeping \( \bar{S}_i \) constant, thereby achieving the same, or higher, period \( t \) revenues while staying feasible and bringing \( D_i(\bar{p}_i) \) down to equal \( \bar{S}_i \). Hence the result.

Proof of Lemma 3.3.1.2:
It was shown in the proof of Theorem 3.2.3.2 that if a smooth, invertible and monotonically decreasing demand function \( D(\cdot) \) satisfies Condition 3.2.3.1, the function \( dD^{-1}(d) \) is concave in \( d \). Lemma 3.3.1.2 now follows in a straightforward manner.
Proof of Lemma 3.3.1.3

We will prove this lemma by contradiction. Suppose it were not true, i.e., suppose there exists a solution \( \{\bar{d}_t, \bar{\pi}_t, \bar{\lambda}\} \) in which, for some \( t = \tau \), we have \( \bar{\pi}_t \neq 0 \). Then it follows from the KKT optimality conditions that \( \bar{d}_t = D_t(0) \).

Consider a new solution \( \{d_t\} \) where \( d_t = \bar{d}_t \) for all \( t \) except \( t = \tau \), and \( d_\tau \) is any positive number smaller than \( D_\tau(0) \). \( \{d_t\} \) is clearly a feasible solution for the problem, and

\[
\sum_{t=1}^{T} d_t D_t'(d_t) - \sum_{t=1}^{T} \bar{d}_t D_t'(\bar{d}_t) = d_\tau D_\tau'(d_\tau) - \bar{d}_\tau D_\tau'(\bar{d}_\tau) = d_\tau D_\tau'(d_\tau) - D_\tau(0)0 = d_\tau D_\tau'(d_\tau) > 0,
\]

contradicting the assumption that \( \{\bar{d}_t\} \) is an optimal solution.

\[\square\]

Proof of Theorem 3.3.1.1

Note that the term on the left hand side is just the partial derivative of \( F(d_1, ..., d_T, \lambda) = \sum_{t=1}^{T} d_t (D_t'(d_t) - \lambda) \) with respect to \( d_t \). By Lemma 3.3.1.2, the function \( \sum_{t=1}^{T} d_t D_t'(d_t) \) is concave in \((d_1, ..., d_T)\), and then it is easy to see that \( F(d_1, ..., d_T, \lambda) = \sum_{t=1}^{T} d_t D_t'(d_t) - \sum_{t=1}^{T} d_t \lambda \) is also concave in \((d_1, ..., d_T)\). Hence, for any \( t \), the partial derivative of \( F(d_1, ..., d_T, \lambda) \) with respect to \( d_t \) must be non-increasing in \( d_t \). Hence, for any fixed \( \lambda \), the solution \( d_t(\lambda) \) to the equation

\[
d_t \frac{\partial D_t'(d_t)}{\partial d_t} + D_t'(d_t) - \lambda = 0
\]
can be derived via a binary search on \( d_t \). Hence, in particular, the solution to case 1 can be found via a binary search process.

We now show that \( d_t(\lambda) \) is non-increasing in \( \lambda \). By definition of \( d_t(\lambda) \), we have

\[
d_t(\lambda) \cdot \frac{\partial D_t'(d_t(\lambda))}{\partial d_t} + D_t'(d_t(\lambda)) - \lambda = 0
\]

Differentiating with respect to \( \lambda \), we get

\[
d_t'(\lambda) \cdot \frac{\partial D_t'(d_t(\lambda))}{\partial d_t} + d_t(\lambda) \cdot \frac{\partial^2 D_t'(d_t(\lambda))}{\partial d_t^2} d_t'(\lambda) + d_t'(\lambda) \cdot \frac{\partial D_t'(d_t(\lambda))}{\partial d_t} - 1 = 0,
\]

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where \( d'(\lambda) = \frac{\partial d_t(\lambda)}{\partial \lambda} \). This yields

\[
d'(\lambda) = \frac{1}{2} \frac{\partial^2 d_t'(d_t(\lambda))}{\partial d_t^2} + d_t(\lambda) \cdot \frac{\partial^2 d_t'(d_t(\lambda))}{\partial d_t^2}.
\]

The term in the denominator is non-positive since \( D_t(\cdot) \) satisfies Condition 3.2.3.1, as shown in the proof of Theorem 3.2.3.2. Hence \( d'(\lambda) \) is non-positive, and so \( d_t(\lambda) \) is non-increasing in \( \lambda \).

Now let \( G(\lambda) = \sum_{t=1}^{T} d_t(\lambda) \). By the above result, \( G(\lambda) \) is non-increasing in \( \lambda \). Hence the solution to the equation \( G(\lambda) = 1 \) can be found through a binary search on \( \lambda \). This shows how the solution to the \( T+1 \) equations in case 2 can be derived via a simple line search on \( \lambda \).

\[ \square \]

Proof of Theorem 3.3.1.2

The single store problem addressed in this theorem is a special case of the multi-store problem addressed in Theorem 5.1 (Chapter 5). Therefore, the proof for Theorem 5.1 applies to this case also.

Proofs for Chapter 5

Proof of Lemma 5.1

Let \( S_u, d_u, \lambda_u \) be a solution for the problem and suppose for some \( \bar{s}, \bar{t} \) we have \( S_{\bar{s}\bar{t}} < d_{\bar{s}\bar{t}} \). We use this solution to derive another feasible solution in which \( d_{\bar{s}\bar{t}} \) is reduced to \( S_{\bar{s}\bar{t}} \) while the other variables are not changed. It is easy to see that this solution will be feasible and will have at least as high an objective function value as the original solution.

Now consider this new solution in which, for all \( s \) and \( t \), \( S_u = d_u \). Suppose that for some \( \bar{s}, \bar{t} \) we have \( S_{\bar{s}\bar{t}} < \lambda_{u}(p_{\bar{t}}) \). If in fact for all \( s \), it is the case that \( S_u < \lambda_u(p_{\bar{t}}) \), then we can use this solution to derive a new solution which has a slightly higher price and the same values for all the \( S_u \)'s and \( d_u \)'s. So now consider the set \( S^- = \{ s : S_{s} = \lambda_u(p_{s}) \} \) and the set \( S^+ = \{ s : S_{s} < \lambda_u(p_{s}) \} \). We
know that $S^c$ is non-empty since $\bar{s} \in S$. Suppose we increase the price $p_i$ a "little". This will cause each of the $\lambda_i(p_i)$'s to decrease. In order to keep the overall sales in period $\bar{t}$ the same, we redistribute some of the inventory from the stores in the set $S^c$ to those in the set $S^r$. This will then allow us to raise the sales revenues (through the increase in the price) while keeping the solution feasible and the distribution costs constant in period $\bar{t}$, thus leading to a better solution. This proves that at the optimal, we must have $S^c = \lambda_i(p_i)$.

Proof of Theorem 5.1

We will prove the result for the POST model. The proof for the other two models can be derived by making some straightforward modifications to the proof below.

We define

- $C = (C_1, ..., C_s)$
- $J = (J_1, ..., J_s)$
- $V_t(I, C) = \text{Optimal expected revenues for the POST model from period } t \text{ onwards when inventory at end of period } t-1 \text{ is } I \text{ at the central stocking site and } C_s \text{ at store } s$.
- $V_t(I, C) = \text{Similar to } V_t(I, C) \text{ for the expected price approximation to the model}$
- $V_t(I, C, p, J) = \text{Optimal expected revenues for the POST model from period } t \text{ onwards when the inventory at the end of period } t-1 \text{ is } I \text{ at the central stocking site and } C_s \text{ at store } s, \text{ the price charged in period } t \text{ is } p, \text{ and the inventory moved from the central stocking site to store } s \text{ at beginning of period } t \text{ is } J.$
- $V_t(I, C, p, J) = \text{Similar to } V_t(I, C, p, J) \text{ for the expected price approximation to the model}$.

We need to show that $V_t(I, 0) \leq V_t(I, 0)$.

We show first by induction that, for all $t$, $V_t(I, C) \leq V_t(I + \sum_{s=1}^{s} C_s, 0) + v \sum_{s=1}^{s} C_s$.
First, let \( t = T \), the last period. Consider any \( p, I, C \) and \( J_s \geq 0 \) such that \( \sum_{s=1}^{S} J_s \leq I \).

\[
V_T(I, C, p, J) = p \sum_{s=1}^{S} E[\min\{C_s + J_s, \lambda_{sT}(p)\}] - \nu \sum_{s=1}^{S} J_s
\]

For any constant \( K \), the function \( \min\{K, X\} \) is concave in \( X \). Therefore, by Jensen's inequality, we have:

\[
\sum_{s=1}^{S} E[\min\{C_s + J_s, \lambda_{sT}(p)\}] \leq \sum_{s=1}^{S} \min\{C_s + J_s, \lambda_{sT}(p)\}.
\]

(A18)

Let \( p^*(p) \) be the price for which \( \sum_{s=1}^{S} E[\min\{C_s + J_s, \lambda_{sT}(p)\}] = \sum_{s=1}^{S} \min\{C_s + J_s, \lambda_{sT}(p^*(p))\} \).

(Such a price must exist since we have assumed in condition 5.1 that for all \( t \), \( \lambda_{a}(p) \rightarrow 0 \) as \( p \rightarrow \infty \).) Then, since \( \lambda_{sT}(p) \) is decreasing in \( p \), (A18) implies that \( p^*(p) \geq p \). So,

\[
V_T(I, C, p, J) = p \sum_{s=1}^{S} E[\min\{C_s + J_s, \lambda_{sT}(p)\}] - \nu \sum_{s=1}^{S} J_s
\]

\[
\leq p^*(p) \sum_{s=1}^{S} E[\min\{C_s + J_s, \lambda_{sT}(p)\}] - \nu \sum_{s=1}^{S} J_s
\]

\[
= p^*(p) \sum_{s=1}^{S} \min\{C_s + J_s, \lambda_{sT}(p^*(p))\} - \nu \sum_{s=1}^{S} J_s
\]

\[
= V_T(I, C, p^*(p), J)
\]

\[
\leq V_T(I + \sum_{s=1}^{S} C_s, 0, p^*(p), J + C) + \nu \sum_{s=1}^{S} C_s .
\]

Define \( L_s = J_s + C_s \) for all \( s \). Note that since \( \sum_{s=1}^{S} J_s \leq I \), we must have \( \sum_{s=1}^{S} L_s \leq I + \sum_{s=1}^{S} C_s \).

Hence, from the above inequality,

\[
\max_{p, L_x \geq 0, \sum_{s=1}^{S} J_s \leq I} \{ V_T(I, C, p, J) \} \leq \max_{p, L_x \geq 0, \sum_{s=1}^{S} L_s \leq I} \{ V_T(I + \sum_{s=1}^{S} C_s, 0, p, L) + \nu \sum_{s=1}^{S} C_s \},
\]

i.e.,

\[
V_T(I, C) \leq V_T(I + \sum_{s=1}^{S} C_s, 0) + \nu \sum_{s=1}^{S} C_s .
\]

Now assume that \( V_{t-1}(I, C) \leq V_{t-1}(I + \sum_{s=1}^{S} C_s, 0) + \nu \sum_{s=1}^{S} C_s \). We show that this implies
\begin{align*}
V_{t}(I, C) & \leq V_{t}(I + \sum_{s=1}^{S} C_{s}, 0) + v \sum_{s=1}^{S} C_{s}.
\end{align*}

Consider any \( p, I, C, \) and \( J_{s} \geq 0 \) such that \( \sum_{s=1}^{S} J_{s} \leq I. \)

\begin{align*}
V_{t}(I, C, p, J) = p \sum_{s=1}^{S} E\left[ \min \{ C_{s} + J_{s}, \lambda_{\nu}(p) \} \right] + \\
E \left[ V_{t+1}\left( I - \sum_{s=1}^{S} J_{s}, C + J - \min \{ C + J, \lambda_{\nu}(p) \} \right) \right] - v \sum_{s=1}^{S} J_{s},
\end{align*}

where \( \min \{ C + J, D_{\nu}(p) \} \) is an \( S \)-vector whose \( s \)-th component is \( \min \{ C_{s} + J_{s}, D_{\nu}(p) \} \).

Let \( A_{s} = E\left[ \min \{ C_{s} + J_{s}, \lambda_{\nu}(p) \} \right] \). Then

\( A_{s} \leq E\left[ \lambda_{\nu}(p) \right] = \lambda_{\nu}(p) \) and \( A_{s} \leq C_{s} + J_{s} \), and so

\( A_{s} = \min \{ A_{s}, \lambda_{\nu}(p) \} \) and also

\begin{align*}
E \left[ \sum_{s=1}^{S} \min \{ C_{s} + J_{s}, \lambda_{\nu}(p) \} \right] = \sum_{s=1}^{S} E\left[ \min \{ C_{s} + J_{s}, \lambda_{\nu}(p) \} \right] = \sum_{s=1}^{S} A_{s}.
\end{align*}

We also have

\begin{align*}
E \left[ V_{t+1}\left( I - \sum_{s=1}^{S} J_{s}, C + J - \min \{ C + J, D_{\nu}(p) \} \right) \right] & \\
& \leq E \left[ V_{t+1}\left( I - \sum_{s=1}^{S} J_{s} + \sum_{s=1}^{S} C_{s} + \sum_{s=1}^{S} J_{s} - \sum_{s=1}^{S} \min \{ C_{s} + J_{s}, D_{\nu}(p) \}, 0 \right) \right] + \\
& \quad E \left[ v \left( \sum_{s=1}^{S} C_{s} + \sum_{s=1}^{S} J_{s} - \sum_{s=1}^{S} \min \{ C_{s} + J_{s}, D_{\nu}(p) \} \right) \right]
\end{align*}

(by induction hypothesis)

\begin{align*}
& \leq V_{t+1}\left( I - \sum_{s=1}^{S} J_{s} + \sum_{s=1}^{S} C_{s} + \sum_{s=1}^{S} J_{s} - E \left[ \sum_{s=1}^{S} \min \{ C_{s} + J_{s}, D_{\nu}(p) \} \right], 0 \right) + \\
& \quad v \left( \sum_{s=1}^{S} C_{s} + \sum_{s=1}^{S} J_{s} - E \left[ \sum_{s=1}^{S} \min \{ C_{s} + J_{s}, D_{\nu}(p) \} \right] \right)
\end{align*}

(by Jensen's inequality, since the function \( V_{t+1}(I, 0) \) is concave in \( I \) from Lemma 5.2)
\[ V_{t+1}(I + \sum_{s=1}^{S} C_s - \sum_{s=1}^{S} A_s, 0) + \nu \sum_{s=1}^{S} (C_s + J_s - A_s) \]

Hence

\[ V_t(I, C, p, J) \leq p \sum_{s=1}^{S} A_s + V_{t+1}(I + \sum_{s=1}^{S} C_s - \sum_{s=1}^{S} A_s, 0) + \nu \sum_{s=1}^{S} (C_s + J_s - A_s) - \nu \sum_{s=1}^{S} J_s \]

\[ = p \sum_{s=1}^{S} \min\{A_s, \lambda_{s}(p)\} + V_{t+1}\left( I + \sum_{s=1}^{S} C_s - \sum_{s=1}^{S} A_s, 0 \right) - \nu \sum_{s=1}^{S} A_s + \nu \sum_{s=1}^{S} C_s \]

\[ = V_t\left( I + \sum_{s=1}^{S} C_s, 0, p, A \right) + \nu \sum_{s=1}^{S} C_s \]

Note that \( \sum_{s=1}^{S} A_s \leq I + \sum_{s=1}^{S} C_s \), since \( A_s \leq C_s + J_s \) and \( \sum_{s=1}^{S} J_s \leq I \). Hence,

\[ \max_{p, A_s \geq 0, \sum J_s \leq I} \{ V_t(I, C, p, J) \} \leq \max_{p, A_s \geq 0, \sum A_s \leq \sum C_s} V_t(I + \sum_{s=1}^{S} C_s, 0) + \nu \sum_{s=1}^{S} C_s. \]

i.e., \( V_t(I, C) \leq V_t(I + \sum_{s=1}^{S} C_s, 0) + \nu \sum_{s=1}^{S} C_s \)

Therefore, by induction, it follows that \( V_t(I, C) \leq V_t(I + \sum_{s=1}^{S} C_s, 0) + \nu \sum_{s=1}^{S} C_s \) for all \( t \).

Hence \( V_t(I, C) \leq V_t(I + \sum_{s=1}^{S} C_s, 0) + \nu \sum_{s=1}^{S} C_s \). In particular, for \( C_s = 0 \) for all \( s \), we have \( V_t(I, 0) \leq V_t(I, 0) \), which is the desired result.
REFERENCES


