MINIMAL ORDER WIENER FILTER FOR A SYSTEM WITH EXACT MEASUREMENTS

by

Violet B. Haas*

ABSTRACT

The minimal order Wiener filter is constructively derived for a linear, time invariant, detectable system some of whose measurements are noiseless, and a separation principle is derived for the general, singular LQG problem.

*Laboratory for Information and Decision Systems, MIT, Cambridge, MA 02139. The author is on sabbatical leave from the School of Electrical Engineering, Purdue University, West Lafayette, IN 47907.

This research was supported by the National Science Foundation under grant no. R-11-8310350.
I. Introduction

Here we consider the system represented by

\[ \dot{x}(t) = Ax(t) + Bw(t), \quad (1.1) \]
\[ y(t) = Cx(t) + v(t), \quad t > 0 \quad (1.2) \]

where \( w(\cdot) \) and \( v(\cdot) \) are sample vector valued functions of zero mean uncorrelated Gaussian white noise processes and the initial state \( x(0) \) is a zero mean Gaussian random variable which is uncorrelated from \( w(t) \) and \( v(t) \) for all \( t > 0 \). The vector \( x(t) \) belongs to \( \mathbb{R}^m \), \( m < n \), \( w(t) \) belongs to \( \mathbb{R}^p \), \( p < n \), the real matrices \( A, B, C \) have appropriate dimensions, \( C \) has full rank, \( (C, A) \) is a detectable pair, \( w(\cdot) \) and \( v(\cdot) \) have intensities \( I_p \) and \( R \) respectively, where \( I_p \) is the identity matrix in \( \mathbb{R}^p \) and \( R \) is real and positive semidefinite. Let \( \text{rank } R = r < m \). Then there is a nonsingular transformation \( T_0 = (U_0', W_0')' \) in the space of measurement variables such that,

\[ T_0^{RT_0} = \begin{pmatrix} R_1 & 0 \\ 0 & 0 \end{pmatrix} \]

where \( R_1 = U_0 R U_0' \) has full rank, and primes denote matrix transposition.

The transformed measurement vector \( T_0 y(t) \) consists of two components, one of which has associated noise with intensity \( R_1 \) and the other of which contains no noise. We write,

\[ y_1 \triangleq U_0 y = U_0 C x + U_0 v, \quad (1.3) \]
\[ y_2 \triangleq W_0 y = W_0 C x. \quad (1.4) \]

In this paper, we show that under certain hypothesis on the matrices \( A, B, C \), there is an \( n-m+r \)th order steady-state unbiased optimal state estimator,
which is the Wiener filter for an \( n-m_r^{th} \) order dynamical system. The state of the filter is a linear functional of the measurements \( y(\cdot) \) and a finite number of time derivatives of the exact, or noiseless, measurements. Our hypothesis is precisely the dual of the strengthened, generalized, Legendre-Clebsch condition for the dual optimal singular regulator. The strengthened generalized Legendre-Clebsch condition guarantees that all optimal controls are either regular or have finite order of singularity. For a definition of order of singularity, see Krener [1].

Our results will apply, not only to systems with some exact measurements, but also to systems whose measurements contain colored noise (signals whose correlation times are not short when compared with times of interest in the system). Colored noise may be simulated by a "shaping filter", whose system equations combine with the original system equations in such a way that the augmented system appears as though it has some noiseless measurements. Bryson and Johansen [2] in 1965 were the first to derive a Kalman filter for such singular systems, but they did not carry their development beyond the case of a "singularity or order one", that is the case when a regular (nonsingular) Kalman filter results from replacing the exact measurements by their first time derivatives. They thought there would always be only a finite number of differentiations and replacements of exact measurements needed to reduce the problem to a nonsingular one, an assumption that is erroneous. A number of authors have attempted to extend the Bryson-Johansen results for both continuous and discrete systems, but none of them understood how to obtain the
In [6] we considered the singular, finite horizon, optimal state estimation problem. There we showed that if the measurement noise intensity matrix is singular then there is no optimal state estimator (Kalman filter) whose state is a linear functional of only the original measurements. We also showed that if when a finite number of time derivatives of exact measurements replace the exact measurements in the system representation and yield a new measurement noise intensity matrix with full rank then there is an optimal state estimator which can be described by the \((m-r)\) exact measurement equations (1.4) together with a dynamical system of order \(n-m+r\). The optimal estimate is a linear functional of the original measurements and a finite number of time derivatives of the exact measurements. In [7] we considered steady-state optimal state estimators and showed that when some of the dynamical state equations are not affected by noise, i.e., when some states are undisturbed by input noise, and when a finite number of time differentiations and replacements of exact measurements result in a full rank new measurement noise intensity matrix then the order of the optimal state estimator may be reduced by the codimension of the "disturbable subspace". However, we did not show how to reduce the order further by using the exact measurements as new states. The difficulty was that detectability may be destroyed by the needed coordinate transformations and reduction of the state space. Detectability is of no concern for estimation over a finite time horizon, but is crucial in the steady state. Lemma 2.2 shows that detectability can
be preserved if we choose our state coordinate transformation wisely.

II. Some Preliminary Lemmas

Lemma 2.1. Let $A$ be a homomorphism of $\mathbb{R}^n$ into itself and let $C: \mathbb{R}^n \to \mathbb{R}^m$ be epic. Then there exists a map $V$ such that the map defined by $(C', V')'$ is an isomorphism and if $(G, H)' = (C', V')^{-1}$ then all eigenvalues of $VAH$ are eigenvalues of $A$.

Proof. First let us suppose that $A$ has $n$ linearly independent eigenvectors. Then there are $(n-m)$ linearly independent eigenvectors $v_i, i = 1, 2, \ldots, (n-m)$ of $A^*$ (the complex conjugate transpose of $A$) for which the matrix $V$, whose $i^{th}$ row is $v_i^*$ satisfies

$$\text{rank } (C^*, V^*) = n$$

Define the (mutually orthogonal) unit vectors $w_i, i = 1, \ldots, n-m$, by

$$v_i^* = w_i^*V. \quad (2.1)$$

Then since $VH = I_{n-m}$ we have

$$w_i^* = v_i^*H. \quad (2.2)$$

For each $i$ there is an eigenvalue, $\lambda_i$, of $A$ satisfying

$$v_i^*A = \lambda_i v_i^*. \quad (2.3)$$

Postmultiplying (2.3) by $H$ yields

$$v_i^*AH = \lambda_i w_i^*. \quad (2.4)$$

and for all $i$ these equations may be written as

$$VAH = \text{diag } (\lambda_1, \ldots, \lambda_{n-m}). \quad (2.5)$$
Thus, the eigenvalues of $V A H$ are precisely $\lambda_1', \ldots, \lambda_{n-m}'$. If $V$ and $H$ are complex-valued then there is a nonsingular transformation $L$ so that $LV$ and $HL^{-1}$ are real. We may then substitute $LV$ for $V$ in the construction described above.

Now suppose that $A^*$ has fewer than $n$ linearly independent eigenvectors. Then we can augment an appropriate set of linearly independent eigenvectors with enough generalized eigenvectors to yield a total of $(n-m)$ linearly independent vectors which we can stack together in rows to form the matrix $V$ in such a way that $(C', V')$ is an isomorphism. If $v_i$ is an eigenvector then (2.3) holds. If $v_j$ is a generalized eigenvector then either we can choose $v_1, \ldots, v_{n-m}$ so that

$$v_j^* A = \lambda_j v_j^* + v_{\ell}^*$$

(2.6)

for some $\ell = 1, \ldots, (n-m)$, or

$$v_j^* A = \lambda_j v_j^* + c^*$$

(2.7)

where $c^*$ is a linear combination of the rows of $C$. In the latter case we have $c^*H = 0$, so that if we again define $w_i$, $i=1,2,\ldots, n-m$ as in (2.1), then from (2.7) we obtain,

$$v_j^* AH = \lambda_j w_j^* .$$

(2.8)

If an equation like (2.7) holds for each $j = 1, \ldots, n-m$, then again $V A H$ is diagonal with its diagonal entries all eigenvalues of $A$. Now suppose that exactly one row of $V$ is a generalized eigenvector of $A^*$ and that (2.6) holds. Without loss of generality we may suppose that $j = n-m$ and $\ell = n-m-1$. In this case we find that
and again the eigenvalues of $V_{AH}$ are eigenvalues of $A$. It is now clear that if more than one row of $V$ is a generalized eigenvector of $A^*$ and if the vectors $w_i$ are defined by (2.1) then $V_{AH}$ is in Jordan normal form and all its diagonal entries are eigenvalues of $A$.

The matrix pair $(C, A)$ is said to be detectable if there is a matrix $F$ such that $A - FC$ is a stable matrix. A square matrix is stable if all its eigenvalues have negative real parts.

**Lemma 2.2.** Suppose $A$ and $C$ satisfy the hypothesis of the previous lemma and suppose that $(C, A)$ is a detectable pair. Let $0 < r < m$ and write $C = (C_1', C_2')'$ where $C_1$ has $r$ rows. Then there exists a map $V$ so that $(C_2', V')$ is an isomorphism, and if $(C_2', V')^{-1} = (G, H)'$ then $(C_1H, VAH)$ is a detectable pair.

**Proof.** Choose $F = (F_1, F_2)$ so that $A - FC = A - F_1C_1 - F_2C_2$ is stable and let $V$ be a matrix whose existence relative to the pair $(A - FC, C_2)$ is asserted in Lemma 3.1. Let $(C_2', V')^{-1} = (G, H)'$. Then the eigenvalues of $V(A - FC)H$ are all in the open left half plane and

$$V(A - FC)H = V(A - F_1C_1)H = VAH - VF_1C_1H.$$

Thus $(C_1H, VAH)$ is a detectable pair.
III. The Optimal Filter in Case of a First Order Singularity

Consider the system described by (1.1) and (1.3)-(1.4) where (C,A) is detectable. Differentiate (1.4) with respect to time noting (1.1) to obtain

\[
\dot{y}_2 = W_0 Cx + W_0 CBw \quad (3.1)
\]

If \( W_0 CB'C'W_0' \) has full rank define \( D_0 = W_0 C \). Let \( z_1 = y_1 \) and \( z_2 = \dot{y}_2 \) be the new set of measurement variables. Our system is now equivalently represented by (1.1) and

\[
\begin{align*}
z_1 &= U_0 Cx + U_0 v \quad (3.2) \\
z_2 &= W_0 Cx + W_0 CBw \quad (3.3)
\end{align*}
\]

Define \( C_1 = U_0 C, C_2 = W_0 C \) and let \((V,H)\) be the matrix pair whose existence is asserted in Lemma 2.2. Define a nonsingular coordinate transformation in the state space by (1.4) and

\[
\xi = Vx, \quad (3.4)
\]

and let \((C'W_0', V')^{-1} = (G,H)'\). Then

\[
x = Gy_2 + H\xi. \quad (3.5)
\]

Pre-multiplying (1.1) by \( V \) and substituting (1.4) and (3.5) into the result and also into (3.2)-(3.3), we obtain

\[
\begin{align*}
\dot{\xi} &= VAH\xi + VAGy_2 + VBw \quad (3.6) \\
\eta_1 &= \Delta z_1 - U_0 CGy_2 = U_0 CH\xi + U_0 v \quad (3.7) \\
\eta_2 &= \Delta z_2 - W_0 CGy_2 = D_0 AH\xi + D_0 Bw. \quad (3.8)
\end{align*}
\]
We must now uncorrelate the state and measurement noise in such a way that the property of detectability is not destroyed. To this end let k=0, let \( Q = B B' \) and define

\[
M_k = Q D_k' (D_k Q D_k')^{-1}.
\]

Add zero to the right hand side of (3.6) in the form

\[
V(I - M_k D_k) A H \xi - D_k B w.
\]

We obtain,

\[
\dot{\xi} = V(I - M_k D_k) A H \xi + V A G y_2 + V M_k \eta_2
\]

\[
+ V(I - M_k D_k) B w,
\]

(3.10)

the matrix pair,

\[
((H'C'U_0', H'A'D_0'), V(I - M_k D_k) A H)
\]

(3.11)

is detectable and the noise signal \( V(I - M_k D_k) B w \) is uncorrelated from the measurement noise, \( (v'u_0', w'B'D_0')' \). Hence, by earlier results [7] the system represented by (3.10), (3.7) and (3.8) has a Wiener filter whose state \( \hat{\xi} \) is described by,

\[
\dot{\hat{\xi}} = V A H \hat{\xi} + V A G y_2 + K_1 (\eta_1 - U_0 C H \hat{\xi})
\]

\[
+ K_2 (\eta_2 - D_k A H \hat{\xi})
\]

(3.12)

where,

\[
K_1 = P H'C'U_0'(U_0 R U_0')^{-1}, \quad K_2 = (P H'A' + V Q V') D_k' (D_k Q D_k')^{-1}
\]

(3.13)
P is the maximal positive semidefinite solution of the algebraic Riccati equation,

\[ VAHP + PH'A'V' + VQV' = K_1 U_0 RU'K_1' + K_2 D_0 D'K_2' \]  \hspace{1cm} (3.14)

and the restriction of the map

\[ (VA - K_1 U_0 C - K_2 D_k A)H \]  \hspace{1cm} (3.15)

to the disturbable (i.e., controllable) subspace of the pair

\[ (V(I - M_k D_k)AH, V(I - M_k D_k)B) \]  \hspace{1cm} (3.16)

is stable.

We have found that if \( D_0 BB'D_0' \) is nonsingular, then there is a linear state estimator for the system described by (1.1)-(1.2) which minimizes

\[ \lim_{t \to \infty} \text{tr} E[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^t]. \]

This estimator is described by (1.4), (3.4) and (3.12) - (3.14) and the estimator dynamics are stable on the disturbable subspace of the \((n-m+r)\) - dimensional state space.

The initial condition \( \hat{\xi}(0^+) \) may be updated from \( Vx(0) = 0 \) by use of the exact measurement vector \( y_2(0^+) \). However, as we shall see in the next section, the best choice of \( \hat{\xi}(0^+) \) may indeed by null.
IV. Further Order Reduction of the Filter

Since the map (3.14) may not be stable it is desirable, for computational purposes, to remove from consideration those states which lie in the undisturbable subspace. To this end, with $k=0$ define maps $A_k$ and $B_k$ by

$$A_k = V(I-M_kD_k)AH, B_k = V(I-M_kD_k)B. \tag{4.1}$$

With these substitutions (3.10) becomes

$$\xi' = A_1 \xi' + VAGy_2 + VM_k \eta_2 + B_k w. \tag{4.2}$$

Let $N = n-m+r$ and define

$$N_k = \bigcap_{i=1}^{N-1} \ker(B_i'(A_i')^{k-1}). \tag{4.3}$$

Let $<A_k | B_k>$ denote the orthogonal complement of $N_k$ in $\mathbb{R}^N$. Then $N_k$ is the undisturbable subspace of the pair $(A_k, B_k)$ and $<A_k | B_k>$ is the disturbable (controllable) subspace of the pair. In a coordinate system compatible with the decomposition

$$\mathbb{R}^N = N_k \oplus <A_k | B_k>, \tag{4.4}$$

we have $\xi' = (\xi'_1, \xi'_2)$,

$$A_k = \begin{pmatrix} A_{1k} & 0 \\ A_{2k} & A_k \end{pmatrix}, B_k = \begin{pmatrix} 0 \\ B_k \end{pmatrix}, V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, H = (H_1, H_2).$$

Then,

$$\xi'_1 = A_{1k} \xi'_1 + V_1 AGy_2 + V_1 M_k \eta_2. \tag{4.5}$$
\[ \dot{\xi}_2 = \overline{A}_k \hat{\xi}_2 + A_{2k} \xi_1 + V_2 AG_2 + V_2 M \eta_2 + \overline{B}_k w \quad (4.6) \]

\[ \xi_1 = \eta_1 - U_0 C_1 \xi_1 = U_0 C_2 \xi_2 + U_0 v \quad (4.7) \]

\[ \xi_2 = \eta_2 - D_k A \xi_1 = D_k A \xi_2 + D_k B w \quad (4.8) \]

Note that (4.5) has no additive noise. The measurements \( y_2 \) and \( \eta_2 \) are known functions and so may be considered as inputs. Hence, the best estimate of \( \xi_1 \) must be given by

\[ \dot{\hat{\xi}}_1 = A_{1k} \hat{\xi}_1 + V_1 AG_2 + V_1 M \eta_2, \quad \hat{\xi}_1(0) = 0. \quad (4.9) \]

Then if we define \( \tilde{\xi}(t) = \xi(t) - \hat{\xi}(t) \), we have,

\[ \dot{\tilde{\xi}}_1 = A_{1k} \tilde{\xi}_1, \]

and since \( E[\tilde{\xi}_1(0)] = 0 \) we must have \( \tilde{\xi}_1(t) = 0 \). Thus, we may suppose that \( \xi_1(\cdot) \) is a known function. With this in mind \( \zeta_1 \) and \( \zeta_2 \) may be considered as new measurement variables. Since the pair \((A_k, B_k)\) is controllable and the pair \( (H_1^C U_0^A, H_1 A^D) \), \( \overline{A}_k \) is detectable (see [8]) then by standard theory [9] the Wiener filter for the system described by (4.6)-(4.8) is represented by

\[ \dot{\hat{\xi}}_2 = \overline{A}_k \hat{\xi}_2 + A_{2k} \hat{\xi}_1 + V_2 AG_2 + V_2 M \eta_2 \]

\[ + K_1 (\zeta_1 - U_0 C_1 \hat{\xi}_1) + K_2 (\zeta_2 - D_k A \hat{\xi}_2), \quad (4.10) \]

where

\[ K_1 = \overline{PH}^T C_0^T (U_0 P U_0^T)^{-1}, \quad K_2 = (\overline{PH}^T A + \overline{B}_k \overline{B}_k^T) D_1^T (D_1 D_1^T)^{-1} \quad (4.11) \]

\( \overline{P} \) is the unique, symmetric, positive definite solution of the algebraic Riccati equation,
\[
\begin{align*}
\Delta_k \frac{P}{K_1} + \frac{PA'}{K_2} + \frac{B_k B'}{K_k} &= K_1 (U_0 RU') K_1' + K_2 (D_k OD_k') K_2', \\
(4.12)
\end{align*}
\]

and the matrix
\[
\bar{A}_k - (K_1 U_0 C + K_2 D_k A) H_2
\]
(4.13)
is stable. The initial condition on \( \hat{\xi}_2 \) may be chosen to satisfy,
\[
\hat{\xi}_2 (0^+) = 0,
\]
(4.14)
for since the matrix (4.13) is stable, the steady state value of \( \hat{\xi}_2 \) is unaffected by the initial condition. The best steady-state estimate \( \hat{x}(t) \) of the original state \( x(t) \) is given by
\[
\hat{x}(t) = G \hat{y}_2(t) + H \hat{\xi}(t).
\]
(4.15)

Contrary to popular belief, there is no information to help us update the estimate \( \hat{\xi}(0) \) once \( y_2(0) \) is known. In references [2] and [10] it is suggested that the initial condition \( x(0) \) be updated by means of a formula that involves the expression
\[
(R + CPC')^{-1}.
\]

We note here that the kernel of \( R \) must be contained in the kernel of \( PC' \) since \( PC' W_0' \) must vanish - there can be no estimation error for those (transformed) state variables which are components of the exact measurement \( W_0 y \). Thus, if \( R \) is singular then so is \( (R + CPC') \).
5. Higher Order Conditions for Optimality: $k>l$.

Suppose $D_0QD_0'$ does not have full rank. Then there exists a sequence
\{T_i\}, $i=1,...,k$ of transformations,
\[ T_i = (U'_i, W'_i), \quad (5.1) \]
such that for $i = 1,2,...,k-1$,
\[ B'(A')^{i-1}C'_0W'_0W'_1...W'_{i-1}U'_i \text{ has full rank} \quad (5.2) \]
\[ B'(A')^{i-1}C'_0W'_0W'_1...W'_i = 0 \quad (5.3) \]
and
\[ B'(A')^{k-1}C'_0W'_0...W'_{k-1} \text{ has full rank} \quad (5.4) \]

For $k>1$ let $U_k$ be an identity map and define $D_k$ by,
\[ D'_k = (C'_0W'_0U'_1, A'C'_0W'_0U'_1,...,(A')^{k-1}C'_0W'_0...W'_{k-1}U'_k) \quad (5.5) \]

If $D_0QD_0'$ does not have full rank, then $k>1$, and from (3.1) we find
\[ U'_1\dot{y}_2 = U'_1W'_0CAx + U'_1W'_0CBw, \]
\[ W'_1\dot{y}_2 = W'_1W'_0CAx \]
and
\[ W'_1\dot{y}_2 = W'_1W'_0CAx^2 + W'_1W'_0CBw \quad (5.6) \]

If now $W'_1W'_0CA$ has maximal rank then the sequence (5.1) stops with $k=1$ and
\[ D'_1 = (C'_0W'_0U'_1, A'C'_0W'_0U'_1). \]
If $D_1QD_1'$ has full rank, we define $z_1$ as before, and we define $z_2 = U'_1\dot{y}_2$ and $z_3 = W'_1\dot{y}_2$ as new measurement vectors. The measurement noise now has a full rank intensity matrix. We define $V, G, H, \eta_1$ and $\xi$ as before, and define new measurements $\eta_2$, $\eta_22$ by
\[ \eta_{21} \triangleq z_2 - W_1W_0CAH_2 = W_1W_0CAH_2 + W_1W_0CBW \quad (5.7) \]

\[ \eta_{22} \triangleq z_3 - W_1W_0CAGy_2 = W_1W_0CAGy_2 + W_1W_0CBW \quad . \quad (5.8) \]

Let \( \eta_2 \triangleq (\eta_{21}, \eta_{22})' \). Then with \( k=1 \) we have,

\[ \eta_2 \triangleq D_kAH_2 + D_kBW \quad . \quad (5.9) \]

Then exactly as before, the system described by (3.10), (3.7) and (5.9) has an optimal steady-state estimator which is described by (1.4), (3.5), (4.9)-(4.12) and (4.14), and the matrix (4.13) is stable. Note that \( D_kQD_k' > 0 \) implies (5.4).

It can easily be shown by induction that if (5.2) - (5.3) hold for \( i=1, \ldots, k-1 \), and if \( D_kQD_k' > 0 \) then the steady state, optimal state estimator is given by (1.4), (3.5), (4.9)-(4.12) and (4.14) and the matrix (4.13) is stable. Note that by the Cayley-Hamilton Theorem, \( D_{n+j} \) has the same number of linearly independent rows as \( D_n \) for \( j \geq 0 \). We have thus proved the following theorem.

**Theorem 5.1.** Let \( A: R^n \rightarrow R^n \), \( C: R^n \rightarrow R^m \), \( B: R^p \rightarrow R^n \), and consider the uncertain system described by (1.1) and (1.3)-(1.4). Suppose the pair (C,A) is detectable and rank \( R=r \). Let \( \{T_1^i\} \), \( i=1,2,\ldots,k-1 \) denote a sequence of isomorphisms satisfying (5.1)-(5.3). If \( D_kQD_k' \) has full rank then \( k<n \), and in a coordinate system compatible with the decomposition (4.4) there exists a Wiener filter with state \( \hat{\xi}(t) \) satisfying (4.9)-(4.12) and (4.14) and the matrix (4.13) is stable. The best estimate \( \hat{x}(t) \) of the original state \( x(t) \) is a linear combination of the exact measurements, \( W_0y(t) \) and of \( \hat{\xi}(t) \) and \( \hat{\xi} \) is described by an \( (n-m+r) \)th order dynamical system. The error vector \( \hat{\xi}(t) \) lives in the disturbable subspace of the pair \( (A_k,B_k) \) where \( A_k \) and \( B_k \) are defined by (4.1).
When $D_k Q D_k$ has full rank, we shall say that our optimal estimator has order of singularity $k+1$. The condition $D_k Q D_k > 0$ is precisely the dual of the condition guaranteeing that each component of the optimal control of a dual singular regulator have order of singularity $k$ or less. This latter condition was derived in earlier work [11].

It may happen that there is no nonnegative integer $k$ for which $D_k Q D_k$ has full rank. In this event the steady-state, optimal state estimator cannot be derived by our methods and we shall say that this estimator has infinite order of singularity.
VI. The Singular Separation Principle

Here we consider the optimal stochastic control problem defined by

\[ \dot{x} = Ax + Bu + w, \quad t > 0. \] (6.1)

\[ y = Cx + v, \quad t > 0. \] (6.2)

We seek a control function \( u^*(\cdot) \) to minimize the cost functional,

\[ J(u) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \left( x'x + 2x'Su + u'Uu \right) dt \right] \] (6.3)

where \( w, v \) and \( x(0) \) are described as before, except that now the semidefinite intensity \( Q \) of the process \( w(\cdot) \) is not \( BB' \). The weighting matrix \( U \) is symmetric and nonnegative definite. When \( U \) is not strictly positive definite and the analogous deterministic control problem has finite order of singularity then there is a finite sequence of nonsingular transformations of the control and state variables which will put the problem into a similar form but with a strictly positive definite control weighting matrix. So without loss of generality we may suppose that \( U \) is strictly positive definite and that the solution of the deterministic optimal control problem exists.

We suppose further that the pair \((C, A)\) is detectable and the pair \((A, B)\) is stabilizable.

Define the transformation \( T_0 = (U_0', W_0')' \) as before and by Lemma 3.2 we can find a matrix \( V \) such that \((C'W_0', V')\) has full rank and if \((C'W_0', V')^{-1} = (G, H)'\), then the pair \((C'U_0, A'C'W_0)' \), \( VAH \), is detectable. We further suppose that the estimation problem has finite order of singularity and that there exists a nonnegative integer \( k \) such that \( D_kQD_k' \) is nonsingular. By standard arguments [12] we obtain a separation principle. The optimal control \( u^* \) is given by,
where $P$ is the maximal symmetric positive semidefinite solution of the algebraic Riccati equation,

$$(A-BU^{-1}S')'P + P(A-BU^{-1}S') + (X-SU^{-1}S') = PBU^{-1}B'P, \quad (6.7)$$

$K_1$ and $K_2$ satisfy

$$K_1 = \pi H'C'U_0'(U_0RU_0')^{-1}, \quad (6.8)$$

$$K_2 = (\pi H'A' + VQV')D_k(D_kQD)_k^{-1}, \quad (6.9)$$

$\pi$ is the maximal symmetric positive semidefinite solution of the algebraic Riccati equation,

$$VAH\pi + \pi H'A'V' + VQV' = K_1(U_0RU_0')K_1' + K_2(D_kQD)_kK_2', \quad (6.10)$$

and $\eta_1$ and $\lambda_2$ are observation variables defined by

$$\eta_1 = U_0CH\xi + U_0v, \quad (6.11)$$

$$\eta_2 = D_kAH\xi + D_kw . \quad (6.12)$$

Note from equations (6.4)-(6.5) that the necessary condition for optimality derived in [13], namely that if the optimal control is given by $u^* = Z_1\hat{\xi} + Z_2y$ then trace $(Z_2'UZ_2R) = 0$, is automatically satisfied since $Z_2R = -U^{-1}(B'P+S')GW_0R = 0$.  

VII. Conclusion.

We have seen that in case there is a nonnegative integer $k$ for which the matrix $D_k C D_k'$ has full rank then a steady-state optimal state estimator of reduced order can be designed for the uncertain system described by (1.1)-(1.2). In this case the order of the optimal estimator is reduced by the number of noise-free measurements, and the error vector lives in an even lower dimensional space when not all states are disturbable by input noise. Furthermore the solution of this singular estimation problem is the dual of the solution of the dual optimal regulator problem if all optimal controls have finite order of singularity. We have also combined the results of the deterministic optimal regulator problem with the singular state estimation problem to solve the general singular LQG problem and state a singular separation principle.
REFERENCES


