Nonlinear acoustics in the presence of an object with sum or difference frequency sensing

by

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Abstract

A general and complete second-order theory of nonlinear acoustics in the presence
of an object is derived and shown to be consistent with experimental measurements.
The total second-order field occurs at sum or difference frequencies of the primary
fields and naturally breaks into (A) nonlinear waves generated by wave-wave inter-
actions, and (B) second-order scattered waves that include the effect of centriodal
motion of the object driven by a complete second-order wave-exciting force. Analy-
lytic expressions for second-order fields due to combinations of planar and spherical
wave-wave interactions are derived. Wave-wave interactions are analytically shown
to always dominate the total second-order field at sufficiently large range and carry
only primary frequency response information about the object. As range decreases,
the dominant mechanism is shown to vary with object size, boundary condition, and
frequencies making it sometimes possible for sum or difference frequency response in-
formation about the object to be measured from second-order fields. Unique opportu-
nities arise for nonlinear sum or difference frequency sensing that differ substantially
from traditional sensing by linear scattering. Analytic proof shows that there is no
scattering of sound by sound outside the region of compact support intersection of
finite-duration plane waves at sum or difference frequencies, to second-order.

Thesis Supervisor: Nicholas C. Makris
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H.1 Quantities to determine corner contributions of $P_{\text{IS}}(r_B)$ in the backscatter direction.
Chapter 1

Introduction

A general and complete second order theory of nonlinear acoustics in the presence of an object is derived in this thesis, where the object centroid can move. A complete second order wave-exciting force is introduced and used to determine the effect of centroidal motion on the second order field. The theory employs consistent asymptotic analysis of the wave equation and time-dependent boundary conditions to second order, following methods developed in fluid dynamics to quantify second order nonlinear waves found in the presence of floating objects [5, 6]. The theory is confirmed by comparison with the experimental measurements of Jones and Beyer [2].

When an object is insonified by two primary frequency incident waves, second order acoustic waves at the sum and difference frequencies of the primary waves arise due to multiple mechanisms. It is found that these generating mechanisms fall into two categories. The first is (A) nonlinear wave-wave interactions of the primary waves, including the interaction between two incident waves (denoted as II), the interaction between two scattered waves (denoted as SS) and the interactions between an incident and scattered wave (denoted as IS and SI). The second is (B) second order linear scattering of nonlinear waves of the first category due to the presence of the object (denoted as S2). Second order acoustic waves of category (A) contain information about the object’s first order response at the primary frequencies but not at the sum or difference frequency. Second order acoustic waves of category (B), on the other hand, contain information about the object’s second order response at the sum or
difference frequency.

We show analytically that wave-wave interactions dominate the total second order field at long range from the object. At shorter ranges, second order scattering may also dominate depending on physical parameters such as object size, boundary condition and primary frequencies. We show that centroidal motion of a rigid object due to its interaction with acoustic waves can significantly affect the second order field at short range. In this case, $S_2$ can be decomposed into scattering from a fixed object at its time-averaged position (denoted as $D_2$), and radiation due to the second order centroidal motion (denoted as $R_2$).

To quantify the scattering of sound by sound inside and outside a bounded interaction region of primary waves with compact support, we derive analytic expressions for the second-order nonlinear field arising from the interaction of plane waves of arbitrary time dependence. In an early investigation of the scattering of sound by sound, Ingard and Pridmore-Brown [7] employed unphysical collimated primary time-harmonic fields, which led to unphysical results as noted by Westervelt [8]. The subsequent time-harmonic plane wave analysis of Westervelt [8] provided no spatial region without interacting primary fields. By analytic proof we show that to second order there is no scattering of sound by sound outside the region of compact support intersection of finite-duration plane waves at sum or difference frequencies in the absence of an object. We also show that non-collinear interaction leads to finite scattering of sound by sound at the primary frequencies only within the region of compact support union through which compact support intersection occurred if the primary waves had zero-frequency spectral components. We rigorously prove that Lamb's collinear [9] and Westervelt's non-collinear [8] time-harmonic solutions at the sum or difference frequency for ideal time-harmonic primary plane wave interactions provide good approximations to the second order field arising from the interaction of narrow-band finite-duration primary plane waves with sufficiently long and smooth envelopes, but only in the region of compact support intersection where the primary fields exist. We show such a time-harmonic approximation can be made in many practical scenarios involving the interaction between scattered waves, as well as scattered
and incident waves of narrow band. Following experimental practice, we employ a time domain formulation of finite-duration narrow-band incident waves and provide analytic solutions for the resulting second order fields in the presence of an object. Finite-duration incident waves enable nonlinear field components that contain information about the object to be separated in time and space from those that do not, such as the II component, for certain sensing geometries.

Since the SS, IS and SI interactions always dominate the field components that carry object information at long range, exact and asymptotic solutions are rigorously obtained for them, which put previous heuristic estimates [10, 1, 2] of SS into perspective. For spherically symmetric interacting waves, our solution is exact and reduces to Baxter's [11] for this special case, where we show that Baxter's solution satisfies the Sommerfeld radiation condition but Dean's [12] does not. Analytic solutions are derived for the second-order field due to the interaction of a plane and spherical wave. These enable determination of IS and SI in the far field of any scatterer.

Unique opportunities then arise for sensing an object at the sum or difference frequency. These are substantially different from traditional sensing via linear scattering at the primary frequency. By ensonifying an object at primary frequencies and sensing at sum or difference frequencies, primary frequency response information about an object can be estimated from wave-wave interactions at long range (SS, IS or SI), while sum or difference frequency response information about an object can sometimes be estimated from second order scattered waves (S2) measured at shorter range. We investigate several practical sensing scenarios in air, water and solid earth for rigid, pressure-release and resonating objects. Approximate analytic solutions for the second order nonlinear wave fields found in the presence of low-impedance contrast inhomogeneities are also derived in Appendix A.

Inspired by 18th-century accounts of difference frequency waves propagating from musical instruments [2], Lamb developed solutions for the II interaction of collinear time-harmonic primary plane waves [9]. Westervelt later solved the time-harmonic non-collinear plane wave case [8] and estimated the second order field of an end-fire array [13]. Then Dean and Baxter obtained time-harmonic solutions for the
omnidirectional cylindrical [12] and spherical [12, 11] wave cases, and Tjotta et al. investigated II interaction for time-harmonic incident beams [14, 15, 16]. Jones and Beyer measured the second-order acoustic field at the sum frequency caused by an object in the overlap region of two primary incident waves [1, 2] and proposed a heuristic formula consistent with an SS-type mechanism based on Dean's solution. Greenleaf et al. proposed measuring an object's difference frequency response from primary wave insonification [17, 18]. Without a complete theoretical formulation for the second order field in the presence of a movable object, it was noted that the interaction of primary scattered waves may dominate the second order field [19, 20, 21] and mask information about the object's difference frequency response contained in weaker field components. Such a complete theory, as presented here, is necessary to investigate the conditions under which an object's difference frequency response may be detectable and when each of the various second order mechanisms may dominate. To treat the case of a movable object, we define a complete and self-consistent second order acoustic wave-exciting force, and find previous formulations [22, 23, 24, 25] missed the effects of some second order field components and overcounted others. A detailed discussion of past work on nonlinear acoustics in the presence of an object is provided in Appendix L.
1.1 Nonlinear interaction of waves

The principle of superposition holds for any linear system. In linear acoustics, if two acoustic waves propagate simultaneously through a common region, the total wave field is the sum of individual waves propagating through the same region separately. However, due to the nonlinear nature of both the fluid dynamic equations and the constitutive relation of the medium, acoustic waves are nonlinear and the total wave field is not exactly the sum of individual waves.

One analytical approach to model the nonlinear problem is to use the perturbation theory. This method is useful especially when the nonlinearity is small so that the solutions can be approximated by only the first few terms in the perturbation series. In this thesis, we restrict ourselves to second order nonlinear acoustics, where acoustics is linear to first order.

As can be seen from the analysis of Chapter 2, the second order field is generated from products of primary fields in the governing equation and the boundary conditions. If the primary fields have two frequency components at angular frequencies $\omega_a$ and $\omega_b$, we have

$$p_1 = \Re\{P_a e^{-i\omega_a t} + P_b e^{-i\omega_b t}\}.$$  \hfill (1.1)

Then $p_1^2$ can be written as

$$p_1^2 = \frac{1}{2} \Re\{|P_a|^2 + |P_b|^2\} + \frac{1}{2} \Re\{P_a^2 e^{-2i\omega_a t} + P_b^2 e^{-2i\omega_b t}\}$$

$$+ \Re\{P_a P_b e^{-i(\omega_a + \omega_b)t} + P_a P^*_b e^{-i(\omega_a - \omega_b)t}\}. \hfill (1.2)$$

which has zero- and double- ($2\omega_a$ and $2\omega_b$) frequency components due to the nonlinear interaction of a field component with itself (self-interaction), and it also has sum- ($\omega_+ = \omega_a + \omega_b$) and difference- ($\omega_-= \omega_a - \omega_b$) frequency components due to the nonlinear interaction of two different field components (cross-interaction). We will mainly focus on cross-interactions, because they are responsible for the generation of sum and difference frequency second order waves, also because self-interaction of the
same wave is the special case of cross-interaction when the two interacting waves are identical.
1.2 Problem definition

The general physical problem is illustrated in figure 1-1. Two primary incident fields $p_{ia}$ and $p_{ib}$ are incident on an object, causing two primary scattered fields, $p_{sa}$ and $p_{sb}$. The second order field $p_2$ generated by wave-wave interactions and second order scattering is measured by a receiver at $r_R$. The purpose of this study is to investigate the generation mechanisms of second order waves at the sum and difference frequencies, to understand their behavior and to quantify their contribution to the total second order field.

![Diagram](image)

Figure 1-1: Two primary incident fields $p_{ia}$ and $p_{ib}$ are incident on an object, causing two primary scattered fields, $p_{sa}$ and $p_{sb}$. A receiver at $r_R$ measures the second order field $p_2(r_R, t)$. 

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Chapter 2

Linear and second order nonlinear formulation

Assuming weak nonlinearity, the total pressure field \( p(r, t) \), density field \( \rho(r, t) \), fluid particle velocity \( v(r, t) \), object boundary displacement \( \xi(r, t) \) and total force on the object \( f(t) \) due to the acoustic field can be described by [26]

\[
\begin{align*}
    p &= p_0 + p_1 + p_2 + \cdots, \\
    \rho &= \rho_0 + \rho_1 + \rho_2 + \cdots, \\
    v &= v_1 + v_2 + \cdots, \\
    \xi &= \xi_1 + \xi_2 + \cdots, \\
    f &= f_1 + f_2 + \cdots,
\end{align*}
\]

(2.1a - 2.1e)

where subscripts 0, 1 and 2 indicate zeroth-, first- and second-order quantities, respectively, with \( \{0\} \gg \{1\} \gg \{2\} \). Here \( p_0 \) and \( \rho_0 \) are ambient pressure and density of the medium without acoustic waves and \( v_0 = \xi_0 = f_0 = 0 \).
2.1 First order formulation

The total first order pressure field \( p_1 \) satisfies the wave equation with source function \( q_1 \), via

\[
\left( \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p_1(r, t) = -q_1(r, t),
\]

(2.2)

where \( c_0 \) is the sound speed and \( \nabla^2 \) is the Laplacian. The boundary conditions for pressure release, rigid immovable, or rigid movable objects without rotation are

- pressure release (Dirichlet): \( p_1 = 0 \) on \( \bar{S} \),
  \[
  \text{(2.3)}
  \]
- rigid immovable (Neumann): \( \mathbf{v}_1 \cdot \mathbf{n} = 0 \) on \( \bar{S} \),
  \[
  \text{(2.4)}
  \]
- rigid movable:
  \[
  \mathbf{v}_1 \cdot \mathbf{n} = \mathbf{u}_{c1} \cdot \mathbf{n} \quad \text{on } \bar{S},
  \text{(2.5)}
  \]

where the fluid velocity is related to the pressure by the linearized momentum equation \( \mathbf{v}_1 = -\rho_0^{-1} \int \nabla p_1 \, dt \), \( \mathbf{u}_{c1} \) is velocity of the object's centroidal motion in response to the acoustic field, \( \bar{S} \) is the mean reference boundary position, and \( \mathbf{n} \) is the surface outward normal vector of the medium. Pressure release and rigid immovable boundaries are two limiting cases of a general locally reacting boundary. It is shown in Appendix B that the solution to the first and second order scattering problems for any locally reacting boundary can be constructed from pressure release and rigid immovable scattering solutions. A rigid movable boundary, however, is a special case of a non-locally reacting boundary because \( \mathbf{u}_{c1} \) depends on the acoustic field on the whole boundary.

The complete solution for \( p_1 \) is given by Green theorem, as [27, 28]

\[
p_1(r, t) = p_{11}(r, t) + p_{S_1}(r, t) + p_{\text{init1}}(r, t),
\]

(2.6)
where

\[ p_{11}(r, t) = \int_{t_{\text{init}}}^{t_{\text{init}}+} dt_0 \iiint dV_0 g(r, t|t_0, t_0)q_1(r_0, t_0), \]  

(2.7)

\[ p_{S1}(r, t) = \int_{t_{\text{init}}}^{t_{\text{init}}+} dt_0 \iiint dS_0 n_0(r_0) \cdot [p_{S1}(r_0, t_0) \nabla_0 g(r, t|t_0) - g(r, t|t_0) \nabla_0 p_{S1}(r_0, t_0)], \]  

(2.8)

\[ p_{\text{init}1}(r, t) = -\frac{1}{c_0^2} \iiint dV_0 \left[ \frac{\partial g(r, t|t_0, t_0)}{\partial t_0} p_1(r_0, t_0) - g(r, t|t_0, t_0) \frac{\partial p_1(r_0, t_0)}{\partial t_0} \right]_{t_0=t_{\text{init}}}. \]  

(2.9)

Here \( p_{11} \) is the known incident wave, \( p_{S1} \) is the scattered wave, which depends on the object boundary condition, and \( p_{\text{init}1} \) represents the transient effect associated with initial conditions at \( t_{\text{init}} \). The time domain Green function is \( g(r, t|t_0, t_0) = \delta(t - t_0 - R/c_0)/(4\pi R) \) where \( \delta \) is the Dirac delta function and \( R = |r - r_0| \).

For pressure release or rigid immovable objects, the scattered wave \( p_{S1} \) can be obtained by solving the integral equation (2.8) with boundary conditions

\[ \text{pressure release:} \quad p_{S1} = -p_{11} \quad \text{on } \bar{S}, \]  

(2.10)

\[ \text{rigid immovable:} \quad v_{S1} \cdot n = -v_{11} \cdot n \quad \text{on } \bar{S}, \]  

(2.11)

where \( v_{S1} = -\rho_0^{-1} \int \nabla p_{S1} dt \) and \( v_{11} = -\rho_0^{-1} \int \nabla p_{11} dt \).

For rigid movable objects, let the total scattered wave \( p_{S1} \) be decomposed as

\[ p_{S1} = p_{D1} + p_{R1}, \]  

(2.12)

where subscript \( D1 \) represents first order scattering from the rigid object whose boundary is fixed, and subscript \( R1 \) represents first order radiation due to centroidal motion of the rigid object. By definition, \( p_{D1} \) is the same as \( p_{S1} \) for the rigid immovable object. The radiated wave \( p_{R1} \) depends on the \( u_{c1} \), and it satisfies the boundary
condition

\[ \mathbf{v}_{R1} \cdot \mathbf{n} = u_{c1} \cdot \mathbf{n} \quad \text{on} \quad \bar{S} \]  

(2.13)

where \( \mathbf{v}_{R1} = -\rho_0^{-1} \int \nabla p_{R1} \, dt \). The centroidal velocity \( u_{c1} \) is determined by the equation of motion

\[ M \frac{du_{c1}}{dt} = \iint_S (p_{11} + p_{D1} + p_{R1}) \, ndS, \]  

(2.14)

where \( M \) is the mass of the object and the integral on the right-hand side of equation (2.14) represents the total force on the object. The inertial term on the left-hand side of equation (2.14) can be written as \( z_m u_{c1} \) with operator \( z_m = Md/dt \). Since the radiated wave is a linear function of the centroidal motion, the force on the body due to radiation can be written as \( \iint_S p_{R1} \, ndS = -z_r u_{c1} \), where the linear operator \( z_r \) depends on the object geometry and the medium properties. Then \( u_{c1} \) becomes

\[ u_{c1} = (z_m + z_r)^{-1} f_{excit}^{1}, \]  

(2.15)

where the wave-exciting force is defined as

\[ f_{excit}^{1} = \iint_S (p_{11} + p_{D1}) \, ndS, \]  

(2.16)

which is completely determined by \( p_{11} \) and \( p_{D1} \). Once \( u_{c1} \) in equation (2.13) is determined, \( p_{R1} \) can be obtained with the same procedure as that for \( p_{D1} \).
For a time harmonic problem at angular frequency $\omega$, it follows that

\begin{align}
q_i(r,t) &= \Re\{Q_i(r)e^{-i\omega t}\}, \quad (2.17a) \\
p_i(r,t) &= \Re\{P_i(r)e^{-i\omega t}\}, \quad (2.17b) \\
p_{11}(r,t) &= \Re\{P_{11}(r)e^{-i\omega t}\}, \quad (2.17c) \\
p_{S1}(r,t) &= \Re\{P_{S1}(r)e^{-i\omega t}\}, \quad (2.17d) \\
p_{D1}(r,t) &= \Re\{P_{D1}(r)e^{-i\omega t}\}, \quad (2.17e) \\
p_{R1}(r,t) &= \Re\{P_{R1}(r)e^{-i\omega t}\}, \quad (2.17f) \\
u_{e1}(t) &= \Re\{U_{e1}(t)e^{-i\omega t}\}, \quad (2.17g) \\
f_1^{\text{excit}}(t) &= \Re\{F_1^{\text{excit}}e^{-i\omega t}\}, \quad (2.17h)
\end{align}

where capital symbols are the complex amplitudes of the corresponding physical quantities denoted by lower case symbols. Substituting $p_i$ and $q_i$ into equation (2.2) yields the Helmholtz equation

\[(\nabla^2 + k^2)P_i = -Q_i, \quad (2.18)\]

where $k = \omega/c_0$. Substituting $p_{S1}$ into equation (2.8) and integrating over $t_0$ gives

\[P_{S1}(r,t) = \int\int dS_0 n(r_0) \cdot [P_{S1}(r_0)\nabla_0 G(r|r_0) - G(r|r_0)\nabla_0 P_{S1}(r_0)], \quad (2.19)\]

where the frequency domain Green function is $G(r|r_0) = e^{ikR}/(4\pi R)$. Solution for $P_{S1}$ for pressure release or rigid immovable objects can be obtained from equation (2.19) with boundary conditions

\begin{align}
\text{pressure release:} & \quad P_{S1} = -P_{11} \quad \text{on } \bar{S} \quad (2.20) \\
\text{rigid immovable:} & \quad V_{S1} \cdot n = -V_{11} \cdot n \quad \text{on } \bar{S} \quad (2.21)
\end{align}

where $V_{S1} = (i\omega \rho_0)^{-1}\nabla P_{S1}$ and $V_{11} = (i\omega \rho_0)^{-1}\nabla P_{11}$.

For rigid movable objects, $P_{D1}$ is given by the rigid immovable object and $P_{R1}$ can
be obtained from equation (2.19) with \( P_{S1} = P_{R1} \) and boundary condition

\[
V_{R1} \cdot n = U_{c1} \cdot n,
\]

(2.22)

where \( V_{R1} = (i\omega \rho_0)^{-1} \nabla P_{R1} \),

\[
U_{c1} = [Z_m(\omega) + Z_r(\omega)]^{-1} F_1^{\text{excit}},
\]

(2.23)

and

\[
F_1^{\text{excit}} = \iint_S (P_{11} + P_{D1}) ndS.
\]

(2.24)

Here \( Z_m(\omega) = -i\omega Z_m \) is the Fourier transform of \( z_m \) and \( Z_r(\omega) \) is the Fourier transform of \( z_r \), which is sometimes referred to as the radiation impedance [29, 26, 24].

If the rigid movable object is attached to a spring and damper in the direction of motion, equation (2.23) becomes

\[
U_{c1} = \left[ Z_m(\omega) + Z_r(\omega) + \frac{M\omega_n^2}{-i\omega} + 2\zeta M\omega_n \right]^{-1} F_1^{\text{excit}},
\]

(2.25)

where \( \omega_n \) is the (undamped) natural frequency, and \( \zeta \) is the dimensionless damping ratio.

To find \( Z_r(\omega) \), three radiated waves \( \hat{P}_{R,j}(\omega), \ j = x, y, z \) at frequency \( \omega \) need to determine from the boundary conditions

\[
\hat{V}_{R,j} \cdot n = \hat{U}_{c,j} \cdot n, \quad \text{for} \ j = x, y, z
\]

(2.26)

where \( \hat{V}_{R,j} = (i\omega \rho_0)^{-1} \nabla \hat{P}_{R,j}, \ j = x, y, z \), given unit velocities \( \hat{U}_{c,x} = (1, 0, 0), \hat{U}_{c,y} = (0, 1, 0) \) and \( \hat{U}_{c,z} = (0, 0, 1) \). These radiated waves can be determined from the integral equation (2.19), in a similar manner as \( P_{S1} \). In terms of \( \hat{P}_R = (\hat{P}_{R,x}, \hat{P}_{R,y}, \hat{P}_{R,z}), Z_r(\omega) \)
is determined from its elements by

\[ \{Z_r(\omega)\}_{ij} = -\int_S n_i \hat{P}_{R,j}(\omega) dS, \quad \text{for } i, j = x, y, z. \quad (2.27) \]

As a special case, for a rigid sphere with radius \( a \) in an ideal fluid with density \( \rho_0 \) and sound speed \( c_0 \), the radiation impedance reduces to the scaler \( Z_r(\omega) = 4\pi \rho_0 c_0 a^2 h_1(ka)/h'_1(ka)/3 \), where \( h_1 \) is the spherical Hankel function of the first kind and \( h'_1 \) is the derivative of \( h_1 \) with respect to its argument.

For an object with centroidal velocity \( U_c = U_{c,x} \hat{U}_{c,x} + U_{c,y} \hat{U}_{c,y} + U_{c,z} \hat{U}_{c,z} \), the total radiated wave is

\[ P_R(\omega) = \hat{P}_R(\omega) \cdot U_c(\omega). \quad (2.28) \]

This relation holds in both first and second order.

### 2.2 Second order formulation

The governing equation for the second order acoustic pressure field \( p_2(r, t) \) in a homogeneous, inviscid medium is [8]

\[ \left( \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p_2 = -q_2, \quad (2.29) \]

where

\[ q_2 = \frac{1}{\rho_0 c_0^4} \frac{B}{2A} \frac{\partial^2 p_1}{\partial t^2} + \nabla \cdot (\nabla \cdot \mathbf{J}_2), \quad (2.30) \]

and \( \mathbf{J}_2 = \rho_0 \mathbf{v}_1 \mathbf{v}_1 \) is the momentum tensor [26], and \( A = \rho_0 (\partial p/\partial \rho)|_{\rho_0} = \rho_0 c_0^2 \) and \( B = \rho_0^3 (\partial^2 p/\partial \rho^2)|_{\rho_0} \) are constants from the equation of state. The second order fluid velocity \( \mathbf{v}_2 \) is related to \( p_2 \) via

\[ \mathbf{v}_2 = -\frac{1}{\rho_0} \int \nabla p_2 dt + \frac{1}{2\rho_0^2 c_0^2} \int \nabla p_1^2 dt - \frac{1}{2} \int \nabla (\mathbf{v}_1 \cdot \mathbf{v}_1) dt. \quad (2.31) \]
To find the second order boundary condition on the body, we expand the boundary condition on the exact boundary $S(t)$ in a Taylor series with respect to $\tilde{S}$ [5]. After substituting the perturbation expansions for $p$, $v$ and $\xi$ into the series and collecting terms at each order, the second order boundary conditions are obtained as

$$p_2 = -\xi_1 \cdot \nabla p_1 \text{ on } \tilde{S}, \quad (2.32)$$

$$v_2 \cdot n = 0 \text{ on } \tilde{S}, \quad (2.33)$$

$$v_2 \cdot n = u_{c2} \cdot n - \xi_{c1} \cdot \nabla (v_1 \cdot n) \text{ on } \tilde{S}, \quad (2.34)$$

where $u_{c2}$ is the second order centroidal velocity for rigid objects without rotation, $\xi_1$ and $\xi_{c1}$ are first order boundary and centroidal displacements, respectively. The second order boundary condition for a general locally reacting boundary is derived in Appendix B, where it is shown that solution to the second order scattering problem for any locally reacting boundary can also be constructed from pressure release and rigid immovable scattering solutions.

As in the first order problem, the total second order wave field is given by Green theorem, as

$$p_2(r, t) = p_{i2}(r, t) + p_{s2}(r, t) + p_{\text{init}2}(r, t), \quad (2.35)$$

where

$$p_{i2}(r, t) = \int_{t_{\text{init}}}^{t+} dt_0 \iint dV_0 \left[ g_0(r, t | r_0, t_0) g_2(r_0, t_0) \right], \quad (2.36)$$

$$p_{s2}(r, t) = \int_{t_{\text{init}}}^{t+} dt_0 \iint dS_0 n(r_0) \cdot \left[ p_{s2}(r_0, t_0) \nabla g_0(r, t | r_0, t_0) - g(r, t | r_0, t_0) \nabla p_{s2}(r_0, t_0) \right], \quad (2.37)$$

$$p_{\text{init}2}(r, t) = -\frac{1}{c_0^2} \iint dV_0 \left[ \frac{\partial g(r, t | r_0, t_0)}{\partial t_0} p_2(r_0, t_0) - g(r, t | r_0, t_0) \frac{\partial p_2(r_0, t_0)}{\partial t_0} \right]_{t_0 = t_{\text{init}}} \quad (2.38)$$

Here $p_{i2}$ is the second order incident wave, which is due to wave-wave interactions of
the first order field, $p_{S2}$ is the second order scattered field determined by the second order boundary condition, and $p_{\text{init2}}$ accounts for the initial conditions.

The governing equation for the second order nonlinear wave field in the presence of low-impedance contrast inhomogeneities are also derived in Appendix A. Compared to equation (2.29) that applies to homogeneous medium, equation (A.14) in Appendix A has extra terms on the right-hand side due to the change of density and compressibility of the medium. The total second order field is then given as a single volume integral as equation (2.36), which includes both wave-wave interactions in space and second order scattering from volume inhomogeneities.

When the primary wave field consists of incident and scattered waves, we have

$$p_1 = p_{i1} + p_{S1},$$

$$v_1 = v_{i1} + v_{S1}.\tag{2.39}$$

Substituting $p_1$ and $v_1$ from (2.39) into equation (2.30) then into equation (2.36) leads to a decomposition of $p_{i2}$ as

$$p_{i2} = p_{i1} + p_{SS} + p_{IS} + p_{SI},\tag{2.40}$$
where

\[
p_{11}(r,t) = \int_{t_{\text{init}}}^{t^+} dt_0 \iint dV_0 g(r,t_0 | r_0, t_0) \left\{ \frac{1}{\rho_0 c_0^2} B \frac{\partial^2}{\partial t_0^2} (p_{11} p_{11}) + \rho_0 \nabla_0 \cdot [\nabla_0 \cdot (v_{11} v_{11})] \right\},
\]

\[
p_{SS}(r,t) = \int_{t_{\text{init}}}^{t^+} dt_0 \iint dV_0 g(r,t_0 | r_0, t_0) \left\{ \frac{1}{\rho_0 c_0^2} B \frac{\partial^2}{\partial t_0^2} (p_{S1} p_{S1}) + \rho_0 \nabla_0 \cdot [\nabla_0 \cdot (v_{S1} v_{S1})] \right\},
\]

\[
p_{IS}(r,t) = \int_{t_{\text{init}}}^{t^+} dt_0 \iint dV_0 g(r,t_0 | r_0, t_0) \left\{ \frac{1}{\rho_0 c_0^2} B \frac{\partial^2}{\partial t_0^2} (p_{11} p_{S1}) + \rho_0 \nabla_0 \cdot [\nabla_0 \cdot (v_{S1} v_{S1})] \right\},
\]

\[
p_{SI}(r,t) = \int_{t_{\text{init}}}^{t^+} dt_0 \iint dV_0 g(r,t_0 | r_0, t_0) \left\{ \frac{1}{\rho_0 c_0^2} B \frac{\partial^2}{\partial t_0^2} (p_{S1} p_{11}) + \rho_0 \nabla_0 \cdot [\nabla_0 \cdot (v_{S1} v_{11})] \right\},
\]

(2.41) (2.42) (2.43) (2.44)

in which \( p_{11}, p_{S1}, v_{11}, \) and \( v_{S1} \) are functions of \( r_0 \) and \( t_0 \), \( \nabla_0 \) is the gradient operator with respect to \( r_0 \), II represents contributions from incident-incident interactions, SS from scattered-scattered interactions, and IS and SI from incident-scattered interactions.

The procedure to obtain \( p_{S2} \) for pressure release or rigid immovable objects is similar to that for \( p_{S1} \) (§2.1), but the boundary conditions are

\[
\text{pressure release:} \quad p_{S2} = -p_{r2} - \xi_1 \cdot \nabla p_1 \quad \text{on } \mathcal{S}, \tag{2.45}
\]

\[
\text{rigid immovable:} \quad v_{S2} \cdot \mathbf{n} = -v_{r2} \cdot \mathbf{n} \quad \text{on } \mathcal{S}, \tag{2.46}
\]

where \( v_{r2} \) is the second order fluid velocity due to \( p_{r2} \), which is given by equation (2.31) with \( p_2 = p_{r2} \). The fluid velocity \( v_{S2} = -\rho_0^{-1} \int \nabla p_{S2} dt \). The pressure component \( \xi_1 \cdot \nabla p_1 \) describes contributions from wave-boundary interactions due to first order boundary motion.

Following the approach used in first order for rigid movable objects, \( p_{S2} \) is decom-
posed as

\[ p_{S2} = p_{D2} + p_{R2}, \]  
\[ (2.47) \]

where \( p_{D2} \) is the second order scattered wave from a rigid object whose centroid is fixed at second order, i.e. \( u_{c2} = 0 \), and \( p_{R2} \) is the second order radiated wave due to any second order centroidal motion \( u_{c2} \).

The procedure to obtain \( p_{D2} \) is similar to those for \( p_{S2} \) and \( p_{S1} \), but the boundary condition is

\[ \mathbf{v}_{D2} \cdot \mathbf{n} = -\mathbf{v}_{12} \cdot \mathbf{n} - \xi_{c1} \cdot \nabla (\mathbf{v}_1 \cdot \mathbf{n}) \quad \text{on } \mathcal{S}, \]
\[ (2.48) \]

where \( \mathbf{v}_{D2} = -\rho_0^{-1} \int \nabla p_{D2} dt \). The velocity component \( \xi_{c1} \cdot \nabla (\mathbf{v}_1 \cdot \mathbf{n}) \) also describes contributions from wave-boundary interactions due to first order boundary motion.

To obtain \( p_{R2} \), the procedure is again similar to that for \( p_{R1} \), but with boundary condition

\[ \mathbf{v}_{R2} \cdot \mathbf{n} = u_{c2} \cdot \mathbf{n} \quad \text{on } \mathcal{S}, \]
\[ (2.49) \]

where \( \mathbf{v}_{R2} = -\rho_0^{-1} \int \nabla p_{R2} dt \). The centroidal velocity \( u_{c2} \) is determined by the second order equation of motion

\[ M \frac{du_{c2}}{dt} = \int_S (p_{12} + p_{D2} + p_{R2} + \xi_{c1} \cdot \nabla p_1) ndS, \]
\[ (2.50) \]

where the second order pressure component \( \xi_{c1} \cdot \nabla p_1 \) arises from Taylor series expansion of the total pressure \( p \) at the instantaneous boundary \( S(t) \) with respect to \( \mathcal{S} \). By defining the second order wave-exciting force as

\[ \mathbf{f}_{2 \text{excit}} = \int_S (p_{12} + p_{D2} + \xi_{c1} \cdot \nabla p_1) ndS, \]
\[ (2.51) \]
\( u_{c2} \) becomes

\[
 u_{c2} = (z_m + z_r)^{-1} T^\text{excit.} \tag{2.52}
\]

When the primary incident field has two components \( p_{11}(r, t) = \Re \{\tilde{p}_{ta}(r, t) + \tilde{p}_{tb}(r, t)\} \), the primary scattered field is \( p_{s1}(r, t) = \Re \{\tilde{p}_{sa}(r, t) + \tilde{p}_{sb}(r, t)\} \), and the total first order field is \( p_1 = \Re \{\tilde{p}_{ta} + \tilde{p}_{tb} + \tilde{p}_{sa} + \tilde{p}_{sb}\} \). This leads to

\[
 p_1^2(r, t) = \Re \left\{ \frac{1}{2} \tilde{p}_{ta}\tilde{p}_{ta} + \frac{1}{2} \tilde{p}_{tb}\tilde{p}_{tb} + \frac{1}{2} \tilde{p}_{ta}\tilde{p}_{tb} + \frac{1}{2} \tilde{p}_{tb}\tilde{p}_{ta} \right\} 
+ \Re \left\{ \frac{1}{2} \tilde{p}_{sa}\tilde{p}_{sa} + \frac{1}{2} \tilde{p}_{sb}\tilde{p}_{sb} + \frac{1}{2} \tilde{p}_{sa}\tilde{p}_{sb} + \frac{1}{2} \tilde{p}_{sb}\tilde{p}_{sa} \right\} 
+ \Re \{\tilde{p}_{ta}\tilde{p}_{sa} + \tilde{p}_{ta}\tilde{p}_{sb} + \tilde{p}_{ta}\tilde{p}_{sb} + \tilde{p}_{sa}\tilde{p}_{tb} + \tilde{p}_{sb}\tilde{p}_{ta} + \tilde{p}_{sb}\tilde{p}_{tb} + \tilde{p}_{sa}\tilde{p}_{tb} + \tilde{p}_{sb}\tilde{p}_{ta}\} \tag{2.53}
\]

and \( v_1v_1 \) can be expressed in the same manner. Substituting \( p_1^2 \) and \( v_1v_1 \) into \( q_2 \) in equation (2.30) and then equation (2.36) leads to

\[
 p_1^2(r, t) = (p_1,aa + p_1,bb + p_1,aa^* + p_1,bb^* + p_1,ab + p_1,ab^*) 
+ (p_{ss,aa} + p_{ss,bb} + p_{ss,aa^*} + p_{ss,bb^*} + p_{ss,ab} + p_{ss,ab^*}) 
+ (p_{ss,aa} + p_{ss,bb} + p_{ss,aa^*} + p_{ss,bb^*} + p_{sI,ab} + p_{sI,ab^*} + p_{sI,bb} + p_{sI,bb^*}). \tag{2.54}
\]

The first parenthetical group of terms in equation (2.54) is due to II interaction, the second to SS interaction, and the last to IS and SI interactions. The II components \( p_{11,aa}, p_{11,aa^*}, p_{11,bb}, \) and \( p_{11,bb^*} \), SS components \( p_{ss,aa}, p_{ss,aa^*}, p_{ss,bb}, \) and \( p_{ss,bb^*} \), and IS and SI components \( p_{ss,aa}, p_{ss,aa^*}, p_{sI,bb}, \) and \( p_{sI,bb^*} \) correspond to self-interactions of each incident wave. The II components \( p_{11,ab} \) and \( p_{11,ab^*} \), SS components \( p_{ss,ab} \) and \( p_{ss,ab^*} \), and IS and SI components \( p_{sI,ab}, p_{sI,ab^*}, p_{sI,ab} \) and \( p_{sI,ab^*} \) are due to cross-interactions between the \( p_{ta} \) and \( p_{tb} \) incident fields.

When \( p_{11} \) has two harmonic components at angular frequencies \( \omega_a \) and \( \omega_b, \omega_a > \omega_b \), the variables \( p_{11}, p_{s1}, v_{11}, v_{s1}, \xi_1 \) and \( \xi_{c1} \) also contain these harmonic components.
i.e.

\[ p_{11} = \Re \{ P_{1a} e^{-i\omega_a t} + P_{1b} e^{-i\omega_b t} \}, \]  
(2.55a)

\[ p_{21} = \Re \{ P_{2a} e^{-i\omega_a t} + P_{2b} e^{-i\omega_b t} \}, \]  
(2.55b)

\[ v_{11} = \Re \{ V_{1a} e^{-i\omega_a t} + V_{1b} e^{-i\omega_b t} \}, \]  
(2.55c)

\[ v_{21} = \Re \{ V_{2a} e^{-i\omega_a t} + V_{2b} e^{-i\omega_b t} \}, \]  
(2.55d)

\[ \xi_1 = \Re \{ \Xi_a e^{-i\omega_a t} + \Xi_b e^{-i\omega_b t} \}, \]  
(2.55e)

\[ \xi_2 = \Re \{ \Xi_{c1} e^{-i\omega_a t} + \Xi_{c2} e^{-i\omega_b t} \}. \]  
(2.55f)

Then \( p_{12}, p_{II}, p_{SS}, p_{IS} \) and \( p_{SI} \) in equation (2.40) contain static and double frequency \((2\omega_a \text{ and } 2\omega_b)\) components due to self-interactions, and sum \((\omega_+ = \omega_a + \omega_b)\) and difference \((\omega_- = \omega_a - \omega_b)\) frequency components due to cross-interactions. Specifically, the \(\omega_\pm\) components for \( p_{12}, p_{II}, p_{SS}, p_{IS} \) and \( p_{SI} \) are

\[ p_{12\pm} = \Re \{ (P_{11\pm} + P_{SS\pm} + P_{IS\pm} + P_{SI\pm}) e^{-i\omega_\pm t} \}, \]  
(2.56)

\[ p_{II\pm} = \Re \{ P_{II\pm} e^{-i\omega_\pm t} \}, \]  
(2.57)

\[ p_{SS\pm} = \Re \{ P_{SS\pm} e^{-i\omega_\pm t} \}, \]  
(2.58)

\[ p_{IS\pm} = \Re \{ P_{IS\pm} e^{-i\omega_\pm t} \}, \]  
(2.59)

\[ p_{SI\pm} = \Re \{ P_{SI\pm} e^{-i\omega_\pm t} \}, \]  
(2.60)
where

\[ P_{11}(r) = \int \int \int G_{\pm}(r|r_0) \left\{ -\frac{\omega_{\pm}^2}{\rho_0 c_0^4} \frac{B}{2A} P_{1a} P_{1b}^{(s)} + \frac{\rho_0}{2} \nabla_0 \cdot \left[ \nabla_0 \cdot (V_{1a} V_{1b}^{(s)} + V_{1b}^{(s)} V_{1a}) \right] \right\} dV_0, \]  
\[(2.61)\]

\[ P_{SS}(r) = \int \int \int G_{\pm}(r|r_0) \left\{ -\frac{\omega_{\pm}^2}{\rho_0 c_0^4} \frac{B}{2A} P_{Sa} P_{sb}^{(s)} + \frac{\rho_0}{2} \nabla_0 \cdot \left[ \nabla_0 \cdot (V_{Sa} V_{sb}^{(s)} + V_{sb}^{(s)} V_{Sa}) \right] \right\} dV_0, \]  
\[(2.62)\]

\[ P_{SS}(r) = \int \int \int G_{\pm}(r|r_0) \left\{ -\frac{\omega_{\pm}^2}{\rho_0 c_0^4} \frac{B}{2A} P_{Sa} P_{sb}^{(s)} + \frac{\rho_0}{2} \nabla_0 \cdot \left[ \nabla_0 \cdot (V_{Sa} V_{sb}^{(s)} + V_{sb}^{(s)} V_{Sa}) \right] \right\} dV_0, \]  
\[(2.63)\]

\[ P_{SL}(r) = \int \int \int G_{\pm}(r|r_0) \left\{ -\frac{\omega_{\pm}^2}{\rho_0 c_0^4} \frac{B}{2A} P_{Sa} P_{sb}^{(s)} + \frac{\rho_0}{2} \nabla_0 \cdot \left[ \nabla_0 \cdot (V_{Sa} V_{sb}^{(s)} + V_{sb}^{(s)} V_{Sa}) \right] \right\} dV_0, \]  
\[(2.64)\]

in which the asterisk denotes complex conjugation that applies to the difference frequency case only. Equations (2.61) - (2.64) are obtained by substituting \( p_{11}, p_{S1}, v_{11} \) and \( v_{S1} \) from (2.55) into equations (2.41-2.44) and integrating over \( t_0 \).

The \( \omega_{\pm} \) components of \( p_{S2}, p_{D2}, p_{R2}, u_{c2} \) and \( f_{2}^{\text{excit}} \) can be written as

\[ p_{S2\pm} = \Re \{ P_{S2\pm} e^{-i\omega_{\pm} t} \}, \]  
\[(2.65a)\]

\[ p_{D2\pm} = \Re \{ P_{D2\pm} e^{-i\omega_{\pm} t} \}, \]  
\[(2.65b)\]

\[ p_{R2\pm} = \Re \{ P_{R2\pm} e^{-i\omega_{\pm} t} \}, \]  
\[(2.65c)\]

\[ u_{c2\pm} = \Re \{ U_{c2\pm} e^{-i\omega_{\pm} t} \}, \]  
\[(2.65d)\]

\[ f_{2\pm}^{\text{excit}} = \Re \{ F_{2\pm}^{\text{excit}} e^{-i\omega_{\pm} t} \}, \]  
\[(2.65e)\]

where the complex amplitudes \( P_{S2\pm}, P_{D2\pm}, P_{R2\pm}, U_{c2\pm} \) and \( F_{2\pm}^{\text{excit}} \) can then be determined by further analysis. Specifically, substituting \( p_{S2\pm} \) into equation (2.37) and integrating over \( t_0 \) yields

\[ P_{S2\pm}(r, t) = \int \int dS_0 n(r_0) \cdot [P_{S2\pm}(r_0) \nabla_0 G_{\pm}(r|r_0) - G_{\pm}(r|r_0) \nabla_0 P_{S2\pm}(r_0)], \]  
\[(2.66)\]

where \( G_{\pm}(r|r_0) = \frac{e^{ik_{\pm} R}}{(4\pi R)} \) and \( k_{\pm} = \omega_{\pm}/c_0 \). The scattered wave \( P_{S2\pm} \) for pres-
sure release or rigid immovable objects can be obtained from equation (2.66) with boundary conditions

\[ P_{S2\pm} = -P_{12\pm} - \frac{1}{2}(\Xi_a \cdot \nabla P_b^{(s)} + \Xi_b^{(s)} \cdot \nabla P_a) \quad \text{on } \hat{S}, \quad (2.67) \]

rigid immovable: \[ V_{S2\pm} \cdot n = -(V_{1\pm} + V_{SS\pm} + V_{IS\pm} + V_{SI\pm}) \cdot n \quad \text{on } \hat{S}, \quad (2.68) \]

where \( V_{1\pm}, V_{SS\pm}, V_{IS\pm} \) \( \text{and } V_{SI\pm} \) are the second order fluid velocities associated with \( P_{1\pm}, P_{SS\pm}, P_{IS\pm} \) \( \text{and } P_{SI\pm} \) respectively, \( P_a = P_{1a} + P_{Sa} \) and \( P_b = P_{1b} + P_{Sb} \) are first order pressure fields at frequencies \( \omega_a \) and \( \omega_b \) respectively, and \( V_{S2\pm} = (i\omega_{\pm}p_0)^{-1}\nabla P_{S2\pm}. \)

For rigid movable objects, \( P_{D2\pm} \) can be obtained from equation (2.66) with \( P_{S2\pm} = P_{D2\pm} \) and boundary condition

\[ V_{D2\pm} \cdot n = -(V_{1\pm} + V_{SS\pm} + V_{IS\pm} + V_{SI\pm}) \cdot n \\
- \frac{1}{2} \left[ \Xi_a \cdot \nabla (V_b^{(s)} \cdot n) + \Xi_b^{(s)} \cdot \nabla (V_a \cdot n) \right] \quad \text{on } \hat{S}, \quad (2.69) \]

where \( V_{D2\pm} = (i\omega_{\pm}p_0)^{-1}\nabla P_{D2\pm}, V_a = V_{1a} + V_{Sa} \) and \( V_b = V_{1b} + V_{Sb} \) are first order fluid velocities at frequencies \( \omega_a \) and \( \omega_b \) respectively.

The wave-exciting force is

\[ F_2^{\text{excit}} = \iint_S \left[ P_{12\pm} + P_{D2\pm} + \frac{1}{2}(\Xi_a \cdot \nabla P_b^{(s)} + \Xi_b^{(s)} \cdot \nabla P_a) \right] ndS. \quad (2.70) \]

The centroidal velocity for a rigid movable object attached to a spring and damper in the direction of motion is

\[ U_{C2\pm} = \left[ Z_m(\omega_{\pm}) + Z_r(\omega_{\pm}) + \frac{M\omega_n^2}{-i\omega_{\pm}} + 2\zeta M\omega_n \right]^{-1} F_2^{\text{excit}}, \quad (2.71) \]

where \( \omega_n \) is the undamped natural frequency, \( \zeta \) is the dimensionless damping ratio, \( Z_m(\omega_{\pm}) = -i\omega_{\pm}M \) is the Fourier transform of \( z_m \) and \( Z_r(\omega_{\pm}) \) is the Fourier transform of \( z_r \), which is sometimes referred to as the radiation impedance [29, 26, 24] and is
determined from its elements by

\[ \{Z_r(\omega)\}_{ij} = -\int_{S} n_i \dot{P}_{R,j}(\omega) dS, \quad \text{for } i,j = x,y,z, \quad (2.72) \]

where \( P_{R,j}(\omega) \), for \( j = x,y,z \) are the first order radiated fields from an object due to unit amplitude velocity oscillation at frequency \( \omega \) in \( x,y,z \) directions. In terms of \( \dot{P}_{R} = (\dot{P}_{R,x}, \dot{P}_{R,y}, \dot{P}_{R,z}) \), the second order radiated field \( P_{R2} \) is

\[ P_{R2\pm} = \dot{P}_{R}(\omega_{\pm}) \cdot \mathbf{U}_{c2\pm}. \quad (2.73) \]

### 2.3 Summary

In summary, the total second order field from any object insonified by acoustic waves is generated by wave-wave interactions, including II, SS, IS and SI, and by second order scattering \( S2 \), as

\[ p_2 = p_{II} + p_{SS} + p_{IS} + p_{SI} + p_{S2}, \quad (2.74) \]

which, for the sum and difference frequency amplitudes, leads to

\[ P_{2\pm} = P_{II\pm} + P_{SS\pm} + P_{IS\pm} + P_{SI\pm} + P_{S2\pm}. \quad (2.75) \]

The interaction components \( p_{II}, p_{SS}, p_{IS} \) and \( p_{SI} \) are given by equations (2.41-2.44). The \( S2 \) component \( p_{S2} \) is given by equation (2.37) with boundary conditions (2.45) and (2.46) for pressure release and rigid immovable objects, respectively. The general solution for objects with non-rigid locally reacting boundaries is given in Appendix B in terms of the pressure release and rigid immovable solutions of this section. For rigid movable objects, \( S2 \) is decomposed into \( D2 \) and \( R2 \). The \( D2 \) component \( p_{D2} \) is given by equation (2.37) with boundary condition (2.48). The \( R2 \) component \( p_{R2} \) is given by equation (2.37) with boundary condition (2.49), where the centroidal motion is determined from the wave-exciting force by equation (2.51).
From the sensing perspective, the II interaction only depends on the primary incident waves and it contains no information about the object. The SS, IS and SI interactions depend on the first order scattered waves $p_{Sa}$ and $p_{Sb}$, and they contain primary frequency scattering information of the object. S2 contains the object’s second order scattering information at the sum or difference frequency. For example for the rigid object, D2 depends on the geometry of the object and R2 depends on the second order motion of the object.

The present theory provides a complete and self-consistent treatment of second order interaction and scattering in the presence of an object, including the effect of second order centroidal motion. Previous theory based on the dynamic radiation force is shown to omit some of the second order effects and overcounted others. More details about the wave-exciting force formulation are provided in Appendix D.
Chapter 3

Solutions for wave-wave interactions

3.1 Direct integration

3.1.1 General approach

Let \( p_{12} \) of equation (2.35) and the associated second order velocity \( v_{12} \) due to wave-wave interactions be further decomposed as [12]

\[
p_{12}(r, t) = p'_2(r, t) + p''_2(r, t) \quad \text{and} \quad v_{12}(r, t) = \frac{1}{\rho_0} \int_{-\infty}^{t} \nabla p'_2(r, \tau) d\tau + v''_2(r, t), \quad (3.1)
\]

where \( p''_2 \) and \( v''_2 \) are nonzero only when the primary field exists and can be directly evaluated from first order pressure \( p_1 \) and velocity \( v_1 \), as

\[
p''_2(r, t) = -\frac{\rho_0 v''_1(r, t)}{2} - \frac{p''_1(r, t)}{2} - \frac{1}{A} \frac{\partial p_1(r, t)}{\partial t} \int_{-\infty}^{t} p_1(r, \tau) d\tau, \quad (3.2)
\]

\[
v''_2(r, t) = \frac{1}{2\rho_0 c_0^2} \frac{\partial}{\partial t} \left[ \nabla \left( \int_{-\infty}^{t} p_1(r, \tau) d\tau \right) \right], \quad (3.3)
\]

and \( p'_2 \) satisfies

\[
\left( \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p'_2 = \frac{\beta}{A} \frac{\partial^2 p_1^2}{\partial t^2}, \quad (3.4)
\]
where \( \beta = 1 + B/(2A) \). As in equation (2.36), solution for \( p'_2 \) is

\[
p'_2(r, t) = \frac{\beta}{Ac_0^2} \int_{t_{\text{init}}}^{t^+} dt_0 \int \int \frac{\partial^2 p^2_2(r_0, t_0)}{\partial t_0^2} g(r, t | r_0, t_0) dV_0.
\]

(3.5)

Finite-duration incident plane waves propagating at directions \( \hat{i}_a \) and \( \hat{i}_b \) with center frequencies \( \omega_a \) and \( \omega_b \) respectively can be written as

\[
p_{ia}(r, t) = \Re \{ P_{ia}(r)e^{-i\omega_a t} w_1(t - \hat{i}_a \cdot r/c_0) \},
\]

(3.6a)

\[
p_{ib}(r, t) = \Re \{ P_{ib}(r)e^{-i\omega_b t} w_1(t - \hat{i}_b \cdot r/c_0) \},
\]

(3.6b)

where

\[
P_{ia} = P_{a0} e^{i\omega_a \hat{i}_a \cdot r/c_0},
\]

(3.7a)

\[
P_{ib} = P_{b0} e^{i\omega_b \hat{i}_b \cdot r/c_0},
\]

(3.7b)

are harmonic plane waves with amplitudes \( P_{a0} \) and \( P_{b0} \), respectively, and \( w_1(t) \) is a window function with compact support of duration \( T \), which is here assumed to have unit height within the window except at each end where smooth transitions to zero occur over time periods much less than \( T \).

When the object is small compared to the spacial extent of the incident plane wave window, the dominant scattering components are within the narrow band of the incident waves, and the window functions are sufficiently smooth, the primary scattered waves can be approximated as

\[
p_{sa}(r, t) \approx \Re \{ P_{sa}(r)e^{-i\omega_a t} w_1(t - r/c_0) \},
\]

(3.8a)

\[
p_{sb}(r, t) \approx \Re \{ P_{sb}(r)e^{-i\omega_b t} w_1(t - r/c_0) \},
\]

(3.8b)

which correspond to harmonic scattered waves \( P_{sa} \) and \( P_{sb} \) at frequencies \( \omega_a \) and \( \omega_b \), where compact support is preserved by moving windows.

Like \( p_{12} \) in equation (2.54), \( p'_2 \) and \( p''_2 \) have self-interaction and cross-interaction components due to II, SS, IS and SI mechanisms. By substituting equations (3.6) and
(3.8) into equation (3.5) and integrating over \( t_0 \), the cross-interaction components for \( p'_2 \) are

\[
p'_{1,ab}(r, t) = -\frac{\beta \omega^2}{A c_0} \mathbb{R} \left\{ e^{-i \omega t} \int \int \int w_{1,ab}(r_0) P_{1a}(r_0) P_{1b}^{(*)}(r_0) G_{\pm}(r|r_0) dV_0 \right\}, \tag{3.9}
\]

\[
p'_{SS,ab}(r, t) = -\frac{\beta \omega^2}{A c_0} \mathbb{R} \left\{ e^{-i \omega t} \int \int \int w_{SS,ab}(r_0) P_{Sa}(r_0) P_{Sb}^{(*)}(r_0) G_{\pm}(r|r_0) dV_0 \right\}, \tag{3.10}
\]

\[
p'_{IS,ab}(r, t) = -\frac{\beta \omega^2}{A c_0} \mathbb{R} \left\{ e^{-i \omega t} \int \int \int w_{IS,ab}(r_0) P_{la}(r_0) P_{Sb}^{(*)}(r_0) G_{\pm}(r|r_0) dV_0 \right\}, \tag{3.11}
\]

\[
p'_{SI,ab}(r, t) = -\frac{\beta \omega^2}{A c_0} \mathbb{R} \left\{ e^{-i \omega t} \int \int \int w_{SI,ab}(r_0) P_{Sa}(r_0) P_{lb}^{(*)}(r_0) G_{\pm}(r|r_0) dV_0 \right\}, \tag{3.12}
\]

where

\[
w_{1,ab}(r, r_0, t) = \left(1 - \frac{1}{\omega^2} \frac{\partial^2}{\partial t^2}\right) \left[ w_1 \left( t - \frac{R}{c_0} - \frac{i_a \cdot r_0}{c_0} \right) w_1 \left( t - \frac{R}{c_0} - \frac{i_b \cdot r_0}{c_0} \right) \right], \tag{3.13}
\]

\[
w_{SS,ab}(r, r_0, t) = \left(1 - \frac{1}{\omega^2} \frac{\partial^2}{\partial t^2}\right) \left[ w_1 \left( t - \frac{R}{c_0} - \frac{r_0}{c_0} \right) w_1 \left( t - \frac{R}{c_0} - \frac{r_0}{c_0} \right) \right], \tag{3.14}
\]

\[
w_{IS,ab}(r, r_0, t) = \left(1 - \frac{1}{\omega^2} \frac{\partial^2}{\partial t^2}\right) \left[ w_1 \left( t - \frac{R}{c_0} - \frac{i_a \cdot r_0}{c_0} \right) w_1 \left( t - \frac{R}{c_0} - \frac{r_0}{c_0} \right) \right], \tag{3.15}
\]

\[
w_{II,ab}(r, r_0, t) = \left(1 - \frac{1}{\omega^2} \frac{\partial^2}{\partial t^2}\right) \left[ w_1 \left( t - \frac{R}{c_0} - \frac{r_0}{c_0} \right) w_1 \left( t - \frac{R}{c_0} - \frac{i_b \cdot r_0}{c_0} \right) \right]. \tag{3.16}
\]

When the primary incident and scattered waves are purely time harmonic, \( w_{1,ab} = w_{SS,ab} = w_{IS,ab} = w_{SI,ab} = 1 \). Equations (3.9)-(3.12) become

\[
p'_{1,ab}(r, t) = \mathbb{R}\{P'_{1\pm}(r)e^{-i\omega t}\}, \tag{3.17}
\]

\[
p'_{SS,ab}(r, t) = \mathbb{R}\{P'_{SS}(r)e^{-i\omega t}\}, \tag{3.18}
\]

\[
p'_{IS,ab}(r, t) = \mathbb{R}\{P'_{IS}(r)e^{-i\omega t}\}, \tag{3.19}
\]

\[
p'_{SI,ab}(r, t) = \mathbb{R}\{P'_{SI}(r)e^{-i\omega t}\}. \tag{3.20}
\]

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where

\[ P'_{\parallel \pm}(r) = -\frac{\beta \omega^2}{Ac_0^2} \iiint P_{la}(r_0)P_{lb}^{(s)}(r_0)G_{\pm}(r|r_0) dV_0, \quad (3.21) \]

\[ P''_{\parallel \pm}(r) = -\frac{\beta \omega^2}{Ac_0^2} \iiint P_{Sa}(r_0)P_{Sb}^{(s)}(r_0)G_{\pm}(r|r_0) dV_0, \quad (3.22) \]

\[ P'_{\perp \pm}(r) = -\frac{\beta \omega^2}{Ac_0^2} \iiint P_{la}(r_0)P_{Sb}^{(s)}(r_0)G_{\pm}(r|r_0) dV_0, \quad (3.23) \]

\[ P''_{\perp \pm}(r) = -\frac{\beta \omega^2}{Ac_0^2} \iiint P_{Sa}(r_0)P_{lb}^{(s)}(r_0)G_{\pm}(r|r_0) dV_0. \quad (3.24) \]

The cross-interaction components of \( p'_2 \) are denoted by \( p''_{\parallel \pm,ab}^{(s)}, p''_{\perp,ab}^{(s)}, p''_{\perp \pm,ab}^{(s)} \) and \( p''_{\parallel \pm,ab}^{(s)} \). They can be determined by substituting appropriate first-order field products into equation (3.2). When the primary fields are time harmonic,

\[ p''_{\parallel \pm,ab}^{(s)}(r,t) = \Re\{ P''_{\parallel \pm,ab}(r)e^{-i\omega_{\pm}t} \}, \quad (3.25) \]

\[ p''_{\perp,ab}^{(s)}(r,t) = \Re\{ P''_{\perp,ab}(r)e^{-i\omega_{\pm}t} \}, \quad (3.26) \]

\[ p''_{\perp \pm,ab}^{(s)}(r,t) = \Re\{ P''_{\perp \pm,ab}(r)e^{-i\omega_{\pm}t} \}, \quad (3.27) \]

\[ p''_{\parallel \pm,ab}^{(s)}(r,t) = \Re\{ P''_{\parallel \pm,ab}(r)e^{-i\omega_{\pm}t} \}, \quad (3.28) \]

where \( P''_{\parallel \pm}, P''_{\perp \pm}, P''_{\perp,ab} \) and \( P''_{\parallel \pm,ab}^{(s)} \) are the complex amplitudes at the sum and difference frequency for the cross-interactions, which again can be directly obtained from equation (3.2).
3.1.2 Confirmation of theory with measurements

The theory presented here (equation (2.75)) is computationally confirmed by comparison with experimental data from Jones and Beyer [1, 2] for second-order sum-frequency pressure at range $r_R = 481.8$ mm due to a large rigid sphere of radius $a = 3.18$ mm insonified by two perpendicular incident beams at frequencies $\omega_a/2\pi = 7$ MHz and $\omega_b/2\pi = 5$ MHz. Excellent quantitative agreement is found between theory and measurements with 0.98 correlation across scattered angle and overall mean square difference of less than 0.3 dB (figure 3-1). Details for the quantitative comparison is in Appendix E.

![Graph](image)

Figure 3-1: Comparison between Jones and Beyer's experiment [1, 2] and theory for the sum frequency second order pressure. Excellent quantitative agreement is found between measurement and theory with 0.98 correlation across scattered angle and overall mean square difference of less than 0.3 dB.
3.2 Analytic solutions for II interaction of plane waves of general time dependence

3.2.1 Existing time-harmonic solutions

For two collinear time harmonic plane waves given by equation (3.7) with \( \hat{\mathbf{i}}_a = \hat{\mathbf{i}}_b = \hat{\mathbf{i}}_z \), \( P_{\pm} \) was derived by Lamb [9], who showed that it grows linearly with range. For zero second order normal velocity at \( z_s \), \( P_{\pm} \) is

\[
P_{\pm}(z) = \frac{\beta P_{a0}P_{b0}^{(\ast)}}{2A} [1 - ik_\pm(z - z_s)] e^{ik_\pm z}.
\] (3.29)

Here the complex conjugate and the "-" sign in "\( \pm \)" apply to the difference frequency case only. For two non-collinear time harmonic plane waves given by equation (3.7) with \( \hat{\mathbf{i}}_a = \hat{\mathbf{i}}_x \) and \( \hat{\mathbf{i}}_b = \hat{\mathbf{i}}_{z'} \), Westervelt [8] found that

\[
P_{\pm}(x, z') = \frac{P_{a0}P_{b0}^{(\ast)}}{2A} \left[ \pm \left( \frac{\beta}{1 - \cos \theta} - 1 \right) \frac{\omega_x^2}{\omega_a \omega_b} + 1 - \cos \theta \right] e^{i(k_a x \pm k_b z')},
\] (3.30)

where \( z' = x \cos \theta + z \sin \theta \) and \( \theta \) is the angle between \( \hat{\mathbf{i}}_x \) and \( \hat{\mathbf{i}}_{z'} \).

3.2.2 Solution for collinear plane waves

These time-harmonic results are here generalized to plane waves with arbitrary time dependence to investigate the scattering of sound by sound and also properly model typical experimental scenarios. For two collinear plane waves of arbitrary time dependence \( p_{\pm} = \Re \{ p_a(t - z/c_0) \} \) and \( p_{\pm} = \Re \{ p_b(t - z/c_0) \} \), the second order fields
due to their cross-interaction are

\[
P_{11,ab(*)}(z,t) = \Re \left\{ \frac{\beta z}{2A_{c0}} \frac{\partial}{\partial t} \left[ \bar{\tilde{p}}_{1a}(t - z/c_0) \tilde{p}_{1b}^{(*)}(t - z/c_0) \right] \right\}
\]

\[
- \Re \left\{ \frac{1}{2A} \bar{\tilde{p}}_{1a}(t - z/c_0) \tilde{p}_{1b}^{(*)}(t - z/c_0) \right\}
\]

\[
- \Re \left\{ \frac{1}{2A} \left( \frac{\partial}{\partial t} \bar{\tilde{p}}_{1a}(t - z/c_0) \right) \int_{-\infty}^{t-z/c_0} \tilde{p}_{1b}^{(*)}(\tau) d\tau \right\}
\]

\[
- \Re \left\{ \frac{1}{2A} \left( \frac{\partial}{\partial t} \bar{\tilde{p}}_{1b}(t - z/c_0) \right) \int_{-\infty}^{t-z/c_0} \tilde{p}_{1a}(\tau) d\tau \right\},
\]

(3.31)

where the complex conjugates only apply to \(p_{11,ab*}\). For narrow-band waves with compact support given by equation (3.6) with \(i_a = i_b = i_z\), and a sufficiently smooth temporal window \(w_1\), equation (3.31) becomes

\[
p_{11,ab(*)}(z,t) \approx \Re \left\{ -\frac{P_{a0} P_{b0}^{(*)}}{2A} \left[ i\beta k_{\pm} z \pm \frac{\omega_z^2}{\omega_{a0} \omega_{b0}} \right] e^{i(k_{\pm} z - \omega_{\pm} t)} \right\} w_1^2(t - z/c_0),
\]

(3.32)

where the \(i\beta k_{\pm} z\) term in the square brackets corresponds to \(p_{11,ab(*)}'\) and the other term in the square brackets corresponds to \(p_{11,ab(*)}''\). It can be seen from equation (3.31) and (3.32) that sum and difference frequency components of \(p_{11,ab(*)}\) only exist within the region of compact support intersection. For narrow-band primary waves with sufficiently smooth windows, the second order field in the intersection corresponds to Lamb's solution with boundary condition \(p_{11,ab(*)} = p_{11,ab(*)}''\) at \(z = z_{s} = 0\).

Derivation for analytic solutions (3.31) and (3.32) is provided in Appendix F.1.

### 3.2.3 Solution for non-collinear plane waves

For two non-collinear intersecting plane waves of arbitrary time dependence \(p_{1a} = \Re \{\bar{\tilde{p}}_{1a}(t - x/c_0)\}\) and \(p_{1b} = \Re \{\bar{\tilde{p}}_{1b}(t - z'/c_0)\}\), the second order fields due to their
cross-interaction are found to be

\[
p_{I I, a b(\ast)}(x, z', t) = \Re \left\{ \frac{1}{2A} \left[ \frac{\beta}{1 - \cos \theta} - 1 \right] \left[ \frac{\partial}{\partial t} \tilde{p}_{I a}^{(\ast)}(t - z'/c_0) \right] \left[ \int_{-\infty}^{t - z'/c_0} \tilde{p}_{I a}(\tau) d\tau \right] \right\}
\]

\[
+ \Re \left\{ \frac{1}{2A} \left[ \frac{\beta}{1 - \cos \theta} - 1 \right] \left[ \frac{\partial}{\partial t} \tilde{p}_{I a}(t - x/c_0) \right] \left[ \int_{-\infty}^{t - x'/c_0} \tilde{p}_{I b}^{(\ast)}(\tau) d\tau \right] \right\}
\]

\[
+ \Re \left\{ \frac{1}{2A} \left[ \frac{2\beta}{1 - \cos \theta} - (1 + \cos \theta) \right] \tilde{p}_{I a}(t - x/c_0) \tilde{p}_{I b}^{(\ast)}(t - z'/c_0) \right\},
\]

which is the full solution including both \( p'_{I I, a b(\ast)} \) and \( p''_{I I, a b(\ast)} \) components. Sum and difference frequency components of \( p_{I I, a b(\ast)} \) only exist within the region of compact support intersection, so that no scattering of sound by sound at sum or difference frequencies is found outside the region of compact support intersection to second order. Primary frequency components in \( p_{I I, a b(\ast)} \) exist within the region of compact support union through which compact support intersection occurred if the primary waves had non-zero zero-frequency components. There is no scattering of sound by sound at any frequency outside the region of compact support union through which compact support intersection occurred. The different regions are illustrated in figure 3-2.

For narrow-band plane waves with compact support given by equation (3.6) with \( \hat{i}_a = \hat{i}_x \) and \( \hat{i}_b = \hat{i}_{x'} \), where window \( w_1 \) is sufficiently long and smooth, equation (3.33) becomes

\[
p_{I I, a b(\ast)}(x, z', t) \approx \Re \left\{ \frac{P_{a 0} P_{b 0}^{(\ast)}}{2A} \left[ \pm \left( \frac{\beta}{1 - \cos \theta} - 1 \right) \frac{\omega_+^2}{\omega_a \omega_b} + 1 - \cos \theta \right] e^{i(k_x x + k_{x'} x' - \omega_+ t)} \right\}
\]

\[
\times \frac{1}{w_1(t - x/c_0)} w_1(t - z'/c_0),
\]

which shows that a time-harmonic approximation involving Westervelt's time harmonic solution (equation (3.30)) can be made in the region of compact support intersection for narrow-band plane waves with sufficiently long and smooth windows.

Detailed derivation for solutions (3.33) and (3.34) is provided in Appendix F.2.
Figure 3-2: Intersecting non-collinear plane waves of compact support duration $T$. The $p_{la}(t - x/c_0)$ wave (blue) is propagating to the right and the $p_{lb}(t - z'/c_0)$ wave (red) is propagating upward. The sum and difference frequency components of $p_{la,ab}$ and $p_{la,ab^*}$ are only nonzero in the region of compact support intersection (purple). There are no conditions under which there is second order sound outside the union of the primary windows. The second order field $p_{ll,ab}$ or $p_{ll,ab^*}$ will have primary frequency components where the primary field exists and the intersecting plane wave has passed through (hatched) if the intersecting field has a non-zero zero-frequency spectral component. For narrow-band plane waves with sufficiently long and smooth windows, a harmonic approximation can be made for the sum or difference frequency field in the region of compact support intersection.
3.2.4 Collinearity and dispersion relation

The fact that second order fields due to interaction plane waves are localized can be understood from the dispersion relation. For two plane waves with vector wave number \( \mathbf{k}_a \) and \( \mathbf{k}_b \) at angular frequencies \( \omega_a \) and \( \omega_b \) propagating in a medium with sound speed \( c_0 \), they satisfy the dispersion relations \( \omega_a = |\mathbf{k}_a|c_0 \) and \( \omega_b = |\mathbf{k}_b|c_0 \) respectively. The sum \( (\omega_+ = \omega_a + \omega_b) \) and difference \( (\omega_- = |\omega_a - \omega_b|) \) frequency second order fields due to their cross-interaction have vector wavenumbers \( \mathbf{k}_{\pm} = \mathbf{k}_a \pm \mathbf{k}_b \). These second order fields propagate when the dispersion relations are satisfied, i.e. \( \omega_\pm = |\mathbf{k}_{\pm}|c_0 \), which only occurs when the two plane waves are collinear, as illustrated in figure 3-3.

**Figure 3-3:** Sum and difference frequency second order fields due to the cross-interaction of two plane waves propagate only when the two plane waves are collinear, such that the dispersion relation \( \omega_\pm = |\mathbf{k}_{\pm}|c_0 \) can be satisfied. Here \( \mathbf{k}_{\pm} = \mathbf{k}_a \pm \mathbf{k}_b \), \( \omega_\pm = \omega_a \pm \omega_b = (|\mathbf{k}_a| \pm |\mathbf{k}_b|)c_0 \).

For collinear plane waves, the second order field is propagating together with the primary plane waves, as if it is localized in the compact support intersection of the primary waves. For non-collinear plane waves, the second order field (due to cross-
interaction) is truly non-propagating and localized in the compact support intersection of the primary waves. Any sum or difference frequency second order field measured experimentally outside the compact support intersection in the non-collinear case is due to the inevitable collinearity of the two primary waves in reality, i.e. they are not perfect plane waves.
3.3 Solution for time-harmonic SS interaction

Let \( r_a \) and \( r_b \) be the far field ranges for the primary scattered fields \( P_{sa} \) and \( P_{sb} \) respectively, such that beyond \( r_a \) and \( r_b \), far field approximations [30] apply

\[
P_{sa}(r) = \frac{P_{a0} S_a(\hat{i}_r)}{r} \quad \text{and} \quad P_{sb}(r) = \frac{P_{b0} S_b(\hat{i}_r)}{r}
\]

where \( r_{a,b} = l^2/\lambda_{a,b}, k_a r_a, k_b r_b \gg 1 \), \( l \) is the length scale of the object, \( \lambda_{a,b} \) are the wavelengths of the incident fields, \( P_{a0} \) and \( P_{b0} \) are the amplitudes of the incident fields, \( S_a \) and \( S_b \) are the far field scatter functions, and \( \hat{i}_r = r/r \).

The integral representation of \( P'_{SS\pm} \), equation (3.22), can then be decomposed into

\[
P'_{SS\pm} = P'^{(1)}_{SS\pm} + P'^{(2)}_{SS\pm}
\]

where \( P'^{(1)}_{SS\pm} \) is due to the SS interaction within \( r_{ref\pm} \) and \( P'^{(2)}_{SS\pm} \) is due to SS interaction beyond \( r_{ref\pm} \), and \( r_{ref\pm} \) satisfies \( k_{\pm} r_{ref\pm} \gg 1 \) and \( r_{ref\pm} \geq r_a, r_b \). The \( P'^{(2)}_{SS\pm} \) term for \( r > r_{ref\pm} \) can then be analytically approximated via a stationary phase or spherical wave expansion approach (Appendix G), yielding

\[
P_{SS\pm}(r) = P_{SS\pm}^{(2)}(r) = \frac{\omega_{\pm}^2 \beta P_{a0} P_{b0}^{(s)}(r)}{2 i A c^2 k_a k_b} \frac{e^{ik_{\pm} r}}{k_{\pm} r} \left[ -E_i(-2ik_{\pm} r_{ref\pm}) S_a(-\hat{i}_r) S_b^{(s)}(-\hat{i}_r) 
\right.

\left. + \log \left( \frac{r}{r_{ref\pm}} \right) S_a(\hat{i}_r) S_b^{(s)}(\hat{i}_r) + e^{-2ik_{\pm} r} E_i(-2ik_{\pm} r) S_a(\hat{i}_r) S_b^{(s)}(\hat{i}_r) \right],
\]

where \( E_1 \) is the exponential integral [31, 11]. For \( k_{\pm} r \to \infty \), since the logarithmic function dominates the exponential integrals in equation (3.37), \( P'^{(2)}_{SS\pm} \) falls off by \( \log(r)/r \). The component \( P'^{(1)}_{SS\pm} \) falls off by \( r^{-1} \) and \( P'^{(2)}_{SS\pm} \) by \( r^{-2} \). The total second order field due to the SS interaction can be approximated by

\[
P_{SS\pm}(r) \approx P'_{SS\pm}(r) \approx P_{SS\pm}^{(2)}(r) \approx \frac{\omega_{\pm}^2 \beta P_{a0} P_{b0}^{(s)}(r)}{2 i A c^2 k_a k_b} \frac{e^{ik_{\pm} r}}{k_{\pm} r} \log \left( \frac{r}{r_{ref\pm}} \right) S_a(\hat{i}_r) S_b^{(s)}(\hat{i}_r),
\]

which is consistent with a general diverging waveform suggested by Westervelt and Radue [10] but without explicit derivation or interpretation.
When the two primary fields radiating from a sphere with radius $a$ are spherically symmetric, $S_a = S_b = \text{constant}$, $r_{ref} = a$ and $P_{SS}^{(1)} = 0$. Equation (3.37) reduces exactly to Baxter’s solution [11]. Compared to Baxter’s solution, it is found that Dean’s earlier solution [12] has an extra term proportional to the spherical Bessel function $j_0(k_{\pm}r)$ so violates the Sommerfeld radiation condition (Append G.3). A further approximation made by Dean for the far field $k_{\pm}r \gg 1$ and a small object $k_{\pm}a \ll 1$, however, satisfies the Sommerfeld radiation condition because it contains only the dominant $e^{ik_{\pm}r} \log(k_{\pm}r)/r$ term. Jones and Beyer [1, 2] developed a heuristic formula based on the radial dependence of Dean’s omnidirectional solution with far field and small object approximations, and applied it to model the sum frequency interaction of two scattered fields from a large sphere ($k_a a = 160$), where they found good agreement in angular dependence with data by introducing an arbitrary scaling factor.

Figure 3-4 compares Baxter’s solution and Dean’s solution at the sum- and difference-frequency. It can be seen that for the SPL at the difference-frequency, Baxter’s solution forms a smooth decaying curve while Dean’s solution has local maxima and minima. The range dependence in Dean’s solution can be explained by the fact that it violates the Sommerfeld radiation condition, i.e. wave coming from infinity and wave propagating out interfere and form a standing wave pattern. At sum frequency, Dean solution appears as a smooth decaying curve. This is due to the fact that it enters its asymptotic region from very close range, given the physical parameters in this example.

To our best knowledge, there is no derivation available in literature for Dean solution. In Appendix G.3, we present a derivation based on the variation of parameters method [32].
Figure 3-4: Sound pressure level of the second order sound wave at the sum- and difference-frequency due to the interaction of two spherically symmetric outward propagating waves. The radius of the object is $a = 1$ mm, frequencies of the primary waves are $f_a = 2$ MHz and $f_b = 1.99$ MHz such that the sum- and difference-frequency are $f_+ = 3.99$ MHz and $f_- = 10$ kHz respectively. The sound pressure level (SPL) is calculated from the boundary of the object up to $r = 1$ m. If the sound speed is assumed to be $c_0 = 1500$ m/s, then the dimensionless object sizes are $k+a \approx 17$ and $k-a \approx 0.04$, and the dimensionless receiver ranges are up to $k+r = 16713$ and $k-r \approx 42$. 
3.4 Solutions for time-harmonic IS and SI interactions

For incident plane waves $P_{\text{la}}$ and $P_{\text{lb}}$ given by equation (3.7) and scattered waves $P_{\text{sa}}$ and $P_{\text{sb}}$ given by equation (3.35), the second order fields $P_{\text{Is}}$ and $P_{\text{Si}}$ in the forward directions are analytically derived in Appendix H.1 from equations (3.23) and (3.24). At large range, $P'_{\text{is}}$ and $P'_{\text{si}}$ have constant magnitude in range while $P''_{\text{is}}$ and $P''_{\text{si}}$ fall off by $r^{-1}$. The total second order fields due to the IS and SI interactions are then

\[
P_{\text{Is}+}(r\hat{\mathbf{i}}_a) \approx P'_{\text{Is}+}(r\hat{\mathbf{i}}_a) \approx -\frac{\omega^2 \beta P_{00}^*}{A \omega^2} \frac{e^{ikr}}{2k_0k_b} \left[ i \log \left( \frac{k_+}{k_b} \right) \right] S_b(\hat{\mathbf{i}}_a), \quad (3.39)
\]

\[
P_{\text{Is}+}(r\hat{\mathbf{i}}_a) \approx P'_{\text{Is}+}(r\hat{\mathbf{i}}_a) \approx -\frac{\omega^2 \beta P_{00}^* P_{00}^*}{A \omega^2} \frac{e^{ikr}}{2k_0k_b} \left[ i \log \left( \frac{k_-}{k_b} \right) + \pi \right] S_b(\hat{\mathbf{i}}_a), \quad (3.40)
\]

\[
P_{\text{Si}+}(r\hat{\mathbf{i}}_b) \approx P'_{\text{Si}+}(r\hat{\mathbf{i}}_b) \approx -\frac{\omega^2 \beta P_{00}^* P_{00}^*}{A \omega^2} \frac{e^{ikr}}{2k_0k_b} \left[ \pm i \log \left( \frac{k_\pm}{k_a} \right) \right] S_a(\hat{\mathbf{i}}_b). \quad (3.41)
\]

Unlike the interaction of collinear plane waves where growth is found along the propagation path, collinearity between planar and spherical wavefronts within an equivalent Fresnel area about the forward direction, together with spreading of the spherical wave, balances out second-order wave growth. For $P_{\text{Is}+}$, the $\pi$ term in equation (3.40) is due to the additional contribution from a stationary phase point at range $rk_b/k_a$ in the forward direction.

The backscatter directions $-\hat{\mathbf{i}}_a$ and $-\hat{\mathbf{i}}_b$ for the IS and SI interactions respectively lack collinearity and stationary phase contributions so that $P'_{\text{is}}$ and $P'_{\text{si}}$ fall off in range by $r^{-1}$ as do $P''_{\text{is}}$ and $P''_{\text{si}}$, and have much lower magnitude than $P'_{\text{is}}$ and $P'_{\text{si}}$ in the forward directions. Detailed derivations for IS and SI interactions are provided in Appendix H for both the field in the forward directions and backscatter directions.
Chapter 4

Sum and difference frequency sensing of an object

4.1 Space-time isolation of second order field components that carry information about an object

Here we consider sum and difference frequency sensing of an object by measurement of the second order nonlinear fields arising from SS, IS and S2 mechanisms. When the primary waves have compact support, we show here for \( p_{SS,ab} \) and \( p_{IS,ab} \) via equations (3.10) and (3.11) and for \( p_{11,ab} \) via equations (3.31) - (3.34) that the second order waves also have compact support (figure 4-1). Given this, we show that it is possible to isolate in time and space field components \( p_{SS} \), \( p_{IS} \) and \( p_{S2} \) that carry information about the object from \( p_{II} \) that does not contain such information by the appropriate choice of receiver location (figure 4-2).

The compact support of \( p_{11,ab} \) as analytically derived in equation (3.33) is confirmed by direct numerical integration of the full time domain Green theorem solution (\( p'_{11,ab} \) from (3.9) plus \( p''_{11,ab} \) from (3.2)) and shown in figure 4-1a. Compact support of \( p_{SS,ab} \) and \( p_{IS,ab} \) components is also shown by direct numerical integration of the respective time domain Green theorem solutions (\( p'_{SS,ab} \) from (3.10), \( p'_{IS,ab} \) from (3.11), and \( p''_{SS,ab} \) and \( p''_{IS,ab} \) from (3.2)) and shown in figure 4-1bcd. Within these regions of
compact support, the $p_{SS,ab^*}$ and $p_{IS,ab^*}$ can be well approximated by a number of time harmonic approximations including the asymptotic solutions (3.37), (3.40), (H.61), and direct numerical integration of the harmonic wave Green theorem solutions (3.22) and (3.23), plus the respective $p''_2$ components from (3.2). The backscattered IS (figure 4-1d) is much smaller than the forward scattered IS component (figure 4-1c) as expected from §3.4, and the $p''_2$ component is zero because the primary incident and scattered waves do not intersect at the receiver in the backscatter direction for this case (figure 4-2).

The II, SS, IS and SI overlap regions of compact support are shown in figure 4-2. If a receiver is placed in any location where the II overlap region (gray in figure 4-2) and the SS overlap region (blue in figure 4-2) do not intersect, as in the example shown in figure 4-2, it will measure $p_{SS} + p_{IS} + p_{SI} + p_{S2}$ between $t = r_R/c_0$ and $t = r_R/c_0 + T$ with no $p_{II}$ component, where $t = 0$ occurs when the fronts of the incident waves simultaneously arrive at the object center. The II overlap region entirely passes the receiver before the SS overlap region arrives for the examples shown in figure 4-2 if $r_R > c_0T/2$ for the collinear case, and if $r_R > c_0T\sqrt{2}/(\sqrt{2} + 1)$ for the perpendicular case. If a receiver is placed in the forward direction in the collinear case, $p_{II}$ cannot be separated from the other components regardless of the duration of the incident waves, and will mask information about the object carried in $p_{SS}, p_{IS}, p_{SI}$ and $p_{S2}$.

Since second order scattering (S2) occurs at the object between $t = 0$ and $t = T$ when all narrow-band primary waves of compact support overlap, the $p_{S2}$ component has compact support via

$$p_{S2}(r, t) \approx \Re\{P_{S2\pm}e^{-i\omega_\pm t}w_2(t - r/c_0)\} \tag{4.1}$$

where $P_{S2\pm}$ is the time harmonic complex-amplitude of the second order scattered wave of equation (2.66) and $w_2$ has compact support, and following $w_1$, has duration $T$ with unit height within the window except at the ends where smooth transitions to zero occur over periods much less than $T$.

More details on the space-time isolation and the harmonic wave approximation
are provided in Appendix I.

Figure 4-1: Second order wave due to the interaction of (a) incident waves of compact support (II), (b) scattered waves (SS) of compact support, (c) incident and scattered waves (IS) of compact support in the forward (c), and backscatter (d) direction by direct numerical integration of the indicated full time domain solutions (solid lines). The second order waves have compact support within which the indicated harmonic approximations are shown to be valid. The top panels show the real pressure and the bottom panels show the pressure amplitude at the difference frequency. Here $\omega_a/(2\pi) = 500$ kHz, $\omega_b/(2\pi) = 300$ kHz, $T = 50$ $\mu$s, $c_0 = 1500$ m/s and $\beta = 3.6$. The pressure is normalized by $2E_0$, where $E_0 = |P_{a0}P_{b0}|/(2\rho_0c_0^2)$. In (a), the pressure is calculated at the origin; in (b)-(d), the pressure is calculated at $r_R = 0.1$ m from the center of a pressure release sphere of radius $a = 10$ $\mu$m and the scattered waves are approximately spherically symmetric with scatter functions $S_{a,b} = -k_{a,b}a$.

4.2 Transitions between dominant mechanisms

At long range, the SS, IS and SI interactions are always the dominant mechanisms, as seen from the asymptotic behavior given by equations (3.37) and (3.39) - (3.41). This shows that only the object’s response at the primary frequencies can be measured from the second order field at long range. Our analysis here (figures 4-3 - 4-7) shows that at shorter ranges, however, depending on frequencies, receiver range and
Figure 4-2: Primary incident and scattered waves of compact support and regions of compact support overlap, at different time instances (a)-(e). The left column is for the collinear incidence case, and the right column is for the perpendicular incidence case. The overlap regions for the incident waves, scattered waves, and incident and scattered wave are volumes in 3-D, and they are projected and shown on a 2-D plane as the gray, blue and green areas, respectively.
the object's scattering properties, the S2 mechanism can also dominate, where the object's response at the sum or difference frequency may be deduced from the sum or difference frequency second order wave. These complicated transitions between dominant mechanisms can be exploited for novel remote sensing applications.

To generalize the analysis, dimensionless parameters \( k_0 a \), \( k_0 r_R \), \( k_- r_R \) and \( k_- a \) are used, where \( k_0 = (\omega_a + \omega_b)/(2c_0) \) is the center frequency wavenumber, \( k_- = \omega_-/c_0 \) is the difference frequency wavenumber, \( a \) is the radius of the object and \( r_R \) is the receiver range. The values for the parameters are chosen based on the following considerations: (1) \( k_- r_R \gg 1 \), so that the II interaction can be time separated from the other mechanisms, (2) the difference frequency is lower than the primary frequencies, and (3) maximum \( k_0 a \) is on the order of 1 for computational efficiency.

Five canonical second order field scenarios are shown at the difference frequency: (i) a rigid immovable sphere with collinear incident waves (figure 4-3); (ii) a rigid movable sphere with collinear incidence (figure 4-4); (iii) a rigid immovable sphere with perpendicular incidence (figure 4-5); (iv) a pressure release sphere with collinear incidence (figure 4-6); and (v) a resonating rigid movable sphere with collinear incidence (figure 4-7). The geometry is shown in figure 4-2. With the harmonic wave approximation, \( p_{II} \) is given by equation (3.29) with \( z_s = 0 \) or (3.30), \( p_{SS} \), \( p_{IS} \) and \( p_{SI} \) are determined from equations (3.22), (3.23) and (3.24) respectively, and \( p_{S2} \) from spherical wave expansions (Appendix K). The results represent difference frequency wave amplitudes in the time domain between \( t = r_R/c_0 \) and \( t = r_R/c_0 + T \) for narrow-band incident fields, as discussed in §4.1.

General patterns of behavior for various mechanisms and transitions between dominant mechanisms are observed at the difference frequency when the receiver range is not asymptotically large:

For rigid objects, it is found that: (1) \( p_{SS} \) is dominant at large primary frequencies and for large objects, as seen in figures 4-3ab and 4-4ab for collinear incidence, and in figures 4-5ab for perpendicular incidence; (2) \( p_{SS} \) is most sensitive to the changes of primary frequency and object size, because it is a function of the product of primary scattered fields \( P_{Sa} \), \( P_{Sb} \), which depends strongly on the dimensionless sizes \( k_a a \) and \( k_b a \);
(3) when the primary frequency or object size decreases, the dominant mechanism transitions from SS to D2, IS and SI. Specifically, for movable objects, $p_{D2}$ due to the wave-boundary interaction becomes dominant as seen in figure 4-4ab. For immovable objects, $p_{D2}$ and $p_{IS} + p_{SI}$ are comparable, as seen in figure 4-3ab and figure 4-5ab; (4) $p_{IS} + p_{SI}$ is dominant at large difference frequency, as seen in figure 4-3d, 4-4d and 4-5d; and (5) $p_{R2}$, which arises from the second order centroidal motion, has very little contribution to the total second order field, as seen in figure 4-4. For resonating objects, however, the centroidal motion can be amplified such that $p_{R2}$ becomes dominant when the difference frequency is close to the resonance frequency, as shown in figure 4-7.

For pressure release objects, it is found that (1) $p_{S2}$ is dominant at small primary frequencies, for small objects or at small difference frequency, as seen in figure 4-6abd, and is mainly due to $p_2'$ (equation (3.2)) and the wave-boundary interaction (equation (2.67)); (2) $p_{SS}$ is sensitive to the change in object size but not to the primary frequency for small objects, as seen in figure 4-6ab, because $P_{S}a$ and $P_{S}b$ do not depend on frequency for small pressure release objects; and (3) $p_{S2}$ is insensitive to the change in difference frequency when it is small, as seen in figure 4-6d, because $p_2'$ and the wave-boundary interaction depend on the primary frequencies, which converge to the center frequency as the difference frequency decreases.

For all rigid, pressure release and resonating cases, $p_{SS}$ falls off in range less than $p_{IS} + p_{SI}$ (off the forward direction) and $p_{S2}$ do as seen in figures 4-3c, 4-4c, 4-5c and 4-6c. This is consistent with the asymptotic behavior of each component discussed in §3. When the primary incident waves are collinear, $P_{II\pm}$ is proportional to $k_\pm z_s$ for $k_\pm z_s \gg 1$, as seen in equation (3.29). In examples (i), (ii), (iv) and (v), $k_{+-} = k_{-+} = 0$ is used. For larger $k_\pm z_s$, $p_{S2}$ increases causing the range for which the SS interaction becomes dominant to increase.

In these examples, the maximum $k_0 a$ is on the order of one. Frequencies and object size were chosen to demonstrate transitions between dominant mechanisms for a practical amount of computation effort. When comparing with Jones and Beyer's experiment (figure 3-1), where $k_0 a \approx 90$ and $k_0 a \approx 60$, our theory is shown to be
accurate at very large $k_0a$. The second order pressure at each angle in figure 3-1 took more than a week to compute. This becomes impractical if we need to cover a high dimensional parameter space including object size, primary frequencies, receiver range and boundary conditions.

Figure 4-3: A rigid immovable sphere with collinear incident waves propagating at $\hat{r}_z$. The difference frequency component of $p_2$ is calculated at $r_R = -r_R\hat{r}_z$ as a function of (a) center frequency $k_0a$, (b) object radius $k_-a$, (c) receiver range $k_0r_R$ and (d) difference frequency $k_-r_R$. In (a), $p_{SS}$ dominates for large $k_0a$, while $p_{IS} + p_{SI}$ dominates for small $k_0a$. In (b), $p_{SS}$ dominates for large $k_-a$, $p_{IS} + p_{SI}$ dominates for small $k_-a$. In (c), $p_{SS}$ is dominant and it falls off the least in range. In (d), $p_{SS}$ dominates for small $k_-r_R$ while $p_{IS} + p_{SI}$ dominates for large $k_-r_R$. The nonlinear parameter is $\beta = 3.6$. 

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Figure 4-4: A rigid movable sphere with collinear incident waves propagating at $\hat{\mathbf{r}}_z$. The difference frequency component of $p_2$ is calculated at $\mathbf{r}_R = -r_R \hat{\mathbf{r}}_z$ as a function of (a) center frequency $k_0a$, (b) object radius $k-a$, (c) receiver range $k_0r_R$ and (d) difference frequency $k-r_R$. In (a), $p_{\text{SS}}$ dominates for large $k_0a$, while $p_{\text{D2}}$ dominates for small $k_0a$. In (b), $p_{\text{SS}}$ dominates for large $k-a$ and $p_{\text{D2}}$ dominates for small $k-a$. In (c), $p_{\text{SS}}$ is dominant and it falls off the least in range. In (d), $p_{\text{SS}}$ dominates for small $k-r_R$ and $p_{\text{LS}} + p_{\text{SL}}$ dominates for large $k-r_R$. The nonlinear parameter is $\beta = 3.6$, and the density ratio between the object and medium is 7.6.
Figure 4-5: A rigid immovable sphere with perpendicular incident waves propagating at \( i_x \) and \( i_z \) direction, respectively. The difference frequency component of \( p_2 \) is calculated at \( r_R = -r_R(i_x + i_z)/\sqrt{2} \) as a function of (a) center frequency \( k_a \), (b) object radius \( k_rR \), (c) receiver range \( k_0rR \), and (d) difference frequency \( k_rR \). In (a), \( p_{SS} \) dominates for large \( k_0a \), while \( p_{IS} + p_{SI} \) and \( p_{S2} \) dominate for small \( k_0a \). In (b), \( p_{SS} \) dominates for large \( k_a \), \( p_{IS} + p_{SI} \) and \( p_{S2} \) dominate for small \( k_a \). In (c), \( p_{SS} \) is dominant and it falls off the least in range. In (d), \( p_{SS} \) dominates for small \( k_rR \), while \( p_{IS} + p_{SI} \) and \( p_{S2} \) dominate for large \( k_rR \). The nonlinear parameter is \( \beta = 3.6 \).
Figure 4-6: A pressure release sphere with collinear incident waves propagating in $\hat{r}_z$.

The difference frequency component of $p_2$ is calculated at $r_R = -r_R \hat{r}_z$ as a function of (a) center frequency $k_0a$, (b) object radius $k_-a$, (c) receiver range $k_0r_R$, and (d) difference frequency $k_-r_R$. In (a), $p_{S2}$ is dominant. In (b), $p_{S2}$ dominates for small $k_-a$ and $p_{SS}$ becomes comparable to $p_{S2}$ for large $k_-a$. In (c), $p_{S2}$ is dominant, but $p_{SS}$ will dominate as range increases. In (d), $p_{S2}$ dominates for small $k_-r_R$, and $p_{SS}$ becomes comparable to $p_{S2}$ for large $k_-r_R$. The nonlinear parameter is $\beta = 3.6$. 

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Figure 4-7: A resonating rigid movable sphere with collinear incident waves propagating in \( \hat{\mathbf{i}}_z \) direction. \( p_{2-} \) is calculated in \( \mathbf{r}_R = -r_R \hat{\mathbf{i}}_z \). In (a), total difference frequency component of \( p_2 \) is plotted as a function of difference frequency near resonance for damping ratio \( \zeta = 0.001, 0.01, 1 \). In (b), \( p_2, p_{ss}, p_{ss} + p_{s1}, p_{d2} \) and \( p_{R2} \) are shown near resonance for \( \zeta = 0.001 \). When the object’s resonance is excited at the difference frequency, \( p_{R2} \) is amplified and it becomes dominant for small enough damping. The object radius \( a \), receiver range \( r_R \) and the center frequency are fixed so \( k_0 a = 2.09 \) and \( k_0 r_R = 1047 \). The nonlinear parameter is \( \beta = 3.6 \) and the density ratio between the object and the medium is 2.4.
4.3 Examples in air, water and solid earth

By matching the relevant dimensionless parameters, results presented in §4.2 can be applied to various sensing scenarios in air, water and solid earth. Besides identifying the actual dominant mechanisms, the received sound pressure level (SPL) for the second order difference frequency wave is estimated and compared with typical noise level for each of the cases. It is found that $P_{2-}$ is measurable above noise in most cases. Depending on the actual dominant mechanism, the response of the object at the primary frequencies or difference frequency can then be deduced from $P_{2-}$.

The primary frequencies, difference frequency, object size and receiver range for various scenarios are listed in Table 4.1. Case 1 is in air, where density $\rho_{\text{air}} = 1.2$ kg/m$^3$, sound speed $c_{\text{air}} = 340$ m/s and nonlinear parameter $\beta_{\text{air}} = 1.2$. Cases 2-4 are in water, where $\rho_{\text{water}} = 1000$ kg/m$^3$, $c_{\text{water}} = 1500$ m/s and $\beta_{\text{water}} = 3.6$. Case 5 is in solid earth, where $\rho_{\text{earth}} = 3000$ kg/m$^3$, $c_{\text{earth}} = 3000$ m/s (compressional wave) and $\beta_{\text{earth}} = 1000$ for porous media [4]. Nominal applied values for incident SPL, received SPL at the difference frequency, noise SPL, and signal-to-noise ratio (SNR) for these cases are listed in Table 4.2. For cases where the nominal received SPL is lower than the nominal noise SPL specified in Table 4.2, (e.g. Cases 1 and 4), enhancement of SNR can be achieved by: (1) increasing the incident SPL; (2) transmitting longer pulses and measuring the second order wave for a longer duration, which reduces noises out of band; (3) using a receiver array rather than a single element, which reduces noise by moving it out of beam, as described by the array gain (AG) factor; (4) increasing the number of scatterers in the resolution footprint, so that received SPL can be increased by incoherent superposition, which, however, is only applicable when $S2$ dominates (e.g. Cases 3-5); Otherwise the second order pressure from many objects cannot be superimposed because SS and IS are coherent nonlinear mechanisms; and (5) tuning the difference frequency to the object's resonance frequency. Gas bubbles or fish swim-bladders are known to exhibit resonance, where the scattering level can be increased by orders of magnitude compared to that calculated using the pressure release assumption. For example, the scattering cross section of a bubble can be 900
times larger than the geometric cross section at resonance [33]. In other words, the SNR can increase by 30 dB at resonance. This is applicable to Case 4.

In summary, analysis showed that the second order wave at the difference frequency can be measured in various scenarios, some of which may require a large array gain, a large number of objects and/or resonance effects, in order to overcome the typical environmental noise.
<table>
<thead>
<tr>
<th>Case</th>
<th>Medium</th>
<th>Primary Freq.</th>
<th>Diff. Freq.</th>
<th>Object Size</th>
<th>Receiver Type</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>air</td>
<td>40 kHz</td>
<td>400 Hz</td>
<td>4 mm</td>
<td>RG</td>
<td>bats[34], robots[35]</td>
</tr>
<tr>
<td>2</td>
<td>water</td>
<td>1 MHz</td>
<td>10 kHz</td>
<td>0.5 mm</td>
<td>RG</td>
<td>medical imaging[24, 21]</td>
</tr>
<tr>
<td>3</td>
<td>water</td>
<td>500 kHz</td>
<td>100 kHz</td>
<td>10 μm</td>
<td>PR</td>
<td>medical imaging[36]</td>
</tr>
<tr>
<td>4</td>
<td>water</td>
<td>100 kHz</td>
<td>10 kHz</td>
<td>1 mm</td>
<td>PR</td>
<td>side-scan sonar[37, 38]</td>
</tr>
<tr>
<td>5</td>
<td>solid earth</td>
<td>50 Hz</td>
<td>10 Hz</td>
<td>4 m</td>
<td>PR</td>
<td>exploration seismology[39]</td>
</tr>
</tbody>
</table>

Table 4.1: Example applications in different media. For air, density \( \rho_{\text{air}} = 1.2 \text{ kg/m}^3 \), sound speed \( c_{\text{air}} = 340 \text{ m/s} \) and nonlinear parameter \( \beta_{\text{air}} = 1.2 \). For water, \( \rho_{\text{water}} = 1000 \text{ kg/m}^3 \), \( c_{\text{water}} = 1500 \text{ m/s} \) and \( \beta_{\text{water}} = 3.6 \). For solid earth, \( \rho_{\text{earth}} = 3000 \text{ kg/m}^3 \), \( c_{\text{earth}} = 3000 \text{ m/s} \) (compressional wave) and \( \beta_{\text{earth}} = 1000 \) for porous medium [4]. Type RG represents immovable rigid object and PR represents pressure release object.

<table>
<thead>
<tr>
<th>Case, Medium, Mechanism</th>
<th>Incident SPL</th>
<th>Received Noise SPL</th>
<th>Array Gain</th>
<th>Multiple Resonant</th>
<th>SNR Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (air, SS)</td>
<td>140 dB</td>
<td>-14 dB</td>
<td>36 dB</td>
<td>N/A</td>
<td>0 dB</td>
</tr>
<tr>
<td>2 (water, SS)</td>
<td>240 dB</td>
<td>86 dB</td>
<td>57 dB</td>
<td>0 dB</td>
<td>29 dB</td>
</tr>
<tr>
<td>3 (water, S2)</td>
<td>240 dB</td>
<td>134 dB</td>
<td>65 dB</td>
<td>0 dB</td>
<td>69 dB</td>
</tr>
<tr>
<td>4 (water, S2)</td>
<td>200 dB</td>
<td>9 dB</td>
<td>57 dB</td>
<td>20 dB</td>
<td>30 dB</td>
</tr>
<tr>
<td>5 (solid earth, S2)</td>
<td>240 dB</td>
<td>163 dB</td>
<td>132 dB</td>
<td>10 dB</td>
<td>0 dB</td>
</tr>
</tbody>
</table>

Table 4.2: Nominal primary incident SPL, received second order SPL, noise SPL, adjusted SNR with potential array gain, incoherent superposition of multiple scatterer, and resonance effect for applications considered in Table 4.1. The SPL in air is referenced to 20 μPa, the SPL in water and solid earth is referenced to 1 μPa. The incident SPL is at the object and assumed to be the same for both primary incident waves. The noise SPL is determined by integrating the noise spectral density over a 10 % frequency bandwidth centered at the difference frequency, which corresponds to a measure window of 10 difference frequency cycles. The nominal incident SPL, received SPL and typical noise SPL estimation are provided in the Appendix J for each case.
4.4 Long range sensing using the IS and SS interactions

As seen from equations (3.39)-(3.41), second-order field magnitudes from IS and SI interactions remain constant in the forward direction of the incident wave scaled by the angular dependence of the primary scattered field in this direction. This could be advantageous for stealthy long range sensing of an object's primary frequency response via sum and difference frequency measurements. For example, if an object scatters or radiates an outward propagating wave $P_{sb}$ at angular frequency $\omega_b$, one can transmit a plane wave $P_{ia}$ at a different frequency $\omega_a$ in the direction of the receiver from the object. By continuing this for a receiver moving about the object, one can reconstruct the directivity pattern of $P_{sb}$ from $P_{2\pm}$, which can then be used for primary frequency object classification at the difference frequency. The SPL of $P_{IS\pm}$ in the forward direction is determined from equations (3.39) and (3.40) as

$$\text{SPL}[P_{IS\pm}(\vec{i}_a)] = \text{SPL}(P_{a0}) + \text{SPL}(P_{b0}) + 20 \log_{10} \left( \frac{\beta |S_b(\vec{i}_a)|k^2_{a+b} \rho_{ref}}{2Ak_a k_b} \right)$$

$$+ \begin{cases} 20 \log_{10} \left[ \log(k+/k_b) \right] & \text{dB re } \rho_{ref}, \\ 20 \log_{10} \left[ |i \log(k-_/k_b) + \pi| \right] & \end{cases}$$

which depends on the SPL of incident waves, the frequencies $\omega_a, \omega_b$, the far field scatter function $S_b$ and constants $\beta$ and $A$. In both air and water, $P_{IS\pm}$ is measurable for moderate to high incident SPL, as shown in the online supporting material.

The SPL of $P_{IS\pm}$ in the forward direction is plotted as a function of incident SPL in figure 4-8 for air (a)(c) and water (b)(d), assuming $P_{a0} = P_{b0}$ and $|S_a| = |S_b| = 1$. As seen in figure 4-8(a) and (c) for air, high frequency incident SPL between 120 dB and 140 dB re 20 $\mu$Pa gives sum frequency SPL between 48 dB to 88 dB re 20 $\mu$Pa, and difference frequency SPL between 38 dB and 77 dB re 20 $\mu$Pa respectively, for $\omega_b/\omega_a = 0.6$. Similarly, as seen in (b) and (d) for water, high frequency incident SPL between 170 dB and 190 dB re 1 $\mu$Pa gives sum frequency SPL between 48
dB to 90 dB, and difference frequency SPL between 37 dB and 77 dB respectively, for \( \omega_b/\omega_a = 0.6 \). With moderate to high incident SPLs, these sum and difference frequency second order waves can be measurable in long range.

Figure 4-8: SPL of \( P_{\text{IS} \pm} \) in the forward direction for large range, as a function of the SPL of the incident waves for \( \omega_b/\omega_a = 0.6, 0.9 \) and 0.99. The SPL in air (a) is referenced to 20 \( \mu \)Pa and the SPL in water (b) is referenced to 1 \( \mu \)Pa.

At long range and away from the forward direction, \( P_{\text{SS} \pm} \) is dominant and it can also be used for sensing primary frequency responses from sum or difference frequency measurements. The SPL for \( P_{\text{SS} \pm} \) for \( k_r r \to \infty \) can be determined from equation (3.38), as

\[
\text{SPL}(P_{\text{SS} \pm}) = \text{SPL}(P_{ao}) + \text{SPL}(P_{b0}) + 20 \log_{10} \left[ \frac{\beta |S_b| |k^2_{\text{ref}}| |S_a|}{2A k_a k_b} k_{\pm} r \log\left( \frac{r}{r_{\text{ref} \pm}} \right) \right] \text{dB re } p_{\text{ref}}.
\]

Compared to the SPL of \( P_{\text{IS} \pm} \) in equation (4.2), it can be seen that a higher incident SPL is required for \( P_{\text{SS} \pm} \) to compensate for its falloff. An advantage of using the
$P_{SS\pm}$, however, is that it can be measured in all directions, whereas the $P_{\Sigma\pm}$ is mostly useful in the forward direction.
Chapter 5

Conclusions

A general second order theory of nonlinear interaction and scattering of acoustic waves in the presence of an object is derived and confirmed by comparison with experimental measurements. The theory employs complete and consistent asymptotic analysis of the wave equation and time dependent boundary conditions to second order, including a complete second order wave-exciting force to determine the effect of an object's centroidal motion on the second order field. After object insonification by two primary frequency incident waves, it is found that sum or difference frequency second order acoustic waves arise due to: (A) nonlinear wave-wave interactions of the primary waves, including the interaction between incident waves (denoted as II), the interaction between scattered waves (denoted as SS) and the interaction between an incident and scattered wave (denoted as IS and SI); and (B) second order linear scattering of nonlinear waves of the first category due to the presence of the object (denoted as S2). It is shown that second order acoustic waves due to (A) wave-wave interaction contain information about the object's first order response at the primary frequencies but not at the sum or difference frequency, while (B) waves due to second order linear scattering contain information about the object's second order response at the sum or difference frequency.

It is analytically shown that wave-wave interactions dominate the total second order field at long range from the object. In particular, the SS, IS and SI interactions are found to always dominate the second order field components that carry object in-
formation at long range, via rigorous asymptotic solutions. At shorter ranges, second
order linear scattering may also dominate depending on physical parameters such as
objects size, boundary condition and frequencies, including the effect of centroidal
motion of the object due to its interaction with acoustic waves. Approximate an-
alytic solutions for the second order nonlinear wave fields found in the presence of
low-impedance contrast inhomogeneities are also derived.

Analytic proof shows that to second order there is no scattering of sound by sound
outside the region of compact support intersection of finite-duration plane waves at
sum or difference frequencies in the absence of an object, via generalization of Lamb’s
and Westervelt’s time-harmonic formulations to one of general time dependence. We
also show that non-collinear interaction leads to finite scattering of sound by sound
at the primary frequencies only within the region of compact support union through
which compact support intersection occurred if the primary waves had zero-frequency
spectral components. These findings elucidate issues related to the scattering of sound
by sound in a rigorous manner for interaction regions of compact support. In the re-
gion of compact support intersection, it is found that a time-harmonic approximation
to the second order field is valid for narrow-band plane waves with sufficiently long
and smooth envelopes. It is found that such a time-harmonic approximation can be
made in many practical scenarios involving the interaction between scattered waves,
as well as scattered and incident waves of narrow band. Following experimental prac-
tice, we employ a time domain formulation of finite-duration narrow-band incident
waves and provide analytic solutions for the resulting second order fields in the pres-
ence of an object. Finite-duration incident waves enable nonlinear field components
that contain information about the object to be separated in time and space from
those that do not, such as the II component, for certain sensing geometries.

Unique opportunities are shown to arise for sensing an object at the sum or dif-
ference frequency, which are substantially different from traditional sensing via linear
scattering at the primary frequency. By ensonifying an object at primary frequencies
and sensing at sum or difference frequencies, information about an object’s primary
frequency response information about an object can be estimated from SS, IS and
SI interactions at long range, while sum or difference frequency response information about an object can sometimes be estimated from S2 at shorter range.
Appendix A

Second order nonlinear acoustic wave equation in medium with volume inhomogeneities

Let \( \rho_0 \) be the ambient density without inhomogeneities or the acoustic wave, \( \delta_e \) be the change of density due to inhomogeneities without the acoustic wave, and \( \delta \) be the change of density due to the acoustic wave. The total density \( \rho_e(r, t) \) can be written as

\[
\rho_e(r, t) = \rho_0 + \delta_e(r, t) + \delta(r, t) = \rho_{e0}(r, t) + \delta(r, t).
\]  

(A.1)

Without the acoustic wave, the total derivative of ambient density vanishes,

\[
\frac{d\rho_0}{dt} = \frac{\partial \delta_e}{\partial t} + \mathbf{v} \cdot \nabla \delta_e = 0.
\]  

(A.2)

The exact continuity and momentum equations are

\[
\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{v}) = \frac{\partial}{\partial t}(\delta_e + \delta) + \nabla \cdot [(\rho_0 + \delta_e + \delta)\mathbf{v}] = 0, \quad \text{and}
\]

\[
\frac{\partial}{\partial t}[(\rho_0 + \delta_e + \delta)\mathbf{v}] + \nabla \cdot [(\rho_0 + \delta_e + \delta)\mathbf{v}\mathbf{v}] + \nabla p = 0.
\]  

(A.3)

(A.4)
Applying equation (A.2) to equations (A.3) and (A.4) leads to
\[
\frac{\partial \delta}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} + \nabla \cdot (\delta \mathbf{v}) + \delta_e \mathbf{v} = 0, \quad \text{and} \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial}{\partial t} (\delta \mathbf{v}) + \nabla \cdot [(\rho_0 + \delta) \mathbf{v} \mathbf{v}] + \delta_e \mathbf{v} \cdot (\mathbf{v} \mathbf{v}) + \nabla p = 0. \quad (A.5, A.6)
\]

Now perturbation expansions are introduced to \(\delta, p\) and \(\mathbf{v}\) in equations (A.5) and (A.6). Canceling the first order velocity \(\mathbf{v}_1\) from equations (A.5) and (A.6) leads to
\[
\frac{\partial}{\partial t} \left( \frac{1}{\rho_0} \frac{\partial \delta_1}{\partial t} \right) = \nabla \cdot \left( \frac{1}{\rho_0} \nabla p_1 \right). \quad (A.7)
\]

Similarly, eliminating the second order velocity \(\mathbf{v}_2\) from equations (A.5) and (A.6) yields
\[
\frac{\partial}{\partial t} \left( \frac{1}{\rho_0} \frac{\partial \delta_2}{\partial t} \right) + \frac{\partial}{\partial t} \left[ \nabla \cdot (\delta_1 \mathbf{v}_1) \right] - \nabla \left[ \frac{1}{\rho_0} \nabla p_1 \right] + \nabla \cdot \left[ \nabla \cdot (\mathbf{v}_1 \mathbf{v}_1) \right] + \nabla \cdot \left( \frac{1}{\rho_0} \nabla p_2 \right). \quad (A.8)
\]

For the equation of state, it is assumed that the density is a function of pressure only and can be expressed as a Taylor series expansion about the ambient pressure,
\[
\rho_e = \rho_{e0} + \frac{\partial \rho_e}{\partial p} \bigg|_{p_0} (p - p_0) + \frac{1}{2} \frac{\partial^2 \rho_e}{\partial p^2} \bigg|_{p_0} (p - p_0)^2 + \cdots. \quad (A.9)
\]

Taylor series expansion leads to
\[
\delta_1 = \frac{\partial \rho_e}{\partial p} \bigg|_{p_0} p_1 = c_e^{-2} p_1 = \kappa_e \rho_{e0} p_1, \quad \text{and} \quad \delta_2 = \frac{\partial \rho_e}{\partial p} \bigg|_{p_0} p_2 + \frac{1}{2} \frac{\partial^2 \rho_e}{\partial p^2} \bigg|_{p_0} p_1^2 = c_e^{-2} p_2 + \Gamma_e \rho_{e0}^2 p_1^2, \quad (A.10, A.11)
\]

where the compressibility \(\kappa_e = \rho_{e0}^{-1} \partial \rho_e/\partial p \big|_{p_0} = \rho_{e0}^{-1} c_e^{-2} \) and \(\Gamma_e = \partial^2 \rho_e/\partial p^2 \big|_{p_0} \rho_{e0} c_e^2/2\).

Substituting equation (A.10) into equation (A.7) to eliminate \(\delta_1\) and assuming the
inhomogeneities $\rho_0$ and $\kappa_e$ are stationary in time yields

$$\nabla \cdot \left( \frac{1}{\rho_0} \nabla p_1 \right) - \kappa_e \frac{\partial^2 p_1}{\partial t^2} = 0, \quad (A.12)$$

which is the classical wave equation for space containing inhomogeneities [26].

Substituting equation (A.11) into equation (A.8) to eliminate $\delta_2$ and further assuming $\Gamma_e$ is stationary in time yields

$$-\nabla \cdot \left( \frac{1}{\rho_0} \nabla p_2 \right) + \kappa_e \frac{\partial^2 p_2}{\partial t^2} + \Gamma_e \kappa_e \frac{\partial^2 p_1^2}{\partial t^2} - \nabla \left( \frac{1}{\rho_0} \right) \cdot \frac{\partial}{\partial t} (\delta_1 v_1) - \nabla \cdot [\nabla \cdot (v_1 v_1)] = 0.$$

(A.13)

By adding $\rho_0^{-1} \nabla^2 p_2 - \kappa_0 \frac{\partial^2 p_2}{\partial t^2}$ on both sides and then multiplying by $\rho_0$, equation (A.13) can be rewritten as

$$\nabla^2 p_2 - \frac{1}{c_0^2} \frac{\partial^2 p_2}{\partial t^2} = \frac{\gamma_\kappa}{c_0^2} \frac{\partial^2 p_2}{\partial t^2} + \nabla \cdot (\gamma_\rho \nabla p_2) - \frac{1}{c_0^2} \nabla \left( \frac{\rho_0}{\rho_0} \right) \cdot \frac{\partial}{\partial t} (p_1 v_1)$$

$$+ \Gamma_e \kappa_e \rho_0 \frac{\partial^2 p_1^2}{\partial t^2} - \rho_0 \nabla \cdot [\nabla \cdot (v_1 v_1)], \quad (A.14)$$

where $\gamma_\kappa = (\kappa_e - \kappa_0)/\kappa_0$ and $\gamma_\rho = (\rho_e - \rho_0)/\rho_0$ are the fractional changes in compressibility and density of the medium due to inhomogeneities, $\kappa_0 = \rho_0^{-1} c_0^{-2}$ is the mean compressibility and $c_0$ is the mean sound speed of the medium. For a homogeneous medium, $\gamma_\kappa = \gamma_\rho = 0$, $\rho_0 = \rho_0$, $\kappa_e = \rho$, $\Gamma_e = \Gamma_0 = \frac{\partial^2 p/\partial p^2}{\rho_0 c_0^4/2} = -B/(2A)$, and the above equation reduces to equation (2.29).

Consider time harmonic fields incident on a single target. Let the incident wave consist of two harmonic components, $P_a$ and $P_b$ with frequencies $\omega_a$ and $\omega_b$. The corresponding Helmholtz equation for the second order pressure at the sum or difference frequency is

$$(\nabla^2 + k_{\pm}^2) P_{2\pm} = -Q_{2\pm}^{\text{inhomo}} \quad (A.15)$$
where

\[
Q_{2\pm}^{\text{inhomo}} = \{k_{\pm}^2 \gamma_{\kappa} P_{2\pm}\} - \{\nabla \cdot (\gamma_p \nabla P_{2\pm})\} - \left\{\frac{i \omega_{\pm}}{2c_s^2} \nabla \left( \frac{\rho_0}{\rho_c(\omega)} \right) \cdot (P_a V_b^{(*)} + V_a P_b^{(*)}) \right\} \\
+ \left\{ \Gamma \kappa_{\pm} \rho_0 \omega_{\pm}^2 P_a P_b^{(*)} \right\} + \left\{ \frac{\rho_0}{2} \nabla \cdot [\nabla \cdot (V_a V_b^{(*)} + V_b^{(*)} V_a)] \right\}.
\]

(A.16)

The solution to equation (A.15) is

\[
P_{2\pm}(r) = \iiint Q_{2\pm}^{\text{inhomo}}(r_0) G(r|r_0, \omega_{\pm}) dV_0,
\]

which has five components corresponding to the five parenthetical groups in curly brackets in \(Q_{2\pm}^{\text{inhomo}}\) of equation (A.16). Components in the first three sets of curly brackets in equation (A.16) are nonzero in or on the inhomogeneity and describe second order scattering \(S_2\). As in the linear case, the changes of compressibility in the medium, \(k_{\pm}^2 \gamma_{\kappa} P_{2\pm}\), scatter as monopoles while the changes of density, \(\nabla \cdot (\gamma_p \nabla P_{2\pm})\), scatter as dipoles. Components in the last two sets of curly brackets in equation (A.16) are nonzero where the primary waves are nonzero and describe nonlinear wave-wave interactions, including \(II\), \(SS\) and \(IS\), where only \(II\) exists if the medium has no inhomogeneities.
Appendix B

Locally reacting boundary

The normal velocity \( \mathbf{v} \cdot \mathbf{n} \) is a function of pressure \( p \) on a locally reacting boundary \( S \) as

\[
\mathbf{v} \cdot \mathbf{n} = h(p) \quad \text{on } S(t),
\]

(B.1)

where the nonlinear function \( h \) can be expanded with respect to ambient pressure \( p_0 \) as

\[
\mathbf{v} \cdot \mathbf{n} = h_0(p - p_0) + h_1(p - p_0)^2 + \cdots \quad \text{on } S(t).
\]

(B.2)

This boundary condition on \( S(t) \) can be further expanded with respect to the reference mean boundary \( \bar{S} \), as

\[
\mathbf{v} \cdot \mathbf{n} + \xi \cdot \nabla(\mathbf{v} \cdot \mathbf{n}) + \cdots \\
= h_0(p - p_0) + \xi \cdot \nabla(h_0(p - p_0)) + h_1(p - p_0)^2 + \xi \cdot \nabla(h_1(p - p_0)^2) + \cdots \quad \text{on } \bar{S}.
\]

(B.3)

Substituting the perturbation expansions for \( \mathbf{v}, \xi \) and \( p \) to the above equation and collecting terms of the same order, we obtain first and second order boundary
conditions for the general locally reacting boundary, as

\[ \mathbf{v}_1 \cdot \mathbf{n} = h_0 p_1 \quad \text{on } \bar{S}, \quad (B.4) \]

first order:

\[ \mathbf{v}_2 \cdot \mathbf{n} + \xi \cdot \nabla (\mathbf{v}_1 \cdot \mathbf{n}) = h_0 p_2 + h_0 \xi \cdot \nabla p_1 + h_1 p_1^2 \quad \text{on } \bar{S}. \quad (B.5) \]

second order:

Here \( 1/h_0 \) is the same as the acoustic impedance \( z \) in Ref. [26].

The first and second order fields for a pressure release boundary problem are denoted as \( p_1^{PR} \) and \( p_2^{PR} \) respectively. They satisfy the boundary conditions

\[ p_1^{PR} = 0 \quad \text{on } \bar{S}, \quad (B.6) \]

first order: \( p_1^{PR} \)

\[ p_2^{PR} + \xi \cdot \nabla p_1^{PR} = 0 \quad \text{on } \bar{S}. \quad (B.7) \]

second order:

The first and second order fields for a fixed rigid boundary problem are denoted as \( p_1^{RG} \) and \( p_2^{RG} \) respectively. They satisfy the boundary conditions

\[ \mathbf{v}_1^{RG} \cdot \mathbf{n} = 0 \quad \text{on } \bar{S}, \quad (B.8) \]

first order:

\[ \mathbf{v}_2^{RG} \cdot \mathbf{n} = 0 \quad \text{on } \bar{S}. \quad (B.9) \]

second order:

On \( \bar{S} \), the first and second order waves for a general locally reacting boundary problem are

\[ p_1 = \alpha_1 p_1^{PR} + (1 - \alpha_1) p_1^{RG} \quad \text{on } \bar{S}, \quad (B.10) \]

\[ p_2 = \alpha_2 p_2^{PR} + (1 - \alpha_2) p_2^{RG} \quad \text{on } \bar{S}. \quad (B.11) \]

The coefficients \( \alpha_1 \) and \( \alpha_2 \) are functions of position on \( \bar{S} \), and they can be determined from the first or second order boundary conditions (B.6) and (B.7), as

\[ \alpha_1 = \frac{h_0 p_1^{RG}}{\mathbf{v}_1^{PR} \cdot \mathbf{n} + h_0 p_1^{RG}}, \quad (B.12) \]

\[ \alpha_2 = \frac{h_0 p_2^{RG} + h_1 (p_2^{RG})^2}{\mathbf{v}_2^{PR} \cdot \mathbf{n} + \xi \cdot \nabla (\mathbf{v}_1^{PR} \cdot \mathbf{n}) + h_0 p_2^{RG} + h_1 (p_2^{RG})^2}. \quad (B.13) \]
The first order scattered wave $p_{s_1}$ from a general locally reacting boundary can be obtained by evaluating equation (2.8), with equations (B.10) and (B.12). The second order scattered wave $p_{s_2}$ from a general locally reacting boundary can be obtained by evaluating equation (2.37) with equations (B.11) and (B.13).
Appendix C

Extension to Fluid-filled and elastic objects

Objects with locally reacting boundaries and objects that are rigid and movable are treated thoroughly in this thesis. Our theory, however, can also be extended to general fluid-filled or elastic objects, where first and second order acoustic fields inside and outside the object need to be determined simultaneously via boundary conditions on the interface. Regardless the complexity of the second order field inside the object, the second order boundary conditions can still be obtained via Taylor series expansion, and the second order field outside the object still consists of II, SS, IS, SI and S2 components only. The conclusion that SS, IS and SI always dominate at large range still remains and the fact that SS, IS and SI carries primary frequency response information about the object while S2 carries sum and difference frequency information about the object still holds.
Starting from $f = \iint p \, dS$, the total second order force $f_2$ acting on an object can be written as [23]

$$f_2 = \rho_0 \frac{d}{dt} \left( \iint_{\delta V} v_1 dV \right) + \rho_0 \frac{d}{dt} \left( \iint \phi_2 n \, dS \right) - \iint L \, n \, dS + \rho_0 \iint v_1 v_1 \cdot n \, dS,$$

where $\delta V$ is the difference between the volume occupied by the exact boundary $S(t)$ and time-averaged boundary $\bar{S}$. $L = T - V$ is the Lagrangian density, where $T = \rho_0 v_1^2 / 2$ is the kinetic energy density and $V = p_1^2 / (2A)$ is the potential energy density. Components of the second order potential $\phi_2$ are explicitly written out following our convention as

$$\phi_2 = \phi_{II} + \phi_{SS} + \phi_{IS} + \phi_{SI} + \phi_{D2,II} + \phi_{D2,SS} + \phi_{D2,IS} + \phi_{D2,SI} + \phi_{R2,II} + \phi_{R2,SS} + \phi_{R2,IS} + \phi_{R2,SI}.$$  \hspace{1cm} (D.2)

Similarly, $L = L_{II} + L_{SS} + L_{IS} + L_{SI}$.

The first two terms on the right-hand side of equation (D.1) have zero zero-frequency component due to the time derivative, so the static radiation force $f_2^{\text{static}}$ is determined by the zero-frequency component of the last two terms. This, for example, enables King [40] to calculate $f_2^{\text{static}}$ on a rigid movable sphere for plane wave incidence.
without solving $\phi_2$. The radiation force here has its counterpart in hydrodynamics as the second order drifting force on an object [41].

In the dynamic case, all terms in equation (D.1) must be included. As in equation (2.50), the centroidal motion of the object, $u_{c2}$, is determined by the equation of motion

$$M \frac{du_{c2}}{dt} = f_2.$$  

(D.3)

Substituting equations (D.1) and (D.2) into equation (D.3) and combining the contribution of $R_2$ as the radiation impedance $z_r$, we obtain

$$(z_m + z_r)u_{c2} = f_2^{\text{excit}},$$  

(D.4)

where

$$f_2^{\text{excit}} = \rho_0 \frac{d}{dt} \left( \iiint_{V} v_1 dV \right) + \rho_0 \frac{d}{dt} \left( \iint_{S} \phi_2^{\text{excit}} n dS \right) - \iint_{S} L n dS + \rho_0 \iint_{S} v_1 v_1 \cdot n dS$$

(D.5)

is equivalent to equation (2.51), and

$$\phi_2^{\text{excit}} = \phi_{II} + \phi_{SS} + \phi_{IS} + \phi_{SI} + \phi_{D2,II} + \phi_{D2,SS} + \phi_{D2,IS} + \phi_{D2,SI},$$  

(D.6)

which contains the wave-wave interactions and the corresponding D2 components.

The object's centroidal velocity $u_{c2}(t)$ can be determined from equations (D.4) - (D.6). A derivation along these lines found in references [22, 23, 24, 25], however, uses $\phi'$ instead of $\phi_2^{\text{excit}}$ where $\phi'$ omits terms $\phi_{SS}$, $\phi_{IS}$, $\phi_{SI}$, $\phi_{D2,SS}$, $\phi_{D2,IS}$ and $\phi_{D2,SI}$, and adds an extraneous $\phi_{R2,II}$, and so does not equal $\phi_2^{\text{excit}}$ of equation (D.6) as it must to quantify all second order effects in a general manner. Low difference frequency measurements [24] show that $f_2^{\text{excit}}$ converges to $f_2^{\text{static}}$, which also occurs if $\phi'$ is used. Using $\phi'$ instead of $\phi_2^{\text{excit}}$ in equation (D.5), however, does not generally lead to the correct relation between $f_2^{\text{excit}}$ and centroidal velocity $u_{c2}$ as in equation (D.4) across
frequency.
Jones and Beyer’s experiment

Jones and Beyer [1, 2] measured the sum frequency second order pressure due to an object insonified by two perpendicular incident beams at 7 MHz and 5 MHz respectively. The total second order pressure is determined via equations (2.66), (2.75), (3.22) - (3.24) as a function of angle at a fixed range about the forward direction of the 7 MHz beam (figure E-1) and found to be consistent with these measurements as shown in figure 3-1.

Figure E-1: Geometry of the Jones and Beyer measurement, where a rigid sphere is located at the center of the overlap region of the two incident beams. Not to scale.

For quantitative comparisons between theory $X_i$ and measured data $Y_i$ for second order pressure amplitude at the sum frequency, the correlation coefficients and mean square error are determined, where $i = 1, 2, \cdots, N$ for $N$ angular positions. The
correlation coefficient is defined as

\[ r_{X,Y} = \frac{\sum_{i=1}^{N} (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{N} (X_i - \bar{X})^2 \sqrt{\sum_{i=1}^{N} (Y_i - \bar{Y})^2}}} \]  

(E.1)

where \( \bar{X} \) and \( \bar{Y} \) are the average values for \( X_i \) and \( Y_i \), respectively. The correlation coefficient \( r_{X,Y} \) is found to be 0.98 for the \( N = 21 \) points measured by Jones and Beyer between \( \pm 0.2 \) rad. The mean square error (MSE) is defined in decibels as

\[ \text{MSE} = 10 \log_{10} \left[ 1 + \frac{\sum_{i=1}^{N} (X_i - Y_i)^2}{\sum_{i=1}^{N} Y_i^2} \right] \text{ dB}, \]  

(E.2)

which is found to be 0.23 dB.

Since the incident waves used by Jones and Beyer were narrow-band finite-duration pulses of 15 \( \mu \)s, we use the harmonic wave approximation to estimate the second order pressure amplitude at the sum frequency. The \( P_{SS+} \) component is found to be dominant, with the \( P_{IS+} + P_{SI+} \) amplitude no more than roughly 3% that of \( P_{SS+} \) according to our calculations. This is due to the narrowness of the incident beams generated by Jones and Beyer's circular transducers (radius 0.9525 cm) [1], which makes a plane wave a poor approximation to the spatially varying incident field necessary to calculate \( P_{IS+} \) and \( P_{SI+} \) by the extended IS interaction dictated by their experimental geometry. Jones and Beyer, however, showed that the measured primary scattered fields are in excellent match to those due to incident plane waves [1, 2] (especially at local maxima), which is consistent with fact that the object was well within the beams. Given this and the fact that they did not specify the transducer tapers which determine details of the beams, we use their suggested equivalent plane wave fields at the object to calculate the primary scattered fields required for determination of \( P_{SS+} \) in figure 3-1. When evaluating the volume integrals for \( P'_{SS+}, \ P'_{IS+} \) and \( P'_{SI+} \), of equations (3.22)-(3.24), a spatial step size of 1/20 of the sum frequency wavelength is used to ensure convergence and accuracy. The amplitudes of the plane waves for SS computations are made to be consistent with the primary scattered fields reported by Jones and Beyer, while the amplitudes of the beams for IS and SI calculations
are made to be consistent with measured values (2.45 × 10^5 Pa and 3.36 × 10^5 Pa) that were specified at 0.2189 m from the sources for the 7 MHz and 5 MHz beams respectively [2], averaged over a 0.2 cm [1] radius to account for the finite receiver size [42]. Calculations using equation (2.66) show the second order scattered field \( P_{s2+} \) be on the order of 1 Pa at the receiver, which is negligibly small compared to \( P_{ss+} \) (figure 3-1). The differences between the theoretically calculated and experimentally measured second order field seen in figure 3-1, are again likely due to the uncertainties in the taper functions of the transducers used by Jones and Beyer, which were not specified but had to be inferred from photographs and discussions provided in [1].
Appendix F

Derivation of the second-order field arising from the nonlinear interaction of plane waves of arbitrary time dependence

Let the first order incident field be the sum of two incident waves of arbitrary time dependence, \( p_1(\mathbf{r}, t) = \Re \{ \tilde{p}_{1a}(\mathbf{r}, t) + \tilde{p}_{1b}(\mathbf{r}, t) \} \), then

\[
p_1^2 = \frac{1}{2} \Re \{ \tilde{p}_{1a}\tilde{p}_{1a}^* + \tilde{p}_{1b}\tilde{p}_{1b}^* + \tilde{p}_{1a}\tilde{p}_{1b}^* + \tilde{p}_{1b}\tilde{p}_{1a}^* \} + \Re \{ \tilde{p}_{1a}\tilde{p}_{1b} + \tilde{p}_{1b}\tilde{p}_{1a}^* \}, \tag{F.1}
\]

and the solution to equation (3.5) can then be decomposed as

\[
p_{11}' = p_{11,aa} + p_{11,aa}^* + p_{11,bb} + p_{11,bb}^* + p_{11,ab} + p_{11,ab}^*. \tag{F.2}
\]
where each component satisfies

\[
\nabla \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p_{11,\alpha\alpha} = -\frac{\beta}{2A c_0^2} \frac{\partial^2}{\partial t^2} \Re \{ \tilde{p}_{1a}^2 \} \quad \text{(F.3a)}
\]

\[
\nabla \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p_{11,\alpha\alpha}^* = -\frac{\beta}{2A c_0^2} \frac{\partial^2}{\partial t^2} \Re \{ \tilde{p}_{1a} \tilde{p}_{1a}^* \} \quad \text{(F.3b)}
\]

\[
\nabla \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p_{11,bb} = -\frac{\beta}{2A c_0^2} \frac{\partial^2}{\partial t^2} \Re \{ \tilde{p}_{1b}^2 \} \quad \text{(F.3c)}
\]

\[
\nabla \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p_{11,bb}^* = -\frac{\beta}{2A c_0^2} \frac{\partial^2}{\partial t^2} \Re \{ \tilde{p}_{1b} \tilde{p}_{1b}^* \} \quad \text{(F.3d)}
\]

\[
\nabla \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p_{11,ab} = -\frac{\beta}{A c_0^2} \frac{\partial^2}{\partial t^2} \Re \{ \tilde{p}_{1a} \tilde{p}_{1b} \} \quad \text{(F.3e)}
\]

\[
\nabla \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p_{11,ab}^* = -\frac{\beta}{A c_0^2} \frac{\partial^2}{\partial t^2} \Re \{ \tilde{p}_{1a} \tilde{p}_{1b}^* \} \quad \text{(F.3f)}
\]

where \( p'_{1,\alpha\alpha} \), \( p'_{1,\alpha\alpha}^* \), \( p'_{1,bb} \), and \( p'_{1,bb}^* \) correspond to self-intersections of \( p_{1a} \) or \( p_{1b} \), and \( p'_{1,ab} \) and \( p'_{1,ab}^* \) correspond to cross-interactions between \( p_{1a} \) and \( p_{1b} \).

Self-interaction component \( p'_{1,\alpha\alpha} \) for \( \alpha = a, b \) contains sum frequencies of all frequency components in wave \( p_{1\alpha} \). If \( p_{1\alpha} \) is harmonic at frequency \( \omega_{\alpha} \), then \( p'_{1,\alpha\alpha} \) contains the double frequency \( 2\omega_{\alpha} \). Self-interaction component \( p'_{1,\alpha\alpha}^* \) for \( \alpha = a, b \) contains difference frequencies of all frequency components in wave \( p_{1\alpha} \) which will include a zero-frequency component from the difference of each frequency with itself. If \( p_{1\alpha} \) is harmonic at frequency \( \omega_{\alpha} \), then \( p'_{1,\alpha\alpha}^* \) is at zero frequency and it is zero.

The cross-interaction component \( p'_{1,ab} \) due to \( \tilde{p}_{1a} \tilde{p}_{1b} \) contains sum frequencies of all frequency components between \( p_{1a} \) and \( p_{1b} \). If \( p_{1a} \) and \( p_{1b} \) are harmonic at frequencies \( \omega_{a} \) and \( \omega_{b} \) respectively, then \( p'_{1,ab} \) is at the sum frequency \( \omega_{a} + \omega_{b} \). The cross-interaction component \( p'_{1,ab}^* \) due to \( \tilde{p}_{1a} \tilde{p}_{1b}^* \) contains difference frequencies of all frequency components between \( p_{1a} \) and \( p_{1b} \). If \( p_{1a} \) and \( p_{1b} \) are harmonic at frequencies \( \omega_{a} \) and \( \omega_{b} \), then \( p'_{1,ab}^* \) is at the difference frequency \( \omega_{a} - \omega_{b} \).

For the collinear case, we solve for the cross-interactions \( p'_{1,ab} \) and \( p'_{1,ab}^* \) in section F.1. Self-interaction components \( p'_{1,aa} \) and \( p'_{1,bb} \) can be obtained from \( p_{11,ab} \) by letting \( \tilde{p}_{1b} = \tilde{p}_{1a} \), and letting \( \tilde{p}_{1a} = \tilde{p}_{1b} \), respectively in equation (3.31). Self-interaction components \( p'_{1,aa}^* \) and \( p'_{1,bb}^* \) can be obtained from \( p'_{1,ab}^* \) by letting \( \tilde{p}_{1b} = \tilde{p}_{1a}^* \), and letting \( \tilde{p}_{1a}^* = \tilde{p}_{1b} \), respectively in equation (3.31).
For the non-collinear case, we solve for the cross-interactions $p'_{1,ab}$ and $p'_{1,ab*}$ in section F.2. Self-interaction components in this case are the same as those in the collinear case given in section F.1. Specifically, $p'_{1,aa}$, $p'_{1,bb}$ can be obtained from equation (F.19a) and $p'_{1,aa*}$ and $p'_{1,bb*}$ can be obtained from equation (F.19b).

The $p''_1$ component is given in equation (3.2). When $p_{la}$ and $p_{lb}$ are plane waves propagating in directions $\hat{i}_a$ and $\hat{i}_b$ respectively, the first order velocity is

$$v_1(r, t) = \frac{1}{\rho_0 c_0} \Re \left\{ \tilde{p}_{la}(r, t) \hat{i}_a + \tilde{p}_{lb}(r, t) \hat{i}_b \right\},$$

(F.4)

then

$$v_1^2 = v_1 \cdot v_1 = \frac{1}{2\rho_0^2 c_0^2} \Re \left\{ \tilde{p}_{la} \tilde{p}_{la} + \tilde{p}_{la} \tilde{p}_{la}^* + \tilde{p}_{lb} \tilde{p}_{lb} + \tilde{p}_{lb} \tilde{p}_{lb}^* \right\}$$

$$+ \frac{1}{\rho_0^2 c_0^2} \Re \left\{ \tilde{p}_{la} \tilde{p}_{lb} + \tilde{p}_{la} \tilde{p}_{lb}^* \right\} \cos \theta,$$

(F.5)

where $\cos \theta = \hat{i}_a \cdot \hat{i}_b$, and

$$\frac{\partial p_1}{\partial t} \int_{-\infty}^{t} p_1 \, d\tau = \frac{1}{2} \Re \left\{ \frac{\partial \tilde{p}_{la}}{\partial t} \int_{-\infty}^{t} \tilde{p}_{la} \, d\tau + \frac{\partial \tilde{p}_{la}}{\partial t} \int_{-\infty}^{t} \tilde{p}_{la}^* \, d\tau \right\}$$

$$+ \frac{1}{2} \Re \left\{ \frac{\partial \tilde{p}_{lb}}{\partial t} \int_{-\infty}^{t} \tilde{p}_{lb} \, d\tau + \frac{\partial \tilde{p}_{lb}}{\partial t} \int_{-\infty}^{t} \tilde{p}_{lb}^* \, d\tau \right\}$$

$$+ \frac{1}{2} \Re \left\{ \frac{\partial \tilde{p}_{la}^*}{\partial t} \int_{-\infty}^{t} \tilde{p}_{la} \, d\tau + \frac{\partial \tilde{p}_{la}^*}{\partial t} \int_{-\infty}^{t} \tilde{p}_{la}^* \, d\tau \right\},$$

(F.6)

Then components of $p''_1$ due to self-interactions and cross-interactions are

$$p''_1 = p''_{1,aa} + p''_{1,aa*} + p''_{1,bb} + p''_{1,bb*} + p''_{1,ab} + p''_{1,ab*},$$

(F.7)
where

\[
\begin{align*}
  p_{11,aa}''(r, t) &= -\Re \left\{ \frac{\tilde{p}_{1a}(r, t)\tilde{p}_{1a}(r, t)}{2A} \right\} - \Re \left\{ \frac{1}{2A} \frac{\partial \tilde{p}_{1a}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1a}(r, \tau) d\tau \right\}, \quad (F.8) \\
  p_{11,aa}''(r, t) &= -\Re \left\{ \frac{\tilde{p}_{1a}(r, t)\tilde{p}_{1a}^*(r, t)}{2A} \right\} - \Re \left\{ \frac{1}{2A} \frac{\partial \tilde{p}_{1a}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1a}^*(r, \tau) d\tau \right\}, \quad (F.9) \\
  p_{11,bb}''(r, t) &= -\Re \left\{ \frac{\tilde{p}_{1b}(r, t)\tilde{p}_{1b}(r, t)}{2A} \right\} - \Re \left\{ \frac{1}{2A} \frac{\partial \tilde{p}_{1b}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1b}(r, \tau) d\tau \right\}, \quad (F.10) \\
  p_{11,bb}''(r, t) &= -\Re \left\{ \frac{\tilde{p}_{1b}(r, t)\tilde{p}_{1b}^*(r, t)}{2A} \right\} - \Re \left\{ \frac{1}{2A} \frac{\partial \tilde{p}_{1b}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1b}^*(r, \tau) d\tau \right\}, \quad (F.11) \\
  p_{11,ab}''(r, t) &= -\Re \left\{ \frac{\tilde{p}_{1a}(r, t)\tilde{p}_{1b}(r, t)(1 + \cos \theta)}{2A} \right\} \\
  &\quad - \Re \left\{ \frac{1}{2A} \left( \frac{\partial \tilde{p}_{1a}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1b}(r, \tau) d\tau + \frac{\partial \tilde{p}_{1b}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1a}(r, \tau) d\tau \right) \right\}, \quad (F.12) \\
  p_{11,ab}''(r, t) &= -\Re \left\{ \frac{\tilde{p}_{1a}(r, t)\tilde{p}_{1b}^*(r, t)(1 + \cos \theta)}{2A} \right\} \\
  &\quad - \Re \left\{ \frac{1}{2A} \left( \frac{\partial \tilde{p}_{1a}(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1b}^*(r, \tau) d\tau + \frac{\partial \tilde{p}_{1b}^*(r, t)}{\partial t} \int_{-\infty}^{t} \tilde{p}_{1a}(r, \tau) d\tau \right) \right\}. \quad (F.13)
\end{align*}
\]

For the collinear case, \( p_{11,aa}'' \), \( p_{11,aa}''^* \), \( p_{11,bb}'' \), \( p_{11,bb}''^* \), \( p_{11,ab}'' \) and \( p_{11,ab}''^* \) for both self-interactions and cross-interactions appear as plane waves propagating with the primary plane waves. For the non-collinear case, self-interaction components \( p_{11,aa}'' \) and \( p_{11,aa}''^* \) for \( \alpha = a, b \) are the same as those in the collinear case, while cross-interaction components \( p_{11,ab}'' \) and \( p_{11,ab}''^* \), are nonzero only in the union of two primary waves through which intersection occurred.
F.1 Second order field for collinear plane waves of arbitrary time dependence, equation (3.31)

Let \( \tilde{p}_{1a} \) and \( \tilde{p}_{1b} \) be two collinear plane waves of arbitrary time dependence propagating in the positive \( \hat{z} \) direction

\[
\tilde{p}_{1a}(r, t) = \tilde{p}_{1a}(t - z/c_0), \quad (F.14)
\]
\[
\tilde{p}_{1b}(r, t) = \tilde{p}_{1b}(t - z/c_0). \quad (F.15)
\]

The second order fields \( p'_{11,ab}(x, z', t) = \Re \{ \tilde{p}'_{11,ab}(x, z', t) \} \) and \( p'_{11,ab^*}(x, z', t) = \Re \{ \tilde{p}'_{11,ab^*}(x, z', t) \} \) due to cross_interaction must satisfy

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \tilde{p}_{11,ab}(z, t) = -\frac{\beta}{A c_0^2} \frac{\partial^2}{\partial t^2} \left[ \tilde{p}_{1a}(t - z/c_0)\tilde{p}_{1b}(t - z/c_0) \right], \quad (F.16a)
\]
\[
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \tilde{p}'_{11,ab^*}(z, t) = -\frac{\beta}{A c_0^2} \frac{\partial^2}{\partial t^2} \left[ \tilde{p}_{1a}(t - z/c_0)\tilde{p}'_{1b}(t - z/c_0) \right]. \quad (F.16b)
\]

Let \( \tilde{p}_{1a}(t) \leftrightarrow \Psi_a(\omega) \) and \( \tilde{p}_{1b}(t) \leftrightarrow \Psi_b(\omega) \), so that \( \tilde{p}_{1a}(t - z/c_0) \leftrightarrow e^{i\omega z/c_0} \Psi_a(\omega) \), \( \tilde{p}_{1b}(t - z/c_0) \leftrightarrow e^{i\omega z/c_0} \Psi_b(\omega) \) and \( \tilde{p}'_{1b}(t - z/c_0) \leftrightarrow e^{i\omega z/c_0} \Psi_b^*(-\omega) \). Fourier transformation of equations (F.16a) and (F.16b) leads to the Helmholtz equations

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c_0^2} \right) \tilde{P}'_{11,ab}(z, \omega) = \frac{\beta \omega^2}{A c_0^2} e^{i\omega z/c_0} \Psi_a(\omega) * \Psi_b(\omega), \quad (F.17a)
\]
\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c_0^2} \right) \tilde{P}'_{11,ab^*}(z, \omega) = \frac{\beta \omega^2}{A c_0^2} e^{i\omega z/c_0} \Psi_a(\omega) * \Psi_b^*(-\omega), \quad (F.17b)
\]

where \( \tilde{P}'_{11,ab}(z, \omega) \leftrightarrow \tilde{p}'_{11,ab}(z, t) \), \( \tilde{P}'_{11,ab^*}(z, \omega) \leftrightarrow \tilde{p}'_{11,ab^*}(z, t) \), and \( * \) is the convolution over frequency \( \omega \). The solutions to equations (F.17a) and (F.17b) can be found as

\[
\tilde{P}'_{11,ab}(z, \omega) = -\frac{i\omega \beta}{2A c_0} \Psi_a(\omega) * \Psi_b(\omega) e^{i\omega z/c_0}, \quad (F.18a)
\]
\[
\tilde{P}'_{11,ab^*}(z, \omega) = \frac{i\omega \beta}{2A c_0} \Psi_a(\omega) * \Psi_b^*(-\omega) e^{i\omega z/c_0}. \quad (F.18b)
\]

Taking the inverse Fourier transform of equations (F.18a) and (F.18b) and keeping
the real part yields

\[
p_{11,ab}(z,t) = \Re \left\{ \frac{\beta z}{2A c_0} \frac{\partial}{\partial t} \left[ \tilde{p}_{1a}(t - z/c_0) \tilde{p}_{1b}(t - z/c_0) \right] \right\} \tag{F.19a}
\]

\[
p_{11,ab}^*(z,t) = \Re \left\{ \frac{\beta z}{2A c_0} \frac{\partial}{\partial t} \left[ \tilde{p}_{1a}(t - z/c_0) \tilde{p}_{1b}^*(t - z/c_0) \right] \right\} \tag{F.19b}
\]

By adding equations (F.12) and (F.13) into equations (F.19a) and (F.19b) respectively, we obtain the complete solution for the cross interaction of collinear plane waves of arbitrary time dependence as

\[
p_{11,ab}(z,t) = \Re \left\{ \frac{\beta z}{2A c_0} \frac{\partial}{\partial t} \left[ \tilde{p}_{1a}(t - z/c_0) \tilde{p}_{1b}(t - z/c_0) \right] \right\} - \Re \left\{ \frac{1}{2A} \tilde{p}_{1a}(t - z/c_0) \tilde{p}_{1b}(t - z/c_0) \right\} - \Re \left\{ \frac{1}{2A} \left( \frac{\partial}{\partial t} \tilde{p}_{1a}(t - z/c_0) \right) \int_{-\infty}^{t-z/c_0} \tilde{p}_{1b}(\tau) d\tau \right\} - \Re \left\{ \frac{1}{2A} \left( \frac{\partial}{\partial t} \tilde{p}_{1b}(t - z/c_0) \right) \int_{-\infty}^{t-z/c_0} \tilde{p}_{1a}(\tau) d\tau \right\}, \tag{F.20a}
\]

\[
p_{11,ab}^*(z,t) = \Re \left\{ \frac{\beta z}{2A c_0} \frac{\partial}{\partial t} \left[ \tilde{p}_{1a}(t - z/c_0) \tilde{p}_{1b}^*(t - z/c_0) \right] \right\} - \Re \left\{ \frac{1}{2A} \tilde{p}_{1a}(t - z/c_0) \tilde{p}_{1b}^*(t - z/c_0) \right\} - \Re \left\{ \frac{1}{2A} \left( \frac{\partial}{\partial t} \tilde{p}_{1a}(t - z/c_0) \right) \int_{-\infty}^{t-z/c_0} \tilde{p}_{1b}^*(\tau) d\tau \right\} - \Re \left\{ \frac{1}{2A} \left( \frac{\partial}{\partial t} \tilde{p}_{1b}^*(t - z/c_0) \right) \int_{-\infty}^{t-z/c_0} \tilde{p}_{1a}(\tau) d\tau \right\}, \tag{F.20b}
\]

When boundary conditions are given, additional homogeneous solutions (plane waves) need to be added to the above solutions, but these homogeneous solutions are not due to the nonlinear interactions.

When \( \tilde{p}_{1a}(t) \) and \( \tilde{p}_{1b}(t) \) are narrow-band plane waves with compact support between \( t = 0 \) and \( t = T \), they can be written as the product of harmonic plane waves
with a real moving window \( w_1(t - z/c_0) \) as

\[
\tilde{p}_{1a}(t - z/c_0) = P_a e^{-i\omega_a(t - z/c_0)} w_1(t - z/c_0), \tag{F.21}
\]
\[
\tilde{p}_{1b}(t - z/c_0) = P_b e^{-i\omega_b(t - z/c_0)} w_1(t - z/c_0). \tag{F.22}
\]

For a sufficiently long and smooth window \( w_1 \), its time derivatives are negligible, such that

\[
\frac{\partial}{\partial t} \tilde{p}_{1a}(t - z/c_0) \approx -i\omega_a P_a e^{i(k_\alpha z - \omega_a t)} w_1(t - z/c_0), \tag{F.23}
\]
\[
\frac{\partial}{\partial t} \tilde{p}_{1b}(t - z/c_0) \approx -i\omega_b P_b e^{i(k_\alpha z - \omega_b t)} w_1(t - z/c_0), \tag{F.24}
\]

and the time integrals of \( \tilde{p}_{1a} \) and \( \tilde{p}_{1b} \) can be approximated by the contributions from the end point \( t - z/c_0 \) [43], such that

\[
\int_{-\infty}^{t-z/c_0} e^{-i\omega_a \tau} w_1(\tau) d\tau \approx \frac{1}{-i\omega_a} w_1(t - z/c_0) e^{-i\omega_a(t - z/c_0)}, \tag{F.25}
\]
\[
\int_{-\infty}^{t-z/c_0} e^{-i\omega_b \tau} w_1(\tau) d\tau \approx \frac{1}{-i\omega_b} w_1(t - z/c_0) e^{-i\omega_b(t - z/c_0)}. \tag{F.26}
\]

Substituting equations (F.23) - (F.26) into equations (F.19a) and (F.19b) yields

\[
p_{11,ab}(z, t) \approx \Re \left\{ -\frac{P_a P_b}{2A} \left[ i\beta k_\pm z + \frac{\omega_\pm^2}{\omega_a \omega_b} \right] e^{i(k_\pm z - \omega_\pm t)} w_1^2(t - z/c_0) \right\}, \tag{F.27a}
\]
\[
p_{11,ab^*}(z, t) \approx \Re \left\{ -\frac{P_0^* P_0}{2A} \left[ i\beta k_\pm z - \frac{\omega_\pm^2}{\omega_a \omega_b} \right] e^{i(k_\pm z - \omega_\pm t)} w_1^2(t - z/c_0) \right\}. \tag{F.27b}
\]

where the terms linear in \( z \) in equations (F.27a) and (F.27b) correspond to Lamb's respective sum and difference frequency second order fields caused by the interaction of two collinear harmonic plane waves [9].

By setting \( \tilde{p}_{1b} = \tilde{p}_{1a} \) in equations (F.19a) and (F.19b) then dividing the results by two, we can obtain the self-interaction components \( p_{11,aa} \) and \( p_{11,aa^*} \) respectively. Similarly by setting \( \tilde{p}_{1a} = \tilde{p}_{1b} \) in equations (F.19a) and (F.19b) then dividing by 2, we can obtain the self-interaction components \( p_{11,bb} \) and \( p_{11,bb^*} \), respectively. It can be seen that all self-interactions components are non-zero only inside the compact
support of the corresponding primary waves.

F.2 Second order field for non-collinear plane waves of arbitrary time dependence, equation (3.33)

Let $\tilde{p}_{ia}$ and $\tilde{p}_{ib}$ be two non-collinear plane waves of arbitrary time dependence propagating in $\hat{i}_x$ and $\hat{i}_z$ directions respectively,

$$\tilde{p}_{ia}(x, z', t) = \tilde{p}_{ia}(t - x/c_0), \quad \text{ (F.28)}$$
$$\tilde{p}_{ib}(x, z', t) = \tilde{p}_{ib}(t - z'/c_0), \quad \text{ (F.29)}$$

where $z' = x \cos \theta + z \sin \theta$ and $\theta \neq 0$ is the angle between $\hat{i}_x$ and $\hat{i}_z$.

The second order fields $p'_{1ab}(x, z', t) = \Re \{\tilde{p}'_{1ab}(x, z', t)\}$ and $p'_{1ab^*}(x, z', t) = \Re \{\tilde{p}'_{1ab^*}(x, z', t)\}$ due to cross-interactions must satisfy

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}\right) \tilde{p}'_{1ab}(x, z', t) = -\beta \frac{\partial^2}{\partial t^2} \left[\tilde{p}_{ia}(t - x/c_0)\tilde{p}_{ib}(t - z'/c_0)\right], \quad \text{ (F.30a)}$$
$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}\right) \tilde{p}'_{1ab^*}(x, z', t) = -\beta \frac{\partial^2}{\partial t^2} \left[\tilde{p}_{ia}(t - x/c_0)\tilde{p}_{ib}(t - z'/c_0)\right]. \quad \text{ (F.30b)}$$

Let $\tilde{p}_{ia}(t) \leftrightarrow \Psi_a(\omega)$ and $\tilde{p}_{ib}(t) \leftrightarrow \Psi_b(\omega)$, it follows that $\tilde{p}_{ib}(t) \leftrightarrow \Psi^*(-\omega)$. Fourier transformation of equations (F.30a) and (F.30b) leads to the Helmholtz equations

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right) \tilde{P}'_{1ab}(x, z', \omega) =$$
$$\frac{\beta \omega^2}{A_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega x/c_0} \Psi_a(\Omega) e^{i(\omega - \Omega)z'/c_0} \Psi_b([-\omega - \Omega]) d\Omega, \quad \text{ (F.31a)}$$

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right) \tilde{P}'_{1ab^*}(x, z', \omega) =$$
$$\frac{\beta \omega^2}{A_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega x/c_0} \Psi_a(\Omega) e^{i(\omega - \Omega)z'/c_0} \Psi_b^*(-\omega) d\Omega, \quad \text{ (F.31b)}$$

where $\tilde{P}'_{1ab}(x, z', \omega) \leftrightarrow \tilde{P}_{1ab}(x, z', t)$, $\tilde{P}'_{1ab^*}(x, z', \omega) \leftrightarrow \tilde{P}'_{1ab^*}(x, z', t)$, and the convolution theorem is used. Equations (F.31a) and (F.31b) can be solved analytically.
Spatial Fourier transformation of (F.31a) and (F.31b) leads to

\[
\left( -k_z^2 - k_z^2 + \frac{\omega^2}{c_0^2} \right) \int_{-\infty}^{\infty} \hat{P}_{11,ab}(x, z', \omega) e^{jk_x x + ik_z z} dx \, dz = \\
\beta \frac{\omega^2}{A c_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\Omega x / c_0} \hat{\Psi}_a(\Omega) e^{i(\omega - \Omega)z' / c_0} \hat{\Psi}_b[(\omega - \Omega)] d\Omega e^{ik_z z} dx dz,
\]

(F.32a)

\[
\left( -k_z^2 - k_z^2 + \frac{\omega^2}{c_0^2} \right) \int_{-\infty}^{\infty} \hat{P}_{11,ab'}(x, z', \omega) e^{ik_z z} dx \, dz = \\
\beta \frac{\omega^2}{A c_0^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\Omega x / c_0} \hat{\Psi}_a(\Omega) e^{i(\omega - \Omega)z' / c_0} \hat{\Psi}_b^*[-(\omega - \Omega)] d\Omega e^{ik_z z} dx dz.
\]

(F.32b)

Dividing by \((-k_z^2 - k_z^2 + \omega^2/c_0^2)\) and taking the inverse spatial Fourier transform of equations (F.32a) and (F.32b) leads to

\[
\int_{-\infty}^{\infty} \hat{p}_{11,ab}(x, z', t) e^{i\omega t} dt = \frac{\beta \omega^2}{A} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega x / c_0} \hat{\Psi}_a(\Omega) e^{i(\omega - \Omega)z' / c_0} \hat{\Psi}_b[(\omega - \Omega)] d\Omega
\]

(F.33a)

\[
\int_{-\infty}^{\infty} \hat{p}_{11,ab'}(x, z', t) e^{i\omega t} dt = \frac{\beta \omega^2}{A} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega x / c_0} \hat{\Psi}_a(\Omega) e^{i(\omega - \Omega)z' / c_0} \hat{\Psi}_b^*[-(\omega - \Omega)] d\Omega
\]

(F.33b)

With the aid of partial fractions, the above expressions become

\[
\int_{-\infty}^{\infty} \hat{p}_{11,ab}(x, z', t) e^{i\omega t} dt = \frac{\beta \omega e^{i\omega z' / c_0}}{2A(1 - \cos \theta)} \times \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega x / c_0} \hat{\Psi}_a(\Omega) \hat{\Psi}_b(\omega - \Omega) \left( \frac{1}{\Omega} + \frac{1}{\omega - \Omega} \right) d\Omega,
\]

(F.34a)

\[
\int_{-\infty}^{\infty} \hat{p}_{11,ab}(x, z', t) e^{i\omega t} dt = \frac{\beta \omega e^{i\omega z' / c_0}}{2A(1 - \cos \theta)} \times \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega x / c_0} \hat{\Psi}_a(\Omega) \hat{\Psi}_b^*[-(\omega - \Omega)] \left( \frac{1}{\Omega} + \frac{1}{\omega - \Omega} \right) d\Omega.
\]

(F.34b)
Taking the inverse Fourier transform and keeping the real part yields

\[
p_1^{1,ab}(x, z', t) = \Re \left\{ \frac{i\beta}{2A(1 - \cos \theta)} \frac{\partial}{\partial t} \left[ I_{ab}^{(1)}(x, z', t) + I_{ab}^{(2)}(x, z', t) \right] \right\}, \tag{F.35a}
\]

\[
p_1^{1,ab}(x, z', t) = \Re \left\{ \frac{i\beta}{2A(1 - \cos \theta)} \frac{\partial}{\partial t} \left[ I_{ab}^{(1)}(x, z', t) + I_{ab}^{(2)}(x, z', t) \right] \right\}, \tag{F.35b}
\]

where

\[
I_{ab}^{(1)}(x, z', t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega(x-z'/c_0)} \Psi_a(\Omega) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(\omega - \Omega)}{\omega - \Omega} e^{-i\omega(t-z'/c_0)} d\omega \right] d\Omega, \tag{F.36a}
\]

\[
I_{ab}^{(2)}(x, z', t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega(x-z'/c_0)} \Psi_a(\Omega) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(\omega - \Omega)}{\omega - \Omega} e^{-i\omega(t-z'/c_0)} d\omega \right] d\Omega, \tag{F.36b}
\]

\[
I_{ab}^{(1)}(x, z', t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega(x-z'/c_0)} \Psi_a(\Omega) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(\omega - \Omega)}{\omega - \Omega} e^{-i\omega(t-z'/c_0)} d\omega \right] d\Omega, \tag{F.36c}
\]

\[
I_{ab}^{(2)}(x, z', t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega(x-z'/c_0)} \Psi_a(\Omega) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(\omega - \Omega)}{\omega - \Omega} e^{-i\omega(t-z'/c_0)} d\omega \right] d\Omega, \tag{F.36d}
\]

after switching the order of integration.

The bracketed integrals in equations (F.36a) and (F.36c) are inverse Fourier transforms that can be written as \(e^{-i\Omega(t-z'/c_0)}\hat{\rho}_b(t - z'/c_0)\) and \(e^{-i\Omega(t-z'/c_0)}\hat{\rho}_b(t - z'/c_0)\), respectively. Then \(I_{ab}^{(1)}\) and \(I_{ab}^{(1)}\) become

\[
I_{ab}^{(1)}(x, z', t) = \hat{\rho}_b(t - z'/c_0)J_a(t - x/c_0), \tag{F.37a}
\]

\[
I_{ab}^{(1)}(x, z', t) = \hat{\rho}_b(t - z'/c_0)J_a(t - x/c_0), \tag{F.37b}
\]

where

\[
J_a(t - x/c_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_a(\Omega)}{\Omega} e^{-i\Omega(t-x/c_0)} d\Omega \tag{F.38}
\]

is the inverse Fourier transform of \(\Psi_a(\Omega)/\Omega\), evaluated at \(t - x/c_0\). It follows from
the integration property of Fourier transforms [44] that

\[ J_a(t - x / c_0) = -i \int_{-\infty}^{t-x/c_0} \bar{\tilde{p}}_a(\tau) d\tau + \frac{i\Psi_a(0)}{2}, \]  

(F.39)

and the time derivative of \( J_a \) is

\[ \frac{\partial}{\partial t} J_a(t - x / c_0) = -i \tilde{p}_a(t - x / c_0). \]  

(F.40)

The procedure to simplify \( I_{ab}^{(2)}(t) \) and \( I_{ab}^{(2)}(t) \) is similar. Substituting \( \eta = \omega - \Omega \) in the bracketed integrals of equations (F.36b) and (F.36d) leads to

\[ I_{ab}^{(2)}(x, z', t) = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_a(\Omega) e^{-i\Omega(t-x/c_0)} d\Omega \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(\eta)}{\eta} e^{-in(t-z'/c_0)} d\eta \right], \]  

(F.41a)

\[ I_{ab}^{(2)}(x, z', t) = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_a(\Omega) e^{-i\Omega(t-x/c_0)} d\Omega \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(-\eta)}{\eta} e^{-in(t-z'/c_0)} d\eta \right], \]  

(F.41b)

which simplify to

\[ I_{ab}^{(2)}(x, z', t) = \tilde{p}_{ab}(t - x / c_0) J_b(t - z'/c_0), \]  

(F.42a)

\[ I_{ab}^{(2)}(x, z', t) = \tilde{p}_{ab}(t - x / c_0) J_b^*(t - z'/c_0), \]  

(F.42b)

where

\[ J_b(t - z'/c_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b(\eta)}{\eta} e^{-in(t-z'/c_0)} d\eta, \]  

(F.43a)

\[ J_b^*(t - z'/c_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi_b^*(-\eta)}{\eta} e^{-in(t-z'/c_0)} d\eta. \]  

(F.43b)

It can be seen that \( J_b \) has the same form as \( J_a \) in equation (F.38) and \( J_b^* = -J_b^* \).

With the integration property of Fourier transforms, equations (F.43a) and (F.43b)
become

\[ J_b(t - z'/c_0) = -i \int_{-\infty}^{t-z'/c_0} \tilde{P}_{ib}(\tau) d\tau + \frac{i\Psi_b(0)}{2}, \quad (F.44a) \]

\[ J_{b*}(t - z'/c_0) = -i \int_{-\infty}^{t-z'/c_0} \tilde{P}_{ib}^*(\tau) d\tau + \frac{i\Psi_{b*}(0)}{2}. \quad (F.44b) \]

The time derivatives of \( J_b \) and \( J_{b*} \) are

\[ \frac{\partial}{\partial t} J_b(t - z'/c_0) = -i \tilde{P}_{ib}(t - z'/c_0), \quad (F.45a) \]

\[ \frac{\partial}{\partial t} J_{b*}(t - z'/c_0) = -i \tilde{P}_{ib}^*(t - z'/c_0), \quad (F.45b) \]

Substituting equations (F.37), (F.39), (F.42) and (F.44) into equations (F.35a) and (F.35b) yields particular solutions

\[ P'_{1,ab}(x, z', t) = \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \left[ \frac{\partial}{\partial t} \tilde{P}_{ia}(t - x/c_0) \right] \left[ -i \int_{-\infty}^{t-x/c_0} \tilde{P}_{ib}(\tau) d\tau + \frac{i\Psi_a(0)}{2} \right] \right\} \]

\[ + \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \left[ \frac{\partial}{\partial t} \tilde{P}_{ia}(t - x/c_0) \right] \left[ -i \int_{-\infty}^{t-x/c_0} \tilde{P}_{ib}(\tau) d\tau + \frac{i\Psi_a(0)}{2} \right] \right\} \]

\[ + \Re \left\{ \frac{\beta}{A_1 - \cos \theta} \tilde{P}_{ia}(t - x/c_0) \tilde{P}_{ib}(t - z'/c_0) \right\}, \quad (F.46a) \]

\[ P'_{1,ab*}(x, z', t) = \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \left[ \frac{\partial}{\partial t} \tilde{P}_{ia}^*(t - x/c_0) \right] \left[ -i \int_{-\infty}^{t-x/c_0} \tilde{P}_{ib}^*(\tau) d\tau + \frac{i\Psi_{a*}(0)}{2} \right] \right\} \]

\[ + \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \left[ \frac{\partial}{\partial t} \tilde{P}_{ia}^*(t - x/c_0) \right] \left[ -i \int_{-\infty}^{t-x/c_0} \tilde{P}_{ib}^*(\tau) d\tau + \frac{i\Psi_{a*}(0)}{2} \right] \right\} \]

\[ + \Re \left\{ \frac{\beta}{A_1 - \cos \theta} \tilde{P}_{ia}(t - x/c_0) \tilde{P}_{ib}^*(t - z'/c_0) \right\}. \quad (F.46b) \]

The physical solutions for \( P'_{1,ab}(x, z', t) \) and \( P'_{1,ab*}(x, z', t) \) are obtained by adding homogeneous solutions \( \Re \left\{ \psi_{ab}^{(1)}(t - x/c_0) \right\} \) and \( \Re \left\{ \psi_{ab}^{(2)}(t - z'/c_0) \right\} \) to equation (F.46a), and \( \Re \left\{ \psi_{ab}^{(1)}(t - x/c_0) \right\} \) and \( \Re \left\{ \psi_{ab}^{(2)}(t - z'/c_0) \right\} \) to equation and (F.46b) respectively, and imposing the causality condition that the second order field from intersecting plane waves must be zero in space before any wave arrives there. The homogeneous
solutions are determined as

\[ \psi_{ab}^{(1)}(t - x/c_0) = -\frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \frac{\partial}{\partial \tau} \mathcal{P}_{la}(t - x/c_0), \quad (F.47) \]

\[ \psi_{ab}^{(2)}(t - z'/c_0) = -\frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \frac{\partial}{\partial \tau} \mathcal{P}_{lb}(t - z'/c_0), \quad (F.48) \]

\[ \psi_{ab}^{(1)}(t - x/c_0) = -\frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \frac{\partial}{\partial \tau} \mathcal{P}_{la}(t - x/c_0), \quad (F.49) \]

\[ \psi_{ab}^{(2)}(t - z'/c_0) = -\frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \frac{\partial}{\partial \tau} \mathcal{P}_{lb}(t - z'/c_0), \quad (F.50) \]

such that the physical solutions for \( p^{l}_{1ab}(x, z', t) \) and \( p^{l}_{1ab}^*(x, z', t) \) are

\[
P^{l}_{1ab}(x, z', t) = \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \left[ \frac{\partial}{\partial \tau} \mathcal{P}_{lb}(t - z'/c_0) \right] \left[ -i \int_{-\infty}^{t - x/c_0} \mathcal{P}_{la}(\tau) d\tau \right] \right\} 
+ \Re \left\{ \frac{\beta}{A_1 - \cos \theta} \mathcal{P}_{la}(t - x/c_0) \mathcal{P}_{lb}(t - z'/c_0) \right\}, \quad (F.51a) \]

\[
P^{l}_{1ab}^*(x, z', t) = \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \left[ \frac{\partial}{\partial \tau} \mathcal{P}_{lb}(t - z'/c_0) \right] \left[ -i \int_{-\infty}^{t - x/c_0} \mathcal{P}_{la}(\tau) d\tau \right] \right\} 
+ \Re \left\{ \frac{i\beta}{2A_1 - \cos \theta} \frac{1}{2} \left[ \frac{\partial}{\partial \tau} \mathcal{P}_{la}(t - x/c_0) \right] \left[ -i \int_{-\infty}^{t - x/c_0} \mathcal{P}_{lb}(\tau) d\tau \right] \right\} 
+ \Re \left\{ \frac{\beta}{A_1 - \cos \theta} \mathcal{P}_{la}(t - x/c_0) \mathcal{P}_{lb}^*(t - z'/c_0) \right\}. \quad (F.51b) \]

By adding equations (F.12) and (F.13) into equations (F.51a) and (F.51b) respectively, we obtain the complete solution for the cross interaction of non-collinear plane
waves of arbitrary time dependence

\[ p_{11,ab}(x, z', t) = \mathcal{R} \left\{ \frac{1}{2A} \left[ \frac{\beta}{1 - \cos \theta} - 1 \right] \left[ \frac{\partial}{\partial t} \tilde{p}_{1a}(t - x/c_0) \right] \left[ \int_{-\infty}^{t - z'/c_0} \tilde{p}_{1a}(\tau) d\tau \right] \right\} 
\]

\[ + \mathcal{R} \left\{ \frac{1}{2A} \left[ \frac{\beta}{1 - \cos \theta} - 1 \right] \left[ \frac{\partial}{\partial t} \tilde{p}_{1b}(t - z'/c_0) \right] \left[ \int_{-\infty}^{t - z'/c_0} \tilde{p}_{1b}(\tau) d\tau \right] \right\} 
\]

\[ + \mathcal{R} \left\{ \frac{1}{2A} \left[ \frac{2\beta}{1 - \cos \theta} - (1 + \cos \theta) \right] \tilde{p}_{1a}(t - x/c_0)\tilde{p}_{1b}(t - z'/c_0) \right\}, \quad \text{(F.52a)} 
\]

\[ p_{11,ab^*}(x, z', t) = \mathcal{R} \left\{ \frac{1}{2A} \left[ \frac{\beta}{1 - \cos \theta} - 1 \right] \left[ \frac{\partial}{\partial t} \tilde{p}_{1b}^*(t - x/c_0) \right] \left[ \int_{-\infty}^{t - z'/c_0} \tilde{p}_{1a}^*(\tau) d\tau \right] \right\} 
\]

\[ + \mathcal{R} \left\{ \frac{1}{2A} \left[ \frac{\beta}{1 - \cos \theta} - 1 \right] \left[ \frac{\partial}{\partial t} \tilde{p}_{1b}(t - x/c_0) \right] \left[ \int_{-\infty}^{t - z'/c_0} \tilde{p}_{1b}(\tau) d\tau \right] \right\} 
\]

\[ + \mathcal{R} \left\{ \frac{1}{2A} \left[ \frac{2\beta}{1 - \cos \theta} - (1 + \cos \theta) \right] \tilde{p}_{1a}(t - x/c_0)\tilde{p}_{1b}^*(t - z'/c_0) \right\}. \quad \text{(F.52b)} 
\]

When boundary conditions are given, additional homogeneous solutions need to be added to the above solutions, but these homogeneous solutions are not due to the nonlinear interactions.

It can be seen from equations (F.52a) and (F.52b) that sum and difference frequency components due to cross-interaction between two intersecting plane waves \( p_{1a}(t - x/c_0) \) and \( p_{1b}(t - z'/c_0) \) with compact support between \( t = 0 \) and \( t = T \) only exist in the region of compact support intersection (purple region in figure 3-2), so that no scattering of sound by sound at sum or difference frequencies is found outside the region of compact support intersection to second order. There will be a component at the primary frequency of a given plane wave (\( p_{1a} \) or \( p_{1b} \)) due to cross-interaction only where the given primary field exists and the intersecting plane wave (\( p_{1b} \) or \( p_{1a} \)) has passed through (hatched region in figure 3-2), if the intersecting plane wave has a non-zero zero-frequency component.

There will be non-zero \( p_{11,aa} \) and \( p_{11,ab^*} \) inside the compact support of \( p_{1a} \) due to self-interaction, which gives rise to sum and difference frequency components from \( p_{1a} \). Similarly, there will be non-zero \( p_{11,bb} \) and \( p_{11,bb^*} \) inside the compact support of \( p_{1b} \) due to self-interaction, which gives rise to sum and difference frequency components.
from \( p_{nb} \). There is no scattering of sound by sound due to cross-interaction or self-interaction of two finite-duration plane waves at any frequency outside the region of compact support union through which compact support intersection occurred.

When \( p_{na}(t-x/c_0) \) and \( p_{nb}(t-z'/c_0) \) are narrow-band plane waves with compact support, they can be written as the product of harmonic plane waves and real moving windows \( w_1(t-x/c_0) \) or \( w_1(t-z'/c_0) \) as

\[
\tilde{p}_{na}(t-x/c_0) = P_{a0}e^{-i\omega_a(t-x/c_0)}w_1(t-x/c_0), \quad (F.53)
\]

\[
\tilde{p}_{nb}(t-z'/c_0) = P_{b0}e^{-i\omega_b(t-z'/c_0)}w_1(t-z'/c_0). \quad (F.54)
\]

For a sufficiently long and smooth window \( w_1 \), its time derivatives are negligible, such that

\[
\frac{\partial}{\partial t}\tilde{p}_{na}(t-x/c_0) \approx -i\omega_a P_{a0}e^{i(k_a x - \omega_a t)}w_1(t-x/c_0), \quad (F.55)
\]

\[
\frac{\partial}{\partial t}\tilde{p}_{nb}(t-z'/c_0) \approx -i\omega_b P_{b0}e^{i(k_b z' - \omega_b t)}w_1(t-z'/c_0), \quad (F.56)
\]

and the time integrals of \( \tilde{p}_{na} \) and \( \tilde{p}_{nb} \) can be approximated by the contributions from the respective end points \( t-x/c_0 \) and \( t-z'/c_0 \) [43], such that

\[
\int_{-\infty}^{t-x/c_0} e^{-i\omega_a \tau}w_1(\tau)d\tau \approx \frac{1}{-i\omega_a} w_1(t-x/c_0)e^{-i\omega_a(t-x/c_0)}, \quad (F.57)
\]

\[
\int_{-\infty}^{t-z'/c_0} e^{-i\omega_b \tau}w_1(\tau)d\tau \approx \frac{1}{-i\omega_b} w_1(t-z'/c_0)e^{-i\omega_b(t-z'/c_0)}. \quad (F.58)
\]

Substituting equations (F.55) - (F.58) into equations (F.52a) and (F.52b) yields

\[
p_{il,ab}(x, z', t) \approx \Re \left\{ \frac{P_{a0}P_{b0}}{2A} \left[ \left( \frac{\beta}{1-\cos \theta} - 1 \right) \frac{\omega_a^2}{\omega_a \omega_b} + 1 - \cos \theta \right] \right. \times e^{i(k_a x + k_b z' - \omega_b t)}w_1(t-x/c_0)w_1(t-z'/c_0) \left\} . \quad (F.59a)
\]

\[
p_{il,ab^*}(x, z', t) \approx \Re \left\{ \frac{P_{a0}P_{b0}^*}{2A} \left[ - \left( \frac{\beta}{1-\cos \theta} - 1 \right) \frac{\omega_a^2}{\omega_a \omega_b} + 1 - \cos \theta \right] \right. \times e^{i(k_a x - k_b z' - \omega_b t)}w_1(t-x/c_0)w_1(t-z'/c_0) \left\} . \quad (F.59b)
\]
The sum and difference frequency components due to cross-interaction of two intersecting narrow-band plane wave pulses are again seen to only exist in the region of compact support intersection of the two plane waves (purple region in figure 3-2), where they agree with Westervelt’s respective sum and difference frequency second order fields resulting from the interaction of two non-collinear time-harmonic plane waves [12]. This shows that a time-harmonic approximation can be made in the region of compact support intersection for narrow-band finite-duration plane waves with sufficiently long and smooth windows.
Appendix G

Asymptotic solution for SS interaction, equation (3.37)

We begin with a spherical object assumption to illustrate how general asymptotic expressions can be obtained for the SS interaction in terms of the far field scatter functions of any object. For a spherical object, the two scattered fields can be expressed as

\[ P_{sa}(r) = P_{a0} \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} a_{lm} h_l(k_ar)Y_l^m(\theta, \phi), \]  
\[ P_{sb}(r) = P_{b0} \sum_{k=0}^{k_{max}} \sum_{n=-k}^{k} b_{kn} h_k(k_br)Y_k^n(\theta, \phi), \]

where \( a_{lm} \) and \( b_{kn} \) are constant coefficients determined by the boundary condition.

The product \( P_{sa}P_{sb}^{(*)} \) can be expanded in spherical harmonics:

\[ P_{sa}(r)P_{sb}^{(*)}(r) = P_{a0}P_{b0}^{(*)} \sum_{L=0}^{L_{max}} \sum_{M=-L}^{L} Q_L^{M(*)}(k_ar, k_br)Y_L^M(\theta, \phi), \]

where \( Q_L^{M(*)} \) is a quadruple summation involving Clebsch-Gordan coefficients for every degree \( L \) and order \( M \) [45]. When the expansions for \( P_{sa} \) and \( P_{sb} \) are truncated at \( l_{max} \) and \( k_{max} \) respectively, the expansion for \( P_{sa}P_{sb}^{(*)} \) is limited to \( L_{max} = l_{max} + k_{max} \).
The free space Green function can also be expanded as [26]

\[
G_{\pm}(r|r_0) = i\kappa_0 \sum_{K=0}^{\infty} \sum_{N=-K}^{K} Y_{K}^{N*}(\theta_0, \phi_0) Y_{K}^{N}(\theta, \phi)j_{K}(k_{\pm} r_{<})h_{K}(k_{\pm} r_{>}), \tag{G.4}
\]

where \(j_{K}\) is the spherical Bessel function of order \(K\), \(r_{<} = \min(r, r_0)\) and \(r_{>} = \max(r, r_0)\). With the orthogonal property of the spherical harmonics, \(P_{ss\pm}'\) in equation (3.22) can be written as

\[
P_{ss\pm}'(r) = \frac{-i\beta k_{\pm}^3 P_{a0}'P_{b0}^{'(s)}}{A} \sum_{L=0}^{L_{\text{max}}} \sum_{M=-L}^{L} Y_{L}^{M}(\theta, \phi)
\times \left[h_{L}(k_{\pm} r) \int_{r_{a}}^{r_{ref}} Q_{L}^{M(*)} j_{L}(k_{\pm} r_{0})r_{0}^2dr_{0} + j_{L}(k_{\pm} r) \int_{r_{ref}}^{\infty} Q_{L}^{M(*)} h_{L}(k_{\pm} r_{0})r_{0}^2dr_{0}\right], \tag{G.5}
\]

where the first integral represents the interaction within range \(r\) and the second integral represents the interaction beyond range \(r\). Equation (G.5) is exact and can be directly evaluated analytically or numerically, yet the number of terms in \(Q_{L}^{M(*)}\) increases dramatically as \(L_{\text{max}}\) increases [45]. To proceed, we define a range \(r_{ref\pm} \leq r\) and decompose the first integral in equation (G.5) into two integrals: one from \(a\) to \(r_{ref\pm}\) and one from \(r_{ref\pm}\) to \(r\), such that

\[
P_{ss\pm}' = P_{ss\pm}'^{(1)} + P_{ss\pm}'^{(2)} \tag{G.6}
\]

where

\[
P_{ss\pm}'^{(1)}(r) = \frac{-i\beta k_{\pm}^3 P_{a0}'P_{b0}^{'(s)}}{A} \sum_{L,M} Y_{L}^{M}(\theta, \phi)h_{L}(k_{\pm} r) \int_{a}^{r_{ref\pm}} Q_{L}^{M(*)} j_{L}(k_{\pm} r_{0})r_{0}^2dr_{0}, \tag{G.7}
\]

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and

\[ P_{SS\pm}^{(3)}(r) = -\frac{i\beta k^3}{A} \sum_{L,M} Y_L^M(\theta, \phi) \]

\[ \times \left[ h_L(k\pm r) \int_{r_{\text{ref}}}^r Q_L^{M(*)} j_L(k\pm r_0) r_0^2 dr_0 + j_L(k\pm r) \int_r^\infty Q_L^{M(*)} h_L(k\pm r_0) r_0^2 dr_0 \right]. \quad (G.8) \]

The \( P_{SS\pm}^{(1)} \) term of equation (G.7) can be numerically evaluated because the integration domain is finite (from \( a \) to \( r_{\text{ref}} \)). For any finite \( r_{\text{ref}} \), \( P_{SS\pm}^{(1)} \) falls off by \( r^{-1} \) as \( r \to \infty \) so becoming small compared to \( P_{SS\pm}^{(2)} \).

### G.1 Spherical wave expansion and approximation to \( P_{SS\pm}^{(2)} \)

Let \( r_a \) and \( r_b \) be the far field ranges for the primary scattered fields \( P_{Sa} \) and \( P_{Sb} \), respectively, such that beyond \( r_a \) and \( r_b \), far field approximations [30] apply

\[ P_{Sa}(r) = \frac{P_{a0} S_a(\hat{i}_r)}{k_a}, \quad (G.9) \]

\[ P_{Sb}(r) = \frac{P_{b0} S_b(\hat{i}_r)}{k_b}, \quad (G.10) \]

where \( r_{a,b} = l^2/\lambda_{a,b}, k_a r_a, k_b r_b \gg 1, l \) is the length scale of the object, \( \lambda_{a,b} \) are the wavelengths of the incident fields, \( P_{a0} \) and \( P_{b0} \) are the amplitudes of the incident fields, \( S_a \) and \( S_b \) are the far field scatter functions, and \( \hat{i}_r = r/r \).

The product \( S_a S_b(*) \) can be expanded in spherical harmonics with coefficients \( q_L^{M(*)} \) as

\[ S_a(\theta, \phi) S_b(*) (\theta, \phi) = \sum_{L=0}^\infty \sum_{M=-L}^L q_L^{M(*)} Y_L^M(\theta, \phi). \quad (G.11) \]
Comparing equation (G.3) with equation (G.11) for \( r \geq r_{\text{ref}} \), we have

\[
Q^{M(*)}_L = \frac{e^{ikr}}{k_{a_k} k_{b_k} r^2} q^{M(*)}_L , \quad \text{for } L = 0, 1, \cdots, L_{\text{max}}. \tag{G.12}
\]

When \( k_{\pm} r_{\text{ref}} \gg 1 \), the spherical Bessel and Hankel functions follow their asymptotic behaviors [31] for \( r > r_{\text{ref}} \)

\[
j_L(k_{\pm} r) \approx \frac{e^{i(k_{\pm} r - L\pi/2)} - e^{-i(k_{\pm} r - L\pi/2)}}{2ik_{\pm} r}, \tag{G.13}
\]

\[
h_L(k_{\pm} r) \approx \frac{e^{i(k_{\pm} r - L\pi/2)}}{ik_{\pm} r}. \tag{G.14}
\]

Substituting equations (G.12)-(G.14) into equation (G.8) yields

\[
P^{(2)}_{\text{SS}}(r) = \frac{i\beta k^2 p_{ao} p_{bo}}{2Ak_a k_{b_k} r} \left[ e^{ikr} \int_{r_{\text{ref}}}^{\infty} \frac{e^{ikr}}{r} dr_0 \left( \sum_{L,M} q^{M(*)}_L Y^M_L (\theta, \phi) (-1)^L \right) \right.
\]

\[\left. - e^{-ikr} \int_{r_{\text{ref}}}^{r} \frac{dr_0}{r} \left( \sum_{L,M} q^{M(*)}_L Y^M_L (\theta, \phi) \right) \right. \]

\[\left. - e^{-ikr} \int_{r_{\text{ref}}}^{\infty} \frac{e^{ikr}}{r} dr_0 \left( \sum_{L,M} q^{M(*)}_L Y^M_L (\theta, \phi) \right) \right]. \tag{G.15}
\]

Since \( \sum_{L,M} q^{M(*)}_L Y^M_L (\theta, \phi) = S_a(\theta, \phi) S\bar{b}^{(*)}(\theta, \phi) = S_a(\hat{i}_r) S\bar{b}^{(*)}(\hat{i}_r) \) as given by equation (G.11) where \( \hat{i}_r := r/r \), it can be shown, by using the relation \( P^M_L(-x) = (-1)^{L+M} P^M_L(x) \) for associated Legendre function \( P^M_L \), that \( \sum_{L,M} (-1)^L q^{M(*)}_L Y^M_L (\theta, \phi) = S_a(\pi - \theta, \pi + \phi) S\bar{b}^{(*)}(\pi - \theta, \pi + \phi) = S_a(-\hat{i}_r) S\bar{b}^{(*)}(-\hat{i}_r) \). Equation (G.15) then becomes identical to equation (3.37).

### G.2 Stationary phase approximation to \( P^{(2)}_{\text{SS}} \)

Without loss of generality, we choose a spherical coordinate system \((r_0, \alpha_0, \beta_0)\) such that the zenith direction coincides with \( r \). In equation (G.8), \( P^{(2)}_{\text{SS}} \) can be rewritten
in this coordinate system as

\[ P_{SS}^{(2)}(r) = -\frac{\beta k_2^2}{A} \int_{r_0 \geq r_{ref}} P_{Sa}(r_0, \alpha_0, \beta_0) P_{Sb}^{(s)}(r_0, \alpha_0, \beta_0) \]
\[ \times \frac{e^{ik_\pm R}}{4\pi R} \int_0^{2\pi} r_0 dr_0 \sin \alpha_0 d\alpha_0 d\beta_0. \]  

(G.16)

where \( \exp(ik_\pm R)/(4\pi R) \) is the free space Green function and \( R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \alpha_0} \).

To remove the singularity in the Green function when \( R = 0 \), we use \( R \) instead of \( \alpha_0 \) as dummy variable, then

\[ P_{SS}^{(2)}(r) = -\frac{\beta k_2^2}{4\pi Ar} \int_{r_{ref}}^{\infty} r_0 dr_0 \int_0^{2\pi} d\eta_0 \]
\[ \times \int_{|r-r_0|} e^{ik_\pm R} \hat{P}_{Sa}(r_0, R, \beta_0) \hat{P}_{Sb}^{(s)}(r_0, R, \beta_0) e^{ik_\pm R} dR. \]  

(G.17)

Assuming \( k_\pm r \geq k_\pm r_{ref} \gg 1 \), and \( \hat{P}_{Sa} \) and \( \hat{P}_{Sb} \) are slow varying functions in \( R \), we find that the integrand for the \( \xi_0 \) integral in equation (G.17) is rapidly oscillating with a linear function of \( R \) in its phase, suggesting that the leading order contribution comes from the two end points [43] at \( R = |r-r_0| \) and \( R = |r+r_0| \). Integrating over \( R \) then \( \beta_0 \) yields

\[ P_{SS}^{(2)}(r) \approx -\frac{\beta k_+}{2iAr} \left[ e^{ik_+ r} \int_{r_{ref}}^{\infty} P_{Sa}(r_0, -\hat{i}_r) P_{Sb}^{(s)}(r_0, -\hat{i}_r) e^{ik_\pm r_0} r_0 dr_0 \right. \]
\[ - e^{ik_+ r} \int_{r_{ref}}^{r} P_{Sa}(r_0, \hat{i}_r) P_{Sb}^{(s)}(r_0, \hat{i}_r) e^{-ik_\pm r_0} r_0 dr_0 \]
\[ - e^{-ik_+ r} \int_{r}^{\infty} P_{Sa}(r_0, \hat{i}_r) P_{Sb}^{(s)}(r_0, \hat{i}_r) e^{ik_\pm r_0} r_0 dr_0 \].  

(G.18)

If \( r_{ref} \geq r_a, r_b \) so that the far field approximations for \( P_{Sa} \) and \( P_{Sb} \) apply, substituting equations (G.9) and (G.10) into equation (G.18) yields equation (3.37).
G.3 Dean’s solution

For two spherical waves $P_{Sa}$ and $P_{Sb}$ given by

$$P_{Sa}(r) = P_{a0} \frac{e^{ik_ar}}{k_ar} \quad \text{and} \quad P_{Sb}(r) = P_{b0} \frac{e^{ik_br}}{k_br}, \quad (G.19)$$

Dean [12] provided a solution of the inhomogeneous Helmholtz equation corresponding to equation (3.4) for their cross-interaction outside a sphere of radius $a$

$$P_{\pm}^{Dean} = -\frac{i\beta k_{\pm}}{2Ak_{a}k_{b}}P_{a0}P_{b0}^{(\ast)} \frac{e^{ik_{\pm}r}}{r} \left[ \log \left( \frac{r}{a} \right) - e^{-2ik_{\pm}r} \int_{a}^{r} \frac{e^{2ik_{\pm}r_{0}}}{r_{0}} dr_{0} \right]. \quad (G.20)$$

This solution does not satisfy the Sommerfeld radiation condition [26] in the far field since

$$r \left( \frac{\partial}{\partial r} - i k_{\pm} \right) P_{\pm}^{Dean} \propto e^{-i k_{\pm}r} \int_{a}^{r} \frac{e^{2ik_{\pm}r_{0}}}{r_{0}} dr_{0} + O(r^{-1}) \quad (G.21)$$

does not vanish but approaches a finite value as $r \to \infty$.

The difference between Dean’s and Baxter’s [11] solutions is a term proportional to the spherical Bessel function $j_{0}(k_{\pm}r)$. By adding $\alpha j_{0}(k_{\pm}r)$ to Dean’s solution, the resulting solution $P_{\pm}^{Dean} + \alpha j_{0}(k_{\pm}r)$ satisfies the Sommerfeld radiation condition with constant $\alpha$ given by

$$\alpha = -\frac{\beta k_{\pm}^{2}}{Ak_{a}k_{b}}P_{a0}P_{b0}^{(\ast)} \int_{a}^{\infty} \frac{e^{2ik_{\pm}r_{0}}}{r_{0}} dr_{0}. \quad (G.22)$$

The solution $P_{\pm}^{Dean} + \alpha j_{0}(k_{\pm}r)$ is then identical to Baxter’s solution.

Dean did not provide any information on how to derive his solution. Here we present a derivation based on the variation of parameters method [32]. We consider the inhomogeneous Helmholtz equation for the interaction of two spherical waves $P_{Sa}$ and $P_{Sb}$ given by equations (G.19),

$$(\nabla^{2} + k_{\pm}) P_{\pm}' = -Q_{\pm}(r), \quad (G.23)$$
This problem is spherically symmetric. The Helmholtz equation is in fact a second order inhomogeneous ordinary differential equation. According to the method of variation of parameter, a particular solution \( \hat{P}_\pm'(r) \) can be constructed from two known linearly independent homogeneous solutions \( P^{(1)}_\pm(r) \) and \( P^{(2)}_\pm(r) \), as \[ (G.24) \]

\[
\hat{P}_\pm'(r) = P^{(1)}_\pm(r) \int \frac{P^{(2)}_\pm(r)Q'_\pm(r)}{W_\pm(r)} \, dr - P^{(2)}_\pm(r) \int \frac{P^{(1)}_\pm(r)Q'_\pm(r)}{W_\pm(r)} \, dr \tag{G.25}
\]

where \( W_\pm(r) \) is the Wronskian of the two homogeneous solutions,

\[
W_\pm(r) = P^{(1)}_\pm(r) \frac{dP^{(2)}_\pm(r)}{dr} - P^{(2)}_\pm(r) \frac{dP^{(1)}_\pm(r)}{dr}. \tag{G.26}
\]

We choose the spherical Hankel functions of the first and second kind as the homogeneous solutions,

\[
P^{(1)}_\pm(r) = h_0^{(1)}(k_\pm r), \tag{G.27}
\]

\[
P^{(2)}_\pm(r) = h_0^{(2)}(k_\pm r), \tag{G.28}
\]

and their Wronskian is \( W_\pm(r) = -2i(k_\pm r)^{-2} \). Substituting equations (G.24), (G.26), (G.27) and (G.28) into equation (G.25), and changing the indefinite integral to a definite integral from \( a \) to \( r \), we obtain

\[
\hat{P}_\pm'(r) = -\frac{i\beta k_\pm P_{ao}P_{b0}^{(*)} e^{ik_\pm r}}{2Ak_\pm k_b} \int_a^r \frac{1}{r_0} \, dr_0 + \frac{i\beta k_\pm P_{ao}P_{b0}^{(*)} e^{-ik_\pm r}}{2Ak_\pm k_b} \int_a^r \frac{e^{2ik_\pm r_0}}{r_0} \, dr_0, \tag{G.29}
\]

which is identical to Dean’s solution in equation (G.20). We note that the particular solution \( \hat{P}_\pm'(r) \) obtained by this method is not unique and it depends on the choice of the homogeneous solutions \( P^{(1)}_\pm(r) \) and \( P^{(2)}_\pm(r) \). As we discussed above, the choice of \( P^{(1)}_\pm = h_0^{(1)} \) and \( P^{(2)}_\pm = h_0^{(2)} \) leads to Dean’s solution that violates the Sommerfeld
radiation condition.
Appendix H

Asymptotic solution for IS and SI interactions

Consider equations (3.23) and (3.24) with incident waves given by equation (3.7) and scattered waves given by equation (3.35) in the far field.

H.1 Forward direction, equations (3.39) - (3.41)

We first consider the case when the receiver at range $r$ is in the forward direction of the incident wave and focus on the spherically symmetric scattered waves ($S_a = S_b = 1$). Generalization to arbitrary scattered waves valid in the far field are provided later in terms of the their far field scatter functions. We define a spherical coordinate system $(r_0, \theta_0, \phi_0)$ so that the zenith direction coincides with $\hat{\mathbf{i}}_a$ or $\hat{\mathbf{i}}_b$. Equations (3.23) and (3.24) become

$$P'_{IS+}(r\hat{i}_a) = -\frac{\omega^2 \beta P_{ao} P_{bo}}{Ac_0^2} \int \int \int e^{ik_a r_0 \cos \theta_0} e^{ik_b r_0} e^{ik_+ R} \frac{r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0}{r_0^2} $$  \hspace{1cm} (H.1)

$$P'_{SI+}(r\hat{i}_b) = -\frac{\omega^2 \beta P_{ao} P_{bo}}{Ac_0^2} \int \int \int e^{ik_b r_0 \cos \theta_0} e^{ik_b r_0} e^{ik_+ R} \frac{r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0}{r_0^2} $$  \hspace{1cm} (H.2)

$$P'_{IS-}(r\hat{i}_a) = -\frac{\omega^2 \beta P_{ao} P_{bo}}{Ac_0^2} \int \int \int e^{ik_a r_0 \cos \theta_0} e^{-ik_b r_0} e^{-ik_+ R} \frac{r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0}{r_0^2} $$  \hspace{1cm} (H.3)

$$P'_{SI-}(r\hat{i}_b) = -\frac{\omega^2 \beta P_{ao} P_{bo}}{Ac_0^2} \int \int \int e^{-ik_b r_0 \cos \theta_0} e^{ik_a r_0} e^{-ik_+ R} \frac{r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0}{r_0^2} $$  \hspace{1cm} (H.4)
In the \((r_0, \theta_0, \phi_0)\) coordinate system, \(R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta_0}\), so \(r_0 \cos \theta_0 = (r^2 + r_0^2 - R^2)/(2r)\), \(r_0 \sin \theta_0 d\theta_0 = RdR/r\), and the integrals of (H.1) - (H.4) become

\[
P_{IS+}(\hat{r}_a) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i k_a r \frac{r^2 - R^2}{2r^2} + i k_0 r_0 + ik R dR dr_0}, \quad (H.5)
\]
\[
P_{SI+}(\hat{r}_b) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i k_0 r_0 + ik_0 r \frac{r^2 - R^2}{2r^2} + ik R dR dr_0}, \quad (H.6)
\]
\[
P_{IS-}(\hat{r}_a) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i k_0 r_0 - ik_0 r \frac{r^2 - R^2}{2r^2} - ik R dR dr_0}, \quad (H.7)
\]
\[
P_{SI-}(\hat{r}_b) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i k_0 r_0 - ik_0 r \frac{r^2 - R^2}{2r^2} - ik R dR dr_0}, \quad (H.8)
\]

where the integration domain is \(D = \{r_0, R : |r - r_0| \leq R \leq |r + r_0|; r_0 \geq a\} \) (figure H-3).

We define a new coordinate system \((\xi_1, \xi_2)\) as a rotation of \((r_0, R)\), such that

\[
r_0 = \frac{1}{\sqrt{2}}(\xi_1 + \xi_2), \quad (H.9)
\]
\[
R = \frac{1}{\sqrt{2}}(-\xi_1 + \xi_2), \quad (H.10)
\]
\[
r_0^2 - R^2 = 2\xi_1 \xi_2, \quad (H.11)
\]
\[
dr_0 dR = d\xi_1 d\xi_2. \quad (H.12)
\]

Equations (H.5) - (H.8) can be written as

\[
P_{IS+}(\hat{r}_a) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i(k_a \xi_1 \xi_2 + k_0 \frac{\xi_1 + \xi_2}{\sqrt{2}} + k_0 \frac{-\xi_1 + \xi_2}{\sqrt{2}})} d\xi_1 d\xi_2, \quad (H.13)
\]
\[
P_{SI+}(\hat{r}_b) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i(k_0 \frac{\xi_1 + \xi_2}{\sqrt{2}} + k_0 \frac{\xi_1 + \xi_2}{\sqrt{2}} + k_0 \frac{-\xi_1 + \xi_2}{\sqrt{2}})} d\xi_1 d\xi_2, \quad (H.14)
\]
\[
P_{IS-}(\hat{r}_a) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i(k_0 \frac{\xi_1 + \xi_2}{\sqrt{2}} + k_0 \frac{-\xi_1 + \xi_2}{\sqrt{2}} + k_0 \frac{-\xi_1 + \xi_2}{\sqrt{2}})} d\xi_1 d\xi_2, \quad (H.15)
\]
\[
P_{SI-}(\hat{r}_b) = -\frac{\omega^2 \beta P_{a0} P_{b0}}{A \gamma^2} \int_D e^{i(k_0 \frac{\xi_1 + \xi_2}{\sqrt{2}} - k_0 \frac{\xi_1 + \xi_2}{\sqrt{2}} + k_0 \frac{-\xi_1 + \xi_2}{\sqrt{2}})} d\xi_1 d\xi_2. \quad (H.16)
\]

Since the leading order contribution comes from the region near the line segment \(r_0 + R = r\) between \(r_0 = a\) and \(r_0 = r\), we approximate domain \(D\) in the \((\xi_1, \xi_2)\) coordinate system as \(D' = \{\xi_1, \xi_2 : |\xi_1| \leq r/\sqrt{2}; \xi_2 \geq r/\sqrt{2}\}\), which introduces an
error proportional to \(1/r\) which rapidly becomes negligible as \(r\) increases. Integrals (H.13) - (H.16) can then be evaluated by integrating over \(\xi_1\) first then \(\xi_2\). Integrating over \(\xi_1\) leads to

\[
P'_{IS+}(r\hat{i}_a) = -\frac{\omega^2_+ \beta P_{a0} P_{b0}}{A c_0^2} e^{ik_a \sqrt{2}} \nonumber
\]

\[
\times \lim_{\xi_2 \to \infty} \int_{r/\sqrt{2}}^{L} \frac{e^{ik_a (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2} - e^{-ik_a (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2}}{ik_a (\frac{x^2}{r} - \frac{1}{\sqrt{2}})} e^{i(k_a \sqrt{2} + k_b \sqrt{2}) \xi_2} d\xi_2, \quad (H.17)
\]

\[
P'_{SI+}(r\hat{i}_b) = -\frac{\omega^2_+ \beta P_{a0} P_{b0}}{A c_0^2} e^{ik_b \sqrt{2}} \nonumber
\]

\[
\times \lim_{\xi_2 \to \infty} \int_{r/\sqrt{2}}^{L} \frac{e^{ik_b (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2} - e^{-ik_b (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2}}{ik_b (\frac{x^2}{r} - \frac{1}{\sqrt{2}})} e^{i(k_a \sqrt{2} + k_b \sqrt{2}) \xi_2} d\xi_2, \quad (H.18)
\]

\[
P'_{IS-}(r\hat{i}_a) = -\frac{\omega^2_- \beta P_{a0} P_{b0}}{A c_0^2} e^{ik_a \sqrt{2}} \nonumber
\]

\[
\times \lim_{\xi_2 \to \infty} \int_{r/\sqrt{2}}^{L} \frac{e^{ik_a (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2} - e^{-ik_a (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2}}{ik_a (\frac{x^2}{r} - \frac{1}{\sqrt{2}})} e^{i(k_a \sqrt{2} - k_b \sqrt{2}) \xi_2} d\xi_2, \quad (H.19)
\]

\[
P'_{SI-}(r\hat{i}_b) = -\frac{\omega^2_- \beta P_{a0} P_{b0}}{A c_0^2} e^{-ik_b \sqrt{2}} \nonumber
\]

\[
\times \lim_{\xi_2 \to \infty} \int_{r/\sqrt{2}}^{L} \frac{e^{ik_b (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2} - e^{-ik_b (\frac{x^2}{r} - \frac{1}{\sqrt{2}}) \xi_2}}{ik_b (\frac{x^2}{r} - \frac{1}{\sqrt{2}})} e^{i(k_a \sqrt{2} - k_b \sqrt{2}) \xi_2} d\xi_2. \quad (H.20)
\]
Let $\xi_2/r - 1/\sqrt{2} = \eta$, then $d\xi_2 = r d\eta$ and equations (H.17) - (H.20) become

\[
P'_{IS+}(\hat{\mathbf{r}}_z) = \frac{\omega^2 \beta P_0 P_{B0} e^{ik_0 r/2}}{A_0^2 k_0 r} \times \lim_{L \to \infty} \int_0^{L/r} \frac{e^{i k_0 \eta \sqrt{2}} - e^{-i k_0 \eta \sqrt{2}}}{i k_0 \eta} e^{i(k_0 \sqrt{2} + k_0 \sqrt{2})(\eta + \frac{\eta}{\sqrt{2}})} \, d\eta,
\]  
(\text{H.21})

\[
P'_{SI+}(\hat{\mathbf{r}}_z) = -\frac{\omega^2 \beta P_0 P_{B0} e^{ik_0 r/2}}{A_0^2 k_0 r} \times \lim_{L \to \infty} \int_0^{L/r} \frac{e^{i k_0 \eta \sqrt{2}} - e^{-i k_0 \eta \sqrt{2}}}{i k_0 \eta} e^{i(k_0 \sqrt{2} - k_0 \sqrt{2})(\eta + \frac{\eta}{\sqrt{2}})} \, d\eta,
\]  
(\text{H.22})

\[
P'_{IS-}(\hat{\mathbf{r}}_z) = -\frac{\omega^2 \beta P_0 P_{B0} e^{i k_0 r/2}}{A_0^2 k_0 r} \times \lim_{L \to \infty} \int_0^{L/r} \frac{e^{-i k_0 \eta \sqrt{2}} - e^{i k_0 \eta \sqrt{2}}}{-i k_0 \eta} e^{i(k_0 \sqrt{2} - k_0 \sqrt{2})(\eta + \frac{\eta}{\sqrt{2}})} \, d\eta,
\]  
(\text{H.23})

\[
P'_{SI-}(\hat{\mathbf{r}}_z) = \frac{\omega^2 \beta P_0 P_{B0} e^{-ik_0 r/2}}{A_0^2 k_0 r} \times \lim_{L \to \infty} \int_0^{L/r} \frac{e^{-i k_0 \eta \sqrt{2}} - e^{i k_0 \eta \sqrt{2}}}{-i k_0 \eta} e^{i(k_0 \sqrt{2} + k_0 \sqrt{2})(\eta + \frac{\eta}{\sqrt{2}})} \, d\eta,
\]  
(\text{H.24})

which can be written as

\[
P'_{IS+}(\hat{\mathbf{r}}_a) = -\frac{\omega^2 \beta P_0 P_{B0} e^{ik_0 r}}{A_0^2 k_0} \lim_{L \to \infty} \int_0^{L/r} e^{i\sqrt{2k_0 \eta} - i\sqrt{2k_0 \eta}} \, d\eta,
\]  
(\text{H.25})

\[
P'_{SI+}(\hat{\mathbf{r}}_b) = -\frac{\omega^2 \beta P_0 P_{B0} e^{ik_0 r}}{A_0^2 k_0} \lim_{L \to \infty} \int_0^{L/r} e^{i\sqrt{2k_0 \eta} - i\sqrt{2k_0 \eta}} \, d\eta,
\]  
(\text{H.26})

\[
P'_{IS-}(\hat{\mathbf{r}}_a) = -\frac{\omega^2 \beta P_0 P_{B0} e^{i k_0 r}}{A_0^2 2k_0 k_0} \lim_{L \to \infty} \int_0^{L/r} e^{i\sqrt{2k_0 \eta} - i\sqrt{2k_0 \eta}} \, d\eta,
\]  
(\text{H.27})

\[
P'_{SI-}(\hat{\mathbf{r}}_b) = -\frac{\omega^2 \beta P_0 P_{B0} e^{ik_0 r}}{A_0^2 k_0} \lim_{L \to \infty} \int_0^{L/r} e^{i\sqrt{2k_0 \eta} - i\sqrt{2k_0 \eta}} \, d\eta.
\]  
(\text{H.28})
Let $\sqrt{2r\eta} = \zeta$, then equations (H.25) - (H.28) become

\begin{align*}
P'_{\text{IS}+}(\mathbf{r}_a) &= -\frac{\omega^2 \beta P_0}{A_0^2} \frac{\text{e}^{ik_+r}}{2k_a k_b} \lim_{L \to \infty} \int_0^L \frac{\text{e}^{ik_+\zeta} - \text{e}^{ik_\zeta}}{i\zeta} d\zeta, \quad (\text{H.29}) \\
P'_{\text{ST}+}(\mathbf{r}_b) &= -\frac{\omega^2 \beta P_0}{A_0^2} \frac{\text{e}^{ik_+r}}{2k_a k_b} \lim_{L \to \infty} \int_0^L \frac{\text{e}^{ik_+\zeta} - \text{e}^{ik_\zeta}}{i\zeta} d\zeta, \quad (\text{H.30}) \\
P'_{\text{IS}-}(\mathbf{r}_a) &= -\frac{\omega^2 \beta P_0^*}{A_0^2} \frac{\text{e}^{ik_-r}}{2k_a k_b} \lim_{L \to \infty} \int_0^L \frac{\text{e}^{ik_-\zeta} - \text{e}^{-ik_\zeta}}{i\zeta} d\zeta, \quad (\text{H.31}) \\
P'_{\text{ST}-}(\mathbf{r}_b) &= -\frac{\omega^2 \beta P_0^*}{A_0^2} \frac{\text{e}^{ik_-r}}{2k_a k_b} \lim_{L \to \infty} \int_0^L \frac{\text{e}^{ik_\zeta} - \text{e}^{-ik_\zeta}}{i\zeta} d\zeta. \quad (\text{H.32})
\end{align*}

The limits in equations (H.29) - (H.32) can be determined analytically with the
cosine integral \( \text{Ci}(x) \) [31], as

\[
\lim_{L \to \infty} \int_0^L \frac{e^{ik_L \zeta} - e^{ik_0 \zeta}}{i \zeta} d \zeta = i \lim_{L \to \infty} \int_0^L \frac{1 - \cos k_L \zeta}{\zeta} d \zeta - i \lim_{L \to \infty} \int_0^L \frac{1 - \cos k_0 \zeta}{\zeta} d \zeta + \int_0^\infty \frac{\sin k_L \zeta}{\zeta} d \zeta - \int_0^\infty \frac{\sin k_0 \zeta}{\zeta} d \zeta
\]

\[
= i \lim_{L \to \infty} \left[ \gamma + \log(k_L L) - \text{Ci}(k_L L) \right] - i \lim_{L \to \infty} \left[ \gamma + \log(k_0 L) - \text{Ci}(k_0 L) \right] + \pi/2 - \pi/2
\]

\[
= i \log(k_L/k_0), \tag{H.33}
\]

\[
\lim_{L \to \infty} \int_0^L \frac{e^{ik_0 \zeta} - e^{-ik_L \zeta}}{i \zeta} d \zeta = i \lim_{L \to \infty} \int_0^L \frac{1 - \cos k_0 \zeta}{\zeta} d \zeta - i \lim_{L \to \infty} \int_0^L \frac{1 - \cos k_L \zeta}{\zeta} d \zeta + \int_0^\infty \frac{\sin k_0 \zeta}{\zeta} d \zeta + \int_0^\infty \frac{\sin k_L \zeta}{\zeta} d \zeta
\]

\[
= i \lim_{L \to \infty} \left[ \gamma + \log(k_0 L) - \text{Ci}(k_0 L) \right] - i \lim_{L \to \infty} \left[ \gamma + \log(k_L L) - \text{Ci}(k_L L) \right] + \pi/2 + \pi/2
\]

\[
= i \log(k_0/k_L) + \pi, \tag{H.34}
\]

\[
\lim_{L \to \infty} \int_0^L \frac{e^{ik_0 \zeta} - e^{-ik_0 \zeta}}{i \zeta} d \zeta = i \lim_{L \to \infty} \int_0^L \frac{1 - \cos k_0 \zeta}{\zeta} d \zeta - i \lim_{L \to \infty} \int_0^L \frac{1 - \cos k_0 \zeta}{\zeta} d \zeta + \int_0^\infty \frac{\sin k_0 \zeta}{\zeta} d \zeta - \int_0^\infty \frac{\sin k_0 \zeta}{\zeta} d \zeta
\]

\[
= i \lim_{L \to \infty} \left[ \gamma + \log(k_0 L) - \text{Ci}(k_0 L) \right] - i \lim_{L \to \infty} \left[ \gamma + \log(k_L L) - \text{Ci}(k_L L) \right] + \pi/2 - \pi/2
\]

\[
= -i \log(k_0/k_L), \tag{H.35}
\]

where \( \gamma \) is the Euler-Mascheroni constant [31].

We then obtain the second order nonlinear fields in the forward directions due to
interaction between a plane wave and a spherical wave, as

\[
P'_{IS+}(r\mathbf{\hat{a}}) = -\frac{\omega^2 \beta P_{a0} P_{b0} e^{ik_{+}r}}{A c_0^2} 2k_a k_b \left[ i \log \left( \frac{k_{+}}{k_{b}} \right) \right], \quad (H.37)
\]

\[
P'_{SI+}(r\mathbf{\hat{b}}) = -\frac{\omega^2 \beta P_{a0} P_{b0} e^{ik_{+}r}}{A c_0^2} 2k_a k_b \left[ i \log \left( \frac{k_{+}}{k_{a}} \right) \right], \quad (H.38)
\]

\[
P'_{IS-}(r\mathbf{\hat{a}}) = -\frac{\omega^2 \beta P_{a0} P_{b0}^* e^{ik_{-}r}}{A c_0^2} 2k_a k_b \left[ i \log \left( \frac{k_{-}}{k_{b}} \right) + \pi \right], \quad (H.39)
\]

\[
P'_{SI-}(r\mathbf{\hat{b}}) = -\frac{\omega^2 \beta P_{a0} P_{b0}^* e^{ik_{-}r}}{A c_0^2} 2k_a k_b \left[ -i \log \left( \frac{k_{-}}{k_{a}} \right) \right]. \quad (H.40)
\]

As seen in equations (H.37) - (H.40), the IS and SI field magnitudes are constant with range. Unlike the interaction of collinear plane waves where growth is found along the propagation path, collinearity between planar and spherical wavefronts within an equivalent Fresnel area about the forward direction, together with spreading of the spherical wave, balances out second-order wave growth. For \( P'_{IS-} \), there is an additional contribution from a stationary phase point at range \( rk_b/k_a \) in the forward direction. To see this, we rewrite equation (H.7) for \( P'_{IS-} \) as

\[
P'_{IS-}(r\mathbf{\hat{z}}) = -\frac{\omega^2 \beta P_{a0} P_{b0}^* e^{ik_{-}r/2}}{A c_0^2} 2k_{\mathbf{\hat{z}}} \int \int_D e^{ik_{-}r_\text{forward}(r_0,R)} dR dr_0, \quad (H.41)
\]

where

\[
\varphi_{\text{forward}}(r_0, R) = \frac{k_a r_0^2 - R^2}{R - \frac{R - k_a r_0}{2r^2}} \left( \frac{k_a r_0 - k_a r}{r} \right), \quad (H.42)
\]

The stationary phase point can be found by letting \( \partial/\partial r_0 \varphi_{\text{IS-}} = 0 \) and \( \partial/\partial R \varphi_{\text{IS-}} = 0 \), which gives \( (r_0, R) = (rk_b/k_a, r_{k-}/k_a) \). Applying a two-dimensional stationary phase approximation [46] to equation (H.41) for \( P'_{IS-} \) leads to

\[
P'_{\text{IS- stationary phase}}(r\mathbf{\hat{a}}) = -\frac{\omega^2 \beta P_{a0} P_{b0}^* e^{ik_{-}r}}{A c_0^2} 2k_a k_b \pi, \quad (H.43)
\]

which corresponds to the \( \pi \) term contribution of the full solution for \( P'_{IS-} \) in equation (H.39). The stationary phase points for \( P'_{IS+}, P'_{SI+} \) and \( P'_{SI-} \) can be found in the same
way but they are outside of the domain $D$, i.e. they do not exist in physical domain. The leading order contributions to $P'_{IS+}$, $P'_{SI+}$ and $P'_{SI-}$ come from an equivalent Fresnel width along the forward path between the object and the receiver, as found in equations (H.37), (H.38) and (H.40).

Analytic solutions (H.37) - (H.40) are verified by direct numerical integration of equations (H.1) - (H.4), as shown in figure H-1(a)-(d). We truncate the numerical integration at radius $r_{0\text{max}}$. It can be seen that the numerical results agree very well with the analytic solutions as long as the integrations are truncated beyond the receiver radius $r$. It can also be seen the integral for $P'_{IS-}$ has a significant contribution near $0.6r$, which corresponds to the stationary phase point at $r_0 = r_{ka}/k_a$. The physical parameters are $\omega_a/2\pi = 500$ kHz, $\omega_b/2\pi = 300$ kHz, $a = 1$ mm and $r = 1$ m.

The range dependence of $P'_{IS-}$ in the forward direction is shown in figure H-2. The frequencies and the object radius are the same as before. It can be seen that as $k_- r$ increases, the amplitude of $P'_{IS-}$ calculated via direct volume integration equation (3.23) approaches the value given by analytic solution (H.39). When evaluating the volume integral numerically, we truncate the upper range to $r_{0\text{max}}$ using a smooth taper function with a width of two difference frequency wavelengths, or $4\pi/k_-$. This reduces but cannot eliminate the artificial contribution from the outer edge of the volume. The slight difference between the two curves for large $k_- r$ is due to such artifact.

If the scattered waves are not spherically symmetric the solutions in (H.37) - (H.40) for sufficiently large range become

\begin{align*}
P'_{IS+}(\hat{r}_a) &\approx -\frac{\omega_2^2 \beta P_0 P_0}{\mathcal{A}_a^2} \frac{e^{ik_- r}}{2k_0^2 k_b} \left[ i \log \left( \frac{k_+}{k_b} \right) \right] S_\delta(\hat{r}_a), \quad (H.44) \\
P'_{SI+}(\hat{r}_b) &\approx -\frac{\omega_2^2 \beta P_0 P_0}{\mathcal{A}_a^2} \frac{e^{ik_- r}}{2k_0^2 k_b} \left[ i \log \left( \frac{k_+}{k_a} \right) \right] S_\delta(\hat{r}_b), \quad (H.45) \\
P'_{IS-}(\hat{r}_a) &\approx -\frac{\omega_2^2 \beta P_0 P_0}{\mathcal{A}_a^2} \frac{e^{ik_- r}}{2k_0^2 k_b} \left[ i \log \left( \frac{k_-}{k_b} \right) + \pi \right] S_\delta(\hat{r}_a), \quad (H.46) \\
P'_{SI-}(\hat{r}_b) &\approx -\frac{\omega_2^2 \beta P_0 P_0}{\mathcal{A}_a^2} \frac{e^{ik_- r}}{2k_0^2 k_b} \left[ -i \log \left( \frac{k_-}{k_a} \right) \right] S_\delta(\hat{r}_b). \quad (H.47)
\end{align*}

which are scaled by the corresponding far field scatter functions $S_a$ and $S_b$ of the
Figure H-1: Verification of analytic solutions (H.37) - (H.40) with direct numerical integration of equations (H.1) - (H.4). Excellent agreements are found as long as the integrations are truncated at a range $r_{0\text{max}}/r$ that is larger than receiver range $r$. The physical parameters are $\omega_a/2\pi = 500$ kHz, $\omega_b/2\pi = 300$ kHz, $a = 1$ mm and $r = 1$ m.
Figure H-2: Range dependence for the $P_{\text{IIB}}(r\hat{i}_a)$. The direct volume integration, solid line, calculated via equation (3.23) agrees very well with the analytic solution, dashed line, calculated via equation (H.39) for large $k-r$

scattered waves in the respective forward directions $\hat{i}_b$ and $\hat{i}_a$.

The second order field components $P_{\text{IIS}}$ and $P_{\text{SI}}$ due to IS and SI interactions in the forward directions can then be approximated by equations (H.44) - (H.47) for large range because $P_{\text{IIS}}$ and $P_{\text{SI}}$ all fall off by $r^{-1}$.

### H.2 Backscatter direction

Now consider the case when the receiver at range $r$ is in the backscatter direction $-\hat{i}_a$ or $-\hat{i}_b$ of the incident waves for plane wave and spherical wave interaction. We define another spherical coordinate system $(r_0, \theta_0, \phi_0)$ so that the zenith direction coincides
with \( \hat{\text{i}}_a \) or \( \hat{\text{i}}_b \). Equations (3.23) and (3.24) become

\[
P'_{\text{IS}+}(-\hat{\text{i}}_a) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int e^{-ikr_0 \cos \theta_0} e^{ikr_0} e^{ik_0 R} \frac{k_0 r_0}{4 \pi R} r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0, \quad (H.48)
\]

\[
P'_{\text{SI}+}(-\hat{\text{i}}_b) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int e^{-ikr_0 \cos \theta_0} e^{ikr_0} e^{ik_0 R} \frac{k_0 r_0}{4 \pi R} r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0, \quad (H.49)
\]

\[
P'_{\text{IS}+}(-\hat{\text{i}}_a) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int e^{-ikr_0 \cos \theta_0} e^{ikr_0} e^{ik_0 R} \frac{k_0 r_0}{4 \pi R} r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0, \quad (H.50)
\]

\[
P'_{\text{SI}+}(-\hat{\text{i}}_b) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int e^{-ikr_0 \cos \theta_0} e^{ikr_0} e^{ik_0 R} \frac{k_0 r_0}{4 \pi R} r_0^2 \sin \theta_0 dr_0 d\theta_0 d\phi_0. \quad (H.51)
\]

In the \((r_0, \theta_0, \phi_0)\) coordinate system, \( R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta_0} \), so \( r_0 \cos \theta_0 = (r^2 + r_0^2 - R^2)/(2r) \), \( r \) \( \sin \theta_0 \) \( d\theta_0 = RdR/r \), and the integrals of (H.48) - (H.51) become

\[
P'_{\text{IS}+}(-\hat{\text{i}}_a) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int \frac{e^{ik_0 R}}{2k_0 r} dD r_0, \quad (H.52)
\]

\[
P'_{\text{SI}+}(-\hat{\text{i}}_b) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int \frac{e^{ik_0 R}}{2k_0 r} dD r_0, \quad (H.53)
\]

\[
P'_{\text{IS}+}(-\hat{\text{i}}_a) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int \frac{e^{ik_0 R}}{2k_0 r} dD r_0, \quad (H.54)
\]

\[
P'_{\text{SI}+}(-\hat{\text{i}}_b) = -\frac{\omega^2 \beta P_{\theta 0} P_{\theta 0}}{A c^2} \int \int \frac{e^{ik_0 R}}{2k_0 r} dD r_0. \quad (H.55)
\]

where the integration domain is \( D = \{ r_0, R : |r - r_0| \leq R \leq |r + r_0| ; r_0 \geq a \} \) (figure H-3), and

\[
\varphi_{\text{IS}+}(r_0, R) = \frac{k_a r_0^2 - R^2}{k_+ r} + \frac{k_+ r_0}{k_+ r} + \frac{R}{r}, \quad (H.56)
\]

\[
\varphi_{\text{SI}+}(r_0, R) = \frac{k_a r_0}{k_+ r} - \frac{k_+ r_0^2 - R^2}{k_+ r} + \frac{R}{r}, \quad (H.57)
\]

\[
\varphi_{\text{IS}+}(r_0, R) = \frac{k_a r_0}{k_- r} - \frac{k_+ r_0^2 - R^2}{k_- r} + \frac{R}{r}, \quad (H.58)
\]

\[
\varphi_{\text{SI}+}(r_0, R) = \frac{k_a r_0}{k_- r} + \frac{k_+ r_0^2 - R^2}{k_- r} + \frac{R}{r}. \quad (H.59)
\]
The stationary points in \( \varphi_{IS+}^{\text{back}}, \varphi_{SI+}^{\text{back}}, \varphi_{IS-}^{\text{back}}, \) and \( \varphi_{SI-}^{\text{back}} \) are

\[
(r_0, R) = \begin{cases} 
(r_{k_b}/k_a, -r_{k+}/k_a) & \text{for } \varphi_{IS+}^{\text{back}}, \\
(r_{k_a}/k_b, -r_{k+}/k_b) & \text{for } \varphi_{SI+}^{\text{back}}, \\
(-r_{k_b}/k_a, -r_{k-}/k_a) & \text{for } \varphi_{IS-}^{\text{back}}, \\
(-r_{k_a}/k_b, r_{k-}/k_b) & \text{for } \varphi_{SI-}^{\text{back}}, 
\end{cases}
\] (H.60)

none of which are inside domain \( D \). Unlike the case in the forward direction, the phases \( \varphi_{IS+}^{\text{back}}, \varphi_{SI+}^{\text{back}}, \varphi_{IS-}^{\text{back}}, \) and \( \varphi_{SI-}^{\text{back}} \) also vary along the path between the object and the receiver in the backscatter direction. The leading order contribution to the integrals in equations (H.52) - (H.55) then comes from the all finite corners in domain \( D \) for \( k - r \gg 1 \) [47]. As shown in figure H-3, there are three corners: (1) \((r_0, R) = (a, r + a)\), (2) \((r_0, R) = (a, r - a)\), and (3) \((r_0, R) = (r, 0)\). As an example, when \( P_{Sb} \) is a spherical wave, corner contributions to the integral (H.54) can be determined via

\[
\int_D e^{ik_{-}r\varphi_{IS-}^{\text{back}}} dR dr_0 
\approx \sum_{\text{corners } 1, 2, 3} \frac{2\cot \alpha e^{ik_{-}r\varphi_{IS-}^{\text{back}}}}{k^2 r^2[(\frac{\partial \varphi_{IS-}^{\text{back}}}{\partial r_0} \cos \beta + \frac{\partial \varphi_{IS-}^{\text{back}}}{\partial R} \sin \beta)^2 - (\cot \alpha)^2(-\frac{\partial \varphi_{IS-}^{\text{back}}}{\partial r_0} \sin \beta + \frac{\partial \varphi_{IS-}^{\text{back}}}{\partial R} \cos \beta)^2]}.
\] (H.61)

where the variables being summed are \( \alpha, \beta, \varphi_{IS-}^{\text{back}}, \partial \varphi_{IS-}^{\text{back}}/\partial r_0 \) and \( \partial \varphi_{IS-}^{\text{back}}/\partial R \), whose values are listed in Table H.1 for each corner. As \( r \to \infty \), \( \partial \varphi_{IS-}^{\text{back}}/\partial r_0 \) and \( \partial \varphi_{IS-}^{\text{back}}/\partial R \) behave as \( r^{-1} \) as seen in Table H.1, so the integral (H.61) approaches constant magnitude for large \( r \). The backscatter magnitude \( P_{IS-}^{\text{back}}(-r_i a) \) in equation (H.54) then falls off by \( r^{-1} \) as \( r \to \infty \). Results for \( P_{IS+}^{\text{back}}, P_{SI+}^{\text{back}} \) and \( P_{SI-}^{\text{back}} \) can be obtained in a similar manner using the formulas in Ref. [47]. When including \( P_{IS\pm}^{\text{back}} \) and \( P_{SI\pm} \) that also fall off by \( r^{-1} \), the second order field components due to IS and SI interactions in the backscatter direction have an overall range dependence of \( r^{-1} \), which become insignificant for large range because the dominant component \( P_{SS\pm} \) due to SS interaction has
Figure H-3: Integration domain $D$ (blue shaded) in equations (H.5) - (H.8) and (H.52) - (H.55) for the IS and SI interactions. For a receiver in the forward direction, the dominant contribution comes from the region near the line segment between the object to the receiver (corresponding to the red segment). The difference between domain $D$ and $D'$ (enclosed by the solid boundary) become negligible as $r \to \infty$. For a receiver in the backscatter direction, the dominant contribution comes from the three corners, which can be determined via equation (H.61) and Table H.1 for $P'_{IS}(-r_\xi a)$.

For the case that both primary incident and scattered waves are pulses of finite support, if the pulse length $T$ is long enough so that the incident pulse and scattered pulse overlap at the receiver in the backscattered direction, the time domain solution $p'_{IS\pm}(-r_\xi a)$ calculated by equation (3.11) can be approximate by frequency domain solution $P'_{IS\pm}$ with all three corner contributions included. However, in the case where the pulses are relatively short so that the incident and scattered pulses do not overlap at the receiver, the time domain solution corresponds to the frequency
domain solution with only two corner points that are at the object. To exclude the third corner contribution when numerically evaluating equation (3.23), it is necessary to smoothly taper off the outer edge of the integration volume, so the artificial corner contribution can be reduced.

Figure H-4 shows the amplitude of $P'_{aS-}(rB)$ evaluated by truncating integral (3.23) at different $r_{\text{omax}}$ using a smooth function (blue solid line) and a sharp step function (red dashed line). The frequencies and object radius are the same as those used in figure H-1, and $r/a = 1000$. It can be seen that $P'_{aS-}$ with the smooth truncation has three different values. For $0.02 < r_{\text{omax}}/r < 1$, it agrees (with less than 2% relative difference) with the lower dashed line, which corresponds to the contribution from the two corners near the object determined via equation (H.61). For $r_{\text{omax}}/r > 1$, it agrees (with less than 2% relative difference) with the upper dashed line, which corresponds to the contribution from all three corners in equation (H.61). For $r_{\text{omax}}/r < 0.02$, the integration volume is too small to capture to full contribution from the two corners near the object. For the truncation using the sharp step function, artificial finite corners are introduced whose contributions are of the same order as the contributions from the three real corners, making the numerical integration very sensitive to the truncation range $r_{\text{omax}}$.

The range dependence of $P'_{aS-}(rB)$ is shown in figure H-5. The frequencies and the object radius are the same as before. It can be seen that as $k_-r$ increases, the amplitude of $P'_{aS-}$ calculated by direct volume integral equation (3.23) approaches the analytic solution, equation (H.61). This is also as expected because the condition for equation (H.61) to be valid is $k_-r \gg 1$. Figure H-5 also shows that $P'_{aS-}(rB_a)$ falls off as $r^{-1}$ in range, which is different from the $P'_{aS-}(rB_a)$ in the forward direction.
Figure H-4: Truncating the incident-scattered interaction in the backscatter direction at different $r_{0\text{max}}$ with a smooth function (blue solid line) and a sharp step function (red dashed line). For both $r_{0\text{max}} < r$ and $r_{0\text{max}} > r$, volume integration truncated with a smooth function agrees with the analytic solution (black dashed lines) determined via equation (H.61).

Figure H-5: In the backscatter direction, $P_{s_3}^r(-r\hat{n}_a)$ falls off as $1/r$ in range. The analytic solution, dashed line, calculated via equation (H.61) agrees with the direct volume integration, solid line, calculated via equation (3.23) for large $k_r r$. 
Appendix I

More on space-time isolation of sum and difference frequency field components containing object information §4.1

For the collinear case, the II overlap region is an infinite slab moving in the propagating direction of the primary incident waves in 3-D; for the perpendicular case, it is the intersection of two slabs and it moves diagonally. These II overlap regions (grey) are projected on a 2-D plane in figure 4 of the main text. The SS overlap region is a 3-D spherical shell that appears as a 2-D ring (blue) in the 2-D plane in figure 4 of the main text. IS or SI overlap region is a spherical cap moving in the direction of the incident wave in 3-D, which is shown as a circular segment in green in figure 4 of the main text.
The following window function $w_1(t)$ is used in the computation:

$$w_1(t) = \begin{cases} \frac{t}{t_1} - \frac{1}{2\pi} \sin \left( \frac{2\pi t}{t_1} \right), & 0 < t \leq t_1 \\ 1, & t_1 < t \leq T - t_1 \\ \frac{T - t}{t_1} - \frac{1}{2\pi} \sin \left( \frac{2\pi (T - t)}{t_1} \right), & T - t_1 < t \leq T \\ 0, & \text{otherwise,} \end{cases}$$

(I.1)

where $T = 20\pi/w_-$ is the duration of the window, which contains 10 difference frequency cycles, and $t_1 = 4\pi/w_-$ is the duration of the transition regions, which contain 2 difference frequency cycles.

For the interaction of waves of compact support, time domain Green theorem solutions (3.9) - (3.12) are numerically evaluated for $p_{1I,ab^*}$, $p_{SS,ab^*}$ and $p_{IS,ab^*}$. The integrations are over finite volumes in space defined by the compact support of $w_{1I,ab^*}$, $w_{SS,ab^*}$ and $w_{IS,ab^*}$ of equations (3.13) - (3.16), respectively. These volumes are functions of time. For example, the volume corresponding to the SS interaction is defined by $w_1(t - R/c_0 - r_0/c_0)$. It is empty for $t < r_R/c_0$, a line segment connecting the origin and the receiver for $t = r_R/c_0$, a prolate spheroid for $r_R/c_0 < t \leq r_R/c_0 + T$ and a prolate spheroidal shell for $t > r_R/c_0$. The total fields $p_{1I,ab^*}$, $p_{SS,ab^*}$ and $p_{IS,ab^*}$ are obtained by including $p''_{1I,ab^*}$, $p''_{SS,ab^*}$ and $p''_{IS,ab^*}$ from equation (3.2) with appropriate products of the primary fields.

Frequency domain Green theorem solutions (3.21) - (3.24) are used in the harmonic wave approximations. For the II, SS and forward direction IS interactions, we can either integrate the infinite space where the primary fields exist, or integrate over a finite space defined by the compact support of $w_{1I,ab^*}$, $w_{SS,ab^*}$ and $w_{IS,ab^*}$ at a time instance $\bar{t}$ between $t = r_R/c_0$ and $t = r_R/c_0 + T$. The time $\bar{t} = r_R/c_0 + T/2$ was used in our computation, but it is found that the results are insensitive to the choice of $\bar{t}$ as long as the window duration $T$ is sufficiently long and $\bar{t}$ is at the center of the constant region within the window $w_1(t - r_R/c_0)$. For the backscatter direction IS interaction if the incident and scattered waves never overlap at the receiver, $p_{IS,ab^*}$ is generated by the IS interaction that took place earlier between $t = 0$ and $t = T$ when
the incident and scattered waves overlapped near the object. We have to integrate over the finite space defined by the compact support of $w_{1S,ab}^*$ at time instance $\tilde{t}$ between $t = \frac{r_R}{c_0}$ and $t = \frac{r_R}{c_0} + T$ in the harmonic wave approximation. When using equation (H.61) for this case, we only sum over the contributions from corners 1 and 2. The total fields $P_{11-}$, $P_{SS-}$ and $P_{IS-}$ are obtained by including $P_{11-}^\prime\prime$, $P_{SS-}^\prime\prime$ and $P_{IS-}^\prime\prime$ from equation (3.2) with appropriate products of the primary fields.

From the sensing perspective, the $p_{II}$ is undesirable because it contains no information about the object and it can mask out other field components. If the receiver is placed near the backscatter directions, as shown in figure 4 of the main text, it is possible to separate the II component. There is, however, one more constraint that requires attention. Ideally, the time interval between $t = \frac{r_R}{c_0}$ and $t = \frac{r_R}{c_0} + T$ is arrival time for $p_{SS}$, $p_{IS}$ and $p_{S2}$, which is also the available window for these field components. For a very long window or at very close range, during the early part of this measuring window, the receiver may still be inside the II overlap region. If this is the case, the available measuring window is reduced to between $t = -\frac{r_R}{c_0} + T$ and $t = \frac{r_R}{c_0} + T$, with a duration $\Delta T = 2\frac{r_R}{c_0}$. To measure steady state response at the difference or sum frequency, $\Delta T$ should contain at least a few difference or sum frequency cycles, which imposes a lower limit on the normalized receiver range $k_{\pm}r_R$. For example if $N = 10$ difference or sum frequency cycles are desired, then $\Delta T > 2\pi N/\omega_\pm$ or $k_{\pm}r_R > N\pi$. 

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Appendix J

Incident level, received level and noise level in Table 4.2

J.1 Incident level

Case 1 describes a sensing scenario in air. The 40 kHz primary frequency is found in bats [34] and ultrasound sensors in autonomous robots [35]. It is reported that SPL for bats can reach 140 dB re 20 \(\mu\)Pa at 0.1 meter from its mouth [48]. With the same source but further away from the object, the incident SPL reduces due to the spreading loss.

Cases 2 and 3 describe medical ultrasound imaging of hard and soft objects, respectively. For an imaging system, the incident energy intensity at the target was reported to be 2240 J/m\(^{-3}\) [21], which corresponds to a pressure amplitude of 1 MPa or 240 dB re 1 \(\mu\)Pa.

Case 4 describes sensing of gas bubbles in the ocean. The primary frequency of 100 kHz is commonly used in side-scan sonar systems [37, 38]. For a 2 MW source, the source level is 234 dB re 1 \(\mu\)Pa and 1 m. The SPL at 500 m reduces to 180 dB re 1 \(\mu\)Pa due to spherical spreading loss. If the source is closer to the target, the same incident SPL can be achieved by a source with lower power.

Case 5 describes local seismic sensing in the solid earth with compressional waves. The 240 dB re 1 \(\mu\)Pa incident SPL can be generated by a vibrator of 150 kN peak
force output at 30 meters away. This incident SPL is estimated from the 0.1 m/s measured particle velocity at 15 Hz [49] (assuming the seismometer has sensitivity 5 volt/(inch/s) [50]), multiplied by $\rho_{\text{earth}} = 3000 \text{ kg/m}^3$, $c_{\text{earth}} = 3000 \text{ m/s}$. If the source is 500 m from the object, a much stronger source is required to achieve similar incident SPLs.

J.2 Received level

Results in figures (4-3) - (4-7) are calculated using $\beta = 3.6$, which is a typical value for water. Special attention is needed to properly scale the results to different values of $\beta$. As seen in the decomposition in equation (3.1), $p'_2$ is linearly proportional to $\beta$ (i.e. depends on the nonlinear property of the medium), while $p''_2$ is independent of $\beta$. The portion of the second order incident field $p'_2$ and its associated scattered field can be scaled by $\beta$. The portion of the second order incident field $p''_2$ and its associated scattered field, as well as the effects from the body-wave interactions (i.e. quadratic terms in the second order boundary conditions (2.45) and (2.48) and the quadratic term in the second order wave-exciting force (2.51)) should not be scaled. Following this scaling principle, results for air with $\beta = 1.2$ and for solid earth with $\beta = 1000$ can be obtained. Results in air corresponding to figures 4-3, 4-4 and 4-6 are shown in figures J-1, J-2 and J-3. Results in solid earth corresponding to figures 4-3, 4-4 and 4-6 are shown in figures J-4, J-5 and J-6.

Case 1 corresponds to the top right point in figure J-1(b), where $P_2/(2E_0) = 1.44 \times 10^{-5}$. The incident level of 140 dB re 20 $\mu$Pa corresponds to 200 Pa, which gives $2E_0 = 0.3$. The second order pressure is then found to be 4 $\mu$Pa or -14 dB re 20 $\mu$Pa.

Cases 2-4 are for water, so there is no need to scale $\beta$ and we can use the figures in the main text directly. Case 2 corresponds to the top right point in 4-3(b). The incident level of 240 dB re 1 $\mu$Pa corresponds to 1 MPa, which gives $2E_0 = 444$ Pa. The second order pressure is found to be 0.02 Pa or 86 dB re 1 $\mu$Pa.

Case 3 corresponds to a point on the curve in figure 4-6(b), where $P_2/(2E_0) =
Figure J-2: Same as figure 4-4 except that the nonlinear parameter is $\beta = 1.2$ for air.
Figure J-3: Same as figure 4-6 except that the nonlinear parameter is $\beta = 1.2$ for air.

0.012. The second order pressure is then 5 Pa or 134 dB re 1 $\mu$Pa.

Case 4 corresponds to a point on the curve in figure 4-6(c), where $P_2/(2E_0) = 3.6 \times 10^{-5}$. The incident level of 200 dB re 1 $\mu$Pa corresponds to 10 kPa, which gives $2E_0 = 0.044$ Pa. The second order pressure is then found to be 9 Pa or 63 dB re 1 $\mu$Pa.

Case 5 can be determined by extrapolating the results shown in figure J-6(c) to $k_0r = 100$. This is possible because the S2 mechanism remains dominant and $P_{S2}$ varies in range as $r^{-1}$. It gives $P_2/(2E_0) = 3.75$. The incident level of 240 dB re 1 $\mu$Pa corresponds to 1 MPa, which gives $2E_0 = 37$ Pa. The second order pressure is then found to 140 Pa or 163 dB.
Figure J-4: Same as figure 4-3 except that the nonlinear parameter is $\beta = 1000$ for solid earth.
Figure J-5: Same as figure 4-4 except that the nonlinear parameter is \( \beta = 1000 \) for solid earth.
Figure J-6: Same as figure 4-6 except that the nonlinear parameter is $\beta = 1000$ for solid earth.

### J.3 Noise level

#### J.3.1 In air

In the air, the 1/3 octave band noise spectrum given in Ref.[3] is shown in figure J-7. At 400 Hz, the spectral level is about 8 dB re 20 $\mu$Pa in Hermit Basin, Grand Canyon National Park and about 40 dB re 20 $\mu$Pa in quiet residential environments. This is a 93 Hz frequency band between 356 Hz and 449 Hz. If we assume the spectral density is flat within this band, we can estimate the SPL for a 40 Hz frequency band centered at 400 Hz to be 4 dB re 20 $\mu$Pa in Hermit Basin and 36 dB re 20 $\mu$Pa in quiet residential areas.
Figure J-7: Ambient noise spectrum in air, extracted from figure 3 in Ref.[3].
J.3.2 In water

Figure J-8 shows the nominal high and low ambient noise in the ocean. The values are extracted from figure 2 in Ref.[3].

![Graph showing ambient noise spectral density in the ocean](image)

Figure J-8: Ambient noise spectral density in the ocean [3].

At 10 kHz, the spectral density level is about 27 dB re 1 μPa²/Hz for the nominal low noise condition according to figure J-8. The noise level for a 1 kHz band centered at 10 kHz is estimated to be 57 dB re 1 μPa with the following formula

$$10 \log_{10} \left(10^{27/10} \times 10^8 \times \int_{10512}^{9512} \frac{1}{f^2} df\right) \approx 57 \text{ dB re } 1 \mu\text{Pa}.$$  

Medical ultrasound imaging experiments are usually done in water tanks. But there are very few noise spectral measurements available for water tanks near 10 kHz. One example is shown in Ref.[20], where the noise floor for the frequency band between 1 kHz and 50 kHz was measured to be 86 dB re 1 μPa. According to figure J-8, the noise in the ocean for the same frequency band is between 77 dB (nominal
low) and 104 dB re 1μPa (nominal high). The noise in Ref.[20] is even stronger than the noise in open water in low noise condition. This may be due to contamination by other noises in the experimental tank or low sensitivity hydrophones. We use 57 dB re 1 μPa as an estimation of the noise level in the ocean as well as in the water tank.

At 100 kHz, thermal noise due to random motion of water molecules becomes dominant when the environmental noise is low, as shown in the lower right corner of figure J-8. The power spectral density for the thermal noise increases as 6 dB/octave [51]. At 100 kHz, the spectral density level for thermal noise is about 25 dB re 1 μPa²/Hz. For a 10 kHz band centered at 100 kHz, the thermal noise SPL is estimated to be 65 dB re 1 μPa with the following formula

\[ 10 \log_{10} \left( \frac{10^{25/10}}{10^{10}} \times \int_{95125}^{105125} f^2 df \right) \approx 65 \text{ dB re 1 } \mu \text{Pa}. \] (J.2)

### J.3.3 In solid earth

Figure J-9 shows the ambient seismic noise spectral densities from Ref.[52], which are referred to as the New High Noise Model (NHNM) and New Low Noise Model (NLNM). The noise in terms of surface velocity for a 1 Hz frequency band between 9 Hz and 10 Hz is found to be between \(7 \times 10^{-11} \text{ m/s}\) and \(4 \times 10^{-7} \text{ m/s}\). With the plane wave impedance relation and assuming \(\rho = 3000 \text{ kg/m}^3\) and \(c_0 = 3000 \text{ m/s}\), we estimate the seismic ambient noise in pressure (normal stress) to be between 0.0006 Pa and 4 Pa, or equivalently between 56 dB and 132 dB re 1 μPa.
Figure J-9: Ambient seismic noise spectral density. The New High Noise Model (NHNM) and New Low Noise Model (NLNM) are given in terms of surface velocity square per Hz.
Appendix K

Scattered field from a sphere

The spherical wave expansion is used to calculate the scattered field from a sphere of radius $a$. In a spherical coordinate system $(r, \theta, \phi)$, the first or second order scattered field pressure and normal velocity can be written as

$$P_s(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} h_l(kr) Y_l^m(\theta, \phi), \quad (K.1)$$

$$n \cdot V_s(r, \theta, \phi) = \frac{1}{i\rho_0 c_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} h'_l(kr) Y_l^m(\theta, \phi), \quad (K.2)$$

where $a_{lm}$ are the coefficients determined by the boundary condition, $h_l$ is spherical Hankel function of the first kind of order $l$, $h'_l$ is the derivative of $h_l$ with respect to its argument, $Y_l^m$ is the Laplace spherical harmonics of degree $l$ and order $m$, $k = \omega / c_0$ is the wavenumber, $\rho_0$ and $c_0$ are the ambient density and sound speed.

When the boundary condition is given in terms of pressure as $P_s(a, \theta, \phi) = \tilde{P}_s(\theta, \phi)$, it is a Dirichlet boundary condition, which includes first and second order pressure release conditions (2.10) and (2.32). We expand the known $\tilde{P}_s(\theta, \phi)$ in spherical harmonics as

$$\tilde{P}_s(\theta, \phi) = \sum_{l,m} c_{lm} Y_l^m(\theta, \phi), \quad (K.3)$$
where

\[ c_{lm} = \int_{2\pi} d\phi \int_{\pi} \bar{P}_S(\theta, \phi) Y_l^{m*}(\theta, \phi) \sin \theta d\theta. \]  \hfill (K.4)

The coefficients \( a_{lm} \) for \( P_S \) can then be determined by equating (K.3) with (K.1) evaluated at \( r = a \), as

\[ a_{lm} = \frac{c_{lm}}{h_l(ka)}. \]  \hfill (K.5)

When the boundary condition is given in terms of normal velocity as \( \mathbf{n} \cdot \mathbf{V}_S(a, \theta, \phi) = \bar{V}_{Sn}(\theta, \phi) \), it is a Neumann boundary condition, which includes first and second order rigid boundary conditions (2.11), (2.33) and (2.34). We expand the known function \( \bar{V}_{Sn}(\theta, \phi) \) in spherical harmonics as

\[ \bar{V}_{Sn}(\theta, \phi) = \sum_{l,m} d_{lm} Y_l^m(\theta, \phi) = \frac{1}{i\rho_0 c_0} \sum_{l,m} a_{lm} h_l'(ka) Y_l^m(\theta, \phi), \]  \hfill (K.6)

where

\[ d_{lm} = \int_{2\pi} d\phi \int_{\pi} \bar{V}_{Sn}(\theta, \phi) Y_l^{m*}(\theta, \phi) \sin \theta d\theta. \]  \hfill (K.7)

The coefficients \( a_{lm} \) for \( P_S \) can then be determined by equating equation (K.6) with (K.2) evaluated at \( r = a \), as

\[ a_{lm} = \frac{i\rho_0 c_0 d_{lm}}{h_l'(ka)}. \]  \hfill (K.8)

The expansion series of equation (K.1) is truncated at \( l_{\text{max}} \) when evaluating the scattered fields numerically. An empirical formula \( l_{\text{max}} = ka + 4(ka)^{1/3} + 2 \) [53] is found to give satisfactory convergence for wide range of \( ka \) from \( ka \ll 1 \) to \( ka \gg 1 \) in our applications.
Appendix L

Absence of a complete, self-consistent second order nonlinear acoustic theory in the presence of an object in the past work

As summarized in §2.3, the total second order field has components $p_{II}$, $p_{SS}$, $p_{IS}$ and $p_{SI}$ due to wave-wave interactions, and $p_{S2}$ due to second order scattering. For rigid movable objects, $p_{S2}$ can be further decomposed into $p_{D2}$ from the rigid fixed object and $p_{R2}$ due the second order motion of the object. The second order scattered field $p_{S2}$ or $p_{D2}$ has contribution from the all II, SS, IS and SI interactions, as well as wave-boundary interactions (the quadratic terms in the second order boundary conditions (2.45) and (2.46)) if the first order boundary motion is non-zero. The second order radiated wave $p_{R2}$ is determined via the second order wave-exciting force.

Jones and Beyer [1, 2] conducted an experiment and used the angular dependence of the measured sum frequency second order field to conclude that only the SS mechanism was measured. They did not explain conditions in which the SS or other mechanisms would dominate the field as noted in Appendix E.

In the series papers by Fatemi and Greenleaf [17, 18], Mitri et al. [22], Silva et al.
[23], Chen et al. [24] and Silva et al. [25] on vibro-acoustics and dynamic radiation force, only $p_{R2}$ was theoretically considered away from the object and assumed to always be dominant. All other second order components, however, can be dominant as shown here and need to be carefully checked before measurements can be theoretically explained. Also in references [22, 23, 24, 25] the dynamic radiation force used to determine $p_{R2}$ is found to be missing some components and adding extraneous components as shown in Appendix D. The conclusions based on theories with these missing terms need to be carefully checked.

In the context of vibro-acoustic measurements [17, 18], the importance of $p_{SS}$ was noted by Makris et al. [19] and further analyzed by Thierman [20] who considered spherically symmetric primary scattered fields. Thierman also considered a special type of IS/SI interaction, suggested by Westervelt in personal communication, that is due to multiple reflection of the primary incident and scattered waves between the object and the transducer, and concluded that it was dominant in his experiments followed by $p_{SS}$. In solving for $p_{S2}$, Thierman only included II, omitted SS, SI and SI and omitted the quadratic terms in the second order boundary condition, so his results and conclusions need to be checked.

Silva and Mitri [21] considered $p_{SS}$ for non-spherically symmetric primary scattered fields, and neglected $p_{IS}$, $p_{SI}$ and all $p_{S2}$ components except for that due to SS. By considering only rigid immovable objects, they did not consider or develop a quadratic second order boundary condition, or a $p_{R2}$ component which is necessary for more general boundary conditions and movable objects. The conclusion that SS is the dominant mechanism in any vibro-acoustic type experiment, which is apparently implied in [21], is not generally the case from the analysis presented here but is certainly a special case for asymptotically large range and possibly other special combinations of frequencies, object size, boundary condition and range for non-asymptotically large range. Similarly Silva and Bandeira [45] only considered rigid immovable objects and did not consider or develop a quadratic second order boundary condition, or a $p_{R2}$ component, which is necessary for more general movable objects.

Following [19], the current thesis differs from other past work by identifying sec-
ond order scattering, $S_2$, as a fundamentally different mechanism from the wave-wave interaction mechanisms, $II$, $SS$, $SI$ and $IS$ because we show $S_2$ carries sum or difference frequency response information about the object while $II$, $SS$, $SI$ and $IS$ do not but only carry primary frequency response information about the object. At long range, wave-wave interactions dominate so the sum and difference frequency response information about the object will be masked as noted in the introduction and conclusions of the thesis.
Bibliography


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