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Linear-Quadratic Optimal Control Revisited

by

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ABSTRACT

We offer a new proof of the relationship between the solution of a matrix Riccati equation and the optimal solution of a linear-quadratic regulator problem in the presence of linear terminal constraints.

Key words. Matrix Riccati equation, hamiltonian system, terminal hyperplane.

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1. *INTRODUCTION*

In a previous paper, Coppel [1] presented some new proofs of the relationship between the solution of a matrix Riccati equation and the optimal trajectory of a linear-quadratic regulator problem in case the target is a fixed point and also in case it is free. He did not discuss the same problem in case the target is partially constrained, which is the topic of this paper. In the sequel, primes as superscripts indicate matrix or vector transposition.

We consider the problem of minimizing the cost functional,

$$J(t_1,\xi;u)=x'(t_f)Gx(t_f)+\int_{t_1}^{t_f}[x'Q(t)x+2x'S(t)u+u'R(t)u]dt, \tag{1}$$

subject to the constraints,

$$\dot{x}=A(t)x+B(t)u,\quad t_0\leq t_1<t_f, \tag{2}$$

$$x(t_1)=\xi, \tag{3}$$

$$Dx(t_f)=0, \tag{4}$$

where the coefficient matrices, A, B, Q, S, and R are continuous on $[t_0,t_f]$, the rows of the constant qxn matrix D are linearly independent, and the control u is to selected from the class U of appropriately dimensioned vector-valued functions which are piecewise continuous on $[t_1,t_f]$. The state vector x(t) belongs to R^n and the control vector u (t) belongs to R^m. Without loss of generality, we may suppose that G is symmetric and that Q(t) and R(t) are symmetric for each t. Let $U_0(t_1,\xi)$ denote the subclass of U whose members steer the point ξ to the hyperplane (4) on the time interval $[t_1,t_f]$. Let $\Phi(t,t_0)$ denote the state transition matrix for the system (2). It was shown in [2] that the class U_0 is not empty if and only if

$$\int_{t_1}^{t_f}D\Phi(t_f,t)B(t)B'(t)\Phi'(t_f,t)D'\,dt>0. \tag{5}$$

2. *THE MAIN THEOREM*

Theorem. Suppose,

(i) $R(t)>0$ for $t_0\leq t\leq t_f$,

(ii) Relation (5) holds for all $t_1\in[t_0,t_f)$

(iii) $J(t_0,0;u)>0$ for all nontrivial u in U_0.

Let $X(t),\Lambda(t)$ denote a matrix solution of the hamiltonian system,

$$S'x+B'\lambda+Ru=0 \tag{6}$$

$$\dot{x}=(A-BR^{-1}S')x-BR^{-1}B'\lambda \tag{7}$$

$$\dot{\lambda}=-(Q-SR^{-1}S')x-(A-BR^{-1}S')'\lambda \tag{8}$$

for which

$$DX(t_f)=0, \tag{9}$$

$$\Lambda(t_f)=GX(t_f)+D'D, \tag{10}$$

and where the matrix $(D',X'(t_f))$ has full rank. Then X(t) is nonsingular for all $t\epsilon[t_0,t_f)$. Moreover,

$$\min_{u\epsilon IU} J(t_1,\xi;u)=\xi'\Lambda(t_f)X^{-1}(t_1)\xi,\quad t_1\epsilon(t_0,t_f). \tag{11}$$

The minimum is attained for the unique input,

$$u_0(t)=-R^{-1}(t)[S'(t)X(t)+B'(t)\Lambda(t)]X^{-1}(t_1)\xi. \tag{12}$$

If we set $P(t)=\Lambda(t)X^{-1}(t)$, then P(t) is defined on $[t_0,t_f)$,and is symmetric. Furthermore,

$$u_0(t)=-R^{-1}(t)[S'(t)+B'(t)P(t)]x_0(t), \tag{13}$$

where x_0 is the solution of (2) - (3) corresponding to the input u_0, and P satisfies the matrix Riccati equation,

$$\dot{P}+A'P+PA+Q=(PB+S)R^{-1}(B'P+S') \tag{14}$$

and terminal condition,

$$\lim_{t\to t_f^-} X'(t)[P(t)-G]X(t)=0. \tag{15}$$

Proof. Let η be a fixed vector in R^n and let

$$x_0(t)=X(t)\eta,\quad \lambda_0(t)=\Lambda(t)\eta,\quad u_0(t)=-R^{-1}(t)(S'(t)x_0+B'(t)\lambda_0).$$

Then

$$Dx_0(t_f)=0.$$

Define,

$$\omega(x,u)=x'Qx+2x'Su+u'Ru.$$

Then

$$\omega(x_0,u_0)=x'_0(Q-SR^{-1}S')x_0+\lambda_0'BR^{-1}B'\lambda_0.$$

By Lemma 1 of [1],

$$\int_{t_1}^{t_f}\omega(x_0,u_0)dt=x_0'(t_1)\lambda_0(t_1)-x_0'(t_f)\lambda_0(t_f).$$

Then,

$$J(t_1,X(t_1)\eta;u_0) = x_0'(t_f)Gx_0(t_f)+\int_{t_1}^{t_f}\omega(x_0,u_0)dt=x'_0(t_1)\lambda_0(t_1). \tag{16}$$

Suppose $X(t_1)\eta=0$. Then $x_0(t_1)=0$ and from (16) we find that $J(t_1,0;u_0)=0$. Hypothesis (iii) then implies that $u_0=0$. Hence, from (2), x_0 must vanish identically. From (6) and (8) we find,

$$B'\lambda_0=0 \text{ and } \dot{\lambda}_0=-A'\lambda_0, \quad t\epsilon[t_1,t_f], \tag{17}$$

and from (10) we find that

$$\lambda_0(t_f)=D'D\eta. \tag{18}$$

It was shown in [2] that hypothesis (iii) implies that the only solution of (17) - (18) is $\lambda_0(t)\equiv 0$ and $D\eta=0$. Hence, $\eta'(X'(t_f),D')=0$, and so $\eta=0$. Thus, $X(t_1)$ is nonsingular. If (iii) holds for all $t_1\in[t_0,t_f)$ then $X(t_1)$ is nonsingular for all $t_1\in[t_0,t_f)$.

Let x_0,u_0 be defined as above and let $\eta\neq 0$. Define $\xi=X(t_1)\eta$. Then $\xi\neq 0$,

$$x_0(t)=X(t)X^{-1}(t)\xi, \quad \lambda_0(t)=\Lambda(t)X^{-1}(t_1)\xi,$$

and u_0 satisfies (12).

Let (x,u) be any admissible solution of (2) satisfying (3) and (4). Then as in [1],

$$J(t_1,\xi;u)=x'(t_f)Gx(t_f)+\int_{t_1}^{t_f}\omega(x,u)dt$$

$$=J(t_1,\xi;u_0)+(x(t_f)-x_0(t_f))'G(x(t_f)-x_0(t_f))+\int_{t_1}^{t_f}\omega(x-x_0,u-u_0)dt\geq J(t_1,\xi;u_0),$$

and equality holds if and only if $u\equiv u_0$. Furthermore, from (16) we find,

$$J(t_1,\xi;u_0) = x_0'(t_1)\lambda_0(t_0) = \xi'\Lambda(t_1)X^{-1}(t_1)\xi,$$

and if we define $P(t)=\Lambda(t)X^{-1}(t)$ then

$$J(t_1,\xi;u_0)=\xi'P(t_1)\xi.$$

From the definition of P(t) and from (7) and (8) we obtain (14). We see from (10) that

$$X'(t)(P(t)-G)X(t)=X'(t)(\Lambda(t)-GX(t)).$$

Thus,

$$\lim_{t\to t_f^-} X'(t)(P(t)-G)X(t) = X'(t_f)D'D = 0.$$

To see that P is symmetric, we note that it follows from (6) - (7) that

$$\frac{d}{dt}(X'(t)\Lambda(t)-\Lambda'(t)X(t))=0. \tag{19}$$

Since $X'(t_f)\Lambda(t_f)=X'(t_f)GX(t_f)$then (19) implies that

$$X'(t)\Lambda(t)=\Lambda'(t)X(t). \tag{20}$$

Multiplying (20) by $X^{-1}(t)$ on the right and $[X'(t)]^{-1}$ on the left we find that

$$\Lambda(t)X^{-1}(t)=[X'(t)]^{-1}\Lambda'(t),$$

and this completes the proof.

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