OPTIMUM RETRO-THRUST TRAJECTORIES

by

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ABSTRACT

The calculus of variations is used to determine the thrust program and the trajectory which will minimize the propellant necessary for a space vehicle to effect a soft landing on an airless moon or planet. In establishing the equations of motion, this moon or planet is approximated by a flat plane with a constant gravitational field.

The trajectory is shown to consist of two portions; the first is a free fall from the initial altitude to the altitude where the retrorocket is ignited. The second portion is the powered phase which slows the vehicle to a stop just as the ground is reached. The retrorocket is shown to operate at maximum thrust during the powered phase and the tangent of the thrust attitude angle is shown to be a linear function of time.

The two portions of the trajectory are both solved in closed form. However, due to the transcendental nature of the closed form solution of the powered portion of the trajectory with time-varying thrust attitude, the total optimum trajectory which meets all of the boundary conditions must be pieced together from the trajectory equations of the two phases by trial and error.

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SYMBOLS

A, B, C, D	Constant of the thrust attitude program
a, b	Nondimensional constants of the thrust attitude program
c	Rocket Exhaust Velocity
c_1, c_2, c_3, c_4, c_5	Constants of integration
F	Fundamental function
g	Gravitational acceleration
G	Nondimensional gravitational acceleration
I	The function (H) of the boundary conditions to be minimized
K	A constant
m	Mass
$\dot{m}$	Mass flow
MR	Mass ratio
T	Thrust
t	Time
Δt	Burning time
v_x, v_y	Horizontal and vertical velocity
V_x, V_y	Nondimensional horizontal and vertical velocity
V_1	Total initial velocity
V_3	Total velocity after one burning period (See Fig. 2)
$\Delta V_a, \Delta V_b$	Changes in vehicle momentum/unit mass
$\Delta v_x, \Delta v_y$	Increments of velocity during retrorocket operation

SYMBOLS (Cont'd.)

x	Horizontal distance
X	Nondimensional horizontal distance
y	altitude
Y	Nondimensional altitude
$\beta \equiv -\dot{m}$	
ρ	Thrust inclination angle
$\eta \equiv \sqrt{\beta_{MAX} - \beta}$	
$\lambda_1 \rightarrow \lambda_7$	Lagrange multipliers
$\xi \equiv \sqrt{\beta}$	
τ	Nondimensional time
$\varphi_1 \rightarrow \varphi_7$	Constraint equations
$\psi_1 \rightarrow \psi_{12}$	Physical variables which appear in the transversality condition

Subscripts

x	Horizontal direction
y	Vertical direction
max	Maximum value
1	Initial point on trajectory
2	Ignition point, a corner in the trajectory
3,4	Possible additional corners in the trajectory (See Fig. 2)
0	Landing point

Miscellaneous

$(\dot{\ })$	Time derivative $\dot{z} = \frac{dz}{dt}$
$\ln$	Natural logarithm

OBJECT

The object of this analysis is to determine the thrust program and the trajectory which must be followed by a space vehicle so that a soft landing can be made on an airless moon or planet using a minimum amount of retro-rocket propellant.

INTRODUCTION

The accomplishment of a soft landing by a space vehicle on a planet or moon requires a mechanism which can dissipate the large relative energy of the vehicle. If the soft landing is to be made on a moon or planet which is not surrounded by an atmosphere or for which the atmospheric effects are negligible, present day technology provides only retrorockets as a means of accomplishing this mission. A problem now arises. If the velocity of the vehicle is known at a given altitude, what is the path which must be followed and how must the retrorocket thrust be programmed, both in magnitude and in the inclination of the thrust vector, so that a minimum amount of propellant is required to bring the vehicle to a landing with nearly zero velocity.

Although similar to the problem of optimizing rocket take-off, which has been widely treated in the literature, it is apparent that the landing problem will have a different solution than simply the reverse of the take-off trajectory. For rocket take-off, the weight of the vehicle is greatest on the ground and grows lighter as propellant is burned and altitude is gained. The acceleration for a given thrust will therefore increase with altitude. For retrorocket landing, the acceleration for a given thrust will be least at the ignition altitude and increase as altitude is lost. Aerodynamic drag, if effectively contributing to the trajectories, would accentuate this difference because drag will always act opposite to the direction of motion.

The problem treated here is limited to that of a vehicle approaching an airless planet or moon on a steep trajectory and effecting a landing entirely by retrorocket. By nature, this problem lends itself to solution by the use of the calculus of variations.

BASIC APPROXIMATIONS AND EQUATIONS OF MOTION

Two approximations concerning the nature of the target planet are made to facilitate the variational solution. The landing planet can be approximated by a flat plane of infinite extent provided the horizontal velocity is small compared to the satellite velocity and provided the horizontal distance traversed is reasonably small. The gravitational field strength can be approximated as a constant with altitude provided the altitudes are small compared with the planet radius. These approximations will work best when the vehicle is coming in on a steep (close to the vertical) approach path which will intersect the planet surface as in a hyperbolic approach. The approximations do not appear satisfactory for the case of a landing from a satellite orbit.

The desired trajectory will lie in a vertical plane since the initial velocity vector will identify that plane. Any motion out of this plane would not aid in the vehicle deceleration. The problem of guidance and maneuvering to a specific landing point is excluded from consideration here.

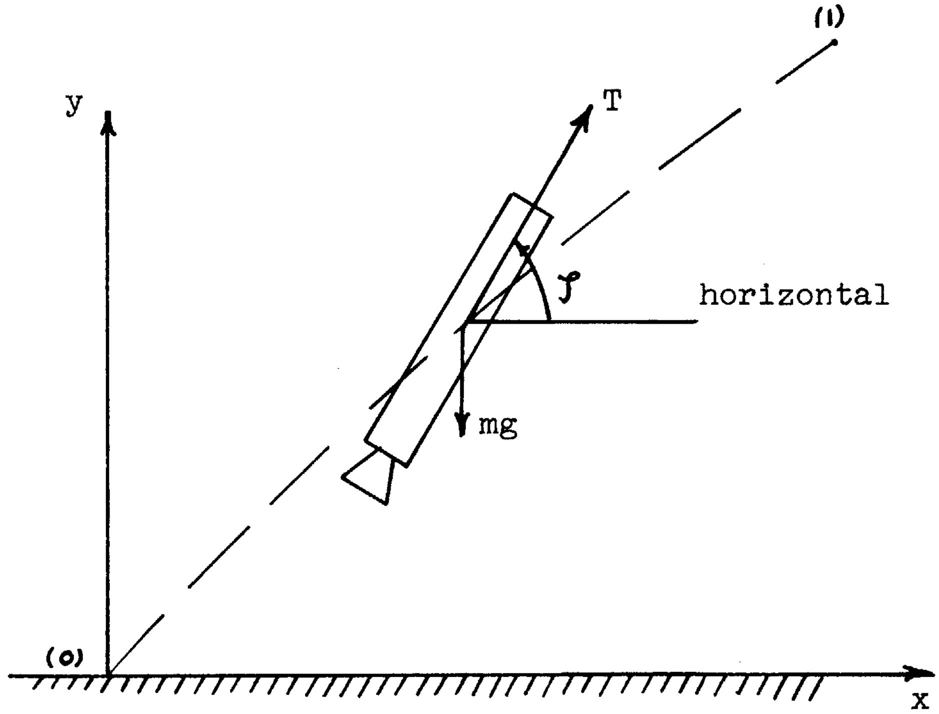


Figure 1.

The equations of motion for a rocket vehicle in a vertical plane near this idealized planet are:

$$T \cos \gamma = m \dot{v}_x \tag{1}$$

$$T \sin \gamma - mg = m \dot{v}_y \tag{2}$$

The thrust is proportional to the exhaust velocity c and to the mass flow $(\dot{m})$. In the absence of atmospheric pressure the exhaust velocity is assumed to be essentially constant. The mass flow, however, is considered to be a variable for this solution and is always negative. The solution is facilitated by defining a new function (β) which is always positive:

$$\beta = -\dot{m} \tag{3}$$

The equation for thrust becomes :

$$T = \beta c \tag{4}$$

The mass flow which can be obtained from a given retrorocket will be limited to some maximum value. The mass flow must therefore be limited in the variational procedure to :

$$0 \le \beta \le \beta_{MAX}$$

By substituting for thrust and dividing by the instantaneous mass, the equations of motion become:

$$\dot{v}_x = \frac{\beta c}{m} \cos \varphi \tag{5}$$

$$\dot{v}_y = \frac{\beta c}{m} \sin \varphi - g \tag{6}$$

VARIATIONAL PROCEDURE

The Constraint Equations

The steps in derivation of a solution by variational calculus are outlined briefly below. This mathematical method has received considerable attention recently because of its importance in optimization of rocket thrust programs and as a result is well-documented in the literature (1,2,3,4).

The constraint equations for the variational procedure in Mayer's form are:

$$\varphi_1 = \dot{v}_x - \frac{\beta c}{m} \cos \psi = 0 \tag{7}$$

$$\varphi_2 = \dot{v}_y - \frac{\beta c}{m} \sin \psi + g = 0 \tag{8}$$

$$\varphi_3 = v_x - \dot{x} = 0 \tag{9}$$

$$\varphi_4 = v_y - \dot{y} = 0 \tag{10}$$

$$\varphi_5 = \beta + \dot{m} = 0 \tag{11}$$

$$\varphi_6 = \beta - \xi^2 = 0 \tag{12}$$

$$\varphi_7 = \beta_{max} - \beta - \eta^2 = 0 \tag{13}$$

Constraint equations φ_1 and φ_2 are simply the equations of motion (5) and (6). Equations φ_3 , φ_4 , and φ_5 define v_x , v_y and β respectively. Constraint equations φ_6 and φ_7 define the variables ξ and η respectively and serve the purpose of limiting the mass flow as follows (3):

$$0 \leq \beta \leq \beta_{MAX}$$

Fundamental Function

The "fundamental function" F which appears in the Mayer form of the variational calculus apparatus is:

$$F = \lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \lambda_3 \varphi_3 + \lambda_4 \varphi_4 + \lambda_5 \varphi_5 + \lambda_6 \varphi_6 + \lambda_7 \varphi_7$$

The Lagrangian multipliers λ_1 through λ_7 are, in general, undetermined functions of time. Substituting the constraint equations (7)-(13),

$$\begin{aligned} F = {} & \lambda_1 \left(\dot{v}_x - \frac{\beta c}{m} \cos\rho \right) + \lambda_2 \left(\dot{v}_y - \frac{\beta c}{m} \sin\rho + g \right) \\ & + \lambda_3 (v_x - \dot{x}) + \lambda_4 (v_y - \dot{y}) + \lambda_5 (\beta + \dot{m}) \\ & + \lambda_6 (\beta - \xi^2) + \lambda_7 (\beta_{MAX} - \beta - \eta^2) = 0 \end{aligned} \qquad (14)$$

<u>Euler Lagrange Equations</u>

Time is the independent variable in this analysis while x, y, v_x, v_y, m, β, ρ, η, and ξ are the dependent variables. The Euler Lagrange equations are therefore:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 = -\frac{d}{dt}(-\lambda_3) = 0 \qquad (15)$$

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 = -\frac{d}{dt}(-\lambda_4) = 0 \qquad (16)$$

$$\frac{\partial F}{\partial v_x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}_x}\right) = 0 = \lambda_3 - \frac{d}{dt}(\lambda_1) = 0 \qquad (17)$$

$$\frac{\partial F}{\partial v_y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}_y}\right) = 0 = \lambda_4 - \frac{d}{dt}(\lambda_2) = 0 \qquad (18)$$

$$\frac{\partial F}{\partial m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{m}}\right) = 0 = \lambda_1 \frac{\beta c}{m^2} \cos\rho + \lambda_2 \frac{\beta c}{m^2} \sin\rho - \frac{d}{dt}(\lambda_5) = 0 \qquad (19)$$

$$\frac{\partial F}{\partial \beta} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\beta}}\right) = 0 = -\lambda_1 \frac{c}{m} \cos\rho - \lambda_2 \frac{c}{m} \sin\rho + \lambda_5 + \lambda_6 - \lambda_7 = 0 \qquad (20)$$

$$\frac{\partial F}{\partial \rho} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\rho}}\right) = 0 = \lambda_1 \frac{\beta c}{m} \sin\rho - \lambda_2 \frac{\beta c}{m} \cos\rho = 0 \qquad (21)$$

$$\frac{\partial F}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\eta}}\right) = 0 = -2\eta\lambda_7 = 0 = \eta\lambda_7 = 0 \tag{22}$$

$$\frac{\partial F}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\xi}}\right) = 0 = -2\xi\lambda_6 = 0 = \xi\lambda_6 = 0 \tag{23}$$

Time does not appear explicitly in the fundamental function. Therefore the alternate equation $\frac{d}{dt}\left(F - \dot{x}\frac{\partial F}{\partial \dot{x}} - \dot{y}\frac{\partial F}{\partial \dot{y}} - \cdots\right) = \frac{\partial F}{\partial t}$ integrates to

$\left(F - \dot{x}\frac{\partial F}{\partial \dot{x}} - \dot{y}\frac{\partial F}{\partial \dot{y}} - \cdots\right) = K$ a constant

Since $F = 0$ the alternate equation becomes:

$$\dot{x}\lambda_3 + \dot{y}\lambda_4 - \dot{v}_x\lambda_1 - \dot{v}_y\lambda_2 - \dot{m}\lambda_5 = K \tag{24}$$

Integration of Euler Lagrange equations (15) and (16) yields:

$$\lambda_3 = A \tag{25}$$

$$\lambda_4 = B \tag{26}$$

Since λ_3 and λ_4 are constants, equations (17) and (18) can be integrated to yield:

$$\lambda_1 = At + C \tag{27}$$

$$\lambda_2 = Bt + D \tag{28}$$

Equation (21) yields two solutions:

$$\beta = 0 \qquad \text{or} \qquad \tan\beta = \frac{\lambda_2}{\lambda_1}$$

When $\beta = 0$ the thrust angle has no meaning, therefore the important solution is:

$$\tan \beta = \frac{\lambda_2}{\lambda_1} = \frac{Bt + D}{At + C} \tag{29}$$

Equations (22) and (23) along with constraints φ_6 and φ_7 give three possible solutions.

Case I. $\lambda_7 = 0$ $\quad\xi = 0$ $\quad\beta = 0$

Case II. $\lambda_7 = 0$ $\quad\lambda_6 = 0$ $\quad\beta = \beta(t)$

where $\beta = \beta(t)$ is a variable within the limit $0 \le \beta \le \beta_{max}$.

Case III. $\eta = 0$ $\quad\lambda_6 = 0$ $\quad\beta = \beta_{max}$

Transversality Condition

The transversality condition is expressed by the following relation:

$$\Big[\Big(F - \dot{x}\frac{\partial F}{\partial \dot{x}} - \dot{y}\frac{\partial F}{\partial \dot{y}} - \dot{v}_x\frac{\partial F}{\partial \dot{v}_x} - \dot{v}_y\frac{\partial F}{\partial \dot{v}_y} - \dot{m}\frac{\partial F}{\partial \dot{m}}\Big)dt + \frac{\partial F}{\partial \dot{x}}dx + \frac{\partial F}{\partial \dot{y}}dy + \frac{\partial F}{\partial \dot{v}_x}dv_x + \frac{\partial F}{\partial \dot{v}_y}dv_y + \frac{\partial F}{\partial \dot{m}}dm\Big]_0^1 + dh = 0 \tag{30}$$

Minimization of propellant expenditure corresponds to minimizing the initial mass when the final mass is known. Therefore the following relation holds:

$$I = h(x_0, y_0, m_0, \cdots t_0, x_1, y_1, m_1, \cdots t_1) = m_1 \tag{31}$$

$$dh = dm_1 \tag{32}$$

The physical variables which appear in the transversality condition are the following:

$\psi_1 = t_1$ unknown $\qquad \psi_7 = t_0$ given

$\psi_2 = x_1$ unknown $\qquad \psi_8 = x_0 = 0$

$\psi_3 = y_1$ given $\qquad \psi_9 = y_0 = 0$

$\psi_4 = v_{x_1}$ given $\qquad \psi_{10} = v_{x_0} = 0$

$\psi_5 = v_{y_1}$ given $\qquad \psi_{11} = v_{y_0} = 0$

$\psi_6 = m_1$ unknown $\qquad \psi_{12} = m_0$ given

Only the unknown variables have differentials. Therefore:

$$dt_1 \neq 0$$

$$dx_1 \neq 0$$

$$dm_1 \neq 0$$

But:

$$dy_1 = 0$$

$$dy_0 = 0$$

$$dv_{x_1} = 0$$

etc.

The transversality condition becomes:

$$K\, dt_1 - \lambda_3\, dx_1 + \lambda_{5_1}\, dm_1 + dm_1 = 0 \qquad (33)$$

Since t_1 , x_1 and m_1 are independent:

$$K = 0 \tag{34}$$

$$\lambda_3 = A = 0 \tag{35}$$

$$\lambda_{5_1} = -1 \tag{36}$$

Because $A = 0$ by the transversality condition, $\lambda_1 = C$.
Therefore:

$$\tan \rho = \frac{Bt + D}{C} \tag{37}$$

That is, the tangent of the thrust angle is a linear function of time.

THE TRAJECTORY

Subarcs of the Trajectory

The optimum trajectory (that is the one which minimizes propellant expenditure) will be composed of a series of subarcs which correspond to the possible solutions of Euler-Lagrange equations (22) and (23), Cases I, II and III. Case I arcs will be the portions of the trajectory which occur when the mass flow and hence the thrust is zero. This is the condition of free fall. Arcs of Case II solutions correspond to possible portions of the trajectory where the mass flow and thrust are allowed to vary with time within the limits, $0 \leq \beta \leq \beta_{MAX}$. Solutions of this type will be shown to be unacceptable. Case III arcs are the arcs where the thrust and mass flow are maximum. During these powered flight portions of the trajectory, the thrust vector must be rotated so that $\tan \rho = \frac{Bt + D}{C}$. This rotation allows the initial altitude (y_1) boundary condition to be met. (If this boundary condition is removed so that y_1 is unknown, the transversality condition will dictate that $\lambda_y = B = 0$, making the tangent of the thrust angle constant. Some combinations of the initial altitude and velocity can also be expected to require arcs of constant thrust attitude.)

The Arrangement of the Trajectory

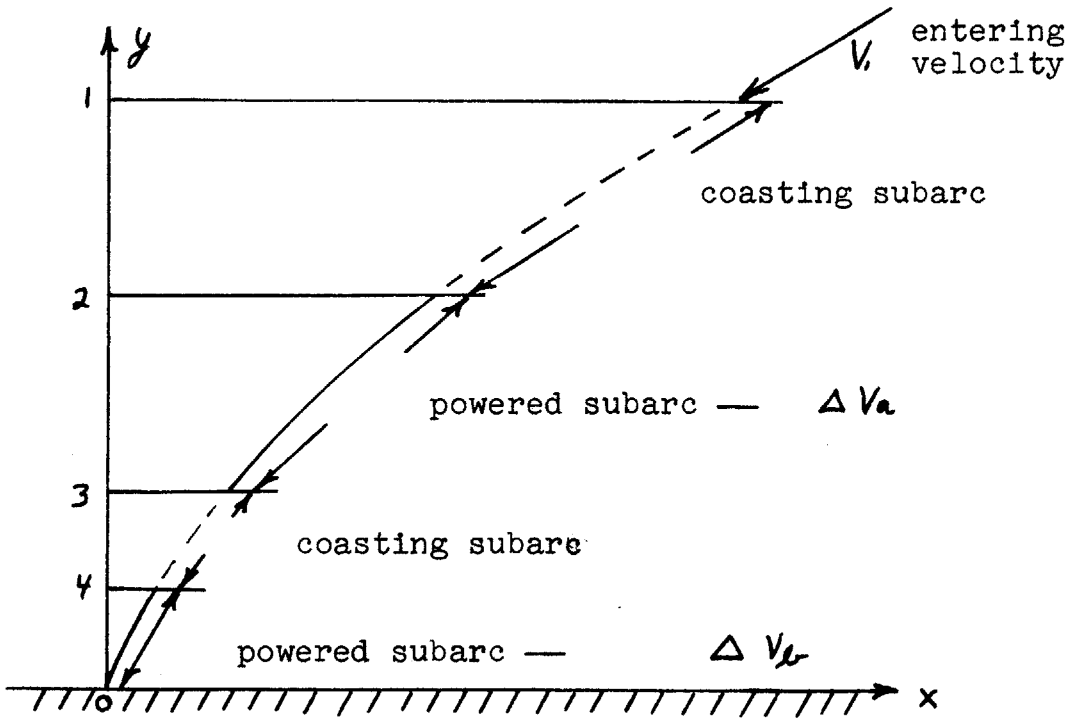


Figure 2.

The vehicle has an initial amount of energy, both kinetic and potential, which has to be removed in order to meet the boundary condition of zero velocity at landing. This energy is removed during the portions of the trajectory where thrust is applied. The assumed trajectory shown above contains a general combination of coasting and burning subarcs which might yield an optimum trajectory. By identifying the change in momentum of the vehicle, per unit mass, during the two burning periods in terms of the energy/unit mass actually removed from the vehicle, the velocity increments for the two periods of burning are, respectively:

$$\Delta V_a = \sqrt{V_1^2 + 2g(y_1 - y_3) - V_3^2} \tag{38}$$

$$\Delta V_b = \sqrt{V_3^2 + 2gy_3} \tag{39}$$

The sum of the velocity increments is a measure of the mass of propellant which must be expended to bring the vehicle to rest at zero altitude. When the sum of the velocity increments is a minimum, the propellant expenditure will be minimum and hence the trajectory will be optimum. Summing equations (38) and (39):

$$(\Delta V_a + \Delta V_b) = \sqrt{V_1^2 + 2gy_1 + 2\sqrt{V_3^2 + 2gy_3}\sqrt{V_1^2 + 2g(y_1 - y_3) - V_3^2}} \tag{40}$$

The initial conditions fix the values of V_1 and y_1 so that the only variables which can be changed to minimize ($\Delta V_a + \Delta V_b$) are V_3 and y_3 . Since $\sqrt{V_3^2 + 2gy_3}\sqrt{V_1^2 + 2g(y_1 - y_3) - V_3^2}$ is never negative, ($\Delta V_a + \Delta V_b$) will be minimized when this term is zero.

This term can be zero only if:

(A) $$V_3^2 = -2 g y_3$$

(B) $$V_3^2 - V_1^2 = 2g\,(y_1 - y_3)$$

(C) or $V_3 = 0$ and $y_3 = 0$

Case (A) is clearly not possible since V_3^2 is always positive and $-2gy_3$ is always negative. Case (B) occurs when the gain in kinetic energy between points 1 and 3 is exactly equal to the loss in potential energy between these two points. Therefore ΔV_a must be equal to zero and the vehicle coasts from point (1) to point (4) with no burning period in between. Case (C) occurs when the burnout conditions at the end of the first burning period exactly meet the boundary conditions of zero velocity at zero altitude. Cases (B) and (C) both show that the minimizing trajectory will be made up of one coasting subarc and one portion where thrust is applied, in that order. Burnout will occur at zero altitude and zero velocity. Cases (B) and (C) are clearly different statements of the same trajectory.

This analysis does not show the composition of the period of burning; that is, the powered portion of the trajectory could be made up of combinations of subarcs with maximum thrust and subarcs with variable thrust as long as no coasting occurred between these possible powered subarcs. This possibility is investigated later.

Coasting Flight

When the thrust is zero, as it is in the subarc corresponding to Case I, the equations of motion reduce to:

$$\dot{v}_x = 0 \tag{41}$$

$$\dot{v}_y = -g \tag{42}$$

These equations contain all the information needed for the integration to be carried out.

The first integration yields:

$$v_x = c_1 \tag{43}$$

$$v_y = -gt + c_2 \tag{44}$$

The second integration yields:

$$x = c_1 t + c_3 \tag{45}$$

$$y = -\frac{1}{2} g t^2 + c_2 t + c_4 \tag{46}$$

The coasting subarc is the initial portion of the trajectory. Therefore the boundary conditions of initial altitude and velocity are the initial conditions of the coasting subarcs; that is, when $t = t_1$:

$$v_x = v_{x_1}$$

$$v_y = v_{y_1}$$

$$y = y_1$$

Applying these initial conditions to the coasting subarc equations (45) and (46) yields

$$v_x = v_{x_1} \tag{47}$$

$$x = v_{x_1} t + C_3 \tag{48}$$

$$v_y - v_{y_1} = -g(t - t_1) \tag{49}$$

$$y - y_1 = -\frac{1}{2} g (t - t_1)^2 + v_{y_1} (t - t_1) \tag{50}$$

Time can be eliminated from the altitude equations as follows:

$$t - t_1 = -\frac{(v_y - v_{y_1})}{g} \tag{51}$$

$$y_1 - y = \frac{1}{2g} \left[v_y^2 - v_{y_1}^2 \right] \tag{52}$$

The two remaining Euler Lagrange equations (19) and (20) must be investigated to prevent a contradiction to the equations of coasting flight. For Case I, $\beta = 0$ and $\lambda_7 = 0$. Therefore equation (19) and (20) reduce to:

$$\frac{d}{dt}(\lambda_5) = 0 \tag{53}$$

$$\lambda_6 = \lambda_1 \frac{c}{m} \cos\varphi + \lambda_2 \frac{c}{m} \sin\varphi - \lambda_5 \tag{54}$$

These equations serve only to evaluate the Lagrangian multipliers λ_5 and λ_6 , which are inconsequential to the integration. Therefore these equations do not contradict the coasting flight solution.

<u>Variable Thrust Subarc</u>

As in the case of the coasting flight, the unintegrated Euler Lagrange equations (19) and (20) must be investigated. For the variable thrust subarc, Case II:

$$\lambda_7 = 0 \quad \text{and} \quad \lambda_6 = 0$$

Euler-Lagrange equation (2) becomes:

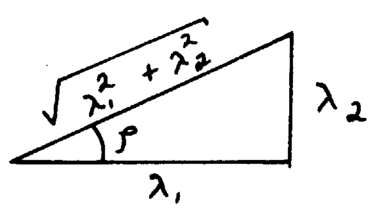
$$-\lambda_1 \frac{c}{m} \cos\rho - \lambda_2 \frac{c}{m} \sin\rho + \lambda_5 = 0 \tag{55}$$

or

$$\lambda_5 = \frac{c}{m} \left(\lambda_1 \cos\rho + \lambda_2 \sin\rho \right) \tag{56}$$

But:

$$\tan\rho = \frac{\lambda_2}{\lambda_1} \tag{29}$$



Therefore:

$$\cos \varphi = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{57}$$

$$\sin \varphi = \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{58}$$

Substituting and combining terms:

$$\lambda_5 = \frac{c}{m}\sqrt{\lambda_1^2 + \lambda_2^2} = \frac{c}{m}\sqrt{C^2 + (Bt + D)^2} \tag{59}$$

But from Euler-Lagrange equation (19):

$$\frac{d\lambda_5}{dt} = \frac{\beta c}{m^2}\sqrt{C^2 + (Bt + D)^2} \tag{60}$$

Differentiating (59) yields:

$$\frac{d\lambda_5}{dt} = \frac{\beta c}{m^2}\sqrt{C^2 + (Bt + D)^2} + \frac{c}{m}\frac{(Bt + D)\, B}{\sqrt{C^2 + (Bt + D)^2}} \tag{61}$$

Equating (60) and (61):

$$0 = \frac{c}{m}\frac{(Bt + D)\, B}{\sqrt{C^2 + (Bt + D)^2}} \tag{62}$$

Or:

$$B = 0 \tag{63}$$

But this is constant attitude thrust [Ref. equation (37)].

Three questions now arise:

(1) Do the boundary conditions permit a constant attitude thrust as the general case?

(2) If constant attitude thrust is the solution for a special set of boundary conditions, does a varying thrust magnitude yield an optimum trajectory?

(3) Can the burning period be composed of combinations of constant attitude, varying magnitude thrust arcs and varying attitude, maximum thrust arcs?

The answer to the first question can be found mathematically and by physical reasoning. Since the differential equations are all first order and linear, the number of boundary conditions must equal the number of differential equations. There are eight physical boundary conditions and two boundary conditions due to the transversality condition. These ten boundary conditions are:

1. x_0 given
2. $y_0 = 0$
3. $v_{x_0} = 0$
4. $v_{y_0} = 0$
5. m_0 given
6. y_1 given
7. v_{x_1} given

8. v_y, given

9. $\lambda_3 = A = 0$ } from the transversality condition

10. $\lambda_{5_f} = -1$

The other transversality result, $K = 0$, pertains to the alternate equation.

There are also ten first-order linear differential equations: constraint equations φ_1, φ_2, φ_3, φ_4, φ_5 and Euler-Lagrange equations (15), (16), (17), (18), and (19).

In considering the variable thrust subarc when the thrust attitude is constant, equations (19) and (20) were shown to be equivalent in equation (62), but equation (20) is not a differential equation. There are, therefore, ten boundary conditions and only nine equations. As a result, the boundary conditions cannot be met in general by a constant attitude, variable magnitude thrust, trajectory. Variable magnitude thrust might still be a solution if a constant attitude thrust program is a natural result of the boundary conditions. A physical feel for this problem can be obtained by considering just the burning phase. (A similar argument holds for the complete trajectory.) If the thrust attitude was constant the thrust magnitude and inclination could be adjusted to just cancel the initial velocity vector, but the altitude where the cancellation occurred could not be expected to be correct.

The answer to question two can be found from a further look into the energy conditions. For a single burning period:

$$\Delta V^2 = V_1^2 + 2 g y_1 \tag{64}$$

But ΔV and V_1 will each be composed of two perpendicular components. Therefore:

$$\Delta V^2 = \Delta v_x^2 + \Delta v_y^2 \tag{65}$$

$$V_1^2 = v_{x_1}^2 + v_{y_1}^2 \tag{66}$$

Substituting:

$$\Delta v_x^2 + \Delta v_y^2 = v_{x_1}^2 + v_{y_1}^2 + 2 g y_1 \tag{67}$$

Δv_x will remove the horizontal velocity component and Δv_y will remove the initial vertical velocity plus the velocity gained in free fall. Therefore:

$$\Delta v_x = v_{x_1} \tag{68}$$

$$\Delta v_y = \sqrt{v_{y_1}^2 + 2 g y_1} \tag{69}$$

For the case of constant attitude, variable thrust, the equations of motion can be easily integrated to yield Δv_x and Δv_y . The equations of motion are:

$$\dot{v}_x = \frac{\beta c}{m} \cos \varphi \tag{5}$$

$$\dot{v}_y = \frac{\beta c}{m} \sin \varphi - g \tag{6}$$

Integrating :

$$\Delta v_x = c \cos\varphi \int \frac{\beta}{m}\, dt \tag{70}$$

$$\Delta v_y = c \sin\varphi \int \frac{\beta}{m}\, dt - g\,\Delta t \tag{71}$$

where Δt is the burning time.

But : $\beta = - \dot{m} = - \frac{dm}{dt}$

Therefore :

$$\Delta v_x = - c \cos\varphi \ln m \Big]_{m_2}^{m_o} \tag{72}$$

$$\Delta v_x = c \cos\varphi \ln MR \tag{73}$$

where MR = mass ratio $= \frac{m_2}{m_o} > 1$

$$\Delta v_y = c \sin\varphi \ln MR - g\,\Delta t \tag{74}$$

For the constant attitude, variable thrust case therefore:

$$c \cos\varphi \ln MR = v_{x_1} \tag{75}$$

$$c \sin\varphi \ln MR - g\,\Delta t = \sqrt{v_{y_1}^2 + 2gy_1} \tag{76}$$

Clearly the mass ratio and hence the required propellant is minimized when the burning time Δt is minimized. The integration, $\int \frac{\beta}{m} dt$ is independent of the nature of $\beta(t)$ and $m(t)$ so the variable thrust solution is non-unique. Since the minimum propellant utilization will occur when the burning time is minimum, from equation (76), setting $\beta(t) = \beta_{MAX}$ will yield the required optimum trajectory. But this is just a special case of the maximum thrust solution, Case III.

Considering now the third question, if the burning portion of the trajectory is to be composed of constant attitude subarcs and variable attitude subarcs, these subarcs must be pieced together at "corners." A condition which applies at these junctures is the Erdman-Weierstrass corner condition, which states that

$\frac{\partial F}{\partial \dot{z}_i}$ and $\sum \dot{z}_i \frac{\partial F}{\partial \dot{z}_i}$ (where $\dot{z}_i = \dot{x}$, or $\dot{y}$, or $\dot{v}_x$ etc.)

are continuous across the corner. Therefore:

$$\lambda_3^+ = \lambda_3^- \tag{77}$$

$$\lambda_4^+ = \lambda_4^- \tag{78}$$

$$\lambda_1^+ = \lambda_1^- \tag{79}$$

$$\lambda_2^+ = \lambda_2^- \tag{80}$$

$$\lambda_5^+ = \lambda_5^- \tag{81}$$

and $K^+ = K^-$ since $\sum \dot{z}_i \frac{\partial F}{\partial \dot{z}_i} = K$ (82)

The plus and minus superscripts refer to conditions just before and just after the corner. Equations (77) and (82) are automatically satisfied since $\lambda_3 = 0$ and $K = 0$. Equation (81) is of no consequence since λ_5 has no significance. Equations (78), (79), and (80) show that B, C, and D are continuous across any corner, and hence $\tan \varphi$ must be a single function of time throughout the powered flight portion of the trajectory. Thus the powered portion of the trajectory is made up of only one subarc with one thrust attitude program, which could be constant or varying, but not a combination of both.

These corner conditions must also hold across the corner between the coasting arc and the thrust arc but thrust attitude has no meaning during the coasting phase.

The conclusions that one can draw are:

(1) The thrust attitude program must be a varying function of time in general to meet the boundary conditions.

(2) The optimum trajectory will consist of only two subarcs; namely, (a) a coast from altitude to where burning is initiated, and (b) a burning period at maximum thrust with thrust attitude defined by a single function. Burnout occurs at the instant velocity and altitude are simultaneously brought to zero.

(3) The variable thrust case is shown to be non-unique and of no interest in finding the optimum trajectory. (This is proved by G. Leitmann in Reference 2.)

Maximum Thrust Subarc

The powered portion of the trajectory will correspond to the Case III solutions:

$$\beta = \beta_{MAX} \qquad \eta = 0 \qquad \lambda_6 = 0$$

As before the Euler Lagrange equations (19) and (20) must be satisfied. But Euler Lagrange equation (19) evaluates λ_5 and equation (20) evaluates λ_7. Lagrange multipliers λ_5 and λ_7 have no physical significance so (19) and (20) are satisfied but need not be solved.

The equations of motion for maximum thrust and varying thrust attitude are :

$$\dot{v}_x = \frac{\beta_{MAX}\, c}{m} \cos \varphi \tag{83}$$

$$\dot{v}_y = \beta_{MAX}\, c \sin \varphi - g \tag{84}$$

The mass becomes a known function of time when $\beta = \beta_{MAX}$. The $\sin \varphi$ and $\cos \varphi$ are also known functions of time from the Euler Lagrange equations. Therefore $\dot{v}_x$ and $\dot{v}_y$ are completely known functions of time and no additional information is needed to complete the integration.

The time function of mass can be found from integration of constraint equation φ_5 with constant mass flow:

$$\dot{m} = -\beta_{MAX} \tag{85}$$

$$m = -\beta_{MAX} \int dt \tag{86}$$

$$m = -\beta_{MAX}\, t + C_5 \tag{87}$$

When $t = t_0$:

$m = m_0$

Therefore:

$$m = m_0 - \beta_{MAX}(t - t_0) \tag{88}$$

The initial conditions for the powered portion of the trajectory will depend on the coasting portion. The final conditions are all known, however, from the boundary conditions. For this reason it is more convenient to integrate the equations of motion in reverse. This integration would start on the ground with zero velocity and with burnout mass and then proceed backwards in time and with increasing mass until the initial conditions are met. A convenient time reference for this procedure is $t_0 = 0$ with time negative throughout the integration. Therefore:

$$m = m_0 - \beta_{MAX}\, t \tag{89}$$

Nondimensionalization Procedure

Nondimensionalizing the equations of motion provides a concise form which facilitates the integration. The important ratios which can be used to nondimensionalize the equations can be found using the Buckingham Pi Theorem (Ref. 6).

Variables	Dimensions
x or y	L
v_x or v_y	LT^{-1}
m	M
β	MT^{-1}
c	LT^{-1}
g	LT^{-2}
t	T

Table 1

Where:

L = length

M = mass

T = time

Nondimensionalizing with respect to the known quantities c, β_{MAX} and m_0 :

$$X = \frac{\beta_{MAX}}{c \; m_0} \; x \tag{90}$$

$$Y = \frac{\beta_{MAX}}{c \; m_0} \; y \tag{91}$$

$$V_x = \frac{v_x}{c} \tag{92}$$

$$V_y = \frac{v_y}{c} \tag{93}$$

$$G = \frac{m_0}{c \, \beta_{MAX}} \; g \tag{94}$$

$$\tau = \frac{\beta_{MAX}}{m_0} \; t \tag{95}$$

Nondimensionalization proceeds by first writing the equations of motion as known functions of time:

$$\frac{d v_x}{d t} = \frac{\beta_{MAX} \, c}{m_0 - \beta_{MAX} t} \; \frac{C}{\sqrt{C^2 + (Bt + D)^2}} \tag{96}$$

$$\frac{d v_y}{d t} = \frac{\beta_{MAX} \, c}{m_0 - \beta_{MAX} t} \; \frac{Bt + D}{\sqrt{C^2 + (Bt + D)^2}} - g \tag{97}$$

Then substituting the nondimensionalizing ratios:

$$\frac{d\, V_x\, c}{d\, \frac{m_o \tau}{\beta_{MAX}}} = \frac{\beta_{MAX}\; c}{m_o - \frac{\beta_{MAX}\, m_o}{\beta_{MAX}}\, \tau} \; \frac{1}{\sqrt{1 + \left(\frac{B\, m_o}{C\, \beta_{MAX}}\, \tau + \frac{D}{C}\right)^2}} \tag{98}$$

$$\frac{d\, V_x}{d\, \tau} = \frac{1}{(1-\tau)} \; \frac{1}{\sqrt{1 + (a\tau + b)^2}} \tag{99}$$

where $a = \frac{B\, m_o}{C\, \beta_{MAX}}$ and $b = \frac{D}{C}$ are nondimensional constants. Similarly:

$$\frac{d\, V_y}{d\, \tau} = \frac{1}{(1-\tau)} \; \frac{(a\, \tau + b)}{\sqrt{1 + (a\, \tau + b)^2}} - G \tag{100}$$

The Integration

The integration of $\frac{d\, V_x}{d\, \tau}$ using dummy upper limits:

$$\int_0^{V_x} d\, V_x = \int_0^{\tau} \frac{d\, \tau}{(1-\tau)\sqrt{(b^2+1) + 2ab\tau + a^2\tau^2}} \tag{101}$$

is obtained from equation 200, ref. 5, where

$$b^2 + 1 + 2ab + a^2 = (a+b)^2 + 1 \geqslant 1,$$

as follows:

$$V_x = \frac{1}{\sqrt{(a+b)^2+1}} \ln\Big[2(a+b)^2 + 2 - 2a(a+b)(1-\tau) + \tag{102}$$
$$+ 2\sqrt{(a+b)^2+1}\;\sqrt{(a\tau+b)^2+1}\Big]\Big|_0^{\tau}$$

Evaluating at the limits and combining terms:

$$V_x = \frac{1}{\sqrt{(a+b)^2+1}} \ln \left\{ \frac{(a+b)(a\tau+b) + 1 + \sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}}{(1-\tau)\left[b^2 + ab + 1 + \sqrt{(a+b)^2+1}\,\sqrt{1+b^2}\right]} \right\} \quad (103)$$

The second integration yields the horizontal distance traversed:

$$\sqrt{(a+b)^2+1}\int_0^X dX = \int_0^\tau \ln\left[(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}\right]d\tau$$

$$-\int_0^t \ln(1-\tau)\,d\tau \;-\ln\left[b^2+ab+1+\sqrt{(a+b)^2+1}\,\sqrt{1+b^2}\right]\int_0^\tau d\tau \quad (104)$$

The second term of this equation can be integrated as follows, Eq. 442, Ref. 5:

$$-\int_0^\tau \ln(1-\tau)\,d\tau = +\left[(1-\tau)\ln(1-\tau) - (1-\tau)\right]\Big|_0^\tau = \left[(1-\tau)\ln(1-\tau) + \tau\right] \quad (105)$$

The third term integrates directly to yield:

$$-\ln\left[b^2 + ab + 1 + \sqrt{(a+b)^2+1}\,\sqrt{1+b^2}\right]\tau \quad (106)$$

The first term is first integgated by parts to give

$$\int_0^t \ln\left[(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}\right] d\tau =$$

$$= \tau \ln\left[(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}\right]\Big|_0^t -$$

$$-\int_0^t \frac{\left[a(a+b)+\sqrt{(a+b)^2+1}\;\dfrac{a(a\tau+b)}{\sqrt{(a\tau+b)^2+1}}\right]\tau\, d\tau}{\left[(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}\right]} \qquad (107)$$

Multiplying the numerator and denominator by the conjugate of the denominator:

$$(a+b)(a\tau+b)+1-\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1} \qquad (108)$$

and combining terms:

$$\int_0^\tau \ln\left[(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}\right] d\tau =$$

$$= \tau \ln\left[(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}\right] +$$

$$+\frac{1}{a}\int_0^\tau \frac{\tau}{[1-\tau]^2}\left[a(1-\tau)-\frac{a\sqrt{(a+b)^2+1}\,[1-\tau]}{\sqrt{(a\tau+b)^2+1}}\right] d\tau \qquad (109)$$

$$= \tau \ln \left[(a+b)(a\tau+b) + 1 + \sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1} \right] +$$

$$+ \int_0^\tau \frac{\tau\, d\tau}{[1-\tau]}^{*} + \sqrt{(a+b)^2+1} \int_0^\tau \frac{d\tau}{\sqrt{(a\tau+b)^2+1}}^{**} -$$

$$- \sqrt{(a+b)^2+1} \int_0^\tau \frac{d\tau}{(1-\tau)\sqrt{(a\tau+b)^2+1}} \qquad (110)$$

Integration of these terms yields :

$$\int_0^\tau \ln \left[(a+b)(a\tau+b) + 1 + \sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1} \right] d\tau =$$

$$= \tau \ln \left[(a+b)(a\tau+b) + 1 + \sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1} \right] +$$

$$+ \left[(1-\tau) - \ln(1-\tau) \right] \Big|_0^\tau - \sqrt{(a+b)^2+1}\; V_x +$$

$$+ \sqrt{(a+b)^2+1} \left[\frac{1}{a} \ln \left[a\tau + b + \sqrt{(a\tau+b)^2+1} \right] \right] \Big|_0^\tau \qquad (111)$$

* Ref. 5, Eq. 31.

** Ref. 5, Eq. 128.

Substituting the limits and combining terms:

$$X = \frac{(1-\tau)}{\sqrt{(a+b)^2+1}} \ln\left[\frac{(1-\tau)\left[b^2 + ab + 1 + \sqrt{(a+b)^2+1}\,\sqrt{1+b^2}\right]}{(a+b)(a\tau+b) + 1 + \sqrt{(a+b)^2+1}\,\sqrt{(a\tau+b)^2+1}}\right] +$$

$$+ \frac{1}{a} \ln\left[\frac{(a\tau+b) + \sqrt{(a\tau+b)^2+1}}{b + \sqrt{b^2+1}}\right] \qquad (112)$$

The integration of $\frac{dV_y}{d\tau}$ using dummy upper limits:

$$\int_0^{V_y} dV_y = \int_0^{\tau} \frac{(a\tau+b)\,d\tau}{(1-\tau)\sqrt{(a\tau+b)^2+1}} - \int_0^{\tau} G\,d\tau \qquad (113)$$

$$V_y + G\tau = \int_0^{\tau} \frac{-a(1-\tau) + (a+b)}{(1-\tau)\sqrt{(a\tau+b)^2+1}}\,d\tau \qquad (114)$$

$$V_y - G\tau = (a+b)\int_0^{\tau} \frac{d\tau}{(1-\tau)\sqrt{(a\tau+b)^2+1}} - a\int_0^{\tau} \frac{d\tau}{\sqrt{(a\tau+b)^2+1}}{}^{*} \qquad (115)$$

*Ref. 5, Eq. 128.

$$V_y + G\tau = (a+b)\,V_x - \ln\left[(a\tau+b) + \sqrt{(a\tau+b)^2+1}\right]\Big|_0^{\tau} \qquad (116)$$

$$V_y + G\tau = (a+b)\,V_x - \ln\left[\frac{(a\tau+b) + \sqrt{(a\tau+b)^2+1}}{b+\sqrt{b^2+1}}\right] \qquad (117)$$

The second integration yields the altitude as a function of time for the powered portion of the flight.

$$\int_0^Y dY + \int_0^{\tau} G\tau\,d\tau = (a+b)\int_0^X dX + \ln\left[b+\sqrt{b^2+1}\right]\int_0^{\tau} d\tau -$$

$$-\int_0^{\tau} \ln\left[(a\tau+b) + \sqrt{(a\tau+b)^2+1}\right] d\tau \qquad (118)$$

$$Y + \frac{1}{2}G\tau^2 = (a+b)\,X + \ln\left[b+\sqrt{b^2+1}\right]\tau -$$

$$-\int_0^{\tau} \ln\left[(a\tau+b) + \sqrt{(a\tau+b)^2+1}\right] d\tau \qquad (119)$$

Making a trigonometric substitution, let:

$$(a\tau+b) = \sinh w \qquad (120)$$

$$a\,d\tau = \cosh w\,dw \qquad (121)$$

Then:

$$\int_0^{\tau} \ln\left[(a\tau+b) + \sqrt{(a\tau+b)^2+1}\right] d\tau =$$

$$= \frac{1}{a}\int_0^{\tau} \ln\left[\sinh w + \cosh w\right] \cosh w \, dw \tag{122}$$

Integrating by parts:

$$= \frac{\sinh w}{a} \ln\left[\sinh w + \cosh w\right]\Big|_0^{\tau} - \frac{1}{a}\int_0^{\tau} \sinh w \, dw \tag{123}$$

$$= \frac{\sinh w}{a} \ln\left[\sinh w + \cosh w\right]\Big|_0^{\tau} - \frac{1}{a} \cosh w \Big|_0^{\tau} \tag{124}$$

$$\begin{aligned} &= \frac{(a\tau+b)}{a} \ln\left[(a\tau+b) + \sqrt{(a\tau+b)^2+1}\right] - \\ &- \frac{b}{a}\ln\left(b+\sqrt{b^2+1}\right) - \frac{1}{a}\sqrt{(a\tau+b)^2+1} + \frac{\sqrt{b^2+1}}{a} \end{aligned} \tag{125}$$

Combining terms yields:

$$\begin{aligned} Y &= \frac{(a+b)(1-\tau)}{\sqrt{(a+b)^2+1}} \ln\left\{\frac{(1-\tau)\left[b^2+ab+1+\sqrt{(a+b)^2+1}\sqrt{1+b^2}\right.}{(a+b)(a\tau+b)+1+\sqrt{(a+b)^2+1}\sqrt{(a\tau+b)^2+1}}\right\} + \\ &+ (1-\tau)\ln\left[\frac{(a\tau+b)+\sqrt{(a\tau+b)^2+1}}{b+\sqrt{b^2+1}}\right] + \\ &+ \frac{1}{a}\sqrt{(a\tau+b)^2+1} - \frac{1}{a}\sqrt{b^2+1} - \frac{1}{2}G\tau^2 \end{aligned} \tag{126}$$

Constant Attitude Case

In the limit as $a \to 0$ the trajectory equations should approach the equations of the constant attitude thrust case.

$$\lim_{a \to 0} V_x = \frac{1}{\sqrt{b^2+1}} \ln\left(\frac{1}{1-\tau}\right) \tag{127}$$

$$\lim_{a \to 0} X = \frac{1}{\sqrt{b^2+1}} \left[(1-\tau)\ln(1-\tau) + \tau\right] \tag{128}$$

$$\lim_{a \to 0} V_y = \frac{b}{\sqrt{b^2+1}} \ln\left(\frac{1}{1-\tau}\right) - G\tau \tag{129}$$

$$\lim_{a \to 0} Y = \frac{b}{\sqrt{b^2+1}} \left[(1-\tau)\ln(1-\tau) + \tau\right] - \frac{G\tau^2}{2} \tag{130}$$

These are the equations for constant attitude thrust which can be found by integrating equations (99) and (100) when $a = 0$.

DISCUSSION

The trajectory equations for each of the two subarcs have been solved in closed form. These two portions must be pieced together to obtain the total optimum trajectory for a given mission. The factors which adjust to give this optimum trajectory are this burning time of the retrorocket and the constants a and b of the thrust attitude program. The transcendental nature of the trajectory equations prevents the direct solution of these factors. As a result, the trajectory must be pieced together by trial and error.

As an example of how this trial and error piecing can be carried out, for a specific mission, start by assuming values for a, b and the retrorocket burning time. Working backwards from the ground up, the burning time will dictate the retrorocket ignition time (τ_2). This is also the time of the transition between the two subarcs. The velocity and altitude conditions at the junction point (corner) between the subarcs corresponding to these assumptions can then be found from the trajectory equations for the burning phase (eqs. 103, 117, 126). The trajectory equations of the coasting phase (eqs. 47, 52) can then be used to transfer the velocity and altitude conditions at the junction point to the initial altitude (y_1) where the error in velocity can be noted. Changes in a, b, and burning time are then made and the process repeated until all the boundary conditions are met. The horizontal distance traversed can be determined but does not aid in determining the unknowns. This trial and error solution might also be accomplished graphically.

CONCLUSIONS

1. The optimum trajectory will consist of two portions, a free fall subarc and a powered subarc in that order.
2. The retrorocket must be operated at maximum thrust during the powered portion of the trajectory.
3. The thrust inclination angle varies in such a manner that its tangent is a linear function of time.
4. The equations of motion of each of the two subarcs have been solved in closed form.
5. The total optimum trajectory can be pieced together from the closed form solutions of the two subarcs by trial and error.

REFERENCES

1. Miele, A., "General Variational Theory of the Flight Paths of Rocket-Powered Aircraft, Missiles and Satellite Carriers," Astronautica Acta, Vol. IV, Fasc. 4, pp. 264-288, 1958.

2. Leitmann, G., "On a Class of Variational Problems in Rocket Flight." Journal of the Aero/Space Sciences, Vol. 26, pp. 586-591, September 1959.

3. Miele, A., "Stationary Condition for Problems Involving Time Associated with Vertical Rocket Trajectories." Journal of the Aero/Space Sciences, Vol. 25, pp. 467-9, July 1958.

4. Bliss, G.A. <u>Lectures on the Calculus of Variations</u>, Chicago, Univ. of Chicago Press, 1946.

5. Peirce, B.O. and Foster, R.M., <u>A Short Table of Integrals</u> 4th ed., Boston, Ginn and Company, 1958.

6. Binder, R.C. <u>Fluid Mechanics</u>, 3rd ed. Englewood Cliffs, N.J., Prentice-Hall, 1955.