Stability Robustness to Unstructured Uncertainty for Nonlinear Systems under Feedback Linearization

by

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Abstract

Feedback linearization is the use of coordinate transformation and state feedback to algebraically transform the input-output map of a nonlinear system to a linear one. Once the nonlinear system has been suitably transformed, the resulting linearized system can be controlled through additional state feedback using well established linear control methodologies. Consequently, feedback linearization has been an attractive control design method for nonlinear systems. However, since feedback linearization is predicated on the exact cancellation of dynamics via feedback, issues of robustness remain as a chief drawback for this methodology. In particular, the resulting feedback system will remain nonlinear due to the uncertainty.

This research addresses stability robustness to linear unstructured uncertainty at the input of the model for nonlinear systems under feedback linearization. The model is assumed to be exact feedback linearizable. Using a small gain argument, two sufficient conditions for small-signal finite-gain stability in $L_2$ emerge. The first condition is based on the relation between the $L_2$ induced norm of a nonlinear system and the solution of the Hamilton Jacobi Inequality corresponding to that system. Since the solution of the Hamilton Jacobi Inequality is computationally complex, this result can only be of theoretical interest. The second condition is based on the relation between the $L_2$ induced norm of a nonlinear system and the $H_{\infty}$ norm of its linearization. Specifically, this condition states that there exist a neighborhood in which stability robustness for the resulting nonlinear feedback system is guaranteed if its linearization is robustly stable with margin. Since robust stability for linear time invariant systems is well understood, this condition can be readily verified.

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Chapter 1

Introduction

1.1 Motivation

Feedback control is the use of measurements or feedback as a means to correct and to force a physical process, or plant, to conform to a desired behavior. One such desired behavior is quantified in terms of stability where stability refers to the boundedness of signals in some sense. Given a model of the plant, which is a set of equations describing the behavior of the physical process, a controller is designed to operate on the feedback signals so that the closed loop system satisfies this desired behavior. Since an exact representation of the plant is rarely known, the controller must be designed so as to achieve the desired behavior for a set of possible representations of the plant. In other words, the controller must be designed to cover, or to be robust to, a certain level of uncertainty in the model. For the class of linear time invariant models, this design process is well established and there exists a widely accepted framework, [9], [10], under which the analysis and synthesis of the feedback system is performed. For the more general class of nonlinear models, time-invariant or time-varying, such a unifying framework does not exist. As a result, the design process is dependent on the type of nonlinear system encountered and still relies substantially on engineering art.

One method for nonlinear feedback design, that has gained much favor over the past decade, is feedback linearization, or dynamic inversion. Feedback linearization
is the use of coordinate transformations and state feedback to algebraically transform the dynamics of a nonlinear plant to a linear time-invariant one. There are two forms of feedback linearization. First, given any set of outputs, the input-output map can be made linear through coordinate transformation and state feedback. This is called input-output linearization, or partial feedback linearization, since the operator from the inputs to the states are, in general, still nonlinear. Second, under certain necessary and sufficient conditions on the dynamics, the input to state map can be made linear. This is called input-state linearization or exact feedback linearization. One advantage of exact feedback linearization is that once the nonlinear system has been suitably transformed, quite possibly in another set of coordinates, the resulting linearized system can be stabilized through additional state feedback using the well-established linear time invariant framework mentioned above. In addition, the conditions for exact linearization are readily computable and have been shown to be satisfied by many physical systems of interest. Furthermore, there exists a precise method for the computation of the coordinate transformation and state feedback for the design. As a result, feedback linearization can be applied to a large class of physical systems of interest and eliminates the need for engineering art found in other, more traditional, nonlinear design methodologies such as gain scheduling and synthesis using Lyapunov’s direct method. In fact, feedback linearization shields the control engineer from the unwielding world of nonlinear control since once it is determined that the nonlinear system is exact feedback linearizable, the design problem becomes a familiar linear time invariant control problem in designing the additional state feedback. Consequently, feedback linearization has been heralded as the emerging framework for nonlinear feedback design.

While there are obvious advantages to feedback linearization, there are also certain disadvantages. Chief among them is that since feedback linearization is predicated on the exact cancellation of dynamics via feedback, it requires an exact knowledge of these dynamics. Therefore, issues of robustness and sensitivity to uncertainty loom large. The uncertainty in a model of the plant is generally classified into two types. The first type is parametric uncertainty, which represents imprecision of parameters
within the model, which is assumed to have the same structure as the actual plant. The second type is unstructured uncertainty, which represents unmodeled dynamics. In this case, we plead a total lack of knowledge of the order of the actual plant, and therefore, the uncertainty refers to the difference in the dynamics between the model and the actual plant, which can be infinite dimensional.

Over the last decade, much research has been done to address the robustness of feedback linearization to parametric uncertainty. This work has centered around the application of feedback linearization to robotic manipulators. For example, Slotine and Sastry [27], Samson [22], Spong et. al. [29], and Spong and Vidyasagar [30] have all proposed different synthesis schemes to guarantee robustness under bounded parametric uncertainties. In each case, the overall control consists of nominal feedback linearization followed by additional state feedback designed to guarantee robustness. Subsequently, Spong [28] extended some of these schemes to general feedback linearizable systems by imposing structured matching conditions on the parametric uncertainties.

In contrast to parametric uncertainty, robustness to dynamic uncertainty has not been widely dealt with in the literature. In the few works that mention dynamic uncertainty, the problem has been largely relegated to a side issue and subsequent results given in a qualitative manner. For example, Slotine [25] places rule of thumb type restrictions on the desired control "bandwidth." In Kravaris, [17], a sufficient condition for stability robustness to multiplicative dynamic uncertainty at the plant output is proposed. However, the result reduces to well-known robustness results for linear time invariant systems, [8], since the uncertainty is assumed to be outside of the feedback linearization loop, i.e. the state feedback is assumed not to be corrupted by the uncertainty, and thus does not capture physical reality. Finally, in a recent paper by Enns et al. [12], the issue of robustness to multiplicative dynamic uncertainty at the plant input is examined in the application of feedback linearization to flight control.

This thesis addresses the stability robustness to unstructured uncertainty for nonlinear systems under feedback linearization. For many physical systems of interest,
the dominant nonlinear dynamics are well known. For example, in the case of robot manipulators, the joint dynamics for fixed and known inertias are precisely known. However, one can never model all of the dynamics. Therefore, we are motivated toward this research by the fact that unmodelled dynamics, such as actuator dynamics, structural modes, and transport delays, are always present and that they affect the stability of the feedback system. Another motivation for this research is that feedback linearization is beginning to find applications in aircraft flight control and spacecraft attitude control. Therefore, a major goal of this research is to provide a practical framework that addresses the unstructured uncertainty problem for certain classes of nonlinear systems from which the design of feedback control using feedback linearization can be performed.

1.2 Previous Work and Related Literature

The first group of results that is relevant to this research is in the general area of feedback linearization or geometric control. The use of feedback to cancel dynamics for nonlinear systems is not new, and many cases exist where it is applied to specific physical systems. For example, in the field of robotics, the technique is known as the computed torque method, [1]. However, the study of a general class of nonlinear systems that can be so transformed through the use of coordinate transformation and state feedback is due to the pioneering work of Brockett [2]. Subsequently, necessary and sufficient conditions for feedback linearizability as well as a general method for constructing the required coordinate transformation and state feedback are independently and simultaneously developed by Jacubczyk and Respondek [15], and Hunt, Su, and Meyer [13]. As good references concerning the basics of feedback linearization and its application to physical systems, there are the standard texts on nonlinear system and control by Slotine and Li [26], and Vidyasagar [34]. For a more rigorous treatment of the subject including the underlying differential geometry upon which it is based, there is the text by Isidori [14].

As mentioned above, there has been some research done on the problem of robust
stability to parametric uncertainties. These approaches share a common thread in that in each case, the overall control scheme consists of a nominal feedback linearization control law and additional state feedback used to guarantee stability robustness given a certain bound on the uncertainty. In Slotine and Sastry [27], this additional state feedback is based on sliding mode theory with a linear "boundary layer" to avoid chattering. On the other hand, Samson [22] uses high gain nonlinear feedback to achieve boundedness of the tracking error. In addition, Spong, Thorp, and Kleinwaks [29] design a saturating nonlinear state feedback controller whose robust stability is established using Lyapunov's direct method. Furthermore, Spong and Vidyasagar [30] use a linear dynamic compensator as the additional state feedback, which is designed using the method of stable factorizations in which robust stability is established using a small gain argument. In the latter two cases, the issue of sensor noise in perturbing the state feedback is also addressed. Finally, Spong [28] extends the schemes of [22], [29], and [30] to the case of general feedback linearizable systems, those not in normal form, using structural matching conditions on the uncertainties.

Another approach in stabilizing a nonlinear system with parametric uncertainty has been to combine feedback linearization with adaptive control. In general, the adaptation law is chosen so that the time derivative along a trajectory for a suitable Lyapunov function is negative semi-definite. For robotic systems, Slotine and Li [26] have combined this adaptive scheme with sliding mode control. In the works of Taylor, Kokotovic, Marino, and Kanellakopoulos [31], and Sastry and Kokotovic [24], the adaptive approach is applied to general feedback linearizable systems. For systems with a relative degree of one, a simple parameter adaptation law, similar to that for adaptive control of linear systems, can be used. However, for systems with higher relative degree, the adaptation law becomes increasingly complex unless structured matching conditions are placed on the uncertainties. In Kanellakopoulos, Kokotovic, and Morse [16], this difficulty is overcome through the use of "backstepping," which is a recursive method of designing the adaptation control law where at each stage the system seen by the control has a relative order of one. Finally, robustness to dynamic uncertainty is also treated in [31] and [24]. Here, the dynamic uncertainty is assumed
stable and is formulated using singular perturbations techniques. That is, the model of the plant is assumed to be a singularly perturbed limit of the actual plant. In this case, a simple adaptation law can be constructed which stabilizes the system without exciting the uncertainty. However, this is only shown for the case with relative degree of one.

Another group of literature of interest for this thesis is the general area of nonlinear system analysis. In this area, we are primarily interested in analysis of systems using an input/output setting. The pioneering work here is due to Zames [37], [38] and Sandberg [23]. While the number of references in this field are vast, a good beginning reference is Vidyasagar [34]. However, for a rigorous treatment of input/output analysis, there is the classic text of Desoer and Vidyasagar [4]. In addition, we are interested in the application of this input/output theory to the problem of stability and performance robustness. In this area, the papers by Doyle and Stein [8], Doyle [5], Doyle, Wall, and Stein [10] and Doyle and Stein [9] provide much of the modern linear time invariant design framework for both stability and performance robustness in terms of bounded energy signals. For a summary and extension to other signal spaces, the text by Dahleh and Diaz-Bobillo [3] is an excellent reference.

1.3 Contributions of Thesis

As noted in the previous sections, there has been very little research performed on the analysis of the unstructured uncertainty stability robustness problem for nonlinear systems under a feedback linearization control law. Since unmodeled dynamics are always present, the feedback linearization methodology is incomplete without such an analysis. The focus of this thesis is to provide a framework for such an analysis. In this regard, this thesis provides the following contributions.

First, in Chapter 2, an analytic framework is developed which results in feasible solutions to the stability robustness problem. As shown in Chapter 2, if we represent the model uncertainty as multiplicative uncertainty at the input of the model, the resulting closed loop system is the multiplicative error connected in standard feedback
configuration to a nonlinear operator. This nonlinear operator consists of a linear
time invariant dynamic system, which correspond to the nominal feedback linearized
system, with nonlinear static input and output maps. A key to our analytic framework
is the assumption that the model uncertainty can be captured by a multiplicative
error that is a linear time invariant operator whose magnitude bound as a function of
frequency is known and can be represented in terms of stable, proper rational transfer
function matrices, or frequency dependent weights. This assumption allows us to
take advantage of the fact that linear unmodeled dynamics are usually small at low
frequencies and only becomes significant at high frequencies. We recall that for the
analogous stability robustness problem for linear time invariant systems, this fact is
the key to obtaining a meaningful stability robustness condition since it allows for the
tradeoff between the gain of the uncertainty as reflected by the weights and the gain
of the nominal closed loop transfer function matrix across frequency. Our approach
here to obtaining a meaningful stability robustness result is the same. Although we
cannot directly perform this tradeoff due to the nonlinear input and output maps,
we have demonstrated that we can incorporate the frequency dependent weights as
multipliers in reducing the conservatism of a small gain argument thereby obtaining
a feasible solution.

Second, using the above analytical framework, we are able to derive verifiable
sufficient conditions for robust stability. In Chapter 3, we derive such a condition
for a special subclass of exact feedback linearizable systems, namely those that are
in the so-called normal form with a constant normal form input map. Furthermore,
In Chapter 4, we derive a sufficient condition for the general class of exact feedback
linearizable systems. Finally, even though these sufficient conditions are shown to be
conservative, they represent the only guarantees for stability robustness to unstruc-
tured uncertainties for these feedback systems.
1.4 Organization of Thesis

The organization of the thesis is as follows. In Chapter 2, we formulate the unstructured uncertainty stability robustness problem for nonlinear systems under a feedback linearization control law. We start with a discussion of the problem formulation where the class of nonlinear systems under study, the representation of model uncertainty, and the notion of stability are all suitably defined. We then present a preliminary analysis of the problem to understand its difficulties. In particular, we compare this problem to the similar stability robustness problem for linear time invariant systems, whose analysis is well known from [8]. The emphasis here is to understand precisely the differences between the two problems and to identify the difficulties in our problem that result from these differences. Using this knowledge, a strategy for analyzing this stability robustness problem is formulated and a formal problem statement is defined.

In Chapter 3, we turn our attention to a special subclass of exact feedback linearizable systems, namely systems that are in the so-called normal form with a constant normal form input map. For this special class of systems, we derive a sufficient condition for robust stability that is similar in form to that for the linear time-invariant stability robustness problem. In particular, we show that this condition also has a frequency domain interpretation. In addition, we argue that this condition is in some sense the best that we can obtain under a small gain argument. Specifically, we show that we cannot improve the condition using loop transformations. Furthermore, we also present a sufficient condition for instability. We use this instability condition to show that under some additional assumptions on the dynamics of the system, we can construct a destabilizing uncertainty if the sufficient condition for robust stability is violated. Finally, we present some illustrative examples.

In Chapter 4, we address the stability robustness to unstructured uncertainty problem for the general class of exact feedback linearizable systems. Here, we present two sufficient conditions for robust stability. These two conditions are based on the relationship between the $\mathcal{L}_2$-induced norm of a nonlinear system and the dissipativity
of that system with respect to a certain supply rate, which, in turn, is related to the solution of a corresponding Hamilton Jacobi Inequality for that system. The first sufficient condition presented is simply a small gain argument using the relationship above. Unfortunately, this condition is difficult to verify since it requires a solution to the Hamilton Jacobi Inequality, which is computationally complex. As a result, we present a second sufficient condition which relates the robust stability of the non-linear closed loop system to the robust stability of its linearization. The applicability of this condition is then demonstrated using an example of a two link planar manipulator. Finally, we present concluding remarks and suggestions for future research in Chapter 5.
Chapter 2

The Stability Robustness Problem

As discussed in the previous chapter, the problem that is addressed in this report concerns stability robustness to unstructured uncertainty. That is, we want to analyze the closed loop stability about a desired equilibrium point for the actual nonlinear plant with a feedback linearization control law given that the nominal closed loop stability for a certain model of that plant is assured by that control law. In this chapter, we formalize this problem mathematically and present some preliminary insights into its analysis. We start by setting up the problem in Section 2.1. In this section, we first define the class of nonlinear systems that we will consider as the model of the plant to be controlled, namely that of exact feedback linearizable systems, and describe the nominal closed loop system under feedback linearization. Next, we define the representation of uncertainty that we use to describe the modeling errors. We then examine the resulting closed loop system with this representation of uncertainty. Since we are dealing with nonlinear systems, our notion of stability for the problem is also defined. Throughout the discussion, we will refer to the notation and background material on nonlinear systems presented in Appendix A.

Once the problem is defined, some preliminary insights into the difficulties of the problem are presented in Section 2.2. In particular, we compare this problem to the stability robustness problem involving linear time-invariant operators for which the analysis is well known. The emphasis here is to understand precisely how our problem differs from the linear time invariant case and to identify the difficulties that
are introduced as a result of these differences. Using this knowledge, we then present a strategy on how the problem can be analyzed. Finally, a formal problem statement is presented in Section 2.3.

2.1 Problem Formulation

2.1.1 Nonlinear Model Description

For the model of the actual plant, $P_m$, we consider the class of $m$ input, $n$th order nonlinear systems of the form

$$\dot{\xi} = a_m(\xi) + B_m(\xi)u,$$  \hspace{1cm} (2.1)

which are affine in the control $u$. We assume without loss of generality that $\xi = 0$ is the equilibrium point about which we wish to stabilize and that $a_m(0) = 0$. We refer to $\xi$ as the physical coordinates or physical states for the model. In addition, we will assume throughout this report that the model $P_m$ given by (2.1) satisfies the following.

Assumption 2.1

1. $a_m(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B_m(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{nxm}$ are smooth functions, i.e. continuous with continuous partial derivatives of any order with respect to any combination of its arguments.

2. The physical state $\xi$ can be measured perfectly and used as feedback.

3. $P_m$ is exact feedback linearizable.

Refering to [14], Assumption 2.1 implies two things. First, there exists a nonlinear coordinate transformation $x = \Phi(\xi)$, which when applied to the system in (2.1), the
transformed system, in the new coordinates $x$ given by

$$
x = \begin{bmatrix}
x^1 \\
\vdots \\
x^n
\end{bmatrix}
$$

(2.2)

where each $x^i$ is a vector of dimension $r_i$, for $1 \leq i \leq m$, and where $\sum_{i=1}^{m} r_i = n$, is in the so-called normal form given by

$$
\begin{align*}
\dot{x}_1^i &= x_2^i \\
\vdots \\
\dot{x}_{r_i-1}^i &= x_{r_i}^i \\
\dot{x}_{r_i}^i &= f_i(x) + \sum_{j=1}^{m} G_{ij}(x) u_j
\end{align*}
$$

(2.3)

for $1 \leq i \leq m$. Second, the matrix

$$
G(x) = \begin{bmatrix}
G_{11}(x) & \cdots & G_{1m}(x) \\
\vdots & \ddots & \vdots \\
G_{m1}(x) & \cdots & G_{mm}(x)
\end{bmatrix}
$$

(2.4)

is nonsingular for all $x$. We denote $G(x)$ as the normal form input map. We note that $\Phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth diffeomorphism and that we can compute $\Phi(\cdot)$ analytically so that the equilibrium point $\xi = 0$ is mapped to the origin in the $x$ coordinates. As a result, we have $f(0) = 0$ where

$$
f(x) = \begin{bmatrix}
f_1(x) \\
\vdots \\
f_m(x)
\end{bmatrix}
$$

(2.5)

Furthermore, since we know the nonlinearities of the model, $a_m(\cdot)$ and $B_m(\cdot)$, it follows that given $\Phi(\cdot)$, we can analytically compute the nonlinearities, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ and $G(\cdot) : \mathbb{R}^n \to \mathbb{R}^{m \times m}$, of the normal form system. Finally, we note that since $a_m(\cdot)$,
$B_m(\cdot)$, and $\Phi(\cdot)$ are smooth functions, $f(\cdot)$ and $G(\cdot)$ are smooth.

The importance of the normal form is that the nonlinearities, $f(x)$ and $G(x)$, are now directly seen by the control $u$. Since

1. $f(\cdot)$ and $G(\cdot)$ are known functions,

2. $x = \Phi(\xi)$ can be fed back exactly, and

3. The matrix $G(x)$ is invertible for all $x$,

we can define the feedback control law

$$ u = [G(x)]^{-1}(-f(x) + v), \quad (2.6) $$

which, when applied to (2.3), yields the closed loop system

$$
\begin{align*}
\dot{x}_1^i &= x_2^i \\
\vdots \\
\dot{x}_{r_i-1}^i &= x_r^i \\
\dot{x}_r^i &= v_i
\end{align*}
$$

(2.7)

for $1 \leq i \leq m$. We note that the system from $v$ to $x$ is now linear time invariant and that the effect of the control law (2.6) is to algebraically cancel the nonlinearities $f(x)$ and $G(x)$ in (2.3). In fact, (2.7) is also controllable since it is simply $m$ decoupled sets of integrators, each of order $r_i$. Furthermore, we note that this cancellation occurs in the $x$ coordinates, which are, in general, not the physical coordinates for the system. We call these coordinates the linearizing coordinates. Once the system has been suitably transformed, we can impose additional linear, static feedback in the linearizing coordinates of the form

$$ v = -Kx + r \quad (2.8) $$
such that the resulting closed loop transfer function matrix from $r$ to $x$ given by

$$x(s) = \Psi(s)r(x). \tag{2.9}$$

satisfies some desired property such as stability. In particular, we can use (2.8) to place the poles of $\Psi(s)$ in the open left half of the complex $s$-plane. It is clear that $\Psi(s)$ is the nominal closed loop operator from the input $r$ to the linearizing coordinates $x$, and it is denoted the desired linearized transfer function matrix.

From the above discussion, the attractiveness of feedback linearization becomes apparent. Given a nonlinear system that is exact feedback linearizable, the methodology is exact in that it defines a way to compute a state transformation $\Phi(\cdot)$ and linearizing feedback (2.6) that will algebraically transform the system to a linear time invariant, controllable one. Since the linearizing feedback is fixed given the model of the system, the additional state feedback (2.8) becomes our design to shape the resulting nominal closed loop operator. This design can be carried out in the linearizing coordinates using the wealth of linear design methodologies such as eigenstructure assignment, LQR, $\mathcal{H}_\infty$, etc. Although the transformed system is linear time invariant in the linearizing coordinates, it is still, in general, nonlinear in the physical coordinates. To see this, we simply apply our feedback linearization control law, (2.6) and (2.8), to our physical system (2.1) to obtain

$$\dot{\xi} = a_m(\xi) + B_m(\xi) \left[ [G(\Phi(\xi))]^{-1}(-f(\Phi(\xi)) - K(\Phi(\xi)) + r) \right] \tag{2.10}$$

where we have used the coordinate transformation (2.2) to express the feedback linearization control law in terms of physical states. Clearly, the operator defined by this differential equation is nonlinear. Another way to see this is that since the state transformation is invertible, the operator from $x$ to $\xi$ is given by

$$\xi = \Phi^{-1}(x), \tag{2.11}$$
and therefore the operator from \( r \) to \( \xi \) is \( \Phi^{-1}\Psi \), which is clearly nonlinear for a nonlinear coordinate transformation \( \Phi(\cdot) \). Finally, we note that for many applications of feedback linearization, the physical coordinates of the model is such that the model is naturally in normal form. Manipulator dynamics in joint coordinates is such an example. For these cases, the required state transformation is the identity operator. Therefore, the physical coordinates and the linearizing coordinates correspond, and the resulting nominal closed loop operator from \( r \) to \( \xi \) will be linear time invariant.

### 2.1.2 Representation of Uncertainty

We have seen in the previous section that given a model that belongs to the class of feedback linearizable systems, feedback linearization will allow us to transform this system to a linear time invariant one in some linearizing coordinates. It is a fact, however, that any model is only an approximation of the true plant, and therefore, the algebraic transformation of feedback linearization will invariably be imperfect due to the model uncertainty. To analyze the effect of this uncertainty on the closed loop system, we first need to define a mathematical representation of the uncertainty, which is the topic for this section.

For this report, we adopt a set membership representation of model uncertainty where we assume that the actual plant belongs to a set of nonlinear operators. This set of operators is viewed as the set of possible plants that are perturbed from the model by the uncertainty. Towards this end, we define the following representation for the uncertainty in the model.

**Definition 2.1** The actual plant, \( P_a \), belongs to the set

\[
\mathcal{P} = \{ P \mid P = P_m(I + W_i\Delta W_o) \}.
\]  

(2.12)

where

1. \( W_i\Delta W_o \) is a linear time invariant, causal, and stable dynamic operator.

2. \( \sigma_{\max}[W_i(j\omega)\Delta(j\omega)W_o(j\omega)] < l(\omega), \) for all \( \omega \), where \( l(\omega) \) is a known function of
3. $\Delta \in D = \{ \Delta \mid \| \Delta \|_{\mathcal{H}_\infty} < 1 \}$

4. $W_i(s)$ and $W_o(s)$ are stable and proper rational transfer function matrices.

We note that such a representation of unstructured uncertainty is standard when dealing with stability robustness for linear time invariant models, for example [8], and that the above characterization is commonly referred to as multiplicative uncertainty at the input of the model. In our definition, the operator $W_i \Delta W_o$ is known as the multiplicative error. We note that while the magnitude of the multiplicative error is strictly bounded from above as a function of frequency by $l(\omega)$, the direction and phase of this operator is arbitrary for all frequencies. From the definition for $D$, it is clear that the transfer function matrices, $W_i(s)$ and $W_o(s)$, are frequency dependent weights that are chosen to reflect this magnitude bound information. It is clear that the choice for $W_i(s)$ and $W_o(s)$ is not unique and that we only need to choose these weights so that the conditions of the definition are satisfied. That is, we need to choose them so that they are rational, stable and proper with

$$\sigma_{\min}[W_i(j\omega)]\sigma_{\min}[W_o(j\omega)] \geq l(\omega) \ \forall \omega$$

(2.13)

Since the above representation of the uncertainty is a linear time invariant perturbation from the model, it limits us to consider only the cases where all of the nonlinear dynamics of the plant can be captured by the model. While this may seem restrictive at first, we note that for some physical systems, the dominant nonlinear dynamics are well known and can be fully captured by a finite dimensional nonlinear model. For example, in the case of a robotic manipulator, the dynamics in joint coordinates can be modelled exactly given that the inertial properties of the manipulator are known. In these cases, the unmodelled dynamics may be in the form of actuator dynamics or time delays, which are linear or nearly so in the operating range of the system. In fact, we often neglect these fast dynamics on purpose in order to simplify our model. This is the case when using feedback linearization since the inclusion of actuator dy-
namics or time delays in the model will destroy the feedback linearizability of the
model because the required states of these dynamics are not available for feedback.
Finally, we note that our representation for the uncertainty is one that is based on
input-output relations and therefore does not require an explicit realization for the
uncertainty in some finite dimensional state space. As a result, our representation is
able to cover unmodeled dynamics of unknown or infinite order such as time delays.

2.1.3 Actual Closed Loop System

With the description of the model, the representation of uncertainty, and the feedback
linearization control law, a block diagram of the actual feedback system is given in
Figure 2-1 for some \( \Delta \in \mathcal{D} \). It is important to note that Figure 2-1 also represents the
feedback system for any plant belonging to \( \mathcal{P} \). The key here is that if we can show
stability for all \( \Delta \in \mathcal{D} \), for all elements of \( \mathcal{P} \), then we have shown stability robustness
since the actual plant is contained in the set. To analyze this feedback system, we first
reduce it to a standard feedback configuration by isolating the multiplicative error,
\( W_i \Delta W_o \), and examining the resulting operator that interacts with it in feedback. We
denote this operator as \( \mathcal{M} \), and it is the map from the output of \( W_i \Delta W_o \), \( z \), to the
input of \( W_i \Delta W_o \), \( w \). From Figure 2-1,

\[
\dot{\xi} = a_m(\xi) + B_m(\xi) \left[ (G(x))^{-1}(-f(x) - Kx) + z \right]
= a_m(\xi) + B_m(\xi) \left[ (G(\Phi(\xi)))^{-1}(-f(\Phi(\xi)) - K(\Phi(\xi)) + G(\Phi(\xi))z) \right] \tag{2.14}
\]

\[
x = \Phi(\xi) \tag{2.15}
\]

\[
w = (G(x))^{-1}(-f(x) - Kx) \tag{2.16}
\]

Defining

\[
\zeta = G(\Phi(\xi))z = G(x)z, \tag{2.17}
\]

Equation (2.14) becomes

\[
\dot{\xi} = a_m(\xi) + B_m(\xi) \left[ (G(\Phi(\xi)))^{-1}(-f(\Phi(\xi)) - K(\Phi(\xi)) + \zeta) \right], \tag{2.18}
\]
which is identical to the nominal closed loop state equations in the physical coordinates as given in (2.10). Therefore, it follows that the operator from $\zeta$ to $x$, as realized by Equations (2.18) and (2.15), is simply the desired linearized transfer function matrix $\Psi(s)$. Using this fact and Equations (2.16) and (2.17), we conclude that the operator $M$ consists of the linear time invariant operator $\Psi(s)$ with a state dependent input map, $G(x)$, multiplying the input $z$ and a nonlinear output map $w(x)$ as depicted in Figure 2-2. [t] [b]

We note that the input map $G(x)$ of $M$ corresponds to the input map of the normal form dynamics in the linearizing coordinates, (2.3), and that the output map $w(x)$ corresponds to the feedback linearization control law. In addition, we note that in reducing the feedback loop to the standard feedback configuration, we chose to realize the operator $M$ in the linearized coordinates $x$. This is because we want to exploit the fact that without uncertainty, the nominal closed loop map is the desired linearized transfer function matrix $\Psi(s)$. We recall that we control the stability of $\Psi(s)$ through $K$, and therefore, we would like to obtain stability robustness results involving $\Psi(s)$. 

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On the other hand, this approach forces us to deal with the nonlinear functions $f(\cdot)$ and $G(\cdot)$ of the normal form dynamics which, in general, do not correspond to the physical nonlinearities of the model.

### 2.1.4 Definition of Stability

Before we proceed to analyze stability, we must first define what we mean by stability. In this report, as detailed in Appendix A, we adopt an input-output definition for stability in the space of finite energy signals or $\mathcal{L}_2$. For the nominal system, the closed loop operator from $r$ to $x$ is the desired linearized transfer function matrix, $\Psi(s)$. As discussed beforehand, we can arbitrarily assign the poles of this linear system, and therefore we can place all the poles of $\Psi(s)$ in the open left half $s$-plane so that $\Psi(s)$ is strictly stable. From an input-output viewpoint, this implies that the operator $\Psi$ is finite-gain stable in $\mathcal{L}_2$ since for a strictly stable transfer function matrix, the $\mathcal{L}_2$-induced norm is simply its $\mathcal{H}_\infty$ norm, which is finite. This finite gain stability means that for all inputs $r \in \mathcal{L}_2$, the output $x$ is in $\mathcal{L}_2$. However, we recognize that $x$ is the states corresponding to the linearizing coordinates and may not have any physical meaning. For nominal stability, then, we would like to require that for all
inputs $r \in \mathcal{L}_2$, the physical states $\xi$ is in $\mathcal{L}_2$. We note that this assertion requires that
the nonlinear map $\Phi^{-1}$ is finite gain stable since $\xi = \Phi^{-1}(\varphi)$. For the moment we will assume that this is true so that we can claim nominal stability as defined. In the sequel, we will show that this assumption is always true given additional assumptions
on the problem.

For the actual closed loop system, we seek a similar input-output definition for
stability for the feedback configuration in Figure 2-3. We note that in Figure 2-3 we
added two additional inputs, $r_1$ and $r_2$ to the closed loop. These inputs can be viewed
as exogenous inputs to our system and/or as biases introduced by initial conditions.
As a prelude to defining stability, we need to insure that the feedback loop given in
Figure 2-3 is well-posed. In this regard, we make the following assumption.

**Assumption 2.2** The feedback loop is well-posed in the sense that for any input pairs
$(r_1, r_2) \in \mathcal{L}_{2e} \times \mathcal{L}_{2e}$, the solutions for all signals within the loop exist and are unique.

Given that the solutions exist and are unique, we say that the closed loop system
in Figure 2-3 is finite-gain stable if the operator that maps the signal pair $(r_1, r_2)$ to
the signal pair $(e_1, e_2)$ is finite gain stable. We note that this definition only implies
that given $(r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2$, we have $(e_1, e_2) \in \mathcal{L}_2 \times \mathcal{L}_2$. In particular, it does
not imply that the signal $\xi \in \mathcal{L}_2$, or even that $\varphi \in \mathcal{L}_2$, as for the nominal closed loop. Since for robust stability, we want to preserve the same sense of stability from
the nominal closed loop to the actual closed loop, we need to add this additional
requirement of boundedness to our definition. In addition, for robust stability, we need to insure that the closed loop system in Figure 2-2 is stable for all $\Delta \in \mathcal{D}$. Therefore, we make the following definition for robust stability.

**Definition 2.2** The closed loop system in Figure 2-3 is robustly stable if, for all $\Delta \in \mathcal{D}$,

1. The operator that maps the signal pair $(r_1, r_2)$ to the signal pair $(e_1, e_2)$ is finite gain stable.

2. For all input pairs $(r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2$, $\xi \in \mathcal{L}_2$.

### 2.2 Preliminary Insight

In this section, we present some preliminary analysis into the problem. Specifically, we compare our problem to the stability robustness problem for linear time invariant system for which the analysis is well known. The purpose here is to understand the difficulties of our problem so that we may devise the correct manner in which to analyze it.

#### 2.2.1 Linear Time Invariant Stability Robustness Problem

In the case of stability robustness to multiplicative uncertainty at the model input where the model of the plant, the multiplicative error, and the controller are all linear time invariant, the resulting closed loop system in standard feedback configuration is as given in Figure 2-4. As shown in the figure, the operator that is seen by the multiplicative error, $M(s)$, is now a linear time invariant operator. In fact, it is the input complementary sensitivity transfer function matrix. From Appendix A, we can use the Small Gain Theorem to find a sufficient condition for stability robustness. Namely, if

$$
\|\Delta(j\omega)\|_{\mathcal{H}_\infty} \|W_o(j\omega)M(j\omega)W_i(j\omega)\|_{\mathcal{H}_\infty} < 1
$$

(2.19)
for all \( \Delta \in \mathcal{D} \), then the closed loop system is finite-gain stable for all \( \Delta \in \mathcal{D} \). Since \( \|\Delta(j\omega)\|_{\mathcal{H}_\infty} < 1 \) by definition of \( \mathcal{D} \) it follows that (2.19) is equivalent to the condition

\[
\|W_o(j\omega)M(j\omega)W_i(j\omega)\|_{\mathcal{H}_\infty} \leq 1
\]

(2.20)

Using the definition of the \( \mathcal{H}_\infty \) norm and the submultiplicative property of induced norms, a sufficient condition for stability robustness can be expressed as

\[
\sigma_{\text{max}}[M(j\omega)] \leq \frac{1}{\sigma_{\text{max}}[W_o(j\omega)]\sigma_{\text{max}}[W_i(j\omega)]} \quad \forall \omega
\]

(2.21)

The condition in (2.21) is key in that it allows for the tradeoff between the maximum singular values, or magnitudes, of the nominal closed loop transfer function matrix, \( M(s) \), and that of the uncertainty, reflected in \( W_o(s) \) and \( W_i(s) \), across frequency. As is usually the case, the magnitude of the modeling error is small, below unity, at low frequencies and becomes significant, above unity, at high frequencies because of "fast" unmodeled dynamics. Therefore, this tradeoff allows one to design the nominal closed loop system so as to achieve performance at low frequencies and forces one to roll-off the nominal closed loop system at high frequencies to ensure stability robustness. This embodiment of the physical tradeoff between performance and
stability robustness is illustrated in the frequency domain in Figure 2-5. In addition, we note that for the linear time invariant problem, the stability robustness condition is readily verifiable through computation. This comes from the fact that we know how to numerically calculate the $\mathcal{H}_\infty$ or $\mathcal{L}_2$-induced norm of a linear time invariant operator. Finally, we note that the condition in (2.21) is actually necessary as well as sufficient in the sense that if the condition is violated, then we can construct a destabilizing $\Delta$ that belongs to $\mathcal{D}$. Therefore, the condition is not necessarily conservative given that the actual plant is represented by this worst case $\Delta$.

2.2.2 Feedback Linearization Stability Robustness Problem

We now return our attention to our problem at hand where the closed loop system in standard feedback configuration is again given in Figure 2-6. Comparing our problem to the linear time invariant one, we first note that we cannot directly absorb the frequency dependent magnitude bound of the uncertainty, that is reflected in $W_i(s)$
and $W_0(s)$, into the desired linearized transfer function matrix, $\Psi(s)$, due to the non-linear maps, $G(x)$ and $w(\cdot)$, at the input and output of $\Psi$, respectively. Therefore, even though we can control the roll-off of $\Psi(s)$, we cannot directly trade off between the magnitude of the uncertainty with the magnitude of $\Psi(s)$ across frequency. The consequence of this is that if we use the Small Gain Theorem, where we consider $W_i(s)\Delta(s)W_0(s)$ and $M$ as two separate operators connected in feedback, then we necessarily have to consider the worst case magnitude across frequency for the uncertainty as the gain of $W_i(s)\Delta(s)W_0(s)$. Recalling that the magnitude of the modeling errors are generally significant and above unity at high frequencies, this gain will be greater than one. Therefore, the small gain condition will require that the $\mathcal{L}_2$-induced norm of $M$ be less than unity in order to guarantee stability. As will be evident in the sequel, this may not be possible.

Another difficulty with respect to our problem is that we do not know how to calculate the $\mathcal{L}_2$-induced norm of the nonlinear operator $M$. In the interest of getting a robust stability condition using the Small Gain Theorem, we might be tempted to overbound this gain by assuming that there exists finite constants $\gamma_G$ and $\gamma_w$ such
and \( W_0(s) \), into the desired linearized transfer function matrix, \( \Psi(s) \), due to the non-linear maps, \( G(x) \) and \( w(\cdot) \), at the input and output of \( \Psi \), respectively. Therefore, even though we can control the roll-off of \( \Psi(s) \), we cannot directly tradeoff between the magnitude of the uncertainty with the magnitude of \( \Psi(s) \) across frequency. The consequence of this is that if we use the Small Gain Theorem, where we consider \( W_i(s) \Delta(s) W_0(s) \) and \( M \) as two separate operators connected in feedback, then we necessarily have to consider the worst case magnitude across frequency for the uncertainty as the gain of \( W_i(s) \Delta(s) W_0(s) \). Recalling that the magnitude of the modeling errors are generally significant and above unity at high frequencies, this gain will be greater than one. Therefore, the small gain condition will require that the \( L_2 \)-induced norm of \( M \) be less than unity in order to guarantee stability. As will be evident in the sequel, this may not be possible.

Another difficulty with respect to our problem is that we do not know how to calculate the \( L_2 \)-induced norm of the nonlinear operator \( M \). In the interest of getting a robust stability condition using the Small Gain Theorem, we might be tempted to overbound this gain by assuming that there exists finite constants \( \gamma_G \) and \( \gamma_w \) such
that
\[\sigma_{\max}[G(x)] \leq \gamma_G\] (2.22)
\[\|w(x)\| \leq \gamma_w\|x\|\] (2.23)
for all \(x \in \mathbb{R}^n\). It is clear from the above relationships that \(\gamma_G\) and \(\gamma_w\) are bounds on the \(L_2\)-induced norms of the operators \(G(x)\) and \(w(\cdot)\), respectively. As a result, by the submultiplicative property of induced norms, the \(L_2\)-induced norm of \(M\), denoted by \(\gamma_M\) is bounded by
\[\gamma_M \leq \gamma_w \gamma_G \|\Psi(j\omega)\|_{\mathcal{H}_\infty},\] (2.24)
and the resulting small gain condition that guarantees robust stability is
\[\gamma_w \gamma_G \|\Psi(j\omega)\|_{\mathcal{H}_\infty} \|W_i(j\omega)\|_{\mathcal{H}_\infty} \|W_o(j\omega)\|_{\mathcal{H}_\infty} \leq 1.\] (2.25)

We note, however, that this sufficient condition can be overly conservative in the sense that it cannot be satisfied. This is shown by the following example.

**Example 2.1:**

Consider, as the model of the plant, the single input, scalar nonlinear system
\[\dot{x} = \sin(x) + (2 + \cos(x))u.\] (2.26)

It is clear that the system is in normal form and that \(f(x) = \sin(x)\) and \(G(x) = 2 + \cos(x)\). We represent the uncertainty as multiplicative uncertainty at the input of the model where the magnitude of the multiplicative error is bounded by
\[|W_i(j\omega)\Delta(j\omega)W_o(j\omega)| < \left|\frac{2j\omega}{j\omega + 10}\right| \forall \omega\] (2.27)

We choose
\[W_i(s) = \frac{2s}{s + 10}, \quad W_o(s) = 1\] (2.28)
so that $\Delta \in \mathcal{D}$. Using the feedback linearizing control law,

$$u = \frac{1}{2 + \cos(x)}(-\sin(x) - kx), \quad (2.29)$$

the resulting closed loop system can be put into the standard feedback configuration in Figure 2-6 where

$$G(x) = 2 + \cos(x) \quad (2.30)$$

$$\Psi(s) = \frac{1}{s + k} \quad (2.31)$$

$$w(x) = \frac{-1}{2 + \cos(x)}(\sin(x) + kx). \quad (2.32)$$

We note that we need $k > 0$ in order for nominal stability. In addition, overbounding $G(x)$ and $w(x)$ gives

$$|2 + \cos(x)| \leq 3 \quad (2.33)$$

$$\left| \frac{-1}{2 + \cos(x)}(\sin(x) + kx) \right| \leq (1 + k)|x| \quad (2.34)$$

for all $x \in \mathbb{R}$ so that $\gamma_G = 3$ and $\gamma_w = 1 + k$. Furthermore, from (2.28) and (2.31), we have that

$$\|w_c(j\omega)\|_{\mathcal{H}_\infty} = 2 \quad (2.35)$$

$$\|w_a(j\omega)\|_{\mathcal{H}_\infty} = 1 \quad (2.36)$$

$$\|\Psi(j\omega)\|_{\mathcal{H}_\infty} = \frac{1}{k}. \quad (2.37)$$

Substituting everything into the sufficient condition (2.25), we have that stability robustness is guaranteed if

$$\frac{6(1 + k)}{k} \leq 1, \quad (2.38)$$

which clearly cannot be satisfied by any $k > 0$. \(\Box\)

The above example shows that even for a simple scalar system, the sufficient
condition using overbounds is overly conservative and therefore impractical. This conservatism is the result of two factors. First, the overbounding itself gives a conservative bound on the true $L_2$-induced norm of $M$. Second, the sufficient condition using overbounds forces us to consider only the worst case magnitude bound for the uncertainty across frequency, which is greater than one. In particular, it does not allow us to use the frequency dependent magnitude bound information that we have for the uncertainty so as to tradeoff between the magnitude of the uncertainty with the magnitude of $\Psi(s)$ across frequency. As mentioned beforehand, this forces us to require that the $L_2$-induced norm of $M$ be less than one. In the above example, our overbound for this $L_2$ gain is $\frac{3(1+k)}{k}$, which is greater than unity for any $k > 0$.

Still another difficulty for our problem is that, in general, the nonlinear operator $M$ may not even be finite-gain stable. We illustrate this with the following example.

Example 2.2:

For this example let $M$ be the single input, scalar system describe by

\[
\begin{align*}
\dot{x} & = -kx + z, \\
\dot{w} & = x^2
\end{align*}
\]

for some $k > 0$. This is the case, for example, if the input map $G(x)$ is a constant, independent of $x$. We note that the operator from $z$ to $x$ is linear time invariant. Let $z = \tilde{z}(t)$ be any input signal in $L_{2e}$. Clearly, the responses for $x$ and $w$, denoted $\bar{x}(t)$ and $\bar{w}(t)$, are also in $L_{2e}$. By definition, the gain of $M$ with respect to this signal is

\[
\gamma_\varepsilon = \sup_{T \in \mathbb{R}_+} \left\{ \frac{\|\bar{w}\|_{2T}}{\|\bar{z}\|_{2T}} \right\}.
\]

Now, let us double the input signal so that $z = 2\tilde{z}(t)$. This new input signal is still in $L_{2e}$. Since the operator from $z$ to $x$ is linear, the resulting response for $x$ is $2\bar{x}(t)$, and since $w(x) = x^2$, the resulting response for $w$ is $4\bar{w}(t)$. As a result, the gain of
$M$ with respect to this signal is

\[ \gamma_{2e} = \sup_{T \in \mathbb{R}^+} \left\{ \frac{\|4w\|_{2T}}{\|2\tilde{x}\|_{2T}} \right\} = 2\gamma_e. \tag{2.42} \]

Recalling that the $L_2$-induced norm of $M$ is defined by

\[ \gamma_M = \sup_{z \in L_2} \sup_{T \in \mathbb{R}^+} \left\{ \frac{\|w\|_{2T}}{\|z\|_{2T}} \right\}, \tag{2.43} \]

it is clear that the $L_2$-induced norm of $M$ cannot be finitely bounded since we can always double the input, which will still be in $L_2$, to double the gain.

The fact that we can generate simple examples for $M$ that are not finite gain stable is disturbing because for these cases we cannot apply the Small Gain Theorem to obtain a condition for finite-gain stability of the closed loop. We note that in the example above, the reason why $M$ is not finite-gain stable is that the output map $w(x) = x^2$ is not globally Lipschitz. In fact, for any $w(x)$ that is not globally Lipschitz, we would arrive at the same conclusion for the example. In addition, we note that in the previous discussion, we are only able to bound the $L_2$-induced norm of $M$ because we are able to bound the gains of $G(x)$ and $w(x)$ for all $x$ since they are globally Lipschitz in $x$. In retrospect, then, we realize the concept of gain is really not well suited globally for highly nonlinear operators since for most practical purposes, these nonlinearities are not globally Lipschitz.

Finally, we note that any condition that we obtain for this problem using a small gain argument will only be sufficient and not necessary. This is because it is highly unlikely that we can construct a linear time invariant uncertainty $\Delta \in \mathcal{D}$ to destabilize the nonlinear closed loop if the stability robustness condition is violated.

### 2.2.3 Analysis Strategy

From the lessons learned in the preliminary analysis above, we now devise a strategy with which we can adequately analyze our problem. First, we note that although we
cannot directly absorb the frequency dependent magnitude bound of the uncertainty into $\Psi(s)$, we must somehow use this information in the analysis. Failure to do so will lead to overly conservative results as shown in the first example because the worst case gain across frequency for the uncertainty is invariably greater than unity. To use this information, we need to combine the weights, $W_i(s)$ and $W_o(s)$, and $M$ into a single nonlinear operator $M_c$ as shown in Figure 2-7. We note that $M_c$ is simply three linear time invariant operators, $W_i(s)$, $\Psi(s)$, and $W_o(s)$, connected in series by the two static, nonlinear maps $w(x)$ and $G(x)$. We note that since the static maps $w(x)$ and $G(x)$ are nonlinear, they will distribute the frequency contents of their respective input signals at their outputs. However, if there is enough separation in frequency between the high gains of the uncertainty and the high gains of $\Psi(s)$, then there may not be an input signal to $M$ that will be amplified by the worst case gain across frequency for both the uncertainty and for $\Psi(s)$. The key, of course, is that we need to devise a method to obtain a bound on the $L_2$-induced norm of $M_c$ that is tighter than the obvious overbound presented previously. In this regard, we will use the dissipativity theorems of Willems [35], [36].

Second, we need to ensure that the $L_2$-induced norm of $M$ is bounded in some sense so that we can use the Small Gain Theorem to conclude stability. To do so, we
will assume the following.

**Assumption 2.3** For the closed loop system in Figure 2-3, there exists finite constants, \( R_1 > 0 \) and \( R_2 > 0 \), such that for all input pairs \((r_1, r_2) \in \mathcal{L}_{2e} \times \mathcal{L}_{2e}\) where \( \|r_1(t)\| \leq R_1 \) and \( \|r_2(t)\| \leq R_2 \) for all \( t \geq 0 \), the closed loop solution for \( x(t) \) evolves in some bounded open set \( X \) containing the origin for all \( t \geq 0 \).

We now show that under this assumption, the \( \mathcal{L}_2 \) gain of \( M \) is bounded so that \( M \) is finite-gain stable in some sense. We begin with the following lemmas.

**Lemma 2.1** If the matrix function \( H(\cdot) : \mathbb{R}^n \to \mathbb{R}^{m \times m} \) is continuous over the closure of some bounded open set \( X \), then there exists a finite constant \( \gamma_H \) such that

\[
\sigma_{\max}[H(x)] \leq \gamma_H \tag{2.44}
\]

for all \( x \in X \).

**Proof:** Since the closure of \( X \) is compact, it follows from continuity that each element of the matrix \( H(x) \) is bounded by a finite constant over \( X \). Therefore, the result follows. \( \square \)

**Lemma 2.2** If the vector function \( h(\cdot) : \mathbb{R}^n \to \mathbb{R}^m \) is continuously differentiable over the closure of some bounded open set \( X \), then there exists a finite constant \( \gamma_h \) such that

\[
\|h(x)\| \leq \gamma_h\|x\| \tag{2.45}
\]

for all \( x \in X \).

**Proof:** Since the closure of \( X \) is compact, it follows from continuous differentiability that there exists a finite constant \( \beta_f \) such that

\[
\left| \frac{\partial f_i(x)}{\partial x_j} \right| \leq \beta_f \tag{2.46}
\]
for all $x \in X$. Using the Mean Value Theorem, (2.46) is equivalent to local Lipschitz continuity for $f$ over $X$, which implies that there exist a finite constant $\gamma_h$ such that

$$\|h(x_1) - h(x_2)\| \leq \gamma_h \|x_1 - x_2\|$$  \hspace{1cm} (2.47)

for all $x_1, x_2 \in X$. In particular, (2.47) holds for $x_2 = 0 \in X$, and the result follows. \hfill \Box

Using these two lemmas, we show the following.

**Proposition 2.1** Under our assumptions of smoothness and Assumption 2.3, the $L_2$ gain of $M$ is bounded.

**Proof:** We first note that since $f(\cdot)$ and $G(\cdot)$ are smooth functions and $G(x)$ is invertible for all $x \in \mathbb{R}^n$, $w(\cdot)$ as defined by

$$w(x) = [G(x)]^{-1}(-f(x) - Kx)$$ \hspace{1cm} (2.48)

is a smooth function of $x$ for all $x \in \mathbb{R}^n$. Smoothness implies that $G(x)$ is continuous and $w(x)$ is continuously differentiable over the closure of $X$ since the closure of $X$ is bounded. Using Assumption 2.3 and Lemmas 2.1 and 2.2, there exists finite constants $\gamma_G$ and $\gamma_w$ such that

$$\sigma_{\max}[G(x)] \leq \gamma_G$$ \hspace{1cm} (2.49)

$$\|w(x)\| \leq \gamma_w \|x\|$$ \hspace{1cm} (2.50)

for all $x \in X$. Since the nonlinear operator $M = w\Psi G$, we can therefore overbound the $L_2$ gain of $M$ by

$$\gamma_M \leq \gamma_w \gamma_G \|\Psi(j\omega)\|_{\infty} < \infty$$ \hspace{1cm} (2.51)

\hfill \Box

In Assumption 2.3, we assume that the inputs, $r_1$ and $r_2$, are contained for all
time in a closed ball around the origin of the input signal space and that this results in restricting the closed loop solution for the states of $M$, $x(t)$, to be in some bounded open region $X$ containing the origin. As a result, we note that the above proposition does not imply that $M$ is finite-gain stable in a global sense. This is because by Assumption 2.3, we are in essence restricting the size of the input to $M$, $z$, in Figure 2-3. Therefore, the above proposition shows that $M$ is finite-gain stable in the local or small-signal sense as defined in Appendix A. We now note that we can use the Small Gain Theorem under Assumption 2.3 to conclude robust stability in a local or small-signal sense, which we define below.

**Definition 2.3** The closed loop system in Figure 2-3 is small-signal, robustly stable if there exists finite constants, $R_1 > 0$ and $R_2 > 0$, such that for all input pairs $(r_1, r_2) \in L_2 \times L_2$ where $\|r_1(t)\| \leq R_1$ and $\|r_2(t)\| \leq R_2$ for all $t \geq 0$,

1. The operator that maps the signal pair $(r_1, r_2)$ to the signal pair $(e_1, e_2)$ is finite gain stable.

2. $\xi \in L_2$,

for all $\Delta \in D$.

With Assumption 2.3, we are restricting the state space in the linearizing coordinates in which the closed loop solution for $M$ can evolve. We note that restricting the state space in the linearizing coordinates $x$ corresponds to restricting the state space in the physical coordinates $\xi$. That is, the assumption that the closed loop solution for $x(t)$ is contained in $X$ is equivalent to the assumption that the closed loop solution for $\xi$ is contained in $\Xi$ where $\Xi$ is the image of $X$ under $\Phi^{-1}$. Since this coordinate transformation is bijective and continuous, it follows that $\Xi$ is a bounded, connected, open set in $\mathbb{R}^n$ containing the equilibrium point of interest $\xi = 0$. Although bounding the state space in the linearizing coordinates may hold no physical meaning, bounding the state space in the physical coordinates does indeed make physical sense since $\Xi$ can reflect saturation or other operating range limits for the physical system. Therefore, we can view our assumption as one of bounding the state space in the physical
coordinates in terms of physical limitations and then translating this bound to the linearizing coordinates. Finally, we note that we can choose to bound $\Xi$ or $X$ such that $\Xi$ or $X$ is arbitrarily large and still obtain small-signal finite-gain stability for $M$. However, this will imply that the $\mathcal{L}_2$-induced norm for $M$ for solutions $x \in X$ could be correspondingly large, which will violate the small gain condition that guarantees stability. Although this does not necessarily mean that the resulting closed loop system is unstable, it makes physical sense because by allowing large, in terms of infinity norm, signals to exist in the nonlinear closed loop system, we may excite the unmodeled dynamics, which will lead to instability. Finally, by restricting the state space, we are assured that the small-signal $\mathcal{L}_2$-induced norm of $\Phi^{-1}$ is bounded since $\Phi^{-1}$ is smooth. Therefore, under Assumption 2.3, $x \in \mathcal{L}_2$ implies $\xi \in \mathcal{L}_2$.

2.3 Problem Statement

We conclude this chapter with the following statement of the stability robustness problem.

Given a feedback linearization control law and Assumption 2.3, determine conditions under which the closed loop system in Figure 2-3 is small-signal, robustly stable in $\mathcal{L}_2$.

We note that it is a problem of analysis. In the chapters to follow we will provide a solution to this problem based on the strategy presented in the previous section.
Chapter 3

Stability Robustness for Systems with Constant Normal Form

Input Map

This chapter addresses the stability robustness problem for a special subclass of exact feedback linearizable nonlinear systems. In particular, we look at exact feedback linearizable systems whose normal form dynamics are given by

\[
\begin{align*}
\dot{x}_1^i &= x_2^i \\
\vdots \\
\dot{x}_{r_i-1}^i &= x_{r_i}^i \\
\dot{x}_{r_i}^i &= f_i(x) + \sum_{j=1}^{m} G_{ij} u_j,
\end{align*}
\]  

(3.1)

for \(1 \leq i \leq m\), where the normal form input map defined by

\[
G = \begin{bmatrix}
G_{11} & \cdots & G_{1m} \\
\vdots & \ddots & \vdots \\
G_{m1} & \cdots & G_{mm}
\end{bmatrix}
\]  

(3.2)
is constant, independent of \( x \), and invertible. We are motivated to consider such systems since, as noted in the previous chapter, the dependence of the normal form input map on the state \( x \) prevents us from directly trading, across frequency, between the magnitude of the linear time invariant uncertainty and the desired linearized transfer function matrix \( \Psi(s) \), which we control. We recognize this as one of the difficulties in analyzing our stability robustness problem. By considering systems whose normal form input map is constant, we circumvent this difficulty, which leads to a simplified analysis. The cost for this simplified analysis is that it is restricted to a smaller class of nonlinear systems. In particular, since the state transformation which converts the physical system to normal form is in general nonlinear, the analysis presented in this chapter will only apply to physical systems with a constant input map whose dynamics in physical coordinates are in normal form or are a linear transformation away from normal form. Although this class of nonlinear systems may seem prohibitively small, we note that a physical example of such a system is Euler's equations which define the attitude dynamics for a spacecraft.

The organization of this chapter is as follows. In Section 3.1, we formulate the stability robustness problem with respect to systems with constant normal form input maps and derive a sufficient condition for small signal, robust stability using a small gain argument. Analogous to the linear time invariant stability robustness problem, this sufficient condition also has a frequency domain interpretation, which is discussed in Section 3.2. Next, Section 3.3 addresses the conservatism of the derived sufficient condition. We first show, in Section 3.3.1, that due to the total directional and phase uncertainty of \( \Delta \), we cannot improve upon the conservatism of this condition using the standard methods of loop transformations. We then develop a sufficient condition for instability in Section 3.3.2. Using this instability condition, we derive conditions on the system dynamics under which we can conclude instability for some \( \Delta \in \mathcal{D} \) given that the robust stability condition derived in Section 3.1 is violated. Finally, we present some illustrative examples in Section 3.4 and a summary in Section 3.5.
3.1 Sufficient Condition for Stability Robustness

We start with the stability robustness problem as defined in Chapter 2. We note that while the problem statement has not changed, we further define the model of the system under analysis through the following assumption.

Assumption 3.1 The model of the physical system, $P_m$, in the physical coordinates is in normal form given by

$$
\begin{align*}
\dot{x}_1^i &= x_2^i \\
\vdots \\
\dot{x}_{r_i-1}^i &= x_{r_i}^i \\
\dot{x}_{r_i}^i &= f_i(x) + \sum_{j=1}^{m} G_{ij}u_j,
\end{align*}
$$

for $1 \leq i \leq m$, with a constant input map $G$ defined by

$$
G = \begin{bmatrix}
G_{11} & \cdots & G_{1m} \\
\vdots & \ddots & \vdots \\
G_{m1} & \cdots & G_{mm}
\end{bmatrix}
$$

As shown in Chapter 2, the resulting actual closed loop system can be put into the standard feedback configuration given in Figure 3-1. Here we note that the input map of $M$, which correspond to the normal form input map, is now given by $G$, which is constant and independent of the state $x$. In addition, due to Assumption 3.1, the coordinates $x$ now correspond to both the linearizing coordinates as well as the physical coordinates. Furthermore, the output map $w(\cdot)$ is still nonlinear and is given by the feedback linearization control law, which in this case is

$$
w(x) = G^{-1}(-f(x) - Kx) \quad (3.3)
$$

where $f(x)$ and $K$ are as defined before in Chapter 2. Finally, as motivated in Chapter 2, we will operate under Assumption 2.3 where we assume that the inputs
to the closed loop system in Figure 3-1 are suitably bounded so that the closed loop solution \( x(t) \) evolves in some bounded open set \( X \). The resulting robust stability problem is therefore to find a condition that will guarantee small-signal robust stability, as defined in Definition 2.3, for the closed loop system in Figure 3-1.

To begin our analysis, we note that since \( G \) is assumed constant, the operators \( W_o(s)\Delta(s)W_i(s) \), \( G \), and \( \Psi(s) \) can now be combined to form a single linear time invariant operator. The ability to combine these linear time invariant operators is a manifestation of the fact that we can now directly absorb the frequency dependent magnitude bound information of the uncertainty into the desired linearized system \( \Psi(s) \) through the input of \( M \). To take advantage of this fact, we would like to lump all of the magnitude bound information for the uncertainty to the input of \( M \) or to the input weight, \( W_i(s) \). Therefore, we choose \( W_o(s) = I \) and \( W_i(s) \) as a stable and proper rational transfer function matrix such that

\[
\sigma_{\min}[W_i(j\omega)] \geq I(\omega) \quad \forall \omega. \quad (3.4)
\]

Doing so, the resulting closed loop system in standard feedback form is transformed to Figure 3-2 where the weighted desired linearized transfer function matrix \( \Psi_w(s) \) is defined as

\[
\Psi_w(s) = \Psi(s)GW_i(s) \quad (3.5)
\]
We note that in Figure 3-2, the input pair \((q_1, q_2)\) and the output pair \((d_1, d_2)\) are related to the inputs and outputs of Figure 3-1 by

\[
\begin{align*}
q_1 &= r_1 \quad (3.6) \\
q_2 &= \Psi \omega r_2 \quad (3.7) \\
d_1 &= e_1 \quad (3.8) \\
d_2 &= \Psi \omega e_2. \quad (3.9)
\end{align*}
\]

Before we state our stability robustness condition, we first show the following lemma, which relates the finite-gain stability of Figure 3-2 to that of Figure 3-1.

**Lemma 3.1** If the closed loop system in Figure 3-2 is finite-gain stable, then the closed loop system in Figure 3-1 is finite-gain stable.

**Proof:** We first note that since \(\Psi(s)\) and \(\Delta(s)\) are strictly stable and \(G\) is constant, \(\Psi \omega(s)\) is strictly stable, and therefore, from Equations (3.6) and (3.7) we have that \((r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2\) implies \((q_1, q_2) \in \mathcal{L}_2 \times \mathcal{L}_2\). In fact,

\[
\|q_1\|_{2T} = \|r_1\|_{2T} \quad (3.10)
\]
\[ \|q_2\|_{2T} \leq \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty}\|r_2\|_{2T} \] (3.11)

for all \( T \in \mathbb{R}_+ \).

By definition, the finite-gain stability of the closed loop system in Figure 3-2 means that there exist finite constants \( \gamma_{11}, \gamma_{12}, \gamma_{21}, \) and \( \gamma_{22} \) such that

\[
\begin{bmatrix}
\|d_1\|_{2T} \\
\|d_2\|_{2T}
\end{bmatrix} \leq \begin{bmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{bmatrix} \begin{bmatrix}
\|q_1\|_{2T} \\
\|q_2\|_{2T}
\end{bmatrix}, \quad \forall T \in \mathbb{R}_+ \tag{3.12}
\]

Since \( e_1 = d_1 \), we have for all \( T \in \mathbb{R}_+ \),

\[
\|e_1\|_{2T} = \|d_1\|_{2T} \leq \gamma_{11}\|q_1\|_{2T} + \gamma_{12}\|q_2\|_{2T} \\
\leq \gamma_{11}\|r_1\|_{2T} + \gamma_{12}\|\Psi_w(j\omega)\|_{\mathcal{H}_\infty}\|r_2\|_{2T}. \tag{3.13}
\]

In addition, from Figure 3-1, \( e_2 = r_2 + \Delta e_1 \). Therefore, we have that for all \( T \in \mathbb{R}_+ \),

\[
\|e_2\|_{2T} \leq \|r_2\|_{2T} + \|\Delta(J\omega)\|_{\mathcal{H}_\infty}\|e_1\|_{2T} \\
\leq \|\Delta(J\omega)\|_{\mathcal{H}_\infty}\gamma_{11}\|r_1\|_{2T} + (1 + \|\Delta(J\omega)\|_{\mathcal{H}_\infty}\gamma_{12}\|\Psi_w(j\omega)\|_{\mathcal{H}_\infty})\|r_2\|_{2T} \tag{3.14}
\]

Since \( \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} \) and \( \|\Delta(J\omega)\|_{\mathcal{H}_\infty} \) are finite, it follows that the operator from \((r_1, r_2)\) to \((e_1, e_2)\) is finite-gain stable. Therefore, the closed loop system in Figure 3-1 is finite-gain stable.

We remark that the above lemma also holds for small-signal, finite-gain stability since \( r_1 = q_1 \) and the relationship between \( r_2 \) and \( q_2 \), in (3.7) is a strictly stable, linear time invariant operator \( \Psi_w(s) \). As a result, limiting the magnitude of the inputs \((q_1, q_2)\) so that the solution \( x(t) \in X \) for all \( t \geq 0 \) simply translates to a limit on the magnitude of the inputs \((r_1, r_2)\).

We now present the following sufficient condition for small-signal, robust stability.
Proposition 3.1 Under Assumption 2.3, the closed loop system in Figure 3-1 is small-signal, robustly stable if
\[
\gamma_w \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} \leq 1
\]  
(3.15)
where \(\gamma_w\) is the small-signal, \(L_2\)-induced norm of \(w\) over \(X\).

Proof: We first show that the closed loop system in Figure 3-2 is small-signal, finite-gain stable for all \(\Delta \in \mathcal{D}\). Examining Figure 3-2, we note that the Small Gain Theorem guarantees that this closed loop system is small-signal, finite-gain stable if the product of the small-signal, \(L_2\)-induced norms of the two operators, \(\Psi_w(s)\Delta(s)\) and \(w\), is strictly less than unity. Since \(\Psi_w(s)\Delta(s)\) is a strictly stable, linear time invariant operator, its small-signal, \(L_2\)-induced norm is simply its \(\mathcal{H}_\infty\) norm, \(\|\Psi_w(j\omega)\Delta(j\omega)\|_{\mathcal{H}_\infty}\). Since \(\gamma_w\) is the small-signal \(L_2\)-induced norm of \(w\), we can use the Small Gain Theorem to conclude that the closed loop system in Figure 3-2 is small-signal, finite gain stable for all \(\Delta \in \mathcal{D}\) if
\[
\gamma_w \|\Psi_w(j\omega)\Delta(j\omega)\|_{\mathcal{H}_\infty} < 1, \ \forall \Delta \in \mathcal{D}
\]  
(3.16)
From the definition of \(\mathcal{D}\), it is clear that the above condition is equivalent to that of the proposition, and therefore, the operator from \((q_1, q_2)\) to \((d_1, d_2)\) is small-signal, finite gain stable for all \(\Delta \in \mathcal{D}\).

Now, using Lemma 3.1, the above implies that the closed loop system in Figure 3-1 is also small-signal, finite gain stable for all \(\Delta \in \mathcal{D}\). That is, for all \(\Delta \in \mathcal{D}\) and for all input pairs \((r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2\) such that the closed loop solution \(x(t) \in X\) for all \(t \geq 0\), there exists finite constants \(\gamma_{11}, \gamma_{12}, \gamma_{21}, \text{ and } \gamma_{22}\) such that
\[
\begin{bmatrix}
\|e_1\|_{2T} \\
\|e_2\|_{2T}
\end{bmatrix} \leq 
\begin{bmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{bmatrix}
\begin{bmatrix}
\|r_1\|_{2T} \\
\|r_2\|_{2T}
\end{bmatrix}, \ \forall T \in \mathbb{R}_+
\]  
(3.17)
To complete the proof, we must now show that for all input pairs \((r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2\) such that the closed loop solution \(x(t) \in X\) for all \(t \geq 0\), the resulting signal \(x\) is in

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\( \mathcal{L}_2 \) for all \( \Delta \in \mathcal{D} \). From Figure 3.1, we have

\[
x = \Psi GW_i e_2 = \Psi_w e_2.
\]  

(3.18)

Once again, since \( \Psi_w(s) \) is strictly stable, it follows that

\[
\|x\|_{2T} \leq \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} \|e_2\|_{2T} \quad \forall T \in \mathbb{R}_+.
\]  

(3.19)

Substituting (3.17) into (3.18), we have that for all \( \Delta \in \mathcal{D} \) and for all input pairs \((r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2\) such that the closed loop solution \(x(t) \in X\) for all \( t \geq 0 \),

\[
\|x\|_{2T} \leq \gamma_1 \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} \|r_1\|_{2T} + \gamma_2 \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} \|r_2\|_{2T}, \quad \forall T \in \mathbb{R}_+.
\]  

(3.20)

Therefore, the result follows from recognizing that \( \|r_1\|_{2T} \leq \|r_1\|_2, \|r_2\|_{2T} \leq \|r_2\|_2 \) for all \( T \in \mathbb{R}_+ \) and then taking the limit as \( T \to \infty \) on the left side noting that the right side is now bounded from above by a quantity that is independent of \( T \). \( \square \)

In retrospect, Proposition 3.1 is a trivial application of the Small Gain Theorem given the small-signal \( \mathcal{L}_2 \)-induced norm of the nonlinear operator \( w \). Since \( w \) is a memoryless operator, its small-signal \( \mathcal{L}_2 \)-induced norm is equivalent to

\[
\gamma_w = \sup_{x \in X \setminus \{0\}} \frac{\|w(x)\|}{\|x\|},
\]  

(3.21)

which, by Lemma 2.2, is guaranteed to be finite given that \( X \) is bounded. Since the supremum in (3.21) is taken over \( X \), the value of \( \gamma_w \) is dependent on \( X \). In particular, it is clear that given \( X_1 \subseteq X_2 \), then necessarily \( \gamma_1 \leq \gamma_2 \) where \( \gamma_1 \) and \( \gamma_2 \) are the induced norms over \( X_1 \) and \( X_2 \), respectively. Therefore, it follows that the condition in the proposition depends on \( X \).

In addition, as a practical issue, we need a method to calculate \( \gamma_w \). One method is to calculate \( \gamma_w \) by solving the constrained optimization problem in (3.21). However, since the objective function above is, in general, not convex, the above optimization
problem may not yield a simple algorithmic solution. In practice, then, we select a finite set of grid points in the set \( X \), denoted \( X_I \), and approximate \( \gamma_w \) by solving the finite maximization problem

\[
\gamma_w \approx \max_{x \in X_I} \frac{\|w(x)\|}{\|x\|}
\]  

(3.22)

We note that given the smoothness of \( w \), we can approximate \( \gamma_w \) to any degree of accuracy by choosing a fine enough grid.

### 3.2 Frequency Domain Interpretation

In this section, we provide a frequency domain interpretation for the stability robustness condition in Proposition 3.1. Using the submultiplicative property of induced norms and the definition of the \( \mathcal{H}_\infty \) norm, the condition in (3.15) is satisfied if

\[
\sigma_{\text{max}}[\Psi(j\omega)G] \leq \frac{1}{\gamma_w \sigma_{\text{max}}[W_i(j\omega)]} \forall \omega.
\]

(3.23)

We note that (3.23) is similar to the robust stability conditions found in [8] for linear time invariant systems. In particular, (3.23) has a simple interpretation in the frequency domain as illustrated in a singular value plot such as Figure 3-3. Examining Figure 3-3, stability robustness is guaranteed if the separation in (dB), \( \delta(\omega) \), between the maximum singular value of \( \Psi(j\omega)G \) and the bound \( 1/\sigma_{\text{max}}[W_i(j\omega)] \) is greater than \( 20 \log(\gamma_w) \) for all \( \omega \). Recalling that \( \Psi(s) \) is the desired linearized transfer function matrix that we control through the the additional linear feedback gain \( K \) and that \( \sigma_{\text{max}}[W_i(j\omega)] \) reflects the magnitude bound information of the uncertainty, it is clear from the figure that the condition in (3.23) allows us to tradeoff between the gain of \( \Psi(j\omega) \) and the gain of the uncertainty across frequency. That is, we can take advantage of the fact that the magnitude of the uncertainty is usually low at low frequencies and high at high frequencies so that we can allow the magnitude of \( \Psi(j\omega) \) to be high at low frequencies and then roll off the magnitude of \( \Psi(j\omega) \) at high frequencies to maintain stability robustness. This is the same type of tradeoff between stability robustness and achievable bandwidth of the nominal closed loop.
that is found in the linear time invariant case. We note that this similarity is hardly surprising since the assumption that $G$ is constant allows us to directly absorb the frequency dependent magnitude bound of the uncertainty into $\Psi(s)$ just as in the linear time invariant case.

Unlike the linear time invariant results, (3.15) and (3.23) do not provide a means to directly loopshape the nominal loop to achieve stability robustness. To see why, we note that the nonlinear operator $w$, which is the feedback linearization control law, depends on the design gain matrix $K$. Therefore, when we vary $K$ to shape $\Psi(j\omega)$, it also affects $\gamma_w$, which in turn affects the required separation between $\Psi(j\omega)G$ and the uncertainty in our stability robustness condition. Another way of interpreting this is that changing $K$ changes both sides of the condition in (3.23) so that we do not have enough degrees of freedom to loopshape the nominal loop to achieve stability robustness. This is unfortunate since it means that we cannot directly use these conditions for synthesizing $K$ as we could in the linear time invariant case.
3.3 Issue of Conservatism

In this section, we address the conservatism of the condition in Proposition 3.1. We note that this conservatism comes only from the sufficiency of the Small Gain Theorem. The purpose of this section is to show that this conservatism is in a sense unavoidable, and therefore, the condition in Proposition 3.1 is the best that we can obtain under the assumptions of the problem. Towards this end, we first show in Section 3.3.1 that we cannot improve the condition using standard loop transformation techniques. Then, in Section 3.3.2, we show that under certain conditions on the feedback linearization control law $w(x)$, we can conclude instability for some $\Delta \in D$ if the condition in Proposition 3.1 is violated.

3.3.1 Stability Robustness Condition Using Loop Transformations

A standard way to improve the usefulness of the Small Gain Theorem is to use loop transformations. The resulting stability condition is commonly referred to as the Circle Criterion or Circle Theorem in [37], [34], [4], and [21]. The idea is to introduce a static linear gain, $K_w$, into the closed loop system in Figure 3-2 by first subtracting it from $w$ and then adding it back to $w - K_w$ as shown in Figure 3-4. We note that, in general, $K_w$ can be any linear finite-gain stable operator but that for our purposes, we will assume that $K_w$ is simply a gain matrix. In addition, the introduction of $K_w$ does not affect the stability of the original closed loop system in Figure 3-2. Now, through block diagram manipulations, the system can be transformed to that in Figure 3-5 where $K_w$ now acts as a feedback in the forward loop and as a feed-forward in the feedback loop. As we will prove shortly, the (small-signal) finite-gain stability of the closed loop system in Figure 3-5 will imply the (small-signal) finite gain stability of the closed loop system in Figure 3-4 or Figure 3-2. We can therefore apply the Small Gain Theorem to the closed loop system in Figure 3-5 by looking at the $L_2$-induced norms of the operators $\Psi_w(s)\Delta(s)(I - K_w\Psi(s)\Delta(s))^{-1}$ and $w - K_w$. The point here is that the original closed loop system is stable if the small gain condition is satisfied for
the transformed system for some $K_w$. Therefore, we view $K_w$ as an additional degree of freedom that is introduced in order to minimize the open loop gain product of the transformed system so that the small gain condition can be satisfied to conclude stability.

For our purposes, we use loop transformations to arrive at the following small-signal robust stability condition.

**Proposition 3.2** Under Assumption 2.3 and given $K_w$ and $\gamma_2$ such that

$$
\|w(x) - K_w x\| \leq \gamma_2 \|x\|, \quad \forall x \in X
$$

(3.24)

the closed loop system in Figure 3-1 is small-signal, robustly stable if

1. $\gamma_1 \gamma_k < 1$

2. $\frac{\gamma \gamma_k}{1 - \gamma_1 \gamma_k} < 1$

where we define $\gamma_k = \sigma_{\text{max}}[K_w]$ and $\gamma_1 = \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty}$. 

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Proof: We first show that the closed loop system in Figure 3-2 or Figure 3-4 can be transformed to Figure 3-5. We note that the closed loop system in Figure 3-4 can be described by the following equations.

\[ d_1 = q_1 + y_2 \]  \hspace{1cm} (3.25)
\[ d_2 = q_2 + y_1 \]  \hspace{1cm} (3.26)
\[ y_1 = \Psi \Delta d_1 \]  \hspace{1cm} (3.27)
\[ y_2 = w d_2. \]  \hspace{1cm} (3.28)

Now, define a new output

\[ y_4 = y_2 - K_w d_2. \]  \hspace{1cm} (3.29)

Using (3.29) to eliminate \( y_2 \) in (3.25) gives

\[ d_1 = q_1 + y_4 + K_w d_2. \]  \hspace{1cm} (3.30)
Substituting (3.26) into (3.30) and noting that $K_w$ is linear yields

$$d_1 = q_3 + y_4 + K_w y_1$$

(3.31)

where we define

$$q_3 = q_1 + K_w q_2.$$

(3.32)

In addition, substituting (3.28) into (3.29) gives

$$y_4 = (w - K_w) e_2.$$

(3.33)

We now note that Equations (3.26), (3.27), (3.31), and (3.33) describe the closed loop system in Figure 3-5 if we redefine

$$q_4 = q_2, \ d_4 = d_2, \ y_3 = y_1.$$

(3.34)

Next, we show that under the first condition of the proposition, the closed loop operator from $d_3$ to $y_3$ is finite-gain stable for all $\Delta \in \mathcal{D}$. From the figure, we have

$$d_1 = d_3 + K_w y_3.$$

(3.35)

From (3.27), we have

$$\|y_3\|_{2T} = \|y_1\|_{2T} \leq \|\Psi_w(j \omega) \Delta(j \omega)\|_{\mathcal{H}_\infty} \|d_1\|_{2T}, \ \forall \Delta \in \mathcal{D}, \ T \in \mathbb{R}_+.$$

(3.36)

Since $\|\Delta(j \omega)\|_{\mathcal{H}_\infty} < 1 \ \forall \Delta \in \mathcal{D}$, (3.36) becomes

$$\|y_3\|_{2T} \leq \|\Psi_w(j \omega)\|_{\mathcal{H}_\infty} \|d_1\|_{2T} = \gamma_1 \|d_1\|_{2T}, \ \forall T \in \mathbb{R}_+.$$

(3.37)

In addition, we have from (3.35) that

$$\|d_1\|_{2T} \leq \|d_3\|_{2T} + \gamma_k \|y_3\|_{2T}, \ \forall T \in \mathbb{R}_+, \quad (3.38)$$

55
and substituting (3.37) into (3.38) gives

\[ \|d_1\|_{2T} \leq \|d_3\|_{2T} + \gamma_1 \gamma_k \|d_1\|_{2T}, \quad \forall T \in \mathbb{R}_+. \]  

(3.39)

Using the first condition of the proposition that \(\gamma_1 \gamma_k < 1\), it is clear that \(1/(1 - \gamma_1 \gamma_k)\) is finite, and therefore,

\[ \|d_1\|_{2T} \leq \frac{1}{1 - \gamma_1 \gamma_k} \|d_3\|_{2T}, \quad \forall T \in \mathbb{R}_+. \]  

(3.40)

Finally, substituting (3.40) into (3.37) gives

\[ \|y_3\|_{2T} \leq \frac{\gamma_1}{1 - \gamma_1 \gamma_k} \|d_3\|_{2T}, \quad \forall T \in \mathbb{R}_+, \]  

(3.41)

which shows finite gain stability.

Now, from (3.41), an overbound on the \(L_2\)-induced norm of the operator from \(d_3\) to \(y_3\) is \(\gamma_1/(1 - \gamma_1 \gamma_k)\). In addition, using (3.24) and Assumption 2.3, the small-signal \(L_2\)-induced norm of the operator \(w - K_w\) is \(\gamma_2\). Therefore, from the Small Gain Theorem, it follows that the second condition of the proposition is sufficient to conclude that the closed loop system in Figure 3-5 is small-signal, finite gain stable for all \(\Delta \in \mathcal{D}\).

That is, for all \(\Delta \in \mathcal{D}\) and for all input pairs \((q_3, q_4) \in L_{2e} \times L_{2e}\) such that the closed loop solution \(x(t) \in X\) for all \(t \geq 0\), there exists finite constants \(\gamma_{11}, \gamma_{12}, \gamma_{21}, \) and \(\gamma_{22}\) such that

\[ \begin{bmatrix} \|d_3\|_{2T} \\ \|d_4\|_{2T} \end{bmatrix} \leq \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} \|q_3\|_{2T} \\ \|q_4\|_{2T} \end{bmatrix}, \quad \forall T \in \mathbb{R}_+. \]  

(3.42)

To complete the proof, we need to show that this implies that the closed loop system in Figure 3-2 is small-signal finite gain stable for all \(\Delta \in \mathcal{D}\). We start by relating the inputs in Figure 3-5 to the inputs in Figure 3-2. From Equations (3.32) and (3.34), it is clear that

\[ \begin{bmatrix} \|q_3\|_{2T} \\ \|q_4\|_{2T} \end{bmatrix} \leq \begin{bmatrix} 1 & \gamma_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \|q_1\|_{2T} \\ \|q_2\|_{2T} \end{bmatrix}, \quad \forall T \in \mathbb{R}_+. \]  

(3.43)
Therefore, whenever $(q_1, q_2) \in \mathcal{L}_2 \times \mathcal{L}_2$, it follows that $(q_3, q_4) \in \mathcal{L}_2 \times \mathcal{L}_2$. In addition, since the relationship between $(q_1, q_2)$ and $(q_3, q_4)$ is a linear time invariant operator, it follows that limiting the magnitude of $(q_3, q_4)$ in Figure 3-5 so that the solution $x(t) \in X$ for all $t \geq 0$ simply translates to a limit on the magnitude of $(q_1, q_2)$ in Figure 3-2. Now, substituting (3.42) and (3.43) into (3.40) gives

$$
\|d_1\|_{2T} \leq \frac{\gamma_1}{1 - \gamma_1 \gamma_k} \|q_1\|_{2T} + \frac{\gamma_k \gamma_1 + \gamma_2}{1 - \gamma_1 \gamma_k} \|q_2\|_{2T}, \quad \forall T \in \mathbb{R}_+.
$$

(3.44)

Similarly, since $d_2 = d_4$, we have

$$
\|d_2\|_{2T} \leq \gamma_2 \|q_1\|_{2T} + (\gamma_k \gamma_2 + \gamma_2) \|q_2\|_{2T}, \quad \forall T \in \mathbb{R}_+.
$$

(3.45)

Since the (3.44) and (3.45) hold for all $\Delta \in \mathcal{D}$, it follows by definition that the closed loop system in Figure 3-2 is small-signal finite gain stable for all $\Delta \in \mathcal{D}$. From this, we can conclude that the original system in Figure 3-1 is small-signal robustly stable by arguing in the same manner as in Proposition 3.1.

We remark that the first condition of the proposition is simply a small gain condition that guarantees the finite gain stability of the closed loop operator from $d_3$ to $y_3$ for all $\Delta \in \mathcal{D}$. Having guaranteed its stability, an overbound, $\gamma_1/(1 - \gamma_1 \gamma_k)$, on the $L_2$-induced norm of this closed loop operator for all $\Delta \in \mathcal{D}$ is used in the second small gain condition,

$$
\frac{\gamma_1 \gamma_2}{1 - \gamma_1 \gamma_k} < 1,
$$

(3.46)

which guarantees small-signal finite gain stability for the loop. We note, however, that Proposition 3.2 is also only sufficient and therefore is of added benefit only if it can improve upon the sufficient condition in Proposition 3.1, which can be rewritten as

$$
\gamma_w \gamma_1 < 1.
$$

(3.47)

That is, the loop transformation condition is only of benefit if it can be satisfied for some $K_w$ while the original condition in (3.47) is violated; in other words, if there
exists \( K_w \) such that
\[
\frac{\gamma_1 \gamma_2}{1 - \gamma_1 \gamma_k} < \gamma_w \gamma_1 \quad \text{for} \quad \gamma_w \gamma_1 \geq 1.
\] (3.48)

We now show that the above is never true. That is, the second condition in Proposition 3.2 will not be satisfied if the sufficient condition for small-signal robust stability in Proposition 3.1 is violated.

**Proposition 3.3** Given

1. \( \gamma_2, \gamma_w, \) and \( \gamma_k \) are the small-signal \( L_2 \)-induced norms over \( X \) of the memoryless operators \( w - K_w, w, \) and \( K_w, \) respectively

2. \( \gamma_1 \gamma_k < 1, \) and

3. \( \gamma_1 = \| \Psi_w(j \omega) \|_{H_\infty}, \)

there does not exist a linear gain \( K_w \) such that (3.48) is satisfied.

**Proof:** We note that since all \( \gamma_s \) are positive and \( 1 - \gamma_1 \gamma_k > 0, \) (3.48) is equivalent to
\[
\gamma_2 < (1 - \gamma_1 \gamma_k) \gamma_w < \gamma_w - \gamma_k
\] (3.49)

This is not true for any \( K_w \) since from the triangle inequality, we have
\[
\gamma_2 = \sup_{x \in X \setminus 0} \left\{ \frac{\| w(x) - K_w x \|}{\| x \|} \right\} \geq \sup_{x \in X \setminus 0} \left\{ \frac{\| w(x) \|}{\| x \|} \right\} - \sup_{x \in X \setminus 0} \left\{ \frac{\| K_w x \|}{\| x \|} \right\} = \gamma_w - \gamma_k
\] (3.50)

The consequence of the above proposition is that the use of loop transformations does not reduce the conservatism of the sufficient condition for small-signal robust stability in Proposition 3.1. Examining the conditions in the loop transformation proposition, we note that the chief source of additional conservatism is the overbounding of the \( L_2 \)-induced norm for the closed loop operator from \( d_3 \) to \( y_3. \) However, this overbound is necessary since the closed loop operator from \( d_3 \) to \( y_3 \)
contains the uncertainty $\Delta$. As a result, its $L_2$-induced norm cannot be calculated even though it is linear time invariant. In fact, for the scalar case, this overbound is tight in that there exists a $\Delta \in \mathcal{D}$ that achieves the bound.

### 3.3.2 Sufficient Condition for Instability

In this section, we derive a sufficient condition for instability for the closed loop system in Figure 3-1 using Lyapunov's Linearization Method. Before we proceed, we need to first define what we mean by instability.

**Definition 3.1** The closed loop system in Figure 3-1 is unstable if there exists a rational transfer function matrix $\bar{\Delta} \in \mathcal{D}$ such that the equilibrium point of the resulting closed loop system at the origin is not stable in the sense of Lyapunov.

We note that the above definition for instability is related to an equilibrium point for a finite set of state equations rather than to signals from an input-output relationship for the closed loop system. Therefore, it does not complement our definition of stability in Chapter 2. However, we are only interested in the stability of the closed loop for a particular $\bar{\Delta} \in \mathcal{D}$. Since this $\bar{\Delta}$ is a rational transfer function matrix, it can be realized by a finite set of state equations, i.e. a finite dimensional system. In addition, $\bar{\Delta}$ is linear and strictly stable so that its equilibrium point is at the origin of its state space. As a result, the closed loop system with this $\bar{\Delta}$ can be realized in some finite set of state equations with an equilibrium at the origin, and the present definition for instability makes sense. With this in mind, we now state the following sufficient condition for instability.

**Proposition 3.4** Let

$$C = \left. \frac{\partial w(x)}{\partial x} \right|_{x=0} \quad (3.51)$$

be the Jacobian matrix of $w(x)$ evaluated at the equilibrium point $x = 0$. If

$$\| C \Psi_w(j\omega) \|_{\mathcal{H}_\infty} > 1, \quad (3.52)$$

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then there exists a rational transfer function matrix \( \tilde{\Delta} \in \mathcal{D} \) such that the closed loop system in Figure 3-1 is unstable as defined in Definition 3.1.

Proof: We note that if we linearize the nonlinear closed loop system in Figure 3-1 about the equilibrium point at the origin, the resulting linear time invariant system can be expressed by the block diagram in Figure 3-6 where \( C \) is as defined in (3.51). It is well known that for the linear closed loop system, we can construct a rational transfer function matrix \( \tilde{\Delta} \in \mathcal{D} \) that will destabilize the closed loop if

\[
\|C\Psi_w(j\omega)\|_{\infty} > 1. \tag{3.53}
\]

In other words, if (3.53) is satisfied, then \((I + \tilde{\Delta}(s)C\Psi_w(s))^{-1}\) will have at least one pole in the open right half \(s\)-plane. Therefore, by Lyapunov's Linearization Method, the nonlinear closed loop in Figure 3-1 with this \( \tilde{\Delta} \) is unstable as defined in Definition 3.1.  

Proposition 3.4 simply states that if the linearization of the closed loop system violates its small gain criterion, then the nonlinear closed loop system is unstable. We note that the ability to construct a rational \( \tilde{\Delta}(s) \) that destabilizes the linearized closed loop is crucial since it allows us to apply Lyapunov’s Linearization Method on
the resulting closed loop system. With regard to the stability robustness condition in Proposition 3.1, we now state the following corollary to Proposition 3.4 which gives us a condition on the feedback linearization control law $w(x)$ that allows us to conclude instability when this condition for robust stability is violated.

**Corollary 3.1** Under the condition

$$\gamma_w \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} = \|C\Psi_w(j\omega)\|_{\mathcal{H}_\infty},$$  \hspace{1cm} (3.54)

if

$$\gamma_w \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} > 1,$$  \hspace{1cm} (3.55)

then there exists a rational transfer function matrix $\tilde{A} \in \mathcal{D}$ such that the closed loop system in Figure 3-1 is unstable as defined in Definition 3.1.

We note that the condition (3.54) on $w(x)$ is quite restrictive. To see this, we recall that from the definition of the Jacobian, it follows that if we define

$$w_h(x) = w(x) - Cx$$  \hspace{1cm} (3.56)

then

$$\lim_{\|x\| \to 0} \frac{\|w_h(x)\|}{\|x\|} = 0.$$  \hspace{1cm} (3.57)

Using (3.57), we have

$$\gamma_w = \sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|w(x)\|}{\|x\|} = \sigma_{\text{max}}[C]$$  \hspace{1cm} (3.58)

for $\mathcal{X}_\epsilon = \{x \mid \|x\| < \epsilon\}$ where $\epsilon$ is an arbitrarily small positive number since

$$\sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|Cx\|}{\|x\|} \sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|w_h(x)\|}{\|x\|} \leq \sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|w(x)\|}{\|x\|} \leq \sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|Cx\|}{\|x\|} + \sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|w_h(x)\|}{\|x\|}.$$  \hspace{1cm} (3.59)

From

$$\gamma_w = \sup_{x \in \mathcal{X}_1 \setminus \{0\}} \frac{\|w(x)\|}{\|x\|}$$  \hspace{1cm} (3.60)
we have that for any $X$ such that $X_e \subseteq X$, the corresponding $\gamma_w$ is such that

$$\sigma_{\max}[C] \leq \gamma_w \quad (3.61)$$

and that equality occurs only when the supremum in (3.60) is attained in an arbitrarily small neighborhood around the origin. Furthermore, we have

$$\|C\Psi_w(j\omega)\|_{\mathcal{H}_\infty} \leq \sigma_{\max}[C] \|\Psi(j\omega)\|_{\mathcal{H}_\infty}. \quad (3.62)$$

Therefore, in order for the condition (3.54) to hold, we must have that the supremum in (3.60) be attained in an arbitrarily small neighborhood around the origin and that the input maximum singular direction of $C$ corresponds to the output maximum singular direction of $\Psi_w(j\omega)$ at the frequency for which the $\mathcal{H}_\infty$ norm of $\Psi_w(j\omega)$ is attained.

As shown above, the condition in (3.54) is quite restrictive and is not satisfied in general. However, under this restrictive condition, Corollary 3.1 shows that the stability robustness condition in Proposition 3.1 is not conservative in the sense that if this condition is violated, then there exists a rational $\tilde{\Delta} \in \mathcal{D}$ such that the resulting closed loop system is unstable as defined in Definition 3.1. We note that this does not make the condition in Proposition 3.1 necessary and sufficient since instability can only be shown under the restrictive condition (3.54) and since the definition of instability here does not complement our definition of stability in Proposition 3.1. Finally, we note that if (3.54) is not satisfied, then we can only say that the condition in Proposition 3.1 is conservative due to the Small Gain Theorem since in this case we do not know how to construct the destabilizing $\Delta \in \mathcal{D}$.

### 3.4 Examples

In this section, we present some examples to illustrate the concepts developed in this chapter.

**Example 3.1:**
For the first example, we consider, as the model of the plant, the scalar nonlinear system

\[ P_m : \dot{x} = 2x^3 + u. \]  \hspace{1cm} (3.63)

Clearly, \( P_m \) satisfies Assumption 2.1. In fact, it is in normal form, so that \( x \) corresponds to both the physical coordinates as well as the linearizing coordinates, with constant input map \( G = 1 \). Using the feedback linearization control law

\[ u = -2x^3 - Kx + r, \]  \hspace{1cm} (3.64)

the resulting desired linearized transfer function in the absence of uncertainty is

\[ \Psi(s) = \frac{1}{s + K} \]  \hspace{1cm} (3.65)

where we need \( K > 0 \) so that \( \Psi(s) \) is strictly stable.

We assume that the representation of the uncertainty in the model is as defined in Definition 2.1 where the multiplicative error is bounded by

\[ |W_i(j\omega)\Delta(j\omega)W_o(j\omega)| < \left|\frac{2j\omega}{j\omega + 10}\right| \]  \hspace{1cm} (3.66)

Since we want to take advantage of the constant input map, we choose

\[ W_i(s) = \frac{2s}{s + 10}, \quad W_o(s) = 1. \]  \hspace{1cm} (3.67)

so that \( \Delta \in \mathcal{D} \). The resulting closed loop system with \( r = 0 \) for all \( \Delta \in \mathcal{D} \) is shown in Figure 3-7 where

\[ w(x) = -2x^3 - Kx \]  \hspace{1cm} (3.68)

is the feedback linearization control law. We note that \( w(\cdot) \) is not globally Lipschitz. As a result, we must bound the state space in order for the \( L_2 \)-induced norm of \( w \), \( \gamma_w \), to exist. Towards this end, we choose \( X \) as

\[ X = \{ x \mid |x| < x_{\text{max}}, \ 0 < x_{\text{max}} < \infty \}. \]  \hspace{1cm} (3.69)
In addition, for this scalar case, we can calculate $\gamma_w$ analytically. This $\gamma_w$ is dependent on the choice for $X$ and is given by

$$
\gamma_w = \sup_{x \in X \setminus \{0\}} \frac{|-2x^3 - Kx|}{|x|} = \sup_{x \in X \setminus \{0\}} |2x^2 + K| = 2x_{\text{max}}^2 + K
$$

(3.70)

where the last equality holds since $K > 0$.

Evoking Proposition 3.1, a sufficient condition for small-signal robust stability is given by

$$
\gamma_w \left\| \Psi_w(j\omega) \right\|_{\mathcal{H}_\infty} = (2x_{\text{max}}^2 + K) \left\| \frac{2j\omega}{(j\omega + 10)(j\omega + K)} \right\|_{\mathcal{H}_\infty} \leq 1
$$

(3.71)

Since $\gamma_w$ increases monotonically as a function of $x_{\text{max}}$ for $x_{\text{max}} > 0$, we can solve for the neighborhood $X$ that will guarantee small-signal stability robustness for each $K$ using Proposition 3.1 by solving for $x_{\text{max}}$ as a function of $K$ in condition (3.71) where the inequality is replaced by equality. Doing so yields the solid curve in Figure 3-8. We can interpret this plot in two ways. First, given any $K$ the curve gives the maximum neighborhood about the origin in which the closed loop solution $x(t)$ must
Figure 3-8: Maximum Neighborhood for Small-Signal Robust Stability as a Function of $K$

remain for all $t \geq 0$ in order to guarantee small-signal robust stability. Alternatively, given any $X = \{ x \mid |x| < x_{\text{max}}, 0 < x_{\text{max}} < \infty \}$, the curve gives the maximum $K$, which translates to the maximum bandwidth for $\Psi(s)$, that will guarantee small-signal robust stability.

Examining Figure 3-8, we note the following. First, as $K$ increases, the neighborhood $X$ that will guarantee small-signal stability robustness shrinks. This is because the left hand side of condition (3.71) is dependent on both $x_{\text{max}}$ and $K$, and as $K$ increases, $x_{\text{max}}$ must decrease to maintain the small gain condition. In particular, as noted beforehand, the gain of the $w$ is a monotonically increasing function of $x_{\text{max}}$ or how far the closed loop solution $x(t)$ can deviate from the equilibrium for all $t \geq 0$. Therefore, there is a tradeoff between the bandwidth of the desired linearized system and the constraint that we place on how far the closed loop solution $x(t)$ can deviate.
from the equilibrium point. Physically, this tradeoff implies that if we drive the closed loop harder with a higher gain, then we must constrain our external inputs or initial conditions so that our state trajectory \( x(t) \) stays within a smaller region around the equilibrium in order to reduce the gain due to the nonlinearity \( w \). Otherwise, the large gain of the nonlinearity and the high bandwidth of the desired linearized transfer function may serve to excite high frequency unmodelled dynamics. Second, the linearization of the closed loop system in Figure 3-7 gives

\[
C \Psi_w(s) = \frac{2Ks}{(s + 10)(s + K)}
\]

(3.72)

where \( C \Psi_w(s) \) is as defined in Section 3.3.2. We note that for \( K > 10 \),

\[
\|C \Psi_w(j\omega)\|_{\mathcal{H}_\infty} > 1.
\]

(3.73)

Therefore, by Proposition 3.4, the closed loop system in Figure 3-7 is unstable for \( K > 10 \) and some \( \tilde{\Delta} \in \mathcal{D} \). Finally, we note that for \( x_{\text{max}} > 0 \), the condition, \( \gamma_w \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} = \|C \Psi_w(j\omega)\|_{\mathcal{H}_\infty} \), is not satisfied. Therefore, we can only say that the condition in Proposition 3.1 is conservative due to the Small Gain Theorem since in this case we do not know how to construct the destabilizing \( \Delta \in \mathcal{D} \). This conservatism can be viewed as allowing a larger \( x_{\text{max}} \) at each value of \( K \) and still maintaining small-signal robust stability for all \( \Delta \in \mathcal{D} \) as shown by the dashed curve in Figure 3-8. \( \square \)

**Example 3.2:**

For the second example, we consider, as the model of the plant, the scalar nonlinear system

\[
P_m : \dot{x} = \alpha \sin(x) + u
\]

(3.74)

where \( \alpha \in \mathbb{R} \) is a known parameter. We can view \( |\alpha| \) as being a measure of the size of the system nonlinearity. Again, it is clear that \( P_m \) satisfies Assumption 2.1 and is in normal form, so that \( x \) corresponds to both the physical coordinates as well as the linearizing coordinates, with constant input map \( G = 1 \). Using the feedback
linearization control law

\[ u = -\alpha \sin(x) - Kx + r, \quad (3.75) \]

the resulting desired linearized transfer function in the absence of uncertainty is

\[ \Psi(s) = \frac{1}{s + K} \quad (3.76) \]

where we need \( K > 0 \) so that \( \Psi(s) \) is strictly stable.

We assume that the representation of the uncertainty in the model is as defined in Definition 2.1 where the multiplicative error is bounded by

\[ |W_i(j\omega)\Delta(j\omega)W_o(j\omega)| < \left| \frac{100j\omega}{j\omega + 1000} \right| \quad (3.77) \]

Again we want to take advantage of the constant input map so we choose

\[ W_i(s) = \frac{100s}{s + 1000}, \quad W_o(s) = 1. \quad (3.78) \]

so that \( \Delta \in \mathcal{D} \). The resulting closed loop system with \( r = 0 \) for all \( \Delta \in \mathcal{D} \) is shown in Figure 3-9 where

\[ w(x) = -\alpha \sin(x) - Kx \quad (3.79) \]

is the feedback linearization control law. We note that for this case, \( w(\cdot) \) is globally Lipschitz, and we can bound the \( L_2 \)-induced norm of \( w(x) \) over all \( x \). Therefore, in this example we do not need to assume that the closed loop solution \( x(t) \) is confined to some bounded set \( \mathcal{X} \), and the result that we obtain here is one of finite gain stability instead of small-signal finite gain stability. To find the \( l_2 \)-induced norm, and thus the \( L_2 \)-induced norm, of \( w \), we note that \( w(x) \) satisfies

\[ -(\alpha + K)x^2 \leq w(x)x \leq (0.217\alpha - K)x^2 \text{ for } \alpha \geq 0 \quad (3.80) \]

and

\[ (0.217\alpha - K)x^2 \leq w(x)(x) x \leq -(\alpha + K)x^2 \text{ for } \alpha < 0. \quad (3.81) \]
A straightforward calculation using the definition of the $l_2$-induced norm gives

$$
\gamma_w = \max\{|\alpha + K|, | -0.217\alpha + K|\} \quad (3.82)
$$

Evoking Proposition 3.1, a sufficient condition for robust stability is given by

$$
\gamma_w \|\Psi_w(j\omega)\|_{\mathcal{H}_\infty} = \max\{|\alpha + K|, | -0.217\alpha + K|\} \left\| \frac{100j\omega}{(j\omega + 1000)(j\omega + K)} \right\|_{\mathcal{H}_\infty} \leq 1
$$

(3.83)

Using (3.83), we solve for the range of $K$ that guarantees robust stability to all $\Delta \in \mathcal{D}$ for each value of $\alpha$. Doing so yields Figure 3-10.

Examining Figure 3-10, we note the following. First, for $\alpha = 0$, the model is linear, and the range of $K \in (0, 10.101]$ that guarantees stability robustness corresponds to that from a linear time invariant analysis, as expected. For $-10 \leq \alpha \leq 10$, the upper bound for the range of $K$ that guarantees robust stability decreases linearly as $|\alpha|$ increases. This is again as expected since increasing $|\alpha|$ increases $\gamma_w$ so that we must decrease $K$ to maintain the small gain condition. However, for $\alpha \in [-16.537, 10)$, we note that there exists both a lower bound for $K$, above zero, in addition to the anticipated upper bound. Therefore, in this case, we cannot simply reduce the
Figure 3-10: Maximum Neighborhood for Small-Signal Robust Stability as a Function of $\alpha$

bandwidth of the desired linearized system, as in the linear time invariant case, to ensure stability robustness. To see why this is the case, we note that for $\alpha < 0$, the gain $\gamma_w$ will be given by $|\alpha + K|$ for small $K$. As a result, increasing $K$ will at first lower $\gamma_w$ thus helping the small gain condition. Physically, this phenomenon is very real and points to the dangers of using feedback linearization for cases where the nonlinear dynamics of the plant contains highly nonlinear damping terms. Feedback linearization in these cases requires that we cancel these positive nonlinear damping terms using their equivalent negative damping terms. These negative damping terms are essentially large positive feedback which, in turn, may excite the unmodelled dynamics. As a result, we need to add in some large linear damping term $Kx$ as additional feedback in order to reduce the overall gain of the loop.
3.5 Chapter Summary

In this chapter, we have addressed the stability robustness problem for a special subclass of exact feedback linearizable systems where the dynamics in the physical coordinates are in the so-called normal form and the input map for these normal form dynamics is constant, independent of the state. For this class of systems, we have derived a sufficient condition for small-signal robust stability using a small gain argument. We note that this condition has a frequency domain interpretation similar to that found in the stability robustness problem for linear time invariant systems. In particular, stability robustness is ensured if there is adequate separation between the maximum singular value of the desired linearized transfer function matrix $\Psi(s)$ and the uncertainty magnitude bound at each frequency. This separation is dictated by the $L_2$-induced norm of the static nonlinearity $\omega$, which is the nonlinear output map of $M$. However, unlike the stability robustness results for linear time invariant systems, we cannot use this condition to directly loopshape $\Psi(s)$ to ensure stability robustness, and therefore, the condition cannot be directly used for synthesis. Furthermore, the condition suffers from the limitation that it only applies to this restrictive subclass of exact feedback linearizable systems.

In addition, we have explored the conservatism of this sufficient condition. In particular, we have shown that we cannot reduce the conservatism of this condition using loop transformations. As a result, we can say that our sufficient condition is in a sense the best condition that we can derive using a small gain argument. Finally, we have also developed a sufficient condition for instability using Lyapunov's Linearization Method. Using this result, we have derived conditions on the system dynamics under which we can construct a destabilizing uncertainty if our sufficient condition for small-signal robust stability is violated.
Chapter 4

Stability Robustness for General Exact Feedback Linearizable Systems

In the previous chapter, we address the stability robustness problem with respect to a special class of exact feedback linearizable systems whose normal form input map is constant. For this special class of systems, we are able to obtain a simple stability robustness result due to our ability to absorb directly the frequency dependent magnitude bound information of the uncertainty into the desired linearized transfer function matrix $\Psi(s)$. However, we note that this special case is restrictive and does not include many physical systems of interest. In this chapter, we consider the stability robustness problem with respect to general exact feedback linearizable systems whose normal form input map may be dependent on the states in the linearizing coordinates, $x$. As we have seen in Chapter 2, this general problem proves to be difficult since the state dependent input map prevents us from trading between the magnitudes of the uncertainty and $\Psi(s)$ across frequency. The consequence of this is that if we apply a small gain argument in which we consider the uncertainty and $\Psi(s)$ as two separate operators connected in feedback, then necessarily, we have to consider the worst case gains across frequency of both the uncertainty and $\Psi(s)$, which is shown by example to be overly conservative. To overcome this difficulty, we will analyze this
problem by following the strategy outlined in Chapter 2. The idea is to combine the weights, \( W_r(s) \) and \( W_o(s) \), which contain the frequency dependent magnitude bound information of the uncertainty, into the nonlinear operator \( M \), which contains \( \Psi(s) \). The key, then, is to find a bound on the \( L_2 \)-induced norm of this combined operator so that we can apply the Small Gain Theorem. In bounding the \( L_2 \)-induced norm, we will exploit the relationship between the \( L_2 \)-induced norm of a nonlinear system and the solution of a Hamilton Jacobi Inequality that corresponds to that system. We note that this relationship is a result of the fundamental relationship between the finite-gain stability, in \( L_2 \), of a nonlinear system and the notion of dissipativity for that system as shown by Willems, [35].

The organization of the chapter is as follows. In Section 4.1, we derive a sufficient condition for small-signal robust stability using a small gain argument by relating the \( L_2 \)-induced norm of a nonlinear system to the solution of a Hamilton Jacobi Inequality. We first present this relationship in Section 4.1.1 and then apply it in Section 4.1.2 to obtain our robust stability condition. Unfortunately, this condition is difficult to verify since it requires a solution to the Hamilton Jacobi Inequality, which is computationally complex. As a result, we present a second sufficient condition for small-signal robust stability in Section 4.2, which relates the small-signal robust stability of our feedback linearization closed loop system to the robust stability of its linearization. In Section 4.2.1, we first motivate this approach by relating the small-signal \( L_2 \)-induced norm of a nonlinear operator to the \( H_\infty \) norm of its linearization. Then, in Section 4.2.2, we derive a result that guarantees a neighborhood for small-signal robust stability under the condition that the linearization of our closed loop system is robustly stable. Finally, we present some illustrative examples in Section 4.3 and a summary in Section 4.4.
4.1 Stability Robustness Analysis Using the Hamilton Jacobi Inequality

Our starting point is again the problem statement given in Chapter 2. As shown in Chapter 2, the resulting actual closed loop system can be put into the standard feedback configuration given in Figure 4-1. Following the strategy developed in Chapter 2, we combine the weights, $W_i(s)$ and $W_o(s)$, with the nonlinear operator $M$ to form the combined nonlinear operator $M_c$. Since $W_i(s)$ and $W_o(s)$ are rational transfer function matrices and $M$ is described by a finite set of nonlinear differential equations, it follows that we can describe $M_c$ by the state equations

$$
M_c : \begin{cases} 
\dot{x}_c = A_c(x_c) + B_c(x_c)z_c \\
 w_c = w_c(x_c) 
\end{cases} \tag{4.1}
$$

where

$$
x_c = \begin{bmatrix} x_i \\ x \\ x_o \end{bmatrix} \tag{4.2}
$$

and where $x_i \in \mathbb{R}^{n_i}$ and $x_o \in \mathbb{R}^{n_o}$ are the states corresponding to the realizations of the weights, $W_i(s)$ and $W_o(s)$, respectively. Finally, since we are interested in bounding the $L_2$-induced norm of $M_c$, we make the following assumption.

**Assumption 4.1** For the closed loop system in Figure 4-1, there exists finite constants, $R_1 > 0$ and $R_2 > 0$, such that for all input pairs $(r_1, r_2) \in L_2 \times L_2$ where $\|r_1(t)\| \leq R_1$ and $\|r_2(t)\| \leq R_2$ for all $t \geq 0$, the closed loop solution for $x_c(t)$ evolves in some open set $X_c = \mathbb{R}^{n_i} \times X \times \mathbb{R}^{n_o}$ containing the origin for all $t \geq 0$ where $X$ is bounded.

We note that this assumption is, in fact, equivalent to Assumption 2.3 since we choose $X_c$ so that we do not constrain the state $x_c$ along the subspace of the weight states $x_i$ and $x_o$. We state this assumption explicitly here and emphasize it because $x_i$ and $x_o$ represents part of the internal states of the uncertainty. Therefore, constraining $x_i$ and
$x_o$ is equivalent to constraining the uncertainty. We note that by constraining $x$ we, in turn, do constrain $x_i$ and $x_o$ indirectly since we constrain the amount of interaction between the uncertainty and $M$. However, we should not place a priori constraints on $x_i$ and $x_o$ because this will place a priori constraints on the interaction between the uncertainty and $M$, which does not necessarily exist. The resulting robust stability problem is therefore to find a condition that will guarantee the small-signal robust stability for the closed loop system in Figure 4-1 under Assumption 4.1.

4.1.1 Bounding the $L_2$-induced Norm of $M_c$

To begin our analysis, we want to determine a bound on the $L_2$-induced norm of $M_c$. To do so, we use the following theorem, which is basically a restatement of fundamental results in [35].

**Theorem 4.1** Under Assumption 4.1, the small-signal $L_2$-induced norm of $M_c$ is bounded above by a finite constant $\gamma$ if there exists a smooth positive definite function $V(x_c)$ over $X_c$ such that

$$\frac{\partial V(x_c)}{\partial x_c} A_c(x_c) + \frac{1}{4\gamma^2} \frac{\partial V(x_c)}{\partial x_c} B_c(x_c) B_c^T(x_c) \frac{\partial V(x_c)}{\partial x_c} + w_c^T(x_c) w_c(x_c) \leq 0 \ \forall \ x_c \in X_c$$

(4.3)
Proof: Taking the derivative of $V(x_c)$ with respect to $t$ along the solution $x_c(t)$ yields

$$
\frac{d}{dt} V(x_c) = \frac{\partial V}{\partial x_c}(x_c) \dot{x}_c
$$

$$
= \frac{\partial V}{\partial x_c}(x_c)(A_c(x_c) + B_c(x_c)z_c)
$$

$$
= \frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{\partial V}{\partial x_c}(x_c) B_c(x_c)z_c + \gamma^2 z_c^T z_c - \gamma^2 z_c^T z_c
$$

$$
+ \frac{1}{4\gamma^2} \frac{\partial V}{\partial x_c}(x_c) B_c(x_c) B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c)
$$

$$
- \frac{1}{4\gamma^2} \frac{\partial V}{\partial x_c}(x_c) B_c(x_c) B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c)
$$

$$
(4.4)
$$

We note that

$$
-\gamma^2 \left\| z_c - \frac{1}{2\gamma^2} B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c) \right\|^2 = -\gamma^2 z_c^T z_c + \frac{\partial V}{\partial x_c}(x_c) B_c(x_c) z_c
$$

$$
- \frac{1}{4\gamma^2} \frac{\partial V}{\partial x_c}(x_c) B_c(x_c) B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c).
$$

(4.5)

Substituting (4.5) into (4.4) yields

$$
\frac{d}{dt} V(x_c) = \frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x_c}(x_c) B_c(x_c) B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c)
$$

$$
-\gamma^2 \left\| z_c - \frac{1}{2\gamma^2} B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c) \right\|^2 + \gamma^2 z_c^T z_c,
$$

(4.6)

which can be rearranged as follows

$$
\frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x_c}(x_c) B_c(x_c) B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c) = \frac{d}{dt} V(x_c)
$$

$$
+ \gamma^2 \left\| z_c - \frac{1}{2\gamma^2} B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c) \right\|^2
$$

$$
- \gamma^2 z_c^T z_c.
$$

(4.7)

Now, substituting (4.7) into the condition of the theorem gives

$$
\frac{d}{dt} V(x_c) + \gamma^2 \left\| z_c - \frac{1}{2\gamma^2} B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c) \right\|^2 - \gamma^2 z_c^T z_c + w_c^T(x_c)w_c(x_c) \leq 0 \ \forall x_c \in X_c
$$

(4.8)
Since $\gamma^2 \| z_e - \frac{1}{2\gamma} B_e^T(x_e) \frac{\partial T}{\partial x_e}(x_e) \|^2 \geq 0$, (4.8) implies

$$ \frac{d}{dt} V(x_e) - \gamma^2 z_e^T z_e + w_e^T w_e \leq 0 \ \forall \ x_e \in X_e $$

(4.9)

or

$$ \frac{d}{dt} V(x_e) \leq \gamma^2 z_e^T z_e - w_e^T w_e \ \forall \ x_e \in X_e. $$

(4.10)

Now, for all $T \geq 0$ and for all $z_e \in \mathcal{L}_{2e}$ such that the solution $x_e(t)$ is contained in $X_e$ for all $t \geq 0$, we can integrate both sides of (4.10) to obtain

$$ \int_0^T \frac{d}{dt} V(x_e) \leq \gamma^2 \int_0^T z_e^T z_e dt - \int_0^T w_e^T w_e dt \ \forall \ x_e \in X_e $$

(4.11)

Recalling that the $\mathcal{L}_2$-induced norm of an operator is defined under zero initial conditions, we have $x_e(0) = 0$. Therefore, the integral on the left hand side of (4.11) is greater than or equal to zero since $V(x_e)$ is positive definite over $X_e$ and $V(x_e(0)) = V(0) = 0$. Therefore, for solutions that start at 0, for all $z_e \in \mathcal{L}_{2e}$ and for all $T \geq 0$, we have

$$ \int_0^T w_e^T w_e dt \leq \gamma^2 \int_0^T z_e^T z_e dt, $$

(4.12)

which by definition means that the small-signal $\mathcal{L}_2$-induced norm of $M_e$ is bounded by $\gamma$. \hfill \Box

We remark that Theorem 4.1 gives us a sufficient condition for bounding the $\mathcal{L}_2$-induced norm of $M_e$ by $\gamma$. This sufficient condition involves the existence of a positive definite solution to the Hamilton Jacobi Inequality (4.3), which depends on $\gamma$ and the nonlinear dynamics of $M_e$. In fact, this condition is also necessary, [33], under the condition that the state space representation for $M_e$ given by (4.1) is locally reachable over $X_e$. Here necessity means that if the $\mathcal{L}_2$-induced norm of $M_e$ is bounded by $\gamma$, then there exists a positive definite solution for (4.3) over $X_e$. 

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4.1.2 Sufficient Condition for Stability Robustness

Applying Theorem 4.1, we have the following sufficient condition for small-signal robust stability.

Proposition 4.1 Under Assumption 4.1, the closed loop system in Figure 4-1 is small-signal, robustly stable if for some $W_i(s)$ and $W_c(s)$, there exists a smooth positive definite function $V(x_c)$ over $X_c$ such that

$$\frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{1}{4}\frac{\partial V}{\partial x_c}(x_c)B_c(x_c)B_c^T(x_c)\frac{\partial^2 V}{\partial x_c^2}(x_c) + w_c^T(x_c)w_c(x_c) \leq 0 \quad \forall \ x_c \in X_c$$

(4.13)

Proof: Using Theorem 4.1, the condition of the proposition implies that the small-signal $L_2$-induced norm of $M_c$ is less than or equal to one. Therefore, from the Small Gain Theorem, the operator that maps the signal pair $(r_1, r_2)$ to the signal pair $(e_1, e_2)$ is small-signal, finite gain stable. To complete the proof, we need to show that for all input pairs $(r_1, r_2) \in L_2 \times L_2$ such that the resulting closed loop solution $x(t)$ is contained in $X$ for all $t \geq 0$, the closed loop solution for the physical states $\xi(t)$ is in $L_2$. To show this, we note that

$$\xi = \Phi^{-1}\Psi G(x)W_i e_2$$

(4.14)

where $\Phi^{-1}$ is the diffeomorphism that transforms $x$ to $\xi$, $G(x)$ is the state dependent input map of the normal form dynamics, $\Psi$ is the desired linearized transfer function matrix, and $W_i$ is the input weighting transfer function matrix. Since $\Psi$ and $W_i$ are strictly stable, it follows that their $L_2$ induced norms are finite and given by their $H_\infty$ norms. In addition, since $X$ is bounded by assumption and $G(\cdot)$ is smooth, it follows from Lemma 2.1 that there exists a finite constant $\gamma_G$ such that

$$\sup_{x \in \hat{X}} \sigma_{\max}[G(x)] \leq \gamma_G$$

(4.15)
Similarly, since \( \Phi^{-1} \) is a diffeomorphism and is thus continuously differentiable, it follows from Lemma 2.2 that there exists a finite constant \( \gamma_p \) such that

\[
\|\Phi^{-1}(x)\| \leq \gamma_p \|x\| \tag{4.16}
\]

for all \( x \in X \). As a result, we can overbound the \( L_2 \)-induced norm of \( \xi \) by

\[
\|\xi\|_2 \leq \gamma_p \gamma_G \|\Psi(j\omega)\|_{H_\infty} \|W_i(j\omega)\|_{H_\infty} \|e_2\|_2. \tag{4.17}
\]

Therefore, the result follows since for all input pairs \( (r_1, r_2) \in L_2 \times L_2 \) such that the resulting closed loop solution \( x(t) \) is contained in \( X \) for all \( t \geq 0 \), \( e_2 \) is in \( L_2 \). \( \square \)

We note that the above condition for robust stability is dependent on \( W_i(s) \) and \( W_o(s) \). As mentioned beforehand, the choice for these weights is not unique. In fact, we can alternatively view these weights as being multipliers which add additional degrees of freedom to reduce the \( L_2 \)-induced norm of \( M_c \). However, since the problem is nonlinear, the optimal choice for these weights is not clear. In addition, although the above proposition gives a sufficient condition for small-signal robust stability, we note that this condition is not readily verifiable since it requires a solution to the Hamilton Jacobi Inequality. As is well known, solving the Hamilton Jacobi Inequality is computationally complex and suffers from the curse of dimensionality. The problem stems from not only finding a solution to a partial differential equation, but also to ensure that this solution is positive definite.

### 4.2 Relating the Stability Robustness of a Nonlinear System to Its Linearization

In the previous section, we derived a sufficient condition for robust stability, which requires a solution to the Hamilton Jacobi Inequality over some \( X_c \) that we choose as a constraint for the closed loop solution of \( x_c \). Since solving the Hamilton Jacobi
Inequality is computationally complex, this condition can not be readily verifiable. In this section, we take the opposite approach. That is, we define a positive definite solution for the corresponding Hamilton Jacobi Inequality and then check to see over what region of the state space this inequality is satisfied. The advantage of this approach is that we avoid the complexity of having to solve the Hamilton Jacobi Inequality, and yet we can still use Theorem 4.1 to bound the small-signal $L_2$-induced norm of $M_c$. However, now we no longer define the region over which the closed loop solution for $x_c$ is constrained. Instead, this region is defined for us by the solution that we choose. Therefore, the obvious questions here are

1. What is the proper solution to assume for the Hamilton Jacobi Inequality, and

2. Is there a guarantee that the region, over which the Hamilton Jacobi Inequality is satisfied, contains the origin and is nonempty, and how can we obtain an estimate for this region?

### 4.2.1 Relating the $L_2$-induced Norm of a Nonlinear System to the $H_\infty$ Norm of Its Linearization

To begin to answer these questions, we first motivate our approach by presenting the following result from Van der Schaft, [32], which relates the small-signal $L_2$-induced norm of a nonlinear system to the $H_\infty$ norm of its linearization.

**Theorem 4.2** Let $S$ be a nonlinear system described by

\[
S : \begin{cases} 
\dot{x} = f(x) + g(x)u \\
y = h(x)
\end{cases},
\]  \hspace{1cm} (4.18)

and let $\tilde{S}$ be its linearization about its equilibrium point at $x = 0$

\[
\tilde{S} : \begin{cases} 
\dot{\bar{x}} = F\bar{x} + G\bar{u} \\
\bar{y} = H\bar{x}
\end{cases}.
\]  \hspace{1cm} (4.19)
Let \( \bar{S}(j\omega) = G(j\omega I - F)^{-1}H \). If \( \| \bar{S}(j\omega) \|_{\mathcal{H}_\infty} < \gamma \), then there exists a neighborhood \( X \) around \( x = 0 \) and a smooth positive definite function \( V(x) \) over \( X \) that satisfies the Hamilton Jacobi Inequality

\[
\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{4\gamma^2} \frac{\partial V}{\partial x}(x)g(x)g^T(x)\frac{\partial T V}{\partial x}(x) + h^T(x) h(x) \leq 0 \quad \forall x \in X \quad (4.20)
\]

We note that using Theorem 4.1, an immediate implication of Theorem 4.2 is that if the \( \mathcal{H}_\infty \) norm of \( \bar{S} \) is strictly less than \( \gamma \), then the small-signal \( L_2 \)-induced norm of \( S \) is bounded by \( \gamma \) where small-signal means that the solution for the state of \( S \), \( x \), is contained in some neighborhood \( X \) around the equilibrium point at the origin.

Using Theorem 4.2, we derive the following sufficient condition for small-signal robust stability.

**Proposition 4.2** Let

\[
\bar{M}_c : \begin{cases} 
\dot{x}_c = A \bar{x}_c + B \bar{z}_c \\
\bar{w}_c = C \bar{x}_c
\end{cases} \quad (4.21)
\]

be the linearization of \( M_c \) about the equilibrium point at 0.

If \( \| \bar{M}_c \|_{\mathcal{H}_\infty} < 1 \), then there exists a neighborhood \( X_c \) around \( x_c = 0 \) such that the closed loop system in Figure 4-1 is small-signal robustly stable, given that the closed loop solution for \( x_c(t) \) is contained in this \( X_c \) for all \( t \geq 0 \).

**Proof:** Using Theorem 4.2, the condition of the proposition implies that the small-signal \( L_2 \)-induced norm of \( M_c \) is less than or equal to one. Therefore, from the Small Gain Theorem, the operator that maps the signal pair \((r_1, r_2)\) to the signal pair \((e_1, e_2)\) is small-signal, finite gain stable. To show that for all input pairs \((r_1, r_2) \in \mathcal{L}_2 \times \mathcal{L}_2 \) such that the resulting closed loop solution \( x_c(t) \) is contained in \( X_c \) for all \( t \geq 0 \), the closed loop solution for the physical states \( \xi(t) \) is in \( \mathcal{L}_2 \), we use the same argument as in the proof for Proposition 4.1. \( \square \)

We remark that an interpretation for the condition in Proposition 4.2 is that the linearization of the closed loop system in Figure 4-1 is robustly stable with some margin, i.e. we require \( \| \bar{M}_c \|_{\mathcal{H}_\infty} < 1 \) instead of just \( \leq 1 \). We note that in this case,
the conclusion of local robust stability for the nonlinear closed loop system cannot be simply arrived at by Lyapunov’s Linearization Method. This is because robust stability must be with respect to all $\Delta \in \mathcal{D}$, which contain infinite-dimensional operators. As a result, Lyapunov’s Linearization Method does not apply and the conclusion can only be attained through an input-output approach using Theorem 4.2 and the Small Gain Theorem. In addition, unlike the sufficient condition in Proposition 4.1, this sufficient condition for small-signal robust stability does not require a solution to the Hamilton Jacobi Inequality. In fact, given $A$, $B$, and $C$ in (4.21) the condition is readily verifiable since we know how to calculate the $\mathcal{H}_\infty$ norm of $\bar{M}_c$. Furthermore, we note that the region $X_c$ is now a result of the condition and thus is not defined a priori. Recall from Chapter 2 that we choose $\Xi$ and thus $X_c$ so as to reflect some physical limitations in the state space such as saturation or other operating limits, and therefore, we need not look upon the bounds on this set as constraints on the solution of $x_c$. A consequence of this proposition is that now the condition will dictate the size of $X_c$, and thus $X_c$ becomes a constraint on the closed loop solution for $x_c$, which translates to a constraint on how hard we can excite the closed loop through external inputs. Of particular concern is that this neighborhood $X_c$ can be very small. As shown by an example in [33], the neighborhood over which the small-signal $\mathcal{L}_2$-induced norm for the nonlinear system is bounded by $\gamma$ can shrink to a point as the $\mathcal{H}_\infty$ norm of its linearization approaches $\gamma$. In the context of Proposition 4.2, this means that $X_c$ can shrink to the origin as $\|\bar{M}_c\|_{\mathcal{H}_\infty} \to 1$. On the other hand, it can also be conjectured that the size of $X_c$ will increase as the the margin for robust stability for the linearization increases.

### 4.2.2 Guaranteed Neighborhood for Small-Signal Robust Stability

While Proposition 4.2 provides a verifiable condition for small-signal robust stability, it does not adequately answer the questions posed earlier. In particular, Proposition 4.2 only provides for the existence of some neighborhood $X_c$ and therefore does
not give any a priori information on the size of $X_c$. In fact, following the motivation for Assumption 4.1, we would like a result which states that the neighborhood which constrains $x_c$ can be made to be unbounded in the subspace of the weight states so as to not place any a priori constraints on the interaction between the uncertainty and $M$. Despite these shortcomings, Proposition 4.2 does provide a motivation for looking at the robust stability of the linearization of the closed loop to conclude about the robust stability of the nonlinear closed loop. It is with this motivation that we propose the following.

**Proposition 4.3** Let $M_c$ be described by the state equations in (4.1) and let $M_c$ be its linearization about the equilibrium point at $x = 0$ as described in (4.21). If there exists $\varepsilon > 0$ such that

$$A^TP + PA + PBB^TP + C^TC = \varepsilon^2 I = 0 \tag{4.22}$$

has a solution $P = P^T > 0$, then there exists a region $X_c = \mathbb{R}^{n_i} \times X \times \mathbb{R}^{n_o}$, where $X$ is a neighborhood around $x = 0$, such that the closed loop system in Figure 4-1 is small-signal robustly stable, given that the closed loop solution for $x(t)$ is contained in this $X$ for all $t \geq 0$.

**Proof:** Our goal is to show that under the condition of the proposition, there exists a region $X_c = \mathbb{R}^{n_i} \times X \times \mathbb{R}^{n_o}$, where $X$ is a neighborhood around $x = 0$, over which the Hamilton Jacobi Inequality

$$\frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{1}{4} \frac{\partial V}{\partial x_c}(x_c)B_c(x_c)B_c^T(x_c) \frac{\partial^TV}{\partial x_c}(x_c) + w_c^T(x_c)w_c(x_c) \leq 0 \tag{4.23}$$

is satisfied by some smooth positive definite function $V(x_c)$. Once we show this, the results of the proposition follows from previous results. To show this, we start by adding and subtracting the quantity $\varepsilon^2 x_c^T x_c$ to and from (4.23) to obtain

$$\frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{1}{4} \frac{\partial V}{\partial x_c}(x_c)B_c(x_c)B_c^T(x_c) \frac{\partial^TV}{\partial x_c}(x_c) + w_c^T(x_c)w_c(x_c) + \varepsilon^2 x_c^T x_c - \varepsilon^2 x_c^T x_c \leq 0. \tag{4.24}$$
For our purposes, we choose the solution

\[ V(x_c) = x_c^T P x_c, \quad (4.25) \]

where \( P = P^T > 0 \) is as defined by the condition in the proposition. Clearly, \( V(x_c) \) is a smooth positive definite function of \( x_c \). Applying (4.25) to (4.24), we have

\[ 2x_c^T P A_c(x_c) + x_c^T P B_c(x_c) B_c^T(x_c) P x_c + w_c^T(x_c) w_c(x_c) + \varepsilon^2 x_c^T x_c - \varepsilon^2 x_c^T x_c \leq 0. \quad (4.26) \]

Now, we expand the dynamics of \( \bar{M}_c \) such that

\[
\begin{align*}
A_c(x_c) &= A x_c + A_h(x_c) x_c \\
B_c(x_c) &= B + B_h(x_c) \\
w_c(x_c) &= C x_c + C_h(x_c) x_c
\end{align*}
\quad (4.27) (4.28) (4.29)
\]

where \( A, B, \) and \( C \) represent the linear dynamics of \( \bar{M}_c \) as in Equation (4.21) and where \( A_h(x_c) x_c, B_h(x_c), \) and \( C_h(x_c) x_c \) are higher order terms. Doing so, (4.26) separates into the two parts,

\[ 1. \quad A^T P + PA + P B B^T P + C^T C + \varepsilon^2 I = 0 \quad (4.30) \\
2. \quad 2x_c^T P A_h(x_c) x_c + 2x_c^T P B B_h^T(x_c) P x_c + x_c^T P B_h(x_c) B_h(x_c) P x_c \\
\quad + 2x_c^T C^T C_h(x_c) x_c + x_c^T C_h(x_c) C_h(x_c) x_c - \varepsilon^2 x_c^T x_c \leq 0. \quad (4.31) \]

We note that if both parts above are satisfied, then (4.26) is satisfied, and therefore, the original Hamilton Jacobi Inequality is satisfied for the smooth positive definite function \( V(x_c) = x_c^T P x_c \). From the condition of the proposition, the first part is satisfied. For the second part, we note that a sufficient condition is that the matrix

\[
\begin{align*}
A_h^T(x_c) P + PA_h(x_c) + P B B_h^T(x_c) P + P B_h(x_c) B_h^T(x_c) P + P B_h(x_c) B_h^T(x_c) P \\
\quad + C^T C_h(x_c) + C_h^T(x_c) C + C_h^T(x_c) C_h(x_c) - \varepsilon^2 I
\end{align*}
\quad (4.32) \]
be negative semi-definite. Since \( A_h(x_c) \), \( B_h(x_c) \), and \( C_h(x_c) \) are higher order terms, it follows that
\[
A_h(0) = 0, \quad B_h(0) = 0, \quad C_h(0) = 0. \tag{4.33}
\]

Therefore, by continuity, there exists a neighborhood around the equilibrium \( x_c = 0 \) such that (4.32) is negative semi-definite for all \( x_c \) in that neighborhood. Furthermore, we note that \( A_h(x_c), B_h(x_c), \) and \( C_h(x_c) \) will only depend on \( x \), the states of \( M \). This is because the weights \( W_i(s) \) and \( W_o(s) \) are linear time-invariant operators connected in series with \( M \) to form \( M_c \), which means that their dynamics is fully captured in \( A, B, \) and \( C \). Therefore, the neighborhood around \( x_c = 0 \) for which (4.32) is negative semi-definite is independent of the weight states \( x_i \) and \( x_o \). In other words, we can say that there exists a region \( X_c = \mathcal{R}^{n_i} \times X \times \mathcal{R}^{n_o} \), where \( X \) is a neighborhood around \( x = 0 \), such that (4.32) is negative semi-definite for all \( x_c \in X_c \). As a result, we have shown that under the condition of the proposition, there exists a region \( X_c = \mathcal{R}^{n_i} \times X \times \mathcal{R}^{n_o} \), where \( X \) is a neighborhood around \( x = 0 \), over which the Hamilton Jacobi Inequality
\[
\frac{\partial V}{\partial x_c}(x_c)A_c(x_c) + \frac{1}{4} \frac{\partial V}{\partial x_c}(x_c)B_c(x_c)B_c^T(x_c) \frac{\partial^T V}{\partial x_c}(x_c) + w_c^T(x_c)w_c(x_c) \leq 0 \tag{4.34}
\]
is satisfied by \( V(x_c) = x_c^T P x_c \). \( \square \)

We note that the condition in Proposition 4.3 is equivalent to that in Proposition 4.2 since the existence of a symmetric positive definite solution to (4.22) for some \( \varepsilon > 0 \) is equivalent to \( \| \bar{M}_c \|_{\mathcal{H}_\infty} < 1 \) [39]. However, with Proposition 4.3, we are able to conclude more by specifying a particular solution for the Hamilton Jacobi Inequality, namely \( V(x_c) = x_c^T P x_c \). In particular, we are able to guarantee that the subspace of the weight states is not constrained in \( X_c \). In addition, we note that since (4.32) in the proof of the proposition is only dependent on \( x \), we can use the matrix inequality
\[
\mathcal{M}(x) = A_h^T(x)P + PA_h(x) + PBB_h^T(x)P + PB_h(x)B^T P + PB_h(x)B_h^T(x)P
+ C^T C_h(x) + C_h^T(x)C + C_h^T(x)C_h(x) - \varepsilon^2 I \leq 0 \tag{4.35}
\]
to construct an estimate for the set $X$ since $X$ is the set

$$X = \{ x \mid \mathcal{M}(x) \leq 0 \} \tag{4.36}$$

This estimation for $X$ can be made by selecting a finite set of grid points in the state space and checking to see if (4.35) holds. Finally, we note that we should use the maximum value of $\varepsilon$ that yields a symmetric positive definite solution $P$ in the condition of the proposition. This is because the size of $X$ increases monotonically with $\varepsilon$. That is, if $\varepsilon_1 < \varepsilon_2$ then the corresponding $X_1 \subseteq X_2$. To see this, we note that in the Riccati equation (4.22), the size of $P$ will at most increase at the same order as $\varepsilon$. Therefore, it is clear from (4.35) that if $x^*$ satisfies (4.35) for some $\varepsilon_1$, then $x^*$ will also satisfy (4.35) for $\varepsilon_2 > \varepsilon_1$.

Although Proposition 4.3 provides a verifiable sufficient condition for small-signal robust stability and an estimate of $X$, we note that this estimate is conservative. This is because of two reasons. First, we only use a quadratic solution for the Hamilton Jacobi Inequality; whereas if we are only able to use a higher order solution, we can no doubt get a larger estimate for $X$ due to the added degrees of freedom from the higher order terms. Second, we note that we added conservatism by using the matrix inequality (4.35) instead of the scalar inequality in (4.31). However, this is necessary in order to ensure that (4.35) is dependent only on $x$, the states of $M$. In addition, we note that the results of the proposition does not give any information on how the estimated $X$ depends on the additional full state feedback gain $K$, which is our design. We note from (4.35) that the only dependence for the estimated $X$ on $K$ is through the solution of the Riccati equation $P$. However, since (4.35) is a nonconvex function of $x$, it follows that any estimate for $X$ using (4.35) will depend in a nonconvex manner with respect to $K$. This is unfortunate since it means that we cannot use Proposition 4.3 to determine the optimal $K$ to maximize the estimated $X$, which is the key to a synthesis problem for $K$. Finally, we note that the choice for the weights, $W_i(s)$ and $W_o(s)$, will affect the conservatism of the estimated $X$. This is demonstrated in the examples in the next section.
4.3 Examples

In this section, we present some examples to illustrate the concepts developed in this chapter.

Example 4.1:

For this example, we consider the same scalar nonlinear system as in Example 3.1 where the model of the plant is given by

\[ P_m : \dot{x} = 2x^3 + u \]  
\[ (4.37) \]

and the representation of the uncertainty in the model is as defined in Definition 2.1 where the multiplicative error is bounded by

\[ |W_i(j\omega)\Delta(j\omega)W_o(j\omega)| < \left| \frac{2j\omega}{j\omega + 10} \right|. \]  
\[ (4.38) \]

As in Example 3.1, using the feedback linearization control law

\[ u = -2x^3 - Kx \]  
\[ (4.39) \]

will result in the nominal closed loop dynamics

\[ \dot{x} = -Kx \]  
\[ (4.40) \]

which are strictly stable for \( K > 0 \). With the uncertainty, the actual closed loop system is as shown in Figure 4-2 with

\[
M : \begin{cases} 
\dot{x} = -Kx + z \\
w(x) = -2x^3 - Kx 
\end{cases}
\]  
\[ (4.41) \]

For the weights, \( W_i(s) \) and \( W_o(s) \), we note that their choice is not unique as long as they are stable and proper rational transfer functions that reflect the frequency
dependent magnitude bound of the uncertainty in the sense that

$$|W_i(j\omega)W_o(j\omega)| \geq \left| \frac{2j\omega}{j\omega + 10} \right|. \quad (4.42)$$

For the moment, we will define these weights as generic first order transfer functions

$$W_i(s) = \frac{K_i(s + b_i)}{s + a_i}, \quad W_o(s) = \frac{K_o(s + b_o)}{s + a_o} \quad (4.43)$$

with the following state space representations:

$$W_i: \begin{cases} \dot{x}_i = -a_i x_i + z_c \\ z = K_i(b_i - a_i)x_i + K_iz_c \end{cases} \quad (4.44)$$

$$W_o: \begin{cases} \dot{x}_o = -a_o x_o + w \\ w_c = K_o(b_o - a_o)x_o + K_ow \end{cases} \quad (4.45)$$
Combining the weights with $M$, the combined operator $M_c$ is given by

\[
M_c : \begin{bmatrix} x_i \\ x \\ x_o \end{bmatrix} = \begin{bmatrix} -a_i x_i \\ -K x + K_i (b_i - a_i) x_i \\ -a_o x_o - 2x^3 - K x \end{bmatrix} + \begin{bmatrix} 1 \\ K_i \\ 0 \end{bmatrix} z_c = A_c(x_c) + B_c(x_c) z_c \\
\]

\[
w_c = K_o (b_o - a_o) x_o + K_o (-2x^3 - K x) = w_c(x_c) \quad (4.46)
\]

We note that we can expand the dynamics of $M_c$ as

\[
A_c(x_c) = Ax_c + A_h(x_c) x_c \quad (4.47)
\]

\[
B_c(x_c) = B + B_h(x_c) \quad (4.48)
\]

\[
w_c(x_c) = Cx_c + C_h(x_c) x_c \quad (4.49)
\]

where

\[
A = \begin{bmatrix} -a_i & 0 & 0 \\ K_i (b_i - a_i) & -k & 0 \\ 0 & -k & -a_o \end{bmatrix}, \quad A_h(x_c) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2x^2 & 0 \end{bmatrix} \\
B = \begin{bmatrix} 1 \\ K_i \\ 0 \end{bmatrix}, \quad B_h(x_c) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
C = \begin{bmatrix} 0 & -K_o K & K_o (b_o - a_o) \end{bmatrix}, \quad C_h(x_c) = \begin{bmatrix} 0 & -2K_0 x^2 & 0 \end{bmatrix} \quad (4.50)
\]

Using Proposition 4.3, for each choice of weights, we can solve for the maximum neighborhood around 0 in which we can guarantee small-signal robust stability, given that the closed loop solution for $x(t)$ is contained in $X$ for all $t \geq 0$, as a function of $K$ by following the procedure outlined below.

1. Given the choice for the weights, $W_i(s)$ and $W_o(s)$, and a value for $K$, do a bisection search in $\varepsilon$ to obtain the largest $\varepsilon_{\text{max}} > 0$ that will yield a symmetric
positive definite solution $P$ for the Riccati equation

$$A^TP + PA + PBB^TP + C^TC + \epsilon^2 I = 0 \quad (4.51)$$

2. With this value $\epsilon_{\text{max}}$ and corresponding $P$, perform a line search along $x$ to determine the maximum value of $x$, $x_{\text{max}}$, such that the matrix inequality

$$\mathcal{M}(x) = A_h^T(x)P + PA_h(x) + PBB_h(x)P + PBB_h(x)B_h^TP + PBB_h(x)B_h^T(x)P + C^TC_h(x) + C_h^T(x)C + C_h^T(x)C_h(x) - \epsilon^2 I \leq 0 \quad (4.52)$$

is satisfied.

Using the above procedure for the following three choices of weights,

Case 1. $W_i(s) = \frac{2s}{s + 10}$, $W_o(s) = 1$

Case 2. $W_i(s) = 1$, $W_o(s) = \frac{2s}{s + 10}$

Case 3. $W_i(s) = \frac{20s}{s + 10}$, $W_o(s) = 0.1$,

estimations for the maximum neighborhood about which we can guarantee small-signal robust stability are given in Figure 4-3. The result from Example 3.1, which is represented by the solid curve, is also given. For the first case, we place all of the frequency dependent weight of the uncertainty at the input of $\mathcal{M}$ as we did in Example 3.1. Therefore, we would expect that the results for $x_{\text{max}}$ as a function of $K$ would be similar to that obtained from Example 3.1. Examining Figure 4-3, we note that this is indeed the case. However, we note that for each value of $K$, the estimate for $x_{\text{max}}$ is less, or more conservative, than that obtained in Example 3.1 using Proposition 3.1. For the second case, we place all of the frequency dependent weight of the uncertainty at the output of $\mathcal{M}$. We would expect to obtain more conservative results for this choice of weights than in the first case since this frequency dependent information can no longer be directly absorbed into $\Psi(s)$ due to the nonlinear output map $w$. Examining Figure 4-3, we note that this is indeed the case. We note that
for $K > 6$, the estimate for $x_{max}$ matches that for the first case. This is because the nonlinear output map is given by the feedback linearization control law

$$w(x) = -2x^3 - Kx,$$

and for large enough $K$, the linear term $Kx$ will dominate for small $x$. However, we note that as $K$ decreases, the cubic nonlinearity in $w$ begins to dominate and the estimate for $x_{max}$ decreases, or becomes more conservative. Finally, in the third case, we attempt to decrease the effect of the nonlinear output map by a factor of 10 by setting $W_o(s) = 0.1$. As shown in Figure 4-3, this improves our estimate for $x_{max}$ to approach that obtained in Example 3.1 using Proposition 3.1. This shows that with the proper choice of weights, we can reduce the conservatism of Proposition 4.3 to the level of that in Proposition 3.1 for the constant input map case. \qed
Example 4.2:

In this example, we consider a planar two link manipulator as shown in Figure 4-4. We assume that the relevant inertia parameters and the physical dimensions of the manipulator are known exactly and are given in Table 4.1. A model of the manipulator can be expressed in relative joint coordinates as

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\hat{\theta}} \\
\end{bmatrix} =
\begin{bmatrix}
0_{2 \times 2} \\
0_{2 \times 2} \\
\end{bmatrix}
\begin{bmatrix}
I_{2 \times 2} \\
0_{2 \times 2} \\
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\end{bmatrix}
+ 
\begin{bmatrix}
0_{2 \times 2} \\
[H(\theta)]^{-1} \\
\end{bmatrix}
\begin{bmatrix}
\tau \\
\dot{\tau} \\
\end{bmatrix}
\]

(4.54)

where

\[
\theta =
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\end{bmatrix},
\dot{\theta} =
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix},
\tau =
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\end{bmatrix}
\]

(4.55)

\[
H(\theta) =
\begin{bmatrix}
a_1 + a_2 + 2a_3 \cos(\theta_2) & a_2 + a_3 \cos(\theta_2) \\
a_2 + a_3 \cos(\theta_2) & a_2 \\
\end{bmatrix}
\]

(4.56)

\[
C(\theta, \dot{\theta}) =
\begin{bmatrix}
-a_3 \sin(\theta_2) \dot{\theta}_2 & -a_3 \sin(\theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\
a_3 \sin(\theta_2) \dot{\theta}_1 & 0 \\
\end{bmatrix}
\]

(4.57)

Here, \( \theta \) is the vector of joint positions, \( \dot{\theta} \) is the vector of joint velocities, and the input, \( \tau \), is the vector of joint torques. We note that in the above equations, the matrix \( H(\theta) \) is known as the inertia matrix of the manipulator and is dependent on \( \theta \). For this example, we note that \( H(\theta) \) is only dependent on the position of the second link relative to the first, \( \theta_2 \). As a result, the singular values of this matrix, or the magnitude of this matrix as a gain, will only depend on \( \theta_2 \). This dependency is given in Figure 4-5 where the maximum and minimum singular values of \( H \) are plotted as a function of \( \theta_2 \). We note from the figure that the manipulator inertia is maximized for \( \text{theta}_2 = 0 \) or when the “arm” is outstretched. In addition, the figure shows that the singular values of \( H \) are strictly greater than zero for all \( \theta_2 \). Therefore, the inverse of the inertia matrix, \( [H(\theta)]^{-1} \), exists for all \( \theta_2 \). In fact, from a kinetic energy argument in [27], it is easy to show that the inertia matrix is invertible for all \( \theta \). Therefore, the input map in (4.54) is well defined for all \( \theta \). Furthermore, from the above equations,
the vector \( C(\theta, \dot{\theta})\dot{\theta} \) is commonly referred to as the coriolis and centripetal force terms. We note that these forces reflect the nonlinear dynamic coupling between the two links and are quadratic functions of the joint velocities. Finally, the terms, \( a_1, a_2 \) and \( a_3 \), are constants that are dependent on the inertial and dimensional parameters of the manipulator. For this example, they are given by

\[
a_1 = I_1 + m_1 l_{c1}^2 + m_2 l_1^2 = 2.37
\]  

(4.58)

| \( I_1 \): Inertia of Link 1 (kg-m\(^2\)) | 0.12 \\
| \( m_1 \): Mass of Link 1 (kg) | 1 \\
| \( l_1 \): Length of Link 1 (m) | 1 \\
| \( l_{c1} \): Distance to Centroid of Link 1 (m) | 0.5 \\
| \( I_2 \): Inertia of Link 2 (kg-m\(^2\)) | 0.25 \\
| \( m_2 \): Mass of Link 2 (kg) | 2 \\
| \( l_{c2} \): Distance to Centroid of Link 2 (m) | 0.6 |

Table 4.1: Inertia and Dimensional Parameters for the Planar Two Link Manipulator
Figure 4-5: Inertia Matrix Singular Values as a Function of $\theta_2$

\[
\begin{align*}
  a_2 &= I_2 + m_2l^2_{c_2} = 0.97 \\
  a_3 &= m_2l_1l_{c_2} = 1.2
\end{align*}
\]

Examining (4.54), we note that the dynamics for the manipulator is in normal form in the physical coordinates, $\theta$ and $\dot{\theta}$, with the normal form input map $[H(\theta)]^{-1}$ that is dependent on the state $\theta$. Therefore, using the feedback linearization control law

\[
\tau = C(\theta, \dot{\theta})\dot{\theta} - H(\theta)K_p\theta - H(\theta)K_d\dot{\theta}
\]

the nominal closed loop system is linear time-invariant and is given by

\[
\begin{bmatrix}
  \dot{\theta} \\
  \dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
  0_{2 \times 2} & I_{2 \times 2} \\
  -K_p & -K_d
\end{bmatrix}
\begin{bmatrix}
  \theta \\
  \dot{\theta}
\end{bmatrix}.
\]
We note that $K_p$ and $K_d$ represent the additional linear state feedback that we design to ensure the stability of the nominal closed loop. In this example, we set

$$K_p = \omega_n^2 I, \quad K_d = 2\xi \omega_n I, \quad \xi = \frac{\sqrt{2}}{2}, \quad \omega_n = 1$$ (4.63)

so that the nominal closed loop dynamics are two independent, critically damped, second order systems with bandwidths at 1 rad/sec.

For this example, we assume that for the actual manipulator, there is a 100 msec time delay at each torque input channel. We choose to neglect these dynamics from our model since if we do incorporate them, they will cause our model to lose its exact feedback linearizable property. From [6], we can represent this uncertainty in the model as multiplicative uncertainty at the model’s input, as defined in Definition 2.1, where the multiplicative error is bounded by

$$|W_i(j\omega)\Delta(j\omega)W_o(j\omega)| < \frac{2.1j\omega}{j\omega + 10}$$ (4.64)

It is important to note that this representation of the uncertainty does not just cover our case of 100 msec delays but that it covers a whole set of possible plants perturbed from the model since our case of 100 msec delays is just an element of this set. For simplicity, we choose the weights, $W_i(s)$ and $W_o(s)$, to be diagonal transfer function matrices given by

$$W_i(s) = \begin{bmatrix} w_i(s) & 0 \\ 0 & w_i(s) \end{bmatrix}, \quad W_o(s) = \begin{bmatrix} w_o(s) & 0 \\ 0 & w_o(s) \end{bmatrix}$$ (4.65)

where $w_i(s)$ and $w_o(s)$ are generic first order transfer functions given by

$$w_i(s) = \frac{k_i(s + b_i)}{s + a_i}, \quad w_o(s) = \frac{k_o(s + b_o)}{s + a_o}.$$ (4.66)
Therefore, we can represent $W_i(s)$ and $W_o(s)$ as

\[
W_i : \begin{cases}
\dot{x}_i = -A_i x_i + z_c \\
z = K_i (B_i - A_i) x_i + K_i z_c
\end{cases} \quad (4.67)
\]

\[
W_o : \begin{cases}
\dot{x}_o = -A_o x_o + w \\
w_c = K_o (B_o - B_o) x_o + K_o w
\end{cases} \quad (4.68)
\]

where

\[
A_i = \begin{bmatrix} a_i & 0 \\ 0 & a_i \end{bmatrix} \quad (4.69)
\]

and $B_i$, $K_i$, $A_o$, $B_o$, and $K_o$ are likewise defined. As before, we note that the choice for $W_i(s)$ and $W_o(s)$ is not unique and that this choice will affect the results of the stability robustness analysis. We note that in order to reflect the frequency dependent magnitude bound of the uncertainty, we must choose $w_i(s)$ and $w_o(s)$ so that

\[
|w_i(j\omega) w_o(j\omega)| \geq \left| \frac{2.1j\omega}{j\omega + 10} \right|. \quad (4.70)
\]

Now, in the presence of this uncertainty, the resulting actual closed loop system is again as shown in Figure 4-2 where $M$ is given by

\[
M : \begin{cases}
\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ -K_p & -K_d \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0_{2\times 2} \\ [H(\theta)]^{-1} \end{bmatrix} z \\
\end{cases} \\
\begin{bmatrix} w(\theta, \theta) \end{bmatrix} = \begin{bmatrix} -H(\theta) K_p & C(\theta, \dot{\theta}) - H(\theta) K_d \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}. \quad (4.71)
\]

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Combining the weights with \( M \) yields the combined operator

\[
M_c : \begin{bmatrix}
    x_i \\
    \dot{\theta} \\
    \ddot{\theta} \\
    x_o
\end{bmatrix} = \begin{bmatrix}
    -A_i x_i \\
    \dot{\theta} \\
    -K_p \theta - K_d \ddot{\theta} + K_i (B_i - A_i) x_i \\
    -A_o x_o + w(\theta, \dot{\theta})
\end{bmatrix} + \begin{bmatrix}
    I_{2 \times 2} \\
    0_{2 \times 2} \\
    K_i [H(\theta)]^{-1} \\
    0_{2 \times 2}
\end{bmatrix} z_c
\]

\[
= A_c(x_c) + B_c(x_c) z_c
\]

\[
w_c = K_o w(\theta, \dot{\theta}) + K_o (B_o - A_o) x_o
\]

\[
w_c(x_c) = w_c(x_c)
\]

(4.72)

We note that we can expand the dynamics of \( M_c \) as

\[
A_c(x_c) = A x_c + A_h(x_c) x_c
\]

(4.73)

\[
B_c(x_c) = B + B_h(x_c)
\]

(4.74)

\[
w_c(x_c) = C x_c + C_h(x_c) x_c
\]

(4.75)

where

\[
A = \begin{bmatrix}
    -A_i & 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\
    0_{2 \times 2} & 0_{2 \times 2} & I_{2 \times 2} & 0_{2 \times 2} \\
    K_i (B_i - A_i) [H(\theta)]^{-1} & -K_p & K_d & 0_{2 \times 2} \\
    0_{2 \times 2} & -H(0) K_p & -H(0) K_d & -A_o
\end{bmatrix}
\]

\[
A_h(x_c) = \begin{bmatrix}
    0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\
    0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\
    K_i (B_i - A_i) ([H(\theta)]^{-1} - [H(0)]^{-1}) & 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\
    0_{2 \times 2} & -(H(\theta) - H(0)) K_p & C(\theta, \dot{\theta}) - (H(\theta) - H(0)) K_d & 0_{2 \times 2}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
    I_{2 \times 2} \\
    0_{2 \times 2} \\
    K_i [H(0)]^{-1} \\
    0_{2 \times 2}
\end{bmatrix}
\]

\[
B_h(x_c) = \begin{bmatrix}
    0_{2 \times 2} \\
    0_{2 \times 2} \\
    K_i ([H(\theta)]^{-1} - [H(0)]^{-1}) \\
    0_{2 \times 2}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
    0_{2 \times 2} & -K_o H(0) K_p & -K_o H(0) K_d & K_o (B_o - A_o)
\end{bmatrix}
\]

\[
C_h(x_c) = \begin{bmatrix}
    0_{2 \times 2} & -K_o (H(\theta) - H(0)) K_p & K_o (C(\theta, \dot{\theta}) - (H(\theta) - H(0)) K_d) & 0_{2 \times 2}
\end{bmatrix}
\]

(4.76)
Given a choice for the weights, we can use Proposition 4.3 to estimate a neighborhood, $X$, around the equilibrium point at the origin in which we can guarantee small-signal robust stability, given that the closed loop solution

$$x(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$  \hspace{1cm} (4.77)

is contained in $X$ for all $t \geq 0$. From Proposition 4.3, we note that an estimation for $X$ is simply the set of points for which the matrix inequality

$$\mathcal{M}(x) = A^T_h(x)P + PA_h(x) + PBB^T_h(x)P + PB_h(x)B^T_h(x)P + \varepsilon^2 I \leq 0$$  \hspace{1cm} (4.78)

is satisfied. We first note that the $X$ is unbounded in the coordinate $\theta_1$. This is because the higher order terms $A_h(x)$, $B_h(x)$, and $C_h(x)$ are independent of $\theta_1$ as evident in (4.76). This is as expected since the nonlinearities in the model of the manipulator are independent of $\theta_1$. In addition, we note that the choice for the weights will affect the estimate for $X$. To see this, we first look at the term $B_h(x)$ in (4.76) and the term $PB_h(x)B^T_h(x)P$ in (4.78). From Figure 4-5, we see that as $|\theta|$ increases, the maximum singular value of $B_h(x)$ will increase since the difference between the maximum singular values of $[H(\theta)]^{-1}$ and $[H(0)]^{-1}$, or the difference between the inverses of the minimum singular values of $H(\theta)$ and $H(0)$, will increase. Therefore, the maximum eigenvalue of $PB_h(x)B^T_h(x)P$ will increase, which will eventually lead to the failure of (4.78). Now, $B_h(x)$ is also dependent on $K_i$, the gain on the input weight. By setting $K_i$ small, we can reduce the eigenvalues of the terms containing $B_h(x)$ in (4.78). In particular, with all other terms fixed, we can increase $|\theta_2|$ further before (4.78) fails. Therefore, our estimate for the size of $X$ along the $\theta_2$ coordinate should increase as we decrease $K_i$. However, we note that by decreasing $K_i$, we necessarily increase $K_o$, the gain of the output weight, since the weights must be chosen to satisfy (4.70). This will increase the maximum singular value of $C_h(x)$. Since $C_h(x)$ contains the coriolis and centripetal term $C(\theta, \dot{\theta})$, which is dependent on
the joint velocities, we have by a similar argument that increasing $K_i$ will decrease the size of the estimated $X$ along the coordinates of the joint velocities. As a result, we see that by tuning $K_i$ and $K_o$ we can tradeoff between obtaining an estimate for $X$ that cover a large range in $\theta_2$ and an estimate for $X$ that covers a large range in the joint velocity coordinates $(\dot{\theta}_1, \dot{\theta}_2)$. Finally, we note that since Proposition 4.3 is sufficient, the estimate for $X$ given any choice of weights is sufficient, and therefore, the union of all $X$s from all the choices for the weights is an estimate of the neighborhood in which we can guarantee small-signal robust stability.

To demonstrate this numerically, we choose four sets of weights. For each set of weights we estimate $X$ by first finding the largest $\varepsilon$ such that the Riccati equation

$$A^T P + PA + PBB^T P + C^T C + \varepsilon^2 I = 0 \quad (4.79)$$

yields a symmetric, positive definite solution $P$. With this maximum value for $\varepsilon$ and the corresponding $P$, we estimate $X$ by dividing the state space into a finite set of grid points and checking (4.78) for each point. For simplicity, we use a polar grid for the coordinates $\dot{\theta}_1$ and $\dot{\theta}_2$ and report the estimate for $X$ in terms of $||\dot{\theta}||$ and $\theta_2$ only. The four sets of weights considered are

Case 1. $w_i(s) = \frac{0.87(s + 11.5)}{s + 10}, \quad w_o(s) = \frac{2.41s}{s + 11.5}$
Case 2. $w_i(s) = \frac{0.50(s + 20.0)}{s + 10}, \quad w_o(s) = \frac{4.20s}{s + 20.0}$
Case 3. $w_i(s) = \frac{0.20(s + 50.0)}{s + 10}, \quad w_o(s) = \frac{10.5s}{s + 50.0}$
Case 4. $w_i(s) = \frac{0.12(s + 83.3)}{s + 10}, \quad w_o(s) = \frac{17.5s}{s + 83.3}$

and the results are given in Figures 4-6 thru 4-9, respectively. From the figures, it is clear that as we reduce $K_i$ from 0.87 to 0.12, therefore increasing $K_o$ from 2.41 to 17.5, we increase the range in $|\theta_2|$ while decreasing the range in $||\dot{\theta}||$ in the estimate for $X$. In addition, we note that for the first three cases, there is a maximum value for $\theta_2$ at which the matrix inequality (4.78) is not satisfied for any values of $\dot{\theta}$. This
Figure 4-6: Estimation of $X$ for Case 1

is because for these weights and for this maximum value of $\theta_2$, the terms involving $B_h(x)$, which depends only on $\theta_2$, causes one of the eigenvalues of the matrix $M$ to be positive. For the fourth case, we have reduced $K_i$ sufficiently so that this does not happen for any $|\theta_2| \leq \pi$. Finally, we can combine these four estimates of $X$ since they are all conservative. Doing so yields Figure 4-10 where we have plotted the boundary of the union of the four estimates of $X$.

To see if our estimate for $X$ is conservative, we can check to see if our closed loop system remain stable even if we excite our system sufficiently so that our closed loop solution $x(t)$ exceeds $X$. Through simulation of the manipulator dynamics under the feedback linearization control law as specified above and with a time delay of 100msec on each input channel, we are able to destabilize the closed loop by specifying an initial joint velocity of 10.5 rad/sec at each joint as shown in Figure 4-11. If we reduce the initial joint velocities to 10 rad/sec, the closed loop response is stable as shown in Figure 4-12 with $x(t)$ within $\|\dot{x}\| < 60$ rad/sec and $|\theta_2| < 2\pi$ for all time. We note that these simulations do not mean that the actual $X$ that guarantees small-signal robust
Figure 4-7: Estimation of $X$ for Case 2

Figure 4-8: Estimation of $X$ for Case 3
Figure 4-9: Estimation of $X$ for Case 4

Figure 4-10: Combined Estimate of $X$
stability is as large as \( \|\dot{\theta}\| < 60 \text{ rad/sec} \) and \( |\theta_2| < 2\pi \text{ rad} \). This is because we cannot simulate all possible initial conditions to see if we can destabilize the closed loop. In addition, our condition for robust stability is a guarantee for all \( \Delta \in \mathcal{D} \). For these simulations, we only consider one element of \( \mathcal{D} \), namely the one that corresponds to our modeling error due to the 100 msec delays. However, these simulations do suggest that we can excite the system so that its closed loop solution \( x(t) \) exceeds \( X \), in fact exceeding it by an order of magnitude in the joint velocities, and still maintain closed loop stability in the face of 100 msec time delay at the input channels. Finally, we can simulate the system with initial joint velocities of 1 rad/sec. The results are shown in Figure 4-13. We note that in this case, the closed loop solution for \( x(t) \) is within \( |\theta_2| < 0.5 \text{ rad} \) and \( \|\dot{\theta}\| < 1.1 \text{ rad/sec} \), which is within \( X \), and the closed loop response is stable as guaranteed by Proposition 4.3.

\[ \square \]

4.4 Chapter Summary

In this chapter we have addressed the stability robustness problem with respect to the general class of exact feedback linearizable systems. Our approach is based on the analysis strategy developed in Chapter 2. The idea is to use the frequency dependent magnitude bound information of the weights by combining \( W_i(s) \) and \( W_c(s) \) into the nonlinear operator \( M \), which contains the desired linearized transfer function matrix \( \Psi(s) \), to form the combined operator \( M_c \). In Section 4.1.1, we present a relationship between the \( L_2 \)-induced norm of a nonlinear system and the solution of a Hamilton Jacobi Inequality that corresponds to that system. We then use this relationship in Section 4.1.2 to derive a sufficient condition for small-signal robust stability. Unfortunately, this condition is difficult to verify because it requires a solution to the Hamilton Jacobi Inequality, which is computationally complex.

In the interest of obtaining a verifiable condition, we take the alternative approach of assuming a quadratic positive definite solution for the Hamilton Jacobi Inequality. This results in the sufficient condition for small-signal robust stability in Section 4.2.2.
Figure 4-11: Closed Loop Response for Initial Joint Velocities of 10.5 rad/sec
Figure 4-12: Closed Loop Response for Initial Joint Velocities of 10.0 rad/sec
Figure 4-13: Closed Loop Response for Initial Joint Velocities of 1.0 rad/sec
The result basically says that if the linearization of $M_c$ has $\mathcal{H}_\infty$ norm strictly less than one, in other words if the linearization of the closed loop system is robustly stable with margin, then the closed loop system is small-signal robustly stable. We note that this condition is readily verifiable since the $\mathcal{H}_\infty$ norm for linear time invariant systems, which in the condition is equivalent to the solution of a certain Riccati equation, can be calculated. In addition, the condition guarantees that there exists a non-empty region of the state space over which small-signal robust stability is attained and that this region is unbounded in the subspace of the weight states. Furthermore, this condition provides a methodology by which we can estimate this region. Since we only assume a quadratic solution for the Hamilton jacobi Inequality, this estimate for the region of small-signal robust stability could be quite conservative. This is demonstrated in the application of this condition in analyzing the stability robustness of a two link planar manipulator under feedback linearization where the model uncertainty is in the form of unmodelled time delays at the joint torque inputs. The results of this example also show that the estimated region for small-signal robust stability is affected by the choice of weights.
Chapter 5

Conclusions

In this chapter, we summarize our results and present some ideas for future research.

5.1 Summary of Results

The focus of this thesis was to develop a framework for the analysis of the unstructured uncertainty stability robustness problem for nonlinear systems under a feedback linearization control law. In Chapter 2, we formulated the problem and presented a preliminary analysis. The results of this analysis showed that one of the difficulties of the problem, when compared to the analogous stability robustness problem for linear time invariant systems, was that we were no longer able to directly absorb the frequency dependent magnitude bound information that is contained in the weights, $W_i(s)$ and $W_o(s)$, into our desired linearized transfer function matrix $\Psi(s)$ due to the nonlinear input and output maps of $M$. The consequence of this was that we were not able to tradeoff between the magnitude of the uncertainty with the magnitude of $\Psi(s)$ across frequency, which led to an overly conservative stability robustness condition. This conservatism resulted since we necessarily had to consider the worst case magnitude across frequency of $W_i(s)\Delta(s)W_o(s)$ as the gain for the uncertainty, which was typically greater than unity. In addition, we noted that unlike linear time invariant operators, we did not know how to calculate the $\mathcal{L}_2$-induced norm of a nonlinear dynamic operator, and we showed by example that the operator $M$ may not
even be finite gain stable if its nonlinear output map is not globally Lipschitz. These results from the preliminary analysis gave us the necessary insights in developing a framework by which to analyze the problem. Specifically, we noted that even though we could not directly incorporate the frequency dependent magnitude bound of the uncertainty into $\Psi(s)$, we must still use this information by combining the weights, $W_1(s)$ and $W_2(s)$, and $M$ into a single nonlinear operator $M_c$. The key then was to develop a methodology by which we could calculate a good bound on the $L_2$-induced norm of $M_c$. In this regard, we used the relationship between the general concept of dissipativity, which is related to the solution of a Hamilton Jacobi Inequality, and the $L_2$-induced norm of an operator. Finally, in order to ensure that the $L_2$-induced norm of $M$ was bounded so that we could use the Small Gain Theorem to conclude stability, we restricted ourselves to a local definition for input-output stability, namely that of small-signal, finite gain stability.

In Chapter 3, we addressed the stability robustness problem for a special subclass of exact feedback linearizable systems where the dynamics in the physical coordinates are in the so-called normal form and the input map for these normal form dynamics is constant, independent of the state. We were motivated to consider such systems since it allowed us to circumvent some of the difficulties of the problem as highlighted in Chapter 2. Specifically, since the input map to $M$ is now constant, we could directly absorb the frequency dependent magnitude bound information of the uncertainty into $\Psi(s)$ through the input of $M$, which resulted in a simplified analysis. In Section 3.1, we derived a sufficient condition for small-signal robust stability using a small gain argument. In Section 3.2, we noted that this condition has a frequency domain interpretation similar to that found in the stability robustness problem for linear time invariant systems. In particular, stability robustness was ensured if there was adequate separation between the maximum singular value of the desired linearized transfer function matrix $\Psi(s)$ and the uncertainty magnitude bound at each frequency. This separation was dictated by the $L_2$-induced norm of the static nonlinearity $\omega$, the nonlinear output map of $M$. However, unlike the stability robustness results for linear time invariant systems, we could not use this condition to directly loopshape $\Psi(s)$ to
ensure stability robustness, and therefore, the condition could not be directly used for synthesis. In Section 3.3, we explored the conservatism of this sufficient condition. In particular, we have shown that we cannot reduce the conservatism of this condition using loop transformations. As a result, we can say that our sufficient condition is in a sense the best condition that we can derive using a small gain argument. Finally, we have also developed a sufficient condition for instability using Lyapunov’s Linearization Method. Using this result, we have derived conditions on the system dynamics under which we can construct a destabilizing uncertainty if our sufficient condition for small-signal robust stability is violated.

Finally, in Chapter 4, we addressed the stability robustness problem with respect to the general class of exact feedback linearizable systems using the analysis strategy developed in Chapter 2. In Section 4.1.1, we presented a relationship between the $L_2$-induced norm of a nonlinear system and the solution of a Hamilton Jacobi Inequality that corresponded to that system. This relationship gave us a methodology by which to bound the $L_2$-induced norm of $M_e$. We then used this relationship in Section 4.1.2 to derive a sufficient condition for small-signal robust stability. Unfortunately, this condition was difficult to verify because it required a solution to the Hamilton Jacobi Inequality. In the interest of obtaining a verifiable condition, we used an alternative approach of assuming a quadratic positive definite solution for the Hamilton Jacobi Inequality. This resulted in the sufficient condition for small-signal robust stability in Section 4.2.2. The result basically stated that if the linearization of $M_e$ had $H_\infty$ norm strictly less than one, in other words if the linearization of the closed loop system was robustly stable with margin, then the closed loop system was small-signal robustly stable. We note that we could readily verify this condition since the $H_\infty$ norm for linear time invariant systems, which in the condition was equivalent to the solution of a certain Riccati equation, could be calculated. In addition, the condition guaranteed that there existed a non-empty region of the state space over which small-signal robust stability was attained and that this region was unbounded in the subspace of the weight states. Furthermore, this condition provided a methodology by which we could estimate this region. It was argued that this estimate could be

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quite conservative since we only used a quadratic solution for the Hamilton Jacobi Inequality. This was demonstrated in the application of this condition in analyzing the stability robustness of a two link planar manipulator under feedback linearization where the model uncertainty was in the form of unmodelled time delays at the joint torque inputs. The results of this example also showed that the estimated region for small-signal robust stability was affected by the choice of weights.

5.2 Directions for Future Research

The focus now turns toward future research. One important area for future work is in developing an algorithmic solution to the Hamilton Jacobi Inequality. As noted in Chapter 4, our inability to solve the Hamilton Jaobi Inequality prevents us from verifying the sufficient condition for robust stability that is provided by Proposition 4.1. In addition, it forces us to only consider a quadratic solution for the Hamilton Jacobi Inequality in Proposition 4.3, thereby leading to a conservative estimate for $X$. The main issue here is not simply to solve the partial differential equation, as can be done through polynomial approximation [19], [11], [20], but also to insure that the resulting solution is positive definite over the region of the state space of interest. Recently, research by Lu and Doyle [18] points to finding a positive definite solution to the Hamilton Jacobi Inequality through the use of the so-called Nonlinear Matrix Inequalities. The idea here is that at each point $x$ in the state space, the Hamilton Jacobi Inequality is convex in terms of $\partial V(x)/\partial x$. Therefore, one can solve for $\partial V(x)/\partial x$ at selected grid points of the state space and then piece these solutions together to approximate the gradient of $V(x)$ using interpolation. The problem here, however, is that there are no guarantees that the resulting gradient of $V(x)$ is integrable so that $V(x)$ exists. Therefore, additional research is needed here to insure that $V(x)$ exists.

Another area for future research is in terms of synthesis. As noted in Chapter 3, the sufficient condition in Proposition 3.1 is not conducive to synthesis since we cannot directly loopshape the nominal loop to achieve stability robustness. This
is because varying our design gain $K$ affects both the desired linearized transfer function matrix $\Psi(s)$ and the nonlinear output map $w$ in our condition. As a result, to use this condition for synthesis, we must use an interactive approach where we first fix $w$ with respect to $K$ in order to synthesize $K$ by loopshaping $\Psi(s)$, and then checking the condition afterwards to insure stability robustness. The convergence of such an iterative scheme is of issue and requires further research. In addition, for Proposition 4.3, we note that we can design a linear dynamic compensator using the full information $\mathcal{H}_\infty$ synthesis solution in [7] to insure that the $\mathcal{H}_\infty$ norm of the linearization of $M_c$, $\bar{M}_c$, is strictly less than one. However, we note that since we cannot weight the controller states a priori to the synthesis process, we cannot ensure that there exists an $\varepsilon > 0$ such that the Riccati equation in the condition for Proposition 4.3 is satisfied. Therefore, we cannot call upon the proposition to estimate $X$. In addition, as noted in the discussion following Proposition 4.3, the dependence of the estimated $X$ depends in a nonconvex manner with respect to changes in $K$. Therefore, even if we can use the proposition to estimate $X$, we do not know how to vary the weights in the $\mathcal{H}_\infty$ synthesis problem to maximize this estimate for $X$.

Finally, we would like to extend the stability robustness results in this thesis to nonlinear systems that are not exact feedback linearizable. For these systems we can always choose a set of outputs and use feedback linearization to achieve input-output or partial linearization where only part of the dynamics, the part pertaining to the input to output relationship, is linearized by feedback. The rest of the dynamics, unobservable from the output, are, in general, still nonlinear and are referred to as the zero dynamics. With the presence of unstructured uncertainty, the first issue is how the uncertainty will affect the stability of these zero dynamics. In particular, we need to develop conditions under which the actual zero dynamics are stable given that the nominal zero dynamics, without uncertainty, are stable. The second issue is how these zero dynamics will affect the stability robustness of the linearized portion of the dynamics. We note that this second issue can be analyzed using the framework developed in this thesis. However, in this case, the operator $\Psi$ will no longer be linear time invariant.
Appendix A

Input-Output Stability Definitions

In this appendix, we present the notation and definitions that are used throughout the thesis. The treatment is quite standard, and the reader is referred to [34] and [4] for further details.

A.1 General Notation and Definitions

In general, upper case symbols will denote matrices while lower case symbols will denote vectors, unless otherwise noted. If $x$ is a vector, then $x_i$ is the $i$th component of $x$. If $X$ is a matrix, then $X_{ij}$ is the $(i,j)$th component of $X$. We let $\mathbb{R}^n$ denote the usual $n$ dimensional vector space over the reals that is endowed with the Euclidean or $l_2$ norm. If $x \in \mathbb{R}^n$ then we denote this norm as

$$\|x\| = [x'x]^\frac{1}{2}$$  \hspace{1cm} (A.1)

where ' denotes transpose. If $H$ is an $n \times n$ matrix over $\mathbb{R}$, then we denote the induced norm of the matrix operator $H$ over $l_2$ as

$$\|H\| = \sup_{\|x\|=1} \frac{\|Hx\|}{\|x\|} = \sigma_{\text{max}}[H].$$  \hspace{1cm} (A.2)
where $\sigma_{\text{max}}[H]$ denotes the maximum singular value for the matrix $H$. Finally, we denote $R_+$ as the nonnegative reals.

**A.2 $L_2$ Spaces**

For this thesis, we will be dealing with finite energy signals or signals that are square integrable. In this regard, we make the following definition.

**Definition A.1** Let the signal $f : \mathbb{R}_+ \to \mathbb{R}^n$ be a measurable function of time $t$, The signal $f$ is said to be in the $L_2$ space if

$$
\|f\|_2 = \left[ \int_0^\infty \|f(t)\|^2 dt \right]^{\frac{1}{2}} \quad (A.3)
$$

is finite.

We note that all signals in $L_2$ go to zero as $t \to \infty$. Therefore, in order to study stability and signals that may not go to zero, we need the following extension for $L_2$.

**Definition A.2** Let the signal $f : \mathbb{R}_+ \to \mathbb{R}^n$ be a measurable function of time $t$, The signal $f$ is said to be in the extended $L_2$ space, or the $L_{2e}$ space, if $\|f_T\|_2$ is finite for all $T \in \mathbb{R}_+$ where $f_T$ is the truncation of $f$ defined by

$$
f_T(t) = \begin{cases} 
  f(t) & 0 \leq t \leq T \\
  0 & t > T,
\end{cases} \quad (A.4)
$$

For convenience, we will use the notation $\|f\|_{2T}$ for $\|f_T\|_2$.

We note that the extended $L_2$ space is very general and only excludes signals that goes to infinity in finite time.

**A.3 Input-Output Stability**

At this point, we are ready to define stability from an input-output viewpoint.
Definition A.3 An operator $H$ that maps signals from $L_{2e}$ to $L_{2e}$ is finite gain stable if there exists a finite constant $\gamma$ such that

$$\|Hu\|_{2T} \leq \gamma\|u\|_{2T} \quad (A.5)$$

for all $T \in \mathbb{R}_+$ and for all $u \in L_{2e}$. The smallest such $\gamma$ that satisfies Equation (A.5) is referred to as the $L_2$-induced norm of the operator $H$.

We note that finite-gain stability implies that if the input signal is in $L_2$ then the output signal will be in $L_2$.

Of particular interest is when the operator $H(s)$ is a linear time invariant rational transfer function matrix. In this case, we have the following theorem whose proof is found in [34].

Theorem A.1 If $H(s)$ is stable in the sense that it has no poles in the closed right half $s$-plane, then the operator $H$ is finite gain stable and the $L_2$-induced norm of $H$ is given by

$$\gamma_H = \sup_\omega \{\sigma_{\text{max}}[H(j\omega)]\} = \|H(j\omega)\|_{\mathcal{H}_\infty} \quad (A.6)$$

In addition, we also have a local version for finite gain stability.

Definition A.4 An operator $H$ that maps signals from $L_{2e}$ to $L_{2e}$ is small-signal finite gain stable if there exist finite constants $U > 0$ and $\gamma$ such that for all $u \in L_{2e}$ where $\|u(t)\| \leq U$ for all $t \geq 0$,

$$\|Hu\|_{2T} \leq \gamma\|u\|_{2T} \quad (A.7)$$

for all $T \in \mathbb{R}_+$. The smallest such $\gamma$ that satisfies Equation (A.7) is referred to as the small-signal $L_2$-induced norm of the operator $H$.

We note that the above definitions are for a single operator in an “open loop” sense. For operators connected in the standard feedback configuration in Figure A-1, we have the following definition.
Definition A.5 The feedback system in Figure A-1 is finite gain stable if the operator that maps the signal pair \((v_1, v_2)\) to the signal pair \((z, w)\) is finite gain stable.

Likewise, we also have the following local version of the above.

Definition A.6 The feedback system in Figure A-1 is small-signal finite gain stable if the operator that maps the signal pair \((u_1, u_2)\) to the signal pair \((e_1, e_2)\) is small-signal finite gain stable.

Finally, we have the following Small Gain Theorem which provides a sufficient condition to conclude stability for the feedback system in Figure A-1.

Theorem A.2 Consider the feedback system in Figure A-1. Under the assumptions:

1. The feedback system is well posed in the sense that for any input pairs \((u_1, u_2) \in \mathcal{L}_{2e} \times \mathcal{L}_{2e}\), the solutions for \((e_1, e_2)\) exists and are unique

2. \(G_1\) and \(G_2\) are finite gain stable with \(\mathcal{L}_2\)-induced norms \(\gamma_1\) and \(\gamma_2\), respectively,

the feedback system is finite gain stable if

\[
\gamma_1 \gamma_2 < 1 \tag{A.8}
\]
**Proof:** We note that the feedback system in Figure A-1 can be described by the feedback equations

\[ e_1 = u_1 + G_2 e_2 \quad (A.9) \]
\[ e_2 = u_2 + G_1 e_1. \quad (A.10) \]

Taking norms and using the triangle inequality gives for all \( T \in \mathbb{R}_+ \)

\[ \|e_1\|_{2T} \leq \|u_1\|_{2T} + \|G_2 e_2\|_{2T} \quad (A.11) \]
\[ \|e_2\|_{2T} \leq \|u_2\|_{2T} + \|G_1 e_1\|_{2T}. \quad (A.12) \]

Applying the fact that \( G_1 \) and \( G_2 \) are finite gain stable gives

\[ \|e_1\|_{2T} \leq \|u_1\|_{2T} + \gamma_2 \|e_2\|_{2T} \quad (A.13) \]
\[ \|e_2\|_{2T} \leq \|u_2\|_{2T} + \gamma_1 \|e_1\|_{2T}. \quad (A.14) \]

Substituting (A.14) into (A.13) gives

\[ \|e_1\|_{2T} \leq \|u_1\|_{2T} + \gamma_2 (\|u_2\|_{2T} + \gamma_1 \|e_1\|_{2T}), \quad (A.15) \]

or

\[ \|e_1\|_{2T} \leq (1 - \gamma_1 \gamma_2)^{-1} (\|u_1\|_{2T} + \gamma_2 \|u_2\|_{2T}). \quad (A.16) \]

Similarly, substituting (A.13) into (A.14) gives

\[ \|e_2\|_{2T} \leq (1 - \gamma_1 \gamma_2)^{-1} (\|u_2\|_{2T} + \gamma_1 \|u_1\|_{2T}). \quad (A.17) \]

We note that by definition (A.16) and (A.17) implies that the feedback system is finite gain stable. Furthermore, we have that if \((u_1, u_2) \in \mathcal{L}_2 \times \mathcal{L}_2\), then \((e_1, e_2) \in \mathcal{L}_2 \times \mathcal{L}_2\) and that

\[ \|e_1\|_2 \leq (1 - \gamma_1 \gamma_2)^{-1} (\|u_1\|_2 + \gamma_2 \|u_2\|_2) \quad (A.18) \]
\[ \|e_2\|_2 \leq (1 - \gamma_1 \gamma_2)^{-1}(\|u_2\|_2 + \gamma_1 \|u_1\|_2). \]

We note from the proof of the Small Gain Theorem that we can use it to conclude the small-signal finite gain stability of the feedback system in Figure A-1 as long as the product of the $L_2$-induced norms of $G_1$ and $G_2$ under the small-signal condition are less than unity. Here, the small-signal condition means that there exists constants $U_1 > 0$ and $U_2 > 0$ such that $\|u_1(t)\| \leq U_1$ and $\|u_2(t)\| \leq U_2$ for all $t \geq 0$. 
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