Equilibrium Analysis of Topological Interlocking for Structural Assemblies

by

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Submitted to the Department of Civil and Environmental Engineering
in partial fulfillment of the requirements for the degree of
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Abstract

This thesis presents an exploration of topological interlocking systems of structural assemblies. By analysing existing topological structures, a mathematical theory is constructed to design different topological interlocking systems based on various existing design techniques. A structural equilibrium analysis method is presented for the designed structural assemblies through a detailed analysis of the collapse mechanism. The method also features a novel way to transform indeterminate problems into determinate problems using the geometric relations, by an implementation of the half-edge data structure in mesh manifold computation. The thesis also briefly introduces a mapping strategy based on conformal mapping on NURBS surface to design freeform topological structural assemblies.

Thesis Supervisor: Caitlin Mueller
Title: Assistant Professor of Architecture and Civil and Environmental Engineering
To Professor Caitlin Mueller
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Chapter 1

Introduction

1.1 Introduction

Throughout the building history of mankind, the way we use certain kind of material improves as the understanding of the properties of the material improves – how a specific material take forces and moments. As it is both inefficient and almost impossible to build an infinite scale monolithic structure in one piece, a discretization process is needed to segment the overall structure into pieces to fabricate, and assemble to form a whole.

A key question thus exist in nearly every assembly process – how to connect the parts to form a whole?

Jointing techniques was invented based on different material properties. One of the guideline for joint design is: the performance of joint should never be lower than the performance of the elements. Under this principle, some jointing techniques such as masonry for stone and tile, interlocking joints for wood, screwing for steel, etc. (Figure 1-1) was replaced gradually by fastening materials such as glue and welding, because some of these techniques joint two parts by cutting the material into parts thus breaking the transmission of the force inside the material (i.e. wood); or because of the emergence of new composite materials (i.e. carbon fibre). However, the beauty of connecting different pieces without any additional fastening material still has its place in architecture, especially when more new analysis methods and design tools are invented, such as Topological Interlocking Joinery (Figure 1-3, Kaijima, Conti, and Dunn 2015). Some modern
extensions of masonry techniques also features similar contributions, such as the Thrust Network Analysis (Figure 1-2, P. Block and Ochsendorf 2007).

Figure 1-1: different materials and its typical connections

Figure 1-2: shell designed with Thrust Network Analysis

The idea of connecting multiple pieces without fastening material is very closely related to the geometry of the unit. In other words, the geometries of the units will govern the way units are connected as well as the structural performance of the overall assembly.

1.2 Topological interlocking system

the segmentation of a plate-like structure into specifically shaped elementary blocks constrained by the neighbouring elements. One of the distinctive feature of TI is that both in-plane and out-of-plane movement of a block is hindered by its neighbours and no connectors are needed to hold the blocks in place. The whole structure only require a global constraint, such as a foundation, an external frame, tension wires, to maintain the overall structure.

Dyskin, Estrin and colleagues have shown all five platonic polyhedra (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) and their derivatives allow for TI (Molotnikov et al. 2007; A. V. Dyskin et al. 2003). Studies on beam assembly and plate assembly as also conducted (Feng et al. 2015; Thomas Siegmund et al. 2016), with the establishment of stiffness-scaling relationships (Khandelwal, T. Siegmund, R. J. Cipra, et al. 2013) and the development of thrust line models (Khandelwal, T. Siegmund, R. Cipra, et al. 2012) to describe mechanical behaviour of such assemblies. The contact surface was further developed into no-planar surfaces, and was referred to as osteomorphic blocks (A. Dyskin et al. 2003) due to their similarity to the shape of a bone. The surface can be described mathematically and must meet symmetry requirements for interlocking as well as periodical requirements to fill a plane.

TI system also features structural stability with a high damage tolerance and damage confinement (Thomas Siegmund et al. 2016). This ensures that the structure is more compliant than a monolithic one and performs better to absorb vibrational energy dissipated by frictional losses (Arcady V. Dyskin, Elena Pasternak, and Yuri Estrin 2012). An example showing structure integrity can still be maintained when approximately 25% of the
Figure 1-4: Cited image from previous papers 01 (A. V. Dyskin et al. 2003)

Figure 1-5: a TI system based on ostemorphic brick (Y. Estrin, A. Dyskin, and E. Pasternak 2011)
elements fails randomly was presented by Molotnikov et al. (Molotnikov et al. 2007)

Other features of TI system such as possibilities for extraterrestrial/unmanned aerial vehicle /robotic construction (A. Dyskin et al. 2005) and transformation and deformation of TI systems on on-plate structures (Tessmann 2012), was also proposed by some scholars, but not fully discussed and researched.

Besides the finite element method used to analysis TI system in the papers mentioned above, there is another approach using discrete element method (DEM) to simulate the behaviours of interlocking systems. The DEM is a method to numerically compute and simulate the motion and effect of numerous small particles which are usually modelled as sphere-like objects (Lu and McDowell 2006; Harkness 2009). However, some analyses of masonry structures (Lemos 2007; Tóth, Orbán, and Bagi 2009) using DEM also provide potential numerical simulation methods on TI structures.

1.3 Problem statement

Research conducted that is related to TI system never state clearly why some polyhedra can be assembled into a TI system while some can not, why the case of icosahedron was different from the other four cases and what type of polyhedron allows TI in general, or whether there are systematic ways of assembling polyhedra into a TI system. Theories that generalise these questions are needed for general applications of TI system.

There also exist a gap in consistency between the geometry of the TI systems and the overall structural performance. Though researchers have conducted load tests on different TI systems and different materials, the tested geometries used in these research are quite limited. It is because these research all follow a bottom-up process: generating the individual and overall geometries first and then test the structural performance of the overall geometries. However, a structurally inefficient overall geometry will always result in bad performance however it is composed. Thus, we need more innovative methods to develop TI systems that comply with a structurally efficient geometry, which may involve deformation and transformation of the individual units.

Additionally, as shown in most of the research cited above, assembly feasibility was
seldom considered. The assembly process of these TI systems still need supporting framework during the construction process, which will produce huge cost for large scale construction. A discussion on whether it is possible to take advantage of the assembly properties of TI systems to optimise the assembly process and reduce the amount of the framework during construction is needed for the further development of TI systems.

1.4 Outline of sections

This thesis presents a possible construction system with the idea of topological interlocking. This system allows a reduction of the framework during construction process and provide more bending capacity to the funicular-compression-only structure.

The thesis is divided into two parts. The first part, contained in Chapter 2, presents a mathematical theory to design TI systems using 2D tessellations is discovered. Several methods of customising and detailing the TI units are also introduced in Section 2.2.

The second part presents the topological interlocking structure assemblies and the corresponding structural equilibrium analysis.

Section 3.1 to Section 3.3 introduce the structural properties of a TI system and present a detailed analysis of the collapse mechanism using a derived version of non-directional blocking graph in assembly planning field and equilibrium equations with static friction involved.

Using the framework in Section 3.1, Section 3.4 and 3.5 present the instant on-the-fly structural equilibrium analysis during an assembly process and the corresponding algorithm required for the analysis through the implementation of the "half-edge" data structure.

Section 3.6 presents the application of TI system on non-planar surface structure and a mapping strategy based on conformal mapping on NURBS surface to design freeform topological structural assemblies. Section 3.7 gives a brief introduction of 3D TI systems.

Appendix will include the implementation of some key solutions and algorithms in developing this thesis.
This chapter focuses on the geometric properties of TI systems and the mathematical theory that supports the design of similar systems. First, an overview of the fragmented properties of TI system throughout the time will be made by looking at different TI compositions and unit designs. Second, a deep examination on certain platonic polyhedrons will be conducted to find out the relationship between two-dimensional tessellation systems and the TI systems. Third, we will discuss different methods for customising the design of the units based on the theory we developed and provide more variations to the field.

2.1 Layered topological interlocking system based on 2D regular tessellation

Dyskin et al. first illustrated systematically (A. V. Dyskin et al. 2003) how a TI system can be composed with five platonic polyhedra (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) respectively. He also found out the plane cutting through the middle position of these one-layer plate-like TI structures resulted in two-dimensional assembly of regular polygons – square for tetrahedron, hexagon for cube, octahedron,
and dodecahedron, and decagon for icosahedron. However, this discover was not comprehensive as there can be more than one type of assembly for certain polyhedra, and the two-dimensional assembly of the middle plane section will also vary due to the different arrangements of the three-dimensional polyhedra. Details will be illustrated in the later part of this section.

By examining the middle section plane mentioned in Dyskin et al.’s paper as well as section planes above and below, we can find a geometric transformation of the polygons cut by these planes through each polyhedron (Figure 2-1). As shown in the case of cube and octahedron, it is clear to see that different polyhedral assembly could share same type of middle section plane. In other words, there must be some more fundamental rules that governs whether certain polyhedron can be assembled into a topological interlocked system.

![Transformation of sectional polygons in tetrahedron, cube, octahedron](image)

*Figure 2-1: Transformation of sectional polygons in tetrahedron, cube, octahedron*

Actually, the assemblies Dyskin et al. mentioned in his paper are called tessellations – tiling of regular polygons (in two dimensions), polyhedra (three dimensions), or poly-
Figure 2-2: Regular tessellations described in Schlӓfli symbol or vertex configuration and created automatically through certain algorithms. It is also the theory how certain type or types of polygons can be assembled repeatedly to fully cover a planar area. Tessellations that are composed of only one type of polygons are called regular tessellations (Weisstein n.d.(a)) and there are only three regular tessellations in total (Figure 2-2). Tessellations of the plane by two or more regular polygons such that the same polygons in the same order surround each polygon vertex are called semi-regular tessellations. This paper will not develop in detail on this subtopic, but expressions and more discussions on the basis of semi-regular tessellation can be found in (Watson 1973).

One thing we can see from Figure 2-1 is that the geometric transformation sequences of the polygons in each of the polyhedra share a same property. The number of sides of the starting and ending shape is half of the number of sides of the middle regular polygon:

\[ N_{\text{middle regular polygon}} = 2N_{\text{starting/ending regular polygon}} \]  

(2.1)

In the case of cube and octahedron, both starts and ends in regular triangle and the middle regular polygon is hexagon; in the case of tetrahedron, the middle regular polygon is square and the starts and ends shape is a line, which can be seen as an extreme case of a polygon with two equal size. For the case of decagon-based assembly of dodecahedra described in (A. V. Dyskin et al. 2003), the relationship between the starting and ending polygons and the middle regular polygon still satisfies Equation 2.1.

Another properties of this geometric transformation process is that every two neigh-
bouring polyhedra have reciprocal transformation directions in the transformation sequence, as shown with the circumstance of tetrahedron in Figure 2-3. The whole tessellation then can be divided into two groups with reciprocal transformation directions. This property also applies to other tessellation cases such as cubes, dodecahedra and octahedra.

Figure 2-3: Transformation direction of sectional polygons in tetrahedron tessellation

This is how topological interlocking is achieved, and the position of this polyhedron will be locked by its neighbouring polyhedra because of the reciprocal transformation directions. In the case of tetrahedron, each unit have four neighbouring units each of which has direct surface contact with the four faces of the central unit (Figure 2-4). These four surrounding units blocked the four movement directions in the plane as well as the two directions perpendicular to the plane. In the case of cube or octahedron, as the interlocking are based on hexagon tessellation, each unit will be interlocked by six neighbouring units (Figure 2-5).

Figure 2-6a shows the transformation in sectional planes produces pairs of inclined planes on each of the 3D units. For every unit, the two groups of reciprocal directions make the transformation conducted inwards and outwards respectively. Thus, each unit has a group of parallel faces – top face and bottom face, and several groups of reciprocal faces (in the case of cube/octahedron, the number is 2.) – same inclination compared
Figure 2-4: a tetrahedron unit locked by neighbouring units

Figure 2-5: a cube/octahedron unit locked by neighbouring units

(a) sectional plane transformation  (b) faces inclined inwards and outwards

Figure 2-6: Transformation of sectional planes forming pairs of reciprocal planes (octahedron)
to the perpendicular planes (Figure 2-6b). It is noticeable in Figure 2-4 that the locking mechanism is a *full lock*, which means the unit cannot move in any direction. This is achieved through two parts: movements inside the plane are blocked by its neighbouring units – this is the same as the circumstance in a two-dimensional tessellation; movements outside the plane (upwards and downwards) are blocked by the two pairs of inclined planes. The units on the borders are not fully locked and certain boundary conditions are needed to keep the whole system in position.

In a comparison between the transformations in the case of tetrahedrons, cubes, and octahedrons (Figure 2-7), We are also surprised to find that besides the different inclination angles in the three different cases, for the cubic tessellation, the top and bottom pyramid parts were neither involved in the transformation of the planes nor related to the contact area between neighbouring units (Figure 2-8). Thus, we could conclude that these parts were not essentially necessary for the composition of the TI system. This discovery also indicates that the TI system composed by cubes and octahedrons are actually using the same principle of hexagon tessellation. Platonic solids used in (A. V. Dyskin et al. 2003; A. Dyskin, Y. Estrin, Kanel-Belov, et al. 2003) are just special cases in TI systems. We can also conclude that for a icosahedron, since the middle sectional polygon for tessellation is a regular decagon which is not possible to construct a tessellation by itself, there are no ways to construct a layered TI system without any gaps or holes. The gaped version has been discovered in (A. V. Dyskin et al. 2003), but not fully explained. And for the case of dodecahedron, two regular polygons can be found by connecting the middle
points of certain edges – one hexagon, one decagon. That means a TI system constructed by purely dodecahedra can either use a hexagon tessellation, as for cubes or octahedrons, or follow the way in the icosahedron composition which of course will have gaps or holes (Figure 1-4).

Figure 2-8: unrelated parts in a cubic topological interlocking system

Figure 2-9: two assembly of dodecahedra based on the hexagon tessellation

Here, The core of layered TI is arrived: a group of individual units generated through reciprocal transformations based on a regular two-dimensional tessellation, where the positions of each unit are locked geometrically by its neighbouring units. We should
be aware that partial lock may exist if we change the transformation type and base tessellation, but an algorithm that can describe the transformations clearly is needed for designing and generating a TI system parametrically.

2.2 Topological interlocking unit customisation

By re-examining the Dyskin et al.’s theory (A. V. Dyskin et al. 2003) about TI in platonic polyhedra with the discovery we’ve made and theory we’ve developed in the last section, we now realise that the key to build a TI system is actually not necessarily using platonic solids, but to find a regular two-dimensional tessellation of regular polygons with even number (≥ 4) of edges.

2.2.1 Topological interlocking based on non-regular tessellation

Though we’ve only discussed the regular tessellation which formalised the theory of TI system with platonic solids, there are actually more variations and possibilities in the composition of a TI system based on semi-regular or non-regular tessellation. Semiregular tessellation are tessellations of the plane by two or more convex regular polygons such that the same polygons in the same order surround each polygon vertex (Weisstein n.d.(b)). There are other kinds of tessellations such as demiregular tessellation, uniform tessellation etc. Some of them can be formed systematically and some can not. The whole subject of these tessellations can be categorised by Euclidean tilings by convex regular polygons (Grünaubm and Geoffrey C. Shephard 1977; Grünbaum and G. C. Shephard 1978; Debroey and Landuyt 1981). As the content of this paper does not have too much relation with these difference, we will not go into details about them.

Research in (Weizmann, Amir, and Grobman 2016) has shown potentials of using various types of some of the semiregular and non-regular tessellations to create complex geometries for topological interlocking compositions. However, the distortion methods to fit the existing TI system to the curved surface was not described and remained unclear, as well as whether the contact faces remained planar or not. The 3D printing fabrication method they chose to reduce much complexity and the feasibility of the really
construction possibility. Besides, they choose to converge to simplified compositions for assemblies in the real world, considering the construction feasibility of the overall structure.

With the theory we developed in the last section, we can easily develop tessellations based on semiregular tessellations with similar strategies, as shown in Figure 2-10. The mixed use of octagons and squares (Figure 2-10a) or squares, hexagons, and dodecagon (Figure 2-10b) results in a mixed TI system with different types of polyhedrons. The notation used here is called vertex configuration a similar notation system as a generalised version of Schlafli symbol. The sequence of the number represents the sides of the polygons that surround a vertex in the tessellation. Regular tessellations described in Schläfli symbol (i.e. 6, 3) can also be described in this way (i.e. \{6.6.6\} or \{6_3\}). Since polygons are no longer limited to the three types as it is in regular tessellations, a vast amount of compositions could be generated.

2.2.2 Deformation of the contact face

Another type of TI system developed in (Molotnikov et al. 2007) shows us a different direction of deforming the geometry of the units (Figure 2-11). The TI system composed by the so-called osteomorphic block developed in the paper use a shifted scaled-square tessellation grid (tessellation of rectangle) as the middle sectional plane grid, but did not transform into segments of lines or smaller rectangle, as the way was done in tetrahedron tessellation, but instead transformed the contact face into non-planar surface. Unlike the methods described in previous chapters, this method does not result in inclined faces that
can lock the movements of neighbouring units, but rather result in a non-planar face that have the same effect. The local curvature of each curvy face will block the movement in both up and down directions and interlocking of the overall assembly is achieved by the shifting of osteomorphic blocks on each row, and the close fit between the curved surface of the two units was guaranteed by certain type of periodical functions. There was no description of how the curvature was generated, but after a close observation, it is very likely that these curves on the edges are trigonometric functions, one of the simplest periodical curve functions, and the surfaces was generated by interpolation between the two trigonometric functions, or a "lofting" process.

Similar to the osteomorphic block shown in (Molotnikov et al. 2007), various type of blocks can be generated in similar ways (Figure 2-12, 2-13). This transformation can also be applied in addition to inclined contact face the TI system we’ve already discussed above (Figure 2-14), but may have limited effects depend on the shared area between the two contact faces.

This type of transformation brought the interlocking problem into local relationship between two units, which may reduce the complexity of the overall design, but also may cause overall instability as it requires even higher material strength in the composition of the units. It may also require a relatively larger contact surface for the transformation to happen, thus increase the weight of the units as the thickness of the unit increases.
On the other hand, for a unit that have a certain volume limits, this type of units may have limited number of faces (limited number of edges in planar tessellation) due to the contact surface area requirements.

The periodical properties of the trigonometric functions are also limiting the TI system in a way that the shifting distance have to comply with the phase of the sine or cosine functions, and creating more difficulties in more complex tessellations. However, we could apply the curvature feature only without caring about the shifting problems. Thus,
a local deformation of the contact face is achievable and the curvature we created will not be limited to trigonometric functions. Recent research has also shown potentials of using this technique as a jointing method (Weir, Moult, and Fernando 2016), by increasing the periods and the amplitude of the trigonometric functions and creating a special type of finger joints (Figure 2-15).

Figure 2-15: Curved joint design (Weir, Moult, and Fernando 2016).

2.2.3 Deformation of the tessellation grid

Till this point, the polygons types used in the different tessellations are regular polygons, as described in Section 2.2.1. However, if an overall deformation is applied to the tessellation grid so that every regular polygon in the grid is deformed into non-regular polygon while the tessellation relationship is maintained, what will happen to the geometries of the 3D units based on these polygons? Will the TI system still be achievable?

Figure 2-16: grid deformation with shared vertices
In general, a deformation algorithm applied to a grid will change the coordinates of every vertex in the grid while maintain the topological relationship. Figure 2-16 shows a deformation algorithm creating a *folding fan* effect of the grid. By connecting the same vertices through lines, a new grid is achieved. However, when edges in the grid does not share the same vertices, circumstances in Figure 2-17 may happen. This is because the co-linear vertices of each of the two partially overlapping edges are no longer co-linear after the transformation. To avoid this circumstance, the topological relationship of an edge need to include all the points it passes, and may result in polylines or interpolated curves depend on the connecting method (Figure 2-18).

The 3D units created using the modified algorithm is shown in Figure 2-19. The edge curves was created using the interpolation method rather than the polyline method is shown in Figure 2-18. It is worth noticing that in the projection diagram, the parallel
property in the original unit is not maintained in the deformed unit, thus the side face will no long be planar. This may add more friction and interlocking abilities, but will increase the degree of difficulty in manufacturing. The producing of the non-planar side face here is different from those in Section 2.2.2, since that is a deliberate action and here is an unavoidable result.

![Image of 3D units and projections created by modified algorithm]

*Figure 2-19: A typical 3D units and its projection created by the modified algorithm*

### 2.3 Chapter Summary

In this chapter, a bridge across 2D tessellations and their corresponding 3D topological interlocking systems as well as 3D variations is built after evaluating various different examples in existing published papers. The TI systems shares the same extendability as their base 2D tessellations, and can only be extended in the third dimension through layering, in which the interlocking properties will be lost. Customisation of the TI system is also discussed in this chapter, featuring both the local customisation for individual units and the global customisation for the overall assemblies. A change of geometric relation based on the deformation algorithm is discovered in the latter customisation, and the reason behind is explained.
Chapter 3

Equilibrium Analysis of Topological Interlocking Structural Assemblies

In this chapter, we will connect the geometric properties of TI system with its structural properties, and we discuss the ability of a TI system to transfer structure failure to mechanical failure. We will discuss the circumstance of choosing and assembling a TI system using a hexagon-based regular tessellation (${6^3}$) without limited framework by checking and maintaining equilibrium states at every step of the assembly process. We will also discuss the deformation of tessellation on both planar surface and 3D non-planar surface, and the effect of these deformations will have on the corresponding TI systems.

3.1 Structure failure V.S. material failure

As we stated in Section 1.3, the gap between geometry and structure has been barely filled and there are enormous amounts of possibilities to explore for how to apply the system structurally. Though efforts of both numerical analysis and cast or 3D printed assemblies has be made by existing research (Weizmann, Amir, and Grobman 2014, Khandelwal, T. Siegmund, R. Cipra, et al. 2012, Khandelwal, T. Siegmund, R. J. Cipra, et al. 2013) in extending TI systems to some basic building elements, such as beams and floor systems, but most of them focus on the properties of finished assemblies, with a fixed frame boundary that ensures all the units in position and reduce the possibility of any structural failure.
A very interesting property of a TI system is that once the boundary is fixed, the whole system is also fully geometrically locked and prevent any movement of any unit inside the system to move. In other words, if neither the units nor the boundary is broken, the TI system is stable in any loading condition, theoretically. (Actually research on the damage tolerance of TI system shows that even some units is broken, the structural could be still stable, as we have mentioned in Section 1.2.)

One of the most interesting feature of this property, if we examine carefully, is that it ensures a transformation of the failure of a structure from structural failure to material failure. Besides small movements and displacements of construction tolerance, the assembled TI structure performs like one whole piece of material, and the only way to cause a failure in the structure is the failure of the materials – cracks, stress failure, etc.

### 3.2 Choice of the unit geometry

As we stated in Section 2.1 and in Figure 2-4, 2-5, one of the most prominent features of a TI system is that the movement of each unit with the system is blocked by its neighbouring units or the boundary conditions (usually a frame). In return, it also means the position of each unit is held or supported by its neighbouring units. If every new unit added to the existing structure can be held by its neighbouring units and the overall structure will maintain the stable state after adding the additional piece during a whole construction process, there will be no need for a extra supporting framework for the construction.

The correspondence between the polygons in a tessellation and the polyhedral units in the TI system derived from the tessellation system shows that the number of the edges of the polygons equals to the number of the side faces of the polyhedral units (inclined both inwards and outwards). And the number of the edges of the polygons also need to be a even number, so that there will be equal number of side faces inclined inwards or outwards, both equals half of the total number of side faces. Similar to Equation 2.1, we have Equation 3.1.
(a) equilibrium: removal one upwards-blocking neighbouring units; (b) unequilibrium: removal of one downwards-blocking neighbouring units; (c) unequilibrium: removal of two neighboured neighbouring units.

Figure 3-1: equilibrium or unequilibrium state of tetrahedra TI system

\[ N_{\text{middle regular polygon}} = 2N_{\text{side face inclined inwards}} = 2N_{\text{side face inclined outwards}} \quad (3.1) \]

It is clear to infer that the number of the edges of the middle regular polygon need to be larger than four \((N \geq 4)\) for producing a possible TI system (When \(N = 4\), the TI system is composed of tetrahedra, as shown in Figure 2-4). However, the choice of a TI system that can be assembled without framework (at least partial) also need to satisfy another requirement that during the assembly process, the newly added unit need to be stable as well as extensible. In other words, the unit should reach an equilibrium state even without all of its neighbouring units. And realistically, it is fairly to assume that the assembly sequence begins from the bottom foundation and the structure is assembled upwards.

Under these assumptions, we find that for a tetrahedra unit, since there are two faces blocking the downwards direction and two blocking the upwards direction, an equilibrium state can only be reached when only one neighbouring unit that blocking the upwards direction was removed. As shown in Figure 3-1a. The removal of the other type neighbouring unit or more than one neighbouring units will cause the unit to fall (Figure 3-1b, 3-1c). Thus, we need at least a polyhedron that have a planar corresponding polygon owns at least six edges.
Among hexagon, octagon, dodecagon, etc., hexagon is the most illustrative and preferred case not only because there is a tessellation by itself \(6^2\), but also because under certain setup (Figure 3-2), every new added unit during the assembly process have exactly three contact faces that touched the neighbouring units. This ensures each added unit is in a determinate state and can be solved without compatible equations. Thus we can make the assumption that all the units are rigid bodies, a common assumption when dealing with hard material construction such as stone or concrete. For other type of polygons, the tessellation will either involve square or triangle (\(4.8\), \(4.6.12\), \(12.12.3\)) and result in tetrahedra or empty spaces in the TI system, or involve indeterminate state for the new added units during assembly process.

![Figure 3-2: forces from contact faces of a unit](image)

3.3 Single-unit equilibrium analysis and assembly configuration setup

3.3.1 Basic assumption

Before we step into the analysis of any assembly process for any aggregated structures, it is essential to set up the framework for checking the stability state of a single unit in different positions. As shown in Figure 3-2, the unit type we have chosen can be supported by three faces (the number could be two at certain special cases). To formalise
question, we made one assumption:

**Assumption 1** *All units are rigid bodies.*

This assumption ensures there is no geometric deformation caused by stress between the units, thus avoid the circumstance when elastic deformation is involved and results in additional forces that require using compatibility equations. This of course will reduce the possibility to allow the system to handle more complex conditions, but are fairly enough for the unit type we have chosen, and extremely simplifies the problem we are facing, as explained in Section 3.2. It also eliminates the circumstance that a gap between the contact faces may cause concentrated stress and result in a totally different loading condition (Figure 3-3).

![Figure 3-3: concentrated stress caused by a small gap.](image)

Generally, the equilibrium state of an individual unit is achieved under two conditions:

\[
\begin{align*}
\sum F &= 0 \\
\sum M &= 0
\end{align*}
\]  

(3.2a)  

(3.2b)

As we have already stated, for a determinate state, Equation in (3.2a) could be easily solved through the equilibrium between the three force vectors and the gravity vector of the unit with a set of linear equations. However, as the direction of the force vectors depend on the inclination of the contact faces and may not pass the centroid of the unit, Equation (3.2b) is essential for preventing the unit from overturning. Frictional forces in
these circumstances may also contribute to the equilibrium when the compression forces cannot reach equilibrium thus cannot be ignored either. Sometimes frictions are even used as a main source in maintaining an equilibrium position, such as the research using frictional forces to maintain self-support for discrete-element assemblies (Frick, Mele, and Philippe Block 2015).

3.3.2 A detailed analysis of collapse state

While static friction is needed for the equilibrium described in Equation (3.2), the direction of the static friction forces are still unknown as the intended movement directions have not been determined. In this section, we will demonstrate a detailed analysis of the collapse state of one unit, and determine the direction of the static friction on certain contact faces.

We first implemented a similar concept of Non-Directional Blocking Graph (NDBG) in the assembly planning field to decide the motion space of a unit. The method was first mentioned in a doctoral dissertation (Wilson 1992). Unlike the NDBG where a sphere represents a total blocking of the movement, we use a sphere to represent a free object in space and a partition of the sphere to represent the free movement in certain directions. Figure 3-4 shows different circumstances when a unit is not blocked, partial blocked, or total blocked.

The tool shows that under our setup, if a newly added unit cannot maintain a stable state and collapses under the gravity load, the initial direction of movement is determined to fall into the partition of the sphere that corresponding to the motion space of the unit.
no matter what the following collapse mechanism is. Furthermore, it also shows that the minimum number of neighbouring units to achieve a full block is 4 instead of 6, as we described in Section 2.1.

![Figure 3-5: possible initial direction of movement in a collapse](image)

Two different state of a unit with three neighbouring units as well as its corresponding freedom graph is shown in Figure 3-5. The side view on the right shows the lowest movement directions of the two states that the unit could most possible to move under gravity load. As is known, a movement initiated by gravity must be along the direction which has a gravity component. Thus, if a unit is in the first state in Figure 3-5, there is no intersection between the movement sphere and the motion space under gravity. This means no collapse mechanism will happen in this state under Assumption 1. If the unit is in the second stage, the most possible direction of initial movement of the unit will be along the arrow shown in the figure, following the minimum energy principle. Thus, an equilibrium state can only be achieved when static friction provide enough forces in the opposite direction of the movement to counterbalance the gravity component.

A demonstration of the possible forces involved in maintaining the equilibrium state with static friction is shown in Figure 3-6. Notice that when frictional forces is involved, the compression force of one of the contact faces will automatically disappear because of the geometric relations.

Once the initial movement happens, different circumstance may happen to the following movements in the collapse mechanism. However, this is not related to the discussion
here as we only need to avoid the initial causes to prevent a collapse to happen.

3.3.3 Equilibrium computation

As we discussion in Section 3.3.2, the geometry properties of the TI system can already guarantee a equilibrium stage if the lowest movement direction of the unit is above the horizontal plane. Otherwise, static friction will be involved to keep the unit in equilibrium if possible, depending on how much static friction the system can provide. In this section, we will show how to decide whether static friction is involved and if involved, how to parametrically compute the minimum required friction coefficients of the contact faces using Equation (3.2).

Both Figure 3-4 and Figure 3-5 show that the lowest movement direction is actually the intersection of the two planes that trim the sphere of the motion space. It is obvious to see these planes are parallel to the two corresponding supporting contact faces. Thus, the inclination angle of the lowest movement direction is decided simultaneously by the inclination angle of the side face and the position of the unit in the space. A process of computing the direction need to be first implemented to decide if further equilibrium analysis is needed, as shown in Figure 3-7.

Once static friction is needed to maintain the equilibrium state of the unit, Equation (3.2) can be expended into (3.3), in which the number in the subscripts is based on the numbering of side faces shown in Figure 3-8. The bottom side face correlates with the number 0. Compare to Equation (3.4) which represents the circumstance with no friction
Figure 3-7: static friction decision making process.
involved, the added parts correlate with the newly involved frictional forces.

![Figure 3-8: numbering of side faces of a unit (top view)](image)

\[
\begin{aligned}
\sum_{i=0,1,5} (\mu_{\text{min}_i} n_i \cdot \mathbf{F}_i + \mathbf{F}_i) &= 0 \\
\sum_{i=0,1,5} (r_{\text{friction}_i} \times \mu_{\text{min}_i} \mathbf{F}_i + r_{\text{compression}_i} \times \mathbf{F}_i) &= 0
\end{aligned}
\] (3.3a) (3.3b)

\[
\begin{aligned}
\sum_{i=0,1,5} \mathbf{F}_i &= 0 \\
\sum_{i=0,1,5} r_{\text{compression}_i} \times \mathbf{F}_i &= 0
\end{aligned}
\] (3.4a) (3.4b)

In (3.3), coefficients $\mu_{\text{min}_i}$ is the required minimum static friction coefficient of each side face to prevent sliding and vector $n_i$ is the direction of the friction (the lowest movement direction) of each side face. Vector $r$ is the position vector (from any point in the space) corresponding to the frictional forces or compression forces, based on the subscripts. Notice that in reality, the coefficient $\mu_{\text{min}_i}$ is usually a constant number depend on the surface condition, both the geometry and the material, of the two contact surfaces. The direction of the static frictional forces will always be in the plane of the contact surfaces.

Notice that we have already known the directions of all the compression forces $\mathbf{F}_i$. We also know the directions of the frictional forces are against the potential direction of the movement, which is the intersection of the planes where side face 1 and 5 reside. For Equation (3.3), we have six unknowns (three force norms and three friction coefficients) and six equations so that we could easily get the result of the three $\mu_{\text{min}}$ and the norms of the three compression forces. The maximum $\mu_{\text{min}}$ is the one we need to achieve when deciding the surface condition of the contact faces in reality in order to prevent collapse.
during the assembly process. Notice that the result of (3.3) will contain one zero friction coefficient and one zero force because of the geometric relationship, as discussed in 3.3.2.

Figure 3-9 demonstrate a parameterized model of a single unit with three different typical solutions. These examples show how the position, inclination angle of the unit, and inclination angle of the side faces will affect the force distribution and friction coefficients.

3.4 Multiple-unit equilibrium analysis during assembly

This process need to be top-down or simultaneous that structural performance of the overall geometry need to be taken into consideration at the first stage. And few of these research really take advantage of the assembly properties of the TI systems.

Conventionally, The process of structural analysis of a given geometry is relative mature and have been implemented in many commercial FEA software on different platform, such as SAP2000, Abaqus, and Karamba on Grasshopper. However, this process shows one possible stress state in the material, but a linear elastic analysis does not say anything about the stability or collapse of the arch (P. Block and Ochsendorf 2007). The stability analysis of an assembled structure is relatively immature because of the discontinuity between assembled units generate difficulties in setting the boundary conditions for discrete elements and is beyond the capability of many FEA software. The Block Research Group provided one possible solution to similar problems by implementing TNA or force
density method to check the force distribution and displacements in every unit. This method bypasses the discontinuity problem through the continuity of forces which will always exist unless the structure collapse.

Compared to the process of finished-state structural stability analysis shown above, the structural stability analysis of an assembly process need an additional step to analyse the stability of the structure at every step of the assembly. This step includes: 1) the stability check of the newly added unit and 2) the stability check of the overall structures after adding the new units as the overall stability may be broken by the additional forces added by the new units even the newly added units may be in an equilibrium state at the defined position. While the whole structure may be stable in the final stage, it may be not able to stand without a supporting framework during the construction phase. Thus, this step is essential to understand how the structure will perform during the assembly process and find out the stability of different area of a structure in order to decide precisely if we need a framework, where to put framework, and how much framework is need.

In some type of structures, there is also additional sudden changes in the force distribution of a structure when finishing the structure or sub-structure(s) of the overall structure. However, as a non-general circumstance, certain case-by-case study is required for different specific construction sequence. For instance, if connecting span with two beams from both sides, the force distribution of the upper and lower surface may suddenly reverse the moment when two pieces are connected (Figure 3-10). Discussions about these special cases will not be included in this thesis.

For an assembly in general, an instant on-the-fly structural stability analysis work-
Figure 3-11: a simple work-flow of instant stability analysis
flow follows an iterative process until the equilibrium state cannot be achieved for certain unit. As we have explained in Section 3.2, the geometry of the basic unit we have chosen ensures a determinate analysis of the newly added units. However, besides those who are on the top rows, units are all constrained by at least four neighbouring units. Though this means we do not need to involve static friction in the equilibrium analysis, it seems not possible to compute the forces without compatibility equations.

Surprisingly, this issue can be solved by an equilibrium analysis of each individual units in the reverse sequence of the assembly process. The forces between any two contacted surfaces share the same magnitude and opposite directions. Once the units in the top rows are computed, the supporting forces and frictional forces will be updated to the corresponding neighbouring units. When the iteration arrives at the units surrounded by more than three faces, the forces coming from the side faces that are contacted with the units above has been computed already and only forces coming from the three lower side faces are unknown. Thus, the number of unknowns are still equivalent to the number of equilibrium equations and the forces can be computed. The iteration cycle will continue (Figure 3-12) until equilibrium of some unit cannot be achieved or the analysis finishes – a sign of equilibrium of the whole assembly. A general work-flow is demonstrated in Figure 3-11.

![Figure 3-12: iteratively updating forces for solving every force with determinate equations](image)

A physical assembly test using 3D printed model is shown in Figure 3-14. The cantilever achieved by utilising the TI property can be seen clearly from the side view of the model.
(a) overall stability achieved  (b) failure in overall stability

Figure 3-13: overall stability achieved/unachieved (image need revision)

Figure 3-14: 3D printed physical assembly
3.5 Data structure application: half-edge data structure in the equilibrium analysis of multiple units

3.5.1 Half-edge data structure formation

We’ve already shown the analysis process of calculating the force vector in every unit during the assembly process in Section 3.4, but a question remains that how could we find the corresponding supporting units under each newly added unit, especially when the basic tessellation changes (increase or decrease in number, rotation in position, etc.) thus the assembly sequence changes. Normal list or tree data structure that stores the units in certain sequence cannot satisfy the complex changes during the calculation as they cannot store the neighbouring information. Even with certain techniques to use additional lists or hashing table to store the neighbouring information, the stored information still relies on the assembly sequence thus when the assembly sequence changes while the assembly maintains the same, the neighbouring information will still change and need additional updates.

To solve this problem and create a sequence-free & topological-related-only information storage data structure, we implemented a customised version of the "Half-edge data structure", a data structure designed for manifold representation in CAD software (Weiler 1988). The implementation of this data structure with certain algorithms enables us to generate the neighbouring information based on the assembly topology, or to be more precise the corresponding tessellation of the TI system, at the beginning. This information can be used in later equilibrium analysis for different assembly sequences.

As shown in Figure 3-15 a small section of a half-edge representation, a half-edge is a half of an edge and is constructed by splitting an edge down its length. We’ll call the two half-edges that make up an edge a pair. Half-edges are directed and the two edges of a pair have opposite directions. The vertices define the two ends of an edge and thus the two different sequences of the two vertices represent the two different half-edge in a pair.

As can be seen in Figure 3-15, the hexagon surrounded by six sequential half-edges
which forms a circular list around its border are called the corresponding face of these half-edges. In our hexagon tessellation, each hexagon has six corresponding half-edges and the shared edge with one of the neighbouring hexagon are a pair of half-edges. This is where the power of half-edge data structure emerges – we can get to a unit’s neighbouring units from the shared edges between any of the neighbouring unit and the unit itself.

The vertices in the half-edge data structure stores their coordinates \((x, y, z)\) as well as a link to exactly one of the half-edges which uses the vertex as its starting/ending point. And the faces only need to store one of the half-edges that borders it. For access feasibility, an extra link to the main hexagon is stored in the class of vertices, half-edges, and faces.

Pseudo python codes of the class structure of the hexagon, vertex, half-edge, and face are provided below for a quick reference.

Listing 1: main hexagon class setup (pseudo code)

```python
class Hex:
    def __init__(self):
        self.vert = [a list of Hex_vert class object]
        self.edge = [a list of Hex_edge class object]
        self.face = class Hex_face
        self.bound = None
        self.exist_force = rg.Vector3f(0, 0, 0)
```
Listing 2: vertex class setup (pseudo code)

```python
class Hex_vert:
    def __init__(self):
        self.pt = pt
        self.x = pt.X
        self.y = pt.Y
        self.z = pt.Z
        self.edge = class Hex_edge
```

Listing 3: edge class setup (pseudo code)

```python
class Hex_edge:
    def __init__(self):
        self.hex = mainHex
        self.vert = class Hex_vert
        self.pair = class Hex_edge
        # paired edge in the neighbouring hexagon
        self.face = class Hex_face
        self.next = class Hex_edge
```

Listing 4: face class setup (pseudo code)

```python
class Hex_face:
    def __init__(self):
        self.hex = mainHex
        self.edge = class Hex_edge
```
3.5.2 Adjacent units’ information queries

With the data structure illustrated above, queries for most information about local and neighbouring units can be directly accessed through a series of class objects.

```
local_hex.vert_1 = local_hex.edge.vert_1
local_hex.face = local_hex.edge.face
neighbouring_hex.vert = local_hex.edge.pair.vert
neighbouring_hex.face = local_hex.edge.pair.face
```

An iteration over all the half-edges of a face can be conducted through a loop since these half-edges formed a circular linked list. And the iteration over all the half-edges that surrounds a vertex can be conducted similarly through paired edges:

```
edge = local_hex.face.edge

while (edge != local_hex.face.edge):
    edge = local_hex.edge.next
```

```
edge = local_hex.face.vert.edge

while (edge != local_hex.vert.edge):
    edge = local_hex.edge.pair.next
```
3.6 Curved surface topological interlocking

Very limited research has been conducted on the deformation of TI systems on 3D surfaces as in (Tessmann 2012). As we have already discussed in Section 2.2.3, a deformed grid in which some hexagons are deformed will result in non-planar contact face, thus bring in more complex contact conditions and fabrication difficulties.

In general, purely geometric mapping of a planar tessellation to a non-planar surface and keeping each polygon to be still planar is possible. However, in order to maintain the equilibrium of the overall structure, special attention should be paid to the inclination angle of the side faces, since the middle polygons of the units are no longer co-planar and the angles between these polygons may be different, the inclination angle of each side face in a unit as well as same side face in different unit may differ from each other.

![Figure 3-16: side face inclination in planar/non-planar mapping](image)

Locally, circumstances may happen that, the inclination of the side face may be too large that result in a sharp thin edge and reduce the strength of the material. Figure 3-16 shows a section view of a typical case that a sharper angle appears in order to achieve enough carrying ability of the upper unit, as well as the misalignment of the top/bottom surface between the two units.

Globally, certain mapping strategies should be chosen to create systematic pipeline of mapping a hexagon grid onto free surface while maintaining the structural properties, such as the one shown in Figure 3-17. However, it is not intuitive to modify the target mapping as the only way to modify it is through changing the original planar grid.

A relative better mapping is the dual polygon. Since hexagons and triangles are the dual polygon meshes to each other, it is possible to create a meshing strategy of the
target surface such that the dual mesh will fit the required border conditions. Though this method will eliminate the need of a conformal mapping, it need to be combined with certain optimisation process providing the objective functions for border conditions, etc, as shown in Figure 3-18. Besides, it is still not an intuitive process as we cannot locally control the meshing process.

Mathematically, including the first strategy above, most of the similar mapping strategies belong to the conformal mapping category. A modified panellization method using periodic conformal maps proposed by Rörig et al. (Rörig et al. 2014) is used here without discussing the mathematical details, which is not the aim of this thesis. The method bypasses the difficulty of finding the planar pattern that can give an ideal border condition (Figure 3-17) after mapping to the target surface by build a middle layer cone-shape mapping. By modifying the projection lines we are able to vary the density of different location while still using the same planar pattern (Figure 3-19). After achieving the
desired mapping pattern for the assemblies, we only need to run a simple optimisation to minimise the distance between the hexagons on the border and the border of the target geometry, using least-square method or just evolutinal method, as shown in Figure 3-20. The last step is to the target assemblies and the border frame based on this mapping.

![Figure 3-19: using the projection line to control the target mapping](image)

![Figure 3-20: minimising the distance between border hexagons and geometry borders](image)

Figure 3-19: using the projection line to control the target mapping

Figure 3-20: minimising the distance between border hexagons and geometry borders

Figure 3-21 shows some other mapping created by this mapping strategies. There are some questions left at this moment for planarization process before the algorithm can be applied since it requires inputs of planar hexagons. As the question we’re encountering here is a open-ended question and there are many potential algorithms and mapping
strategies out there depends on the condition of the surface. The discussion of these strategies are beyond the scope of this thesis and the exploration of pros and cons of different mapping methods will be left for the future work.

Figure 3-21: various mapping results

3.7 Beyond surface structure: 3D tessellation system

The TI systems we’ve touched as well as those in the previous research are actually all two-dimensional TI systems. Though we’ve extended the applications to three-dimensional spaces, the techniques we used are just surface mapping, through which we "deform" the space the structure exist in. In other words, we’re still dealing with TI systems that are based on 2D tessellations projected on 3D surfaces. Without scaling, the thickness of these TI systems is limited. Even though we can layer these TI systems to increase the thickness, there are interlocking mechanisms between the layers. We may intentionally invent interlocking mechanisms by modifying the upper and bottom surface of the geometry, the shape we achieved will not be the inherited geometric properties of the geometry, and there will be a lack of consistency between the interlocking mechanism within the layers and that between the layers.

However, there exist real three-dimensional tessellations which are polyhedra accumulated together to occupy a three-dimensional space. Unlike the layered TI systems
transformed from 2D tessellations, these 3D tessellations are TI systems by themselves and are called space-filling polyhedra or plesiohedra.

More types of 3D tessellations such as Andreini tessellation (a three-dimensional equivalent of a semiregular tessellation (Weisstein n.d.(c))), different variations of cubic honeycomb (Conway, Burgiel, and Goodman-Strauss 2016) (Figure 3-22) (i.e. rectified cubic honeycomb, truncated cubic honeycomb, cantellated cubic honeycomb), and 3D voronoi tessellations also exist (Figure 3-23), or even irregular polyhedra like Ruggero Gabbrielli’s space-filling polyhedra with 13 faces (Gabbrielli and O’Keeffe 2008), but are mathematically more complex and have few applications in architectural and structural fields due to the complexity both in theory and construction.

This thesis will not expend on these topics as they are beyond the scope of this thesis. References could be found with a search on the above mentioned terms.
Figure 3-23: a 3D voronoi tessellation example
4.1 Contributions

This thesis presented the topological interlocking assembly system with dynamic analysis of the equilibrium state, and features four major contributions:

- First, it computationally and systemically develops a method to construct TI system between two-dimensional tessellations and the TI geometries through various existing fragmented techniques. The method sets up a mathematical basis of TI generation through tessellation patterns described in Schläfli symbols after giving a thorough literature review of the fragmented theories to design different TI systems in the field. The systematic description of the designing method for different kinds of forms was first of its kind.

- Second, it developed a mapping mechanism to apply one of the tessellation patterns \{6, 3\} (Schläfli symbol) to 3D surface structures, and it develops the theoretical basis for how the assembly can be conducted without extensive framework to reduce the construction complexity, a difficulty appeared generally in large scale spatial structures.

- Third, it introduces a new method to study the equilibrium of topological interlocking surface structures. The method utilises the geometric properties of the
topological interlocking system where each unit interlocked geometrically by its neighbours, the overall structure theoretically eliminates the possibility of mechanism failure but only rely on the material strength. Thus, it reduces the need of extremely precision control of the structural geometry as a purely funicular shape.

- Fourth, it presented a sophisticated on-the-fly equilibrium analysis for assembly process involving frictional forces, in which an indeterminate analysis is conducted with determinate method through the utilisation of geometric relations of the assemblies.

Though the method can be applied to any surface theoretically, the freedom it provides compared with a strictly funicular shape will largely depend on the material properties of the construction material – tension capacity, surface friction, density, etc. The thesis aims to extend the geometric variety of surface structures through TI, featuring some freedom between the geometry and the force, the beauty and the order.

4.2 Future work

4.2.1 General Indeterminate topological interlocking system

The main TI system presented in this thesis is based on regular hexagon tessellations. Planar TI systems based on tessellations with more than one type of polygons are presented, but the structural performance of these systems is not analysed. This is because under our rigid body assumption, the dynamic equilibrium check for the assembly process on the presented system can only solve determinate states or simple indeterminate presented in this thesis. For more complex situations, elastic assumption and compatibility equations need to be involved for maintaining the equilibrium state.

4.2.2 Bending capacity analysis

Once elastic assumption is made, local deformation could be calculated for the unit materials. Thus, we can further decide the bending capacity of the TI system on targeted free
surface.

4.2.3 Construction optimisation and form-finding

The analysis method presented in this thesis are still using checking algorithms and cannot provide feedback for improving assembly sequence or modifying the form. Advanced algorithms are needed to provide design guide and close the design-analysis-feedback-redesign loop.

4.3 Potential Impact

Echoing what was proposed in Chapter 1, the discretization of a monolithic structure is one of the key roads to constructability from the past to the future. The corresponding jointing techniques require not only structural stability but also architectural aesthetics. The designing method and the corresponding analysis framework proposed and developed in this thesis provide an exciting extension to the field of topological interlocking system and structural assemblies. By bridging the two fields, this thesis emphasises on and rethinks the importance of geometric properties in structure performance and stability. It also provides a new angle to look at simultaneous design & analysis process, and proposes a new direction of structural assemblies without fastening materials for future researchers.
References


Dyskin, Arcady V., Elena Pasternak, and Yuri Estrin (2012). “Mortarless Structures Based on Topological Interlocking”. In: Frontiers of Structural and Civil Engineering nil.nil, nil.


Appendix A

Appendix

We will introduce related design and computation methods used in this thesis. The platform used for the implementation of these algorithms and codes are Rhinoceros v5.0, Grasshopper v1.0, and the ghPython component inside Grasshopper platform.

A.1 Various tessellation examples

Very limited tessellations are shown in Section 2.2.1. Below are some of the semi-regular and non-regular tessellations with relative more complex compositions.
Figure A-1: various type of tessellations
A.2 Typical deformation algorithms

In general, any deformation algorithm that can apply on point grids or line segments grids is applicable on the tessellations presented in this paper. The algorithm used to create the folding fan effect in Section 2.2.3 is shown as an example below:

Listing 8: adjacency queries (pseudo code)

```python
ig = input_grid
pln = plane

hex = []
output_grid = []

def gen_grid(crv):
    pts = []
    hex_pts = []
    # if open angle is not 120 degree, deform the grid.
    if open_degree != 120:
        dif_angle_ratio = open_degree/120
        for pt3d in crv:
            v_new = gf.subtraction(pt3d, pln.Origin)
            old_angle = gf.angle(v_new, pln.YAxis)
            cond_angle = gf.angle(v_new, pln.XAxis)
            if cond_angle < math.radians(90):
                rot_angle = old_angle * (1-dif_angle_ratio)
            else:
                rot_angle = - old_angle * (1-dif_angle_ratio)

                (math.cos(rot_angle), -math.sin(rot_angle), 0),
                (math.sin(rot_angle), math.cos(rot_angle), 0),
                (0, 0, 1))
```

```
```python
sum_vec = tran_matrix.Multiply(sum)
newpt = rg.Point3d(*gf.add(pln.Origin, list(sum_vec)))
pts.append(newpt)

return rg.PolylineCurve(pts)
else:
    return crv

for crv in ig:
    output_grid.append(gen_grid(crv))
```

Three required input fields are:

- **input grid**: the grid to apply on. Each cells of the grid need to be closed polylines.
- **open degree**: the desired open angle of the final grid. The range of this degree is 0~180.
- **plane**: the plane which the deformation would like to happen. The origin point of the plane will be the folding centre.