FREE SURFACE WAVES

ON

ELECTRICALLY CONDUCTING FLUIDS

by

CHARLES WESLEY ROOK, Jr.

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ABSTRACT

The propagation of free surface waves on an inviscid, incompressible, electrically conducting fluid in the presence of an applied magnetic field are analyzed. The volume motion equations are developed, and then are constrained by two classes of boundary conditions. One class leads to free surface gravity waves; the other to surface tension waves on a slab or column.

The bulk equations and boundary conditions are linearized for small perturbations of the free surface. Dispersion relations are obtained for wave motions on the free surface. The limiting forms of the dispersion equations for large and small electrical conductivity are obtained by a perturbation expansion in a magnetic Reynolds number based on wave phase velocity and wave length.

An infinite number of natural frequencies are found to be associated with a hydromagnetic free surface disturbance of a given wave length. In the limit as the fluid depth becomes infinite, a continuum of natural frequencies is required to describe the motion of the surface.

The effects of the magnetic field on hydrodynamic disturbances in the examples considered are (1) to damp oscillatory motion and (2) to retard the growth of instabilities.

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Title: Associate Professor of Electrical Engineering

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Mrs. Rita Albee typed the entire manuscript, correcting many of the suthor's spelling and grammatical errors. Mr. R. Ludeman prepared the sketches and graphs.

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Chapter I

Introduction.

1.1. Historical Background.

The surface wave motion of incompressible fluids is among the oldest and most carefully investigated topics in hydrodynamics. A measure of the importance of the subject and the quantity of work that has been done is the space devoted to this topic in the classic texts on hydrodynamics. Almost a third of Lamb[1] and nearly half of Basset[2] are devoted to surface motions. The modern text of Milne-Thompson[9] contains a similar proportion of material. The book by Stoker[6] on water waves is an excellent summary of more recent developments in the field, and a source of mathematical methods that have proved to be of value in such problems.

The interest in hydromagnetic surface waves, where the fluid is electrically conducting and is immersed in a magnetic field, is more recent; contributions have come from geophysicists, applied mathematicians, and engineers.

Investigations were made by Hide[11] of the closely related problem of the stability of a viscous fluid of variable density, and of the stability of an interface between two fluids, under the influence of gravity and a vertical magnetic field.

Subsequently, Roberts and Boardman[4] investigated the motion of the free surface of a viscous, incompressible fluid of uniform density under the influence of gravity and a vertical magnetic field.

Fraenkel[13] analysed the motion of a weakly conducting fluid, under the influence of gravity and a weak magnetic field, for disturbances whose wavelengths are large when compared with the depth of the fluid. Both linear and non-linear solutions were obtained. Lenn[14] later extended the work of Fraenkel[13] to cover a wider range of conductivity and field and conducted experiments with mercury as a working fluid.

Surface motions of a perfectly conducting fluid of infinite depth with the magnetic field normal or tangential to the free surface have been analysed by Chandrasekhar.[3]

Melcher[5] analysed the motion of the surface of a perfectly conducting fluid under the influence of an electric field normal to the surface and a magnetic field tangential to the surface. He shows a close mathematical relationship between the two problems, which he calls antiduality. He also performed experiments for the electric field problem.

Later, Sakurai[8] investigated a magnetic field problem similar to that treated by Melcher, with closely related results.

The stabilization of the surface tension instability of a fluid column by a magnetic field was theoretically investigated by Chandrasekhar.[3] No experimental work on this problem has been reported.

The theoretical analysis of surface waves on a current carrying column was originally motivated as a simple model to study the instabilities of the plasma pinch. The theory of such motions for perfectly conducting fluids is extensively developed and the literature is growing rapidly. A summary of the earlier basic work may be found in Rose and Clark,[15] and Chandrasekhar.[3] The motion of a current carrying fluid of finite electrical conductivity has been studied less extensively. The theoretical analyses are apparently motivated by an experiment on a current-carrying jet of mercury falling under gravity, performed by Dattner, Lenhert, and Lundquist.[16] The fact that the volume equations of motion are solvable in terms of Bessel functions was first pointed out by Taylor.[17] Taylor[17] and Murty[18] produced results in the limit of zero fluid conductivity.

1.2. Object and Scope of the Investigation.

The object of this investigation is to analyse in a unified manner the propagation of small surface disturbances of the free surface of an incompressible, electrically conducting fluid in the presence of an applied D.C. magnetic field. Two basic configurations are treated. The first is the motion of the fluid in a large reservoir or tank (the surface is considered infinite in extent) under the influence of gravity. The second is the motion of a fluid slab or column with gravity absent, where the fluid is confined by its surface tension in the absence of a magnetic field. A unified treatment is obtained by the derivation of volume motion equations that are common to both problems. The difference between the two situations is in the boundary conditions on the volume motion solutions. Emphasis is also given to volume motions which, by themselves, do not disturb the free surface, but are essential in providing a complete solution to the problem of an initial surface disturbance.

Primary attention is given to problems in which the conductivity, as characterized by an appropriate dimensionless parameter, may be considered small; however, some consideration of high conductivity situations is undertaken.

The scope of this investigation is restricted to small amplitude disturbances of the free surface of an inviscid, incompressible fluid of finite, scalar, uniform electrical conductivity. In addition, the applied magnetic field is considered time invariant and spatially uniform. Further, with the exception of Chapter 5, sections 5.3 and 5.4, the surface of the fluid is considered infinite in extent, and no time-average flow is considered.

Chapter II developes the bulk motion equations and boundary condition for the fluid. In Chapters III and IV these equations are applied to gravity wave motions on an infinite sea of fluid. In Chapter V, a simple problem on a finite fluid surface is examined.

Chapter VI treats the modification of surface tension waves on a planar slab by an applied magnetic field. Chapter VII considers the same problem, but on a circular column. In Chapter VIII a D.C. volume current density is introduced as a perturbation to the solutions of Chapter VII.

1.3. Physical Constants for Liquid Metal.

The principal motivation for this thesis is a theoretical description of the surface motions of liquid metals in laboratory magnetic fields. The analysis may have some application in geophysical or astrophysical situations.*
The purpose of this section is to give numerical values for the characteristic lengths and times that appear in the analysis, using the physical constants of mercury and sodium-potassium alloy (NaK).

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* The analysis of this thesis is applicable when the following conditions hold:

1) The fluid is incompressible and of uniform density.
2) The magnetic field is uniform in space.
3) The electrical conductivity is a scalar quantity.
4) Surface forces are representable by a scalar surface tension.

	Hq	NaK
Density (kg/m^3)	1.35×10^4	0.86×10^3
Conductivity (mho/m)	1.04×10^6	2.35×10^6
Viscosity (kg/m.-sec.)	1.55×10^{-3}	1.0×10^{-3}
Surface Tension(newton/m)	0.48	0.1

all data at 20^oC

Table 1.

Several combinations of the above quantities appear repeatedly throughout the analysis. The following Tables are intended to provide the reader with an idea of the magnitudes of these quantities which may be expected to occur in laboratory situations.

A characteristic decay time for free surface motions is

$$\frac{\rho}{\sigma B_o^2} = T.$$

This quantity appears in section 3.2 of Chapter III and section 4.2 of Chapter IV as a time constant for the decay of deep fluid gravity waves in the weak conductivity limit..

$B(weber/m^2)$	T(Hg)	T(NaK)
0.00	∞	∞
0.02	0.92 sec	32.0 sec
0.04	0.23	8.1
0.06	0.102	3.6
0.08	0.057	2.0
0.10	0.037	1.3

Table 2.

The magnetic diffusion time given by

$$\frac{\sigma\mu_o}{k^2} \quad \text{or} \quad \frac{\sigma\mu_o\lambda^2}{4\pi^2}$$

is a frequently recurring factor in the analysis. For example, in Chapter III, section 3.2, it is shown that, for the weak conductivity approximation to hold, the diffusion time based on the wavelength must be short compared with the period of the wave motion. Some characteristic values of diffusion time appear in Table 3.

	Diffusion Time	
Length	Hg	NaK
0.01 m.	3.3×10^{-6} sec.	7.5×10^{-6} sec.
0.05	8.3×10^{-5}	1.87×10^{-5}
0.10	3.3×10^{-4}	7.5×10^{-4}
0.20	1.32×10^{-3}	3.0×10^{-3}
1.00	3.3×10^{-2}	7.5×10^{-2}

Table 3.

In order for high conductivity approximations to hold, the fluid skin depth must be short compared with the fluid depth and the wavelength of the disturbance. Table 4 gives some values of skin depth at low frequencies in mercury and sodium-potassium.

$$d = (2\pi\sigma\mu_o f)^{-1/2}$$

Frequency	Skin Depth	
	Hg	NaK
1 c.p.s.	0.33 m	0.23 m
3	0.19	0.13
10	0.105	0.074
30	0.061	0.042
100	0.033	0.028

Table 4.

It is useful in interpreting the results of Chapters III and IV to know the radian frequency of gravity waves as a function of wavelength. A few values are found in Table 5.

$$\omega = \left(\frac{2\pi g}{\lambda}\right)^{1/2}$$

λ (meters)	ω $\frac{1}{\text{sec.}}$
0.05	35.10
0.10	24.82
0.20	17.55
0.40	12.41
0.80	8.77
1.00	7.85

Table 5.

As an example of the use of these brief Tables, consider the low conductivity approximation to the hydromagnetic gravity wave problem discussed in Chapter III, section 3.2. It is shown there that the approximation is valid if

$$\frac{\omega\sigma\mu_o}{k^2} < < 1$$

and

$$\frac{\sigma\mu_o v_A^2}{\omega} < < 1.$$

Consider mercury as a working fluid at a wavelength of 40 cm. Table 5 shows

$$\omega = 12.41.$$

Table 3 shows the diffusion time is

$$T = 1.8 \times 10^{-3} \text{ seconds}$$

and since

$$T = \frac{\sigma\mu_o}{k^2}$$

the first condition is satisfied. Table 2 shows the decay time at 0.1 webers/m^2 to be 1.3 seconds. The product of the decay time and the radian frequency is about 15. The second condition on the validity of the solution is that the inverse of this number be small compared with one. At 0.1 webers/m^2 this condition is marginally satisfied.

Chapter II.

Fluid Equations and Boundary Conditions.

The motion of a free surface of a fluid is determined by first solving the relevant equations of motion in the fluid bulk, and then applying the necessary boundary conditions. Both the equations of motion and the boundary conditions are nonlinear, and will be linearized considering the fluid motion to be a small perturbation of the equilibrium state.

To further simplify the equations of motion, the fluid is taken to be incompressible, invisicid, and of uniform density with a uniform scalar conductivity. It might be pointed out here that assuming the fluid to be inviscid is not necessary in order to obtain dispersion relations of the problems discussed below. An example of such an analysis may be found in the work of Roberts and Boardman.[4] The viscous losses in liquid metals are unimportant in comparison to electrical losses for laboratory scale experiments, and consequently the analysis may be simplified by neglecting viscosity.

The problem to be considered in this section may be described as follows. An incompressible fluid of infinite extent in the x-z plane has a free surface at $y=o$ and is bounded by a solid, flat bottom at the plane $y = -d$.

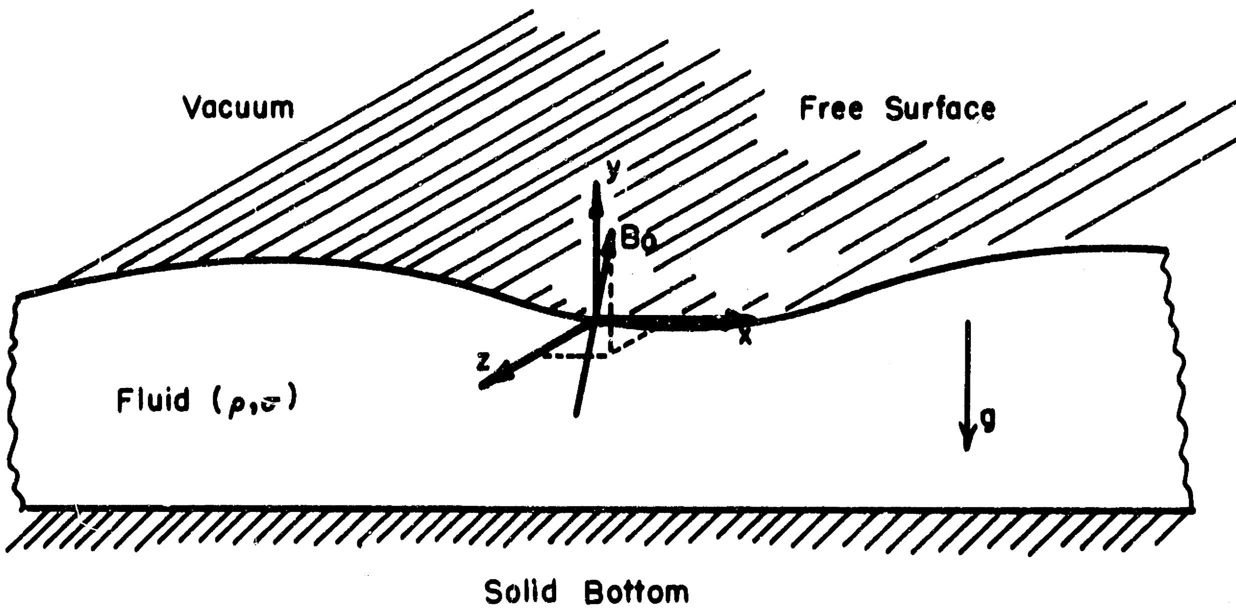

Figure 2.1. FREE SURFACE MHD GRAVITY WAVE GEOMETY

(In later sections the co-ordinate system may be shifted a bit for convenience).

A uniform magnetic field, supplied by outside sources, permeates the fluid. Arbitrary orientation of the field will be considered. Most attention will be paid to the special cases of vertical and horizontal fields. The acceleration of gravity acts along the y-axis in the direction of increasing negative y.

After examining in detail the motions of the infinite sea of fluid, we shall investigate certain simple bounded problems to which the infinite sea results may be directly applied.

2.1. Equations of Motion in the Fluid Region.

In this section, it will be shown that there are two distinct modes of motion in the fluid which are coupled only by the boundary conditions. In addition, it will be seen that we need consider only two-dimensional motion. For two-dimensional motion, it is shown that the component of the impressed magnetic field perpendicular to the plane of motion does not affect the motion. Finally, it will be shown that the impressed magnetic field in the plane of motion must be tangential to a surface in order for a surface current to be present in the high conductivity limit.

2.1.1. The Equations for Motion in the Fluid.

The necessary equations of motion are the inviscid

Navier-Stokes equation, the continuity of fluid equation in the incompressible limit, Maxwell's equations in the non-relativistic form applicable to moving media, and the constituant relation between the current and fields for a conductor in motion.

$$\rho \frac{D\bar{v}}{Dt} = - \nabla P + \bar{J} \times \bar{B} + \rho\bar{g} \tag{2.1}$$

$$\nabla \cdot \bar{v} = 0 \tag{2.2}$$

$$\nabla \times \bar{E} = - \frac{\partial}{\partial t} \bar{B} \tag{2.3}$$

$$\nabla \times \bar{B} = \mu_o \bar{J} \tag{2.4}$$

$$\nabla \cdot \bar{B} = 0 \tag{2.5}$$

$$\bar{J} = \sigma(\bar{E} + \bar{V} \times \bar{B}) \tag{2.6}$$

The solutions to some of the following problems will contain an apparent inconsistency. Note that nothing is said about divergence of E in the above equations. It is not properly part of the set of equations to be solved. In some cases, the electric field will have a non-zero, time varying divergence indicating a changing volume charge distribution. It is clear from Eq. 4, however, that the divergence of the current is zero. The resolution of this equation lies in the fact that the additional current needed to supply these changes is of relativistic order compared to the total current in the fluid. The approximation inherent in Eq. 4 allows errors of just this magnitude.

2.1.2. Equilibrium Solution.

The equilibrium situation, about which we will consider small perturbations, consists of the fluid at rest. The state is then specified by the solution to Eq. 2.1.

The steady state solution to Eq. 1 is

$$P_o = - \rho g y + \pi \qquad (7)$$

where π is a constant of integration to be determined later, when the geometry of the problem is fully specified.

2.1.3. Linearization.

Equations 2.1 and 2.6 are non-linear. The only field quantity present in the equilibrium situation is the magnetic field B_o, which we shall take to be uniform and, for the moment, of arbitrary orientation.

The equations are linearized in the manner usually employed in the analysis of water waves (see Stoker,[6] Chapter I).

The linearized equations which describe small perturbations of the rest state are

$$\rho \frac{\partial \overline{v}}{\partial t} = - \nabla p + \overline{j} \times \overline{B}_o \qquad (2.8)$$

$$\nabla \cdot \overline{v} = 0 \qquad (2.9)$$

$$\nabla \times \overline{e} = - \frac{\partial}{\partial t} \overline{b} \qquad (2.10)$$

$$\nabla \times \overline{b} = \mu_o \overline{j} \qquad (2.11)$$

$$\nabla \cdot \overline{b} = 0 \qquad (2.12)$$

$$\overline{j} = \sigma(\overline{e} + \overline{v} \times \overline{B}_o) \qquad (2.13)$$

All equilibrium quantities are indicated by a subscript zero.

2.1.4. Assumed Form of the Solution.

All quantities in the perturbation solution are assumed to have the following dependence on space and time

$$f(y)e^{j(k_1x + k_2z - \omega t)} .$$

The propagation constants k_1 and k_2 are assumed real, hence ω must be complex, as dissipative terms are present in the motion equation.

Since the equations which will be considered are linear, solutions may be superposed. This leads to great simplifications in the manner of solution as we may, without loss of generality, consider two-dimensional motion of the fluid.

In particular, the fluid will be assumed to move in the x-y plane, and k_2 will be zero. It will be seen that the perturbation magnetic field vector will lie in the x-y plane while the electric field and current will lie along the z-axis.

It is important to note that there is another form of two dimensional motion which we shall not treat here. The other form has only a z directed velocity, with current and electric field in the x-y plane. As it does not perturb the free surface, we shall not consider it further here.

2.1.5. Introduction of Vector Potentials.

The linearized equations of motion are considerably

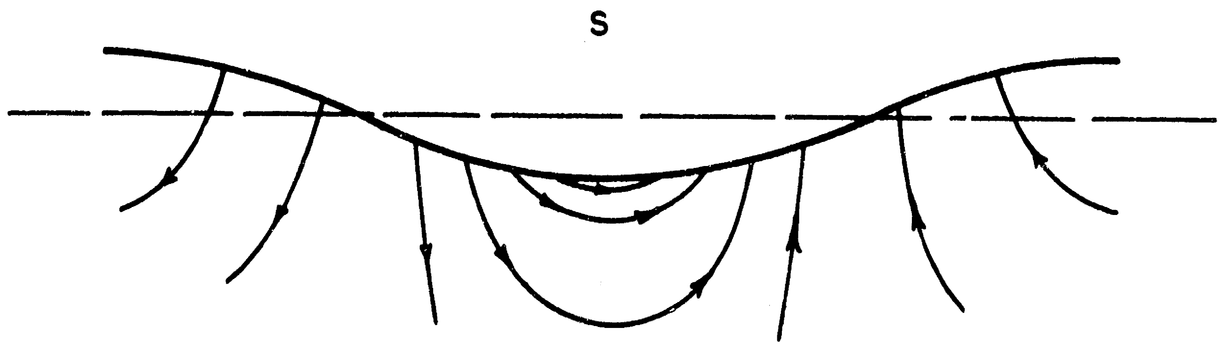

Figure 2.2a. FORM OF THE MAGNETIC AND VELOCITY FIELDS

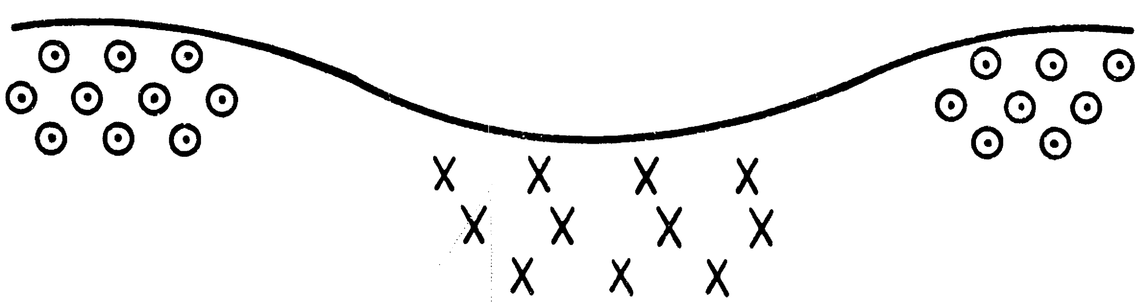

Figure 2.2b. FORM OF THE ELECTRIC AND CURRENT FIELDS

simplified by the introduction of vector potentials for the velocity and magnetic field

$$\nabla \times \overline{\psi} = \overline{v} \quad (2.14)$$

$$\nabla \cdot \overline{\psi} = \overline{0} \quad (2.15)$$

$$\nabla \times \overline{A} = \overline{b} \quad (2.16)$$

$$\nabla \cdot \overline{A} = \overline{0} \quad (2.17)$$

The use of vector potentials automatically insures the divergence conditions on the magnetic and velocity fields will be satisfied.

Since the motion is confined to the x-y plane, $\overline{\psi}$ has only a z component. We shall show that $\overline{A}$ also has only a z component.

Let the impressed magnetic field be specified by

$$\overline{B}_o = B_o(\alpha\overline{i}_x + \beta\overline{i}_y + \gamma\overline{i}_z) \quad (2.18)$$

Substitution of the vector potential into Eq. 2.11 yields

$$\overline{j} = -\frac{1}{\mu_o} \nabla^2\overline{A} \quad (2.19)$$

Equation 2.8 becomes

$$\frac{\partial}{\partial t} (\nabla \times \overline{\psi}) = -\nabla p + \overline{j} \times \overline{B}_o \quad (2.20)$$

Taking the curl of each quantity in Eq. 2.20 eliminates the pressure

$$-\frac{\partial}{\partial t} \nabla^2\overline{\psi} = \nabla \times (\overline{j} \times \overline{B}_o) \quad (2.21)$$

And, since B_o is independent of spatial co-ordinates,

$$- \frac{\partial}{\partial t} \nabla^2 \bar{\psi} = \frac{1}{\rho} (\bar{B}_o \cdot \nabla) \bar{j} \tag{2.22}$$

An examination of Eqs. 2.22 and 2.19 shows that, since $\bar{\psi}$ has only a z component, $\bar{A}$ has only a z component.

For the electric field, we have, as a consequence of Eq. 2.10

$$\bar{e} = - \frac{\partial \phi}{\partial x} \bar{i}_x - \frac{\partial \phi}{\partial y} \bar{i}_y - \frac{\partial A}{\partial t} \bar{i}_z \tag{2.23}$$

where ϕ is a scalar potential which we will evaluate using the linearized constituant relation, Eq. 2.13. This equation now takes the form.

$$- \nabla^2 \bar{A} = \sigma\mu_o \bar{e} + \sigma\mu_o [(\nabla \times \bar{\psi}) \times \bar{B}_o]. \tag{2.24}$$

In the cartesian co-ordinates being employed, the vector Eq. 2.24 represents the following three scalar equations

$$0 = \sigma\mu_o \left[\frac{\partial \phi}{\partial x} + \gamma B_o \frac{\partial \psi}{\partial x}\right] \tag{2.25a}$$

$$0 = \sigma\mu_o \left[\frac{\partial \phi}{\partial y} + \gamma B_o \frac{\partial \psi}{\partial y}\right] \tag{2.25b}$$

$$\nabla^2 A - \sigma\mu_o \frac{\partial A}{\partial t} = - \sigma\mu_o B_o \left[\alpha \frac{\partial \psi}{\partial x} + \beta \frac{\partial \psi}{\partial y}\right] \tag{2.25c}$$

Equation 2.25c is one of the pair of equations which determine the bulk motion of the fluid. The other is obtained by combining Eq. 2.22 with Eq. 2.19.

$$+ \frac{\partial}{\partial t} \nabla^2 \psi = + \frac{B_o k}{\rho\mu_o} \left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} \right) A \qquad (2.26)$$

In the above equations, the symbols A and ψ, without the vector overstroke, indicate the z components of the vectors A and ψ.

2.1.6. The Component of B_o Perpendicular to the Plane of Motion.

The equations of motion, Eq. 2.25c and Eq. 2.26, do not contain a dependence on the z component of the applied magnetic field. Therefore, we may safely ignore it in the succeeding sections. It is interesting to note just what effect this component does have. We see from Eq. 2.25a and Eq. 2.25b that it causes a charge distribution in the fluid which creates an electrostatic potential ϕ

$$\phi = - \gamma B_o \psi. \qquad (2.27)$$

2.1.7. Solution of the Bulk Motion Equations.

In order to solve the bulk motion equations, we introduce the assumed form of the solution

$$f(y)e^{j(kx-\omega t)}$$

Henceforth, in this section, the subscript notation for derivatives will be used. The equations to be solved are

$$A_{(xx+yy)t} - \sigma\mu_o A_t = - \sigma\mu_o B_o [\alpha\psi_x + \beta\psi_y] \qquad (2.28)$$

$$\psi_{(xx+yy)t} = \frac{B_o}{\rho\mu_o} \quad jk\alpha(A_{yy} - k^2 A) + \beta(A_{yyy} - k^2 A) \qquad (2.29)$$

With the assumed x and t dependence, these equations become

$$A_{yy} - k^2 A + j\omega\sigma\mu_o A = - \sigma\mu_o B_o [jk\alpha\psi + \beta\psi_y] \qquad (2.30)$$

$$- j\omega[\psi_{yy} - k^2\psi] = \frac{B_o}{\rho\mu_o} [jk\alpha(A_{yy} - k^2 A) + \beta(A_{yyy} - k^2 A_y) \qquad (2.31)$$

Simple analysis shows that for this pair of coupled ordinary differential equations there are four independent solutions. By inspection, we see that potential motion provides two of the required solutions.

(1) $\psi = \psi_1 e^{j(kx-\omega t)}\ e^{ky}$

$$A = - \frac{B_o k}{\omega} (\alpha - j\beta)\psi$$

(2) $\psi = \psi_2 e^{j(kx-\omega t)}\ e^{-ky}$

$$A = - \frac{B_o}{\omega} (\alpha + j\beta)\psi$$

ψ_1 and ψ_2 are constants to be determined later by the boundary conditions. To proceed, we may reduce differential Eq. 29 to the form

$$- j\omega\psi = \frac{B_o}{\rho\mu_o} [jk\alpha A + \beta A y] \qquad (2.32)$$

This is equivalent to factoring the differential operator and removing that part of it for which we have the solution. As Eqs. 2.30 and 2.32 are linear with constant co-efficients, their solutions will be exponentials.

Let

$$A_y = qA.$$

Then we find q must satisfy the relationship

$$q^2 - k^2 + j\omega\sigma\mu_o \left[1 - M^2 \left(\alpha - j \frac{\beta q}{k} \right)^2 \right] \tag{2.33}$$

where

$$M^2 = \frac{k^2 v_A^2}{\omega^2} \tag{2.34}$$

and

$$v_A^2 = \frac{B_o^2}{\rho\mu_o} \tag{2.35}$$

Then q is given by the expression

$$\frac{q}{k} = \frac{- jR_M M^2 \alpha\beta \pm \sqrt{(1 - jR_M)(1 - jR_M M^2 \beta^2) + jR_M M^2 \alpha^2 (1 + 2jR_M M^2 \beta^2)}}{(1 + jR_M M^2 \beta^2)} \tag{2.36}$$

where we have defined the parameter R_M to be the magnetic Reynolds number based on phase velocity and wavelength

$$R_M = \frac{\omega\sigma\mu_o}{k^2}$$

As there are two values of q defined by Eq. 2.36, the remaining two solutions have been found. They are

3,4) $\psi = \psi_{3,4} \, e^{qy} \, e^{j(kx-\omega t)}$

$$A = - \frac{kB_o}{\omega} \frac{1}{\left(M^2 \alpha - \frac{j\beta q}{k} \right)} .$$

Upon examination of the foregoing relationships, the following facts become evident. First, there is no volume current density associated with the two potential flow solutions, as both the velocity field and the magnetic field are curl free. Second, the relationship for the transverse propagation constant q becomes much simplified when the impressed field is either purely horizontal or purely vertical.

Horizontal Case ($\alpha = 1$, $\beta = 0$)

$$\left(\frac{q}{k}\right)^2 = 1 - j\,\frac{\omega\sigma_o}{k^2}\left[1 - \frac{k^2 v_A^2}{\omega^2}\right] \tag{2.37}$$

Vertical Case ($\alpha = 0$, $\beta = 2$)

$$\left(\frac{q}{k}\right)^2 = \frac{1 - j\,\dfrac{\omega\sigma\mu_o}{k^2}}{1 + j\sigma\mu_o\,\dfrac{v_A^2}{\omega}} \tag{2.38}$$

2.1.8. Formation of Surface Currents and the High Conductivity Form of the Solution.

As noted above, the potential flow solution does not depend on the fluid conductivity, so we may concern ourselves with the second, current carrying solution.

Let us first examine the possibility of a surface current layer. For a layer of this type to form, q must grow as the square root of the conductivity for large conductivity. Notice that this occurs in Eq. 2.37 but not in Eq. 2.38. In fact, direct examination of the general expression for q shows that the surface current layer does not exist for any non-zero value of B, since, as the conductivity approaches infinity in Eq. 2.36, q approaches a finite limit.

$$\lim_{\sigma \to \infty} \frac{q}{k} = \pm \frac{j \sqrt{1 - 2M\alpha^2} - M\alpha}{M\beta} \qquad (2.39a)$$

Therefore, the surface current layer is then properly only a feature of those situations for which B_o is purely horizontal.

When the field is purely normal ($\alpha = 0$) Eq. 2.39a is even further simplified

$$\lim_{\sigma \to \infty} q = \pm j \frac{\omega}{v_A}$$

which is the expected propagation constant for Alfren waves along the field lines.

We might then expect that Eq. 2.37 is the equation for the propagation constant of Alfren waves in a lossy media. It is, and a detailed discussion of its behavior as a function of real ω may be found in a recent paper by Kliman.[7] Unfortunately, as we shall be interested in behavior for real k and complex ω, the interpretation offered there has limited application to the problem at hand.

Further discussion of the high conductivity behavior of Eq. 2.37 is not profitable since, as will be shown later, these surface disturbances are not governed by a simple dispersion relation between k and ω. The high conductivity behavior of Eq. 2.36 is of interest, however. The approximate form for q is

$$q = j\omega\sigma\mu_o \left[1 - \frac{k^2 v_A^2}{\omega^2}\right]^{1/2} . \qquad (2.39b)$$

This is very nearly the expression obtained for the propagation constant in the classical skin depth problem. The modifying term in brackets is a result of fluid motion in the skin depth layer. We may easily see its effect. It is known (see Melcher)[5] that the propagation constant for gravity waves on a perfectly conducting fluid of infinite depth is given by

$$\omega^2 = gk + 2k^2 v_A^2 .$$

Therefore

$$0 < \frac{k^2 v_A^2}{\omega^2} < \frac{1}{2} . \qquad (2.40)$$

Thus the skin depth may be up to 1.4 times as great as that in a solid of the same conductivity. Hence, the first order loss may be as little as half that which would be estimated on the basis of the standard skin depth formula.

2.1.9. Low Conductivity Limit.

As the fluid conductivity approaches zero, the fluid motion must closely approach the ordinary hydrodynamic behavior. Mathematically, this will be seen as

$$\lim_{\sigma \to 0} \left|\frac{q}{k}\right| = 1. \tag{2.41}$$

When the conductivity is small, but non-zero

$$\frac{q_+}{k} = 1 - \frac{jR_M}{2} \left[1 - M^2(\alpha^2 + 2\alpha\beta - \beta^2)\right] \tag{2.42}$$

$$\frac{q_-}{k} = -1 + \frac{jR_M}{2} \quad 1 - M^2(\alpha^2 - 2\alpha\beta - \beta^2) \tag{2.43}$$

We see that, as expected, the solutions differ from irrotational ones by a small amount linearly dependent on the conductivity.

2.2. Boundary Conditions.

In this section, boundary conditions at the free surface and at a solid bottom will be considered. Discussion of end walls will be deferred until Chapter 5, when bounded motions are discussed.

2.2.1. Bottoms.

When the fluid depth is not so great that the motion at the surface is independent of the bottom, it is necessary to take into account the electromagnetic and hydrodynamic boundary conditions at the bottom. When the depth is much greater than the wavelength it is only necessary to sepcify that the motion die out with increasing negative y. This is not possible when waves may propagate along the y axis.

When the bottom is important, the following conditions apply.

1) The vertical velocity is zero.

2) The magnetic field is continuous across the boundary.

When the bottom is perfectly conducting, only the normal field need be continuous and consequently zero. For simplicity, perfectly conducting or non-conducting bottoms will be treated in the work to follow.

2.2.2. The Free Surface.

The boundary conditions applicable at the free surface may be briefly stated as:

1) The free surface is always composed of the same fluid particles.

2) The magnetic field is continuous at the boundary.

3) The discontinuity in the stress tensor is balanced by the surface forces.

The boundary conditions are all non-linear and must be linearized.

Let

$$F(x,y,t) = 0 \tag{2.44}$$

be the equation of a free surface. The mathematical statement of the first condition is then

$$\frac{D}{Dt} F \bigg|_{F=0} = 0. \qquad (2.45)$$

For this problem, F is conveniently written as

$$F = y - \eta(x,t) \qquad (2.46)$$

Equation 2.45 becomes

$$v \cdot \nabla (y - \eta(x,t)) \bigg|_{F=0} - \eta_t = 0. \qquad (2.47)$$

The linearized form is then

$$\bar{v} \cdot \bar{i}_y = \eta_t. \qquad (2.48)$$

The second boundary condition is easily linearized. Since B_o is automatically continuous, we need consider only the perturbation magnetic field, and this becomes a linear boundary condition when we refer the continuity to the equilibrium boundary.

The final boundary condition concerns the fluid stress tensor. The magnetic terms are continuous across the boundary by virtue of the previous condition. The remaining off-diagonal terms are zero for an inviscid fluid. The balance to be effected then is between the pressure discontinuity and the surface forces, which will be caused by the surface tension.

The surface tension produces a surface force density which is inversely proportional to the radius of curvature and directed toward the center of curvature. The linearized relationship is

$$F_s = T\eta_{xx}. \tag{2.49}$$

For the pressure at the free surface, in linearized form, it is necessary to sum the perturbation pressure of the equilibrium boundary with the equilibrium pressure at the perturbed boundary. The final equation is

$$T\eta_{xx} + p - \rho g\eta = 0. \tag{2.50}$$

More care must be taken with the surface discontinuity of the stress tensor when one considers ideal fluids When the fluid conductivity is infinite, the condition on the tangential magnetic field is no longer applicable, and the condition on the tangential portion of the total stress tensor must be invoked.

2.3. Separate Excitation of the Modes of Fluid Motion.

We have seen that the bulk motion consists of two modes of motion, which are independent in the fluid bulk, but will in general, be coupled at the fluid boundaries. As the resultant combined motion of the fluid is somewhat complicated, it is useful to consider rather artificial boundary conditions which allow the volume modes to be excited separately.

The independent excitation of the two modes may be accomplished by removal of a boundary condition at the free surface, namely the condition that the magnetic field be continuous.

One may picture this being accomplished by placing a thin, weightless, perfectly flexible sheet with perfect electrical conductivity over the surface of the fluid. This results in the possibility that a surface current may exist on the fluid, hence the component of the magnetic field which is tangential to the free surface need not be continuous. For simplicity, the depth of the fluid will be taken as infinite and surface tension will be neglected.

2.3.1. The Irrotational Mode.

Let us first take up the motion of the irrotational mode. As we have previously shown, there is no volume current flow associated with this mode. Therefore, since our imposed artificial boundary is lossless, and the fluid motion is also lossless, one may expect that associated with each real value of the propagation constant k will be associated a real value of the square of the radian frequency ω. The situation under examination is clearly stable, hence ω will be a purely real number.

The case for the magnetic field parallel to the equilibrium free surface is now quite straightforward. When the field is normal to the surface, the force on the conducting skin we have placed over the fluid is tangential to the fluid surface and we must introduce a further artifice, namely a constraint that the skin be constrained to move only in a vertical direction and hence not buckle.

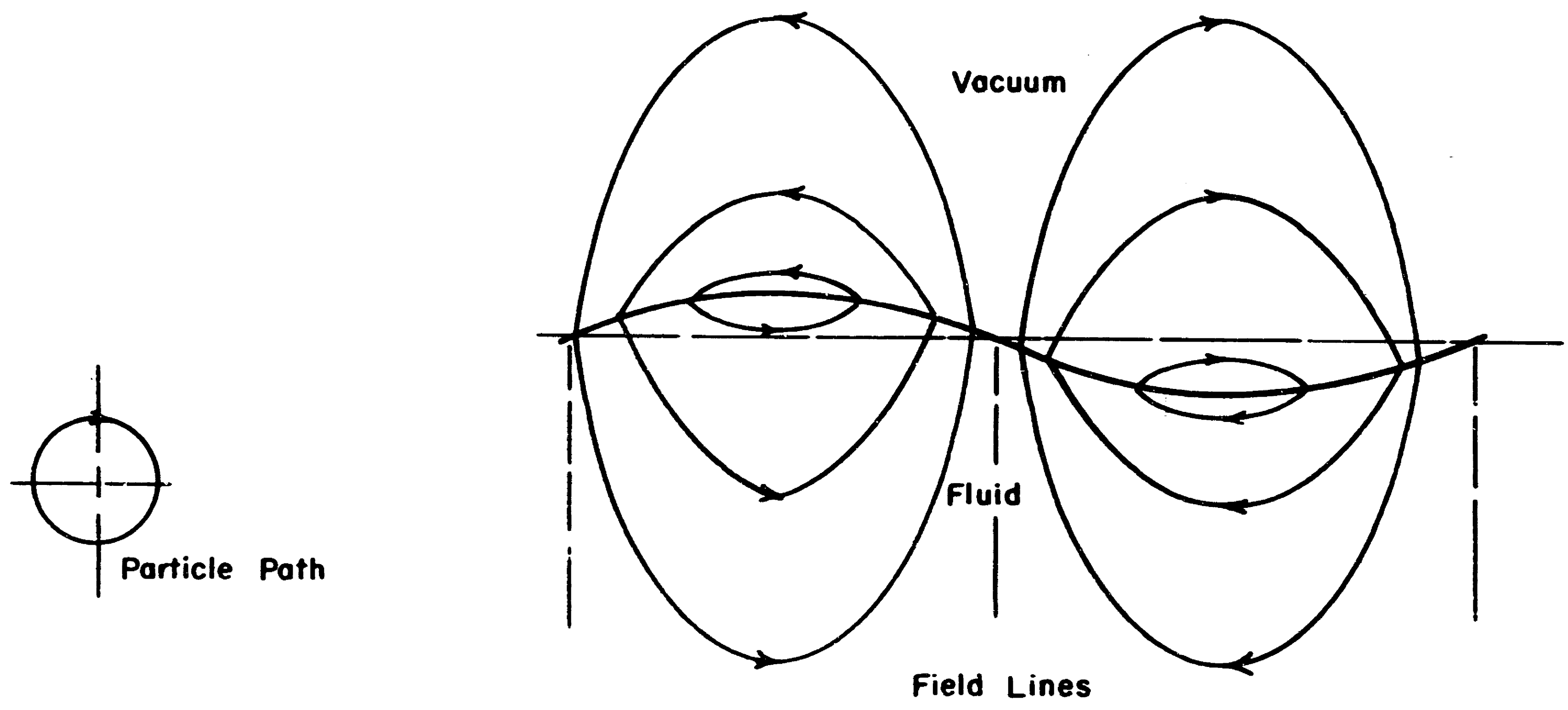

Figure 2.3. PARTICLE PATHS AND FIELD LINES LOSSLESS MODE

Now the form of the two stream functions in the fluid is the same for both orientations of the magnetic field,

$$\psi = \psi_1 \, e^{k y} \, e^{j(kx-\omega t)} \tag{2.51}$$

$$A = -\frac{kB_o}{\omega} \psi. \tag{2.52}$$

The fluid motion is that of ordinary hydrodynamic waves in which the motion of the fluid consists of circular partical paths in the x-y plane, the radius of these paths having an exponential decay with distance in the x-y plane.

There is no current in the bulk of the fluid in this mode, consequently the motional induction in the fluid must be exactly counteracted by the time rate of change of the magnetic field. The source of the magnetic field must be a surface current layer in the conducting skin.

Applying the pressure boundary condition at the free surface, we find

$$\omega^2 = g\,k + 2k^2 v_A^2 \tag{2.53}$$

for the applied magnetic field tangential to the free surface and

$$\omega^2 = g\,k \tag{2.54}$$

for the field normal to the surface. Figure 2.3 shows partical paths and magnetic field lines for a traveling wave of this type.

2.3.2. The Lossy Mode.

The other mode is somewhat more complicated. The stream functions are

$$\psi = \psi_2 e^{qy}\ e^{j(kx-\omega t)} \tag{2.55}$$

$$A = -\frac{kB_o}{\omega}\ \frac{\omega^2}{k^2 v_A^2}\ . \tag{2.56}$$

Because of the loss mechanism, the particle paths spiral inward. They may conveniently be pictured as ellipses whose size is decreasing in time at an exponential rate determined by the imaginary part of ω.

Figure 2.4 shows particle path ellipses and magnetic field contours for such a mode when the applied field is parallel to the surface. Figure 2.5 indicates motion with a vertical applied field. Plots are shown for low conductivity where the motion is a slight modification of irrotational flow, and for a higher conductivity, where the Alfren structure is more developed.

Like a irrotational mode, the lossy rotational mode, when excited in this way, has a characteristic radian frequency ω associated with each k. For the applied field parallel to the free surface

$$gk - \omega^2 \left[1 + 2\left(1 - j\,\frac{\omega\sigma\mu_o}{k^2} + \frac{j\sigma\mu_o v_A^2}{\omega}\right)^{1/2}\right] \tag{2.57}$$

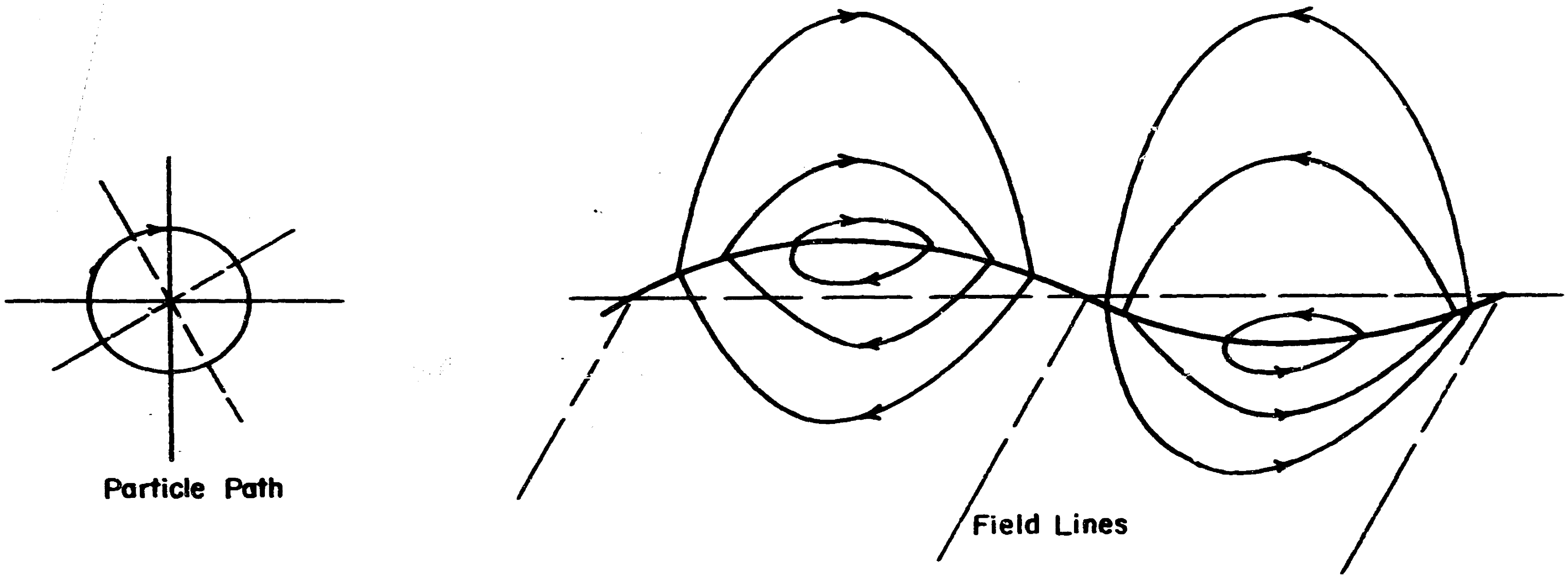

Figure 2.4. PARTICLE PATH AND FIELD LINES LOSSY MODE – B TANGENTIAL

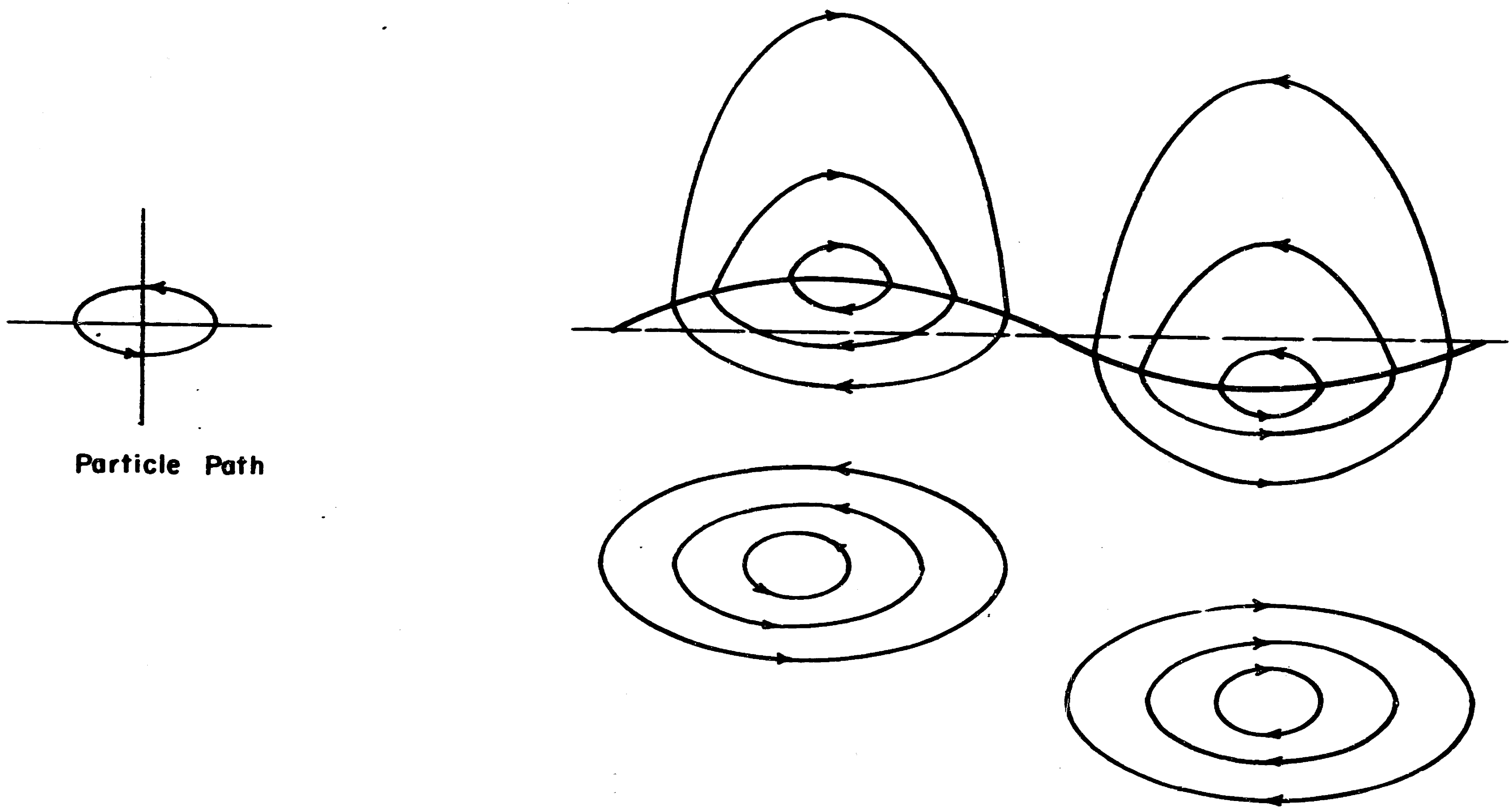

Figure 2.5. PARTICLE PATH AND FIELD LINES
LOSSY MODE – B VERTICAL – HIGH CONDUCTIVITY

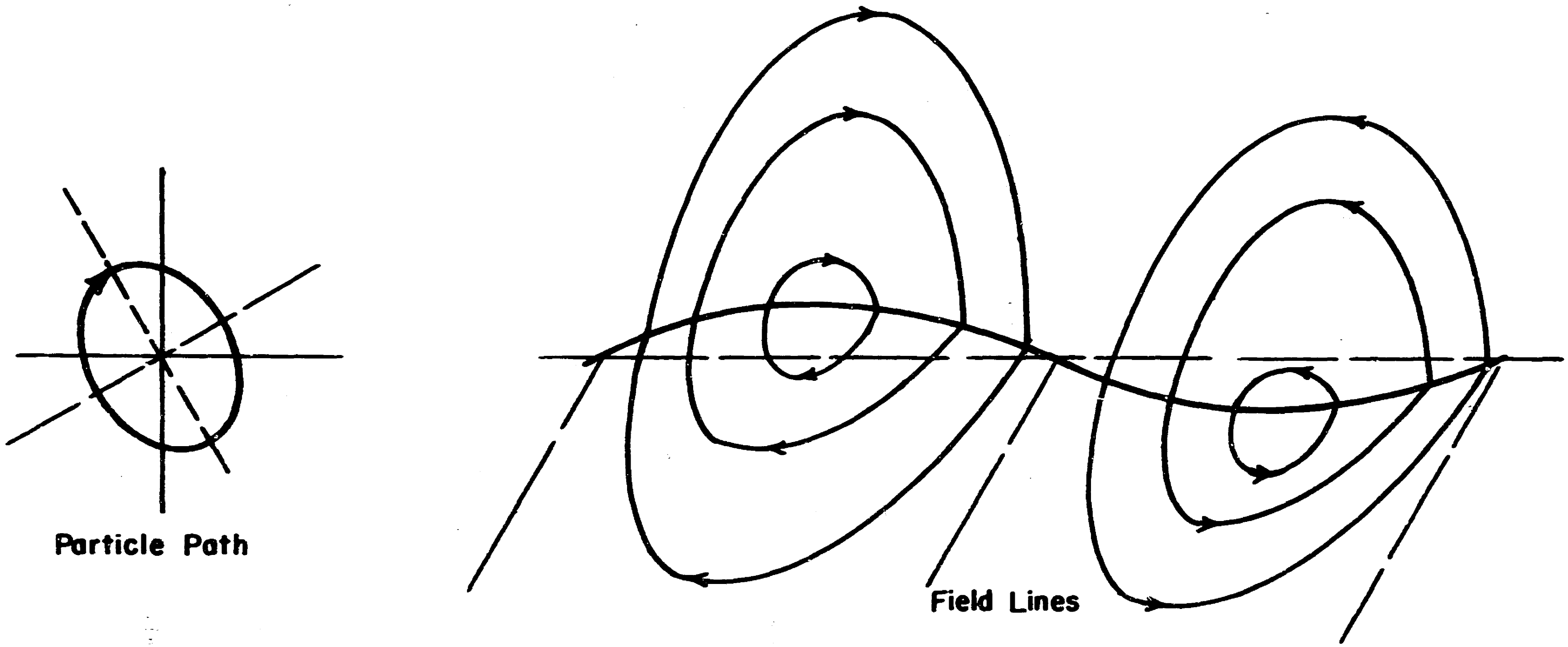

Figure 2.5 . PARTICLE PATHS AND FIELD LINES
LOSSY MODE — B VERTICAL — LOW CONDUCTIVITY

and for the field perpendicular to the surface

$$gk - \omega^2 \left[\frac{1 + \frac{j\sigma\mu_o v_A^2}{\omega}}{1 - \frac{j\omega\sigma\mu_o}{k^2}} \right]^{1/2} = 0. \tag{2.58}$$

2.3.3. Removal of the Artificial Surface Constant.

As a method for making a transition from this set of boundary conditions to a more natural set with no conducting skin on the surface, let us consider driving the two modes with an externally applied surface pressure. Let the applied surface pressure to the irrotational mode be

$$P_1(x,t) = P_o \; e^{j(kx-\omega t)} \tag{2.59}$$

where ω may be complex. Let us drive the dissipative mode with

$$P_2(x,t) = - P_o \; e^{j(kx-\omega t)} \tag{2.60}$$

Now, if it is possible to find some complex ω such that the conducting sheet over the irrotational mode carries a surface current just opposite that which is found in the sheet on the dissipative mode, we may sum the two solutions to obtain the solution for a free surface deep fluid wave without conducting sheets or driving pressures.

This leads to a convenient picture of energy being transported from the lossless irrotational mode to the lossy mode by virtue of the coupling required by the boundary conditions.

Chapter III

Gravity Wave Motion in a Horizontal Magnetic Field.

In the preceeding chapter, it was seen that the volume motion of the fluid was representable by two independent modes, coupled only by the boundaries. In the succeeding sections, the above general considerations will be applied to situations such as that of Fig. 3.1.

Two dimensional motion of the fluid is considered. The magnetic field lies in the plane of motion and parallel to the free surface. The fluid has a surface tension. The lower boundary of the fluid is a solid bottom of arbitrary electrical conductivity.

The simplest situation, namely the case where the bottom has been removed to infinity, will be considered first.

3.1. Wave Motion on a Fluid of Infinite Depth.

The removal of the lower boundary of the fluid to infinity makes the free surface motion of the fluid independent of the nature of the bottom. This results in a considerable simplification of the dispersion relation for traveling and standing waves but introduces some additional difficulties. In Section 3.3, it will be seen that, for a sinusoidal surface disturbance, there is a continuum of natural frequencies associated with other fluid processes in addition to a natural frequency describing surface wave motion. In the present section, only surface wave motion will be treated and the continuum will be ignored.

3.1.1. Solutions to the Bulk Motion Equations.

The geometry under consideration is shown in Fig. 3.1. A fluid of electrical conductivity σ and density ρ lies below the plane $y = 0$, which is the equilibrium free surface. A magnetic field B_o, caused by a current distribution outside the fluid lies along the x axis.

The bulk motion equations are formulated and solved in section 2.1.7. The required solutions are those which vanish at $y = -\infty$.

$$\psi = e^{j(kx-\omega t)} \left[\psi_1 e^{|k| y} + \psi_2 e^{qy}\right] \qquad (3.1)$$

$$A = -\frac{kB_o}{\omega} \left[\psi_1 e^{|k| y} + M_A^2 \psi_2 e^{qy}\right] \qquad (3.2)$$

where

$$M_A^2 = \frac{\omega^2}{\omega_A^2} \qquad (3.3)$$

$$\omega_A^2 = k^2 \frac{B_o^2}{\rho\mu_o} \qquad (3.4)$$

$$q = |k| \left[1 - j \frac{\omega\sigma\mu_o}{k^2} \left(1 - \frac{\omega_A^2}{\omega^2}\right)\right]^{1/2} \qquad (3.5)$$

and in Eq. 3.5, the root with the positive real part is intended. The above expressions for ψ and A are those solutions

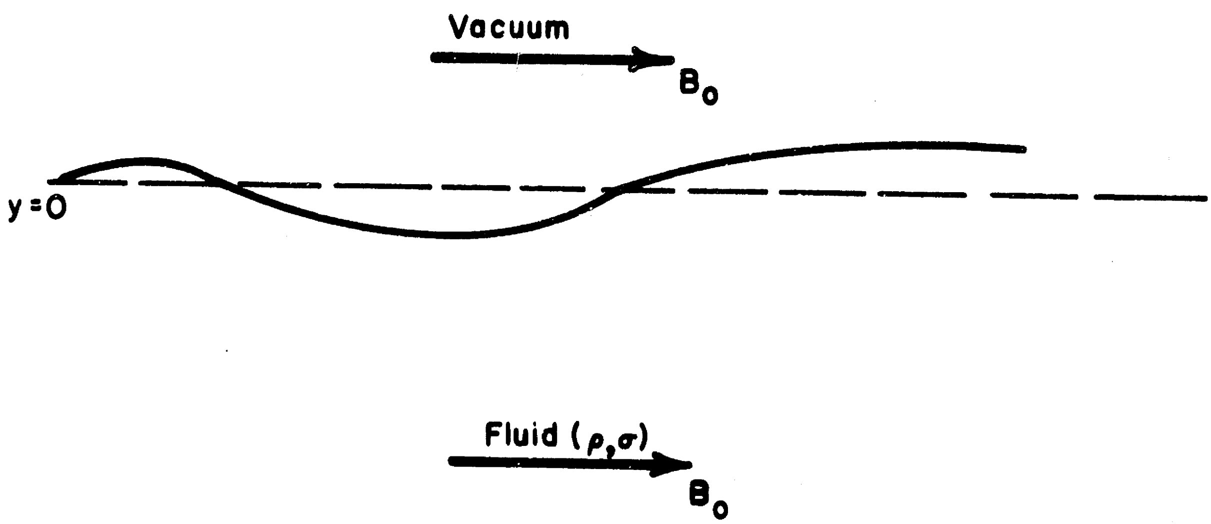

Figure 3.1. GEOMETRY OF SURFACE WAVE ON INFINITE DEPTH FLUID

of Eq. 2.30 and Eq. 2.31 which vanish at $-\infty$.

3.1.2. The External Magnetic Field and the Magnetic Boundary Condition.

The region above the fluid is current free, hence the perturbation magnetic field must be curl free. The vector potential for the external magnetic field is then

$$A = A_o e^{j(kx-\omega t)} e^{-|k|y}. \qquad (3.6)$$

The normal and tangential components of the magnetic field are continuous at the boundary if the vector potential and its y derivative are continuous at the plane y = 0.

Because of the form of the solution for the external magnetic stream function in Eq. 3.6, the condition on the internal magnetic stream function at the free surface is

$$[Ay + |k| A]_{y=0} = 0. \qquad (3.7)$$

Upon substitution of the expression for A from Eq. 3.1 and Eq. 3.2 into the above equation, we obtain

$$2\psi_1 + M_A^2 \left[1 + \frac{q}{|k|} \right] \psi_2 = 0. \qquad (3.8)$$

This is one of the two equations required to determine the dispersion relation. The other equation is obtained by application of the condition of pressure continuity at the free surface.

3.1.3. The Boundary Condition on the Surface Pressure.

Equation 2.30 expresses the condition on the perturbation pressure p in terms of the elevation of the free surface $\eta(x,t)$. Equation 2.48 relates the elevation of the surface to the vertical velocity at the surface. Making use of the stream function Eq. 2.50 becomes

$$\eta = \frac{k}{\omega}\psi. \tag{3.9}$$

It remains to express the perturbation pressure in terms of ψ. This is accomplished by means of the x-component of the Navier-Stokes equation, Eq. 2.8

$$p = \frac{\omega\rho}{k}\psi_y. \tag{3.10}$$

Substitution of the above two results into the expression for the surface pressure boundary condition, Eq. 2.50, yields

$$0 = \left[\omega^2 - |k|\left(g - \frac{Tk^2}{\rho}\right)\right]\psi_1 + \left[\omega^2\frac{q}{|k|} - |k|\left(g - \frac{Tk^2}{\rho}\right)\right]\psi_2. \tag{3.11}$$

Equations 3.8 and 3.11 determine the dispersion relation for traveling and standing waves.

3.1.4. The Dispersion Equation and Some Simple Limiting Forms.

Direct combination of Eqs. 3.8 and 3.11 gives the dispersion relation. It is

$$M_A^2(\omega^2 - \omega_g^2)\left(1 + \frac{q}{|k|}\right) - 2\left(\frac{q}{|k|}\ \omega^2 - \omega_g^2\right) \tag{3.12}$$

where

$$\omega_g^2 = |k|\left(g - \frac{Tk^2}{\rho}\right) \tag{3.13}$$

is the characteristic frequency associated with the propagation of surface waves in the absence of a magnetic field.

The dispersion relation must reduce to hydrodynamic behavior when the magnetic field is removed, or the electrical conductivity vanishes. When the magnetic field becomes small, M_A becomes very large

$$\lim_{B_o \to 0} \frac{1}{M_A^2} = 0.$$

Substitution of this into Eq. 3.12 gives

$$\omega^2 = \omega_g^2. \tag{3.14}$$

When the electrical conductivity becomes small the transverse wave number q approaches k. It is seen from Eq. 3.5 that

$$\lim_{\sigma \to 0} \quad q = |k|.$$

Substitution of the above result into Eq. 3.12 again gives Eq. 3.14.

As the conductivity becomes infinite, Eq. 3.12 becomes

$$\frac{\omega^2}{k^2} = \frac{\omega_g^2}{k^2} + 2v_A^2. \tag{3.15}$$

This result has been noted by Melcher[5] and Sakurai.[8]

A spurious root of Eq. 3.12 is

$$\omega^2 = k^2 v_A^2.$$

For at this frequency, Eq. 3.5 indicates that

$$q = |k|$$

and substitution of the above values of ω and q into the dispersion relation, Eq. 3.12, shows that this is a solution. However, substitution of the above expressions for ω and k into either Eq. 3.8 or Eq. 3.11 yield

$$\psi_1 + \psi_2 = 0.$$

The stream function is thus identically zero for this solution. The spurious solution represents no net motion of the system.

3.1.4. Low Conductivity Behavior.

Henceforth in the consideration of the limiting forms of the dispersion equation, the wavelength will be considered long enough to enable surface tension effects to be ignored. This simplifies the expression for the dispersion relation and decreases the number of dimensionless parameters needed to describe the system.

Approximate solutions to the dispersion equation in the limit of small electrical conductivity will be examined. Before doing this, it is useful to make the equations dimensionless.

Let

$$k' = Lk$$

$$\omega' = T\omega$$

$$q' = q/k$$

be the set of dimensionless variables where

$$L = \frac{v_A^2}{g}$$

$$T = \frac{v_A}{g}$$

define the characteristic length and time of the system.

Then the dispersion relation, Eq. 3.14, becomes

$$0 = \frac{\omega'^2}{k'^2} (\omega'^2 - |k'|)(1 + q') - 2(q'\omega'^2 - |k'|) \qquad (3.16)$$

and Eq. 3.5

$$q' = \left[1 - jR_M \frac{\omega'}{k'^2} \left(1 - \frac{k'^2}{\omega'^2} \right) \right]^{1/2} \qquad (3.17)$$

where

$$R_M' = \frac{\sigma\mu_o \, v_A^3}{g}$$

is a characteristic magnetic Reynolds number for the system.

Substituting Eq. 3.17 into Eq. 3.16, and properly grouping and squaring the resulting expression, the following polynomial expression is obtained.

$$0 = 4(\omega'^4 - k'^2) + j\,\frac{R'_M \omega'^3}{k'^4}\,[\omega'^2 - k'(1 + 2k')]^2 \qquad (3.18)$$

This polynomial has roots other than those of the original dispersion relation. These spurious roots are a result of the squaring operation and are roots of the dispersion equation with q' replaced by $-q'$. The spurious roots discussed in the preceding section have been algebraically divided out of the above expression. There are seven roots to Eq. 3.18. For zero R'_M, there are 4 finite roots of Eq. 3.18.

1) $\omega = k^{1/2}$

2) $\omega = -k^{1/2}$

3) $\omega = jk^{1/2}$

4) $\omega = -jk^{1/2}$

The remaining three roots diverge to infinity as the conductivity becomes small. Their asymptotic forms are

5) $\omega = \frac{j + \sqrt{3}}{2} \left(\frac{4k^4}{R'_M}\right)^{1/3}$

6) $\omega = \frac{j - \sqrt{3}}{2} \left(\frac{4k^4}{R'_M}\right)^{1/3}$

7) $\omega = -j \left(\frac{4k^4}{R'_M}\right)^{1/3}$

Of these seven roots, only roots 1 and 2 belong to the original dispersion equation, Eq. 3.16. Figure 3.2 indicates the initial path of these roots in the complex ω plane as R_M is increased from zero. In order to find an analytic expression for the roots of the dispersion equation a power series expansion in the magnetic Reynolds number may be made for ω'

$$\omega' = \omega_o' + R_M\omega_1' + R_M^2\omega_2' + \cdots \tag{3.19}$$

When a series of this form is substituted into Eq. 3.18 the first three terms are

$$\omega_o' = k'^{1/2}$$

$$\omega_1' = - j/4$$

$$\omega_2' = \frac{4 - 7\omega_o^2}{32\ \omega_o^2} .$$

The first term represents the hydrodynamic velocity. The second term represents simple exponential damping. Removing the normalization,, it is seen that all modes decay as

$$e^{-\sigma\mu_o v_A^2 t}$$

which is independent of wavelength.

The third term is real and represents a correction to the rate of oscillation. The feature of note here is that for a certain range of wave numbers, the frequency increases with the increase of conductivity, while for the remaining wave

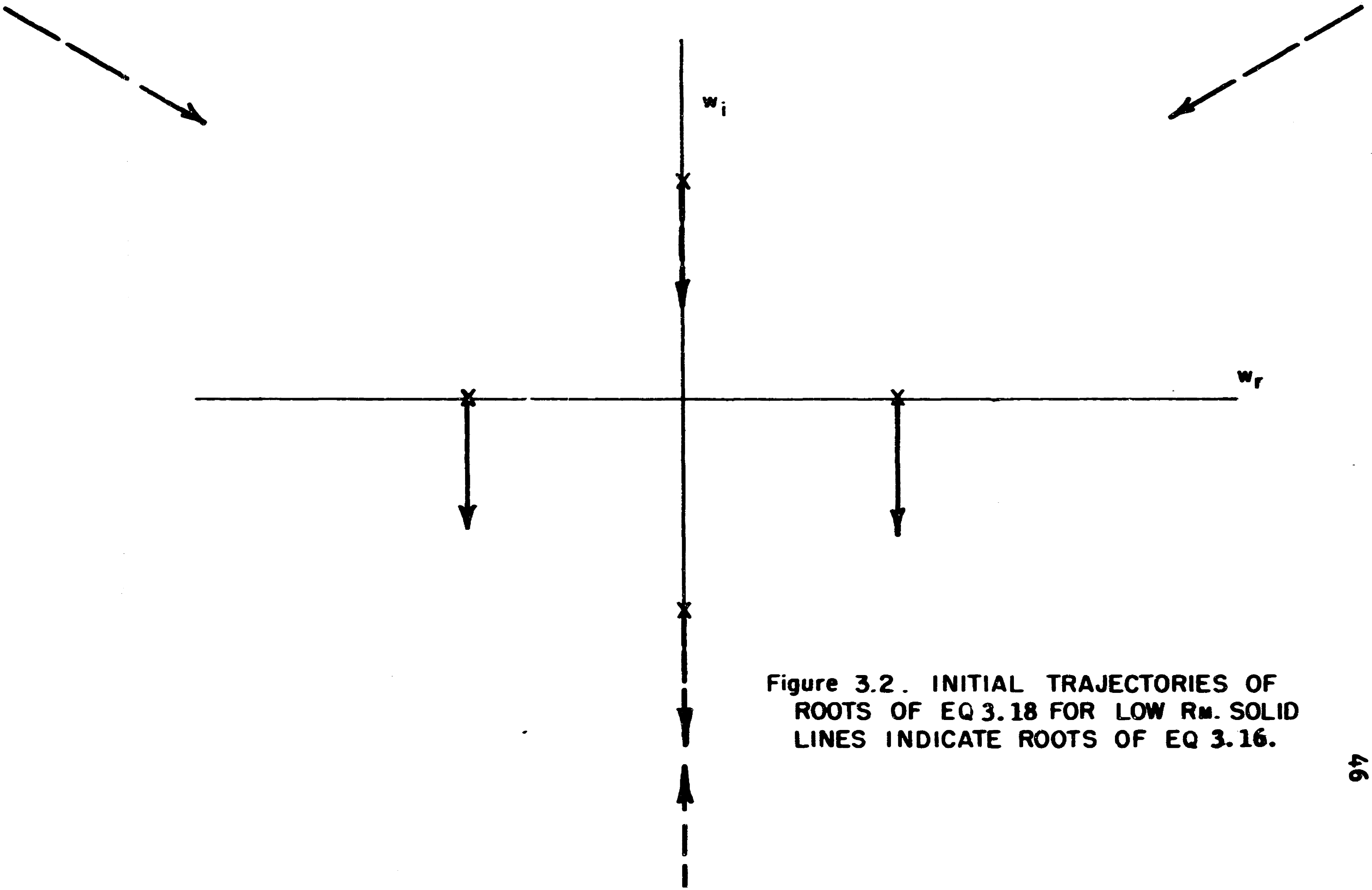

Figure 3.2. INITIAL TRAJECTORIES OF ROOTS OF EQ 3.18 FOR LOW R_M. SOLID LINES INDICATE ROOTS OF EQ 3.16.

numbers it decreases. For

$$k > \frac{4}{7} \frac{g}{v_A^2}$$

the second correction is a decrease in the real part of ω, while for

$$k < \frac{4}{7} \frac{g}{v_A^2}$$

the term represents an increase in the oscillatory frequency. The physical mechanism for this lies in the phase angle between the current and the vertical component of velocity. When the current is in phase with the velocity, the effect is damping of the oscillation. When the phase of the current leads that of the velocity, the effect is essentially that of increased restoring force and the rate of oscillation increases. When the current lags the velocity, the restoring force of gravity is partially counteracted and the frequency of oscillation is decreased. An examination of the expressions for current and velocity show the former effect at long wavelengths and the latter at short wavelengths. From Eqs. 3.1 through 3.5 an approximate expression for the phase angle at low conductivity and at the free surface may be obtained

$$0 = \frac{\omega\sigma\mu_o}{16\ k^2} \left[1 - \frac{|k|\ v_A^2}{g} \right]$$

This has the above noted behavior at high and low wave length, but the wavelength associated with the sign change is not the same.

3.1.5. High Conductivity Behavior.

When the conductivity is very large, the following situation is approached

1,2	$\omega = [k(1 + 2k)]^{1/2}$	double root
3,4	$\omega = - [k(1 + 2k)/^{1/2}$	double root
5,6,7	$\omega = 0$	triple root

All these roots satisfy both Eqs. 3.16 and 3.18. To analyse the behavior in this region, it is necessary to find the root trajectories as $(1/R_M)^{1/2}$ approaches zero.

Roots 5, 6, and 7 have asymptotic forms as follows:

5) $\omega = jA$

6) $\omega = - \dfrac{j + \sqrt{3}}{2} A$

7) $\omega = - \dfrac{(j - \sqrt{3})}{2} A$

where

$$A = \left[\frac{4k^6}{R_M[k(1 + 2k)]^2} \right]^{1/3}$$

Roots 6 and 7 are solutions to the dispersion relation, while root 5 is not. The other roots split in the following manner:

$$1)\quad \omega = \omega_a - \frac{1+j}{\sqrt{2}} \left(\frac{1}{R_M}\right)^{1/2} \omega_b$$

$$2)\quad \omega = \omega_a + \frac{1+j}{\sqrt{2}} \left(\frac{1}{R_M}\right)^{1/2} \omega_b$$

$$3)\quad \omega = -\omega_a + \frac{1-j}{\sqrt{2}} \left(\frac{1}{R_M}\right)^{1/2} \omega_b$$

$$4)\quad \omega = -\omega_a - \frac{1-j}{\sqrt{2}} \left(\frac{1}{R_M}\right)^{1/2} \omega_b$$

where

$$\omega_a = [k(1+2k)]^{1/2}$$

$$\omega_b = \frac{k^4(1+k)}{\omega_a^{7/2}}$$

Initial root trajectories for high conductivity are shown in Figure 3.3. Roots 1 and 3 are solutions of the dispersion equation, while roots 2 and 4 are not, having been introduced by the squaring operation.

There are four solutions to the dispersion equation for high conductivity while for low conductivity there are only two.

Two of these roots are simple modifications of the usual perfectly conducting solution caused by the existence of a surface current boundary layer of finite non-zero width. It is a characteristic of the skin effect that the deviation

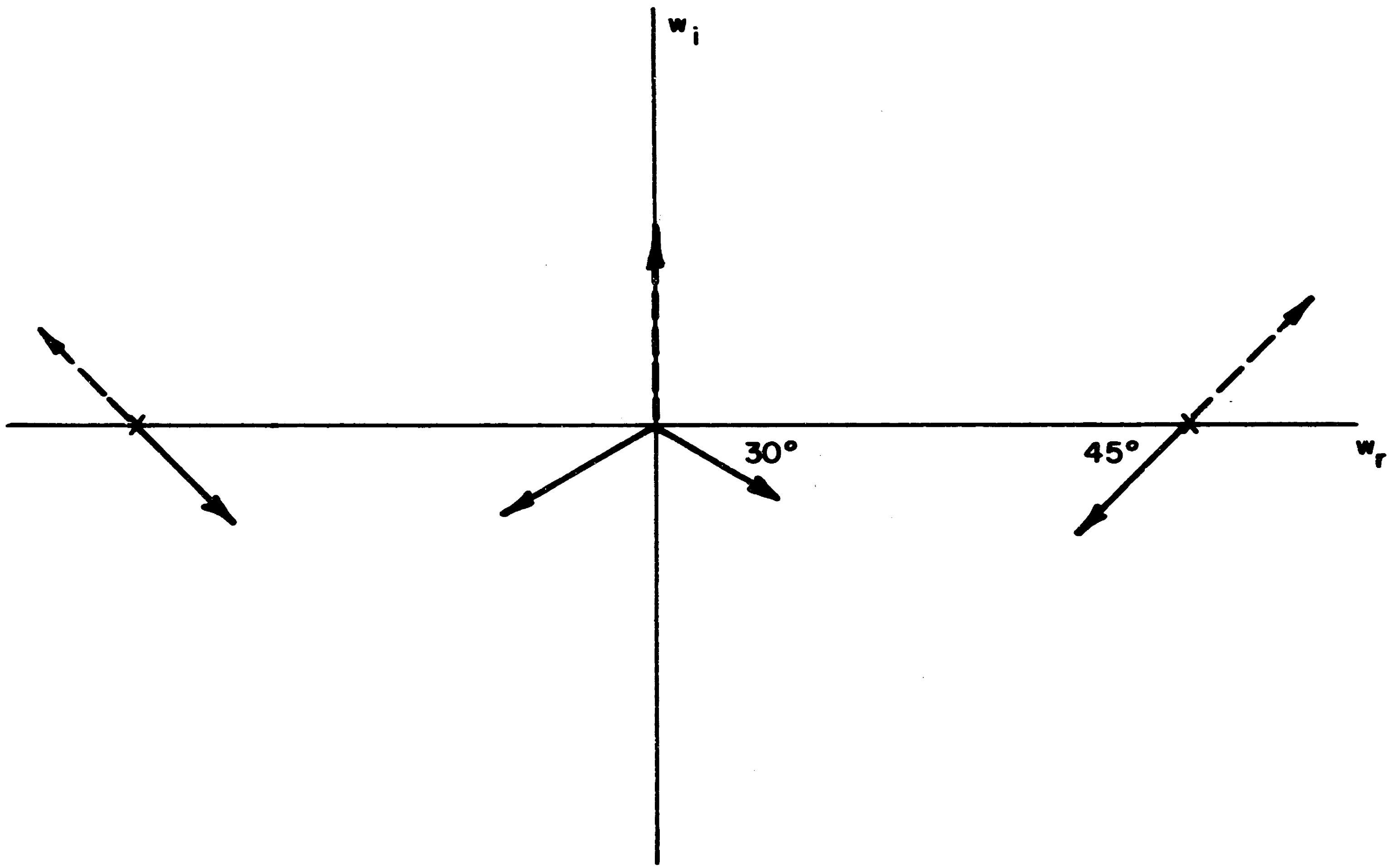

Figure 3.3 . INITIAL TRAJECTORIES OF ROOTS OF EQ 3.18 FOR HIGH R_M. SOLID LINES INDICATE ROOTS OF EQ 3.16

from the perfectly conducting situation consists of equal parts damping and frequency shift.

The other two roots represent the relaxation of an initial fluid elevation supported by the magnetic field.

3.1.5. An Expansion for Small Magnetic Fields.

The dimensionless representation of the previous section is not useful for finding the fluid behavior at small magnetic fields as the characteristic velocity chosen was the Alfven velocity.

For the purpose of obtaining an expansion valid for arbitrary conductivity but small magnetic field the following characteristic quantities may be introduced.

$$L = \frac{1}{k}, \qquad T = \frac{1}{(gk)^{1/2}} .$$

The pair of equations to be solved become

$$(1 + q')\omega'^2(\omega'^2 - 1) - v_A'^2(\omega'^2 q' - 1) = 0 \tag{3.20}$$

$$q' = \left[1 - j\omega' R_M \left(1 - \frac{v_A'^2}{\omega'^2} \right) \right]^{1/2} \tag{3.21}$$

where the characteristic magnetic Reynolds number is

$$R_M' = \frac{\sigma\mu_o g^{1/2}}{k^{3/2}} \tag{3.22}$$

and the normalized Alfven velocity is defined by

$$v_A'^2 = \frac{v_A^2 k}{g} \tag{3.23}$$

It is now possible to expand in a series for small values of the magnetic field and arbitrary conductivity.

$$\omega' = \omega_o' + v_A'^2\omega_1' + v_A'^4\omega_2' + \cdots \tag{3.24}$$

Substitution of the above series in Eqs. 3.20 and 3.21 yields

$$\omega_o' = 1 \tag{3.25}$$

$$\omega_1' = \frac{q_o - 1}{q_o + 1} \tag{3.26}$$

where

$$q_o = (1 - jR_M')^{1/2} \tag{3.27}$$

It is seen that the angle the initial root trajectory makes with the real ω axis varies from $\theta = 0$ to $\theta = \pi/2$ as the characteristic Reynolds number of Eq. 3.22 goes from zero to infinity. A graph of angle vs. magnetic Reynolds number is shown in Figure 3.4. This angular function of the magnetic Reynolds number of Eq. 3.22 serves to conveniently delineate high and low conductivity, for when $\theta(R_M)$ is approximately $\pi/2$ a low conductivity approximation is to be used, while for $\theta(R_M)$ near zero, it is possible to use the high conductivity approximation. The angle is approximately $\pi/4$ when the magnetic Reynolds number is unity, so that this alternate method of specifying the importnace of the fluid conductivity corresponds well with the usual one based on R_M.

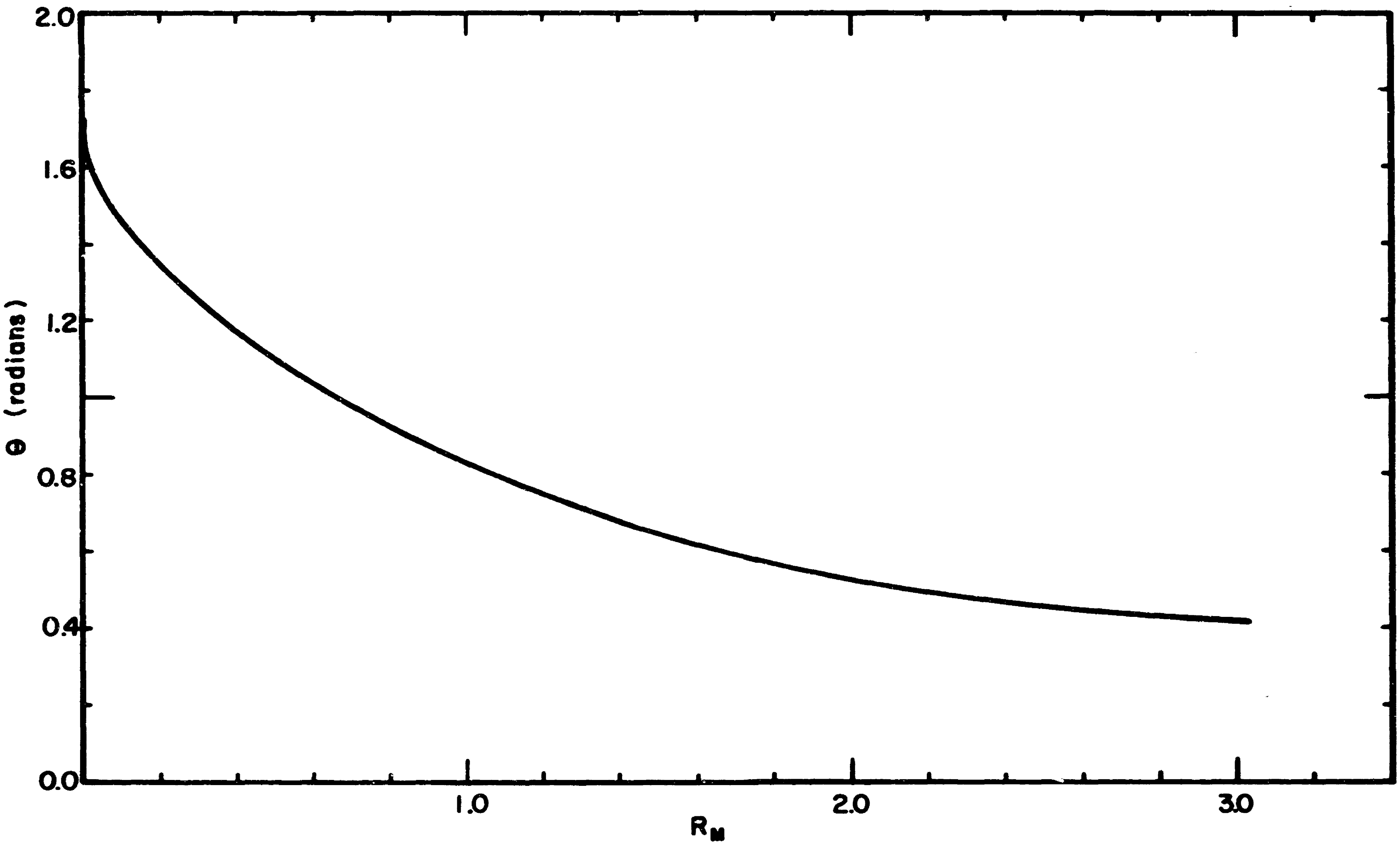

Figure 3.4. ROOT TRAJECTORY ANGLE vs MAGNETIC REYNOLDS NUMBER

3.2. Wave Motion in a Fluid of Finite Depth.

The problem of the previous section will now be extended to include the presence of a solid bottom at a finite distance below the surface. When considering wave motion on a fluid of finite depth, it is convenient to arrange the co-ordinate system to be that shown in Figure 3.5 with the origin for y being the bottom of the fluid, rather than at the free surface, and this will be done throughout this section.

3.2.1. Bulk Solutions and Boundary Conditions.

The solutions of the motion equations for the magnetic and velocity stream functions obtained in section 2.1.7 may be combined in a simple manner to give

$$\psi = e^{j(kx-\omega t)}[\psi_1 \sinh ky + \psi_2 \cosh ky + \psi_3 \sinh qy + \psi_4 \cosh qy] \tag{3.28}$$

$$A = -\frac{kB_0}{\omega}[\psi_1 \sinh ky + \psi_2 \cosh ky + M_A^2\psi_3 \sinh qy + M_A^2\psi_4 \cosh qy] \tag{3.29}$$

The application of the condition that the vertical fluid velocity be zero at the bottom gives the simple result

$$\psi_2 + \psi_4 = 0. \tag{3.30}$$

This is the first of four homogeneous equations needed to determine the dispersion relation.

The second boundary condition involves the continuity of the magnetic field at the bottom. The behavior of the magnetic stream function in the bottom is governed by the equation

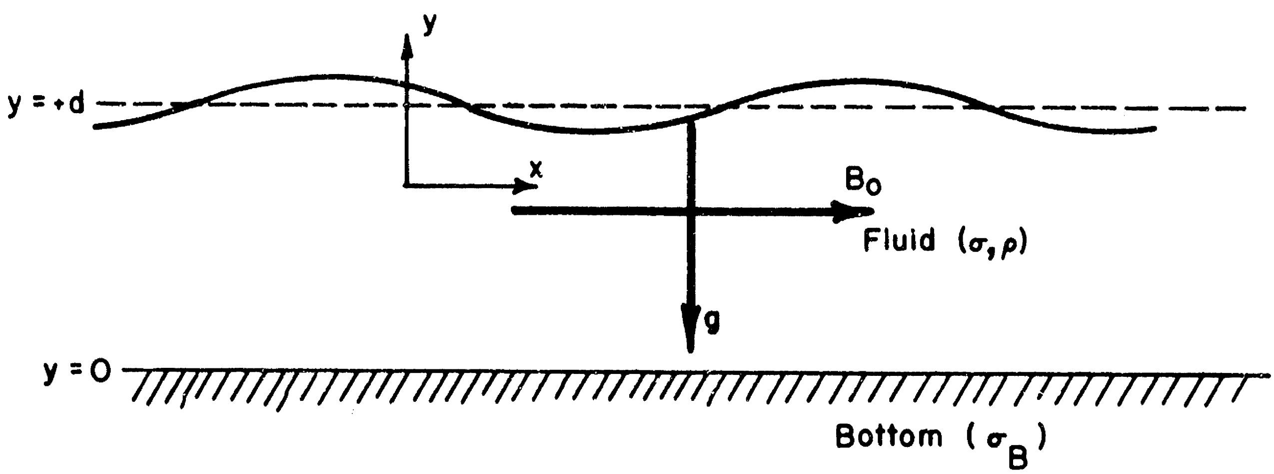

Figure 3.5. GEOMETRY OF SURFACE WAVE MOTION ON FINITE DEPTH FLUID.

$$A_{(xx+yy)} - \sigma\mu_o A_t = 0 \tag{3.31}$$

hence

$$A = A_o e^{j(kx-\omega t)} \; e^{\gamma y} \tag{3.32}$$

where

$$\gamma = k\left(1 + \frac{j\omega\sigma\mu_o}{k^2}\right)^{1/2} \tag{3.33}$$

where the square root with a positive real part is the one to be used. This is the usual skin depth behavior.

Continuity of the magnetic field across the boundary is established when the magnetic stream function and its y derivative are continuous.

$$\psi_1 - \frac{\gamma}{k}\psi_2 + \frac{q}{k}M_A^2\psi_3 - \frac{\gamma}{k}M_A^2\psi_4 = 0. \tag{3.34}$$

If the bottom is non-conducting, the equation is

$$\psi_1 - \psi_2 + \frac{q}{k}M_A^2\psi_3 - M_A\psi_4. \tag{3.35}$$

When the bottom is perfectly conducting, the equation becomes

$$\psi_2 + M_A^2\psi_4 = 0, \tag{3.36}$$

For this case, it is seen that Eqs. 3.30 and 3.35 result in the particularly simple condition

$$\psi_2 = \psi_4 = 0. \tag{3.37}$$

The boundary conditions on the free surface are the same as those of section 3.2, and the method of obtaining the boundary equations is substantially the same. The condition on the magnetic field at the free surface becomes

$$(\psi_1 + \psi_2)e^{kd} + \psi_3 M_A^2 \left(\frac{q}{k} \cosh qd + \sinh qd \right) + \psi_4 M_A^2 \left(\cosh qd + \frac{q}{k} \sinh qd \right) = 0. \quad (3.38)$$

And the condition on the free surface pressure discontinuity gives the equation

$$\begin{aligned} &\psi_1 [\sinh ky - M_c^2 \cosh ky] \\ &+ \psi_2 [\cosh ky - M_c^2 \sinh ky] \\ &+ \psi_3 [\sinh qy - M_c^2 \tfrac{q}{k} \cosh qy] \\ &+ \psi_4 [\cosh qy - M_c^2 \tfrac{q}{k} \sinh qy] \\ &= 0 \end{aligned} \quad (3.39)$$

where

$$M_c^2 = \frac{\omega^2}{k^2 v_H^2} \quad (3.40)$$

and

$$v_H^2 = \frac{g}{k} + \frac{Tk}{\rho} \quad (3.41)$$

which is the square of the velocity of free surface waves on a fluid of infinite depth with no hydromagnetic effects. Equations 3.30, 3.34, 3.38 and 3.39 determine the dispersion relation which is given in determinental form in Eq. 3.42. Equations 3.43 and 3.44 represent the special cases of non-conducting, conducting and perfectly conducting bottoms, respectively.

$$\left| C_{ij} \right| = 0$$

$$\begin{aligned}
C_{11} &= 0 & C_{31} &= e^{kd} \\
C_{12} &= 1 & C_{32} &= e^{kd} \\
C_{13} &= 0 & C_{33} &= M_A^2 \left[\frac{q}{k} \cosh qd + \sinh qd \right] \\
C_{14} &= 1 & C_{34} &= M_A^2 \left[\cosh qd + \frac{q}{k} \sinh qd \right] \\
C_{21} &= 1 & C_{41} &= \sinh kd - M_c^2 \cosh kd \\
C_{22} &= - \frac{\gamma}{k} & C_{42} &= \cosh kd - M_c^2 \sinh kd \\
C_{23} &= M_A^2 \frac{q}{k} & C_{43} &= \sinh qd - M_c^2 \frac{q}{k} \cosh qd \\
C_{24} &= - \frac{\gamma}{k} M_A^2 & C_{44} &= \cosh qd - M_c^2 \frac{q}{k} \sinh qd
\end{aligned} \tag{3.42}$$

$$\left| C_{ij} \right| = 0$$

all co-efficients are as in Eq. 3.42 with the exceptions

$$C_{21} = 1 \qquad C_{22} = - 1 \qquad C_{23} = \frac{q}{k} M_A^2 \qquad C_{24} = - M_A^2 \tag{3.43}$$

$$\left| A_{ij} \right| = 0$$

$$\begin{aligned}
A_{11} &= C_{31} & A_{21} &= C_{41} \\
A_{12} &= C_{33} & A_{22} &= C_{43}
\end{aligned} \tag{3.44}$$

The limiting forms of Eq. 3.42 obtained as the magnetic field vanishes and as the electrical conductivity vanishes must correspond to the hydrodynamic results. As the magnetic field vanishes, M_A^2 approaches infinity and the determinant is easily expanded by minors of the dominant terms. The result is

$$\sinh ky - M_c^2 \cosh ky = 0 \tag{3.45}$$

which is perhaps more easily recognized when rewritten

$$\frac{\omega^2}{k^2} = \left[\frac{g}{k} + \frac{Tk}{\rho} \right] \tanh kd. \tag{3.46}$$

As the conductivity vanishes q/k approaches one, and again the limiting form is seen to be Eq. 3.46, the well-known expression for the velocity of water waves in a fluid of finite depth.

As the conductivity becomes infinite, q/k becomes infinite, and a careful reduction of Eq. 3.42 gives

$$\frac{\omega^2}{k^2} = v_H^2 \tanh kd + v_A^2(1 + \tanh kd). \tag{3.47}$$

This corresponds to a result established by Melcher[5] for waves on a perfectly conducting fluid of finite depth.

The detailed solution of the above determinental dispersion relations will not be undertaken in this thesis. They will be examined in section 3.4 for a number of approximate solutions. Further, the mathematically simple and physically interesting long wave limit will be examined.

3.2.2. Shallow Fluid Limit.

When the fluid depth is small enough that the following relationships hold,

$$|kd| << 1, \qquad |qd| << 1$$

a simplification of Eq. 3.42 may be effected by replacing the hyperbolic functions by the initial terms of their power series.

When only the first term in each power series is taken, the result is

$$\frac{\omega^2}{k^2} = \left[\frac{g}{k} + \frac{Tk}{\rho} \right] kd \tag{3.48}$$

which is easily seen to be the shallow depth approximation to Eq. 3.46. There is no effect due to the magnetic field as, to first order in kd, the fluid motion is parallel to the field lines. To see the field effects, it is necessary to use a higher order expansion.

When the first two terms in each power series are taken, the result is

$$\omega = \omega_o \left[1 - \frac{(kd)^2}{6} \right] - j\, \frac{(kd)^2}{6}\ \sigma\mu_o v_A^2 \tag{3.49}$$

where ω_o is the solution to Eq. 3.48. The damping is thus inversely proportional to the square of the wavelength in this long wave limit.

One of the more interesting features of the above expression is that the conductivity of the bottom does not enter into the solution.

Further, although the damping factor is of second order in the expansion parameter, the size of the term

$$\sigma\mu_o v_A^2$$

may be made quite large with liquid conductors and laboratory magnetic fields. Thus the damping effect of the magnetic field, which now in the long wave approximation does depend on the wavelength, may be easily observable at wavelengths where the change in real frequency caused by finite depth, a hydrodynamic phenomenon, is not observable.

3.3. An Initial Value Problem for the Infinitely Deep Fluid.

The complexity of the dispersion relation for waves on a fluid of finite depth contrasts strongly with the relative simplicity of the dispersion relation for the fluid of infinite depth. The finite depth dispersion relation can be shown to have an infinite number of roots. This strongly indicates that there are fluid motions, excitable by outside forces, which are not associated with solutions of the dispersion relation for a fluid of infinite depth.

A formal solution of an initial value problem for free surface motion on a fluid of infinite depth but finite electrical conductivity will now be undertaken. In this manner, a mathematical expression for the additional fluid motions will be obtained. Figure 3.1 again shows the situation under discussion. As before, a sinusoidal dependence upon the x co-ordinate will be assumed.

The stream functions for the velocity and magnetic fields are assumed to be of the following form

$$\psi(x,y,t) = \tilde{\psi}(y,t)\ e^{jkx} \qquad (3.50)$$

$$A(x,y,t) = \tilde{A}(y,t)\ e^{jkx}. \qquad (3.51)$$

The single-ended Laplace transform will be employed here

$$\phi\ (y,s) = \int_0^\infty \psi(y,t)\ e^{-st}\ dt. \qquad (3.52)$$

The volume equations of motion, Eq. 2.25c and Eq. 2.26 may be combined to give a single partial differential equation for the velocity stream function.

$$[\psi_{(xx+yy)t} - \sigma\mu_o\psi_{tt} + \sigma\mu_o v_A^2\psi_{xx}]_{(xx+yy)} = 0. \qquad (3.53)$$

Employing the above transform and the assumed x dependence from Eq. 3.50, and representing the derivative with respect to y by capital D, we obtain for the transform of Eq. 3.53

$$(D^2 - k^2)[(D^2 - k^2)s\phi - \sigma\mu_o(s^2 + k^2v_A^2)\phi]$$

$$= (D^2 - k^2)[(D^2 - k^2)\tilde{\psi}(y,0) - \sigma\mu_o(s\tilde{\psi}(y,0) + \tilde{\psi}'(y,0))]$$

(3.54)

If there is initial motion in the fluid which is not a zero of the differential operator on the right-hand side of

Eq. 3.54, a particular integral of the initial terms will be needed in addition to the homogeneous solution of the equation. The differential operator on the right hand side of Eq. 3.54 is a combination of the Laplacian and the operator for magnetic diffusion in cartesian co-ordinates. Initial disturbances outside the fluid will be considered here and only the homogeneous solution will be required.

The possibility of particular solutions to the motion equation is not entirely due to hydromagnetic effects. Upon removing magnetic effects from the above equation, one obtains an equation for hydrodynamic motions and it is seen that a particular solution needs to be added if the fluid motion initially contained vorticity. This is a well known result in the hydrodynamic theory of surface wave motions.

The homogeneous solution to Eq. 3.54 is

$$\phi = C_1 e^{ky} + C_2 e^{q(s,k)y} \tag{3.55}$$

where

$$q = k \left[1 + \frac{\sigma \mu_o s}{k^2} \left(1 + \frac{k^2 v_A^2}{s^2} \right) \right]^{1/2} \tag{3.56}$$

and the branch of the square root with the real part of q non-negative is required.

The boundary condition on the magnetic field at the free surface has been previously established as:

$$\left. (A_y + k\ A) \right|_{y=0} = 0. \tag{3.57}$$

A manipulation of Eq. 2.25c and Eq. 2.26 yield the following expression for A in terms of ψ, which contains no y derivatives of A and may therefore be substituted into Eq. 3.57,

$$A_{xt} = B_o \psi_{xx} + \frac{B_o}{\sigma \mu_o v_A^2} \psi_{(xx+yy)t} . \tag{3.58}$$

The transformed boundary condition is then

$$(D + k) \left[s(D^2 - k^2) \frac{B_o}{\sigma \mu_o v_A^2} \phi - k^2 B_o \phi \right] \Bigg|_{y=0}$$

$$= (D + k) \left[(D^2 - k^2) \frac{B_o}{\sigma \mu_o v_A^2} \tilde{\psi}(0,0) - jk\tilde{A}(0,0) \right] \tag{3.59}$$

The transformed pressure condition at the free surface may be obtained from the following equations

$$(P - \rho g \eta) \Big|_{y=0} = 0 \tag{3.60}$$

$$\psi_{yt} = \frac{1}{\rho} D_x \tag{3.61}$$

$$\eta_t = - \psi_x . \tag{3.62}$$

The result is

$$s^2 D\phi + gk^2 \phi = sD\tilde{\psi}(0,0) - jkg\tilde{\eta}(0) . \tag{3.63}$$

The transformed boundary conditions together with a specification of the initial conditions determine the constants C_1 and C_2 in Eq. 3.55, the expression for the transformed velocity stream function.

Consider the special case of an initial elevation of the fluid at time $t = 0$. Let the initial elevation of the fluid be specified by

$$\eta\big|_{t=0} = \delta e^{jkx}. \tag{3.64}$$

The boundary Eq. 3.59 and 3.63 become

$$(D + k)\left[s(D^2 - k^2) \frac{B_o}{\sigma\mu_o v_A^2} \phi - k^2 B_o \phi \right] = 0 \tag{3.65}$$

$$s^2 D\phi + gk^2\phi = - jkg\delta. \tag{3.66}$$

The constants C_1 and C_2 of Eq. 3.55 may now be evaluated

$$C_1 = \frac{jg\delta s^2(k + q)}{k^2 v_A^2 P} \tag{3.67}$$

$$C_2 = - \frac{2kjg\delta}{P} \tag{3.68}$$

and

$$P = \frac{s^2}{k^2 v_A^2} (k + q)(s^2 + gk) + 2(s^2 q + gk^2). \tag{3.69}$$

The zeros of P are the roots of the dispersion relation Eq. 3.14 obtained in section 3.1. The zeroes of P at

$$s^2 + k^2 v_A^2 = 0 \tag{3.70}$$

are also zeros of the numerator of the transformed solution, hence the transformed solution has no poles there. This is true for arbitrary initial conditions, not just for the special case examined here, as may be seen by introducing arbitrary driving functions on the right-hand side of Eqs. 3.65 and Eq. 3.66.

When the inversion integral is performed on the transformed solution in the usual manner, the result consists of contributions due to singularities of the transformed solution. The possible singularities are poles and branch points.

The poles of the transformed solution are the zeros of P, with the exception of those indicated in Eq. 3.70. These are the discrete set of natural frequencies associated with the wave solutions discussed in section 3.1.

The transformed stream function ϕ is multi-valued on the s-plane. The multi-valued nature of ϕ arises from the fact that q is a two-valued function of s, as it is defined in Eq. 3.56 with a square root. Now the boundary condition at $y = -\infty$ specifies which of the values for q, and consequently ϕ, is to be taken. However, in order to perform the inversion integral by contour integral methods, it is necessary to introduce branch cuts into the complex plane to make ϕ a single-valued function.

The branch cuts terminate on the branch points of q and follow the path along which the real part of q vanishes. The branch of the function q to be taken is that with a positive real part.

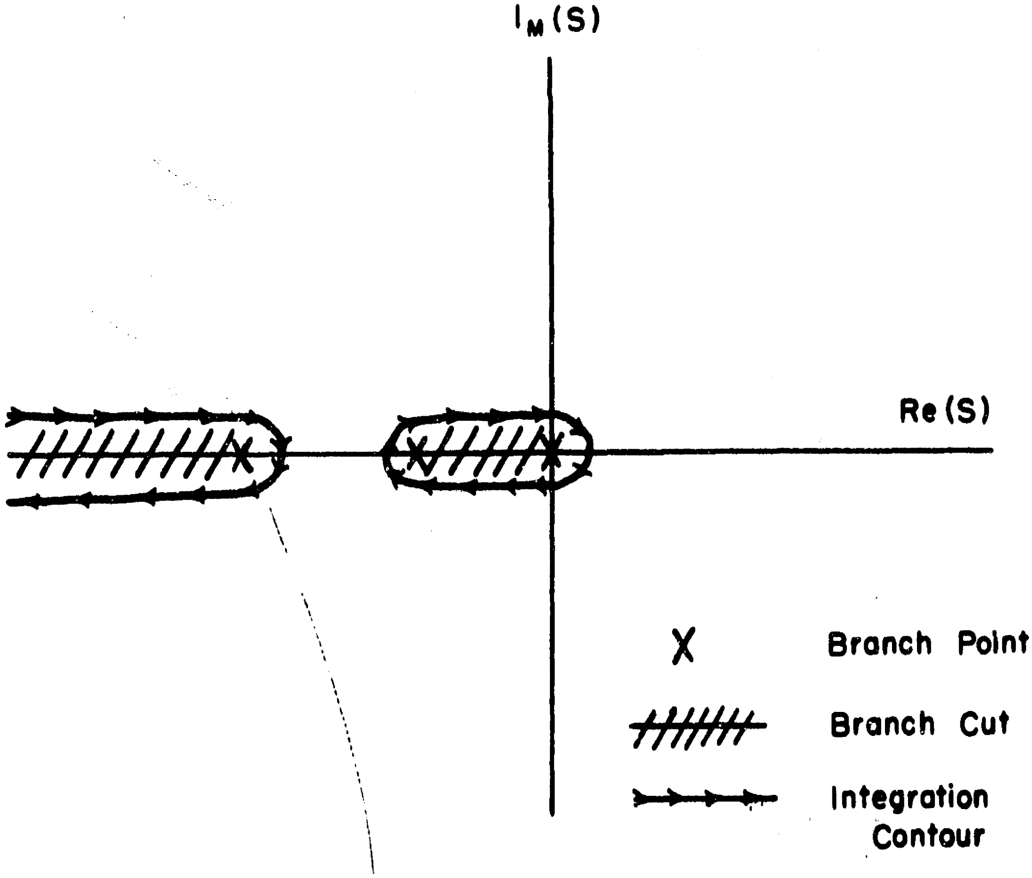

Figure 3.6. BRANCH CUTS AND INTEGRATION CONTOURS — LOW R_M.

The branch points of q are the points in the s plane where $q = 0$ and $1/q = 0$. From Eq. 3.56, it is seen that there are four such points

$$1)\quad s = 0,\ \frac{1}{q} = 0 \tag{3.71a}$$

$$2)\quad s = \infty,\ \frac{1}{q} = 0 \tag{3.71b}$$

$$3)\quad s = -\frac{k^2}{2\sigma\mu_o} + \left[\left(\frac{k^2}{2\sigma\mu_o}\right)^2 - k^2 v_A^2\right]^{1/2},\ q = 0 \tag{3.71c}$$

$$4)\quad s = -\frac{k^2}{2\sigma\mu_o} - \left[\left(\frac{k^2}{2\sigma\mu_o}\right)^2 - k^2 v_A^2\right]^{1/2},\ q = 0. \tag{3.71d}$$

Branch points 3 and 4 lie along the negative real axis for

$$k > 2\sigma\mu_o v_A.$$

For this situation, the branch cuts are shown in Figure 3.6. The branch points 3 and 4 occur at complex conjugate values of s with negative real parts for

$$k < 2\sigma\mu_o v_A.$$

The branch cuts for this situation are shown in Fig. 3.7. The cut consists of the entire negative real axis and a portion of the circle

$$|s| = kv_A.$$

As the conductivity approaches infinity, the branch cut consists of the negative real axis and that half of the above

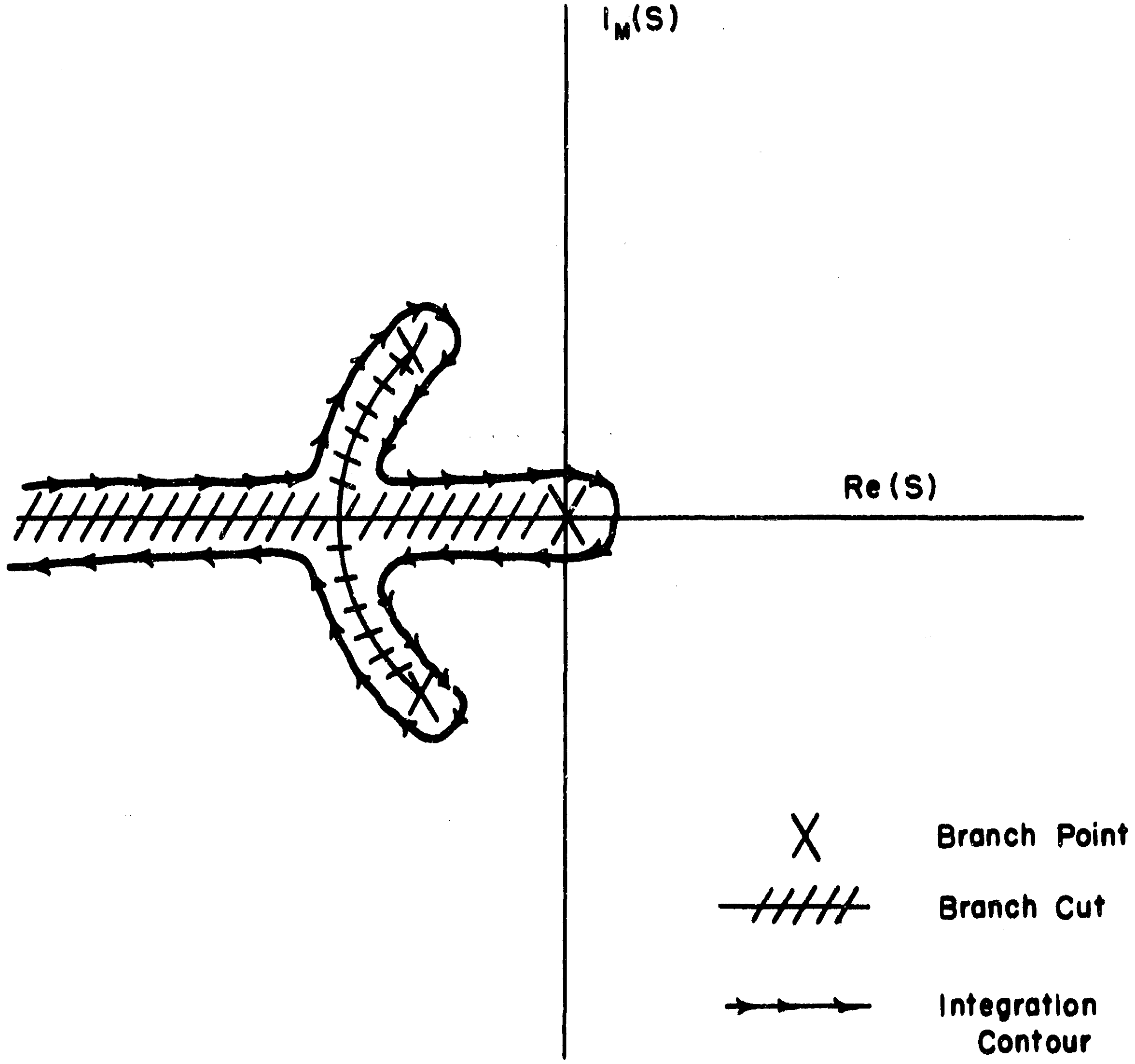

Figure 3.7. BRANCH CUTS AND CONTOUR INTEGRATION FOR HIGH R_M.

circle which lies in the left half of the s-plane.

The branch cuts required always lie in the left half plane for all values of conductivity, wave-length, and magnetic field.

The solution for the velocity stream function is then

$$\psi(y,t) = \sum_{k=1}^{m} (s - s_k)\phi(s_k)e^{s_k t} + \frac{1}{2\pi j} \int_C \phi(s)e^{st}\, ds. \tag{3.72}$$

The contours around which the integral is to be taken are indicated in Fig. 3.6 and Fig. 3.7. The summation is to be taken over the m poles of the function $\phi(s)$.

The nature of the fluid motion associated with the poles was the subject of section 3.1. Section 3.4 will indicate a connection between the branch cut integral associated with the infinite depth solution and the infinity of solutions to the dispersion relation of the finite depth solution.

3.4. Fluid Motion Associated with the Branch Cut Integral.

The dispersion equation for surface wave motion of finite depth, Eq. 3.42, has, for a given wavelength of disturbance, an infinite number of solutions corresponding to natural frequencies which lie in the s-plane on or near the branch cuts of Figs. 3.6 and 3.7. As the fluid depth tends to infinity, the spacing between the solutions tends to zero. Although a complete proof of this assertion has not been carried out, the subsequent analysis of approximate solutions makes this assertion plausible and indicates the types of fluid motion associated with these solutions.

Equation 3.42 yields approximate solutions easily near critical points of the transverse propagation constant q. These critical points are the locations in the s-plane for which q is either zero or infinite. They are also branch points of the transformed solution and are indicated in Eqs. 3.71. The approximate solutions will be shown to be primarily bulk motions with little associated surface disturbance.

First, consider the critical point of Eq. 3.71a. In this region

$$M_A^2 \sim O(s^2)$$

$$M_c^2 \sim O(s^2).$$

Then both of the above quantities are small, and a power series expansion may be made of Eq. 3.42 in terms of increasing powers of M_A^2. When this is done, and only the first term is taken, Eq. 3.42 becomes approximately

$$\sinh qd = 0 \tag{3.73}$$

and since q is purely imaginary and approaches infinity as s approaches zero along the negative real axis, there are an infinite number of solutions along this path for any value of δ. It is seen from Eqs. 3.30 and 3.38 that ψ_1, ψ_2, and ψ_4 are approximately zero. Then from Eq. 3.39 it is seen that the condition of Eq. 3.73 is approximately the requirement that volume mode ψ_3 satisfy the free surface pressure condition. Equation 3.29 shows the magnetic field associated with this motion is of $0(M_A^2)$ compared to the velocity, and M_A^2 is the expansion parameter. Consequently there is no magnetic field to zero order in the expansion associated with this motion. The stream function, Eq. 3.28, determines the motion. Since q is purely imaginary, the motion is sinusoidally periodic in the x direction. As q approaches infinity in this region

$$\frac{q}{k} >> 1.$$

Consequently, the velocity is primarily along the x axis, since the ratio of the velocity components of the motion is

$$\frac{v_x}{v_y} = \frac{q}{k}$$

The approximate solutions in this region are thin shear flows which interact weakly with the applied field. For large values of imaginary q

$$s = \frac{\sigma\mu_o v_A^2 k^2}{q^2} . \tag{3.74}$$

This represents a slow decay in time with no oscillation.

In the region of the critical point defined by Eq. 3.71b, M_A^2 and M_c^2 are large, and a power series expansion of Eq. 3.42 may be made in powers of $1/M_A^2$, Equation 3.42 becomes approximately

$$\cosh qd = 0. \tag{3.75}$$

Equations 3.30 and 3.39 show that again ψ_1, ψ_2, and ψ_4 are approximately zero. Since M_A^2 is a large number, Eq. 3.29 shows that the velocity is now of the order of the expansion parameter times the magnetic field and consequently negligible. Equation 3.38 indicates that the requirement that the magnetic fields be continuous, expanded in the above series, with q/k large, leads to Eq. 3.75. In this region

$$s = \frac{q^2}{\sigma \mu_o} .$$

This is the decay rate for an initial sinusoidal distribution of magnetic field in a solid conductor.

The critical points of Eqs. 3.71c and 3.71d occur at finite, non-zero, values of s. Consequently an expansion of Eq. 3.42 can be made in powers of the dimensionless parameter q/k which is small in the neighborhood of these points. The result is Eq. 3.75. The parameters M_c^2 and M_A^2 are independent of q/k in this region, consequently both magnetic field and velocity are important. Equation 3.34 shows again that ψ_1, ψ_2, and ψ_4 are approximately zero, and Eqs. 3.38 and 3.39, expanded in the above power series show that the condition of Eq. 3.75 is, in these cases, equivalent to both the free surface pressure constraint and the free surface magnetic field constraint. For small values of q/k, it should be noted that Eq. 3.75 demands large values of kd for a solution. Examination of the stream function of Eqs. 3.28 and 3.29 with q/k small show the velocity and perturbation magnetic field to be y directed, and approximately uniform in y, with sinusoidal periodicity in x. The approximate values of s associated with this solution are roots of the equation

$$\frac{s^2}{k^2} + \frac{s}{\sigma\mu_o} + v_A^2 = 0. \qquad (3.76)$$

The approximate value of s given by the above equation is that associated with the propagation of Alfven waves through a lossy conducting fluid.

The branch cut integral of section 3.3 and the infinity of solutions to Eq. 3.42 are thus seen, in the regions where approximate solutions were obtained, to represent three bulk processes in the fluid. These are persistent vortex motion, magnetic diffusion and Alfven wave propagation.

Chapter IV

Gravity Waves Under a Vertical Magnetic Field.

In this chapter, a theory of hydromagnetic waves under the influence of a vertical applied magnetic field is developed.

Two major previous investigations have been made in this field. The investigation by Roberts and Boardman[4] treated the motions of a viscid incompressible homogeneous fluid of infinite depth under the influence of a vertical magnetic field. A dispersion relation for surface wave motion was obtained and a number of limiting cases examined. The inviscid limit is that which is relevant to the present work.

Fraenkel[13] developed a shallow water theory of hydromagnetic surface wave motion over a non-conducting bottom. Both linear and non-linear motions were treated and the linear response to initial impulsive excitations was obtained.

In section 4.1, the results obtained by Roberts and Boardman[4] are developed using the methods and notation established in Chapter 3. No original work is presented.

In section 4.2, the dispersion equation for wave motion in a fluid of finite depth is obtained. It is shown that the limiting form of the equation for long wavelength disturbances is equivalent to the expressions obtained by Fraenkel.[13]

In section 4.3, an initial value problem for motion on a fluid of infinite depth is treated. A continuum of natural frequencies, represented by a branch cut integral, is shown to be necessary in addition to the discrete natural frequencies of the dispersion relation.

In section 4.4, the continuum of natural frequencies of section 4.3 is shown to be related to an infinite number of discrete solutions to the finite depth dispersion equation.

4.1. Wave Motion on a Fluid of Infinite Depth.

The basic equations for the velocity and magnetic stream function are obtained from Eq. 2.30 and Eq. 2.31 with the magnetic field purely vertical ($\alpha = 0$, $\beta = 1$).

$$A_{(xx+yy)} - \sigma\mu_o A_t = - \sigma\mu_o B_o \psi_y \tag{4.1}$$

$$\psi_{(xx+yy)t} = \frac{B_o}{\rho\mu_o} A_{(xx+yy)y} \tag{4.2}$$

The four solutions to this pair of equations, obtained by the methods of section 2.1.7, are

1) $$\psi = \psi_1 e^{j(kx-\omega t)} e^{ky} \tag{4.3}$$

$$A = \frac{jkB_o}{\omega} \psi \tag{4.4}$$

2) $$\psi = \psi_2 e^{j(kx-\omega t)} e^{-ky} \tag{4.5}$$

$$A = - \frac{jkB_o}{\omega} \psi \tag{4.6}$$

3) $$\psi = \psi_3 e^{j(kx-\omega t)} e^{qy} \tag{4.7}$$

$$A = -j\frac{k}{q}\,\frac{kB_o}{\omega}\,M_A^2\,\psi \qquad (4.8)$$

4) $$\psi = \psi_4\,e^{j(kx-\omega t)}\,e^{-qy} \qquad (4.9)$$

$$A = j\frac{k}{q}\,\frac{kB_o}{\omega}\,M_A^2\,\psi \qquad (4.10)$$

where

$$M_A^2 = \frac{\omega^2}{k^2 v_A^2} \qquad (4.11)$$

the ratio of the phase velocity to the Alfven velocity, and q is the transverse propagation constant defined by Eq. 2.38.

When the depth of the fluid is such that

1) $e^{-kd} << 1$

2) $\left|e^{-qd}\right| << 1$

the bottom becomes unimportant and the boundary condition that all motion vanish as y approaches minus infinity is sufficient.

It should be noted that as the conductivity grows large, the second condition becomes increasingly more difficult to satisfy since

$$\lim_{1/\sigma \to 0} \mathrm{Re}\left|q\right| = 0. \qquad (4.12)$$

When the above conditions do apply, the appropriate forms of the volume solutions are

$$\psi = e^{j(kx-\omega t)}\left[\psi_1 e^{|k|y} + \psi_2 e^{qy}\right] \qquad (4.13)$$

$$A = j \frac{|k| \; B_o}{\omega} e^{j(kx-\omega t)} \left[\psi_1 e^{|k|y} - \frac{|k|}{q} M_A^2 \psi_2 e^{qy} \right] \qquad (4.14)$$

In this chapter, the surface force caused by surface tension is ignored. The application of the boundary conditions, Eq. 2.48 and Eq. 2.50, together with the requirement of continuity of the magnetic field at the fluid surface lead to the following expressions

$$2\psi_1 - M_A^2 \left(1 + \frac{q}{|k|}\right) \psi_2 = 0 \qquad (4.15)$$

$$\psi_1(\omega^2 - g|k|) + \psi_2 \left[\frac{q}{|k|} \omega^2 - g|k| \right] = 0 \qquad (4.16)$$

The above equations directly yield the dispersion relation which is

$$M_A^2 \left(1 + \frac{q}{|k|}\right) (\omega^2 - g|k|) + 2 \left(\frac{q}{|k|} \omega^2 - g|k| \right) = 0. \qquad (4.17)$$

This is the equation obtained by Roberts and Boardman[4] as a limiting form for a fluid of zero viscosity.

This equation has two spurious roots, as did the dispersion equation in the case of a tangential field. They occur at

$$\omega^2 = - k^2 v_A^2. \qquad (4.18)$$

Equation 4.17 may be rewritten as

$$M_A^2(\omega^2 - g|k|) - 2g|k| = - \frac{q}{|k|} \left[M_A^2(\omega^2 - g|k|) + 2\omega^2 \right] \qquad (4.19)$$

In this form, the equation may be squared, the expression for q^2 from Eq. 2.38 introduced, and a ninth order polynomial in ω obtained. Algebraic removal of the spurious roots of Eq. 4.18 reduces the polynomial to seventh order. The roots of the polynomial are either roots of Eq. 4.19 or of

$$M_A^2(\omega^2 - g|k|) - 2g|k| = + \frac{q}{|k|}\left[M_A^2(\omega^2 - g|k|) + 2\omega^2\right] \qquad (4.20)$$

Roberts and Boardman[4] have extensively studied the above mentioned polynomial, both computationally and analytically. An interesting feature of their analysis are regions in which all solutions to the polynomial are solutions of 4.20. There are then no solutions to the dispersion equation.

A simple form of Eq. 4.17 was obtained in the limit as the conductivity approaches zero. It is

$$\omega = (g|k|)^{1/2} - j\,\frac{\sigma\mu_o v_A^2}{4} \qquad (4.21)$$

The rate of damping is seen to be the same as that obtained for the horizontal magnetic field case.

Roberts and Boardman[4] point out that the dispersion equation obtained above is not applicable in the perfectly conducting limit, and that an initial value analysis is necessary to determine the motion in this limit.

4.2. Wave Motions on a Fluid of Finite Depth.

The dispersion relation for wave motion on a fluid of finite depth will now be developed.

Consider the situation shown in Fig. 4.1, with a fluid of uniform depth d at rest over a solid bottom. The electrical conductivity of the bottom is σ_B. It is convenient to consider the bottom to be defined by the plane $y = 0$.

Equations 4.3 through 4.11 may be combined to give

$$\psi = e^{j(kx-\omega t)}\Big[\psi_1 \sinh ky + \psi_2 \cosh ky + \frac{q}{|k|}\,\psi_3 \sinh qy + \psi_4 \cosh qy\Big] \qquad (4.22)$$

$$A = \frac{jkB_o}{\omega}\, e^{j(kx-\omega t)}\Big[\psi_1 \cosh ky + \psi_2 \sinh ky - M_A^2\,\psi_3 \cosh qy - \frac{k}{q} M_A^2 \sinh qy\Big] \qquad (4.23)$$

The application of the boundary condition on the vertical velocity at the bottom gives

$$\psi_3 + \psi_4 = 0. \qquad (4.24)$$

The second boundary condition involves the behavior of the magnetic field in the bottom. The equation governing the magnetic stream function A in the bottom is

$$A_{(xx+yy)} - \sigma_B \mu_o A_t = 0 \qquad (4.25)$$

where σ_B is the conductivity of the bottom. The solution for A is that of the well-known skin effect.

$$A = A_B\, e^{j(kx-\omega t)}\, e^{k\gamma y} \qquad (4.26)$$

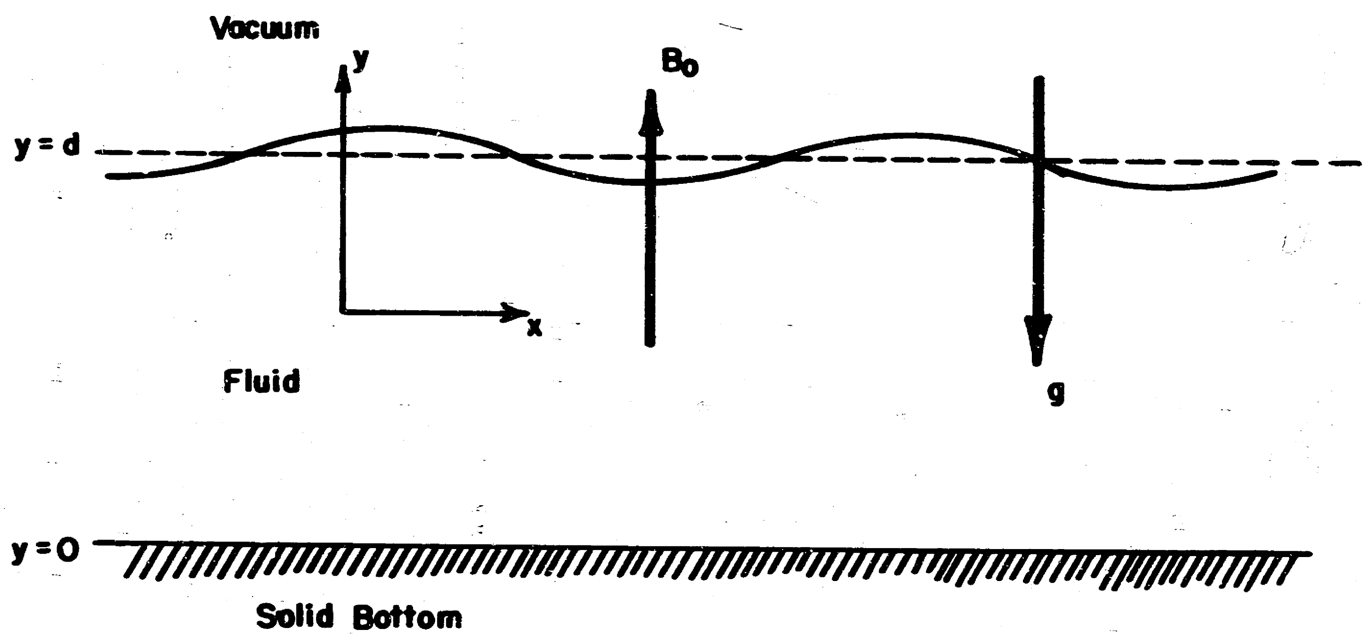

Figure 4.1

GEOMETRY OF SURFACE WAVE MOTION ON FINITE DEPTH FLUID

where

$$\gamma = \left[1 - j\,\frac{\omega \sigma_B \mu_o}{k^2}\right]^{1/2} . \qquad (4.27)$$

The continuity of the normal and tangential magnetic field is established when A and its normal derivative are continuous. The resultant condition on the constants in the volume solution is

$$\gamma\,\psi_1 - \psi_2 - \gamma\,M_A^2\psi_3 + M_A^2\psi_4 = 0. \qquad (4.28)$$

If the fluid lies over a non-conducting bottom, Eq. 4.28 becomes

$$\psi_1 - \psi_2 - M_A^2(\psi_3 - \psi_4) = 0. \qquad (4.29)$$

When the bottom is perfectly conducting, Eq. 4.28 becomes

$$\psi_1 - M_A^2\psi_3 = 0. \qquad (4.30)$$

The methods of section 3.1.2 yield the following equation for magnetic field continuity at the free surface.

$$(\psi_1 + \psi_2)\,e^{|k|d} - M_A^2\left(\cosh qd + \frac{q}{|k|}\sinh qd\right)\psi_3$$

$$- M_A^2\left(\cosh qd + \frac{|k|}{q}\sinh qd\right)\psi_4 = 0 \qquad (4.31)$$

The pressure in the fluid is determined by the y component of the motion equation, Eq. 2.8

$$-\psi_{xt} = -\frac{1}{\rho}\,P_y . \qquad (4.32)$$

Substitution of the expression for ψ from Eq. 4.22 gives

$$P = \rho\omega e^{j(kx-\omega t)} \Big[\psi_1 \cosh ky + \psi_2 \sinh ky + \psi_3 \cosh qy + \psi_4 \frac{|k|}{q} \sinh qy \Big] \qquad (4.33)$$

The pressure condition at the free surface is given by Eq. 2.50. Use of the above expression for the perturbation pressure in Eq. 2.50 yields further constraint on the constants of the volume solution.

$$\psi_1 \left[\sinh ky - M_c^2 \cosh ky \right] + \psi_2 \left[\cosh ky - M_c^2 \sinh ky \right] + \psi_3 \left[\frac{q}{|k|} \sinh qy - M_c^2 \cosh qy \right] + \psi_4 \left[\cosh qy - M_c^2 \frac{|k|}{q} \sinh qy \right] = 0 \qquad (4.34)$$

where, since surface tension has been ignored,

$$M_c^2 = \frac{\omega^2}{g|k|} \qquad (4.35)$$

Equations 4.24, 4.28, 4.31 and 4.34 yield the dispersion relation for this case, which is:

$$\left| C_{ij} \right| = 0 \tag{4.36}$$

$C_{11} = 0$

$C_{12} = 1$

$C_{13} = 0$

$C_{14} = 1$

$C_{21} = \gamma$

$C_{22} = -1$

$C_{23} = -\gamma M_A^2$

$C_{24} = M_A^2$

$C_{31} = e^{kd}$

$C_{32} = e^{kd}$

$C_{33} = -M_A^2 \left[\cosh qd + \frac{q}{k} \sinh qd\right]$

$C_{34} = -M_A^2 \left[\cosh qd + \frac{k}{q} \sinh qd\right]$

$C_{41} = \sinh kd - M_c^2 \cosh kd$

$C_{42} = \cosh kd - M_c^2 \sinh kd$

$C_{43} = \frac{q}{|k|} \sinh qd - M_c^2 \cosh qd$

$C_{44} = \cosh qd - M_c^2 \frac{|k|}{q} \sinh qd$

The dispersion relation here will be examined in the limit of long wavelengths. It will be of use in section 4.4 where some approximate solutions will be obtained.

4.2.1. Long Wave Behavior.

When the conductivity of the fluid, and the magnetic field are both sufficiently small, a shallow water theory similar to that used in ordinary hydrodynamics may be established. This has been done by Fraenkel[13] who thoroughly discussed the problem when the fluid lies over a non-conducting bottom. The results developed here are valid for arbitrary conductivity of the bottom.

The behavior of small amplitide, shallow fluid waves will be established by considering the limiting forms of Eq. 4.36. Under the assumptions

a) $\left| kd \right| << 1$

b) $\left| qd \right| << 1$

c) $\left| \frac{q^2}{k^2} \; kd \right| << 1$

the hyperbolic functions in the dispersion relation may be replaced by the initial forms in their Taylor series.

The dispersion relation is

$$\left[\frac{q^2}{k^2} + M_A^2 \right] \; |k| d = (1 + M_A^2) M_c^2. \qquad (4.37)$$

Let

$$s = - j\omega.$$

Then the values of q^2, M_A^2, and M_c^2 from Eqs. 2.38, 4.11, and 4.35 may be introduced into 4.37 with the result

$$s^2 + \sigma\mu_o v_A^2 s + gk^2 d = 0. \qquad (4.38)$$

This is the result obtained by Fraenkel for the propagation of shallow fluid waves in a vertical magnetic field. In contrast to the shallow depth limit when the magnetic field is tangential to the free surface, the damping effect of the field on the wave motion is quite strong. This is because, for wavelengths long compared with the fluid depth, the fluid motion is primarily horizontal; consequently when the field is vertical, strong interaction results.

4.3. An Initial Value Problem in the Infinitely Deep Fluid.

It was mentioned in section 4.1 that, for certain values of wavelength and magnetic field, Roberts and Boardman had shown that the dispersion relation, Eq. 4.17, had no roots. It was also indicated that the dispersion relation could not explain the behavior of the perfectly conducting fluid. Evidently, a solution to an initial value problem is of interest in this situation.

For this analysis, the dependence of the stream functions on the x co-ordinate is assumed to be sinusoidal and is expressed as

$$\psi(x,y,t) = \tilde{\psi}(y,t)\, e^{jkx} \tag{4.39}$$

$$A(x,y,t) = \tilde{A}(y,t)\, e^{jkx}. \tag{4.40}$$

Equations 4.1 and 4.2 can be combined into a single partial differential equation in A.

$$(A_{(xx+yy)t} - \sigma\mu_o A_{tt} + \sigma\mu_o v_A^2 A_{yy})_{(xx+yy)} = 0 \tag{4.41}$$

The Laplace transform of $\tilde{A}$ is

$$\hat{A} = \int_0^\infty \tilde{A}\,(y,t)\, e^{-st}\, dt. \tag{4.42}$$

Making use of the assumed x dependence and indicating the y derivative by a capital D, the transform of Eq. 4.41 may be written

$$(D^2 - k^2)[(D^2 - k^2)s\hat{A} - \sigma\mu_o s^2\hat{A} + \sigma\mu_o v_A^2 D^2\hat{A}]$$

$$= (D^2 - k^2)[(D^2 - k^2)\tilde{A}(0,y) - \sigma\mu_o(s\tilde{A}(0,y) + \tilde{A}_t(0,y))] \tag{4.43}$$

As was shown in the tangential field case, Eq. 4.43 is found to be homogeneous for an important class of initial conditions, namely those motions arising from disturbance outside the fluid. The solution to Eq. 4.43 is then

$$\hat{A} = C_1 e^{|k|y} + C_2 e^{q(s,k)y} \tag{4.44}$$

where C_1 and C_2 are constants to be determined by the initial conditions and

$$q = \left\{\frac{k^2 s + \sigma\mu_o s^2}{s + \sigma\mu_o v_A^2}\right\}^{1/2} \tag{4.45}$$

with the square root to be chosen such that

$$\mathrm{Re}\ |q| \geq 0. \tag{4.46}$$

The boundary condition on the magnetic field gives the condition

$$(D + k)\hat{A} = 0. \tag{4.47}$$

It is necessary to express the free surface pressure condition

$$(P - \rho g\eta)\Big|_{y=0} = 0 \tag{4.48}$$

in terms of A and initial quantities. Now

$$\eta_{tt} = -\frac{1}{\rho} P_y. \tag{4.49}$$

Thus Eq. 4.48 can be written

$$(P_{tt} + gPy)\Big|_{y=0} = 0. \tag{4.50}$$

The equation necessary to express P in terms of A is

$$A_{(xx+yy)t} - \sigma\mu_o A_{tt} + \sigma\mu_o v_A^2 A_{(xx+yy)} = \frac{\sigma\mu_o B_o}{\rho} P_x. \tag{4.51}$$

The resulting transfomed boundary equation is

$$\begin{aligned}(s^2 + gD)[(D^2 - k^2)(s + \sigma\mu_o v_A^2)\hat{A} - \sigma\mu_o s^2 \hat{A}] \\ = [(D^2 - k^2)s^2 - \sigma\mu_o s^3 + \sigma\mu_o v_A^2(D^2 - k^2)s \\ + gD(D^2 - k^2) - gD\sigma\mu_o s]\tilde{A}(0,0) \\ +[(D^2 - k^2)s - \sigma\mu_0 s^2 + \sigma\mu_o v_A^2(D^2 - k^2) - g\sigma\mu_o D]\tilde{A}_t(0,0) \\ + [D^2 - k^2 - \sigma\mu_o s]\tilde{A}_{tt}(0,0) \\ - \sigma\mu_o\tilde{A}_{ttt}(0,0).\end{aligned} \tag{4.52}$$

Equations 4.47 and 4.52, along with an appropriate set of initial conditions, determine the behavior of the free surface.

Consider, in particular, the case of an initial unit pressure impulse at t = 0. The boundary condition, Eq. 4.48, is replaced by

$$(P - \rho g \eta)\Big|_{y=0} = \delta(t) e^{jkx}. \tag{4.53}$$

Hence, Eq. 4.52 becomes

$$(s^2 + gD)[(D^2 - k^2)(s + \sigma\mu_o v_A^2) - \sigma\mu_o s^2]\, \hat{A} = \frac{jk\sigma\mu_o B_o}{\rho} \quad s^2. \tag{4.54}$$

For this particular problem, the transformed solution for the fluid motion may now be obtained. Substitution of Eq. 4.55 into Eqs. 4.47 and 4.54 yields

$$2k\alpha + (k + q)\beta = 0 \tag{4.55}$$

and

$$(s^2 + gk)C_1 + \frac{v_A^2 k^2}{s^2}(s + gq)\; C_2 = - \frac{jkB_o}{\rho} \tag{4.56}$$

Evaluation of the constants from the above equation gives

$$\hat{A} = \frac{jkB_o}{\rho P} \quad [(k + q)e^{ky} - 2ke^{qy}] \tag{4.57}$$

where

$$P = 2k\, \frac{v_A^2 k^2}{s^2}\; (s + qg) - (k + q)(s + qk) \tag{4.58}$$

The zeros of P are the solutions to the previously obtained dispersion equation. The transformed solution is multi-valued in the complex frequency plane. It remains, therefore, to select the appropriate branch of the function defined in Eq. 4.57 and introduce branch cuts to make it single-valued before contour integral methods can be used to perform the inverse transformation.

The double-valued nature of $\hat{A}$ is caused by the square root operation involved in the definition of q. Boundary conditions at infinity require that

$$\text{Re } q > 0.$$

The branch points for $\hat{A}$ are the branch points for q, and these occur for

$$q^2 = 0 \tag{4.59}$$

or

$$q^2 = \infty. \tag{4.60}$$

Reference to Eq. 4.45 shows the branch points defined by Eq. 4.59 are located at

$$s = 0 \tag{4.61}$$

and

$$s = -\frac{k^2}{\sigma\mu_o} \tag{4.62}$$

Those defined by Eq. 4.60 are located at

$$s = \infty \tag{4.63}$$

$$s = -\sigma\mu_o v_A^2. \tag{4.64}$$

The cuts to be made in the s-plane are along a curve defined by the relationships

$$\text{Im } |q^2| = 0 \tag{4.65}$$

$$\text{R } |q^2| < 0 \tag{4.66}$$

The branch cut is always found along the real axis. Starting from s = 0, it proceeds along the real negative axis to the next branch point, which will be the smaller of the two values in Eqs. 4.62 and 4.64. The cut resumes at the next branch point and continues along the negative real axis to infinity, as in Fig. 4.2.

It is to be noted, in performing the contour integration that no contribution to the integral occurs due to the zeros of P at

$$s^2 = k^2 v_A^2 \tag{4.67}$$

since these are also zeros of the numerator of the fraction defining $\hat{A}$. This is a general property of all initial value solution, not just this particular one, as may be seen by placing arbitrary driving terms in both Eqs. 4.55 and 4.56. The other zeros of P correspond to poles of $\hat{A}$.

The solution then is of the form

$$A(y,t) = \sum_{i=1}^{n} \lim_{s \to s_k} (s - s_k)\, A\,(y,s)\, e^{s_k t}$$

$$+ \frac{1}{2\pi j} \int_C A\,(y,s)\, e^{st}\, ds. \tag{4.68}$$

where the contour C encloses the branch cut, as indicated in Fig. 4.2.

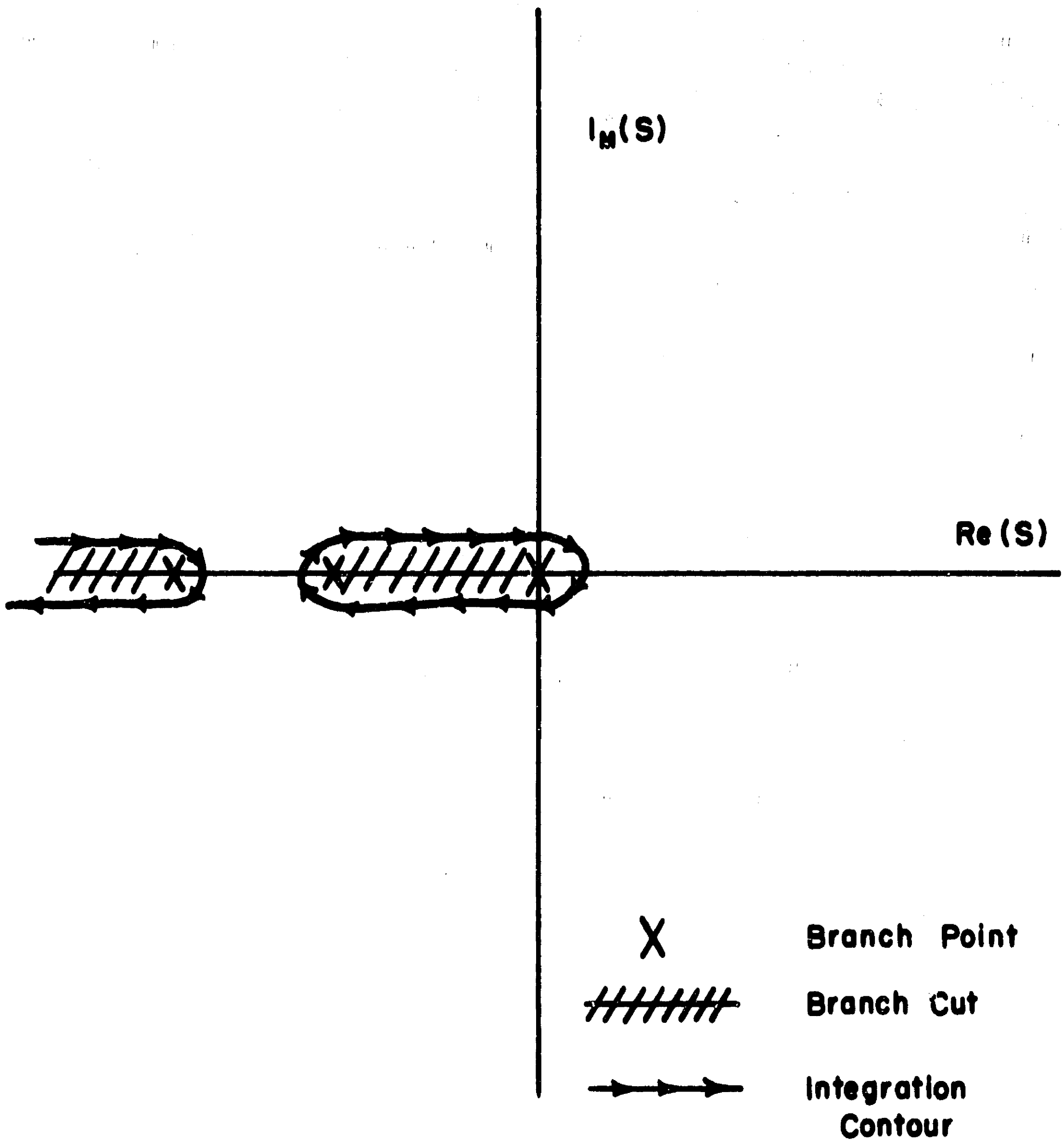

Figure 4.2 . BRANCH CUT AND INTEGRATION CONTOUR IN THE COMPLEX PLANE.

4.4. Interpretation of the Branch Cut Integral.

As was found in the gravity wave problem with a tangential magnetic field, the continuum of natural frequencies represented by the branch cut integral has as a counterpart in the finite depth problem, an infinity of discrete natural frequencies.

It is again possible to determine approximate solutions to the dispersion relation of Eq. 4.36 near critical values of the transverse propagation constant q which represent branch points in the infinite depth solution. These points are

1) $q = 0, \qquad s = 0$ (4.69)

2) $q = 0, \qquad s = -\dfrac{k^2}{\sigma\mu_o}$ (4.70)

3) $\left|\dfrac{1}{q}\right| = 0, \qquad s = \infty$ (4.71)

4) $\left|\dfrac{1}{q}\right| = 0, \qquad s = -\sigma\mu_o v_A^2.$ (4.72)

Approximate solutions of Eq. 4.36 near the point defined by Eq. 4.69 are determined by expanding Eq. 4.36 in a power series in M_A^2 and keeping only the first term. The result is

$$\cosh qd = 0. \qquad (4.73)$$

This equation can be satisfied only for purely imaginary values of q, which occur on the negative real axis. Since the magnitude of q is small compared with k in this region, a large value of kd is required before a solution can exist in this region. Expansion of Eq. 4.24 and Eq. 4.28 show all co-efficients but ψ_3 to be zero to zero order in the expansion parameter. Examination of the velocity stream function of Eq. 4.22 for q very small shows the velocity field to be approximately uniform in y and periodic in x, with the velocity primarily in the y direction. This is principally a shear flow. Examination of the magnetic stream function of Eq. 4.23 shows that for small M_A^2, there is negligible magnetic field associated with the motion. The approximate relationship between q and s is

$$s = \sigma\mu_o v_A^2 \frac{q^2}{k^2} \tag{4.74}$$

which, for q small, represents a slow decay of the motion in time.

For the region around the point defined by Eq. 4.70, s is non-zero and finite, but q is very small. The dispersion equation, Eq. 4.36, may be expanded in powers of q/k and the result when only the zero order term is taken is

$$\sinh qd = 0. \tag{4.75}$$

This equation is also soluble only when q is imaginary and consequently s is real and negative. The depth again must be large for a solution to exist since q is small. Expansion of Eq. 4.24 and Eq. 4.33 shows that to zero order in kd, ψ_3 is the only non-zero constant in the volume solution. The equations for the stream functions, Eq. 4.22

and Eq. 4.33, show that the velocity is a first order quantity in the expansion while the magnetic field is of zero order. The approximate value for s in these solutions is

$$s = \frac{-k^2}{\sigma\mu_o} \tag{4.76}$$

This is the time constant for decay of an initially periodic distribution of magnetic field in a solid conductor. The magnetic field in these solutions is primarily oriented in the y direction and is approximately uniform in y, with a sinusoidal dependence on x.

Near the critical point of Eq. 4.71, s is very large and an expansion of Eq. 4.36 may be made in inverse powers of M_A^2. Equation 4.36 then reduces to

$$\sinh qd = 0. \tag{4.77}$$

Expansion of Eq. 4.24 and Eq. 4.28 again shows ψ_3 to be the important constant in the expressions for the stream functions, with the other constants approximately zero. Since q approaches infinity, for large s and is imaginary for large negative real s, there exists an infinite number of solutions to Eq. 4.77 for any value of depth. Expansion of the equations for the velocity and magnetic stream functions Eq. 4.22 and Eq. 4.23, shows the magnetic field to be a zero order quantity and the velocity field a first order quantity and consequently negligible. The approximate relationship between s and q in this region is

$$s = \frac{q^2}{\sigma\mu_o} . \tag{4.78}$$

This is a [illegible] he decay rate for a magnetic field distribution in an immobile conducting medium. The magnetic field in this case is x-directed, and has a sinusoidal y dependence, with very slow variation in the x-direction.

In the neighborhood of the point defined by Eq. 4.72 s is finite and non-zero, while q is very large. Equation 4.36 may be expanded in a power series in k/q. The zero order term is

$$\cosh qd = o. \tag{4.79}$$

Expansion of Eq. 4.24 and 4.28 show that ψ_3 is the only constant which is zero order in the expansion parameter, the others being of first order. Since q is imaginary and approaches infinity on the negative real axis in this region, there are an infinite number of solutions to Eq. 4.79 in this region. The relationship for s is approximately

$$s = \sigma\mu_o v_A^2 . \tag{4.80}$$

Expansion of Eq. 4.22 and Eq. 4.23 show both the velocity and the magnetic field to be of zero order in the expansion parameter. Both the velocity and the field are approximately uniform in the x direction, are principally x-directed and have rapid sinusoidal dependence on y.

In order to locate in the mathematical expressions an effect identifiable as Alfven wave motion it is simplest to return to the integral expression of Eq. 4.68.

For

$$\sigma\mu_o > \frac{k}{v_A}$$

the branch cut extends from

$$s = 0$$

to

$$s = -\frac{k^2}{\sigma\mu_o}$$

and from

$$s = -\sigma\mu_o v_A^2$$

to

$$s = -\infty.$$

It is this second part of the branch cut to which attention is directed.

In the infinite conductivity limit, the branch cut integral takes a very simple form since

$$\lim_{\sigma\to\infty} q = \frac{s}{v_A} \qquad (4.81)$$

The branch cut has become an essential singularity at infinity. In this limit

$$P = -\frac{s}{v_A}\ (s - kv_A)[s^3 + 2|k|\ v_A s^2 + (g|k| + 2k^2v_A^2)s + 2gkv_A^2] \qquad (4.82)$$

and

$$A = \frac{jkB_o\pi s^2}{\rho P}\left[\left(|k| + \frac{s}{v_A}\right)e^{|k|y} - 2|k|e^{s/v_A\ y}\right] \qquad (4.83)$$

Let s_i be a solution of

$$0 = s^2 + 2\,k\;\; v_A s^2 + (g\,k\;\; + 2k^2 v_A^2)s + 2gk^2 v_A^2 \qquad (4.84)$$

Then the solution is of the form

$$0 = \sum_{i=1}^{3} A_i \mu_{-1}(t) e^{|k|\,y} e^{s_i t} + \sum_{i=1}^{3} B_i \mu_{-1}\left(t + \frac{y}{v_A}\right) e^{s_i\left(t + \frac{y}{v_A}\right)}$$

(4.85)

The second set of terms in the above summation clearly shows the propagation of a discontinuity in the motion at the Alfven velocity along the y axis.

This same set represents the limiting form of the integral along the branch cut from $-\infty$ to $-\sigma\mu_o v_A^2$ in the original problem, since the singularity at infinity which gave rise to these terms is formed by the conjunction of the two branch points which terminated the above branch cut.

The above perfect conductivity solution was first proposed by Roberts and Boardman.[4] Unfortunately, they evaluated the inverse transform incorrectly, attributing part of the solution to a singularity at $s = kv_A$, which does not exist.

The solutions to the infinite depth dispersion relation give the important time variation of the solution above, but the most interesting feature is the discontinuity propagating at the Alfven velocity.

Thus, the branch cut integrals in this vertical fluid problem have been shown to represent the same bulk fluid

Chapter V

Motions of a Bounded Surface.

The dispersion relations of the preceding sections have been derived for motion of an unbounded surface of fluid. In section 5.1 of this chapter, certain special end-walls are discussed, namely those which may be inserted into a standing wave solution on the unbounded surface without disturbing the motion.

In section 5.2 more general end wall constraints are discussed, along with suggestions for methods of solution in these cases.

Section 5.3 treats a bounded surface motion in cylindrical geometry. Numerical results are obtained by approximate computational methods.

Section 5.4 contains a brief discussion of possible experiments associated with the above theoretical results.

5.1. Simple End Wall Constraints.

In this section, the properties of an end wall which can be inserted into the unbounded fluid without disturbing the motion of the fluid will be investigated. For simplicity, all end boundaries considered here will be solid walls. Thus all such boundaries are to be introduced into a standing wave solution at a point of zero transverse velocity. In a standing wave motion, this is a point of maximum vertical velocity. Consequently, the fluid and the surface must slide freely along the wall.

This wall condition is frequently used in hydrodynamics.* In the hydromagnetic problem it is necessary in addition to specify the electrical properties of the wall.

Two cases will be considered. One is motion under a horizontal impressed magnetic field. The other is motion in a vertical field.

When the impressed magnetic field is horizontal, a standing wave disturbance has, at a maximum of vertical velocity, a finite horizontal magnetic field and a zero vertical magnetic field. An end wall to be inserted at this point must be of infinite magnetic permeability in addition to having the above mechanical properties. A wall of soft iron faced with some material to protect it from the liquid metal would adequately approximate such a boundary in a laboratory experiment.

When the impressed magnetic field is vertical, a zero tangential electric field and zero normal magnetic field are found at a maximum of vertical velocity. The tangential magnetic field in the fluid is finite. A perfect conductor is required for the boundary here. Kliman[10] has shown that a copper wall represents a good approximation to such a condition

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* See Milne-Thompson,[9] Article 14.14.

when the fluid is NaK and a very good approximation when the fluid is mercury. The latter combination, however, is chemically unsatisfactory.

5.2. Comments on More General Boundary Constraints.

In the preceding section it was shown that certain boundaries may be inserted into a standing wave solution on an unbounded surface without disturbing the form of the solution. When such boundaries are satisfactorily approximated by a physical situation, dispersion relations previously obtained directly yield the natural frequencies of surface disturbances in the container.

When the boundary conditions are of other than this simple type, a more involved approach is needed. The container and fluid are represented by a system of differential equations, combined with appropriate boundary conditions. Such a system may be characterized by a number of eigenvalues, in this case the natural frequencies of the system. An eigenfunction, in this case the vector function describing the velocity field or the magnetic field, will be associated with each such eigenfunction. Suppose the container has a bottom boundary and a natural frequency is known. Then from the appropriate dispersion relation, suitably modified, an infinite sequence of velocity functions may be found. A linear combination of these which satisfies the end wall conditions is the required eigenfunction.

Two points need further discussion. The first concerns the dispersion relations in Chapters III and IV which were all derived for real values of k. To make them valid for all k, it is only necessary to replace $|k|$ by $(k^2)^{1/2}$ where the root with the positive real part is to be taken.

Secondly, in principle, one finds the natural frequencies ω of the system from the dispersion equation by determining for which values of ω the infinite number of velocity function can form an infinite series which satisfies the boundary condition. In practice, this method would seem to be restricted to limiting cases in which the dispersion relation has a particularly simple form.

Another method deserves brief mention. If the boundaries are rectangular, the dimensional dependence in the surface plane may be expanded in an Fourier series. A series solution for the eigenvalues and eigenfunctions may then be obtained. This process is, however, not without difficulties. In the theory of shallow water hydrodynamic waves, an approximate solution is obtainable for a situation in which, in addition to the usual constraint that the fluid velocity be tangential to the end walls the additional constraint that the end point of the free surface not move is imposed. When an attempt is made to solve this problem by Fourier expansion methods, it is found that upon removal of the first harmonic term, the remainder of the solution is concentrated near the end walls. Such concentrated disturbances require a large number of terms for adequate representation by a Fourier series.

The author suggests two methods for proceeding in such problems. The first is the computation of the large number of terms required in the Fourier series expansion. The second is the possibility of patching an approximate solution near the boundary to a simple sinusoidal standing-wave solution in the center. The latter, if done carefully, would seem to provide a useful physical picture of the motion near the walls, which a series solution tends to obscure.

Only the ideal wall conditions of section 5.1 will be treated in this thesis.

5.3. A Hydromagnetic Surface Wave Resonator.

In this section, the motion of a finite surface of conducting fluid under the influence of an impressed vertical magnetic field, gravity, and surface tension is considered. The boundary walls are of perfect electrical conductivity and allow the fluid surface to move freely at the wall. This is the simple end wall constraint discussed in section 5.1.

A simple situation, both geometrically and mathematically, occurs when the boundary walls form a right circular cylinder.

5.3.1. The Unperturbed System.

An externally applied magnetic field is uniform throughout all space, of value B_o, and directed along the positive z axis. The fluid forms a cylindrical segment of radius r_o, bounded by the planes $z = 0$ and $z = d$. The fluid is assumed to be inviscid, incompressible, of uniform density ρ, and to possess a uniform isotropic conductivity σ and a surface tension τ. The semi-infinite cylinder defined by $r < r_o$ and $z > d$ is a vacuum. The rest of space is filled with a rigid material of infinite electrical conductivity. The geomtery of the system is shown in Figure 5.1.

5.3.2. Assumed Form of the Surface Disturbance.

The frequency of oscillation and rate of decay of small perturbations of the free surface are to be determined. Let the perturbed surface be specified by the equation

$$z = d + a(r, \theta, t) \tag{5.1}$$

where

$$a << d \tag{5.2}$$

for all r, θ, and t.

Further, let $a(r, \theta, t)$ be of the form

$$a = \delta \, e^{j(n\theta - \omega t)} \, J_n(kr) \tag{5.3}$$

5.3.3. Equations for Bulk Motion.

The equations for bulk motion linearized to first order in δ are Eq. 2.25c and Eq. 2.26 with the magnetic field purely in the vertical direction.

$$\frac{\partial}{\partial t} \nabla^2 \overline{\psi} = \frac{B_o}{\rho \mu_o} \frac{\partial}{\partial z} \nabla^2 \overline{A} \tag{5.4}$$

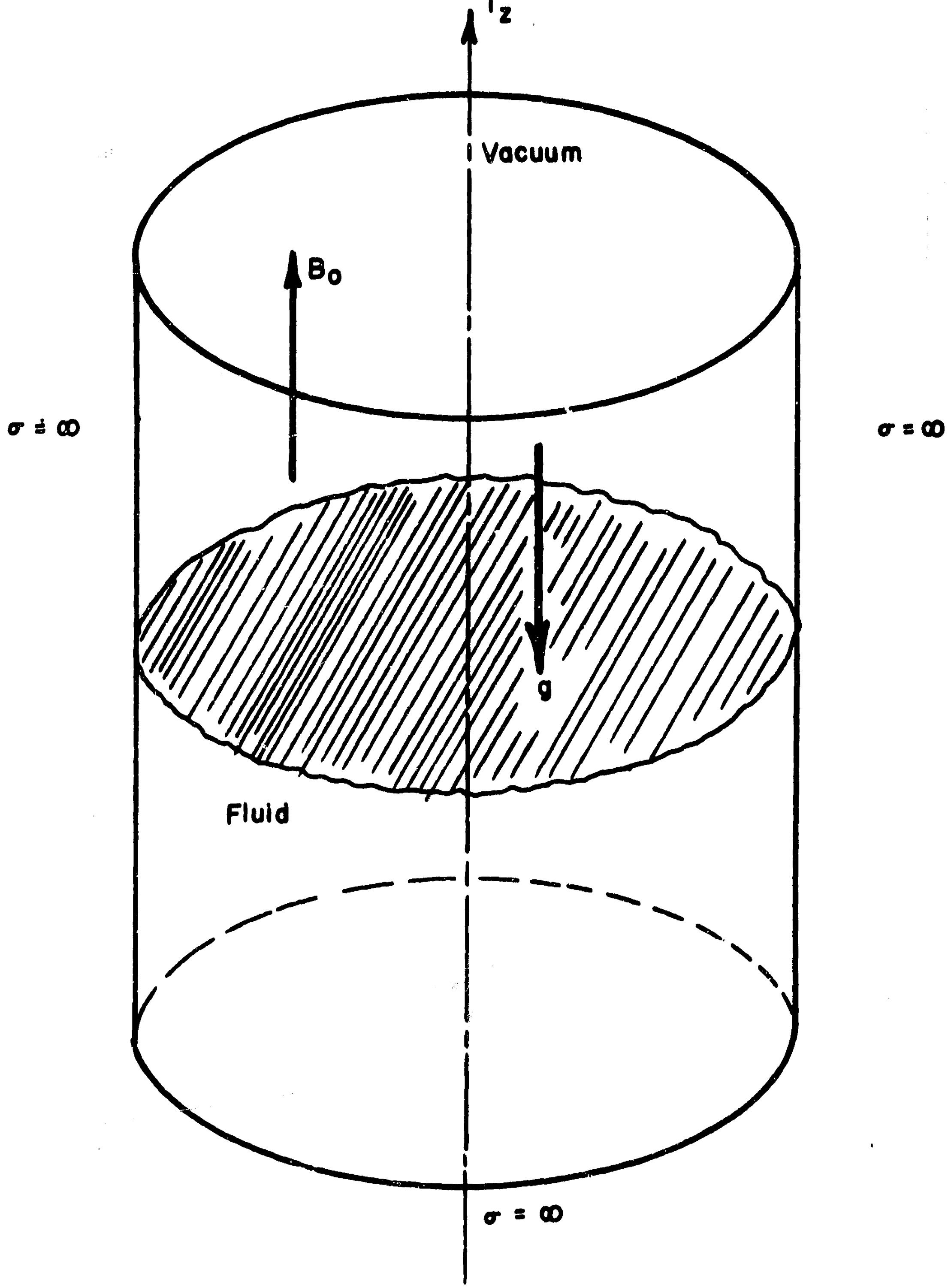


Figure 5.1 . CYLINDRICAL SURFACE WAVE RESONATOR

$$\nabla^2\bar{A} - \sigma\mu_o \frac{\partial\bar{A}}{\partial t} - \sigma\mu_o\nabla\phi = \sigma\mu_o B_o \left[\nabla\psi_z - \frac{\partial}{\partial z}\bar{\psi}\right] \tag{5.5}$$

In view of the assumed form of the surface perturbation, the solutions are obtained as

$$\bar{\psi} = e^{j(n\theta-\omega t)} \; i_r \frac{n}{kr} J_n(kr) + i_\theta j\, J_n'(kr) \quad \left[\psi_1 \sinh kz + \psi_2 \cosh kz + \psi_3 \sinh \beta z + \psi_4 \cosh \beta z \right] \tag{5.6}$$

$$A = \frac{jk\,B_o}{\omega} e^{j(n\theta-\omega t)} \; i_r \frac{n}{kr} J_n(k_r) + i_\theta j\, J_n'(kr) \quad \left[\psi_1 \cosh kz + \psi_2 \sinh kz - \frac{k}{\beta}\left(\frac{\omega^2}{k^2 v_A^2}\right) \psi_3 \cosh \beta z - \frac{k}{\beta}\left(\frac{\omega^2}{k^2 v_A^2}\right) \psi_4 \sin \ \beta z \right] \tag{5.7}$$

where

$$v_A^2 = \frac{B_o^2}{\rho\mu_o} \tag{5.8}$$

and

$$\frac{\beta}{k}^2 = \frac{1 - \dfrac{j\omega\sigma\mu_o}{k^2}}{1 + j\omega\sigma\mu_o \dfrac{v_A^2}{\omega^2}} \tag{5.9}$$

There are additional solutions to Eqs. 5.4 and 5.5, but they do not perturb the free surface, and are not coupled to the above solutions at the boundaries.

5.3.4. Boundary Conditions.

The boundary condition on the velocity of the rigid walls is

$$\bar{v} \cdot \bar{n} = 0 \tag{5.10}$$

where $\bar{n}$ is a vector normal to the rigid surface. This condition at $r = r_o$ gives

$$J_n' \, (kr_o) = 0. \tag{5.11}$$

This equation determines the transverse wave numbers which are permitted.

Applying the above condition at $z = 0$, we find

$$\psi_2 + \psi_4 = 0. \tag{5.12}$$

The perfectly conducting boundaries require that the tangential E field, and hence the tangential component of $\bar{A}$ vanish at those boundaries. The condition implied by Eq. 5.11 fixes A_{tan} equal to zero at $r = r_o$. The boundary at $z = 0$ yields the relationship

$$\psi_1 - \left(\frac{k}{\beta} \right) \left(\frac{\omega^2}{k^2 v_A^2} \right) \psi_3 = 0. \tag{5.13}$$

The condition that the electric and magnetic fields are continuous across the free surface is equivalent to a requirement that $\bar{A}$ and $\partial\bar{A}/\partial z$ be continuous across that boundary. The resulting equation is

$$\psi_1 e^{|k|d} + \psi_2 e^{|k|d} - \psi_3 \left(\frac{\omega^2}{k^2 v_A^2}\right)\left[\frac{|k|}{\beta}\cosh \beta d + \sinh \beta d\right]$$

$$- \psi_4 \left(\frac{\omega^2}{k^2 v_A^2}\right)\left[\frac{|k|}{\beta}\sinh \beta d + \cosh \beta d\right] = 0. \qquad (5.14)$$

The remaining boundary condition is that the discontinuity in the normal stress of the surface be zero, or

$$P\Big|_{z=d} + \frac{T}{R} + \rho g a = 0 \qquad (5.15)$$

where

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \qquad (5.16)$$

R_1 and R_2 are the principal radii of curvature and P is the perturbation pressure.

The expressions for 1/R, P, and a in terms of ψ_1, ψ_2, ψ_3 and ψ_4 are

$$\frac{1}{R} = -\frac{k^3}{\omega} J_n(kr)e^{j(n\theta-\omega t)}\Big[\psi_1 \sinh kd + \psi_2 \cosh kd$$
$$+\psi_3 \sinh \beta d + \psi_4 \cosh \beta d\Big] \qquad (5.17)$$

$$P\Big|_{z=d} = \rho\omega\, J_n(kr)e^{j(n\theta-\omega t)} \Big[\psi_1 \cosh kd + \psi_2 \sinh kd$$
$$+ \frac{k}{\beta}\psi_3 \cosh kd + \frac{k}{\beta}\psi_4 \sinh kd\Big] \qquad (5.18)$$

$$a = \frac{k}{\omega} e^{j(n\theta-\omega t)} J_n(kr) \Big[\psi_1 \sinh kd + \psi_2 \cosh kd$$
$$+ \psi_3 \sinh \beta d + \psi_4 \cosh \beta d\Big] \qquad (5.19)$$

When Eqs. 5.17 through 5.19 are used, Eq. 5.15 becomes

$$\begin{aligned} &\psi_1[M_H^2 \cosh kd - \sinh kd] \\ &+ \psi_2[M_H^2 \sinh kd - \cosh kd] \\ &+ \psi_3[\tfrac{k}{\beta} M_H^2 \cosh \beta d - \sinh \beta d] \\ &+ \psi_4[\tfrac{k}{\beta} M_H^2 \sinh \beta d - \cosh \beta d] = 0 \end{aligned} \tag{5.20}$$

where

$$M_H^2 = \frac{\omega^2}{\frac{Tk^3}{\rho} + gk} . \tag{5.21}$$

The dispersion relation, in determinental form, is obtained by combining Eqs. 5.12, 5.13, 5.14 and 5.20. It is Eq. 5.22.

The dispersion relation of Eq. 5.22 is, when surface tension is neglected, identical to that of Eq. 4.36, which was obtained for two-dimensional motion. The importance of Eq. 5.22, from a practical point of view, is that the volume of fluid to which it applies is bounded in three dimensions. The limitations on construction of such a system in the laboratory are then the problems of approximately realizing the ideal boundary conditions.

5.3.5. Numerical Solutions.

Equation 5.22 was solved by machine computation using the Newton-Raphson method. Fluid parameters of NaK were used and a depth of 5 cm was assumed. The real and imaginary parts of ω for several values of real k are shown in Figure 7.2 as a

$$\left|C_{ij}\right| = 0$$

$$C_{11} = 0 \qquad C_{31} = e^{|k|d}$$

$$C_{12} = 1 \qquad C_{32} = e^{|k|d}$$

$$C_{13} = 0 \qquad C_{33} = -M_A^2\left(\frac{|k|}{\beta}\cosh\beta d - \sinh\beta d\right)$$

$$C_{14} = 1 \qquad C_{34} = -M_A^2\left(\cosh\beta d - \frac{|k|}{\beta}\sinh\beta d\right)$$

$$C_{21} = \frac{\beta}{|k|}M_{A_2} \qquad C_{41} = \sinh kd - M_H^2\cosh kd$$

$$C_{22} = 0 \qquad C_{42} = \cosh kd - M_H^2\sinh kd$$

$$C_{23} = -1 \qquad C_{43} = \sinh\beta d - \frac{|k|}{\beta}M_H^2\cosh\beta d$$

$$C_{24} = 0 \qquad C_{44} = \cosh\beta d - \frac{|k|}{\beta}M_H^2\sinh\beta d$$

(5.22)

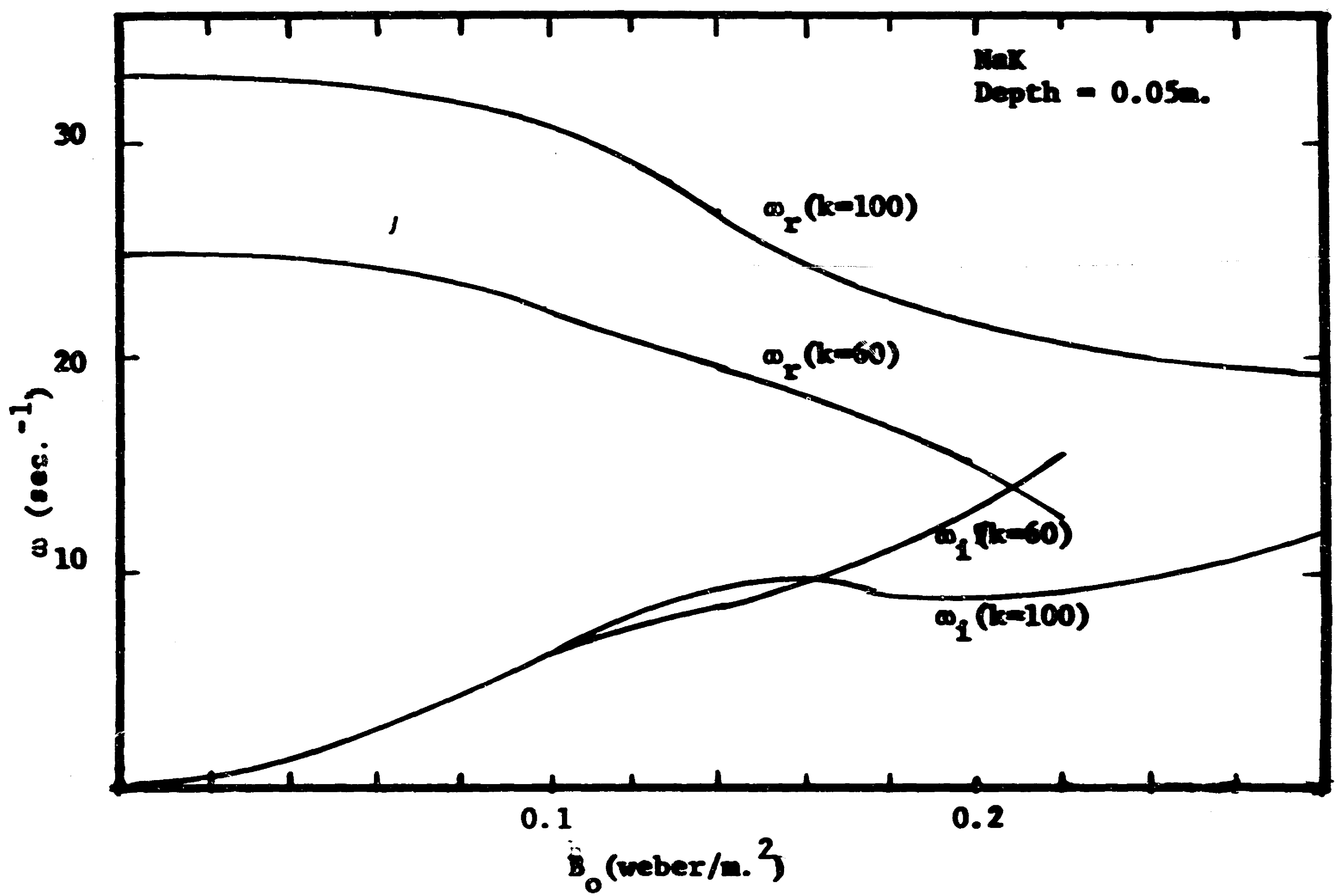


FIG. 5.2 NATURAL FREQUENCIES OF A SURFACE WAVE RESONATOR

function of the magnetic field.

Solutions were obtained in the following manner. At zero magnetic field, Eq. 3.46 gives a value of ω. For each succeeding point, the value of the magnetic field was incremented and an extrapolation of preceding solutions was used to obtain an initial estimate. The accuracy of the estimate was increased by successive applications of the Newton-Raphson technique until three successive estimates of ω differed by less than 10^{-5} times the estimated value of ω.

5.4. Possible Experimental Tests.

The interesting features of the numerical solutions presented above are the frequency shift, and damping of the modes caused by the application of a magnetic field. The wavelengths, fields, and other physical dimensions of the above solutions are obtainable in the laboratory.

An experimental check of the above results may be made if amplitude response vs. frequency of excitation can be obtained for the driven resonator.

Several conditions must be met before this can be accomplished. First, the experiment must be designed so that only one mode of oscillation is strongly excited. Second, that mode must have a reasonably high Q. Third, an excitor which delivers a constant amplitude excitation over a desired range of frequencies must be available. Finally, a method of measuring the amplitude of the response is needed.

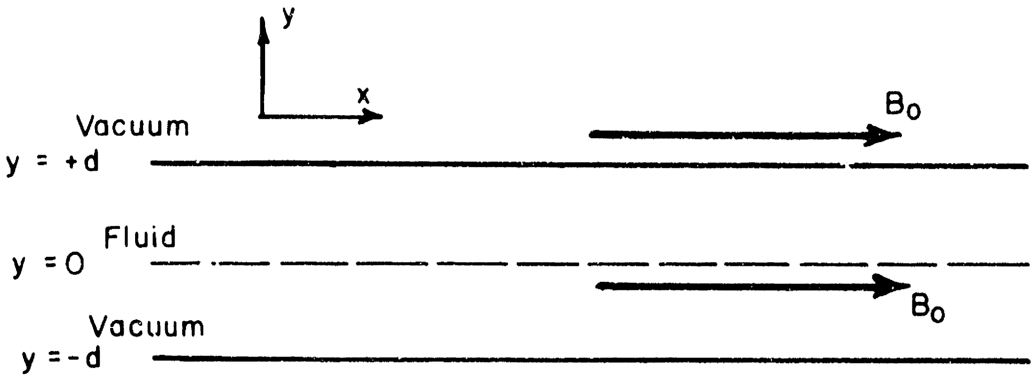
Chapter VI.

Surface Motions of a Slab of Conducting Fluid.

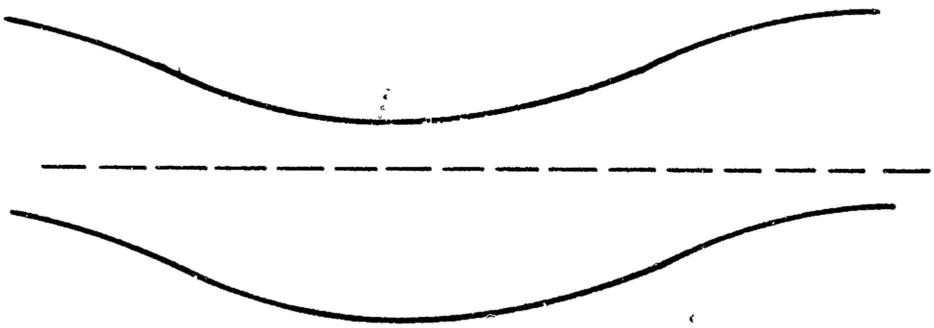
In the preceding sections, emphasis has been placed upon the modification of hydrodynamic gravity waves by the presence of electrical fluid conductivity and an externally applied magnetic field. The subject of surface motions on slabs and cylinders of fluid will now be investigated, where, in the absence of a magnetic field, restoring forces for small surface distrubances are provided by the fluid surface tension.

A convenient introduction to this problem is provided by the following model. Consider an electrically conducting fluid, inviscid and incompressible, infinite in extent in the x and z directions but bound in the equilibrium configuration by the planes $y = \pm d$. A uniform magnetic field lies along the x axis. We consider two-dimensional motion in the x-y plane and all quantities will have zero derivative with respect to z. The geometry under consideration is shown in Fig. 1.

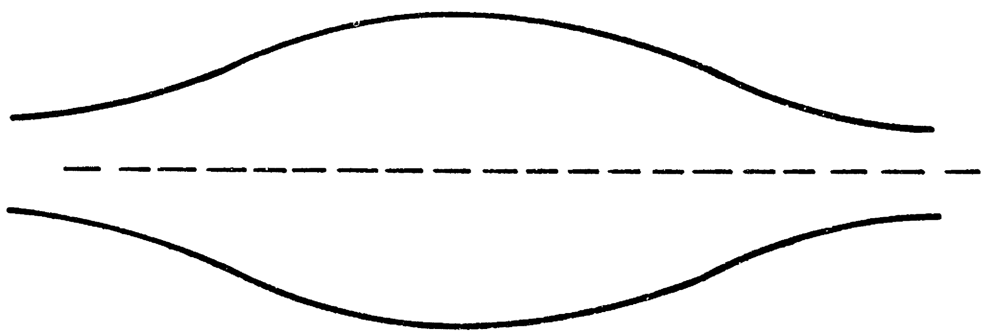
Before entering into the analysis, it is useful to indicate the reasons for undertaking such a problem. By treating two dimensional motion in Cartesian co-ordinates, all velocities and fields can be found in terms of scalar functions of space and time, and these functions will consist of familiar



UNPERTURBED FLUID SLAB



ANTISYMMETRIC PERTURBATION OF THE SLAB



SYMMETRIC PERTURBATION OF THE SLAB

Figure 6.1

WAVE MOTION ON A FLUID SLAB

circular and hyperbolic functions, whose behavior is well known.

Secondly, it is important to emphasize the similarities between the foregoing gravity wave problems and the motion of slabs and cylinders. Since the preceding analysis have been carried out principally in Cartesian co-ordinates, it would seem simplest to begin in the same co-ordinate system in order that physical and mathematical insights may be carried over from one problem to the next.

Finally, it is not uncommon to examine a two dimensional plane-parallel flow in order to gain insight into the related problem in cylindrical co-ordinates. We shall endeavor to point out, in the succeeding chapter both the similarities and the very marked differences between these two geometries.

6.1. Statement of the Problem.

Consider a fluid of infinite extent in the x and z directions. When at rest, the fluid is bounded by the planes $y = \pm d$. The fluid is assumed to be inviscid, incompressible and electrically conducting. This conductivity is uniform and isotropic. The other pertinent characteristics of the fluid are its uniform density ρ and its surface tension T.

The fluid is permeated by a magnetic field, which is uniform throughout space, of value B_o, and directed along the x axis.

Small perturbations of the surface from its equilibrium value will be considered. These perturbations, along with the velocity and magnetic fields, are assumed independent of the z-coordinate, and there will be no z-component of those fields.

The perturbed surfaces are described by the following pair of equations. For the upper surface

$$y = d + \eta_1 (x,t). \tag{6.1}$$

For the lower surface

$$y = - d - \eta_2 (x,t) \tag{6.2}$$

where

$$\eta << d$$

for all x and t.

Let us consider then two cases, an antisymmetric mode

$$\eta_1 = - \eta_2 \tag{6.3}$$

and a symmetric mode

$$\eta_1 = \eta_2. \tag{6.4}$$

The form of the surface perturbation is assumed to be

$$\eta_1 (x,t) = \delta e^{j(kx-\omega t)} \tag{6.5}$$

6.2. Motion In the Fluid Bulk.

The fluid motion equations are those discussed in section 2.1. From section 2.1.7 the required solutions may be obtained. It is then necessary to form linear combinations of those solutions in order to obtain expressions which are symmetric or antisymmetric about the y axis.

To obtain the solution desired for the symmetric mode and the antisymmetric mode, the arguments are the same and the formulas similar. We shall then transcribe the formulas in pairs, the first member of each pair representing antisymmetric motion of the slab, and the second, symmetric motion.

Introducing, as in the previous sections, the stream functions ψ and A for the velocity field and the perturbation magnetic field respectively, the solutions to the bulk motion of the fluid may be written

$$\overline{v} = \psi_y \overline{i}_x - \psi_x \overline{i}_y \tag{6.6}$$

$$\overline{b} = A_y \overline{i}_x - A_x \overline{i}_y . \tag{6.7}$$

For the antisymmetric case

$$\psi_a = [\psi_1 \cosh k_y + \psi_2 \cosh \beta_y] \, e^{j(kx-\omega t)} . \tag{6.8}$$

For the symmetric case

$$\psi_s = [\psi_1 \sinh k_y + \psi_2 \sinh \beta_y] \, e^{k(kx-\omega t)} \tag{6.9}$$

$$A_a = - \frac{kB_o}{\omega} [\psi_1 \cosh k_y + M_A^2 \psi_2 \cosh \beta_y] \, e^{j(kx-\omega t)} \tag{6.10}$$

$$A_s = - \frac{kB_o}{\omega} [\psi_1 \sinh k_y + M_A^2 \psi_2 \sinh \beta_y] \, e^{j(kx-\omega t)} \tag{6.11}$$

where

$$\frac{B^2}{k^2} = 1 - \frac{j\omega\sigma\mu_o}{k^2} \left[1 - M_A^{-2} \right] \tag{6.12}$$

and

$$M_A^2 = \frac{\omega^2}{k^2 v_A^2} \tag{6.13}$$

By separating the solutions, as has been done above, the boundary conditions of symmetry or antisymmetry have been satisfied. It remains only to assure the continuity of the magnetic field and the balance of surface forces at either boundary to complete the problem.

6.3. Boundary Conditions.

Since the motion will be considered either symmetric or antisymmetric about the plane y = 0, it is sufficient to consider only one pair of boundary conditions, and the upper boundary, (y = d), will be chosen.

First consider the stream function for the perturbation magnetic field above the fluid. The space is current free, hence A must be a solution to Laplace's equation which vanishes at large distances from the slab.

$$A = A_o \, e^{j(kx-\omega t)} \, e^{-|k|(y-d)} \tag{6.14}$$

As has been previously established, the magnetic fields are continuous across the boundary when the x and y derivatives of A are continuous. Equations 6.10 and 6.11, along with Eq. 6.14 when subjected to the above constraint, yield a condition on the constants, ψ_1 and ψ_2.

$$\psi_1 e^{|k|d} + M_A^2 \psi_2 \left[\cosh \beta d + \frac{\beta}{|k|} \sinh \beta d \right] = 0 \qquad (6.15)$$

for the antisymmetric case

$$\psi_1 e^{|k|d} + M_A^2 \psi_2 \left[\sinh \beta d + \frac{\beta}{|k|} \cosh \beta d \right] = 0 \qquad (6.16)$$

for the symmetric case.

The remaining boundary condition is the balance of the surface caused by surface tension and the discontinuity in fluid pressure

$$P \Big|_{y=d} = \frac{T}{R} \qquad (6.17)$$

where R is the radius of curvature of the surface. When the surface motion is small then, to a sufficiently accurate approximation,

$$\frac{1}{R} = - \eta_{xx}. \qquad (6.18)$$

The vertical velocity at the free surface is equal to the time rate of increase of the free surface elevation.

$$\eta_t = - \psi_x \Big|_{y=d} \qquad (6.19)$$

The perturbation pressure is most easily obtained from

$$\psi_{yt} = - \frac{1}{\rho} R_x. \qquad (6.20)$$

Combining Eq. 6.17 through 6.20 with the previously found expressions for ψ from Eqs. 6.8 and 6.9 the second relationship between the co-efficients ψ_1 and ψ_2 is obtained

$$0 = \psi_1 \left[\sinh kd - \frac{k^3T}{\rho\omega^2} \cosh kd \right] + \psi_2 \left[\frac{\beta}{k} \sinh \beta d - \frac{k^3T}{\omega^2\rho} \cosh \beta d \right] \tag{6.21}$$

$$0 = \psi_1 \left[\cosh kd - \frac{k^3T}{\rho\omega^2} \sinh kd + \psi_2 \frac{\beta}{k} \cosh \beta d - \frac{k^3T}{\omega^2\rho} \sinh \beta d \right] . \tag{6.22}$$

The pair of equations 6.21 and 6.22 may be combined with the pair of equations 6.15 and 6.16 to give the dispersion relations between k and ω for symmetric and antisymmetric oscillations of the fluid slab. The resulting equations, in determinental form are given by Eqs. 6.23 and 6.24 as:

$$\begin{vmatrix} e^{kd} & M_A^2 \cosh \beta d + \frac{\beta}{k} \sinh \beta d \\ k \sinh kd - \frac{k^4T}{\omega^2\rho} \cosh kd & \beta \sinh \beta d - \frac{k^4T}{\omega^2\rho} \cosh \beta d \end{vmatrix} = 0 \tag{6.23}$$

$$\begin{vmatrix} e^{kd} & M_A^2 \frac{k}{\beta} \sinh \beta d + \cosh \beta d \\ \cosh kd - \frac{k^3T}{\omega^2\rho} \sinh kd & \cosh \beta d - \frac{k^4T}{\omega^2\rho\beta} \sinh \beta d \end{vmatrix} = 0 \tag{6.24}$$

Dispersion relation for antisymmetric and symmetric disturbances respectively.

In the limit as either the magnetic field or the conductivity approaches zero, Eq. 6.23 and Eq. 6.24 become

$$\frac{\omega^2}{k^2} = \frac{kT}{\rho} \coth kd \qquad \text{(antisymmetric)} \tag{6.25}$$

$$\frac{\omega^2}{k^2} = \frac{kT}{\rho} \tanh kd \qquad \text{(symmetric)} \tag{6.26}$$

In the limit as the conductivity becomes infinite, the mode with β as a transverse propagation constant becomes a surface current, and the dispersion relation simplifies to

$$\frac{\omega^2}{k^2} = \frac{kT}{\rho} \coth kd + (1 + \coth kd)\, v_A^2 \quad \text{(antisymmetric)} \tag{6.27}$$

$$\frac{\omega^2}{k^2} = \frac{kT}{\rho} \tanh kd + (1 + \tanh kd)\, v_A^2 \quad \text{(symmetric)} \tag{6.28}$$

Equations 6.25 and 6.26 represent simple hydrodynamic waves, while Eqs. 6.27 and 6.28 represent special cases of the formulae given by Melcher[5] for oscillations of a planar fluid slab.

It may be noted that, just as in the gravity wave problems, we have two modes of fluid motion, independent in the fluid bulk and coupled at the boundary conditions. As before, the mode with transverse propagation constant k is lossless, containing no flow of electrical current. The mode with transverse propagation constant β is then the lossy one and energy is transported from the lossless irrotational mode to the lossy mode by the boundary conditions.

Analysis of three limiting cases which will show the effect of finite non-zero conductivity will now be undertaken. These will be the behavior at long wave lengths, at low conductivity, and at high conductivity.

6.4. The Asymptotic Behavior for Long Wave Length Disturbances.

When the wave length, the fluid conductivity and the impressed magnetic field are small enough to enable the following inequalities to hold:

$$kd << 1$$

$$\left|\frac{\beta^2}{k^2}\right| \quad kd << 1$$

the hyperbolic functions in the dispersion relation may be replaced by the initial terms in their Taylor series expansions. When this is done, we have, for the antisymmetric case

$$\omega^2 + j\omega\sigma\mu_o v_A^2 - \frac{k^2 T}{\rho d} = 0 \tag{6.29}$$

and for the symmetric case

$$\omega^2 - \frac{k^4 T d}{\rho} = 0 \ . \tag{6.3}$$

The lack of effect of the magnetic field on the symmetric oscillations is easily understood if we examine, to the same order of approximation, the velocity field

$$\bar{v} = \psi_y \bar{i}_x - \psi_x \bar{i}_y$$

$$= k \left(\psi_1 + \frac{\beta}{k} \psi_2 \right) \quad e^{j(kx-\omega t)} \bar{i}_x \qquad (6.31)$$

$$+ jk^2 d \left(\psi_1 + \frac{\beta}{k} \psi_2 \right) \quad e^{j(kx-\omega t)} \quad \bar{i}_y .$$

The motion is thus predominantly along the field lines and hence the damping, which is proportional to the square of the transverse velocity component is, to the order of approximation here employed, zero.

In the antisymmetric case, the dominant velocity component is across the magnetic field lines, hence the damping is sensible.

The Eq. 6.30 may be solved directly for ω

$$\omega = - \frac{j\sigma\mu_o v_A^2}{2} \pm \left[\frac{k^2 T}{\rho d} - \left(\frac{\sigma\mu_o v_A^2}{2} \right)^2 \right]^{1/2} \qquad (6.32)$$

Figures 6.2 and 6.3 show curves of complex ω vs. the wave length with NaK and Hg assumed for the working fluids. Some liberties have been taken with our expansion as the curves are considered to be accurate when

$$kd < 0.1$$

$$\frac{\beta^2}{k^2} \quad kd < 0.1.$$

It may be seen from the limited ranges of the curves that this

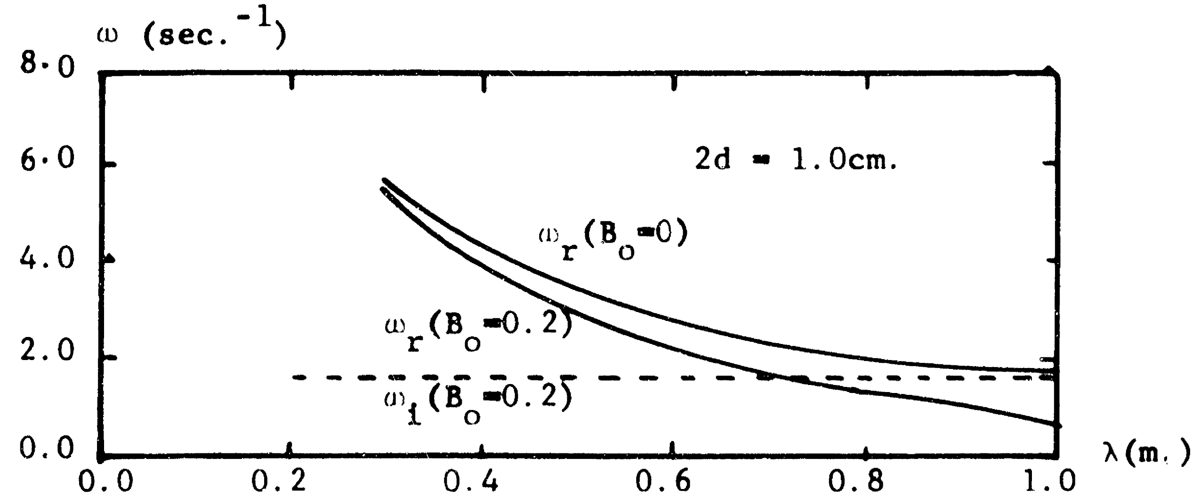


FIG. 6.2 NATURAL FREQUENCY VS. WAVELENGTH
THIN MERCURY SLAB

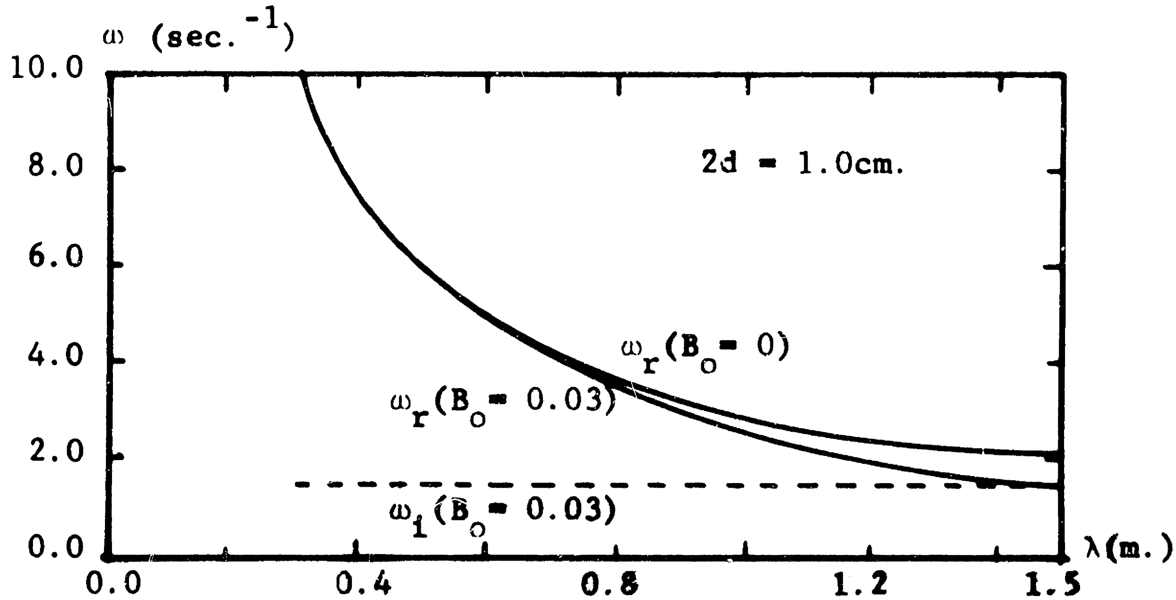


FIG. 6.3 NATURAL FREQUENCY VS. WAVELENGTH
THIN NAK SLAB

expansion suffers a breakdown at quite low fields. This will not be the case in circular geometry, where quite large fields may be used in such an expansion.

6.5. Low Conductivity Limit.

When the fluid conductivity and the magnetic field are low enough so that

$$\frac{\sigma\mu_o\omega}{k^2} << 1, \quad \frac{\sigma\mu_o v_A^2}{\omega} << 1$$

the motion is very nearly irrotational and the dispersion relation simplifies considerably. An expansion of the dispersion relation which begins with the hydrodynamic solution is then easy to perform.

$$\omega = \omega_o + R_M\omega_1 + \cdots \tag{6.33}$$

where for the antisymmetric case

$$\omega_o = \pm \left[\frac{k^3T}{\rho} \coth kd\right]^{1/2} \tag{6.34}$$

and for the symmetric case

$$\omega_o = \pm \left[\frac{k^3T}{\rho} \tanh kd\right]^{1/2} \tag{6.35}$$

and R_M is a characteristic magnetic Reynolds number for the problem

$$R_M = \frac{\omega_o\sigma\mu_o}{k^2} \tag{6.36}$$

The transverse propagation constant β may also be expanded in such a series

$$\beta = \beta_o + R_M \beta_1 . \tag{6.37}$$

It is found that

$$\beta_o = k \tag{6.38}$$

$$\beta_1 = \frac{jk}{2M_A^2} \ (1 - M_A^2) . \tag{6.39}$$

When such series expansions are made of the dispersion equations, it is found that for the antisymmetric case

$$R_M \omega_1 = - \frac{j \sigma \mu_o v_A^2}{4} \ [kd \coth kd + (1 - kd)] \tag{6.40}$$

and for the symmetric case

$$R_M \omega_1 = - \frac{j \sigma \mu_o v_A^2}{4} \quad 1 - \frac{kd}{\tanh kd} \ (1 - \tanh^2 kd) \quad . \tag{6.41}$$

Thus the first order correction terms are proportional to the square of the magnetic field and to the conductivity and represent damping when the term in brackets is a positive quantity. Let

$$A_a(x) = [x \coth x + (1 - x)] \tag{6.42}$$

$$A_s(x) = [1 - x \coth x(1 - \tanh^2 x)] \tag{6.43}$$

Then the nature of the correction depends upon the behavior of $A_a(x)$ and $A_s(x)$. The limiting terms are easily obtained

$$\lim_{x \to 0} \ A_a(x) = 2 \tag{6.44}$$

$$\lim_{x \to 0} A_s(x) = 0 \tag{6.45}$$

$$\lim_{x \to \infty} A_a(x) = 1 \tag{6.46}$$

$$\lim_{x \to \infty} A_s(x) = 1 \tag{6.47}$$

It is easily shown that $A_a(x)$ is a monotonically decreasing function of x and $A_s(x)$ a monotonically increasing function. Graphs of these two functions are shown in Figure 6.4.

6.6. High Conductivity Limit.

When the fluid conductivity is large enough for the perturbation current in the fluid to lie principally in the narrow layer near the free surface, a simple expansion technique yields a direct expression for ω. The dimensionless expansion parameter which it is convenient to use is the product of the longitudinal propagation constant k and the hydromagnetic skin depth. This parameter will be denoted by δ.

$$\delta = k \left[\omega_o \sigma \mu_o \left(1 - \frac{k^2 v_A^2}{\omega_o^2} \right) \right]^{-1/2} \tag{6.48}$$

The characteristic radian frequency ω_o is chosen to be that which describes the oscillation of a perfectly conducting fluid. For the antisymmetric mode

$$\omega_o = \pm k \left[\frac{kT}{\rho} \coth kd + (1 + \coth kd)\, v_A^2 \right]^{1/2} \tag{6.49}$$

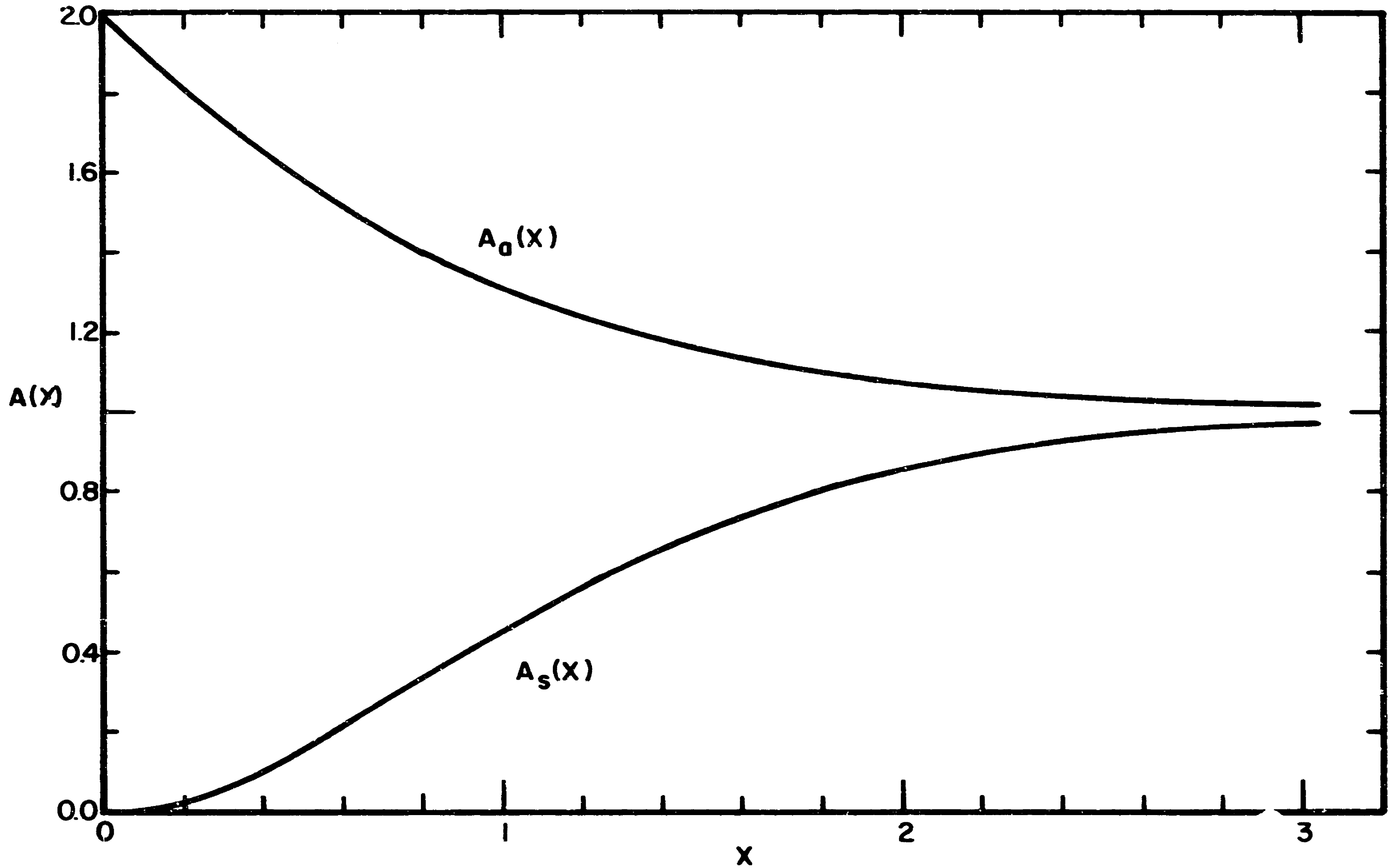


Figure 6.4. DAMPING FACTORS FOR PLANE II SLAB

and for the symmetric mode

$$\omega_o = \pm k \left[\frac{kT}{\rho} \tanh kd + (1 + \tanh kd)\, v_A^2 \right]^{1/2} \tag{6.50}$$

The transverse propagation constant β becomes infinite as the conductivity becomes infinite but $\delta\beta/k$ approaches a finite non-zero value. The radian frequency ω and the transverse propagation constant β may be expended in a power series in δ

$$\omega = \omega_o + \delta\omega_1 + \cdots \tag{6.51}$$

$$\frac{\delta\beta}{k} = \gamma = \gamma_o + \delta\gamma_1 . \tag{6.52}$$

The zero order radian frequency is defined above and ω_1 is the quantity to be found. The quantities γ_o and γ_1 may be determined in terms of ω_o

$$\gamma_o = \frac{1 - j}{\sqrt{2}} \tag{6.53}$$

$$\gamma_1 = \frac{\gamma_o}{2} \left[j + \frac{\omega_1}{\omega_o} \left(\frac{\omega_o^2 + k^2 v_A^2}{\omega_o^2 - k^2 v_A^2} \right) \right] \tag{6.54}$$

The above series expansions of β and ω may be substituted directly into the dispersion relations, Eqs. 6.23 and 6.24. The result is, for the antisymmetric case

$$\omega_1 = - \frac{1 + j}{\sqrt{2}} \frac{k^2 v_A^2}{\omega_o^2} \frac{e^{kd}}{2 \sinh kd} \left(\frac{k^3 T}{\rho\omega_o^2} - 1 \right) \tag{6.55}$$

and for the symmetric case

$$\omega_1 = -\frac{(1 + i)}{\sqrt{2}} \frac{k^2 v_A^2}{\omega_o^2} \frac{e^{kd}}{2 \cosh kd} \quad \frac{k^3 T}{\omega_o^2 \rho} - 1 \qquad (6.56)$$

Thus the radian frequency increment caused by a small deviation from perfect conductivity consists of equal parts damping and frequency decrease.

Chapter VII

Surface Waves on Hydromagnetic Columns.

In this chapter the analysis of Chapter VI will be modified to treat surface disturbances on a circular column of conducting fluid in the presence of an applied axisl D.C. magnetic field. The solution for volume fluid motion will again consist of the two volume modes of section 2.3, now appearing in cylindrical co-ordinates in the mathematical expressions.

The hydrodynamic problem, first analyzed by Lord Rayleigh,* has been extended by Chandrasekhar[3] to include hydromagnetic effects for axisymmetric disturbances and is extended here to include arbitrary surface disturbances.

7.1. The Unperturbed System.

Using cylindrical co-ordinates, the unperturbed system may be described as follows. The externally applied magnetic field is uniform throughout space, of value B_o, and directed along the z-axis. The fluid forms a cylinder of infinite extent along the z-axis, centered on the z-axis, and of radius r_o. The electrical conductivity of the fluid is isotropic and homogeneous and of value σ. Space surrounding the column is a vacuum. This geometry is shown in Fig. 7.1.

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* See Lamb,[1] Article 274.

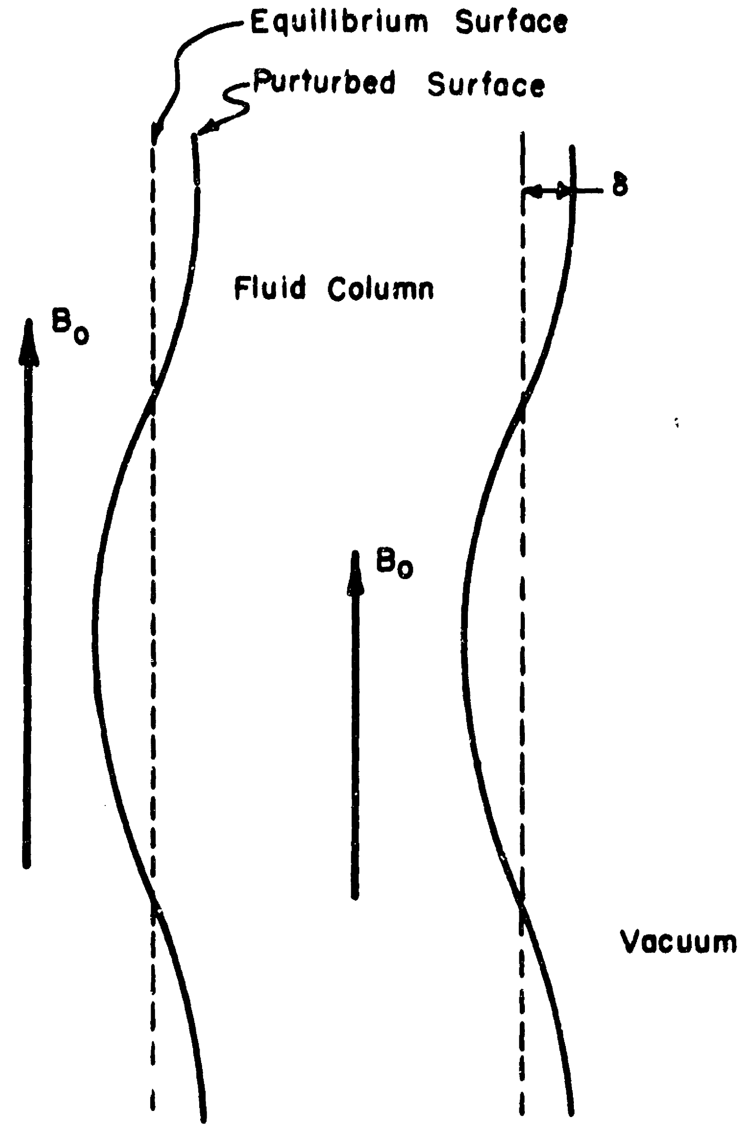

Figure 7.1. CYLINDRICAL FLUID COLUMN

There is a static fluid pressure P_o. It is related to the surface tension T by the equation

$$\frac{T}{r_o} = P_o. \qquad (7.1)$$

7.2. Perturbation of the System.

The propagation of small perturbations of the free surface of the fluid column will be considered. Let the surface of fluid column be specified by the equation

$$r = R(\theta, z, t) \qquad (7.2)$$

where

$$R(\theta, z, t) = R_o + R_1(\theta, z, t) \qquad (7.3)$$

and

$$R_1 << R_o \qquad (7.4)$$

for all θ, z and t.

The following form is assumed for R_1

$$R_1 = R_e[\delta e^{j(n\theta + kz - \omega t)}] \qquad (7.5)$$

In solving the differential equations of this problem, and in applying appropriate boundary conditions, only terms independent of δ or of first order in δ will be considered. Terms of order δ^2 or greater will be considered negligible.

The average radius of the perturbed column, is, to within the above stated limits of accuracy, given by

$$R_o = r_o \qquad (7.6)$$

where r_o is the radius of the unperturbed cylinder.

Quantities characteristic of the unperturbed system will carry a subscript zero. All other terms are perturbation terms of order δ.

3.3. Equations and Solutions for Bulk Motions.

The equations to be solved for the bulk fluid motion and the perturbation magnetic field are Eqs. 5.4 and 5.5.

In view of the form of the perturbation of the surface assumed in Eq. 7.5, all quantities must be of the form

$$f(r)\ e^{j(n\theta+kz-\omega t)}.$$

Under this stipulation, and the further requirement that $\nabla \cdot \bar{\psi} = 0$, and $\nabla \cdot \bar{A} = 0$, there are four independent solutions to Eqs. 5.4 and 5.5.

Solution 1

$$\bar{\psi} = \psi_1\ e^{j(n\theta+kz-\omega t)} \left[\frac{n}{kr} J_n(jkr)\ \bar{i}_r - J_n'(jkr)\bar{i}_\theta \right] \tag{7.7}$$

$$\bar{A} = -\frac{kB_o}{\omega} \bar{\psi} \tag{7.8}$$

$$\phi = 0. \tag{7.9}$$

Solution 2

$$\bar{\psi} = \psi_2\ e^{j(n\theta+kz-\omega t)} \left[\frac{n}{\beta r} J_n(j\beta r)\ \bar{i}_r - J_n'(j\beta r)\bar{i}_\theta \right] \tag{7.10}$$

$$\bar{A} = -\frac{kB_o}{\omega} \quad \frac{\omega^2}{k^2 v_A^2} \quad \bar{\psi} \tag{7.11}$$

$$\phi = 0. \tag{7.12}$$

$$\bar{\psi} = \psi_3 e^{j(n\theta+kz-\omega t)} \left[\frac{k}{\beta} J_n'(j\beta r)\bar{i}_r + \frac{kn}{\beta^2 r} J_n(j\beta r)\bar{i}_\theta + J_n(j\beta r)\bar{i}_z \right] \tag{7.13}$$

$$\bar{A} = - \frac{kB_o}{\omega} \frac{\omega^2}{k^2 v_A^2} \bar{\psi} \tag{7.14}$$

$$\phi = - B_o \psi_3 J_n(j\beta r) \, e^{j(n\theta+kz-\omega t)} \tag{7.15}$$

Solution 4

$$\bar{\psi} = \psi_4 e^{(n\theta+kz-\omega t)} \left[J_n'(jkr)\bar{i}_r + \frac{n}{kr} J_n(jkr)\bar{i}_\theta + J_n(jkr)\bar{i}_z \right] \tag{7.16}$$

$$A = - \frac{kB_o}{\omega} \psi \tag{7.17}$$

$$\phi = - B_o \psi_4 J_n(jkr) \, e^{j(n\theta+kz-\omega t)}. \tag{7.18}$$

In solutions 2 through 4 the parameters β and v_A^2 have been introduced. They are defined as follows:

$$v_A^2 = \frac{B_o^2}{\rho\mu_o} \tag{7.19}$$

$$\beta^2 = k^2 - j\omega\sigma\mu_o \left[1 - \frac{k^2 v_A^2}{\omega^2} \right] \tag{7.20}$$

The quantity v_A^2 is the Alfven velocity and the quantity β is the transverse propagation constant originally introduced in Eq. 2.37.

The fourth solution consists of curl free vector potentials and consequently the velocity and field obtained from it are identically zero. Therefore, it may be discarded.

7.4. External Fields.

The external fields are governed by free space Maxwell's equations with displacement current neglected.

They are given by

$$\bar{B} = B_{ex} e^{j(n\theta+kz-\omega t)} \left[\bar{i}_r H_n'(jkr) + \bar{i}_\theta \frac{n}{kr} H_n(jkr) + \bar{i}_z H_n(jkr) \right] \tag{7.21}$$

$$\bar{E} = E_{ex} e^{j(n\theta+kz-\omega t)} \left[\bar{i}_r H_n'(jkr) + \bar{i}_\theta \frac{n}{kr} H_n(jkr) + \bar{i}_z H_n(jkr) \right] + \frac{\omega}{k} B_{ex} e^{j(n\theta+kz-\omega t)} \left[\bar{i}_r \frac{n}{kr} H_n(jkr) - \bar{i}_\theta H_n'(jkr) \right] \tag{7.22}$$

All Hankel functions in this and the following chapter are of the first kind.

7.5. Internal Field and Fluid Quantities.

In section 4, four independent vector solutions to the coupled set of partial differential equations were found. The general solution is an arbitrary sum of these. The quantities of physical interest are the velocity, pressure, magnetic field, electric field and current.

$$\bar{v} = \nabla \times \bar{\psi}$$
$$= e^{j(n\theta+kz-\omega t)} \left\{ \left[\psi_1 jkJ_n'(jkr) + \psi_2 jkJ_n'(j\beta r) + \psi_3 \left(1 - \frac{k^2}{\beta^2}\right) \frac{jn}{r} J_n(j\beta r) \right] \bar{i}_r \right.$$

$$+ \left[\psi_1 \frac{jn}{r} J_n(jkr) + \psi_2 j \frac{kn}{\beta r} J_n(j\beta r) - \psi_3 \left(1 - \frac{k^2}{\beta^2}\right) j\beta J_n'(j\beta r)\right] \bar{i}_\theta$$

$$+ \left[\psi_1 jkJ_n(jkr) + \psi_2 j\beta J_n(j\beta r)\right] \bar{i}_z \Bigg\} \qquad (7.23)$$

$$\bar{B} = \nabla \times \bar{A}$$

$$= \frac{kB_o}{\omega} e^{j(n\theta+kz-\omega t)} \Bigg\{ \Big[\psi_1 jkJ_n'(jkr) + M_A^2 \psi_2 jkJ_n'(j\beta r)$$

$$+ M_A^2 \psi_3 \left(1 - \frac{k^2}{\beta^2}\right) \frac{jn}{r} J_n(j\beta r)\Big] \bar{i}_r$$

$$+ \Big[\psi_1 \frac{jn}{r} J_n(jkr) + M_A^2 \psi_2 \frac{jkn}{\beta r} J_n(j\beta r)$$

$$- M_A^2 \psi_3 \left(1 - \frac{k^2}{\beta^2}\right) j\beta J_n'(j\beta r)\Big] \bar{i}_\theta$$

$$+ \left[\psi_1 jkJ_n(jkr) + M_A^2 \psi_2 j\beta J_n(j\beta r)\right] \bar{i}_z \Bigg\} \qquad (7.24)$$

$$\bar{J} = -\frac{1}{\mu_o} \nabla^2 \bar{A}$$

$$= \frac{kB_o}{\omega\mu_o} (\beta^2 - k^2) M_A^2 e^{j(n\theta+kz-\omega t)} \Bigg\{ \Big[\psi_2 \frac{n}{\beta r} J_n(j\beta r)$$

$$+ \psi_3 \frac{k}{\beta} J_n'(j\beta r)\Big] \bar{i}_r$$

$$+ \Big[- \psi_2 J_n'(j\beta r)$$

$$+ \psi_3 \frac{kn}{\beta^2 r} J_n(j\beta r)\Big] \bar{i}_\theta$$

$$+ \psi_3 J_n(j\beta r) \quad \bar{i}_z \Bigg\} \qquad (7.25)$$

$$\bar{E} = -\nabla\phi - \frac{\partial\bar{A}}{\partial t}$$

$$= -jkB_o e^{(n\theta+kz-\omega t)} \Bigg\{ \bar{i}_r \Bigg[\psi_1 \frac{n}{kr} J_n(jkr) + M_A^2 \psi_2 \frac{n}{\beta r} J_n(j\beta r) - \psi_3 \frac{\beta}{k} \left(1 - \frac{k^2}{\beta^2} M_A^2\right) J_n'(j\beta r) \Bigg]$$

$$+ \bar{i}_\theta \Bigg[-\psi_1 J_n'(jkr) - \psi_2 J_n'(j\beta r) + \psi_3 \frac{n}{kr} \left(1 - \frac{k^2}{\beta^2} M_A^2\right) J_n(j\beta r) \Bigg]$$

$$+ \bar{i}_z \Bigg[-\psi_3 \left(1 - M_A^2\right) J_n(j\beta r) \Bigg] \Bigg\} \quad (7.26)$$

In the above formulae, the parameter M_A^2, defined by

$$M_A^2 = \frac{\omega^2}{k^2 v_A^2} \quad (7.27)$$

has been introduced. It is the ratio of the phase velocity to the Alfven velocity.

The perturbation pressure is found by substituting the above expressions for $\bar{v}$ and $\bar{J}$ into the linearized Navier-Stokes equation, Eq. 2.8, and is:

$$P = j\omega\rho \left[\psi_1 J_n(jkr) + \frac{\beta}{k} \psi_2 J_n(\beta r) \right] \quad (7.28)$$

7.6. Boundary Conditions.

At the wall of the column, a number of boundary conditions must be satisfied. These are the continuity of the electric and magnetic fields across the boundary, the condition that no current flow through the boundary, and the pressure condition at the free surface. The result of applying these conditions to the general solutions in the fluid and outside it will be a dispersion relation between the k and ω of the assumed form of the surface disturbance.

In order to satisfy the condition that no current flows through the boundaries to within accuracy of terms of order δ, it is only necessary to require that J_r be zero at $r = r_o$.

$$J_r \Big|_{r=r_o} = \frac{kB_o}{\omega\mu_o}(\beta^2-k^2)M_A^2 e^{j(n\theta+kz-\omega t)}\left[\psi_2 \frac{n}{\beta r_o} J_n(j\beta r_o)+\psi_3 \frac{k}{\beta} J_n'(j\beta r_o)\right] = 0. \tag{7.29}$$

Hence

$$\psi_3 = -\frac{n}{kr_o}\frac{J_n(j\beta r_o)}{J_n'(j\beta r_o)}\psi_2 \tag{7.30}$$

It is necessary that all three vector components of the magnetic field be continuous across the boundary of the column since the fluid has finite conductivity and consequently no surface currents may exist.

The continuity of B_z requires

$$-\frac{kB_o}{\omega} e^{j(n\theta+kz-\omega t)}\left[\psi_1 jkJ_n(jkr_o) + M_A^2\, \psi_2 j\beta J_n(j\beta r_o)\right]$$

$$= B_{ex}\, e^{j(n\theta+kz-\omega t)}\, H_n(jkr_o). \tag{7.31}$$

Therefore

$$B_{ex} = -\frac{jk^2B_o}{\omega}\left[\psi_1 \frac{J_n(jkr_o)}{H_n(jkr_o)} + M_A^2\psi_2 \frac{\beta}{k} \frac{J_n(j\beta r_o)}{H_n(jkr_o)}\right] \tag{7.32}$$

The continuity of B_r requires that

$$-\frac{kB_o}{\omega} e^{j(n\theta+kz-\omega t)}\Big[\psi_1 jkJ_n'(jkr_o) + M_A^2\psi_2 jkJ_n'(j\beta r_o)$$

$$+ M_A^2\psi_3\left(1 - \frac{k^2}{\beta^2}\right)\frac{jn}{r_o} J_n(j\beta r_o)\Big]$$

$$= B_{ex}\, e^{j(n\theta+kz-\omega t)}\, H_n'(jkr_o). \tag{7.33}$$

The combination of Eqs. 7.30, 7.32 and 7.33 yields

$$\psi_1 = \frac{\pi kr_o}{2} M_A^2\psi_2\Big[J_n'(j\beta r_o)H_n(jkr_o) - \frac{n^2}{k^2r_o^2}\left(1 - \frac{k^2}{\beta^2}\right)\frac{J_n^2(j\beta r_o)H_n(jkr_o)}{J_n'(j\beta r_o)}$$

$$- \frac{\beta}{k} J_n(j\beta r_o)H_n'(jkr_o)\Big] \tag{7.34}$$

The continuity of B_θ leads to an equation which is already satisfied by Eqs. 7.30, 7.32 and 7.34.

The continuity of E_z leads to the equation

$$j\, kB_o\psi_3(1 - M_A^2)\, J_n(j\beta r_o) = E_{ex}\, H_n(jkr_o) \tag{7.35}$$

or

$$E_{ex} = j\, kB_o\psi_3(1 - M_A^2)\,\frac{J_n(j\beta r_o)}{H_n(jkr_o)} \tag{7.36}$$

The continuity of E_θ leads to an equation which is already satisfied by Eqs. 7.30, 7.32, 7.34 and 7.36. There is a surface charge on the column and E_r is not continuous.

The kinematic free surface condition states that the fluid velocity normal to the free surface is equal to the normal velocity of the surface.

$$v_r \Big|_{r=r_o} = j\omega\delta\, e^{j(n\theta+kz-\omega t)} \quad . \tag{7.37}$$

So

$$\delta = \frac{k}{\omega}\left[\psi_1 J_n'(jkr_o) + \psi_2 J_n'(j\beta r_o) + \psi_3\left(1 - \frac{k^2}{\beta^2}\right)\frac{n}{kr} J_n(j\beta r_o)\right] \tag{7.38}$$

The condition on the pressure at the free surface is

$$[P + P_o]\Big|_{r=r_o} = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{7.37}$$

where T is the fluid surface tension and R_1 and R_2 are the radii of curvature along the principle axes of curvature.

The principle axes of curvature of the perturbed surface are, to within the first order in δ, the same as the principle axes of the unperturbed surface. Keeping only linear terms in δ,

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{r_o} + \delta\left(\frac{n^2 - 1}{r_o^2} + k^2\right) e^{j(n\theta+kz-\omega t)} \tag{7.40}$$

Since

$$\frac{T}{r_o} = P_o$$

$$P_{r=r_o} = T\delta \left(\frac{n^2 - 1}{r_o^2} + k^2 \right) e^{j(n\theta + kz - \omega t)}. \quad (7.41)$$

Equations 7.28, 7.30, 7.34, 7.38 and 7.41 determine the dispersion relation for the system. The combination of Eqs. 7.28, 7.30, 7.38 and 7.41 yields

$$\psi_1 \left[\frac{jk^3T}{\omega^2\rho} \left(\frac{n^2 - 1}{(kr_o)^2} + 1 \right) J_n'(jkr_o) - J_n(jkr_o) \right]$$

$$+\psi_2 \left[\frac{jk^3T}{\omega^2\rho} \left(\frac{n^2 - 1}{(kr_o)^2} + 1 \right) J_n'(j\beta r_o) - \frac{n^2}{(kr)^2} \left(1 - \frac{k^2}{\beta^2} \right) \frac{J_n^2(j\beta r_o)}{J_n'(j\beta r_o)} - \frac{\beta}{k} J_n(j\beta r_o) \right] = 0. \quad (7.42)$$

The requirement that the determinant of the co-efficients of ψ_1 and ψ_2 vanish establishes the dispersion relation.

$$\left| A_{ij} \right| = 0 \quad (7.43)$$

$$A_{11} = \frac{jk^3T}{\omega^2} \left(\frac{n^2 - 1}{(kr_o)^2} + 1 \right) J_n(jkr_o) - J_n(jkr_o)$$

$$A_{12} = \frac{jk^3T}{\omega^2\rho} \left(\frac{n^2 - 1}{(kr_o)^2} + 1 \right) J_n'(j\beta r_o) - \frac{n^2}{(kr)^2} \left(1 - \frac{k^2}{\beta^2} \right) \frac{J_n^2(j\beta r_o)}{J_n'(j\beta r_o)} - \frac{\beta}{k} J_n(j\beta r_o)$$

$$A_{21} = 1$$

$$A_{22} = \frac{\pi k r_o}{2} M_A^2 \left[J_n'(j\beta r_o) H_n(jkr_o) - \frac{n^2}{(kr_o)^2} \left(1 - \frac{k^2}{\beta^2}\right) \frac{J_n^2(j\beta r_o) H_n(jkr_o)}{J_n'(j\beta r_o)} - \frac{\beta}{k} J_n(j\beta r_o) H_n'(jkr_o). \right]$$

This equation will be examined in the three succeeding sections at the limits of long wavelength, low conductivity, and high conductivity.

7.7. The Long Wavelength Limit.

A case for which the dispersion relation assumes a particularly simple form is the long wave limit, where the first few terms of a power series expansion will approximate the Bessel functions closely. The $n = 1$ mode will be considered, as Chandrasekhar has completely treated the $n = 0$ mode.

If one keeps only the first term in the power series expansion, the dispersion relation becomes

$$\frac{\omega^2}{d^2} = \frac{T}{\rho r_o} \tag{7.44}$$

The electromagnetic effects disappear. As an examination of Eq. 7.23 shows that this is because, for small radii, the absolute average current is proportion to the square of the radius.

Keeping the next higher order terms in the expansions of the Bessel functions, the following equation is obtained:

$$(1 - M_A^2) \quad \frac{k^2 T}{\rho r_o} - \omega^2$$

$$+ \frac{(kr_o)^2}{16} \quad \frac{k^2 T}{\rho r_o} \left(4 - \frac{\beta^2}{k^2} - 3M_A^2 \right) + 2\omega^2 \left(M_A^2 - \frac{\beta^2}{k^2} \right) = 0. \quad (7.45)$$

A value of ω which satisfies the above equation to within the accuracy of the approximate equation is

$$\omega = k \quad \frac{T}{\rho r_o} \left\{ 1 + \frac{4 - 3\,\frac{\beta^2}{k^2} - M_A'^2}{1 - M_A'^2} \right\} \quad (7.46)$$

where now

$$M_A'^2 = \frac{v_c^2}{v_A^2} \quad (7.47)$$

and

$$v_c^2 = \frac{T}{\rho r_o} \quad (7.48)$$

Equation 59 simplifies further to

$$\omega = k\,\frac{T}{\rho r_o} \left[1 + \frac{r^2 r_o^2}{32} \left(1 - \frac{3j\ \sigma\mu_o}{k} \quad \frac{v_A^2}{v_c^2} \right) \right] \quad (7.49)$$

The damping constant is then seen to be

$$\alpha = \frac{3}{32}\,(kr)^2 \quad \sigma\ {}_o v_A^2. \quad (7.50)$$

The change in natural frequency is a hydromagnetic effect and the only effect of the magnetic field on the long wavelength disturbances is exponential damping.

7.8. Low Conductivity Behavior.

When the conductivity is small, in a sense which is formalized below, a power series expansion of the dispersion relation may be made in a dimensionless parameter which depends linearly on the conductivity. The first term in the series, which represents the limiting form of the dispersion relation for zero conductivity, must be the classical hydrodynamic solution to the problem. The magnitude and nature of the first order correction term, as well as the applicability of the expansion will now be determined.

The dispersion relation in the limit as σ approaches zero will be examined first

$$\lim_{\sigma \to 0} \beta = k \tag{7.51}$$

Substitution of this result into Eq. 7.42 yields

$$0 = (\psi_1 + \psi_2)\left[\frac{jkT}{\omega^2 \rho}\left(\frac{n^2 - 1}{r_o^2} + k^2\right) J_n'(jkr_o) - J_n(jkr_o)\right] \tag{7.52}$$

Equation 7.34 becomes

$$\psi_1 + M_A^2 \psi_2 = 0. \tag{7.53}$$

The dispersion relation thus has the pair of spurious roots, $\omega^2 = k^2 v_A^2$, plus the pair of roots coming from the equation

$$\frac{jkT}{\omega^2}\left(\frac{n^2 - 1}{r_o^2} + k^2\right) J_n'(jkr_o) - J_n(jkr_o) = 0 \tag{7.54}$$

Since the Bessel functions in this case have purely imaginary arguments, it is useful to introduce the modified, or hyperbolic, Bessel functions.

$$J_n(jx) = j^n I_n(x) \tag{7.55}$$

$$J_n'(jx) = j^{n-1} I_n(x). \tag{7.56}$$

Then

$$\frac{k^3 T}{\rho}\left(1 + \frac{n^2 - 1}{(kr_o)^2}\right)\frac{I_n'(kr_o)}{I_n(kr_o)} = \omega^2. \tag{7.57}$$

Since the modified Bessel function and its first derivation have the property that both are positive and real for positive real arguments, it is easily seen all non-axisymmetric disturbances are stable but all axisymetric (n=0) disturbances of sufficiently long wavelength are unstable.

The values of ω given by Eq. 7.54 will be indicated with a subscript zero. The low conductivity, low field behavior of the system will now be considered. The expansion parameters are

$$R_M' = \frac{\omega_o \sigma \mu_o}{k^2} \tag{7.58}$$

$$M_A'^2 = \frac{\omega_o^2}{k^2 v_A^2}\,. \tag{7.59}$$

The conditions for the low conductivity expansion to be valid are then

$$R'_M << 1$$

and

$$\frac{R'_M}{M'^2_A} << 1.$$

The appropriate expansion is then a power series in R'_M which is a magnetic Reynolds number. The transverse propagation constant β may be represented by the initial terms in its power series

$$\beta = k\left[1 - \frac{jR'_M}{2}\left(1 - \frac{1}{M'^2_A}\right)\right] \tag{7.60}$$

A power series expression for ω may be performed.

$$\omega = \omega_o + R'_M\omega_1 + \cdots \tag{7.61}$$

Upon substitution of the initial terms of the series for ω and β into Eqs. 7.34 and 7.42. an expression for ω_o identical to Eq. 7.54 is obtained ... the resulting equation for ω_1 is

$$\omega_1 = + \frac{j\sigma\mu_o v_A^2}{4}\left[\frac{xI''_n(x)}{I'_n(x)} + \frac{2n^2}{x^2}\,\frac{I^2_n(x)}{I'^2_n(x)} - 1 - \frac{xI'_n(x)}{I_n(x)}\right] \tag{7.62}$$

Let

$$A_n(x) = \left[1 + \frac{xI'_n(x)}{I_n(x)} - \frac{xI''_n(x)}{I_n(x)} - \frac{2n^2}{x^2}\,\frac{I^2_n(x)}{I'^2_n(x)}\right] \tag{7.63}$$

Then

$$\omega_1 = - \frac{j\sigma\mu_o v_A^2}{4}\, A_n(x). \tag{7.64}$$

The following parameter simplifies the expression for $A_n(x)$.

$$R_n(x) = \frac{I_n(x)}{xI_n'(x)} \tag{7.65}$$

Substitution of Eq. 7.65 into Eq. 7.63 yields

$$A_n(x) = 2 + \frac{1}{R_n(x)} \left[1 - (n^2 + x^2)R_n^2(x) - 2n^2R_n^3(x) \right] \tag{7.66}$$

The limiting forms for $R_n(x)$ are

$$\lim_{x \to 0} R_n(x) = \frac{1}{n} \tag{7.67}$$

$$\lim_{x \to \infty} R_n(x) = \frac{1}{x} \quad 1 + \frac{1}{2x} \tag{7.68}$$

Substitution of Eq. 7.67 and Eq. 7.68 into Eq. 7.66 gives the limiting values of $A_n(x)$.

$$\lim_{x \to 0} A_n(x) = 0 \tag{7.69}$$

$$\lim_{x \to \infty} A_n(x) = 1. \tag{7.70}$$

Figure 7.2 gives values of $A_n(x)$ for n = 0 through n = 6 values of x from 0 to 6.

The values of $A_n(x)$ are all positive, consequently Eq. 7.64 indicates that all modes which were oscillatory in the limit of zero conductivity are damped when finite conductivity is present. Modes which grow exponentially in the hydrodynamic problem, grow less rapidly in the presence of a

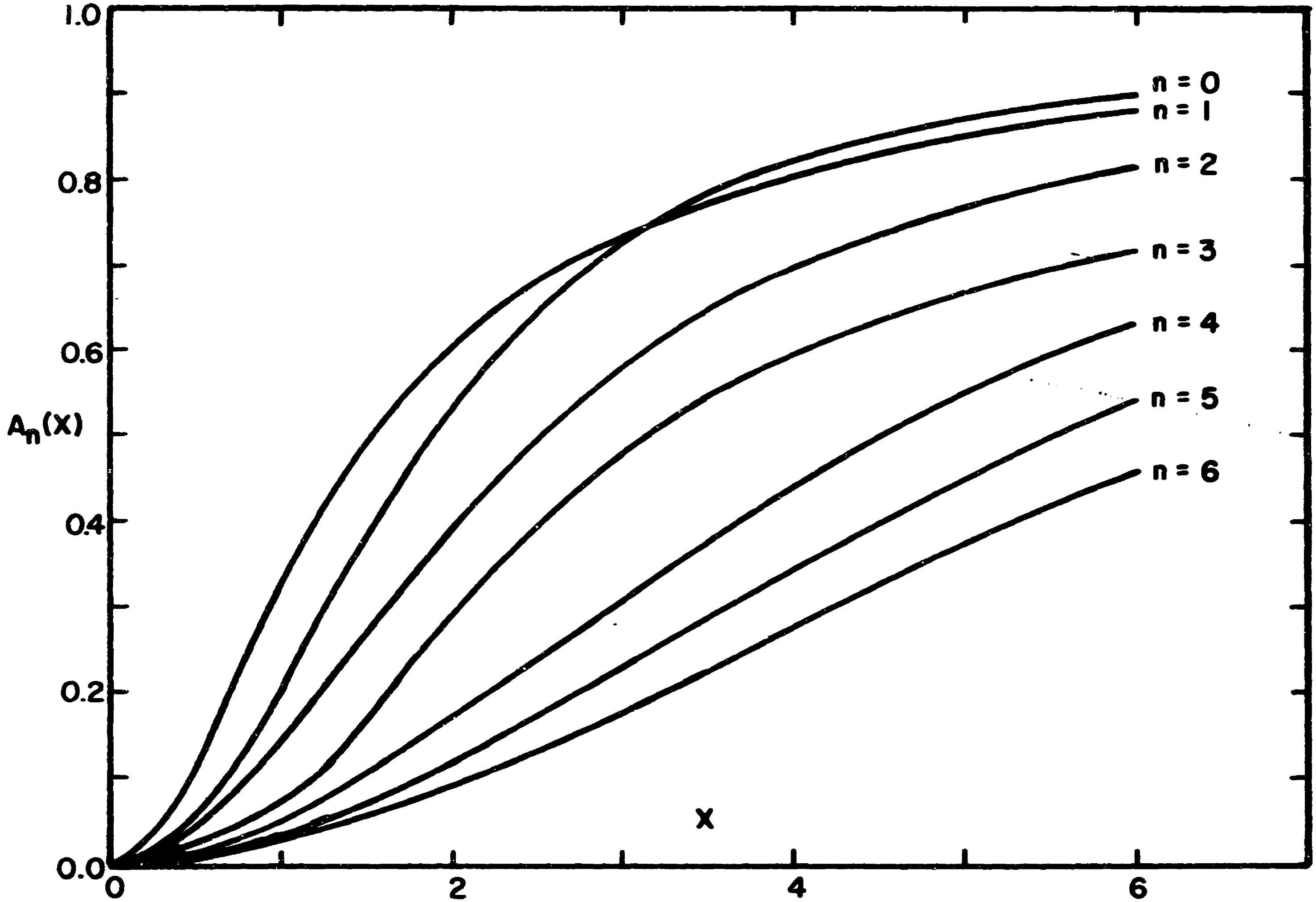

Figure 7.2. FORMS FACTORS FOR THE CYLINDRICAL COLUMN

magnetic field and small finite conductivity. The Eq. 7.64 is invalid when used to predict changes in frequency comparable to the zero order frequency, consequently this theory will not predict transition from instability to stability.

7.9. High Conductivity Behavior.

It has been noted by Chandrasekhar[3] that the varicose capillary instability of a perfectly conducting fluid column may be stabilized by an axial magnetic field. The modification of the perfect conductivity solutions caused by large but finite conductivity will be examined in this section. This will be accomplished by expansion of the dispersion equation, Eq. 7.43, in a parameter which is inversely proportional to the square root of the conductivity.

The transverse propagation constant β, Eq. 7.70, grows as the square root of the conductivity and therefore becomes infinite as the conductivity becomes infinite. The portion of the solution which has this transverse behavior must then be confined to a region near the edge of the column. The width of this region will serve as the basis of the expansion parameter. Let the expansion parameter be given by

$$p = \frac{k}{\beta'} \qquad (7.71)$$

where β' is given by

$$\beta' = \left[j\omega_o \sigma \mu_o \left(1 - \frac{k^2 v_A^2}{\omega_o^2} \right) \right]^{1/2} \qquad (7.72)$$

and ω_o is the value of ω obtained in the limit as the conductivity approaches infinity. This is a legitimate procedure since the value of ω_o may be obtained by direct consideration of a perfectly conducting fluid without recourse to this expansion. It is known from the solution by Chandrasekhar[3] that ω_o is either a purely real or a purely imaginary number for real longitudinal propagation constant k. For ω_o real and $\omega_o^2 > k^2 v_A^2$

$$p = \frac{|k|}{\left[\omega_o \sigma \mu_o \left(1 - \frac{k^2 v_A^2}{\omega_o^2}\right)\right]^{1/2}} . \qquad (7.73a)$$

For ω_o real and $\omega_o^2 = k^2 v_A^2$, the expansion fails.

For ω_o imaginary

$$p = \frac{|k|}{\left[|j\omega_o| \left(1 - \frac{k^2 v_A^2}{\omega_o^2}\right)\right]^{1/2}} . \qquad (7.73b)$$

It will be necessary in the expansion to know

$$\gamma = \lim_{\sigma \to \infty} \frac{p\beta}{k} \qquad (7.74)$$

$$\gamma = 1 \quad ; \quad \omega_o \text{ imaginary, } j\omega > 0 \qquad (7.75a)$$

$$\gamma = -1 \quad ; \quad \omega_o \text{ imaginary, } j\omega < 0 \qquad (7.75b)$$

$$\gamma = \frac{1 - i}{\sqrt{2}} \quad ; \quad \omega_o \text{ real, } \quad \omega_o^2 > k^2 v_A^2 \qquad (7.75c)$$

$$\gamma = \frac{1 + i}{\sqrt{2}} \quad ; \quad \omega_o \text{ real, } \quad \omega_o^2 < k^2 v_A^2 . \qquad (7.75d)$$

The frequency ω will be expanded in a power series in p

$$\omega = \omega_o + p\omega_1 + \cdots \tag{7.76}$$

It is convenient to introduce the modified Bessel functions

$$I_n(x) = j^n J_n(jx) \tag{7.77}$$

$$K_n(x) = \frac{\pi}{2} j^{n+1} H_n^{(1)}(jx) \tag{7.78}$$

The equations determining the dispersion relation, Eq. 7.33 and Eq. 7.42 become

$$0 = \psi_1\left[\frac{k^3T}{\omega^2}\left(1 + \frac{n^2-1}{(kr_o)^2}\right)I_n'(kr_o) - I_n(kr_o)\right]$$

$$+\psi_2\left[\frac{k^3T}{\omega^2}\left(1 + \frac{n^2-1}{(kr_o)^2}\right)I_n'(\beta r_o) + \frac{n^2}{(kr_o)^2}\left(1 - \frac{k^2}{\beta^2}\right)\frac{I_n^2(\beta r_o)}{I_n'(\beta r_o)} - \frac{\beta}{k} I_n(\beta r_o)\right] \tag{7.79}$$

$$0 = \psi_1 + kr_oM_A^2\psi_2\left\{K_n(kr_o)\left[I_n'(\beta r_o) + \frac{n^2}{(kr_o)^2}\left(1 - \frac{k^2}{\beta^2}\right)\frac{I_n^2(\beta r_o)}{I_n'(\beta r_o)}\right] - \frac{\beta}{k} I_n(\beta r_o)K_n'(kr_o)\right\} \tag{7.80}$$

The expansion of Eq. 7.76 may now be substituted into Eqs. 7.79 and 7.80 and the constants ψ_1 and ψ_2 eliminated. The reduction of the resulting expressions is facilitated by the following relationships. Since

$$\text{Re}\,[\beta] > 0$$

then

$$\lim_{\beta \to \infty} \frac{I_n(\beta r_o)}{I_n'(\beta r_0)} = 1. \quad (7.81)$$

The collection of zero and first order terms of the resulting equation gives

$$\omega_o^2 = \frac{k^3 T}{\rho} \left(1 + \frac{n^2 - 1}{x^2}\right) \frac{I_n'(x)}{I_n(x)} - \frac{k^2 v_A^2}{x K_n'(x) I_n(x)} \quad (7.82)$$

$$\frac{\omega_1}{\omega_o} = - \gamma \frac{\omega_A^2}{\omega_o^2} \frac{1}{2x^2} \left(1 - \frac{k^2 v_A^2}{\omega_o^2} \right) \quad (7.83)$$

In obtaining the last expression, use has been made of the identity

$$I_n K_n' - I_n' K_n = - \frac{1}{x}.$$

The implications of Eq. 7.80 are found after considering the various values of γ from Eqs. 7.75a 7.75d with the values of ω_o to which they correspond. The effects of finite but small resistivity may be summarized as:

1) Solutions which are exponentially growing in the zero resistivity limit grow more rapidly.

2) All solutions which are oscillatory in the zero resistivity limit are damped.

For

$$\omega_o^2 < k^2 v_A^2$$

this is accompanied by an increase in the oscillating frequency.

7.10. Application of the Results to a Simple Moving System.

The results of the preceding sections describe the motion of a stationary column of fluid reacting to a disturbance which is periodic in a space variable. In a simple situation of some practical interest, a column of fluid moves at constant velocity past an exciter which is localized in space and sinusoidal in time. An experiment of this type, on electrohydrodynamic free surface waves, has been performed by Melcher.*

The response of the system to a disturbance localized in time and space is

$$\eta(z,t) = \int_{-\infty}^{\infty} e^{(kz-\omega(k)t)}\, dk. \tag{7.84}$$

The exciter is assumed to move along the column in the negative z direction at a velocity v_o and is sinusoidal in time with radian frequency . The response of the system to the excitation is

$$\eta(z,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \cos \Omega_o t_o e^{j[k(x+v_o t_o)-\omega(k)(t-t_o)]} dk dt_o \tag{7.85}$$

---

* See Melcher,[5] Section 6.3, p. 131 and Section 6.4, p. 136.

A stationary phase expansion* of Eq. 7.85 may be performed under the following conditions.

a) $\Omega_o >> \omega\left(\frac{\Omega_o}{v_o}\right)$ (driving frequency much larger than the response)

b) $\frac{\Omega_o}{v_o}(x + v_o t) >> 1$ (the exciter has passed the point of interest several cycles ago).

The stationary phase point of the integral on k in the above conditions occurs for $v_o k = \pm \Omega_o$. Let

$$k_o = \frac{\Omega_o}{v_o}$$

Then the approximate response in time at a fixed position is

$$f(t) = Ae^{+j\omega(k_o)(t-t_o)} + Be^{+j\omega(-k_o)(t-t_o)} . \qquad (7.86)$$

This is the response to an initial disturbance of space propagation constant k_o. Condition (a) is the requirement that the point excitation place on the jet a spatial sinusoidal disturbance over several wave lengths in a time shorter than the characteristic response time of the system.

Returning to a co-ordinate system in which the exciter is fixed, the response, which is

$$\eta(x,t) = \cos \Omega_o t\left[Ae^{j[k_o x-\omega(k_o)v_o x]} + Be^{-j[k_o x+\omega(-k_o)v_o x]}\right] \qquad (7.87)$$

---

* See Erdelyi[20], Section 2.8, page 51.

The response of Eq. 7.87 is valid only when the driving frequency ω_o is large compared with the response frequency. The conditions for a laboratory experiment will now be examined. The parameters used will be those of the experiment by Melcher[5] previously mentioned in this section.

$$\Omega_o = 1500 \text{ rad./sec.}$$

$$v_o = 5 \text{ m./sec.}$$

$$\text{column radius} = 2 \times 10^{-3} \text{ m.}$$

$$k_o = 300 \text{ m.}^{-1}$$

For hydrodynamic motions of long wavelength

$$\omega(k) \approx \frac{T}{\rho r_o}$$

which, for NaK or water, and the above column radius is

$$\rho(k) \approx 5 \times 10^{-3} \text{ sec}^{-1}$$

and the above stipulation on the frequency ratios is satisfied. The expansion of the integral should therefore describe the motion occuring in a laboratory experiment.

Note: The results in sections 7.1 through 7.8 of this chapter have been reported by Nayar and Trehan[19]. The work was performed independently by the author.

Chapter VIII

Surface Waves on a Current Carrying Hydromagnetic Column.

The behavior of a conducting column discussed in Chapter VII is modified by a longitudinal electrical current flowing through the column. The longitudinal electrical current, supplied by a external voltage source is treated here as a perturbation on the solutions of Chapter VII and only the low conductivity limit is considered in detail.

The analysis proceeds in two parts. The first part is a derivation applicable to arbitrary electrical currents and conductivities. An unwieldly dispersion relation results. The equations determining the dispersion relation are then examined in the limit of low conductivity and weak electric current.

8.1. Derivation of the Dispersion Equations.

8.1.1. The Unperturbed System.

The unperturbed system, shown in Fig. 8.1, may be described using cylindrical coordinates. An inviscid incompressible fluid forms a right circular cylinder of radius r_o which extends from $z = -\infty$ to $z = +\infty$. The fluid carries a uniform current density J_o which is directed along the axis in the positive z direction. A uniform magnetic field, $B_o i_z$, fills all space. The fluid density is homogeneous and of

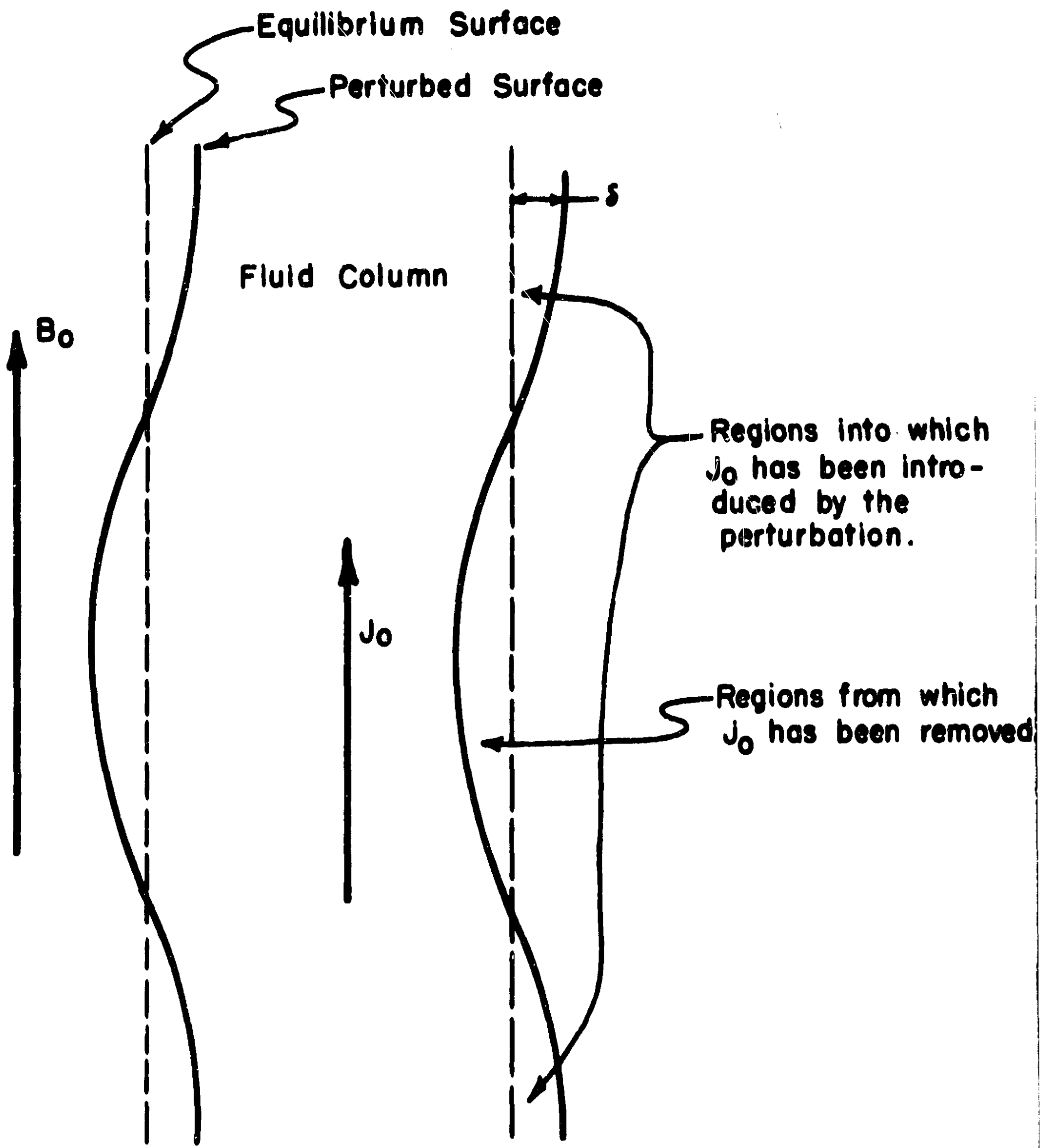

Figure 8.1

CURRENT CARRYING HYDROMAGNETIC COLUMN

value σ. Space surrounding the column is a vacuum. The total magnetic field is

$$\bar{B} = \mu_o J_o r/2 \; \bar{i}_\theta + B_o \bar{i}_z \tag{8.1}$$

inside the column and

$$\bar{B} = \mu_o J_o r_o^2/2r \; \bar{i}_\theta + B_o \bar{i}_z \tag{8.2}$$

outside it.

The steady-state pressure distribution in the fluid column is

$$P = \frac{T}{r_o} + \frac{\mu_o J_o^2}{4} (r^2 - r_o^2) \tag{8.3}$$

where T is the surface tension of the fluid.

8.1.2. Perturbation of the System.

The propagation of small disturbances of the column will be considered. Let the surface of the column be specified by the equation.

$$r = R(\theta, z, t) \tag{8.4}$$

where

$$R = r_o + R_1(\theta, z, t) \tag{8.5}$$

and

$$R_1 << R_o \tag{8.6}$$

for all θ, z, and t.

In particular, perturbations will be of the form

$$R_1 = R_e \; [\delta \; e^{j(n\theta + kz - \omega t)}] \tag{8.7}$$

In the subsequent sections of the chapter, non-linear differential equations will be linearized to contain only constant terms and terms of first order in δ. Complex notation will be used and the operation of taking the real part is to be understood and will not be expressly indicated.

8.1.3. The Surface Current.

In order to formulate the conditions in a simple manner, it is convenient to introduce the artifice of a surface current. This surface current does not represent the limiting form of any physical current distribution, but rather allows us to take into account the effect of the deformation of the conductor through which a d.c. current is passing. It is an idea borrowed from electron beam theory, whereby the effect of perturbation of the boundary of a Brillouin beam is treated.

Let the surface of a current carrying column be specified by

$$r = r_o + \delta\, e^{j(n\theta+\omega t-kz)}. \qquad (8.8)$$

Now the magnetic field resulting from the current flow is no longer that given by Eqs. 8.1 and 8.2. The effects of surface perturbation may be separated into two segments. The first is a first order change in the current in the column. This may be taken into account by assuming that it lies between the unperturbed boundaries, as using the actual boundary introduces only a second-order correction to the field. The second is a first order change in the boundary causing the zero-order current to fill a slightly altered volume of space. It is this effect that will be modeled as a surface current.

Consider Figure 8.1. In some regions, zero-order current now flows where before there was none, while other regions have become current free. Since, by virtue of the assumption of small surface perturbation, these regions are concentrated near the equilibrium surface, they may be taken to represent a first order surface current flowing along the equilibrium surface.

Since

$$J = J_o \bar{i}_z, \tag{8.9}$$

the required surface current is

$$\bar{K}(\theta,z,t) = \bar{J}_o \delta\, e^{j(n\theta+kz-\omega t)}\, \bar{i}_z. \tag{8.10}$$

8.1.4. Equations of Motion in the Fluid.

The equations of motion will be presented in linearized form, where upper case letters will represent steady state quantities and lower case letters represent perturbation quantities. The linearized Navier-Stokes for no zero-order fluid motion equation is

$$\frac{\partial}{\partial t}\ \bar{v} = -\frac{1}{\rho}\nabla_p + \frac{1}{\rho}(\bar{J}x\bar{b}) + \frac{1}{\rho}(\bar{j}\ x\ \bar{B}) \tag{8.11}$$

where

$$\bar{J} = J_o \bar{i}_z$$

and

$$\overline{B} = B_o \overline{i}_z + \frac{\mu_o J_o r}{2} \overline{i}_\theta .$$

Maxwell's equations in the MHD approximation are

$$\nabla \cdot \overline{b} = 0 \tag{8.12}$$

$$\nabla \times \overline{b} = \mu_o \overline{j} \tag{8.13}$$

$$\nabla \times \overline{e} = - \frac{\partial}{\partial t} \overline{b}. \tag{8.14}$$

The conductivity equation is:

$$\overline{j} = \sigma(\overline{e} + \overline{v} \times \overline{B}). \tag{8.15}$$

One exception will be made to the convention that upper case letters represent steady-state quantities. The letter A will represent the vector potential for the perturbation magnetic field. The letter ψ will represent the vector potential for the velocity field.

$$\nabla \times \overline{\psi} = \overline{v} \tag{8.16}$$

$$\nabla \times \overline{A} = \overline{b}. \tag{8.17}$$

The divergence of each vector potentials is chosen to be zero

$$\nabla \cdot \overline{\psi} = 0, \ \nabla \cdot \overline{A} = 0.$$

The equation of motion becomes

$$\frac{\partial}{\partial t} (\nabla \times \overline{\psi}) = - \frac{1}{\rho} \nabla p - \frac{1}{\rho \mu_o} (\nabla^2 \overline{A} \times \overline{B}_o) + \frac{1}{\rho} (\overline{J} \times (\nabla \times \overline{\psi})) \tag{8.18}$$

Maxwell's equations reduce to

$$\nabla^2 A = - \mu_o \overline{j} \tag{8.19}$$

$$\nabla \times \bar{e} = - \frac{\partial}{\partial t} (\nabla \times \bar{A}) \tag{8.20}$$

and the conductivity equation is

$$\nabla^2 \bar{A} = - \sigma \mu_o \, [\bar{e} + (\nabla \times \bar{\psi}) \times \bar{B}] \tag{8.21}$$

In view of the form of perturbation assumed in Eq. 8.7 and the linearity of Eq. 8.18 through 8.21, all perturbation quantities are of the form

$$f(r) \, e^{j(n\theta + kz - \omega t)}$$

and with this spatial dependence, Eqs. 8.18 through 8.21 may be combined to yield

$$- j\omega \nabla^2 \bar{\psi} = \frac{1}{\rho \mu_o} \left[jkB_o + \frac{jn \, \mu_o J_o}{2} \right] \nabla^2 \bar{A} + \frac{jk}{\rho} J_o (\nabla \times \bar{A}) \tag{8.22}$$

and

$$\nabla^2 \bar{A} + j\omega\sigma \, \mu_o \bar{A} = \sigma \mu_o \nabla \emptyset - \sigma \mu_o \left[jkB_o + \frac{jn \, \mu_o J_o}{2} \right] \bar{\psi}$$
$$+ \sigma \mu_o \left[B_o \nabla \psi_z + \frac{\mu_o J_o}{2} \right] \nabla (r \, \psi_\theta) \tag{8.23}$$

The pair of equations, Eq. 8.22 and Eq. 8.23 are solved by an appropriate combination of the longitudinal and transverse solutions to the vector Hemholtz equation in cylindrical coordinates.

Let the vector potential A be given by

$$A = A_o e^{j(n\theta + \omega t - kz)} \left[\bar{i}_r \left| J_n'(j\beta r) - \frac{qn}{\beta r} J_n(j\beta r) \right| + \bar{i}_\theta \left| qJ_n'(j\beta r) + \frac{n}{\beta r} J_n(j\beta r) \right| + \bar{i}_z \left| \frac{\beta}{k} J_n(j\beta r) \right| \right] \tag{8.24}$$

where

$$q^2 = \frac{\beta^2}{k^2} - 1 \quad . \tag{8.25}$$

Let ψ be of the same form; that is,

$$\bar{\psi} = \frac{\psi_o}{A_o} \bar{A}. \tag{8.26}$$

Then

$$\nabla \times \bar{A} = - jkq \bar{A} \tag{8.27}$$

and a similar relation holds for ψ. Upon setting the scalar potential of the electric field to ge given by

$$\phi = - \quad B_o \psi_z + \frac{\mu_o J_o r}{2} \psi_\theta \tag{8.28}$$

Eq. 8.23 becomes

$$\nabla^2 \bar{A} + j\omega\sigma\mu_o \bar{A} = - \sigma\mu_o \left[jkB_o + \frac{jn\mu_o J_o}{2} \right] \bar{\psi} \tag{8.29}$$

and the assumed form of the vector potentials ψ and A represent a solution if q is a root of the cubic equation

$$q^3 + j[R_1 + R_2(1 + n\alpha)]q + 2j\alpha R_2 = 0 \tag{8.30}$$

where R_1 and R_2 are again the two magnetic Reynolds numbers characteristic of the problem

$$R_1 = \frac{\omega\sigma\mu_o}{k^2} \tag{8.31}$$

and

$$R_2 = \frac{\sigma\mu_o v_A^2}{\omega} \tag{8.32}$$

α represents an appropriate ratio of the two steady state magnetic fields.

$$\alpha = \frac{\mu_o J_o}{2kB_o} \tag{8.33}$$

and v_A^2 represents the square of the Alfven velocity

$$v_A^2 = \frac{B_o^2}{\rho\mu_o} \tag{8.34}$$

The constant ψ_o is related to A_o by the equation

$$\psi_o = - \frac{1}{1 + n\alpha} \frac{\omega}{kB_o} \left[\frac{q^2 + jR_1}{jR_1}\right] \tag{8.35}$$

As there are three solutions to Eq. 8.30, three independent solutions to the coupled pair of equations, Eq. 8.22 and Eq. 8.23, have been found. It is seen that the set, Eq. 8.22 and Eq. 8.29, under the constraint that $\nabla \cdot \bar{A} = \nabla \cdot \bar{\psi} = 0$, have four independent solutions. The fourth solution may be found by setting

$$A = \nabla\Theta\ (r,z,t) \tag{8.36}$$

where

$$\Theta(r,z,t) = J_n(jkr)e^{j(n\Theta+kz-\omega t)} \tag{8.37}$$

and relating $\bar{\psi}$ to $\bar{A}$ by Eq. 8.29. $\bar{\psi}$ is also found to be a gradient of a scalar.

Since both $\bar{\psi}$ and $\bar{A}$, in this fourth solution, are gradients of scalar functions, their curls, which represent physical quantities, are zero. Therefore, although this fourth solution is necessary to complete the set of independent solutions for

Eqs. 8.22 and 8.29, only the first three are needed for a complete determination of the physical model which gave rise to those equations.

Thus, the general solution for ψ and A in the fluid column may be written as follows: Let

$$\bar{S}_i = \bar{i}_r \left[J_n'(j\beta r) - \frac{q_i h}{\beta_i r} J_n(j\beta_i r) \right] + \bar{i}_\theta \left[q_i J_n'(j\beta r) - \frac{n}{\beta_i r} J_n(j\beta_i r) \right] + \bar{i}_z \left[\frac{\beta_i}{k} J_n(j\beta_i r) \right] \tag{8.38}$$

Then

$$\bar{A} = \sum_{i=1}^{3} A_i e^{j(n\theta+kz-\omega t)} \bar{S}_i \tag{8.39}$$

and

$$\psi = \sum_{i=1}^{3} \psi_i e^{j(n\theta+kz-\omega t)} S_i \tag{8.40}$$

where the q_i's are the solutions to Eq. 8.30, and the β's are related to the q_i's by Eq. 8.25. The constants ψ_1 are related to the constants A_i by the equation

$$\psi_i = - \frac{1}{1 + n\alpha} \frac{\omega}{kB_o} \left[\frac{q_i^2 + jR_1}{jR_1} \right] A_i. \tag{8.41}$$

8.1.5. Solutions Outside the Fluid Column

An expression for the magnetic field b_{ex} valid outside the fluid column is needed. In this region, the magnetic field is divergence free and curl free. Hence, with the previously assumed θ and z variation,

$$\bar{b}_{ex} = b_{ex} e^{j(n\theta+kz-\omega t)} \left[\bar{i}_r H_n'(jkr) + \bar{i}_\theta \frac{n}{kr} H_n(jkr) + \bar{i}_z H_n(jkr) \right] \tag{8.42}$$

where the Hankel functions intended are those of the first kind, and

$$H_n'(z) = \frac{\partial}{\partial z} H_n(z). \tag{8.43}$$

The constant b_{ex} is related to the A_i's of the preceding equation by the boundary conditions.

8.1.6. Boundary Conditions.

In this section, sufficient boundary conditions will be applied to the previously obtained solutions to produce three equations, linear in the A_i's and homogeneous. The condition that the determinant of the co-efficients of the A_i's be zero results in a trancendental equation which is the dispersion relation.

The following boundary conditions will be employed:

(1) All components of the magnetic field are continuous across the free surface.

(2) The discontinuity in the normal component of the stress tensor is balanced by the surface tension.

Since, for an inviscid fluid, the tangential component of the fluid stress tensor consists only of electromagnetic quantities, condition (1) assures that it also is continuous.

Condition (1) also assures that the boundary condition

$$\bar{J} \cdot \bar{n} = 0$$

is satisfied, where $\bar{J}$ is the total current and $\bar{n}$ is the normal vector to the surface. Similarly, the tangential components of $\bar{e}$ are forced to be continuous. The normal component of $\bar{e}$ is discontinuous at the free surface as there is a surface charge density, time varying and of first order in δ on the fluid surface.

The condition that B_r be continuous is

$$b_{ex} H_n'(jkr_o) = \sum_{i=1}^{3} - jkq_i A_i S_{i,r}(r_o) \qquad (8.44)$$

where $S_{i,r}(r_o)$ is the r component of the vector S_i evaluated at $r = r_o$.

The condition that B_z is continuous is

$$b_{ex} H_n(jkr_o) = \sum_{i=1}^{3} - jkq_i A_i S_{i,z}(r_o). \qquad (8.45)$$

They may be combined to yield

$$0 = \sum_{i=1}^{3} q_i A_i [H_n'(jkr_o) S_{i,z}(r_o) - H_n(jkr_o) S_{i,r}(r_o)] \qquad (8.46)$$

The boundary condition on the Θ component of the magnetic field is a difficult one. The formulation of the boundary condition is simplified if, rather than requiring the continuity of B_θ at the actual boundary, it is required that b_θ be discontinuous by an amount determined by the effective surface current discussed in section 8.1.2. The result of both these approaches is the same, but the latter is much easier to formulate.

In order to write this boundary condition in terms of the A_i's, an expression for δ in terms of the A_i's is needed. The time rate of change of δ is, to first order in δ, equal to the normal velocity of the fluid evaluated at $r = r_o$, the equilibrium surface.

$$j\omega\delta = \sum_{i=1}^{3} - jkq_i \quad \psi_i \; S_{i,r} \; (r_o). \tag{8.47}$$

Hence

$$\overline{K} = \overline{i}_z e^{j(n\theta+kz-\omega t)} \sum_{i=1}^{3} \frac{q_i}{1 + n\alpha} \; \frac{J_o}{B_o} \left(\frac{q_i^2 + jR_1}{jR_1} \right) \; A_i \; S_{i,r} \; (r_o) \tag{8.48}$$

The condition on B_θ is, then

$$b_{ex} \frac{n}{kr} H_n(jkr) = \sum_{i=1}^{3} - jkq_i A_i \; S_{i,\theta} \; (r_o)$$

$$+ \frac{q_i}{1 + n\alpha} \; \frac{\mu_o J_o}{B_o} \left(\frac{q_i^2 + jR_1}{jR_1} \right) A_i \; S_{i,r} \; (r_o) \tag{8.49}$$

b_{ex} may be eliminated by substitution of Eq. 8.45 into Eq. 8.49. Equation 8.45 is used rather than Eq. 8.44 in order to eliminate the Hankel function also. The result is

$$0 = \sum_{i=1}^{3} \frac{jq_i}{r} S_{i,z}(r_o) + jkq_i S_{i,\theta}(r_o) + \frac{q_i}{1 + n\alpha} \frac{\mu_o J_o}{B_o} \left(\frac{q_i^2 + jR_1}{jR_1} \right) S_{i,r}(r_o) \qquad (8.50)$$

In order to apply the second boundary condition, it is necessary to find an expression for the pressure. The perturbation pressure is most easily determined by making use of the z-component of the linearized Navier-Stokes equation, Eq. 8.18. It is:

$$p = j \sum_{i=1}^{3} A_i \omega\rho q_i \frac{1}{1 + n\alpha} \frac{\omega}{kB_o} \left(\frac{q_i^2 + jR_1}{jR_1} \right) S_{i,z} + \frac{kr J_o}{2} q_i^2 S_{i,r} \qquad (8.51)$$

The discontinuity in the perturbation pressure p is set equal to the two surface "forces", one caused by surface tension, the other the result of the effective surface current.

$$p\Big|_{r=r_o} - \frac{r_o\delta}{2} \mu_o J_o^2 - T\delta \left(\frac{n^2 - 1}{r_o^2} + k^2 \right) = 0. \qquad (8.52)$$

Let

$$\varepsilon = \frac{\mu_o J_o^2 r_o}{2} + T \left(\frac{n^2 - 1}{r_o^2} + k^2 \right) . \qquad (8.53)$$

Then

$$p\Big|_{r=r_o} - \varepsilon\delta = 0 \tag{8.54}$$

or

$$0 = \sum_{i=1}^{3} A_i \left[j\omega\rho q_i \frac{1}{1+n\alpha} \frac{\omega}{kB_o} \frac{q_i^2 + jR_1}{jR_1} S_{i,z}(r_o) + j\frac{jkr_oJ_o}{2} q_i^2 S_{i,r}(r_o) - \varepsilon \frac{kq_i}{\omega} \frac{1}{1+n\alpha} \frac{\omega}{kB_o} \frac{q_i^2 + jR_1}{jR_1} S_{i,r}(r_o) \right] \tag{8.55}$$

Equations 8.46, 8.50 and 8.55 are the three homogeneous equations required to determine the dispersion relation

8.2. A Solution in the Low Conductivity Limit.

The preceding development applies to arbitrary electrical co..ductivity and arbitrary D.C. current. It is desired to find a ..olution to the dispersion equation when the conductivi..y is small and the applied longitudinal electric field is of the same order as the electrical field caused by motional induction. In such cases the parameter α of Eq. 8.33 will be replaced by

$$\alpha' R_1$$

where

$$R_M = \frac{\omega_o \sigma \mu_o}{k^2} \left(1 - \frac{k^2 v_A^2}{\omega_o^2}\right) \tag{8.56}$$

and ω_o is given by

$$\omega_o^2 = \frac{Tk^3}{\rho} \left[1 + \frac{n^2 - 1}{(kr_o)^2} \right] \frac{I_n'(kr_o)}{I_n(kr_o)} \tag{8.57}$$

the oscillatory frequency for hydrodynamic disturbances. Then R_M is a magnetic Reynolds number based upon the wavelength and hydrodynamic natural time constant of the system. For the expansion to be valid

$$R_M << 1$$

$$R_M \alpha' << 1$$

consequently α' may be zero in the expansion but it may not become comparable in magnitude to $1/R_M$, the expansion then proceeds in a straightforward manner. From Eq. 8.30, the values of q are, approximately

$$q_1 = 0$$

$$q_2 = + (jR_M)^{1/2}$$

$$q_3 = - (jR_M)^{1/2}.$$

The functions $S_i(r)$ may now be determined and the results substituted in Eq. 8.46, Eq. 8.50 and Eq. 8.55. A power series expansion in R_M is then made. The result is

$$\omega = \omega_o - \frac{j\sigma\mu_o v_A^2}{4} A_n(x) + jnk \frac{JB_o}{\rho\omega_o} R_n(x) \tag{8.58}$$

where $R_N(x)$ is defined in Eq. and $A_n(x)$ is defined in

Eq. 7.66.

It should be noted that the D.C. current has no effect on the axisymmetric disturbances. The D.C. current in the axisymmetric mode interacts only with its self field, and the nature of the expansion undertaken here makes this effect second order in the expansion parameter. It is seen directly from comparison of Eq. 7.64 and Eq. 8.58 that, in the absence of D.C. current the result reduces to that obtained in Chapter VII, section 7.8.

A feature of the term caused by D.C. current in Eq. 8.58 is that the sign may be either positive or negative, depending upon the sign of nk. The surface perturbation described by Eq. 8.8 represents, for n greater than zero, either a left or right helical surface disturbance. The D.C. current tends to cause time growth of those modes which have a helical surface disturbance which is in the same sense as the magnetic field. This result is well known in the perfectly conducting limit.

Chapter IX

Concluding Remarks and Suggestions for Further Investigations.

9.1. Concluding Remarks.

The aim of this investigation was to develop a unified method of determining the effects of a magnetic field on surface wave motions of a fluid with finite electrical conductivity in several geometrical configurations. The development proceeds from the fundamental equations of inviscid incompressible hydromagnetics, linearized for small fluid oscillations, to a general solution for the fluid motion. This solution for the fluid motion is then constrained by the application of various boundary conditions, and the resulting equation determining the behavior in time of the fluid is examined.

The physical configurations examined here are the most commonly treated ones in hydrodynamics. They are the propagation of surface gravity waves and the motion of disturbances on a fluid column.

A brief word is in order concerning the mathematical methods used in this thesis. The solution of the differential equations for volume motion is made simple once it is realized that the linear fourth-order operator in the vector differential equation for volume motion factors in all orthogonal co-ordinate systems into two second-order permutable operators. The formulation of the equations, the boundary conditions, and the dispersion relations is greatly facilitated by the introduction of the vector potentials. The number of mathematical manipulations required in proceeding from problem statement to answer is considerably reduced by these potentials.

The limiting forms, established by expansion of the dispersion relation in some parameter (usually a magnetic Reynolds number), are of interest not only for the information they provide directly, but also for an initial point and a check on digital solutions of the dispersion relation. All dispersion equations are presented in a form the author found useful for machine computation.

The effect of the magnetic field is, in general, a stabilizing one. Motions which are oscillatory in the absence of the field are damped in its presence. The rate of oscillation may also change significantly. The growth rate of hydrodynamic instabilities is reduced.

The nature of the motion in the fluid is complicated by the presence of the field. In the hydromagnetic problem, there are an infinite number of discrete natural frequencies for a disturbance of a given wavelength, while in the hydrodynamic problem, there is only one.

9.2. Suggestions for Experimental Investigations.

An appealing feature of experimental investigations on liquid metal surface motions is that the time and distance scale of the motions involved in laboratory experiments are such that the motions are visually observable.

The gravity wave motion under a vertical magnetic field has received some experimental attention. It would appear that the horizontal field case is more interesting, in view of the results in Chapter III, but no experimental investigations have yet been reported.

Some qualitative experiments, notably those by Dattner,[16] et. al., have been done on freely falling current carrying fluid columns, but no good quantitative results are available either for the current-carrying or the noncurrent-carrying fluid column.

9.3. Theoretical Extensions.

Several possible theoretical extensions of this investigation are concerned with the transition from the idealized physical situations discussed in this work to theoretical situations more closely approximating laboratory experiments.

In the field of gravity waves, theoretical solutions applicable to more general end wall constraints than those of section 5.1 are required.

When the fluid is in motion, passing stationary boundaries, there exists a possibility of energy conversion. Some work on this problem for perfectly conducting fluids has been performed.[12]

The introduction of spatially non-uniform applied fields into the analysis would permit consideration of such problems as an MHD induction generator using a fluid jet.

Appendix A.

A Small-Signal Power Theorem.

The small-signal power theorem presented here is a special case of the theorem for hydromagnetic waves in a moving media given by Cogdell.[12] For the derivation of the theorem and its historical antecedents, see Chapter II of Cogdell.[12]

The small signal power theorem is a relation among products of perturbation quantities in the solution derived from formal mathematical operations on the linearized equations. The terms have the dimensions of power and energy.

The power theorem was used by the author only in the closing stages of his work and as a means of verifying the accuracy of solutions obtained by other methods, and as an aid to the intuition.

The questions of growth or decay of perturbations which arise in Chapter VII can be answered quite easily with the power theorem, but the determination of rate of growth or decay is no simpler than with the methods of Chapter VII.

In the fluid, the power theorem is

$$0 = \frac{J^2}{\sigma} + \nabla \cdot \left[\frac{\bar{E} \times \bar{B}}{\mu_o} + \bar{v}p \right] + \frac{\partial}{\partial t} \left[\frac{1}{2} \rho v^2 + \frac{B^2}{2\mu_o} \right] \qquad (A.1)$$

for the motion of an incompressible, inviscid, fluid of arbitrary conductivity, when the unperturbed solution has zero fluid velocity. The above theory is obtained directly as a limiting form for zero compressibility of the expression given by Cogdell. The expression given by Cogdell, however, was derived under the assumption that the conductivity was large enough that

$$\bar{E} \approx - (\bar{v} \times \bar{B}_o) \qquad \text{(A.2)}$$

This is never a good assumption for any problems considered in this thesis, since Eq. A.2, although approximately correct in the interior of the fluid, is invalid in surface boundary layers where most of the current flows.

Under the assumption of incompressibility, Eq. A.1 may be shown to hold for all conductivity without the aid of the approximation of Eq. A.2.

Small Signal Surface Energy Storage.

The theorem of Eq. A.1 is most frequently used after integration over a volume of interest. In problems of the type considered in this thesis, the volume is one wavelength along assumed direction of wave propagation and extends to infinity in the plane perpendicular to this direction. The surface of the fluid is therefore included in the volume in question. The vector quantity

$$\bar{P} = \frac{\bar{E} \times \bar{B}}{\mu_o} + \bar{v}\, p \qquad \text{(A.3)}$$

may be discontinuous at this surface, which indicates a possibility of energy storage at the surface. In a fluid of finite conductivity, the electric and magnetic fields tangential to the surface are continuous, therefore no discontinuity in

$$\frac{E \times B}{\mu_o}$$

can occur. The term of interest then is the product of discontinuity of the pressure at the free surface and the component of velocity normal to the free surface. The effects of gravity and of surface tension will be treated separately for clarity, though both may be present in a given situation.

The effect of gravity will be considered first. The discontinuity of the perturbation pressure at a free surface under the influence of a normal gravitational acceleration is

$$p = \rho g \eta \tag{A.4}$$

where η is the elevation of the free surface above its equilibrium value. The normal component of velocity is

$$\Delta v_n = \eta_t . \tag{A.5}$$

Consequently (vp), the discontinuity in the perturbation kinetic power flow is given by

$$\Delta (\bar{v}p) = \frac{\partial}{\partial t} \quad \frac{1}{2} \rho g \eta^2 \quad . \tag{A.6}$$

The term on the right-hand side has the physical interpretation of potential energy stored by elevation in a gravitational field.

The effect of surface tension on the small signal energy storage may be found in a similar manner

$$\Delta p = \frac{T}{R} \tag{A.7}$$

where $\frac{1}{R}$ is the sum of the inverse of the principle radii of curvature of the surface. Consider a plane surface (y = 0). Let the perturbation of the surface be

$$\eta(x,t) = f(t) \cos kx$$

then

$$\Delta p = Tk^2 \eta(x,t)$$

and

$$v_n = \eta_t$$

hence

$$\Delta(\overline{vp}) = \frac{\partial}{\partial t} \quad Tk^2 \frac{\eta^2}{2} \quad . \tag{A.8}$$

Now the surface tension represent energy per unit surface area and is often called the surface free energy. A calculation of the change in surface area of the fluid caused by the assumed perturbation shows that the right-hand side of Eq. A.8 represents the product of the surface tension and the change in surface area. In cylindrical geometry, if the surface of the column has the form

$$r = R + \text{Re}\left[\delta e^{j(kz+n\theta)} \right]$$

then

$$\frac{1}{R} = k^2 \delta \quad 1 + \frac{n^2 - 1}{(kr_o)^2} \quad . \tag{A.9}$$

and

$$\Delta(vp) = \frac{\partial}{\partial t} \quad \frac{T}{2} (k\delta)^2 \left(1 - \frac{n^2 - 1}{(kr_o)^2} \right) \tag{A.10}$$

Notice that the term in brackets is negative for axisymmetric disturbances when

$$(kr_o)^2 < 1.$$

The term in brackets is shown to be the product of the surface tension and the change in area by Chandrasekhar[3] in Article III, page 539. The surface tension is thus a negative energy storage on a small signal basis for these disturbances.

Appendix B.

Equations for a Viscous Conducting Fluid.

It was mentioned in Chapter II that the viscous forces could be included in the analysis, with increasing complexity of the mathematical expressions as a consequence. The purpose of this Appendix is to develop the equations necessary to describe a viscous electrically conducting fluid, to indicate the form of the solutions in cartesian and cylindrical co-ordinates, and to indicate the related effects of viscosity and conductivity in wave motions, as characterized by the Reynolds number and the magnetic Reynolds number.

B.1. The Equations of Motion.

The equations that govern the motion, linearized as in Chapter II, section 2.1, are Maxwell's relations

$$\nabla \times \bar{b} = \mu_o \bar{j} \tag{B.1}$$

$$\nabla \times \bar{e} = - \frac{\partial}{\partial t} \bar{b} \tag{B.2}$$

$$\nabla \cdot \bar{b} = 0 \tag{B.3}$$

the constitutive relation

$$\bar{j} = \sigma(\bar{e} + \bar{v} \times \bar{B}_o) \tag{B.4}$$

the Navier-Stokes equation with electrical body forces

$$\frac{\partial}{\partial t} \bar{v} = - \frac{\nabla P}{\rho} + \frac{1}{\rho} (\bar{j} \times \bar{B}_o) + \nu \nabla^2 \bar{v} \tag{B.5}$$

and the incompressibility restriction

$$\nabla \cdot \bar{v} = 0. \tag{B.6}$$

Vector potentials may be introduced for the velocity and magnetic fields by virtue of Eq. B.3 and Eq. B.6

$$\nabla \times \bar{\psi} = \bar{v} \tag{B.7}$$

$$\nabla \cdot \bar{\psi} = 0 \tag{B.8}$$

$$\nabla \times \bar{A} = b \tag{B.9}$$

$$\nabla \cdot \bar{A} = 0. \tag{B.10}$$

Equations B.1, B.9 and B.10 yield

$$\bar{j} = - \nabla^2 \bar{A}. \tag{B.11}$$

Equations B.2 and B.9 yield

$$\bar{e} = - \frac{\partial \bar{A}}{\partial t} - \nabla \phi \tag{B.12}$$

where ϕ is a scalar function of time and position, as yet undetermined.

The magnetic field B_o is assumed to lie along the z-axis of either a rectangular or cylindrical co-ordinate system and to be uniform throughout space. Equations B.5, B.7, and B.8 may be combined, and the curl of that combination taken to eliminate the pressure. The result is

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \bar{\psi} = \frac{B_o}{\rho \mu_o} \frac{\partial}{\partial z} \nabla^2 \bar{A}. \tag{B.13}$$

The results of substituting Eqs. B.11, B.12, B.7, and B.8 into Eq. B.4 is

$$\nabla^2 \bar{A} - \sigma\mu_o \frac{\partial}{\partial t} \bar{A} - \sigma\mu_o \nabla\phi = \sigma\mu_o B_o \left(\nabla\psi_z - \frac{\partial}{\partial z} \bar{\psi} \right). \qquad (B.14)$$

The gradient terms in Eq. B.14 must cancel, hence

$$\nabla\phi = - B_o \nabla\psi_z \qquad (B.15)$$

leaving

$$\left(\nabla^2 - \sigma\mu_o \frac{\partial}{\partial t}\right) \bar{A} = - \sigma\mu_o B_o \frac{\partial}{\partial z} \bar{\psi}. \qquad (B.16)$$

Equations B.13 and B.16 may be combined to give a single partial differential equation for the velocity vector potential

$$\left[\left(\frac{\partial}{\partial t} - \frac{1}{\sigma\mu_o} \nabla^2\right)\left(\frac{\partial}{\partial t} - \nu\nabla^2\right) + v_A^2 \frac{\partial^2}{\partial z^2}\right] \nabla^2\bar{\psi} = 0. \qquad (B.17)$$

B.2. The Equation in Dimensionless Form.

A characteristic length L and a characteristic time T may be introduced into Eq. B.17 to reduce it to dimensionless form.

$$t = \frac{t}{T}$$

$$\nabla^2 = L^2\nabla^2$$

$$\frac{\partial^2}{\partial z^2} = L^2 \frac{\partial^2}{\partial z^2} .$$

The result is

$$\left[\left(\frac{\partial}{\partial t} - \frac{1}{R_M} \nabla^2\right) \left(\frac{\partial}{\partial t} - \frac{1}{R} \nabla^2\right) + M^2 \frac{\partial^2}{\partial z^2}\right] \nabla^2\bar{\psi} = 0 \qquad (B.18)$$

where

$$R_M = \frac{\sigma\mu_o L^2}{T} \qquad \text{(magnetic Reynolds number)} \qquad (B.19)$$

$$R = \frac{L^2}{\nu T} \qquad \text{(Reynolds number)} \qquad (B.20)$$

and

$$M^2 = \frac{T^2 v_A^2}{L^2} \, . \qquad (B.21)$$

The close analogy between the effects of viscosity and conductivity in these motions is illustrated very well by the identical manner in which the two Reynolds numbers enter into Eq. B.18.

B.3. Method of Solution of the Equations.

Equation B.17 can be solved easily in either cartesian or cylindrical co-ordinates for disturbances of the type considered in the thesis. The purpose of this section is to outline a method of solution.

In cartesian co-ordinates, let all components of ψ be of the form

$$f(y)\, e^{j(k_x x + k_z z)}\, e^{st} .$$

Then

$$f(y) = e^{qy}$$

is a solution and substitution into Eq. B.17 yields six values of q.

In cylindrical co-ordinates, assume a solution of the form

$$\overline{\psi} = e^{j(kz+n\theta-\omega t)} \left[J_n'(j\beta r)\,\overline{i_r} - \frac{n}{\beta r} J_n(j\beta r)\,\overline{i_\theta} + \frac{\beta}{k} J_n(j\beta r)\,\overline{i_z} \right]$$

or of the form

$$= e^{j(kz+n\theta-\omega t)} \left[\frac{n}{\beta r} J_n(j\beta r)\,\overline{i_r} - J_n'(j\beta r)\,\overline{i_\theta} \right]$$

Substitution of either form into Eq. B.17 leads to the same six values for β that were obtained for q. This would apparently produce twelve solutions, but examination shows six of them to be redundant.

Appendix C

The Digital Computation.

The numerical calculations presented in Chapter V and Chapter VII of this work were performed by the IBM 7094 Data Processing System at the Computation Center of the Massachusetts Institute of Technology.

The computer programs were written by the author in the FORTRAN programming language. Approximately two and one-half hours of computer time was used in correcting and testing the programs and producing the results.

Biographical Sketch.

Charles Wesley Rook, Jr., was born on August 30, 1938. He attended public schools in Grand Forks, North Dakota, and Lincoln, Nebraska, and was graduated from Lincoln High School in 1956.

He entered M.I.T. in 1956 as a Sloan Scholar and received the S.B. degree in Electrical Engineering in 1960. He then entered the M.I.T. graduate school with a Fellowship from the Alfred P. Sloan Foundation. In 1961 he joined the staff of the Research Laboratory for Electronics, M.I.T., as a Research Assistant. He is married, with no children.

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