Phase-resolved reconstruction and forecast of nonlinear irregular wave field based on direct numerical simulations

by

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Abstract

The problem of phase-resolved reconstruction and forecast of ocean wave field based on measurements is of basic scientific interest and practical importance in ocean science and marine engineering. This capability aids and expands the use and interpretation of field and wave basin measurements, contributing to the study of fundamental wave mechanics. It also expands the operational envelope and improves survivability and efficiency of ships and marine facilities in severe seas. We develop theoretical and computational capabilities to solve this problem, which can be applied to both ocean wave field and wave basin experiments.

Given limited wave measurement data, there exists specific space and time domain(s) (the "predictable zone") where the wave-field can be reconstructed and forecasted. In this thesis, using linearized wave theory and reasonable assumptions of the frequency and directional extent of the wave field, we obtain closed-form expressions for (linear) predictable zone $\mathcal{P}_L$ in terms of set notation involving the individual measurement. We derive and illustrate $\mathcal{P}_L$ obtained for ("probe") measurements at one or more fixed locations over time, for moving probes, for whole-area wave measurements, and combinations of these. We also consider the problem of optimal deployment of these measurements to maximize the volume of $\mathcal{P}_L$ in space-time. For $J$ probes, we show that this volume scales as $J^3$ (in contrast to $J$ when the predictable zones of individual measurements are simply summed).

With the knowledge of the predictable zone, we develop and validate a high-order reconstruction (HOR) method for the phase-resolved reconstruction of nonlinear wave field given a set of wave measurements. HOR optimizes the amplitude and phase of $L$ free-wave components of the nonlinear wave field, accounting for nonlinear wave interactions up to order $M$ in the evolution to obtain a nonlinear wave field that minimizes the reconstruction error between reconstructed wave field and the given measurements. For a given reconstruction tolerance, $L$ and $M$ are provided in the HOR scheme itself. To demonstrate the validity and efficacy of HOR, we perform extensive tests of general two- and three-dimensional wave fields specified by theoretical Stokes waves, nonlinear simulations, and physical wave fields in tank experiments.
The necessary $L$, for general broad banded wave fields, is shown to be relatively small and substantially less than the free and locked modes needed for the nonlinear evolution. We find that, even for relatively small wave steepness, the inclusion of high-order effects in HOR is important for prediction of wave kinematics not in the measurements. For all the cases we consider, HOR converges to the underlying wave field within a nonlinear spatial-temporal predictable zone $\mathcal{P}_{NL}$ (dependent on the measurements and wave nonlinearity). $\mathcal{P}_{NL}$ generally extends in time (and space) beyond the measurements, thus obtaining reliable forecast/predictions of the wave field. For linear waves, $\mathcal{P}_{NL}=\mathcal{P}_L$, verifying the predictable zone theory. With increasing wave nonlinearity, we show that $\mathcal{P}_{NL}$ contains and is generally greater than $\mathcal{P}_L$. Thus $\mathcal{P}_L$ provides a (conservative) estimate of $\mathcal{P}_{NL}$ when the underlying wave field is not known.

For nonlinear steep wave-field, wave breaking plays an important role in the evolution of the wave field. We develop a phenomenological wave breaking model that can be incorporated into the nonlinear evolution engine of HOR to predict breaking onset and simulate proper amount of energy dissipation. Thus HOR can properly reconstruct and forecast nonlinear wave field which may contain breaking events. The breaking model is developed in the spectra domain and based on analysis of simulated two-dimensional wave breaking caused by different wave-wave interaction mechanism, including modulation instability and wave focusing. The developed wave breaking model is calibrated, validated and verified by different wave breaking measurements and excellent agreement is obtained between simulated wave breaking results and measured ones. The wave breaking model can be further used to simulate the locations of breaking events, which is validated statistically by calculating the Phillips statistics.

This thesis does not address the issue of wave-body interaction nor the control problem for scale models in the wave basin, but it provides necessary nonlinear whole-field data for intense CFD analysis of wave-body interaction at a level heretofore not possible. The presence of imposed current or wind is not considered at this stage but can be incorporated in the future using the same framework.

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Chapter 1

Introduction

The problem of phase-resolved reconstruction and forecast of ocean wave field based on measurements is of basic scientific interest and practical importance in ocean science and marine engineering. Scientifically, this capability aids and expands the use and interpretation of field and wave basin measurements, contributing to the study of fundamental wave mechanics (e.g. [22, 29, 75]) such as growth and equilibrium of short gravity waves, nonlinear wave instabilities, bound wave distributions, gravity wave turbulence and rogue wave development. A practical solution to this problem expands the operational envelope and improves survivability and efficiency of ships and marine facilities in severe seas through operational guidance, optimal maneuvering and active control.

This capability can be applied both in the field and in a wave tank/basin. The reconstructed/forecasted whole-field information of the nonlinear wave field can be further used to calculate the nonlinear wave force on structure and ship motions. However, this capability has heretofore never been fully achieved and this thesis is devoted to the development of this capability both theoretically and computationally. In this chapter, the key issues that are related to the thesis topic are briefly introduced and in the end we state the thesis objective, scope and contributions.
1.1 Linear and nonlinear waves

In this work, we focus on the study of nonlinear irregular waves. It is necessary to briefly review the history of wave theory in order to understand the importance of nonlinearity and irregularity in the waves.

The research on water wave theory can be traced back to the seventeenth century when Isaac Newton first attempted a theory of water waves. After his work, in the eighteenth and early nineteenth century, mathematicians and scientists in Europe progressively made contributions to linear and nonlinear wave theories. Among them are French mathematicians Laplace, who first posed the general initial-value problem of linear water waves, Lagrange, who independently derived the linearized governing equations for small-amplitude waves, and Poisson and Cauchy, who addressed the initial-value problem for linearized water waves with sophisticated mathematical skills. In Germany, Gerstner gave the first exact nonlinear solution for waves of finite amplitude on deep water, and the brothers Weber conducted careful laboratory experiments on plane periodic wavetrains in a channel. Later in Britain during 1837-1847, naval engineer Russell conducted a series of wave tank experiments in which he discovered the nonlinear solitary wave, Green, Airy and others made further substantial contributions to water wave theory. Based on these earlier development of water wave theory, Stokes established a definitive theory of linear and weakly nonlinear waves and published his great 1847 paper on water waves. History of the development of regular water wave theories can be found in the review articles of Craik (2004, 2005)[17, 18] and Darrigol (2003)[19]. Comprehensive analysis of the mathematics involved in water wave theories can be found from a number of textbooks such as Whitham (1999)[115], Kundu and Cohen (2004)[51] and Mei et al. (2005a,b)[64, 65].

The water wave theory mentioned above consider regular waves which contain only monochromatic sinusoidal waves, while in the ocean irregular waves which contains wave components of various frequencies are more common. For irregular ocean waves, the now-standard linear wave theory assumes the surface wave is composed of a
large number of independent simple sinusoidal waves with different frequencies and propagation directions, each travels with its own speed determined by the dispersion relation and has no energy transfer with other wave modes. Thus the evolution of a linear wave field is simply the sum of evolution of each wave mode. Linear wave theory works reasonably well only for waves of small amplitude, such as small amplitude swell observed on the sea[100].

For water waves with finite amplitude, linear wave theory is not sufficient any more. Nonlinear wave-wave interaction has to be taken into account for the study of wave evolution. The earliest well-formulated nonlinear wave theory should reside in Stokes’ 1847 paper[105], where he generalized the linear wave theory by including high-order bound-wave effects to sinusoidal waves, obtaining the so-called Stokes waves, which are characterized by sharper crests and flatter troughs compared to sinusoidal waves. Since put forward, the steady and periodic Stokes waves have been considered the prototype of deep-water ocean waves for almost 100 years[111] and used in many practical marine design applications[78]. The Stokes waves are initially derived for regular waves and can be generalized for irregular waves.

Inclusion of bound-wave effects only does not induce energy transfer between wave components in finite-amplitude irregular waves. It was until over 100 years after Stokes that Benjamin and Feir first discovered that periodic Stokes wave trains of finite amplitude are unstable to suitable small side-band disturbances [55, 9, 123], and as a result, cannot exist in permanent form but rather suffer modulations and even more complex distortions[111]. This so-called Benjamin-Feir (modulational) instability has a time scale of $O(T\epsilon^{-2})$ for nonlinear wave energy transfer, where $\epsilon = ak$ is the wave steepness with $a$ and $k$ representing the wave amplitude and wavenumber of the dominant wave, and $T$ is the dominant wave period. An interesting phenomenon related to the modulational instability is the unstable modulations grows to a maximum and decreases to its initial profile and in the absence of energy dissipation, it becomes a series of focusing and de-focusing cycles over a long time evolution, known as the Fermi-Pasta-Ulam recurrence phenomenon[53], which has a time scale of $O(T\epsilon^{-3})$. Although in realistic irregular wave fields this phenomenon has not been
discovered due to the presence of many unstable side bands and wave breaking[13]. For irregular wave field with a broad band of wave frequencies and directions, the growth rate of the modulational instability is reduced and even eliminated [3], while for narrow-band irregular waves the modulational instability still exists.

For gravity waves, the dispersion relation determines that resonant interactions among first-order wave modes happens for four waves, i.e., four-wave (quartet) interaction [84, 63, 65], which causes the energy transfer between wave components with a much longer time scale $O(T\epsilon^{-4})$ compared to Benjamin-Feir instability mentioned above. Benjamin-Feir instability is actually one special example of the quartet interaction with two waves having same wavenumber and two other waves symmetrical to this wavenumber. Thus the evolution of an irregular wave field consists of a relatively fast energy transfer process due to Benjamin-Feir instability and a much slower energy transfer process due to quartet resonance interactions. Comprehensive review on nonlinear deep-water wave-wave interactions can be found in Yuen and Lake (1980)[122] and Hammack and Henderson (1993)[36].

Besides nonlinear wave-wave interaction, there are other dynamic processes that influence the evolution of irregular ocean waves, such as energy input from wind[82, 68] and energy dissipation from wave breaking[28, 50, 90]. To describe the evolution of irregular waves, Sverdrup and Munk (1947)[106] introduced the energy balance equation which states that the evolution of energy density of wave components is determined by a source function consisting of energy input from wind and energy dissipation from wave breaking. Nonlinear wave-wave interaction was added into the source function later[84, 37, 38, 40, 41]. The wave energy balance equation is further generalized to wave action [114, 42] balance equation[39] as in the case of ambient current is present, action density is conserved whereas energy density is not[115].
1.2 Phase-averaged and phase-resolved wave prediction

In this work, we conduct numerical simulations based on phase-resolved methods. The alternative is phase-averaged methods, which neglect the detailed phase of each wave component and only account for the amplitude or energy of each component. Phase-averaged models can predict the evolution of wave energy spectrum by solving the spectral transport equation.

Until recently, ocean wave prediction is based on phase-averaged (statistical) models for the spectral transport. Since the pioneering work of [31], the development of phase-averaged models have matured.

The first generation of phase-averaged models developed in the 1960s and early 1970s assumed that wave components stop growing at a universal saturation level [83] and used prescribed saturation spectrum [109]. The second generation developed in the 1970s still required the spectral shape of the windsea spectrum to be prescribed for frequencies higher than the peak frequency and was unable to properly simulate complex windseas generated by rapidly changing windseas [109]. More details of the shortcomings of the first and second generation models can be found in the SWAMP (1985) wave-model intercomparison study [108]. The state of the art is the third generation models which solve the spectral transport equation without prior assumption of spectral shape, such as WAM model for deep ocean [109], WAVEWATCH model for arbitrary depth, current and wind fields [110] and SWAN model for littoral zones [10]. Parameterizations of the source terms in the transport equation are continuously improved, such as wind input (e.g. [27]), nonlinear energy transfer (e.g. [91]) and energy dissipation (e.g. [4, 92]). Phase-averaged models are mainly used for sea state forecasts as they predict the evolution of wave spectrum as well as properties of wave spectrum, such as wave energy, peak frequency and significant wave height [109].

Despite these successes, many further model refinements are difficult especially for nonlinear processes in the absence of phase-resolved information.

With recent development of fast computing hardware/software and efficient com-
putational algorithms, phase-resolved simulation of nonlinear wave-field evolution (e.g. [24]) is evolving to become an important alternative to complement phase-averaged models for ocean wave prediction. Unlike phase-averaged models, phase-resolved simulation retains the phase of the wave field during the evolution and obtains detailed description of the flow kinematics and dynamics such as wave elevation, particle velocity and acceleration, and fluid pressure. Such phase-resolved simulations allow more accurate modeling of complex physical processes (such as nonlinear wave-wave interactions and wave breaking dissipation), and provides (short-time) deterministic predictions that can be used for operational guidance (e.g. [15]).

Large-scale phase-resolved nonlinear simulations can be performed using initial wave fields generated from specified wave spectra, or using prescribed wave measurements. The former is useful in the study and prediction of nonlinear wave statistics and extreme wave events occurrence (e.g. [118]). It is also used to generate nonlinear ambient wave environments or episodic extreme wave events as necessary input to wave-structure interaction models for ship and marine structure design [121]. The latter is used for short-time deterministic prediction of wave-field evolution in relatively small fields and for analysis and interpretation of detailed wave kinematics/dynamics in wave tank/basin tests. The present study focuses on the latter application involving reconstruction of phase-resolved wave fields based on prescribed wave measurements and forecast of their future evolution.

1.3 Existence of the predictable zone

Reconstruction of phase-resolved random wave fields based on limited wave measurements is a challenging task that requires the solution of the inverse wave dynamics problem involving broadband irregular wave propagation and interactions. Despite the practical importance of the reconstruction problem, there have been few studies on this subject [102, 124, 125].

An important issue of reconstruction is where and when faithful phase-resolved waves in the field can be obtained based on specific/limited measurements. Using
intuitive arguments, [72] first suggested that for a given duration of a point measurement of a unidirectional wave field, there exist a certain zone in the space-time domain where the wave information could be reliably reconstructed; and this predictable zone is determined by the maximum and minimum phase speeds of the wave components in the wave field. By various numerical examples, [116] later showed that the size of the predictable zone is actually correlated to the group speeds of the waves but not the phase speeds (as obtained by [86] for phase-averaged wave prediction). The basic problem of obtaining the phase-resolved predictable zone in space-time for a general set of measurements has not been addressed on a fundamental level. In this thesis we develop a general theoretical framework for determining the phase-resolved predictable zone in space-time for different wave-data conditions.

1.4 Optimal deployment of wave measurements

We study the reconstruction and forecast of wave field from given wave measurements. In practice, there are different kinds of wave measurements, including probes which provide wave-field measurement at discrete locations for a certain period of time, whole-area measurements which provide wave measurement over a certain area at some time instants, such as wave elevation data obtained from X-band marine radar images ([76, 88]), and hybrid measurements which are the combination of the former two. For the purpose of analyzing the wave field, we desire as many measurements as possible. However, in practice there are only limited set of measurements due to the operational and economical cost. For given limited measurement capabilities and space-time domain of interest (the "test" zone), we need to optimize the placement and deployment of measurement tools to maximize the coverage and reliability/accuracy of the predicted wave field based on the predictable zone theory.
1.5 Reconstruction and forecast of irregular waves

Phase-resolved prediction of irregular ocean wave field based on specified wave measurements contains two major tasks: (i) determination of the predictable zone in the space-time domain in which the wave field can be completely predicted; and (ii) phase-resolved reconstruction (within the measurement space-time domain) and forecast (beyond the measurement space-time domain) of the wave field in the predictable zone.

In principle, wave reconstruction assuming linearized theory is straightforward (regardless of whether the underlying waves are linear or not). Any set of measurements in a given $\mathcal{M}$ can in theory be matched by sufficiently large number of (independent) free-wave modes. For example, if the wave elevation $\zeta(t)$ of a unidirectional wave field is measured at some location $x = \xi$ (over a period of time $T$), a Fourier series with sufficient terms could match $\zeta(t \in T)$ to any precision, regardless of whether $\zeta$ measures a linear or nonlinear wave. In Chapter 2, we will show that, for infinitesimal waves, the linearized predicted field $\Phi$ will obtain the underlying wave field $\tilde{\Phi}$ in a space-time domain $\mathcal{P}$ (which we formally obtain in Chapter 2). However, for waves with even relatively small steepnesses commonly encountered in practice, the last statement is generally not true. In the earlier example, if $\zeta(t)$ measures a wave of some finite nonlinearity, the free-wave fitted $\Phi$ obtains $\zeta(t)$ but other quantities such as velocity or pressure under this surface may not be well predicted. We illustrate this later by comparing linear versus nonlinear reconstructed wave fields against synthetic waves (cf. §5.1.1) and tank measurements (cf. §5.1.3).

A general nonlinear wave field contains free propagating wave components as well as "locked" (bound and resonant wave) components due to nonlinear wave-wave interactions. A nonlinear reconstruction must account for all these components in matching the measurements. There are relatively small number of studies that include nonlinearity in reconstruction, for example, [102] and [124, 125], which include up to second-order wave nonlinearity. Since the wave field can be represented in closed form up to second order, the reconstruction is still relatively simple. However, it is
not straightforward to extend such approaches to higher order. In this study (§4.2.7, 5), we show that inclusion of the higher-order nonlinear effects is very important in the reconstruction of steeper wave fields. Recently, [96] considered reconstruction/prediction of sufficiently narrow banded waves using a third-order modified nonlinear Schrodinger equation (MNLS) as the model equation. For long-crested narrow-banded waves, they obtain correct predictions over certain distances, which they show is strongly reduced for short-crested waves with increasing directional spread. The range of applicability of MNLS reconstruction for general broad-banded wave fields is difficult to assess due to the assumptions inherent in the MNLS. Significantly, none of existing work provide the predictable zone \( P \), for a given set of measurements \( M \), within which the (linear or nonlinear) reconstructed wave-field prediction can be assumed correct (without actual knowledge of the underlying wave field).

In this work, we address the problem of nonlinear phase-resolved reconstruction of general two- and three-dimensional wave fields. For a given set of measurements \( M \), the reconstruction problem is a nonlinear inverse problem, in which we aim to reconstruct a (theoretically) continuous nonlinear system using a representation with a finite number of model parameters. Due to data insufficiency, it is strictly impossible to uniquely determine all the details of the original system. For linear inverse problems, however, it has been shown that certain properties, such as local estimation, of the original system can be obtained from the reconstructed model [5]. For nonlinear inverse problems, with complexities due to plateau, forbidden region and multiple minima introduced in the misfit function [99], the model reconstruction becomes much more difficult. For strongly nonlinear problems, there is no systematic way to reconstruct the model and the inverse Monte-Carlo sampling method [73] is probably the only available way to estimate the model.

In this work, we restrict our attention to nonlinear waves which can be (uniquely) expressed as regular perturbation series (of arbitrary high order) in wave steepness. Under this restriction, it has been shown [98] that a high-order (nonlinear) model can be reconstructed to obtain an optimal (and unique) local estimation of the original nonlinear system.
1.6 Phenomenological wave breaking modeling

In the nonlinear phase-resolved wave reconstruction, we need a nonlinear evolution engine to simulate the evolution of the wave field. In principle, in order to simulate well, we need to account for all the main factors that influence the evolution of the wave field, such as energy transfer due to nonlinear wave-wave interaction, energy input from wind force, and energy dissipation due to wave breaking, which altogether keep the energy balance of the wave system (e.g. [39]). In this work, we focus on unforced wave system, thus we neglect the wind input. Nonlinear wave-wave interaction can be well simulated using numerical methods such as the high-order spectral (HOS) method [24]. Thus in order to properly reconstruct and forecast the wave field, especially for steep waves which may contain breaking events in the evolution, it is crucial for the wave simulation model to have the ability of simulating wave breaking effects, so that the simulation can continue to post-breaking wave-field evolution. However, simulation of wave breaking, specifically, simulation of breaking onset, strength and even breaking events locations, have not been addressed well. In this work, we develop a phenomenological wave breaking model that can be incorporated into the nonlinear evolution engine in the wave reconstruction and is expected to be able to capture the main phenomenological effects of wave breaking, including detecting breaking onset at the right time, dissipating appropriate amount of energy and indicating the locations of breaking events.

Historically, lots of efforts have been made on proposing criteria for predicting breaking onset of deep water waves, while very few research has been done on the simulation of wave breaking strength and locations.

Predicting breaking onset of deep water waves has been an elusive problem since the pioneering work of Stokes [105]. The existing breaking onset criteria have been derived based on idealized theoretical models, numerical simulations, laboratory experiments and field observations. These criteria can be classified into three categories of threshold variables: geometric, kinematic and dynamic/energetic. Geometric threshold variables include limiting wave steepness ([105, 112, 77], etc.) and limiting
asymmetric wave form [48]. However, neither limiting wave steepness nor asymmetric wave form criteria are in accord with measurements at sea [44] since breaking is not the consequence of the nonexistence of steeper waves and wave instability may play an important role in breaking inception [111]. Kinematic threshold variables include limiting ratio of water velocity at the crest and phase velocity [66] or group velocity [113] of wave crest and limiting acceleration of water particle at the crest [59, 60, 61]. However, for a wide wave spectrum, it is hard to obtain the actual propagation velocity of the highest wave crest in the wave group [95]. And the criterion based on acceleration needs detailed experimental validation. An overview of wave breaking criteria using geometric/kinematic approaches can be found from [81] which leads to a conclusion that overall, these kinematic criteria do not robustly predict deep water wave breaking onset.

As is realized that wave breaking is intrinsically a dynamic process closely related to the energy transfer in the wave system, people start to study breaking onset criteria based on wave energetics. Tulin and Li (1992) [111] studied the evolution of the intra-group energy flux which causes wave breaking but did not propose a breaking onset criteria. Schultz et al. (1994) [93] proposed a breaking onset criterion as the ratio of potential energy and total wave energy exceeding 0.52. However, this criterion is not valid in rotational flow where the ratio varies significantly as a function of the background vorticity strength [107] and neither in the presence of a surface shear layer which changes the wave shape and hence the potential energy significantly [71]. Banner and Tian (1998) [7] proposed a criterion based on the growth rate of the local mean wave energy and momentum densities, and then later they found this is not a robust indicator for different cases and proposed another criterion based on growth rate of the square of maximum local steepness in 2D wave groups, which is related to the energy convergence rate in the wave group [101, 6]. However, the improved criterion is based on data post-processing, which is not appropriate for simultaneous detection of wave breaking onset. Recently, Derakhti and Kirby (2016) [21] conducted 2D and 3D LES numerical study of unsteady 2D wave packets and proposed a revised threshold growth rate of local energy density based on Song and
Banner (2002) [101] by avoiding data post-processing. Barthelemy et al. (2017) [8] proposed a limiting local ratio of energy flux velocity to crest tip velocity for the tallest wave in the evolving group. So far, all of these energetic criteria are developed in the physical space-time domain. It may not be efficient to apply them on large-domain and/or long-time simulations, for which spectral-based simulations are more efficient. Challenges are that very few research exists on prediction of wave breaking for deep water waves in the spectral (wavenumber-frequency) domain and attempt is made on this topic in this work. Xiao et al. (2013) [118] proposed a wave breaking model incorporated in the phase-resolved simulation, but that model cannot detect breaking onset, nor the different types of wave breaking. It just provided a rough estimation of the overall energy dissipation.

1.7 Thesis objective and scope

The objective of the thesis is to develop both theoretical and computational capabilities for phase-resolved reconstruction and forecast of nonlinear irregular wave field based on wave measurement data.

To achieve this objective, we specifically focus on these scientific issues:

• Investigating the phase-resolved predictability of irregular waves when measurements of the field are provided at discrete spatial point locations (fixed or moving) over some time durations, or over some specified area (fixed or moving) at specific time instants and developing a general theory for determining the predictable zone in space-time for different wave-data conditions;

• Investigating the optimal deployment of different kinds of wave measurement sources in order to maximize the predictable zone;

• Investigating the asymptotic behavior of the predictable zone for large number of wave measurement sources under optimal deployment;

• Developing the theoretical approach for reconstruction and forecast of linear wave field based on simple wave measurement setup;
• Developing the general numerical scheme for reconstruction and forecast of non-linear wave field based on arbitrary measurement setup;

• Verifying the developed predictable zone theory and numerical scheme using both synthetic and experimental data;

• Developing wave breaking onset and strength models which can be incorporated in the phase-resolved wave-field simulations;

• Quantifying wave breaking kinematics and dynamics based on the developed wave breaking models.

This thesis does not address the issue of wave-body interaction nor the control problem for scale models in the wave basin, but it provides necessary nonlinear whole-field data for intense CFD analysis of wave-body interaction (such as the LAMP system [56]) at a level heretofore not possible. The presence of imposed current or wind is not considered at this stage but can be incorporated in the future using the same framework.

1.8 Thesis contributions

The major contributions of this thesis are summarized as follows:

• Development of a general theoretical framework to determine the predictable zone in space-time, obtaining closed-form expressions for these zones in terms of the input measurements, for various wave-data conditions, including unidirectional and multidirectional broadband irregular ambient wave fields based on single/multiple, fixed/moving, probes, whole-area measurements or hybrid measurements. The predictable zone theory also provides a theoretical approach for reconstruction and forecast of linear wave field based on simple wave measurement setup.

• Understanding of the optimal deployment of multiple measurement sources to maximize the predictable zone and the asymptotic behavior of the predictable
zone for large number of optimal deployed measurements.

- Development of an iterative high-order reconstruction (HOR) method for reconstruction and forecast of nonlinear wave field based on general wave measurement setup.

- Development of new wave breaking onset and strength models that can be incorporated into phase-resolved nonlinear wave-field simulations.

- Quantification of wave breaking kinematics and dynamics based on the developed wave breaking models.

1.9 Thesis contents

The rest of the thesis is organized as follows:

Chapter 2 presents the theoretical framework to determine the predictable zone for various wave-data conditions.

Chapter 3 presents the analysis for the optimal deployment of measurements sources to maximize the predictable zone the asymptotic behavior of the predictable zone or large number of optimal deployed measurements.

In Chapter 4, we develop the theoretical approach to reconstruct and forecast linear wave field based on simple wave measurement setup and an iterative high-order reconstruction (HOR) method for reconstruction and forecast of nonlinear wave field based on general wave measurement setup.

Chapter 5 provides the verification of the predictable zone theory and the HOR method using both synthetic and experimental data.

Chapter 6 presents the development of wave breaking onset and strength models which can be incorporated into the phase-resolved nonlinear wave-field simulations.

In Chapter 7, we study the quantification of wave breaking kinematics and dynamics based on the developed wave breaking models.

Finally in Chapter 8, we summarize the conclusions of the thesis and discuss several potential research directions to extend the current work.
Chapter 2

Linear predictable zone

In this chapter, we investigate the phase-resolved predictability of irregular waves when measurements of the field are provided at discrete spatial point locations (fixed or moving) over some time durations, or over some specified area (fixed or moving) at specific time instants. Using linearized wave theory and set notation, we develop a general theoretical framework for determining the phased-resolved predictable zone in space-time, obtaining closed-form expressions for these zones in terms of the input measurements. We consider general unidirectional and multidirectional broadband irregular ambient wave fields, where, without loss of generality and to make the problem well-posed, we assume (most of) the wave energy is confined within some minimum and maximum wave frequency.

We show that in general the total predictable zone $\mathcal{P}$ from multiple measurements is larger than the union of the predictable zones of the individual (or sub-group of) measurements. We generalize the results for moving measurements (following some specified probe trajectories) and show how $\mathcal{P}$ can be greatly increased with optimal trajectories. The results are extended in a straightforward way to the situation where whole-field measurements (over specified areas at given time instants), and when there is a combination of discrete and whole-field measurements. The total predictable zone is further increased when measurements are optimally deployed (in space-time), which is discussed in detail in Chapter 3. Indeed, for $J$ optimally placed discrete measurements, we show that the space-time volume of $\mathcal{P}$ increases as $J^3$. In this
chapter, the objective is to obtain the zone within which the phase-resolved wave field is theoretically predictable (within the context of linearized theory). Our results provide the foundation for Chapter 4, where we address the practically important problem of obtaining the actual phase-resolved waves in the predictable zone, in general for nonlinear waves.

2.1 Problem statement

Our overall objective is to perform phase-resolved reconstruction/forecast of a wave field in some (desired) space-time domain based on given measurements of the field. Given an irregular (original) wave field \( \tilde{\eta}(x, t) \), we assume that we have measurements in specific space-time domains \( \mathcal{M}_j, j=1,2,...,J \) which resolve all the frequency and directional components of \( \tilde{\eta} \) within \( \mathcal{M}_j \). For example, \( \mathcal{M}_j \) could be at a fixed spatial location \( \xi_j \) (hereafter “probe” measurement) over some time duration \( T_j \); or \( \mathcal{M}_j \) could cover an extended spatial domain \( x \in \mathcal{A}_j(t) \) (hereafter “whole-area” measurement) at some instant \( t_j \) or over some time duration \( T_j \). Our interest in Part 1 is to determine the space-time domain \( \mathcal{P} \), the predictable zone, in which a (phase-resolved) continuous wave field \( \eta(x, t) \) can be predicted that matches \( \tilde{\eta} \) according to linearized theory for a given set of \( \mathcal{M}_j \), and indeed to obtain explicit closed-form expressions for \( \mathcal{P} \) in terms of the latter. Clearly \( \mathcal{M} \subset \mathcal{P} \), where \( \mathcal{M} = \bigcup_j \mathcal{M}_j \). For simplicity, we assume no (biased or unbiased) errors in the measurements, although, in the context of linearized theory, the predictable zone \( \mathcal{P} \) itself should not be affected by small values of these.

The original three-dimensional irregular wave field \( \tilde{\eta} \) can be quite general, and can be characterized by say its frequency-directional spectrum \( S(\omega, \theta) \). To avoid ill-posedness, we exclude standing waves, so that \( \theta \in (-\pi/2, \pi/2) \). For ocean gravity waves, energy is confined in frequency, and we assume there are values of frequency \( \omega_a(>0) \) and \( \omega_b(<\infty) \), below and above which respectively the energy is negligible. No other assumption is made regarding say the shape of the underlying spectrum or the stationarity and homogeneity of the wave field. Effects associated with wind input or wave breaking or other dissipation are neglected.
We define a Cartesian coordinate system \((o - xyz)\) with the origin on the mean free surface, \(x \equiv (x, y)\) representing the horizontal coordinates, and \(z\)-axis positive upwards. Let \(\eta(x, t)\) represent the surface elevation of the original wave field. For simplicity, we focus on probes at location \(x = \xi_j(t)\) for \(t \in T = [\tau_j, \tau_j + T]\). The case of whole-area measurements (and combinations of probes and whole-area measurements) is considered in a later section.

2.2 Predictable zone for a unidirectional wave field

To illustrate the basic principle, we first consider a unidirectional (one horizontal dimension) irregular wave field propagating in the positive \(x\)-direction, and one or more probes at fixed spatial locations. Since the wave is unidirectional, a point measurement of the wave elevation itself (over sufficient duration) resolves all the wave components of the original wave field within the space-time domain \(M\).

2.2.1 Measurement at a single fixed location

We consider the elevation measurement \(\zeta(t) = \eta(\xi, t)\) from a single fixed probe located at \(x = \xi\) for \(t \in [0, T]\). From linear wave theory:

\[
\eta(x, t) = \Re \int_{\omega_a}^{\omega_b} A(\omega) e^{i[k(\omega)x - \omega t]} \, d\omega, \quad x, t \in (-\infty, \infty) \tag{2.1}
\]

where \(k(\omega)\) is the wavenumber related to the frequency \(\omega\) by the dispersion relation, and \(A(\omega)\) is the complex amplitude of the wave component of frequency \(\omega\). From (2.1), the given elevation at \(x = \xi\) can be expressed as:

\[
\zeta(t) = \eta(x = \xi, t) = \Re \int_{\omega_a}^{\omega_b} A(\omega) e^{i[k(\omega)\xi - \omega t]} \, d\omega, \quad t \in [0, T]. \tag{2.2}
\]

To find \(P\), we seek the direct relation between \(\eta(x, t)\) in (2.1) and \(\zeta(t)\) in (2.2).

To simplify the integration with respect to \(\omega\) in (2.1), we equally divide the frequency band \([\omega_a, \omega_b]\) into \(N_\omega\) segments. At the center of each segment, the frequency
is \( \omega_n = \omega_a + (n - 1/2)\Delta \omega, \quad n=1, \ldots, N_\omega \), where \( \Delta \omega = (\omega_b - \omega_a)/N_\omega \). With this, (2.1) can be written as:

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_\omega} \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} A(\omega)e^{i[k(x-\xi) - \omega t + k\xi]} \, d\omega .
\]  

(2.3)

Over each segment, \( \omega \in [\omega_n - \Delta \omega/2, \omega_n + \Delta \omega/2] \), we expand \( k(\omega) \) in Taylor series about \( \omega = \omega_n \),

\[
k(\omega) = k_n + \frac{\omega - \omega_n}{C_n} + O(\Delta \omega)^2 \tag{2.4}
\]

where \( k_n \equiv k(\omega_n) \), and \( C_n \equiv \frac{\partial k}{\partial \omega}\big|_{\omega_n} \) is the group velocity at frequency \( \omega = \omega_n \). Making use of (2.4), we have

\[
e^{ik(x-\xi)} = e^{ik_n(x-\xi) + i\frac{\omega - \omega_n}{C_n}(x-\xi) + O(\Delta \omega)^2|\xi|} = e^{i(k_n(x-\xi) + \omega(x-\xi)/C_n)} \{1 + O[(\Delta \omega)^2|x-\xi|]\} \tag{2.5}
\]

for \( \omega \in [\omega_n - \Delta \omega/2, \omega_n + \Delta \omega/2] \). Substitution of (2.5) into (2.3) gives:

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_\omega} e^{i(k_n(x-\xi))} \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} A(\omega)e^{i[k\xi - \omega(t - \Delta t_n)]} \{1 + O[(\Delta \omega)^2|x-\xi|]\} \, d\omega 
\]

\[
= \text{Re} \sum_{n=1}^{N_\omega} e^{i(k_n(x-\xi))} \zeta_n(t - \Delta t_n) \cdot \{1 + O[(\Delta \omega)^2|x-\xi|]\} 
\]

\[
= \text{Re} \sum_{n=1}^{N_\omega} e^{i(k_n-x-\xi))} \zeta_n(t - \Delta t_n) , \quad \text{as} \ N_\omega \to \infty \tag{2.6}
\]

where \( \Delta t_n = (x-\xi)/C_n \). Here \( \zeta_n(t) \) is defined by

\[
\zeta_n(t) = \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} A(\omega)e^{i[k\xi - \omega t]} \, d\omega \tag{2.7}
\]

which represents the wave elevation at \( x = \xi \) due to the contribution from wave components in the narrow frequency band \( \omega \in [\omega_n - \Delta \omega/2, \omega_n + \Delta \omega/2] \). From (2.2), we have that \( \text{Re} \sum_{n=1}^{N_\omega} \zeta_n(t) = \zeta(t) \) for \( t \in [0, T] \). This indicates that \( \zeta_n(t) \) for \( t \in [0, T] \) is known in terms of the given data \( \zeta(t) \). We remark here that, in practice, the process
of $N_\omega \to \infty$, $\Delta \omega \to 0$ in the above analysis imply that $T$ must be sufficient large for the components $\zeta_n$ in (2.7) to be estimated.

Equation (4.2) relates implicitly the wave elevation anywhere in the wave field at any time, $\eta(x, t)$, to the time record of wave elevation at an arbitrary location $(x = \xi)$, $\zeta(t)$. Specifically, $\eta(x, t)$ is the sum of the contributions from wave components, $\zeta_n(\tau)$, evaluated at the shifted time $\tau = t - (x - \xi)/C_n$, $n = 1, \ldots, N_\omega$. Significantly, $\eta(x, t)$ is completely determined from $\zeta(\tau)$ in the duration of time: $t \in [(x - \xi)/C_{\min}, T + (x - \xi)/C_{\max}]$ for $x - \xi > 0$ or $t \in [(x - \xi)/C_{\max}, T + (x - \xi)/C_{\min}]$ for $x - \xi < 0$, where $C_{\max}$ and $C_{\min}$ are the maximum and minimum group velocities of the wave components in the wave field. This provides the basis for the determination of the predictable zone $P$ for given probes.

For a given wave record $\zeta(t)$ with $t \in [0, T]$, we have from (4.2) that in order for a space-time point $(x, t)$ to be inside $P$, it must satisfy the condition $0 \leq t - (x - \xi)/C_n \leq T$ for all values of $n=1, \ldots, N_\omega$. Physically, this requires that all wave components that reach $(x, t)$ must be traced back or forth to somewhere within the duration of given data. For clarity in description, we define $\Pi_n(t)$ as the predictable area at time $t$ for the $n$-th wave component. From the condition $0 \leq t - (x - \xi)/C_n \leq T$, we obtain $\Pi_n(t)$ to be:

$$\Pi_n(t) : (t-T)C_n \leq x - \xi \leq tC_n, \quad n = 1, 2, \ldots, N_\omega . \quad (2.8)$$

In the $x - t$ plane, $\Pi_n(t)$ is given by an infinitely long strip bounded by two parallel lines: $x - \xi = tC_n$ and $x - \xi = (t-T)C_n$, as shown in figure 2-la. The predictable area $\Pi(t)$ at time $t$ for all wave components in the wave field corresponds to the intersection of all $\Pi_n(t)$'s:

$$\Pi(t) = \bigcap_{n=1}^{N_\omega} \Pi_n(t) . \quad (2.9)$$

As $N_\omega \to \infty$, $\Delta \omega \to 0$ and the discrete set series $\Pi_n(t)$, $n = 1, 2, \ldots, N_\omega$, becomes a continuous set series $\Pi_\omega(t)$ with $\omega \in \Omega \equiv [\omega_a, \omega_b]$:

$$\Pi_\omega(t) : (t-T)C_\omega \leq x - \xi \leq tC_\omega, \quad \omega \in \Omega \quad (2.10)$$
where $C_\omega$ is the wave group velocity at frequency $\omega$. According to set theory (e.g. [35]; [45]), for a continuous set series $\Pi_\omega(t)$, $\omega \in \Omega$, where the indexing set $\Omega$ is a continuous range, the intersection of this set series is well defined given by:

$$\Pi(t) = \bigcap_{\omega \in \Omega} \Pi_\omega(t),$$

(2.11)

which corresponds to the predictable area of the wave field at time $t$.

The predictable zone $\mathcal{P}$ of the whole wave field is the combination of $\Pi(t)$ over all predictable time $t \in [T_s, T_e]$:

$$\mathcal{P} : \{\Pi(t), t \in [T_s, T_e]\}$$

(2.12)

where the starting and ending time, $T_s$ and $T_e$, corresponds to the lower and upper boundaries of $\mathcal{P}$ in time, respectively, with $\Pi(T_s) = \Pi(T_e) = \emptyset$.

For unidirectional waves with one fixed probe, $\mathcal{P}$ can be expressed in a closed form:

$$\mathcal{P} : \left\{ \begin{array}{l}
(t - T)C_{\max} \leq x - \xi \leq tC_{\min}, & T < t \leq T_e \\
(t - T)C_{\min} \leq x - \xi \leq tC_{\min}, & 0 < t \leq T \\
(t - T)C_{\min} \leq x - \xi \leq tC_{\max}, & T_s \leq t \leq 0
\end{array} \right\}$$

(2.13)

where

$$T_s = -\beta_u T \quad \text{and} \quad T_e = T + \beta_u T,$$

(2.14)

and we define $\beta_u \equiv C_{\min}/(C_{\max} - C_{\min})$ and $C_{\max} \equiv C_{\omega_s}, C_{\min} \equiv C_{\omega_b}$.

Figure 2-2 displays the predictable zone of a unidirectional broadband irregular waves in the $x - t$ domain, given by (2.13). The point $O$ represents the position of the wave probe. The parallelogram $AOBT$ is the predictable zone $\mathcal{P}$. The slope of the segments $AO$ and $TB$ is $1/C_{\max}$ while that of $AT$ and $OB$ is $1/C_{\min}$. As shown in figure 2-2, $\mathcal{P}$ can be divided into three sub-regions: parallelogram $DOET$ ($0 \leq t \leq T$, $(t - T)C_{\min} \leq x - \xi \leq tC_{\min}$) and triangles $TEB$ ($T \leq t \leq T_e$, $(t - T)C_{\max} \leq x - \xi \leq tC_{\min}$) and $AOD$ ($T_s \leq t \leq 0$, $(t - T)C_{\min} \leq x - \xi \leq tC_{\max}$).
Figure 2-1: Predictable zone $\mathcal{P}$ for a unidirectional wave field, based on one fixed probe measurements at $x = \xi$, with a single wave component of frequency. The segment $[0, T]$ on the $t$-axis represents the duration of time in which the probe measurements is given.
Figure 2-2: Predictable zone $\mathcal{P}$ for a unidirectional wave field, based on one fixed probe measurements at $x = \xi$, with broadband waves of frequencies $\omega \in [\omega_a, \omega_b]$. The segment $[0, T]$ on the $t$-axis represents the duration of time in which the probe measurements is given.
The region $DOET$ is associated with the wave reconstruction problem as it is in the time interval $[0, T]$ of specified wave data. The region $TEB$ ($AOD$) corresponds to the wave forecast (hindcast) problem since it is beyond (before) the interval $[0, T]$ of the data. Note that, the predictable time duration $\mathcal{D} \equiv T_e - T_s = (1 + 2\beta_\omega)T$ is linearly proportional to measurement duration $T$ and $\beta_\omega$ which depends only on $\Omega$.

For a general measurement duration $[\tau, \tau + T]$, the hindcast duration $\mathcal{D}_h \equiv \tau - T_s$ and the forecast duration $\mathcal{D}_f \equiv T_e - (\tau + T)$ are equal, $\mathcal{D}_h = \mathcal{D}_f = \beta_\omega T$.

Note that, while the predicted field is exact inside $\mathcal{P}$, $\eta(x, t) = \tilde{\eta}(x, t)$, $(x, t) \in \mathcal{P}$, $\eta$ deviates (rapidly) from $\tilde{\eta}$ outside $\mathcal{P}$ since $\tilde{\eta}$ contains wave information outside the time interval $[0, T]$, which is not provided. An estimation of the prediction error outside $\mathcal{P}$ (and its increase with distance from $\mathcal{P}$) can be made based on the ratio of the energy contained in the wave components that do not satisfy the condition of $0 < t - (x - \xi)/C_n \leq T$, $n = 1, 2, \ldots, N_\omega$, to the total energy of the wave field.

We remark that, based on intuitive arguments, [72] suggested that $\mathcal{P}$ is governed by the maximum and minimum phase velocities of the wave components in the wave field, a result that follows directly from the above analysis with the use of $k(\omega) = k_n + O(\omega - \omega_n)$, instead of (2.4). This result is incorrect since the resulting residual term in (4.2) becomes $O(1)$ as $N_\omega \to \infty$ and does not vanish. Thus, it is the group velocity rather than phase velocity of the waves that controls the predictability of the wave field based on probe measurements.

### 2.2.2 Measurement at multiple fixed locations

The extension to multiple (fixed) locations follows directly from §2.2.1 which shows that wave information at any space-time point $(x, t)$ is completely determined if all the wave information arriving at this point can be traced back (or forth) in $x$-$t$ to given measurements. For multiple probe data, wave components of different frequency arriving at $(x, t)$ can come from different measurements, which must be taken into account in determining $\mathcal{P}$.

The predictable area at any $t$ for wave component of frequency $\omega$ given multiple $\zeta^j$, $j=1,2,\ldots, J$, is clearly the union of the predictable area of the same wave component.
with each single probe:

\[ \Pi_\omega(t) = \bigcup_{j=1}^{J} \Pi^j_\omega(t) \]  
\hspace{1cm} (2.15)

where \( \Pi^j_\omega(t) \) follows from §2.2.1:

\[ \Pi^j_\omega(t) : (t - \tau_j - T_j)C_\omega \leq x - \xi_j \leq (t - \tau_j)C_\omega. \]  
\hspace{1cm} (2.16)

The predictable area \( \Pi(t) \) of the wave field with waves within a frequency band \( \Omega \) is given by the intersection of the set series \( \Pi_\omega(t) \) with \( \omega \in \Omega \):

\[ \Pi(t) = \bigcap_{\omega \in \Omega} \Pi_\omega(t). \]  
\hspace{1cm} (2.17)

The combination of the set series \( \Pi(t) \) for all predictable \( t \) gives the complete predictable zone \( \mathcal{P} \) in the space-time domain:

\[ \mathcal{P} : \{\Pi(t), t \in [T_s, T_e]\} \].  
\hspace{1cm} (2.18)

Note that the order of union and intersection is in general not commutable. This implies that \( \mathcal{P} \) from the combination of multiple probes, the true solution ( (2.15) to (2.18)), differs from the union \( \tilde{\mathcal{P}} = \bigcup_j \mathcal{P}^j \) of the predictable zones from each measurement. If fact, from set theory, \( \mathcal{P} \) is always larger than (or equal to, when the predictable zones of individual measurement do not overlap) \( \tilde{\mathcal{P}} \).

To illustrate this, we consider a simple case involving two wave components with two probes only and compare the \( \mathcal{P} \) and \( \tilde{\mathcal{P}} \). The predictable area at time \( t \) from the combination of the two probes is:

\[ \Pi(t) = \bigcap_{\omega_1, \omega_2} \bigcup_{j=1}^{2} \Pi^j_{\omega}(t) \]
\[ = (\Pi^1_{\omega_1} \cup \Pi^2_{\omega_1}) \cap (\Pi^1_{\omega_2} \cup \Pi^2_{\omega_2}) \]
\[ = (\Pi^1_{\omega_1} \cap \Pi^1_{\omega_2}) \cup (\Pi^2_{\omega_1} \cap \Pi^2_{\omega_2}) \cup (\Pi^1_{\omega_1} \cap \Pi^2_{\omega_2}) \cup (\Pi^2_{\omega_1} \cap \Pi^1_{\omega_2}) \]
\[ = \tilde{\Pi} \cup (\Pi^2_{\omega_1} \cap \Pi^1_{\omega_2}) \cup (\Pi^1_{\omega_1} \cap \Pi^2_{\omega_2}) \]  
\hspace{1cm} (2.19)
Figure 2-3: Predictable zone \( \mathcal{P} \) for a unidirectional wave field of frequency band \([\omega_a, \omega_b]\) with two fixed probes. The two probes are located at \( \xi_1 \) and \( \xi_2 \) with time durations of data \([\tau_1, \tau_1 + T_1]\) and \([\tau_2, \tau_2 + T_2]\), respectively.

where \( \tilde{\Pi}(t) = (\Pi_{\omega_1}^1 \cap \Pi_{\omega_2}^1) \cup (\Pi_{\omega_1}^2 \cap \Pi_{\omega_2}^2) \) is the union of the predictable areas based on each single probe measurements. Clearly \( \Pi(t) \) is larger than or equal \( \tilde{\Pi}(t) \), indeed, \( \tilde{\Pi} \subseteq \Pi \).

Figure 2-3 shows an example of the predictable zone \( \mathcal{P} \) based on two fixed probes. The predictable zones \( \mathcal{P}^j \) based on each single probe are depicted for comparison. The union of these is clearly smaller than the total predictable zone \( \mathcal{P} \). Note that in order for the \( \mathcal{P} \) to be a single connected domain in \( x-t \), the distance between the two (fixed) probes \( \Delta \xi \) must satisfy the condition \( \Delta \xi \leq C_{\min}(T_2 - \tau_1) \) for \( \tau_1 < \tau_2 < \tau_1 + T_1 \) in this case. For greater values of \( \Delta \xi \), \( \mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2 \) and \( \mathcal{P}^1 \cap \mathcal{P}^2 = \emptyset \).
2.3 Predictable zone for a multidirectional wave field

We extend the algorithm developed in the proceeding section (§2.2) to the case of general three-dimensional irregular wave field containing multidirectional waves. The extension is conceptually straightforward and based on first obtaining the predictable zone \( \mathcal{P}_\theta \) along a specific wave propagation direction \( \theta \), following §2.2, and then obtaining \( \mathcal{P} \) as the intersection of the \( \mathcal{P}_\theta \) over all wave propagation directions \( \theta \in \Theta \equiv [\theta_a, \theta_b] = [\theta_m - \Delta \theta, \theta_m + \Delta \theta] \), where \( \theta_m = (\theta_a + \theta_b)/2 \), and \( \Delta \theta = (\theta_b - \theta_a)/2 \).

The predictable area for waves propagating in the \( \theta \)-direction at any time \( t \), based on multiple fixed probes \( j \), is:

\[
\Pi_\theta(t) = \bigcap_{\omega \in \Omega} \left[ \bigcup_{j=1}^J \Pi_{\omega,j}(t) \right]. \quad (2.20)
\]

In (2.20), the predictable area \( \Pi_{\omega,j}(t) \) for wave component with frequency \( \omega \) and propagation direction \( \theta \) given by the single \( j \)-th measurement is:

\[
\Pi_{\omega,j}(t) : r_1^j(\omega, t) \leq (x - \xi_{x,j}) \cos \theta + (y - \xi_{y,j}) \sin \theta \leq r_2^j(\omega, t) \quad (2.21)
\]

with

\[
r_1^j(\omega, t) = (t - \tau_j - T_j)C_\omega \quad \text{and} \quad r_2^j(\omega, t) = (t - \tau_j)C_\omega. \quad (2.22)
\]

The intersection of \( \Pi_\theta(t) \) given by (2.20) for wave directions over all \( \theta \in \Theta \) gives the predictable area for the multidirectional wave field:

\[
\Pi(t) = \bigcap_{\theta \in \Theta} \Pi_\theta(t) \quad (2.23)
\]

from which the total predictable zone \( \mathcal{P} \) in space-time \((x - t)\) domain is obtained:

\[
\mathcal{P} : \{\Pi(t), t \in [T_s, T_d]\}. \quad (2.24)
\]

Figure 2-4 to 2-9 illustrates a geometric procedure to determine \( \Pi(t) \) for a single fixed probe located at \(((\xi_x, \xi_y) = (0, 0))\) over time duration \( t \in [0, T] \). In 2D, the
Figure 2-4: Illustration of extension of the predictable zone of unidirectional waves with a single fixed probe to multidirectional waves: the predictable zone of unidirectional waves in the $x - t$ plane.
Figure 2-5: Illustration of extension of the predictable zone of unidirectional waves with a single fixed probe to multidirectional waves: the predictable area of unidirectional waves in the two-dimensional $x - y$ plane at time $t=\tau$. 
Figure 2-6: Illustration of extension of the predictable zone of unidirectional waves with a single fixed probe to multidirectional waves: the predictable area of a multidirectional wave field with two propagation directions (in $\theta_a$ and $\theta_b$).
Figure 2-7: Illustration of extension of the predictable zone of unidirectional waves with a single fixed probe to multidirectional waves: the predictable area of a multidirectional wave field with propagation direction band $\Theta = [\theta_a, \theta_b]$. 
Figure 2-8: Illustration of extension of the predictable zone of unidirectional waves with a single fixed probe to multidirectional waves: the detail geometry of predictable area in (d), where $AB$, $BC$, $DE$, $EA$ are linear segments and $CD$ is a circular arc.
Figure 2-9: Illustration of extension of the predictable zone of unidirectional waves with a single fixed probe to multidirectional waves: the predictable area of a multidirectional wave field at different instants: $t_1 < t_2 < 0$ (hindcast), $t_3 = T/2$ (reconstruction), $T < t_4 < t_5$ (forecast) and $T_s$, $T_e$ correspond to the starting and ending time of $P$. 
predictable area at any time \( t = r \) is the line segment \( x_l \leq x \leq x_u \) in the \( x - t \) plane, figure 2-4, see (2.13). In 3D, the corresponding predictable area of an unidirectional wave, \( \theta = 0 \), at time \( t = r \) is a 2D strip, say, \( \Pi_0 \), of the same width in \( x \) in the \( x-y \) plane, figure 2-5. The latter result is clearly generalizable to \( \Pi_\theta \) by rotating \( \Pi_0 \) for any propagation direction \( \theta \). For 3D wave field contains many propagation directions, the predictable area \( \Pi(t) \) is given by the intersection of all such \( \Pi_\theta \) strips. Figure 2-6 shows that the resulting predictable area \( \Pi(t) \) is a parallelogram for a wave field containing waves with two propagation directions, \( \theta_a \) and \( \theta_b \).

For the general case with multiple directional waves with propagating angle \( \theta \in \Theta \), the result is a "fan"-shaped \( \Pi(t) \) as shown in figure 2-7. Figure 2-8 shows the detail geometry of the fan shaped predictable area \( \Pi(t) \) bounded by \( ABCDE \). \( AB, BC, DE, EA \) are linear segments with the vertices given by:

\[
A=(x_A, y_A) = x_l/\cos(\Delta \theta)(\cos(\theta_m), \sin(\theta_m));
B=(x_A - (x_u - x_l) \sin \theta_a/\sin(2\Delta \theta), y_A + (x_u - x_l) \cos \theta_a/\sin(2\Delta \theta));
E=(x_A + (x_u - x_l) \sin \theta_b/\sin(2\Delta \theta), y_A - (x_u - x_l) \cos \theta_b/\sin(2\Delta \theta)).
\]

\( CD \) is a circular arc centered at the origin, \( O \) (of radius \( x_u \)). Figures 2-7,2-8 are clearly a function of time \( t \). Figure 2-9 shows \( \Pi(t) \) at several different instants \( t < 0 \) (hindcast), \( 0 < t < T \) (reconstruction), and \( t > T \) forecast. At a specific time \( t \), each of the predictable area \( \Pi(t) \) is a single region. The specific geometries differ but are obtained in a similar manner by considering the intersection (2.23) of the \( \Pi_\theta(t) \) at the corresponding \( t \). Together these form predictable zone \( \mathcal{P} \) which is a single domain in \( x - t \). The starting and ending vertices of this domain (in \( t \), where \( \Pi(t) \) is a single probe) (indicated in figure 2-9) are \( (x_s, y_s, T_s) \) and \( (x_e, y_e, T_e) \) given by:

\[
x_s = -C_{\min}T_e \cos(\theta_m) \quad \text{and} \quad y_s = -C_{\min}T_e \sin(\theta_m),
\]

\[
x_e = -x_s \quad \text{and} \quad y_e = -y_s,
\]

\[
T_s = -\beta_m T \quad \text{and} \quad T_e = T + \beta_m T,
\]

where we define \( \beta_m \equiv \cos(\Delta \theta)C_{\min}/(C_{\max} - \cos(\Delta \theta)C_{\min}) \). Similar to the unidirectional case in §2.2.1, the predictable time duration \( D = T_e - T_s = (1 + 2\beta_m)T \) is pro-
portional to the measurement duration $T$, while $\beta_m$ depends on $\Omega$ and half direction bandwidth $\Delta \theta$.

So far, we have considered a single (fixed) probe, the general solution for multiple $J$ probes in multi-directional waves follows in a straightforward way. As illustration, we show here an example of $J=3$ measurements at fixed locations $\xi_j$ over duration $\mathcal{T}_j$, $j=1,2,3$. The wave field has wavenumber and propagation direction ranges $[k_a, k_b]$ and $[\theta_a, \theta_b]$ respectively. Figure 2-10 shows the predictable areas $\Pi(t)$ at several different $t$ corresponding to $t \in \mathcal{T}_j$ (reconstruction), and $t <, > \mathcal{T}_j$ (hindcast, forecast). The $\Pi(t)$ shapes are more complicated but resemble that of a single probe (cf. figures 2-9). Figure 2-11 shows $\Pi(t)$ at some particular time compared to $\Pi_j(t)$ for an isolated probe $j$, $j=1,2,3$. Clearly $\Pi(t)$ is greater than the union of $\Pi_j(t)$, as expected from earlier considerations of multiple measurements for unidirectional waves.

2.4 Predictable zone based on moving probes

The above analysis can be extended to the case with moving probe. To illustrate the effect of probe motion on the predictability of a wave field, we first consider the simple case involving a single probe in unidirectional waves. The results are then extended to multiple moving probes in a multi-directional wave field.

2.4.1 Unidirectional waves with a single moving probe

Let $\xi(t)$ represent the instantaneous position of a probe that provides the wave elevation over a measurement duration $t \in [0, T]$. From (2.1), the wave elevation measured by the moving probe in the wave field corresponds to:

$$\zeta(t) \equiv \eta(x = \xi(t), t) = \text{Re} \int_{\omega_a}^{\omega_b} A(\omega) e^{i[k(\omega)\xi(t) - \omega t]} d\omega , \quad t \in [0, T] . \quad (2.28)$$
Figure 2-10: Predictable areas at several predictable instants of a multidirectional wave field based on three fixed probes. The wave field has wavenumber range $[k_a, k_b]=[0.1,1.9]k$, where $k = (k_a + k_b)/2$, and propagation direction range $[\theta_a, \theta_b]=[\pi/12, 5\pi/12]$. The locations of probes are $k\xi_1=(0,0.8)$, $k\xi_2=(-0.8,2.4)$, $k\xi_3=(-1.6,0.8)$ and measurement beginning time $\tau_j=(j-1)\bar{T}/2$, and time duration $T_j=4\bar{T}$, $j = 1, 2, 3$, where $\bar{T} = 2\pi/\sqrt{gk}$. Dimensionless space and time variables $x^* = kx$, $y^* = ky$, $t^* = t/\bar{T}$. The predictable time range is found to be: $[T_s, T_e] = [-1.4, 6.4]\bar{T}$. 

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Figure 2-11: Comparison of the total predictable area based on the combination of three probes and the union of the predictable areas of each single probe at $t^*=3$. (The bullet points represent the position of the probes.)
Following §2.2.1, we seek the direct relation between the elevation of the wave field \( \eta(x, t) \) and \( \zeta(t) \) given in (2.28). We rewrite (2.1) in the form:

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_{\omega}} \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} e^{i(k(\omega)|x-\xi(\tau_n)|)A(\omega)e^{i[k(\omega)\xi(\tau_n)]}d\omega}, \quad x, t \in (-\infty, +\infty)
\]

Using (2.4), (2.29) becomes

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_{\omega}} \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} e^{i[k_n - \omega_n/C_n]|x-\xi(\tau_n)|} A(\omega)e^{i[k(\tau_n) - \omega|x-\xi(\tau_n)|/C_n]}d\omega
\]

as \( N_{\omega} \to \infty \). Defining implicitly

\[
\tau_n = t - \frac{x-\xi(\tau_n)}{C_n},
\]

we can rewrite (2.30) in the form:

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_{\omega}} e^{i[k_n - \omega_n/C_n]|x-\xi(\tau_n)|} \zeta_n(\tau_n)
\]

where

\[
\zeta_n(\tau) \equiv \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} A(\omega)e^{i[k(\tau) - \omega]}d\omega.
\]

Similarly to (4.2) in the fixed probe case, (2.32) relates the elevation at any space-time \( (x, t) \) in the wave field to the wave information from the time record of elevation measured by the moving probe. Specifically, the wave elevation at any \( (x, t) \) of frequency \( \omega_n \) is determined from the moving measurement at time \( \tau_n \) given by (2.31).

The predictable area for frequency component \( \omega_n \) at any time \( t \) is then:

\[
\Pi_n(t) : 0 \leq \tau_n \leq T.
\]

Unlike the case of a fixed probe, the time shift \( t - \tau_n \) here is a function of not only \( x \) and group velocity \( C_n \) but also the measurement trajectory \( \xi(\tau_n) \). Geometric
Figure 2-12: Illustration of the predictable zone $\mathcal{P}_n$ (shaded area) for the frequency component $\omega_n$ of unidirectional waves based on a single moving probe with trajectory $\xi(t), t \in [0,T]$. The predictable zone is the strip between two parallel lines of $t = x/C_n + a_n$ and $t = x/C_n + b_n$. 
inspection of the above condition indicates that $\Pi_n(t)$ is the narrowest strip formed by two parallel lines of slope $x/t = C_n$ in the $x$-$t$ plane that bounds the trajectory $x = \xi(t), \ t \in [0, T]$. For later reference, we define the values of time where these two lines intersect the $t$-axis as $a_n, b_n$. See figure 2-12.

For a general irregular wave field, the predictable area at any time is given by the intersection of $\Pi_\omega(t)$ for all with $\omega \in \Omega$:

$$\Pi(t) = \bigcap_{\omega \in \Omega} \Pi_\omega(t) \quad \text{(2.35)}$$

with

$$\Pi_\omega(t) : (t - b_\omega)C_\omega \leq x \leq (t - a_\omega)C_\omega \ . \quad \text{(2.36)}$$

Then the predictable zone $\mathcal{P}$ is the set of all the $\Pi(t)$ over predictable time $t \in [T_s, T_e]$:

$$\mathcal{P} : \{\Pi(t), t \in [T_s, T_e]\} \quad \text{(2.37)}$$

In the case where the probe moves at a constant speed $\dot{\xi}(t) = U$, $\mathcal{P}$ can be obtained in a closed form:

$$\mathcal{P} : \begin{cases} UT + (t - T)C_{\max} \leq x - \xi(0) \leq tC_{\min} , & T < t \leq T_e \\ UT + (t - T)C_{\min} \leq x - \xi(0) \leq tC_{\min} , & 0 < t \leq T , \\ UT + (t - T)C_{\min} \leq x - \xi(0) \leq tC_{\max} , & T_s \leq t \leq 0 , \end{cases} \quad \text{(2.38)}$$

for $U < C_{\min}$, and

$$\mathcal{P} : \begin{cases} tC_{\max} \leq x - \xi(0) \leq UT + (t - T)C_{\min} , & T < t \leq T_e \\ tC_{\max} \leq x - \xi(0) \leq UT + (t - T)C_{\max} , & 0 < t \leq T , \\ tC_{\min} \leq x - \xi(0) \leq UT + (t - T)C_{\max} , & T_s \leq t \leq 0 , \end{cases} \quad \text{(2.39)}$$
Figure 2-13: Speed effect of the probe motion on the predictable zone of unidirectional waves. The waves propagate in the $x' = x - \xi(0)$ direction. The predictable zone is obtained for a single moving probe at constant speed $U$ with measured data in duration $t \in [0, T]$. The probe trajectories at four different speeds are plotted: $U < 0$ (---), $0 \leq U < C_{\text{min}}$ (----), $C_{\text{min}} \leq U \leq C_{\text{max}}$ (-----), and $C_{\text{max}} < U$ (---); and the associated parallelograms represent the predictable zones. The predictable zone for the fixed probe is also shown for comparison.

for $U > C_{\text{max}}$. For $C_{\text{min}} \leq U \leq C_{\text{max}}$, there is no predictable zone. This is because if $U = C_{\omega}$ for some wave component $\omega$ in the wave field, the predictable zone for that component $P_{\omega} = \emptyset$ (see, e.g., figure 7), and the intersection (see (5.8)) is also null.

Figure 2-13 shows how the predictable zone $P$ changes for different probe speed $U$. The dependence of the volume of $P$ on $U$ can be classified into four cases: (i) for $U < 0$, the volume of $P$ increases with increasing $|U|$; (ii) for $0 \leq U < C_{\text{min}}$, $P$ decreases with increasing $U$; (iii) for $C_{\text{min}} \leq U \leq C_{\text{max}}$, $P = \emptyset$; and (iv) for $U > C_{\text{max}}$, $P$ increases with increasing $U$. Figure 2-13 also shows that $P$ is significantly enlarged
from $U=0$ (fixed probe) with (even a small value of) negative velocity. This can also be obtained by comparing (2.38) with (2.13).

### 2.4.2 Multidirectional waves with multiple moving probes

We generalize the results of §2.4.1 to $J$ moving probes with trajectories $\xi_j(t) = (\xi_{xj}(t), \xi_{yj}(t))$ and measurements over time $t \in T_j$, $j = 1, 2, \ldots, J$. As in the case of fixed probes, $P$ for a multidirectional wave field with multiple moving probes can be obtained in terms of the elementary predictable areas for the $j-$th moving probe, $\Pi_{\phi,\omega}^j$, at time $t$, for wave component of frequency $\omega$ and propagation direction $\theta$.

To find $\Pi_{\phi,\omega}^j(t)$, we define a coordinate system $x'y'$ with the origin located at $(\xi_{xj}(\tau_j), \xi_{yj}(\tau_j))$ and the $x'$-axis along the $\theta-$direction. We project the trajectory of the $j-$th probe, $\xi_j(t)$, into the $x'$-$t$ plane:

$$\xi'_{j\theta}(t) = [\xi_{xj}(t) - \xi_{xj}(\tau_j)] \cos \theta + [\xi_{yj}(t) - \xi_{yj}(\tau_j)] \sin \theta, \quad t \in [\tau_j, \tau_j + T_j].$$  \hspace{1cm} (2.40)

With $\xi'_{j\theta}(t)$ given, we can determine $\Pi_{\phi,\omega}^j(t)$ in the same way as in §2.4.1 for unidirectional waves with a single moving probe. Finally $\Pi_{\phi,\omega}^j(t)$ can be expressed in the form:

$$(t - b_{\phi,\omega}^j)C_\omega \leq [x - \xi_{xj}(\tau_j)] \cos \theta + [y - \xi_{yj}(\tau_j)] \sin \theta \leq (t - a_{\phi,\omega}^j)C_\omega$$  \hspace{1cm} (2.41)

where $b_{\phi,\omega}^j$ and $a_{\phi,\omega}^j$ are the values of time where the two paralleled lines which forms the narrowest strip containing the projected trajectory $\xi'_{j\theta}(t)$ with $t \in [\tau_j, \tau_j + T_j]$ in $x'$-$t$ plane intersect the $t$-axis. The total predictable area of the multidirectional wave field with $J$ moving probes is:

$$\Pi(t) = \bigcap_{\theta \in \Theta} \bigcap_{\omega \in \Omega} \bigcup_{j=1}^J \Pi_{\phi,\omega}^j(t)$$  \hspace{1cm} (2.42)
and the predictable zone in the space-time (x-t) domain is:

\[ \mathcal{P} : \{ \Pi(t), t \in [T_e, T_e] \} . \]

We illustrate the result for a single moving probe in multidirectional waves using the same wave field given in figure 2-10. We consider a linear probe trajectory bisecting \( \Theta \) given by \( \xi = (\xi_x, \xi_y) = Ut(1, 1) \). Figure 2-14 shows the predictable areas \( \Pi(t) \) at several different \( t \) corresponding to \( t \in \mathcal{T} \) (reconstruction), and \( t <, > \mathcal{T} \) (hindcast, forecast). The shapes and sizes of \( \Pi(t) \) resemble those of a fixed probe (cf. figures 2-9). Figure 2-15 compares \( \Pi(t) \) at a specific \( t \) for moving \( (U <0) \) versus fixed measurements \( (U=0) \). It is clear that \( \Pi(t) \) for \( U <0 \) is appreciably larger than that for \( U=0 \).

2.5 Predictable zone based on whole-area measurements

So far we have focussed on (multiple) wave probes at prescribed locations. If whole-area measurements are obtained (say using airborne mapping), in addition to probes, the foregoing analysis for the predictable zone can be extended in a similar way. The derivation of the predictable zone for unidirectional and multidirectional waves follows a similar approach as in §2.2.1. In this section, we derive the solution of predictable zone \( \mathcal{P} \) of both unidirectional and multidirectional irregular waves based on whole-area measurements.

2.5.1 Unidirectional waves

The free surface elevation of a unidirectional wave field can be expressed in terms of wavenumber integration:

\[ \eta(x, t) = \text{Re} \int_{k_a}^{k_b} A(k) e^{i(kx - \omega t)} \, dk , \quad x, t \in (-\infty, +\infty) \quad (2.43) \]
Figure 2-14: Predictable areas at several predictable instants for a multidirectional wave field based on a moving probe. The wave field is the same as in figure 2-10. The arrow points to the direction of the probe motion, which is against \((\theta_a + \theta_b)/2\) and the bullet point denotes the initial position \((0, 0)\) of the probe. The probe moves with constant speed \(|U|/C_{min} = 1/2\) and provides measurement with time duration \(t^* \in [0, 4]\). The predictable time range is found to be: \([T_s, T_e] = [-1.6, 5.6]T\).
Figure 2-15: Comparison of the predictable areas at time $t^* = 2$ obtained with a moving and a fixed probes with the same initial location and time duration.
where \( A(k) \) represent the complex amplitude of the wave component of wavenumber \( k \), and \( k_a \) and \( k_b \) are related to the frequencies \( \omega_a \) and \( \omega_b \), respectively. By applying (2.43) to the given measurements, we have the relation:

\[
\zeta(X) = \text{Re} \int_{k_a}^{k_b} A(k) e^{i(kx - \omega t_0)} \, dk , \quad L_1 \leq X \leq L_2 .
\]  

(2.44)

In order to relate \( \eta(x, t) \) to \( \zeta(X) \) for the determination of \( \mathcal{P} \), we equally divide the wavenumber range \([k_a, k_b]\) into \( N_k \) segments and rewrite (2.43) as:

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_k} \int_{k_n - \Delta k/2}^{k_n + \Delta k/2} A(k) e^{i[kx - \omega(t - t_0) - \omega t_0]} \, dk
\]  

(2.45)

where \( \Delta k = (k_b - k_a)/N_k \) and \( k_n = k_a + (n - 1/2)\Delta k, \, n = 1, \ldots, N_k \). Over each segment, \( k \in [k_n - \Delta k/2, k_n + \Delta k/2] \), we expand \( \omega(k) \) in a Taylor series about \( k = k_n \):

\[
\omega(k) = \omega_n + C_n (k - k_n) + O(\Delta k)^2
\]  

(2.46)

where \( \omega_n = \omega(k_n) \) and \( C_n \) is the group velocity at \( k = k_n \). Making use of (2.46), we rewrite (2.45) in the form:

\[
\eta(x, t) = \text{Re} \sum_{n=1}^{N_k} e^{i[C_n k - \omega_n] (t - t_0)} \zeta_n(x - C_n (t - t_0))
\]  

(2.47)

as \( N_k \to \infty \), where \( \zeta_n(X) \) is defined as:

\[
\zeta_n(X) = \int_{k_n - \Delta k/2}^{k_n + \Delta k/2} A(k) e^{i(kx - \omega t_0)} \, dk .
\]  

(2.48)

It is seen from (2.44) that \( \zeta_n(X) \) represents the narrowband component of \( \zeta(X) \) with center wavenumber \( k_n \). Equation (2.47) relates \( \eta(x, t) \) at any space-time location in the wave field to the whole-area measurement at time \( t_0 \). For limited measurements, there exists only a certain region in the space-time domain, \( \mathcal{P} \), where \( \eta(x, t) \) can be reconstructed based on the given measurements. For the wave component of wavenumber \( k_n \), the predictable zone is clearly a strip in the \( x - t \) domain: \( L_1 + (t - \)
$$t_0)C_n \leq x \leq L_2 + (t - t_0)C_n$$ with $t \in (-\infty, +\infty)$. The predictable zone of the wave field will be the intersection of the predictable zones of all wave components in the wave field. Thus, $\mathcal{P}$ can be formally expressed in the form:

$$\Pi(t) = \bigcap_{\omega \in \Omega} \Pi_\omega(t) \quad (2.49)$$

with

$$\Pi_\omega(t) : L_1 + (t - t_0)C_\omega \leq x \leq L_2 + (t - t_0)C_\omega \quad (2.50)$$

for $t \in [T_s, T_e]$.

### 2.5.2 Multidirectional waves

The wave elevation of a multidirectional wave field can be expressed in the form:

$$\eta(x, y, t) = \text{Re} \int_{\theta_\alpha}^{\theta_\beta} \int_{k_{\alpha}}^{k_{\beta}} A(k, \theta) e^{i(kx'-\omega t)} k \, dk \, d\theta \quad (2.51)$$

where $x'(\theta) = x \cos \theta + y \sin \theta$ represents an axis in the $\theta-$direction. The use of (2.51) for the given elevation measurements gives:

$$\zeta(X, Y) = \text{Re} \int_{\theta_\alpha}^{\theta_\beta} \int_{k_{\alpha}}^{k_{\beta}} A(k, \theta) e^{i[kX'-\omega t_0]} k \, dk \, d\theta \, , \quad (X, Y) \in \mathcal{A} \quad (2.52)$$

where $X'(\theta) = X \cos \theta + Y \sin \theta$. With the same treatment of the $k-$integral as in the case of unidirectional waves, we can rewrite (2.51) as:

$$\eta(x, y, t) = \text{Re} \int_{\theta_\alpha}^{\theta_\beta} \left\{ \sum_{n=1}^{N_k} e^{i(C_n - \omega_0)(t - t_0)} \zeta_n(x' - C_n(t - t_0), \theta) \right\} \, d\theta \quad (2.53)$$

as $N_k \to \infty$, where $\zeta_n(X', \theta)$ is defined as:

$$\zeta_n(X', \theta) = \int_{k_{\alpha} - \Delta k/2}^{k_{\beta} + \Delta k/2} A(k, \theta) e^{i[kX'-\omega t_0]} k \, dk \quad . (2.54)$$
As in the case of unidirectional waves, the predictable zone of the wave component $k_n$ propagating in the $\theta-$ direction is a strip in the $x'-t$ domain: $L_{\theta_1} + (t - t_0)C_n \leq x' \leq L_{\theta_2} + (t - t_0)C_n$ with $t \in (-\infty, +\infty)$, where $L_{\theta_1}$ and $L_{\theta_2}$ represent the lower and upper ends of the projection of $A$ on the $x'$-axis (in the $\theta-$direction), respectively (i.e. this projection falls on the region $[L_{\theta_1}, L_{\theta_2}]$). As (4.15) indicates, the predictable zone of the multidirectional wave field $\mathcal{P}$ is given by the intersection of the predictable zones of all wave components within wavenumber and direction ranges:

$$\Pi(t) = \bigcap_{\theta \in \Theta} \bigcap_{\omega \in \Omega} \Pi_{\theta\omega}(t)$$

with

$$\Pi_{\theta\omega}(t) : L_{\theta_1} + (t - t_0)C_\omega \leq x \cos \theta + y \sin \theta \leq L_{\theta_2} + (t - t_0)C_\omega$$

for $t \in [T_s, T_e]$.

In the general case involving multiple ($J$) whole-area measurements, a similar principle as in the multiple probes case can be applied. We first determine the predictable area $\Pi_{\theta\omega}(t)$ for wave component $\theta\omega$ based on all given measurements:

$$\Pi_{\theta\omega}(t) = \bigcup_{j=1}^{J} \Pi_{\theta\omega}^j(t),$$

where $\Pi_{\theta\omega}^j(t)$ is based on the $j$-th measurement only.

Then we find the intersection of $\Pi_{\theta\omega}(t)$ for all wave components $\omega \in \Omega$ and $\theta \in \Theta$ to obtain $\Pi(t)$.

As illustration, we consider an instantaneous (at $t=0$) wave measurement inside a circular region $A$ of radius $R$ centered at $(x_c, y_c)$: $A=(x-x_c)^2 + (y-y_c)^2 \leq R^2$. From the Appendix, the predictable areas $\Pi(t)$ can be obtained directly. Figure 2-16 shows $\Pi(t)$ at $t=0$ (reconstruction), and $t <, >0$ (hindcast, forecast). Note that the predictable area $\Pi(0)$ at the measurement instant ($t=0$) is larger than the measurement region $A$. This is because the predictable area for a certain wave propagation direc-
Figure 2-16: Predictable areas at different instants of a multidirectional wave field based on given wave elevation measurement inside a circular area (marked by dash-dotted line) at \( t^* = 0 \). The wave field is the same as in figure 2-10. The circular area is centered at \((\bar{k}x_c, \bar{k}y_c) = (1.6, 0)\) and has radius \( \bar{k}R = 6 \). The predictable time range is found to be: \([T_s, T_e] = [-1.6, 1.6]T\).

2.6 Predictable zone based on hybrid measurements

As a further generalization, we consider the case that involves hybrid wave measurements, including fixed and moving probe and whole-area measurements. The similar principle described in the preceding sections can be applied to determine the predictable zone of irregular wave field based on hybrid measurements.

At any time \( t \) in the predictable time range \([T_s, T_e]\), we first determine the elementary predictable area of wave component \( \theta \omega \) for any specific given measurement.
source, using the algorithms described in the preceding sections. Then the total predictable area of the wave field based on all given measurements at time $t$ is determined by:

$$
\Pi(t) = \bigcap_{\theta \in \Theta} \bigcap_{\omega \in \Omega} \bigcup_{j=1}^{J_T} \Pi_{\phi_j}(t)
$$

where $J_T$ denotes the total number of given measurement sources. The predictable zone in $\mathbf{x}$-$t$ is then obtained:

$$\mathcal{P} : \{\Pi(t), t \in [T_s, T_e]\}$$

where $T_s$ and $T_e$ satisfy $\Pi(T_s) = \Pi(T_e) = \emptyset$.

Figure 2-17 to 2-19 shows an example of the predictable zone using hybrid measurements for a multidirectional wave field. The same multidirectional wave field as in figure 2-10 is used. The hybrid measurements used are the combination of measurements in figures 2-10, 2-14 and 2-16 (i.e. three fixed probes, a moving probe, and a circular-area measurement). The predictable areas at three different instants corresponding to $t \in \mathcal{T}_j$ (reconstruction), and $t <, > \mathcal{T}_j$ (hindcast, forecast) are displayed. For comparison, the predictable areas based on each measurement source are also shown. It is again seen that the predictable area based on the combination of hybrid measurements is much larger than the simple union of the predictable areas from each individual measurement source. The predictable time duration based on hybrid measurements is also much longer than any of those by individual measurement source.
Figure 2-17: Predictable areas of a multidirectional wave field determined by use of the hybrid measurements at time $t^* = -1$. The wave field is the same as in figure 2-10. The predictable areas associated with each individual data source are also shown for comparison, where source 1, 2, 3 corresponds to fixed probes as in figure 2-10, moving probe as in figure 2-14 and circular-area measurement as in figure 2-16. The predictable time range is found to be: $[T_s, T_e] = [-2.6, 7.2][T]$. 
Figure 2-18: Predictable areas of a multidirectional wave field determined by use of the hybrid measurements at time $t^* = 1$. The wave field is the same as in figure 2-10. The predictable areas associated with each individual data source are also shown for comparison, where source 1, 2, 3 corresponds to fixed probes as in figure 2-10, moving probe as in figure 2-14 and circular-area measurement as in figure 2-16. The predictable time range is found to be: $[T_s, T_e] = [-2.6, 7.2]T$. 

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Figure 2-19: Predictable areas of a multidirectional wave field determined by use of the hybrid measurements at time $t^* = 5$. The wave field is the same as in figure 2-10. The predictable areas associated with each individual data source are also shown for comparison, where source 1, 2, 3 corresponds to fixed probes as in figure 2-10, moving probe as in figure 2-14 and circular-area measurement as in figure 2-16. The predictable time range is found to be: $[T_s, T_e] = [-2.6, 7.2]T$. 
Chapter 3

Optimal deployment of wave measurements

A problem of significant practical importance is the optimization of the deployment of a sensor suite to obtain the largest predictable zone in some spatial-temporal zone of interest. This inverse problem is quite difficult especially when different combinations of sensors (probe vs. area, fixed vs. moving, high vs. low resolution, ... ) are involved. In many cases, there may be additional constraints or requirements, e.g., to minimize the total power needed for measurements and/or mobility. In this chapter, we discuss the optimal deployment of wave measurements to maximize the predictable zone.

3.1 Maximizing the predictable zone by optimal deployment of fixed probes

In this section, we develop a theoretical framework for obtaining the optimal placement of $J$ probes in a multidirectional wave field. For simplicity, we assume fixed probes obtaining surface measurements over the same time duration $\mathcal{T}=[0, T]$. The generalization to moving probes and different measurement time intervals is relatively straightforward.

To quantify the volume of predictable zone $\mathcal{P}$ in space-time domain, we define a
integral quantity:
\[ V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y, t) dxdydt, \]  

(3.1)

where \( I(x, y, t) \) is an indicator function defined as:

\[ I(x, y, t) = \begin{cases} 
1, & (x, y) \in \Pi(t) \\
0, & \text{otherwise} 
\end{cases} \]  

(3.2)

In addition, we define \( S(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y, t) dxdy \) as the value of predictable area \( \Pi(t) \) at time \( t \).

Consider first the case of two probes, one located at the origin and the other at \( \xi=d(\cos \alpha, \sin \alpha) \) (see figure 3-1). The objective is to optimize \( \xi \) to maximize \( V \).

From the analysis in §2.3 for a single fixed probe, the predictable area \( \Pi_{\omega}(t) \) for wave component \( \theta \omega \) at time \( t \) is an infinitely long strip of width \( C_{\omega}T \) perpendicular to the propagation direction of \( \theta \). In the case of 2 probes, 1 and 2, the predictable area for this wave component is the union of the individual predictable areas: \( \Pi_{\omega}(t) = \Pi_{\omega}^1(t) \cup \Pi_{\omega}^2(t) \) (figure 3-2). The distance between the center lines of these two parallel strips \( \Pi_{\omega}^1(t) \) and \( \Pi_{\omega}^2(t) \) is equal to the length of the projection of \( \xi \) in the \( \theta \)-direction given by \( \ell = d \cos(\theta - \alpha) \). In general, \( \Pi_{\omega}^1(t) \) and \( \Pi_{\omega}^2(t) \) have a strip where they overlap. Obviously \( V \) is maximized when the overlapping region is minimized, or equivalently, \( \ell \) is maximum. In multidirectional waves, the predictable area for all wave components \( \Pi(t) \) is the intersection of \( \Pi_{\omega}(t) \) over \( \theta \in \Theta \), where \( \Pi_{\omega}(t) \) is the intersection of \( \Pi_{\omega}^1(t) \) over \( \omega \in \Omega \). Thus \( V \) is maximized when the minimum of \( \ell \) over all \( \theta \in \Theta \) is maximized. It follows that the optimal value of \( \alpha \) is \( \alpha_{\text{opt}} = \theta_m \).

From practical consideration, it is often desirable for the predictable zone \( \mathcal{P} \) to be a single connected domain in \( x-t \). Assuming this is a requirement, this puts a constraint on the distance \( d \) between the two probes. For \( \mathcal{P} \) to be a single connected domain, \( \Pi_{\omega}^1(t) \) and \( \Pi_{\omega}^2(t) \) must overlap (touch) for all wave components. This requires that \( \ell = d \cos(\theta - \alpha) \leq C_{\omega}T \) for all \( \omega \in \Omega \) and \( \theta \in \Theta \). The optimal distance between the 2 probes is the maximum \( d \) satisfying this constraint given by \( d_{\text{opt}} = C_{\min}T \).
Figure 3-1: Sketch of two probes located at $(0,0)$ and $(\xi_x, \xi_y) = d(\cos \alpha, \sin \alpha)$ in a multidirectional broadband wave field with propagation direction $\theta \in [\theta_a, \theta_b]$. 
Figure 3-2: Sketch of the predictable areas, at any time $t$, of a wave component of frequency $\omega$ propagating in the $\theta$—direction, based on two fixed probes of time duration $[0, T]$. 
The above analysis and procedure for two probes can be simply generalized to \( J \) probes. The optimal placement of \( J \) fixed probes covering time \([0, T]\) that maximizes the space-time predictable volume \( \mathcal{V} \) is a uniformly spaced line array along the mid-\( \Theta \) direction \( \alpha^{opt} = \theta_m \) with distance \( d^{opt} = C_{min}T \) separating adjacent probes.

### 3.2 Asymptotic behavior of the predictable zone

We finally consider the asymptotic behavior of the predictable zone \( \mathcal{P} \) and specifically the predictable volume \( \mathcal{V} \) in the limit of large number \( J \) of (optimally placed) probes. As noted in §2.2.2, the predictable zone \( \mathcal{P}_J \) that is obtained using \( J \) measurements is, in general, bigger that the union of the predictable zones \( \mathcal{P}^j \) associated with isolated measurement \( j \). The interesting question is how the volume of \( \mathcal{P}_J \) scales with (large) \( J \) from an optimally placed probe array. This can be argued based on the preceding theoretical results. For clarity, we use \( \mathcal{V}_J \) to denote the volume of \( \mathcal{P}_J \) and \( \mathcal{V}^j \) for the volume of \( \mathcal{P}^j \) associated with the \( j \)-th measurement.

For specificity, we consider \( J \) probes with identical measurement interval \([0, T]\) in a wave field with \( \theta \in [\theta_a, \theta_b] = [\theta_m - \Delta \theta, \theta_m + \Delta \theta] \). The optimal configuration are probes along the line \( \alpha = \theta_m \) separated by distance \( d = C_{min}T \).

For a single probe, the predictable area at time \( t \) for any wave component \( \theta \omega \), \( \Pi_{\theta \omega}(t) \), is a strip (in \( x-y \)) of width \( w, C_{\omega}T \). Considering all \( \omega \in \Omega \), the intersection is a strip of width \( w \propto C_{\min}T \). Considering all \( \theta \in \Theta \), the intersection is the predictable area \( \Pi(t) \) at time \( t \) of area \( S(t) \propto w^2 \propto C_{\min}^2T^2 \) (see, e.g., figure 3). From (2.27), \( \Pi(t) \) extends in time over a duration \( D = (1 + 2\beta_m)T \), where \( \beta_m \) depends on \( \Omega \) and \( \Delta \theta \). Thus the predictable volume for one probe \( \mathcal{V} = V_1 \) scales as \( w^2D \propto (1 + 2\beta_m)C_{\min}^2T^3 \).

To understand how the predictable volume \( \mathcal{V}_J \) of \( J \) optimally placed probes scales with (large) \( J \), we consider separately the scaling of the predictable area \( S_J(t) \), the predictable time duration \( D_J \), and hence the predictable volume \( \mathcal{V}_J \propto S_JD_J \). For \( S_J(t) \) of \( J \) probes, we follow the procedure in §2.3 to determine \( \mathcal{P}_J \). At any time \( t \), we consider first the union of \( \Pi_{\theta \omega}(t) \) over all probes \( j = 1, ..., J \), which are \( J \) strips of width \( w_{\omega} = C_{\omega}T \) (which may have no intersection or overlap), to obtain \( \Pi_{\theta,\omega}(t) \). If the \( J \)
probes are optimally placed, \( \Pi_{\omega J}(t) \) is a single strip of width \( w_{\omega J} \propto J C_{\omega} T \). Considering all \( \omega \in \Omega \), the intersection is a strip of width \( w_J \propto J C_{\text{min}} T \). Considering all \( \theta \in \Theta \), the intersection is the predictable area \( \Pi_J(t) \) at time \( t \) of area \( S_J(t) \propto w_J^2 \propto J^2 C_{\text{min}}^2 T^2 \) (see figure 3-3).

For the predictable time duration \( D_J \), it can be shown that \( D_J \sim J T \) for \( J \gg 1 \) for optimally placed probes, and in general \( D_1 < D_J \sim J T \) for \( J > 1 \). We first illustrate this for \( J=2 \) probes placed in unidirectional waves (propagating in +x direction). For maximum \( P \), \( d_{\text{opt}} = C_{\text{min}} T \). See figure 3-4 which illustrates the predictable zone \( P \) for 2 optimal placed probes in unidirectional waves. As in §2.2.2, \( P \) is the intersection of \( P_\omega \) over \( \omega \in \Omega \), and bounded by the maximum and minimum wave frequency components. Specifically, \( P_{\omega_{\text{min}}} \) is a strip in \( x-t \) plane with time extent \( 2T \), and \( P_{\omega_{\text{max}}} \) is a strip in \( x-t \) with time extent \( (1 + C_{\text{min}}/C_{\text{max}})T \). The intersection of \( P_\omega \) over \( \omega \in \Omega \) yields a \( x-t \) domain with time extent \( D=D_2=(1 + 4 \beta_u)T \). For \( J \) probes in unidirectional waves, this results generalizes easily: \( D_J=(1 + 2 \beta_u J)T \sim 2 \beta_u J T \) for \( J \gg 1 \). For multidirectional wave field, these results generalized readily: \( D_J=(1 + 2 \beta_m J)T \sim 2 \beta_m J T \) for \( J \gg 1 \). Finally, for \( J \) optimally placed probes, the predictable zone volume \( V_J \propto S_J D_J \sim (J^2 C_{\text{min}}^2 T^2)(2 \beta_m J T) = 2 \beta_m J^3 C_{\text{min}}^2 T^3 \). Thus

\[
\frac{V_J}{V_1} \sim \gamma J^3 \quad \text{for} \quad J \gg 1
\]

for optimally placed probes, where \( \gamma = 2 \beta_m/(1 + 2 \beta_m) \) which is dependent on \( \Omega \) and \( \Delta \Theta \) of the wave field. For large \( J \), (3.3) is thus substantially greater than the sum of the predictable zone volume associated with each individual probe, \( \sum_j V_j \), which scales only as \( J V_1 \).

If the \( J \) probes are not optimally placed, the above result clearly would not obtain. For example, if the \( J \) probes have separation distances all greater than \( C_{\text{min}} T \), then \( \Pi_{\omega J}(t) \) is not a single strip but \( J \) strips of width \( w_\omega = C_\omega T \) and the resulting \( S_J(t) \) scales as simply \( J C_{\text{min}}^2 T^2 \). On the other hand, \( D_J = D_1 \) and does not scale with \( J \). Consequently the total volume of the disjointed predictable zone \( V_J \) scales only as \( J \).

To validate the above asymptotic result, figure 3-5,3-6 show the numerical result

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Figure 3-3: Illustration of the predictable area, at any time $t$, of a multidirectional wave field of direction band $[\theta_a, \theta_b]$ based on the use of $J$ probes. The predictable area $\Pi(t) \sim (JC_{\min}T)^2$. 
Figure 3-4: Illustration of the predictable zone, based on two fixed probes in a duration of $[0, T]$, for wave propagation along the $\theta$—direction. The $x'$—axis is along the $\theta$—direction. The characteristic length in space and time of the predictable zone is increased to $2w$ and $2T$ when two probes are used.
Figure 3-5: Asymptotic dependence of $V_j/V_1$ on the number of probes $J$. Consider the same multidirectional wave field as in figure 2-10 and for probes we set $T/T_1 = 4$. Comparison of the numerical result of $V_j/V_1$ with the $J^3$ asymptotic dependence.

Comparison of the numerical result of $V_j/V_1$ with the $J^3$ asymptotic dependence.

of $V_j/V_1$ and $S_j/S_1$ (at some particular time) as a function of $J$ respectively, which confirms the asymptotic feature of $V_j/V_1 \sim J^3$ and $S_j/S_1 \sim J^2$ at large $J$.

To illustrate the effect of optimal probes placement direction, we compare in figure 3-7 the results of $V_j/V_1$ obtained with different probes placement direction $\alpha$. We consider three placements in each of which $J$ probes are uniformly spaced in a line with the optimal separation distance between adjacent probes but along different directions. The comparison in figure 3-7 shows that for different $\alpha$, the asymptotic behavior of $V_j/V_1$ is similar ($\sim J^3$) but with different increasing rate with respect to $J$. Without loss of generality, we choose $V_{10}/V_1$ to study the effects of placement direction. Figure 3-8 shows that under the optimal placement direction $\alpha^{opt}$, $V_{10}/V_1$ has the largest value, and $V_{10}/V_1$ decreases nonlinearly with $\alpha$ away from $\alpha^{opt}$.
Figure 3-6: Asymptotic dependence of $S_j/S_1$ on the number of probes $J$. Consider the same multidirectional wave field as in figure 2-10 and for probes we set $T/T = 4$. Comparison of the numerical results of $S_j/S_1$ with the $J^2$ asymptotic dependence.
Figure 3-7: Comparison of the numerical results of $V_j/V_1$ with respect to $J$ for three different probes placement direction. $\alpha = (\theta_a + \theta_b)/2$ corresponds to the optimal placement direction.
Figure 3-8: Comparison of $\mathcal{V}_{10}/\mathcal{V}_1$ with different probes placement direction $\alpha$.

### 3.3 Optimal deployment of moving probes

For moving probes, first we need to discuss the optimal path for one moving probe with measurements:

$$\zeta^1(t) = \eta(x = x_1(t), t), \quad t \in [t_1, t_1 + T_1]$$

Suppose the probe speed has a upper limit $U$. Consider a narrow-banded unidirectional wave packet with frequency $\omega$ and propagation direction $\theta$. In order to have the largest predictable zone (maximize $\mathcal{V}$) for this wave packet, the optimal path for the moving probe should be: moving against the wave propagation direction $\theta$ with constant speed $U$, see figure 3-9.

For the general broad-banded multidirectional wave-field, in order to have the largest predictable zone (maximize $\mathcal{V}$), the minimum of the projection length of the probe path $x_1(t)$ onto all wave propagation direction should be maximized. Therefore,
Figure 3-9: Predictable zone of a moving probe for different moving direction for a narrow-banded unidirectional wave packet.

the probe should move against direction \((\theta_a + \theta_b)/2\) with constant speed \(U\), see figure 3-10.

In general, consider \(J\) moving probes with measurements:

\[
\zeta^j(t) = \eta(x = x_j(t), t), \quad t \in [t_j, t_j + T_j], \quad j = 1, 2, \ldots, J
\]  

(3.4)

Let \(\Delta x^j = x_{j+1}(t = t_{j+1}) - x_j(t = t_j)\), \(d_j = |\Delta x^j|\), and \(\alpha_j = \arccos(\Delta x^j \cdot i)\).

Similar as for fixed probes, in order to have the largest predictable zone in space-time domain (maximize \(V\)), the minimum of the projection of \(\Delta x^j\) onto all propagation direction \(\theta \in \Theta\) should be maximized. In order to make the predictable zone a single solid body, there is a critical length for the maximum of the projection of \(\Delta x^j\) onto all propagation direction \(\theta \in \Theta\), see figure 3-11 and 3-12.

Therefore, the optimal deployment for \(J\) moving probes is:

\[
\forall j, \quad \alpha^j_{opt} = \frac{1}{2}(\theta_a + \theta_b),
\]

(3.5)
Figure 3-10: Optimal moving direction of a moving probe in a general wave-field.
Figure 3-11: Optimal distance for two moving probes for $t_j \leq t_{j+1} < t_j + T_j$. 
Figure 3-12: Optimal distance for two moving probes for \( t_j + T_j \leq t_{j+1} < \frac{(U + C_{\min})(T_j + T_{j+1})}{C_{\max} - C_{\min}} + t_j + T_j. \)
\[ c_j^{\text{opt}} = \begin{cases} 
C_{\min}(t_j + T_j - t_{j+1}) + UT_j, & (\text{upwave}), \quad t_j \leq t_{j+1} < t_j + T_j \\
C_{\max}(t_{j+1} - t_j - T_j) - UT_j, & (\text{downwave}), \quad t_j + T_j < t_{j+1} < \frac{(U + C_{\min})(T_j + t_{j+1})}{C_{\max} - C_{\min}} + t_j + T_j 
\end{cases} \]

and all probes should move against direction \((\theta_a + \theta_b)/2\) with constant speed \(U\).

The critical time \(\frac{(U + C_{\min})(T_j + t_{j+1})}{C_{\max} - C_{\min}} + t_j + T_j\) is obtained following similar analysis as for fixed probes.

### 3.4 Optimal deployment of whole-area measurements

For practicality, we only consider the whole-area measurements in a circle area. If we only have wave elevation measured at a certain time, from previous analysis about the predictable zone, we know that the bigger the map radius, the larger the predictable zone. Suppose the biggest radius for a practical wave elevation measurement is \(R\).

From now on, we use \(R\) for the wave elevation map radius.

If we have two wave maps measured at the same time \(t_0\):

\[ \zeta^j(x, y) = \eta(x, y; t = t_0), \quad (x, y) \in A_j, j = 1, 2 \]

where \(A_j\) is a circle with radius \(R\) and center \(x_{cj} = (x_{cj}, y_{cj})\).

Let \(\Delta x_c = x_c - x_{c1}\), \(d_{c1} = |\Delta x_c|\), and \(\alpha_{c1} = \arccos(\Delta x_c \cdot \mathbf{i})\).

In order to make the predictable zone a single solid body, the two maps should be tangent to each other, i.e.,

\[ c_{c1}^{\text{opt}} = 2R \]

In order to have the largest predictable zone (maximize \(V\)), we need to maximize the minimum of the projection of \(\Delta x_{c1}\) onto all propagation direction \(\theta \in [\theta_a, \theta_b]\), see figure 3-13, which leads to:
Figure 3-13: Determination of optimal $\alpha_{c1}$ for two wave maps measured at the same time: the predictable zone in $\theta$ direction is determined by $2R + d_{c1} \cos(\alpha_{c1} - \theta)$, so that in order to have the largest predictable zone for $\theta \in [\theta_a, \theta_b]$, we need to maximize the minimum of $d_{c1} \cos(\alpha_{c1} - \theta)$ for $\theta \in [\theta_a, \theta_b]$, which leads to the optimal $\alpha_{c1}$.

$$\alpha_{c1}^{opt} = \frac{1}{2}(\theta_a + \theta_b)$$

In general, consider $N_m$ wave elevation maps measured at the same time:

$$\zeta_j(x, y) = \eta(x, y; t = t_0), \quad (x, y) \in A_j, j = 1, 2, \ldots, N_m$$

(3.7)

where $A_j$ is a circle with radius $R$ and center $x_{cj} = (x_{cj}, y_{cj})$.

Let $\Delta x_{cj} = x_{cj+1} - x_{cj}$, $d_{cj} = |\Delta x_{cj}|$, and $\alpha_{cj} = \arccos((\Delta x_{cj} \cdot i))$.

The optimal deployment for these $N_m$ wave maps is:

$$\forall j, \quad \alpha_{cj}^{opt} = \frac{1}{2}(\theta_a + \theta_b), \quad d_{cj}^{opt} = 2R$$

(3.8)
Now if the \( N_m \) wave maps are not measured at the same time, but in a sequential time, for example, the maps are measured by a moving ship with constant speed \( V \) and measurement interval \( \Delta t \). Then in order to have a single solid and largest predictable zone, we need to control the path of the ship, so that if the ship moves along the wave propagation direction, see figure 3-14, for a critical measurement interval \( \Delta t_1 \), we have:

\[
R + C_{min} \Delta t_1 + R = V \Delta t_1
\]

(3.9)

from which we can see that \( V \) has to be larger than \( C_{min} \) and \( \Delta t_1 \) satisfies:

\[
\Delta t_1 = \frac{2R}{V - C_{min}}
\]

(3.10)

If the ship moves against the wave propagation direction, see figure 3-15, for a critical measurement interval \( \Delta t_2 \), we have:

\[
R - C_{min} \Delta t_2 + R = V \Delta t_2
\]

(3.11)

and \( \Delta t_2 \) satisfies:

\[
\Delta t_2 = \frac{2R}{V + C_{min}}
\]

(3.12)

Here we define a nondimensional variable \( \tau \) which represents the moving ship measurement property:

\[
\tau = \frac{V \Delta t}{R}
\]

(3.13)

then the critical measurement interval \( \Delta t_1 \) and \( \Delta t_2 \) correspond to:

\[
\tau_1 = \frac{V \Delta t_1}{R} = \frac{2}{1 - \frac{C_{min}}{V}}
\]

(3.14)

and

\[
\tau_2 = \frac{V \Delta t_2}{R} = \frac{2}{1 + \frac{C_{min}}{V}}
\]

(3.15)
Figure 3-14: Wave maps measured by a moving ship: ship moves along wave propagation direction with critical measurement interval $\Delta t_1$. 
Figure 3-15: Wave maps measured by a moving ship: ship moves against wave propagation direction with critical measurement interval $\Delta t_2$. 

$x' = Vt$
Figure 3-16: Determination of ship moving direction, where $\tau$ is the nondimensional variable $\frac{V\Delta t}{R}$.

which are both function of nondimensional quantity $C_{\text{min}}/V$.

Similar as before, in order to have the largest predictable zone, we need to maximize the minimum of the projection of map center distance onto all propagation direction $\theta \in [\theta_a, \theta_b]$. Therefore the ship should move along or against direction $(\theta_a + \theta_b)/2$. Specifically, the moving direction of the ship depends on the nondimensional variable $\tau$:

(1) If $0 < \tau < \tau_2$ for given $C_{\text{min}}/V$ (region A in figure 3-16), the ship should move against the wave direction in order to get a single solid and largest predictable zone.

(2) If $\tau_2 < \tau < \tau_1$ for given $C_{\text{min}}/V$ (region B in figure 3-16), and we want to have a single solid predictable zone, the ship should move along the wave direction; otherwise if we want to have larger predictable zone, the ship should move against the wave direction, but in this case, the predictable zone is not a single solid one.

(3) If $\tau > \tau_1$ for given $C_{\text{min}}/V$ (region C in figure 3-16), no matter the ship moves
along or against the wave direction, the predictable zone is not a single solid one and
the total area is the same.

3.5 Optimal deployment of hybrid measurements

For the most general case which has hybrid measurements, the optimal deployment
is putting all the measurement source along or against wave propagation direction
\((\theta_a + \theta_b)/2\).

We have discussed the optimal distance between the same kind of measurement
source. For different kinds of measurement source, the optimal distance between them
follows similar analysis. Here we discuss 6 basic cases as examples, see figure 3-17,
3-18, 3-19. Specifically,

- For a static wave map in the upwave direction and a fixed probe in the downwave
direction, with measurement time \( t_f < t_s < t_f + T_f \), see figure, the optimal
distance between the wave map center and the fixed probe is \( R + C_{\text{min}}(t_f + T_f -
\ t_s) \).

- For a static wave map in the upwave direction and a moving probe in the
downwave direction, with measurement time \( t_m < t_s < t_m + T_m \), see figure, the optimal
distance between the wave map center and the starting point of the
moving probe is \( R + C_{\text{min}}(t_m + T_m - t_s) + UT_m \). The moving probe will move
against the direction \((\theta_a + \theta_b)/2\) with its maximum speed \( U \). This applies for
the following cases where the moving probe is considered.

- For a moving probe and a fixed probe, with measurement time \( t_f < t_m < t_f + T_f \),
the optimal deployment is to put the moving probe in the upwave direction and
the distance between the fixed probe and the starting point of the moving probe
is \( C_{\text{min}}(t_f + T_f - t_m) \).

- For a fixed probe and a moving probe, with measurement time \( t_m < t_f < \ t_m + T_m \), the optimal deployment is to put the fixed probe in the upwave direction

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Figure 3-17: Optimal distance between a wave map and (a) a fixed probe with measurement time $t_f < t_s < t_f + T_f$; (b) a moving probe with measurement time $t_m < t_s < t_m + T_m$.

and the distance between the starting point of the moving probe and the fixed probe is $C_{\text{min}}(t_m + T_m - t_f) + UT_m$.

- For a fixed probe and a static wave map, with measurement time $t_s > t_f + T_f$, the optimal distance between the fixed probe and the wave map center is $C_{\text{max}}(t_s - t_f - T_f) - R$.

- For a moving probe and a static wave map, with measurement time $t_s > t_m + T_m$, the optimal distance between the starting point of the moving probe and the wave map center is $C_{\text{max}}(t_s - t_m - T_m) - UT_m - R$.

Based on the basic cases, we can deal with more complicated situations. Here we consider a real-world example.

Consider a ocean wave-field $\eta(\vec{x}, t)$ with propagation angle range $\Theta$: $[\theta_s, \theta_e] = [\pi/12, 5\pi/12]$, and frequency range $\Omega$ in every direction: $[\omega_l, \omega_h] = [1, 2]\text{rad/s}$. The group velocity range corresponding to the frequency range is: $[C_{\text{min}}, C_{\text{max}}] = [2.45, 4.9]\text{m/s}$.

Suppose a research group tries to get some measurements of the wave elevation for this wave-field and do prediction. They conduct the scientific investigation by a ship with cruising speed 15 knots ($V = 7.72\text{m/s}$). The ship has mounted a X-band marine
Figure 3-18: Optimal distance between a fixed probe and a moving probe: (a) with measurement time $t_f < t_m < t_f + T_f$ (b) with measurement time $t_m < t_f < t_m + T_m$.

Figure 3-19: Optimal distance between (a) a fixed probe and a wave map with measurement time $t_s > t_f + T_f$; (b) a moving probe and a wave map with measurement time $t_s > t_m + T_m$. 

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radar which can scan ocean surface and generate radar map. The radar map can be further processed to generate wave elevation map \([76, 88]\). Suppose the radius for the generated wave elevation map is \(R = 200m\) and the time interval for two consecutive wave elevation maps is \(\Delta t = 48s\). They also equip with two moving probes and two fixed probes to measure wave elevation. The moving probe has maximum speed \(U = 2m/s\), measurement duration \(T_m = 60s\) and its path can be remote controlled. The fixed probe has the same measurement duration \(T_f = 60s\).

They want to use all of their equipments to obtain the measurements. The question is how to design the operation of measurement to achieve the largest predictable zone for a limited measurement time (say 5 min).

First we need to decide which direction the ship should move towards. Based on the analysis in section 3.4, we need to calculate the nondimensional variable \(\tau\):
\[
\tau = \frac{\sqrt{\Delta t}}{R} = \frac{7.72 \times 48}{200} = 1.85.
\]
The two critical values are:
\[
\tau_1 = \frac{2}{1 - \frac{U}{V_{\text{max}}}} = \frac{2}{1 - \frac{2}{1.52}} = 2.93,
\tau_2 = \frac{2}{1 + \frac{U}{V_{\text{max}}}} = \frac{2}{1 + \frac{2}{1.52}} = 1.52.
\]
Since \(\tau_2 < \tau < \tau_1\), in order to have a single solid predictable zone, the ship should move along the direction \(\alpha = (\theta_a + \theta_b) = \pi/4\).

From the analysis in section 3.3, we know the moving probes should move against the direction \(\alpha = \pi/4\) with maximum speed \(U\). Since the moving probes can be remote controlled, the best strategy is planting moving probes at the optimal location before the fixed probe on the way of ship cruising. Later they will be controlled to move and measure. The fixed probes will start measuring once planted at the optimal location. After planting all the probes the ship can starting running radar measurements at the optimal location until the measurement time is up.

The optimal planting location and start measuring time of the probes is determined according to previous analysis. For the moving probes and the first fixed probe, their measurement path should be connected in order to achieve the largest predictable zone. The second fixed probe is planted on the way of ship cruising and its location is determined to achieve the largest predictable zone. Figure 3-20 shows the moving path of the ship and figure 3-21 shows the optimal measurement path for all measurement sources.

Specifically, the operation of the ship is as follows:
Figure 3-20: Moving path of the ship.
Figure 3-21: The optimal measurement path for all measurement sources.

(1) At the starting point \((x = 0, y = 0, t = 0)\), the ship plants moving probe \(P_1\) to the ocean surface in the standby mode, then the ship moves along direction \(\alpha = \pi / 4\) with speed \(V\).

(2) After traveling distance \(d_1 = UT_m = 120m\) (at time \(t_1 = \frac{d_1}{V} = 15.54s\)), the ship plants moving probe \(P_2\) and fixed probe \(P_3\) to the ocean surface. Probe \(P_3\) starts measuring wave elevation once planted while probe \(P_2\) stays in the standby mode. The ship keeps moving in the direction \(\alpha = \pi / 4\) with speed \(V\).

(3) After traveling distance \(d_2 = \frac{T_f V C_{min}}{V - C_{min}} = 215.34m\) (at time \(t_2 = \frac{d_1 + d_2}{V} = 43.43s\)), the ship plants fixed probe \(P_4\) to the ocean surface and this probe starts measuring wave elevation immediately. The ship keeps moving in the direction \(\alpha = \pi / 4\) with speed \(V\).

(4) During the ship traveling, at time \(t_3 = \frac{UT_m}{V} + T_f = 75.54s\), remote control probe \(P_2\) to start moving against direction \(\alpha = \pi / 4\) with speed \(U = 2m/s\) and measuring wave elevation.

(5) After traveling distance \(d_3 = \frac{V}{V - C_{min}}(d_1 + d_2 + R - C_{min} t_2) - d_1 - d_2 = 293m\) (at time \(t_4 = \frac{d_1 + d_2 + d_3}{V} = 81.39s\)), the ship start running radar measurements (represented
as $M$ in figure) with time interval $\Delta t = 48s$ while keep moving in the direction $\alpha = \pi/4$ with speed $V$.

(6) At time $t_5 = \frac{UT}{V} + T_f + T_m = 135.54s$, remote control probe $P_1$ to start moving against direction $\alpha = \pi/4$ with speed $U = 2m/s$ and measuring wave elevation.

(7) The ship keeps traveling with radar measurements until the measurement time is up (5min).

In this way, the ship will achieve the largest predictable zone within limited measurement time.
Chapter 4

Phase-resolved reconstruction and forecast of irregular waves

Phase-resolved reconstruction and forecast of nonlinear irregular wave field based on specified wave measurements contains two major tasks: (i) determination of the predictable zone in the space-time domain in which the wave field can be completely predicted; and (ii) phase-resolved reconstruction (within the measurement space-time domain) and forecast (beyond the measurement space-time domain) of the wave field in the predictable zone. We have developed the theory to determine the predictable zone in Chapter 2 and 3. In this chapter, we discuss how to reconstruct and forecast waves within the predictable zone.

4.1 Theoretical reconstruction and forecast of linear waves

In this section, we describe the analytic procedure of wave reconstruction/forecast based on the linear wave theory for two relatively simple cases: (i) unidirectional waves with a single probe or two probes; and (ii) multidirectional waves with elevation measurements in a rectangular or circular area. The case of multidirectional waves with multiple probes is doable but the associated analysis becomes more tedious.
especially when a large number of probes are involved. In this case, it is advantageous
to apply the general approach to be described in §4.2.2.

4.1.1 Unidirectional waves based on a single probe measurements

We first consider the case of unidirectional waves with a single fixed probe measurements. The waves are assumed to propagate along the positive $x$ direction. Let $\zeta(t)$ represent the wave elevation measured at the location $x=x_1$. The time duration of the data is $t \in [0, T]$. In terms of the frequency integral, $\zeta(t)$ can be expressed as:

$$
\zeta(t) = \tilde{\eta}(x = x_1, t) = \text{Re} \int_{\omega_a}^{\omega_b} A(\omega) e^{i(kx_1 - \omega t)} \, d\omega, \quad t \in [0, T]
$$

(4.1)

where $A(\omega)$ is the unknown complex wave amplitude of the original wave field, and wavenumber $k(\omega)$ is related to the frequency $\omega$ by the dispersion relation.

In Chapter 2, for determination of the predictable zone $\mathcal{P}$, we derive the general solution of wave elevation anywhere in the wave field at any time, $\eta(x, t)$, given in terms of the wave information at a fixed location ($x = x_1$). In the derivation, the frequency band $\Omega$ is divided into $N_\omega$ equal small segments with $\Delta \omega = (\omega_b - \omega_a)/N_\omega$. At the center of each segment, the frequency is $\omega_n = \omega_a + (n - 1/2) \Delta \omega$, $n=1, 2, \ldots, N_\omega$. The solution is expressed in the form:

$$
\eta(x, t) = \left\{ \text{Re} \sum_{n=1}^{N_\omega} e^{i(kn - \omega n/C_n)(x - x_1)} \zeta_n(t - \Delta t_n) \right\} \left[ 1 + O(\Delta \omega^2 |x - x_1|) \right]
$$

(4.2)

where $\Delta t_n = (x - x_1)/C_n$, $C_n$ is the wave group velocity at $\omega_n$, and $\zeta_n$ is defined as:

$$
\zeta_n(\tau) = \int_{\omega_n - \Delta \omega/2}^{\omega_n + \Delta \omega/2} A(\omega) e^{i(kx_1 - \omega \tau)} \, d\omega
$$

(4.3)

which represents the wave elevation at $x=x_1$ from the contribution in a narrow frequency band $\omega \in [\omega_n - \Delta \omega/2, \omega_n + \Delta \omega/2]$. If the time $\tau$ in (4.3) is limited to within the time duration of given data i.e. $\tau \in [0, T]$, the resulting $\eta(x, t)$ in (4.2) is fully de-
pendent on the given data at \( x = x_1 \). The associated domain in \((x, t)\) is the solution of the predictable zone \( \mathcal{P} \). Thus, \( \mathcal{P} \) follows from the condition: \( 0 \leq t - (x - x_1)/C_n < T \) for all of \( n = 1, 2, \ldots, N_\omega \). This indicates that at any point \((x, t) \in \mathcal{P}\), all wave components contained in \( \eta(x, t) \) must come from somewhere in the time record of given elevation data (i.e. \( \zeta(t) \) with \( t \in [0, T] \)).

Equations (4.2) and (4.3) can be used to reconstruct the wave field for \((x, t) \in \mathcal{P}\) provided that \( A(\omega) \) is known from the measurement. From (4.1), it is seen that \( A(\omega) \), \( \omega \in [\omega_a, \omega_b] \), can be determined by the inverse Fourier transform if \( \zeta^i(t) \) has an infinitely long time record. For a finite record, in practice, we can find an approximation of \( A(\omega) \) by taking a Fourier series expansion of \( \zeta^i(t) \). In this way, the accuracy of the reconstructed wave field inside \( \mathcal{P} \) is controlled by the frequency resolution of the given record of data, which is \( O(T^{-1}) \).

Outside \( \mathcal{P} \), some or all wave components contained in \( \eta(x, t) \) come from the instants not measured in the given data. Thus, \( \eta(x, t) \) outside \( \mathcal{P} \) cannot be fully reconstructed based on the given data. Unreliable portion of the elevation outside \( \mathcal{P} \) is due to the wave components that come from outside the record of given data. From (4.2) and (4.3), this portion of the elevation can be expressed as:

\[
\eta'(x, t) = \text{Re} \sum_{n=1}^{N_\omega} \alpha_n(x, t)e^{ik_n (x-x_1)} \zeta_n(t - \Delta t_n), \quad (x, t) \not\in \mathcal{P} \tag{4.4}
\]

where the coefficient \( \alpha_n = 0 \) (or 1) if the condition \( 0 \leq t - (x - x_1)/C_n < T \) is (or is not) satisfied for \( n = 1, 2, \ldots, N_\omega \). The relative error in the reconstruction of the wave field outside \( \mathcal{P} \) based on the given data is:

\[
\text{err}(x, t) = \frac{|\eta'(x, t)|}{|\eta(x, t)|} = \frac{\sum_{n=1}^{N_\omega} \alpha_n(x, t)|A(\omega_n)|}{\sum_{n=1}^{N_\omega} |A(\omega_n)|}, \quad (x, t) \not\in \mathcal{P} \tag{4.5}
\]

In general, \( \text{err} \) is expected to increase rapidly from \( O(T^{-1}) \) to \( O(1) \) as the distance from the boundary of \( \mathcal{P} \) increases.
4.1.2 Unidirectional waves based on two probes

In the presence of two probes, the analytical procedure of wave reconstruction and forecast is similar. Here we present an analytic solution for linear reconstruction and forecast of unidirectional waves based on two probes: $\zeta^1(t) = \bar{\eta}(x = \xi_1, t)$ and $\zeta^2(t) = \bar{\eta}(x = \xi_2, t)$ with $t \in [0, T]$ and $\xi_2 > \xi_1$. The waves are assumed to propagate in the positive $x$ direction. Figure 4-1 displays the predictable zone $P$ associated with these two probes. To reconstruct the wave field in $P$, we divide $P$ into five regions based on the data with which the waves in $P$ are associated. Specifically, reconstruction of the waves in region $A$ (or $B$) requires the use of data only from $\zeta^1(t)$ (or $\zeta^2(t)$). For region $C$, either $\zeta^1(t)$ or $\zeta^2(t)$ is needed while for $D$ and $E$, both $\zeta^1(t)$ and $\zeta^2(t)$ are needed.

We equally divide the frequency band into $N_\omega$ segments. At the center of each
segment, the frequency is denoted by $\omega_n, n=1, \ldots, N_{\omega}$. We define

$$\zeta_n^1(t) \equiv \int_{\omega_n - \Delta\omega/2}^{\omega_n + \Delta\omega/2} A(\omega) e^{i(k_1 - \omega t)} \, d\omega, \quad t \in [0, T], \quad (4.6)$$

and

$$\zeta_n^2(t) \equiv \int_{\omega_n - \Delta\omega/2}^{\omega_n + \Delta\omega/2} A(\omega) e^{i(k_2 - \omega t)} \, d\omega, \quad t \in [0, T] \quad (4.7)$$

in which $A(\omega)$ can be obtained from $\zeta^1(t)$ or $\zeta^2(t)$ by Fourier transform. Based on the theory in Chapter 2, we have the reconstructed wave fields in region $A$ and $B$ given by:

$$\eta(x, t) = \text{Re} \sum_{n=1}^{N_{\omega}} e^{i(k_n - \frac{\omega_n}{C_n})(x - \xi_1)} \zeta_n^1(t - \frac{x - \xi_1}{C_n}), \quad (x, t) \in A, \quad (4.8)$$

and

$$\eta(x, t) = \text{Re} \sum_{n=1}^{N_{\omega}} e^{i(k_n - \frac{\omega_n}{C_n})(x - \xi_2)} \zeta_n^2(t - \frac{x - \xi_2}{C_n}), \quad (x, t) \in B. \quad (4.9)$$

Equation (4.8) or (4.9) is exactly the analytic solution for linear reconstruction and forecast of unidirectional waves based on one probe. The wave field in region $C$ can be reconstructed either from $\zeta^1(t)$ using (4.8) or from $\zeta^2(t)$ using (4.9).

Since the waves in region $D$ (or $E$) can be traced backward (or forward) to data $\zeta^1(t)$ and $\zeta^2(t)$ with $t \in [0, T]$, both probes are required in reconstruction of the wave fields in $D$ and $E$. To reconstruct the waves in regions $D$ and $E$, we need to identify which probe that contains the information of each wave component $\omega_n, n=1, \ldots, N_{\omega}$, for any $(x, t)$ in $D$ and $E$.

For a point $(x, t)$ in $D$, we define a threshold group velocity $C_c$ by

$$C_c = \frac{x - \xi_2}{t - T} \quad (4.10)$$

which corresponds to the slope of the line between two points $(x, t)$ and $(\xi_2, T)$ (cf. figure 4-1). Among $N_{\omega}$ wave components that arrives at point $(x, t)$, those with $C_n > C_c$ come from the probe $\zeta^1(t)$ while the others are from $\zeta^2(t)$. Let the $J_D$-th wave component have the group velocity $C_{J_D} > C_c \geq C_{J_D+1}$. Then the reconstructed
wave field in region $D$ can be written in the form:

$$
\eta(x, t) = \text{Re} \sum_{n=1}^{J_D} e^{i (k_n - \frac{\omega_n}{c_n})(x - \xi_1)} \zeta_n^1(t - \frac{x - \xi_1}{C_n}) +
\text{Re} \sum_{n=J_D+1}^{N_w} e^{i (k_n - \frac{\omega_n}{c_n})(x - \xi_2)} \zeta_n^2(t - \frac{x - \xi_2}{C_n}), \quad (x, t) \in D.
$$

Similarly for a point $(x, t)$ in region $E$, we define a threshold group velocity corresponding to the slope of the line between $(x, t)$ and $(\xi_1, 0)$:

$$
C_c' = \frac{x - \xi_1}{t}
$$

and find the $J_E$-th wave component that has the group velocity $C_{J_E} > C_c' \geq C_{J_E+1}$.

The wave field in $E$ can be reconstructed by the formula:

$$
\eta(x, t) = \text{Re} \sum_{n=1}^{J_E} e^{i (k_n - \frac{\omega_n}{c_n})(x - \xi_1)} \zeta_n^1(t - \frac{x - \xi_1}{C_n}) +
\text{Re} \sum_{n=J_E+1}^{N_w} e^{i (k_n - \frac{\omega_n}{c_n})(x - \xi_2)} \zeta_n^2(t - \frac{x - \xi_2}{C_n}), \quad (x, t) \in E.
$$

### 4.1.3 Multidirectional waves based on whole-area measurements

Given the wave elevation measured in a specific region $A$ of the wave field at time $t=t_0$, the purpose is to reconstruct and forecast the wave-field evolution in the predictable zone $P$. For simplicity, $A$ is assumed to be a simple connected region. To do that, we first express the elevation data $\zeta(X, Y)$ in terms of the wavenumber integral:

$$
\zeta(X, Y) = \tilde{\eta}(x = X, y = Y, t = t_0) = \text{Re} \int_{\theta_a}^{\theta_b} \int_{k_a}^{k_b} A(k, \theta) e^{i[kX' - \omega t_0]} dk d\theta
$$

for $(X, Y) \in A$, where $X'(\theta) = X \cos \theta + Y \sin \theta$, $A(k, \theta)$ is the unknown complex wave amplitude, and $k_a$ and $k_b$ are wavenumbers associated with frequencies $\omega_a$ and $\omega_b$, respectively. From the analysis in Chapter 2, the wave elevation anywhere in the
wave field at any time, $\eta(x,y,t)$, can be related to $\zeta(X,Y)$ by:

$$
\eta(x,y,t) = \text{Re} \int_{\theta_a}^{\theta_b} \sum_{n=1}^{N_k} e^{i(C_n k_n - \omega_n)(t - t_0)} \zeta_n(x' - (t - t_0)C_n, \theta) \, d\theta \left[ 1 + O(\Delta k^2) \right] \quad (4.15)
$$

as $N_k \to \infty$, where $x' = x \cos \theta + y \sin \theta$ and $\zeta_n(X', \theta)$ is defined as:

$$
\zeta_n(X', \theta) = \int_{k_n - \Delta k/2}^{k_n + \Delta k/2} A(k, \theta) e^{i(kX' - \omega t_0)} \, dk \, d\theta \quad (4.16)
$$

with $\Delta k = (k_b - k_a)/N_k$ and $k_n = k_a + (n - 1/2)\Delta k$, $n = 1, 2, \ldots, N_k$.

The predictable area at time $t$ based on the given wave elevation map (at $t=t_0$) is:

$$
\Pi(t) = \bigcap_{\theta \in \Theta} \bigcap_{\omega \in \Omega(t)} \Pi_{\theta\omega}(t) \quad (4.17)
$$

where $\Pi_{\theta\omega}(t)$ is:

$$
\Pi_{\theta\omega}(t) : L_{\theta_1} + (t - t_0)C_\omega \leq x \cos \theta + y \sin \theta \leq L_{\theta_2} + (t - t_0)C_\omega \quad (4.18)
$$

with $[L_{\theta_1}, L_{\theta_2}]$ being the region on which the projection of $A$ on the axis in the $\theta$-direction falls. The total predictable region $\mathcal{P}$ is the union of predictable areas for all predictable time.

As in §4.1.1, we can use (4.15) to reconstruct and forecast the wave-field evolution provided that $A(k, \theta)$ can be determined, based on (4.14), from the given elevation $\zeta(X,Y)$ in $\mathcal{A}$. When $\mathcal{A}$ is a circular region of radius $R$, $A(k, \theta)$ can be determined by:

$$
A(k, \theta) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^R \zeta(r, \theta') e^{-ikr \cos(\theta - \theta')} r \, dr \, d\theta' \quad (4.19)
$$

for $\theta \in [\theta_a, \theta_b]$ and $k \in [k_a, k_b]$.

We note that when $\mathcal{A}$ is a rectangular region, it is more convenient to use the Cartesian coordinate (instead of the polar coordinates). In this case, by assuming the given measurement $\zeta(X,Y)$ to be doubly periodic in $X$ and $Y$, we can use the
Fourier transform to determine the unknown wave amplitudes $A(k_m, k_n)$:

$$A(k_m, k_n) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \zeta(X_m, Y_n)e^{-i(k_mX_m+k_nY_n)}$$

(4.20)

where $k_m = 2m\pi/L$, $m = 0, \ldots, M-1$; $k_n = 2n\pi/W$, $n = 0, \ldots, N-1$; $X_m = mL/(M-1)$, $m = 0, \ldots, M-1$; and $Y_n = nW/(N-1)$, $n = 0, \ldots, N-1$. Here $L$ and $W$ are the length and width of $A$, respectively. The wave elevation of the reconstructed and forecasted wave field is obtained by:

$$\eta(x, y, t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} A(k_m, k_n)e^{i(k_mx+k_ny-\omega_mnt)}$$

(4.21)

for any $x, y, t \in P$ (that is given by (4.17) and (4.18)), in which $\omega_{mn}$ is determined from the dispersion relation with wavenumber $k_{mn} = (k_m^2 + k_n^2)^{1/2}$.

The error of the reconstructed/forecasted wave field inside $P$ is $O(1/R)$ (or $O(1/L, 1/W)$) for a circular (or rectangular) shaped $A$. The error outside $P$ can be estimated by a formula similar to (4.5). Similarly, the relative error outside $P$ is expected to grow rapidly to $O(1)$ as the distance from the boundary of $P$ increases.

4.2 Nonlinear reconstruction and forecast of irregular waves

For general nonlinear wave field, we develop a high-order phase-resolved reconstruction and to assess its feasibility, validity and efficacy by applying it to a broad ranges of wave field conditions and measurements, including nonlinear wave fields generated numerically and in physical wave experiments. Specifically, we develop an iterative high-order reconstruction (HOR) method which accounts for nonlinear wave interactions up to some order $M$ in wave steepness. HOR optimizes the amplitude and phase of a relatively small number $(L)$ of free wave components, based on which all high-order locked wave components can be determined by solving the nonlinear Euler equations. By iteratively increasing $M$ and $L$, HOR obtains a wave field that
minimizes the reconstruction error between the reconstructed field and given measurements. HOR provides the phase-resolved whole-field description of the nonlinear waves (including wave elevation, velocity and pressure), which we use to compare and evaluate against the larger underlying wave field from which the measurements are taken (beyond $\mathcal{M}$).

HOR requires a computationally efficient general phase-resolved "nonlinear evolution engine" which may be "evaluated" repeatedly in minimizing the reconstruction error (in $\mathcal{M}$) (cf. figure 4-2). To achieve this, we implement the high-order spectral (HOS) method [24] in HOR. Given an initial field (defined by the HOR model parameters), HOS solves the primitive Euler equations using $N$ spectral modes up to a specified nonlinear order $M$ in wave steepness. HOS obtains exponential convergence with $N$ and $M$, and has operation count that scales linearly with $N$ and $M$. In terms of the requirements of the present HOR scheme, HOS is, in many ways, an ideal evolution engine.

We evaluate the validity and performance of HOR. We also address a number of theoretical and practical issues associated with HOR which include: (a) convergence of the HOR scheme to a unique solution $\Phi$ close to the underlying wave field $\tilde{\Phi}$ in some (nonlinear) predictable zone $\mathcal{P}_{NL}$ (beyond the measurements $\mathcal{M}$); (b) the space-time extent of $\mathcal{P}_{NL}$ and its relationship to the linear predictable zone $\mathcal{P}_L$ which can be theoretically obtained given $\mathcal{M}$ (Chapter 2); (c) importance of high-order nonlinear effects in the reconstruction in predicting $\tilde{\Phi}$; and (d) the computational efficacy of HOR, related to the number of HOS "evaluations" $N_{eval}$, which depends on the number of HOR wave-field model parameters $L$, and the nonlinear order $M$ required (all determined by the HOR scheme itself). To address these, we apply HOR systematically to general two- and three-dimensional wave fields, including theoretical Stokes waves (§4.2.7), synthetic wave fields generated computationally (§5.1.1, 5.1.2, 5.2.1, 5.2.2), and measured uni- and multi-directional irregular wave fields in wave tank experiments (§5.1.3, 5.2.3).

We find that, under the conditions/assumptions of HOR, the reconstructed $\Phi$ always converges to a unique wave field. Within some nonlinear predictable zone
\( \mathcal{P}_{NL} \) approaches the underlying \( \tilde{\Phi} \), bounded by tolerance \( \mathcal{E}^* \) for the reconstruction error. Comparing \( \mathcal{P}_{NL} \) with the theoretically obtained \( \mathcal{P}_L \) in Chapter 2 (for the given \( \mathcal{M} \)), we find that for relatively small wave steepness, \( \mathcal{P}_L \) is close to \( \mathcal{P}_{NL} \). As wave steepness increases, \( \mathcal{P}_{NL} \) contains and is generally greater than \( \mathcal{P}_L \). For general applications where \( \tilde{\Phi} \) is unknown and thus \( \mathcal{P}_{NL} \) cannot be obtained, \( \mathcal{P}_L \) provides a (generally) conservative estimate for \( \mathcal{P}_{NL} \) (underlining the importance of the linear predictable zone theory developed in Chapter 2). From extensive tests, we show that, even for relatively small wave nonlinearity, the linear reconstructed wave field \( \Phi_{(1)} \), using wave elevation measurements, say, does not adequately predict underlying quantities such as velocities and pressure (reflecting the significance of locked waves not accounted for in the linear theory). Finally we show that even for somewhat complicated multidirectional wave field, the required \( L \) is much less than the number of modes \( N \) used in HOS. Since the number of wave evolution evaluations \( N_{\text{eval}} \) is typically linearly proportional to \( L \) (and \( M \)), the computational efficiency of HOR is sufficient for realistic practical applications (using high-performance computing).

### 4.2.1 Problem description

Our objective is to develop a general procedure for nonlinear reconstruction of a phase-resolved wave field \( \Phi(x, z, t) \) given a set of wave measurements in spatial-temporal domain \( \mathcal{M} \) of the original wave field \( \tilde{\Phi}(x, z, t) \). The reconstructed \( \Phi \) must satisfy the nonlinear Euler equations, and is required to match \( \tilde{\Phi} \) for \( (x, t) \in \mathcal{M} \) to within some specified small tolerance \( \mathcal{E}^* \).

In Chapter 2, we show, in the context of linear wave theory, that for any given set of measurements in \( \mathcal{M} \) (in \( x-t \)), there exists a (linear) predictable zone \( \mathcal{P}_L \), generally a superset of \( \mathcal{M} \), inside which the underlying wave field can be fully predicted. A general theoretical procedure is developed to find the solution of \( \mathcal{P}_L \) for various measurement combinations including moving point probes and combinations of probes and whole-area measurements. However, Chapter 2 does not discuss the actual method to reconstruct the wave field given \( \mathcal{M} \).

We consider reconstruction of a general nonlinear wave field. Without loss of
generality, we assume that the energy of the wave field is concentrated within some (finite) frequency band \( \Omega = [\omega_a, \omega_b] \) (corresponding to wavenumber band \([k_a, k_b]\)), and wave propagation direction within some range \( \Theta = [\theta_a, \theta_b] \subset (-\pi/2, \pi/2) \). We focus on nonlinear waves which can be represented as a regular perturbation series in wave steepness.

For definiteness and without loss of generality, we assume hereafter that the measured quantity are wave elevations measured at fixed locations over some (finite) durations, i.e., \( \zeta(\xi_j, t) \equiv \tilde{\eta}(x = \xi_j, t), t \in T_j, j = 1, \ldots, J \), where \( \tilde{\eta} \) is the elevation of the underlying wave field, and \( \xi_j \) and \( T_j \) are the measurement location and duration of each record. The measurement domain \( \mathcal{M} \) is simply the union of all \( (x = \xi_j, t \in T_j), j = 1, \ldots, J \). The analysis and approach can be easily extended to more complicated \( \mathcal{M} \), say involving moving probes, measurements over a whole area at specific time instants and/or combinations of such measurements (cf. Chapter 2). We remark here that biased/unbiased measurement errors are not considered in the present development. Using direct evaluations with Monte-Carlo simulations, it is shown that the errors in wave reconstruction are generally bounded by the measurement errors, see [116]. In the present work, we show satisfactory performance of reconstruction using physical tank data (§5.1.3,5.2.3), which clearly also contain measurement errors.

\[ \Phi(x, z, t) = \Phi(M)(x, z, t) = \sum_{p=1}^{M} \Phi^{(p)}(x, z, t), \]  

### 4.2.2 An iterative high-order reconstruction (HOR) method

We consider irrotational gravity waves so that the flow can be described by a velocity potential \( \Phi(x, z, t) \) satisfying Laplace equation within the fluid. We assume that \( \Phi \) (and therefore all related quantities such as surface elevation \( \eta(x, t) \)) can be represented as a (regular) perturbation series in some small parameter \( \epsilon \), which measures the wave steepness. Up to some (arbitrary) order \( M \), we have:

\[ \Phi(x, z, t) = \Phi(M)(x, z, t) = \sum_{p=1}^{M} \Phi^{(p)}(x, z, t), \]
where \((\cdot)^{(p)}\) denotes a quantity of \(O(\varepsilon^p)\). We further represent each \(\Phi^{(p)}\) as an eigenfunction expansion which satisfies all but the nonlinear boundary conditions on the free surface. Thus we write

\[
\Phi^{(p)}(x, z, t) = \sum_{q=1}^{\infty} \Phi_q^{(p)}(t) \Psi_q(x, z),
\]

where, in practice, we truncate (4.23) at a sufficiently large \(N\) to represent all the (free and locked) modes in \(\Phi\). For the purpose of reconstruction, we seek to determine model parameters given in terms of the amplitudes and phases of \(L, L \ll N\) typically) free wave components. Hereafter we denote these \(2L\) model parameters as the vector \(\mathbf{\Upsilon}\).

Nonlinear wave-field reconstruction based on given measurement data is essentially an optimization problem. The objective is to find a (reconstructed) wave field \(\Phi\) which satisfies the nonlinear Euler equations that minimizes the reconstruction error \(E\), i.e., the difference between \(\Phi\) and the measurements of the underlying wave field \(\tilde{\Phi}\) in \(\mathcal{M}\). The definition of the reconstruction error \(E\) depends on the measurement data provided which can be quite general. In the present context where the data is assumed to be point measurements of the wave elevation \(\tilde{\eta}\) at fixed locations \(\xi_j\), over durations \(t \in \mathcal{T}_j, j = 1, \ldots, J\), the reconstruction error for \(\Phi_{(M)}(\mathbf{\Upsilon})\) obtained with model parameters \(\mathbf{\Upsilon}\) up to order \(M\) is defined as:

\[
E(\mathbf{\Upsilon}; M) \equiv \left[ \sum_j \int_{\mathcal{T}_j} [\eta_{(M)}(\mathbf{\Upsilon}) - \tilde{\eta}]^2 dt / \sum_j \int_{\mathcal{T}_j} \tilde{\eta}^2 dt \right]^{1/2}.
\]

Here we develop an iterative high-order reconstruction (HOR) method, which can properly determine \(L\) and \(M\) and obtain a reconstructed wave field with reconstruction error \(E\) satisfying a prescribed tolerance \(E^* (E < E^*)\). Given a predetermined \(N\), we start with a small initial value for \(L\), e.g. \(N/32\), and iteratively increase \(L\) until \(E < E^*\). The proceeding of HOR can be described by three main procedure blocks.

I. Given \(L\), iterate with increasing \(M\) until \(E\) converges. We always start with \(M = 1\) to proceed with procedure block II and iteratively increase \(M\) until \(E\) converges
with increasing $M$, i.e. 
$$\Delta \mathcal{E}(M) \equiv |\mathcal{E}(\Upsilon_{\text{opt}}^{(M)}) - \mathcal{E}(\Upsilon_{\text{opt}}^{(M-1)})| < \tilde{\mathcal{E}},$$
where $\mathcal{E}(\Upsilon_{\text{opt}}^{(M)})$ is the reconstruction error obtained by the optimized model parameters $\Upsilon_{\text{opt}}^{(M)}$ at corresponding order $M$ and $\tilde{\mathcal{E}}$ is the prescribed tolerance.

II. Given $L, M$, obtain optimal $\Upsilon_{\text{opt}}^{(M)}$. We initialize $\Upsilon_{\text{opt}}^{(M)}$ with $\Upsilon_{\text{opt}}^{(M-1)}$ and proceed with procedure block III, the nonlinear evolution engine, to obtain all high-order locked waves and thus the reconstructed wave field and evaluate $\mathcal{E}$. Then we optimize $\Upsilon_{\text{opt}}^{(M)}$ using an efficient optimization scheme, such as the quasi-Newton optimization method (e.g. [12, 33]), until the optimization converges and $\mathcal{E}$ is minimized. Specially, $\Upsilon_{\text{opt}}^{(0)}$ is obtained directly from the Fourier transform of the measurement data.

III. Nonlinear evolution engine. Given $L, M$, and $\Upsilon^{(M)}$, we implement an efficient nonlinear evolution engine to solve the high-order evolution of the wave field, so that the wave elevation $\eta_{(M)}(x, t)$ for $(x, t) \in \mathcal{M}$ can be used to evaluate the reconstruction error using equation (4.24).

This entire process involving these procedures is illustrated in figure 4-2. Outside II, we first check the convergence of $\mathcal{E}$ with increasing $M$. If not, the nonlinear order of the wave model is not sufficient and thus we increase $M$ by 1 and repeat II, otherwise it means further increasing $M$ is not effective and we proceed outside I to check if the reconstruction tolerance is reached. If not, we increase $L$ by appropriate amount (under the condition of $L < N$) and repeat I, otherwise the reconstruction is completed and we obtain a reconstructed wave field of nonlinear order $M$.

The final result of HOR is a nonlinear (up to order $M$) wave field $\Phi$ which minimizes the reconstruction error $\mathcal{E}$ against the given set of measurements in $\mathcal{M}$. Although $\Phi$ extends beyond $\mathcal{M}$ and can be used as initial condition to forecast the evolution to later times, in principle, HOR does not provide the predictable zone $\mathcal{P}_{NL}$. $\mathcal{P}_{NL}$ is the spatial-temporal domain within which $\Phi$ can be expected to predict the underlying wave field $\tilde{\Phi}$ from which the original measurements (in $\mathcal{M}$) are obtained. As we shall show in this work, there is a (nonlinear) predictable zone $\mathcal{P}_{NL}$, which depends on $\mathcal{M}$, within which the difference between the prediction $\Phi$ and $\tilde{\Phi}$ is bounded by $\mathcal{E} < \mathcal{E}^*$. As might be expected heuristically, $\mathcal{P}_{NL}$ should be related to the linear predictable zone $\mathcal{P}_{L}$ that can be obtained theoretically from $\mathcal{M}$ (Chapter
I. Given \( L \), iterate with increasing \( M \) until \( \mathcal{E} \) converges.

II. Given \( L, M \), obtain optimal \( \mathbf{y}^{(M)}_{\text{opt}} \).

III. Nonlinear evolution engine.

Obtain \( \eta_{(M)} \) and evaluate \( \mathcal{E} \).

Optimize \( \mathbf{y}^{(M)} \) by minimizing \( \mathcal{E} \) until optimization converges.

Convergence of \( \mathcal{E} \) with increasing \( M \)?

\[ |\mathcal{E}(\mathbf{y}^{(M)}_{\text{opt}}) - \mathcal{E}(\mathbf{y}^{(M-1)}_{\text{opt}})| < \bar{\mathcal{E}} \]

Reconstruction tolerance reached?

\[ \mathcal{E}(\mathbf{y}^{(M)}_{\text{opt}}) < \mathcal{E}' \]

Yes

No

Increase \( L \)

Reconstruction completed

Figure 4-2: The flow chart for the nonlinear reconstruction scheme using HOR.

2). Significantly, we find that generally \( \mathcal{P}_{NL} \supseteq \mathcal{P}_{L} \) so that \( \mathcal{P}_{L} \) provides a conservative estimate of the space-time domain over which \( \Phi \) predicts (or can forecast) the underlying field \( \Phi \).

4.2.3 Choice of number of model parameters

One of the key parameters in HOR is the number \( L \) (and choice) of the free propagating wave components whose amplitude and phase are the model parameters in the optimization. Sufficient number of free wave components are needed in order to represent the underlying wave field \( \Phi \) as specified by the measurement. Strictly speaking, \( L \) is related to the measurements and not necessarily related to, and could be much smaller than, the total number \( N \) of modes (free and locked) used in the wave evolution model. In HOR, we start with a relatively small \( L = L_0 \), which is iteratively increased. In this approach, it is useful to estimate an upper bound \( L_{\text{max}} \) for
We assume that the reconstructed wave field $\Phi$ can be specified by $L = L_\omega L_\theta$ free propagating wave components, containing $L_\omega$ frequencies $\omega_n$, $n=1,\ldots,L_\omega$, and $L_\theta$ propagation directions, $\theta_m$, $m=1,\ldots,L_\theta$. For the $mn$-th free wave component with amplitude $A_{mn}$ and phase $\psi_{mn} = k_{mn} \cdot x - \omega_n t + \alpha_{mn}$ for the elevation, the first-order elevation is:

$$\eta^{(1)}(x,t) = \text{Re} \sum_{m=1}^{L_\omega} \sum_{n=1}^{L_\omega} A_{mn} e^{i\psi_{mn}},$$

(4.25)

At any order $M$, $\eta^{(M)}$ (containing free and bound waves) is specified by the model parameters: $\omega_n, \theta_m, A_{mn}$ and $\alpha_{mn}$, $n=1,\ldots,L_\omega$, $m=1,\ldots,L_\theta$, which are to be optimized.

For simplicity, in HOR, we specify $\omega_n$ and $\theta_m$ to uniformly span respectively the frequency range, $[\omega_a, \omega_b]$, and directional spreading range, $[\theta_a, \theta_b]$, estimated from the given measurement data $\zeta(\xi_j,t)$, $j=1,\ldots,J$. In general, these can be obtained using Fourier transform or Maximum Likelihood Estimation [119]. For measurements over total duration $T$, the smallest frequency that can be resolved is $\Delta\omega^* = 2\pi/T$, and we specific the upper bound value of $L_\omega$ as $L_{\omega\text{max}} = (\omega_b - \omega_a)/\Delta\omega^*$.

For $L_\theta$, we set the upper bound number $L_{\theta\text{max}}$ equal the number of (evenly spaced) directions $\theta_m$ that can be uniquely determined from the specified measurements. Given the measurement locations $\xi_j$, $j=1,\ldots,J$, this can be obtained by considering the rank of phase function matrices (cf. equation (4.29)) involving the permissible wavenumbers $k_{mn}$ (given $\theta_m$ and $\omega_n$) and the measurement positions $\xi_j$. Details are given in §4.2.4.

In HOR implementation, we generally start with relatively small values of $L_\omega$ and $L_\theta$, and increase these iteratively until $L_\omega = L_{\omega\text{max}}$, $L_\theta = L_{\theta\text{max}}$, or when the reconstruction tolerance is reached.

### 4.2.4 Determination of $L_{\theta\text{max}}$

Now consider the determination of $L_{\theta\text{max}}$ in the reconstruction of a multidirectional wave field with given $J$ probes providing wave elevation measurements. To address
this question, we consider the linear reconstruction of a three-dimensional wave field
with \( L_\theta \) wave directions based on \( J \) probes, since the uniqueness of the nonlinear
solution is assured by the uniqueness of the linear solution.

We can write the linear solution of free surface elevation in (4.25) as

\[
\eta^{(1)}(x,t) = \Re \sum_{n=1}^{L_\omega} \sum_{m=1}^{L_\theta} a_{mn} e^{i(k_{mn} \cdot x - \omega_n t)},
\]

(4.26)

where \( a_{mn} = A_{mn} e^{i\alpha_{mn}} \) is the complex amplitude. Assume two different solutions of
\( a_{mn} \) exist, say \( a_{mn} \) and \( a'_{mn} \). Let \( b_{mn} = a_{mn} - a'_{mn} \). At the measurement location \( \xi_j \),
we must obtain:

\[
\Re \sum_{n=1}^{L_\omega} \sum_{m=1}^{L_\theta} b_{mn} e^{i(k_{mn} \cdot \xi_j - \omega_n t)} = 0, \quad j = 1, \ldots, J \quad \text{for} \quad t \in T_j.
\]

(4.27)

From (4.27), it follows that

\[
\sum_{m=1}^{L_\theta} b_{mn} e^{i(k_{mn} \cdot \xi_j)} = 0, \quad j = 1, \ldots, J
\]

(4.28)

for \( n=1, \ldots, L_\omega \). If any of these homogeneous systems, corresponding to \( n=1, \ldots, L_\omega \),
possesses a nontrivial solution for \( b_{mn} \), the wave reconstruction using (4.26) is non-
unique.

Therefore, in order for the solution of reconstruction to be unique, we require that
nontrivial solution of \( b_{mn} \) does not exist. The existence of nontrivial solution of \( b_{mn} \)
depends on the rank of the \((J \times L_\theta)\) coefficient matrices:

\[
[C]_n = \begin{pmatrix}
e^{ik_{1n} \cdot \xi_j} & e^{ik_{2n} \cdot \xi_j} & \cdots & e^{ik_{L_\theta n} \cdot \xi_j} \\
& & & \\
e^{ik_{1n} \cdot \xi_j} & e^{ik_{2n} \cdot \xi_j} & \cdots & e^{ik_{L_\theta n} \cdot \xi_j}
\end{pmatrix} \quad \text{for} \quad n = 1, \ldots, L_\omega.
\]

(4.29)

A necessary and sufficient condition for (4.28) to have only trivial or no solution is
\([C]_n\) has full column rank, which requires that \( J \geq N_\theta \). This implies that for given \( J \),
the maximum number of wave directions that can be resolved in wave reconstruction
is at most $J$, i.e. $L_{\theta_{\text{max}}} = J$. (In other words, in order to obtain a wave-field reconstruction with $L_{\theta}$ wave directions, we need at least $L_{\theta}$ probes.)

In addition, the condition of $[C]_n$ having full column rank imposes a requirement on the spatial locations of wave probes. As an example, we consider a two-direction wave reconstruction with $L_{\theta}=2$ and $J=2$. It requires that:

\[
\begin{vmatrix}
  e^{ik_{1n}\xi_1} & e^{ik_{2n}\xi_1} \\
  e^{ik_{1n}\xi_2} & e^{ik_{2n}\xi_2}
\end{vmatrix} \neq 0, \quad \forall n, \ n = 1, \ldots, L_{\omega}
\]  

(4.30)

which leads to the condition:

\[
(k_{1n} - k_{2n}) \cdot (\xi_1 - \xi_2) \neq q2\pi, \quad q = 0, \pm 1, \pm 2, \ldots, \quad \forall n, \ n = 1, \ldots, L_{\omega}.
\]  

(4.31)

Physically, this requires that $\xi_1$ and $\xi_2$ can not simultaneously be the node points of the standing wave formed by $k_{1n}$ and $k_{2n}, \ n = 1, \ldots, L_{\omega}$. For $N_{\theta} \geq 3$, no simple solution can be written out from the condition that $[C]_n$ has full column rank. But the general requirement still holds: for all value of $n, \ n = 1, \ldots, L_{\omega}$, there should be at least $N_{\theta}$ probes that are not simultaneously located at the node points of any standing waves that are formed by the wave components $k_{1n}, \ldots, k_{N_{\theta}n}$.

### 4.2.5 Nonlinear evolution engine

HOR requires a nonlinear evolution engine (procedure block III in figure 4-2) to solve the high-order evolution of a wave field $\Phi_{(M)}$ for any given set of model parameters $\Upsilon^{(M)}$. As expected, the number of such "evaluation" $N_{\text{eval}}$ in the HOR procedure increases with $L$ (and $M$), and a relatively large $N_{\text{eval}}$ is required for realistic applications.

In principle, any nonlinear phase-resolved evolution model can be used in HOR (up to second order $M=2$, $\Phi_{(2)}$ is given in closed form (e.g. [124]), and a computational evolution model is not required). For general $M$, we implement here a high-order spectral (HOS) method [24] as the evolution engine. HOS is a modal-decomposition pseudo-spectral method that solves the primitive Euler equations up to arbitrary
high order $M$. HOS obtains exponential convergence with respect to the number of wave modes, $N$, and order $M$; and obtains an operation count that is nearly linearly proportional to $N$ and $M$. In some regards, given the HOR requirements for efficiency and accuracy, HOS is perfectly suited.

In the HOR optimization procedure, HOS is evoked with initial conditions for the wave elevation $\eta(x, t = 0)$ and velocity potential $\Phi_S(x, t = 0) \equiv \Phi(x, z = \eta, t = 0)$ on the surface, given in terms of the optimization parameters $\Upsilon$ consisting of the amplitude $A_{mn}$ and phase $\alpha_{mn}$ of $L_w L_\theta$ free wave components. The computational domain $A$ used by HOS must contain the measurements $M$, and is generally chosen to be (much) larger to cover the (anticipated useful) predictions in a larger domain and to mitigate the (periodic) computational boundary effects of HOS. The number of modes $N$ in HOS is likewise much greater that the number of HOR model modes $L$ to resolve the maximum wavenumber $k_b$ contained in the measurements, over the computational domain $A$. Finally when the HOR nonlinear reconstructed $\Phi$ is obtained, HOS can be simply continued in time (beyond $M$), to predict in forecasting mode.

The computational effort of HOR using HOS can be estimated. For the reconstruction of an irregular wave field in a time interval $T$, containing $N_d$ dominant wave period $T_d$, we typically end up using $L_w = \kappa N_d$ with $\kappa = 3 \sim 5$ and $L_\theta \sim J$ after a few ($\nu \leq 10$) iterations on $L_w$ and $L_\theta$. In the HOS simulations, we use $N = \mu L_w$ or $N = (\mu L_w)^2$ for two- or three-dimensional waves where $\mu \sim O(10)$ in order to cover the range of wave frequencies that we need to optimize. The computational effort per HOS simulation is $O(N_d NM)$. Given the number of HOS evaluations $N_h$ at each $M$ and $L$, the total number of HOS evaluations $N_{\text{eval}} = O(\nu MN_h)$ and the total requisite effort in nonlinear wave-field reconstruction is $O(N_{\text{eval}} N_d NM)$. In the present HOR using the quasi-Newton optimization scheme, we find that $N_h$ typically scales linearly with $L$. Therefore in this case the total computational effort is $O(\nu N_d LN M^2)$.

4.2.6 An illustration example

To illustrate HOR, we consider a synthetic unidirectional second-order wave as an example. For simplicity, the second-order wave is generated based on only two free
Table 4.1: Amplitude and phase of each free wave component in the generation of the synthetic second-order wave.

<table>
<thead>
<tr>
<th>$\omega$ (rad/s)</th>
<th>$A$ (m)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.0</td>
<td>0</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0</td>
<td>$\pi/4$</td>
</tr>
</tbody>
</table>

Table 4.2: Optimized amplitude and phase of each free wave component using linear wave model in the reconstruction of the synthetic second-order wave.

<table>
<thead>
<tr>
<th>$\omega$ (rad/s)</th>
<th>$A$ (m)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.997</td>
<td>2.8624</td>
<td>6.2219</td>
</tr>
<tr>
<td>1.995</td>
<td>1.3982</td>
<td>0.5145</td>
</tr>
</tbody>
</table>

wave components together with all their bound wave components. The amplitude and phase for each free wave component are listed in table 4.1. The generation of the second-order wave follows the formula described in appendix A.

We take the synthetic wave at a fixed point $(x=0)$ within time $t \in [0, T] = [0,6.3]$ s as the measurement data to reconstruct. Figure 4-3 shows the generated second-order wave in the red line. As comparison, the free wave components in the generated wave are also plotted in the black line.

We first try linear wave model for reconstruction. For simplicity, the frequencies in the reconstructed linear wave model are fixed at $\omega_1 = 2\pi/T$ and $\omega_2 = 2\omega_1$ as it is easy to find that the wave energy is concentrated mainly in these two frequencies based on Fourier analysis. The optimized amplitude and phase for each component is shown in table 4.2. The reconstruction error using linear wave model is $\mathcal{E} = 9.13\%$.

As the reconstruction error using linear model is too large, we can conclude that linear wave model is not enough to reconstruct this synthetic wave. Then we replace the linear wave model with the second-order wave model and optimize the same set of model parameters again. This time the optimized amplitude and phase of each free wave component is shown in table 4.3 and the reconstruction error decreases dramatically to $\mathcal{E} = 0.403\%$ which is quite satisfactory.

Figure 4-4 shows the comparison of the original measurements and reconstructed wave using linear and second-order wave models. Note that when we use linear wave model to do reconstruction, we are not estimating the linear components (the black
Figure 4-3: Synthetic first-order and second-order unidirectional waves using two free wave components at fixed point \(x=0\) and within time \(t \in [0, T] = [0, 6.3] \text{s}\).

<table>
<thead>
<tr>
<th>(\omega , (\text{rad/s}))</th>
<th>(A , (\text{m}))</th>
<th>(\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.997</td>
<td>3.0130</td>
<td>6.2760</td>
</tr>
<tr>
<td>1.995</td>
<td>0.9974</td>
<td>0.7714</td>
</tr>
</tbody>
</table>

Table 4.3: Optimized amplitude and phase of each free wave component using second-order wave model in the reconstruction of the synthetic second-order wave.
Figure 4-4: Reconstruction of the original measurement (red dashed line) using linear wave model (green line) and second-order wave model (blue line). The linear components in the original measurement is shown as black dashed line.

dashed line) in the measurements, instead, we are using linear waves to directly estimate the original measurements (red dashed line). The significant difference between the reconstructed linear wave (green line) and the linear components in the original wave (black dashed line) explains this point. Only if we find the best reconstructed result using linear wave model is not satisfactory then we replace the wave model with a higher-order one. No matter which wave model we use to do reconstruction, we always reconstruct the original measurements.

4.2.7 Validation of HOR using Stokes waves

To further validate the performance of HOR, we perform tests to high order $M$ using theoretical exact Stokes waves. Tests using computationally generated synthetic wave fields and experimentally generated waves are discussed in §5.
Table 4.4: $E$, $\Delta E$ and $E^D$ as function of $L_\omega$ and $M$ in HOR for the reconstruction of Stokes wave with steepness $ka=0.20$, where $E^* = 10^{-3}$ and $\tilde{E} = E^*/2$, $D = \{x \in [0, 2\pi], t/T = 10\}$ and the measurement domain $M = \{x = 0, t/T \in [0, 1]\}$. For $M=1, 2$, analytic wave models are used while for $M \geq 3$, HOS model with $N=64$ (corresponding to 128 data points in space) is used to reconstruct the wave field.

For the solution of the Stokes wave, we follow the example used in [24] and solve the nonlinear equations associated with the mapping function (Schwartz’ equations (2.6) [94]) directly using Newton iteration rather than high-order perturbation. The final results are exact to 14 significant figures. For simplicity, we consider deep water waves with a wavelength of $2\pi$, and we take the wave elevation at a fixed point $(x=0)$ with one wave period $T$ as the measurement. Given this Stokes wave measurement, the theoretical predictable zone is the entire $x$-$t$ domain as the Stokes wave is periodic and propagate with permanent form. We define the wave steepness as $ka = \frac{1}{2}k(\eta_{\max} - \eta_{\min})$ and we conduct reconstruction for different values of $ka$. We set the reconstruction tolerance $E^* = 10^{-3}$ and $\tilde{E} = E^*/2$.

When the underlying wave field is known, we define a general prediction error $E^P$ similar as equation (4.24) except that the integration domain is changed from $M$ to a general space-time domain $D$. Specifically if $D$ is the predictable zone $P$, we obtain $E^P$. For Stokes waves, without loss of generality, we consider a space-time domain $D$: $\{x \in [0, 2\pi], t/T = 10\}$ to examine the prediction accuracy and discuss the convergence of prediction.

To illustrate the HOR procedure, table 4.4 shows the detailed results of $E$, $\Delta E$ and $E^D$ as function of $L_\omega$ and $M$ for the reconstruction of Stokes wave with $ka=0.20$. Note
that in HOR for each $L_\omega$ and $M$, once $\Psi^{(M)}$ is optimized, HOS is simply continued in time to calculate $E^D$. We see that for $L_\omega=2$, as $M$ increases, both $E$ and $E^D$ decrease and at $M=6$, $\Delta E$ is below $\tilde{E}$, but $E$ is still larger than $E^*$, which means further increasing $M$ is not effective as the number of free wave components $L_\omega$ is too small. Thus we increase $L_\omega$ to 4 and repeat the iteration on $M$. At $M=5$, both $\Delta E < \tilde{E}$ and $E < E^*$ are satisfied and the reconstruction is regarded finished. The results of $M=6$ are shown just for comparison. The results of $L_\omega=8$ are used to confirm $L_\omega=4$ is sufficient for this case, as the results of $E$ and $E^D$ are quite close for $L_\omega=4$ and 8. Note that using HOS as the nonlinear evolution engine in HOR, it is straightforward to increase $M$ to an arbitrary high order, while using other model equations, it is difficult to consider nonlinear effects beyond the third order.

Figure 4-5 further shows for the reconstruction of Stokes wave with $ka=0.20$, the optimization process for the complex amplitude $(A e^{i\alpha})$ of the first wave component ($\omega = 2\pi/T$). We can see that for given $L_\omega$, as $M$ increases, the optimized value of $A e^{i\alpha}$ moves closer to the theoretical value until the prescribed fitting accuracy is satisfied and reconstruction is finished. And for the same $M$, say $M=6$, reconstruction with larger $L_\omega$ reaches an optimized value closer to the theoretical one.

To show the relationship between $E$ and $E^D$, figure 4-6 plots the change of $E$ and $E^D$ with respect to $M$ for reconstruction of Stokes wave with $ka=0.20$ using different $L_\omega$. We see that for $L_\omega=2$, $E$ decreases relatively slowly as $M$ increases until $\Delta E$ is less than $\tilde{E}$, while for sufficient $L_\omega=4$, $E$ decreases rapidly as $M$ increases until $E$ reaches the prescribed tolerance. It also shows that especially for small value of $M$, $E^D$ is much larger than $E$. Thus nonlinear effects are especially important in order to forecast the wave field well. For $L_\omega=4$, $E^D$ also decreases rapidly as $M$ increases until $E < E^*$ and $E^D$ is constrained by $E$.

To examine the behavior of $E^D$ as $M$ increases for different $ka$, table 4.5 shows the detailed results of $E^D$ for each case. We see that for all cases, $E^D$ decreases exponentially as $M$ increases until it is constrained by $E$. For given $M$, $E^D$ grows rapidly as $ka$ increases. As the wave becomes steeper, higher order of nonlinearity is needed in order for $E^D$ to reach the same level of reconstruction accuracy. For
Figure 4-5: The optimized complex amplitude $Ae^{i\alpha}$ of the first wave component $(\omega = 2\pi/T)$ in the reconstruction of Stokes wave with $k\omega=0.20$ using different $L_\omega$, where the green square represents the initial guess for $M=1$ reconstruction, the blue, black and red lines with markers represent the path for the optimized value using wave models of different $M$, and the red triangle represents the theoretical value in the generation of Stokes wave.
Figure 4-6: Reconstruction error $\mathcal{E}$ and prediction error $\mathcal{E}^D$ as a function of $M$ for the reconstruction of Stokes wave with steepness $k\alpha=0.20$ using $L_\omega=2$ (---) and $L_\omega=4$ (——).
Table 4.5: Prediction error $E'$ as function of $M$ in HOR for the reconstruction of Stokes waves of different steepness $ka$, where $D=\{x \in [0, 2\pi], t/T = 10\}$, $M=\{x = 0, t/T \in [0, 1]\}$, $E^* = 10^{-3}$, $\bar{E} = E^*/2$ and $L_w=4$. For $M=1,2$, analytic wave models are used while for $M \geq 3$, HOS model with sufficient $N=64$ (corresponding to 128 data points in space) is used to reconstruct the wave field.

4.2.8 Optimization scheme

The reconstruction problem is essentially an optimization problem that we optimize the model parameters $(A_{mn}, \alpha_{mn})$ to minimize the reconstruction error $E$. In the HOR method, if we use nonlinear wave model with $M \geq 3$, then each time the model parameters have new values, we need to run HOS simulation in a spatial area covering the measurement locations for the maximum duration of measurements so that we can compare the simulated elevation with the measurements. The computational effort involved is thus extensive and we need an efficient optimization scheme.

In a general optimization problem, we optimize the model parameters to minimize a cost function. In the HOR method, we denote the vector of model parameters as $\mathbf{Y}$, the wave model with nonlinear order $M$ as $\mathbf{F}_M(\mathbf{Y})$ which produces a vector of reconstructed wave elevation at the location and time of measurements, which are denoted here in a vector form $\mathbf{G}$, and the cost function as $C \equiv \frac{1}{2}E^2$.

Common optimization methods include the steepest-descent method (also called gradient method), conjugate gradient method [34, 87, 30], Newton minimization method [33] and related modified Newton methods, and so on. Among these methods, the steepest-descent method chooses search direction as simply along the negative of
the gradient of the cost function and the rate of convergence is known as linear. One way to improve the convergence rate of the steepest descent method is by using a search direction which is a linear combination of the current gradient and the previous search direction and this type of method is called conjugate gradient. As comparison, the full Newton minimization method has quadratic rate of convergence. But one drawback of the full Newton minimization method is that the Jacobian matrix of the reconstruction error with respect to model parameters is needed. In this reconstruction problem, when we use nonlinear wave model with $M \geq 3$, there is no analytic expression for the derivatives of reconstruction error with respect to model parameters, which means that these derivatives should be computed numerically. And since the function evaluation $(F_M(\mathbf{Y}))$ is expensive as mentioned above, then the cost of finite-difference determination of the Jacobian is prohibitive. Thus in the HOR method, we choose a quasi-Newton method which provide cheap approximation to the Jacobian and Hessian matrix [12].

Specifically, the Newton minimization method is based on a local quadratic model of the cost function. The quadratic model is formed by taking the first three terms of the Taylor-series expansion of the cost function around the $j$-th iteration, as follows:

$$C(\mathbf{Y}_j + \mathbf{p}_j) \approx C(\mathbf{Y}_j) + \mathbf{g}^T(\mathbf{Y}_j) \cdot \mathbf{p}_j + \frac{1}{2} \mathbf{p}_j^T \cdot \mathbf{H}(\mathbf{Y}_j) \cdot \mathbf{p}_j,$$  \hspace{1cm} (4.32)

where $\mathbf{p}_j = \mathbf{Y}_{j+1} - \mathbf{Y}_j$ is the step at $\mathbf{Y}_j$ towards the minimum of the cost function $C(\mathbf{Y})$. The vector $\mathbf{g}(\mathbf{Y})$ is the gradient vector of the cost function $C(\mathbf{Y})$:

$$\mathbf{g}(\mathbf{Y}) \equiv \nabla C(\mathbf{Y}) = [g_n = \frac{\partial C}{\partial Y_n}, n = 1, 2, \ldots, N_p] = \mathbf{J}(\mathbf{Y}) \cdot \mathbf{e}(\mathbf{Y}),$$  \hspace{1cm} (4.33)

where $N_p$ is the number of model parameters, $\mathbf{e}(\mathbf{Y}) = F_M(\mathbf{Y}) - \mathbf{G}$ is the vector of error and $\mathbf{J}(\mathbf{Y})$ is the Jacobian matrix given by the following expression:

$$\mathbf{J}(\mathbf{Y}) = [J_{mn} = \frac{e_m}{T_n}, m = 1, 2, \ldots, N_m; n = 1, 2, \ldots, N_p],$$  \hspace{1cm} (4.34)
where \( N_m \) is the number of measurement data points. The matrix \( H(\mathbf{Y}) \) is the Hessian matrix of the cost function \( C(\mathbf{Y}) \) which is a real symmetric \( N_p \times N_p \) matrix (not necessarily positive-definite) given by:

\[
H(\mathbf{Y}) \equiv \nabla^2 C(\mathbf{Y}) = [H_{n,l} \equiv \frac{\partial^2 C}{\partial Y_n Y_l}, n, l = 1, 2, \ldots, N_p].
\] (4.35)

The minimum of the right-hand side of equation (4.32) is achieved if \( \mathbf{p}_j \) is a minimum of the quadratic function

\[
q(\mathbf{p}) = \mathbf{g}^T(\mathbf{Y}_j) \cdot \mathbf{p} + \frac{1}{2} \mathbf{p}^T \cdot H(\mathbf{Y}_j) \cdot \mathbf{p}.
\] (4.36)

This quadratic function \( q(\mathbf{p}) \) has a stationary point at \( \mathbf{p}_0 \), only if the gradient vector of \( q(\mathbf{p}) \) vanishes at \( \mathbf{p}_0 \), i.e.,

\[
\nabla q(\mathbf{p}_0) = H \cdot \mathbf{p}_0 + \mathbf{g} = 0.
\] (4.37)

Thus the stationary point is the solution to the following set of linear equations:

\[
H \cdot \mathbf{p}_0 = -\mathbf{g}.
\] (4.38)

In the full Newton method, the solution of \( \mathbf{p}_0 \) depends on the conditioning (singular or nonsingular nature) and the definiteness (positive/negative definiteness or indefiniteness) of the Hessian matrix \( H \).

In the HOR method, we use quasi-Newton method which provides cheap updating schemes of the Hessian without directly computing the Hessian.

Let \( \mathbf{s}_j = \nu_j \mathbf{p}_j = \mathbf{Y}_{j+1} - \mathbf{Y}_j \) be the step taken from the \( j \)-th iterate, \( \mathbf{Y}_j \), to obtain the \( (j + 1) \)-th iterate, \( \mathbf{Y}_{j+1} \), where \( \nu_j \) is the step length along the search direction \( \mathbf{p}_j \) to make sure this step sufficiently reduces the cost function, which is determined by a line-search algorithm [20]. The line-search algorithm used in the HOR method is described in detail in appendix C.1.

Expanding the gradient vector \( \mathbf{g} \) about the \( j \)-th iterate \( (\mathbf{Y}_j) \) in a Taylor series, we obtain:
\[ g(\mathbf{Y}_j + \mathbf{s}_j) = g(\mathbf{Y}_j) + \mathbf{H}(\mathbf{Y}_j) \cdot \mathbf{s}_j + \ldots, \]  

(4.39)

or

\[ g_{j+1} = g_j + \mathbf{H}_j \cdot \mathbf{s}_j + \ldots. \]  

(4.40)

Instead of directly computing the Hessian, we use a certain update formulas. The updated Hessian, denoted by \( \mathbf{U}_{j+1} \), is required to satisfy equation (4.40) approximated by the first two terms in the Taylor series expansion, i.e.,

\[ g_{j+1} = g_j + \mathbf{U}_{j+1} \cdot \mathbf{s}_j, \]  

(4.41)

or

\[ \mathbf{U}_{j+1} \cdot \mathbf{s}_j = \mathbf{d}_j = g_{j+1} - g_j. \]  

(4.42)

This condition (4.42) is referred to as the quasi-Newton condition. The updated Hessian, \( \mathbf{U}_{j+1} \), is obtained by updating the previous approximate Hessian, \( \mathbf{U}_j \), to take into account the newly acquired information on the curvature of the cost function (contained in equation (4.42)). The Hessian \( \mathbf{U}_j \) is the approximate Hessian at the beginning of the \( j \)-th iteration which reflects the curvature information that has already been accumulated. \( \mathbf{U}_j \) is used to determine the \( j \)-th Newton search direction through equation (4.38):

\[ \mathbf{U}_j \cdot \mathbf{p}_j = -g_j \quad \text{or} \quad \mathbf{U}_j \cdot \mathbf{s}_j = -\nu_j g_j. \]  

(4.43)

In the HOR method, we use the Broyden symmetric rank-one update formula [12] to update \( \mathbf{U}_{j+1} \) from \( \mathbf{U}_j \). Specifically, since the Hessian matrix is symmetric, \( \mathbf{U}_{j+1} \) is constructed from \( \mathbf{U}_j \) by adding a symmetric matrix of rank one:

\[ \mathbf{U}_{j+1} = \mathbf{U}_j + \mathbf{u}\mathbf{u}^T, \]  

(4.44)

for some vector \( \mathbf{u} \). Substituting into the quasi-Newton condition (4.42), we get:
\[(u^T \cdot s_j)u = d_j - U_j \cdot s_j, \]  
(4.45)

from which we further deduce that:

\[u = \frac{1}{|s_j^T \cdot (d_j - U_j \cdot s_j)|^{1/2}}(d_j - U_j \cdot s_j), \]  
(4.46)

and hence, the update formula for \(U_{j+1}\) is given by:

\[U_{j+1} = U_j + \frac{1}{s_j^T \cdot (d_j - U_j \cdot s_j)}(d_j - U_j \cdot s_j)(d_j - U_j \cdot s_j)^T \]  
(4.47)

where we assume that \(U_j \cdot s_j \neq d_j\) and \(s_j^T \cdot (d_j - U_j \cdot s_j) \neq 0\).

There are also a few rank-two matrix update formulas for the Hessian, see details in appendix C.2.

Although it is possible to start the approximation of the Hessian using simply the identity matrix, in the implementation we prefer using finite-difference approximation through function evaluations to initialize the Hessian. One should note that since \(U_j\) is not the exact Hessian, we are not guaranteed that \(p_j\) is a descent direction for the cost function (4.32). Thus the line search algorithm can fail to return a suitable step if \(U_j\) wanders far from the true Hessian. In this case we reinitialize \(U_j\) by finite-difference approximation through function evaluations.

In the HOR method, given the nonlinear order \(M\) of the wave model, the process of optimizing model parameters stops if one of the following conditions occurs first:

- The reconstruction error \(E\) reaches a prescribed accuracy \(\epsilon^*\), i.e., \(E \leq \epsilon^*\), where \(\epsilon^*\) is a pre-determined \textit{a priori} information that has to be provided by the user.

- The differences between two successive iterates, \(j\)-th and \((j+1)\)-th, of the model parameters are within a prescribed tolerance factor, \(tol\), for example, of
the current iterate:

\[ \sum_{l=1}^{N_p} |Y_{l,j+1} - Y_{l,j}| \leq tol \times \sum_{l=1}^{N_p} |Y_{l,j+1}|. \tag{4.48} \]

- The differences between the cost function at two successive iterates, \( j \)-th and \( (j + 1) \)-th, of the model parameters is within a prescribed tolerance factor, \( tol \), of the cost function at the current iterate:

\[ |C(Y_{j+1}) - C(Y_j)| \leq tol \times C(Y_{j+1}). \tag{4.49} \]

- The number of iterations exceeds a prescribed maximum.
Chapter 5

Verification of the predictable zone and HOR

In this chapter, we illustrate HOR using both synthetic wave fields and laboratory measurements. We consider uni- and multi-directional irregular waves without restrictions on frequency and directional band width. These tests demonstrate the performance of HOR, address the importance of including higher-order nonlinearity in the reconstruction, and assess the usefulness of such nonlinear reconstruction in terms of the space-time domain in which the underlying wave fields can be predicted.

5.1 Reconstruction and forecast of unidirectional waves

5.1.1 Synthetic waves with a single point measurement

We consider here reconstruction of unidirectional waves based on a single point measurement. The original wave elevation \( \tilde{\eta}(x, t) \) is generated by HOS simulation (with \( M=3 \)) from a JONSWAP wave spectrum \( S(\omega) \) with peak-enhancement parameter \( \gamma=3.3 \) and peak frequency \( \omega_p=0.52 \text{ rad/s} \), propagating in the \( +x \) direction. We define the effective wave steepness of the irregular waves to be \( (ka)_e \equiv 4\pi\sigma/\lambda_p \), where \( \sigma \) is the root-mean-square of elevation and \( \lambda_p \) is the peak wave length. We assume the wave energy outside the frequency band of \( \Omega=[\omega_a, \omega_b]=[0.7, 2.0]\omega_p \) is negligible. Note
that in this example, as well as later cases, with \(\gamma = 3.3\), the wave field is not narrow banded. We use \(\zeta(t) \equiv \tilde{\eta}(x = 0, t)\), for \(t/T_p \in [0, 4.17]\) as the measurement.

As an initial test, we consider a very small \((ka) \approx 0.04\). Figure 5-1 displays the point-to-point prediction error \(e(x, t) \equiv |\eta(x, t) - \tilde{\eta}(x, t)|/\sigma\) in the x-t domain obtained with HOR which converges with \(M=1, L_\omega=32\) (with HOS \(N=512\)). The contour of \(e(x, t)\), say defined by \(e(x, t) \leq \mathcal{E}^* = 0.01\), indicates the region(s) of predictability \(\mathcal{P}\). In this case, the wave field is approximately linear, and \(\mathcal{P}\) is close to the linear predictable zone \(\mathcal{P}_L\) obtained in Chapter 2 defined by the parallelogram (with maximum x-t point given by \(x/\lambda_p=1.57\) and \(t/T_p=6.42\)) indicated in figure 5-1. Outside \(\mathcal{P} \approx \mathcal{P}_L\), \(e(x, t)\) is continuous but increases to \(O(1)\) within a distance/time of \(x/\lambda_p, t/T_p \approx O(1)\) from \(\mathcal{P}_L\).

In this case, the linearized \(\eta(x, t)\) can be worked out analytically, see appendix 4.1.2 for the case of two point measurements (which reduces directly to single point measurement case here). The error \(e(x, t)\) from HOR \((M=1)\) is indistinguishable from that using the analytic solution for \(\eta(x, t)\). These results offer a direct numerical verification of \(\mathcal{P}_L\) from Chapter 2 in the case of very small wave steepness.

We now consider a realistic case of moderate \((ka) \approx 0.11\). The JONSWAP spectrum used to generate the high-order \((M=3)\) HOS wave field is otherwise the same as before, and the location/duration of the single point elevation measurement \(\mathcal{M}\) is the same as before.

We apply nonlinear HOR reconstruction with \(N=512\) in HOS, obtaining convergent HOR parameters of \(L_\omega=32\) and \(M=3\) for specified reconstruction tolerance \(\mathcal{E}^* = 0.01\). Figure 5-2 plots the direct comparison of the wave elevation history at the measurement location between the given data and those of the reconstructed wave fields with \(M=1, 2, 3\). Figure 5-3 shows the point-to-point prediction error \(e_{(M)}(x, t) \equiv |\eta_{(M)}(x, t) - \tilde{\eta}(x, t)|/\sigma\) for \(M=1, 2, 3\) in the x-t domain. As a reference, the linear predictable zone \(\mathcal{P}_L\) is also indicated. Note that for this case HOR converges with \(M=3\). We confirm (not shown) that \(e_{(3)}\) and \(e_{(4)}\) are graphically indistinguishable.

From figure 5-2, we see that the reconstructed wave elevation with \(M=1, 2, 3\) all
Figure 5-1: Point-to-point prediction error $e(x,t)$ for an unidirectional wave-field reconstructed by HOR with $L_o=32$, $M=1$ and $N=512$ based on a single point measurement, $\zeta(x=0, t/T_p \in [0, 4.17])$ (with peak period $T_p=12$ s and peak wavelength $\lambda_p=227.72$ m). The region enclosed by the white solid line is the linear predictable zone $P_L$ and the dashed line indicates the measurement duration and location.

Figure 5-2: (a) Comparison of the wave elevation histories at the measurement location ($x=0$) from the reconstructed wave fields using HOR with $M=1$ (---), $M=2$ (•••), and $M=3$ (- - -) and the measurement data of the synthetic nonlinear wave-field (-----), where $\sigma$ is the root-mean-square of the measurement elevation. (b) Close up of the comparison around $t/T_p=2.55$ (enlarged view of the dashed square in (a)).
Figure 5-3: Contours of $e(x,t)$ for the reconstructed/forecasted wave fields with (a) linear, (b) second-order, and (c) nonlinear HOS ($M=3$) wave models. Parallelogram region enclosed by the solid line is the predictable zone $P$ predicted from the linear theory. The parallelogram by the dash line in (c) is the predictable zone obtained by use of the nonlinear wave group velocity.
recover the original data well for most of the measurement time except at wave crests and troughs, where reconstruction with $M=1$ and 2 have an error of up to 18% and 5% respectively. Figure 5-3 further shows that $e_3$ within $\mathcal{P}_L$ is much smaller than $e_1$ and $e_2$. Specifically, for this example, the maximum $e_3$ inside $\mathcal{P}_L$ is less than 3% while $e_1$ and $e_2$ can be as large as 20%. $\mathcal{E}^{\mathcal{P}_L}$ for $M=1, 2, 3$ are 8.5%, 7.9%, and 1.3%, respectively. These results indicate that inclusion of high-order wave effects is of importance in the reconstruction of the nonlinear wave field within the whole predictable zone.

Note that in figure 5-3(c) where the reconstruction converges, the actual (nonlinear) predictable zone $\mathcal{P}_{NL}$, which corresponds to approximately the blue region where $e(x, t) \leq 0.01$, is actually larger than $\mathcal{P}_L$ due to the nonlinear effects. Since there is no closed-form expression for the wave field based on fully nonlinear wave theory, it is hard to derive the closed-form expression for $\mathcal{P}_{NL}$. We provide an estimate of $\mathcal{P}_{NL}$ by simply extending the linear predictable zone theory to include nonlinear wave effects on the wave group velocity. In nonlinear wave-field evolution, the wave dispersion relation can be affected by resonant wave-wave interactions, wave group interactions, and bound wave effects depending on the time/space scale of the problem considered. For short-time phase-resolved wave-field reconstruction, the second-order bound wave effects are of importance. For a unidirectional wave field with $N_\omega$ free wave components, the second-order nonlinear dispersion relation takes the form ([58]; [43]):

$$\omega_n = \sqrt{gk_n[1 + \frac{1}{2}(k_nA_n)^2 + \sum_{\ell=1}^{n-1}(\frac{k_n}{k_\ell})^{1/2}(k_\ell A_\ell)^2 + \sum_{\ell=n+1}^{N_\omega}(\frac{k_n}{k_\ell})^{3/2}(k_\ell A_\ell)^2]} \quad (5.1)$$

where $k_i < k_{i+1}$ is assumed. By definition of the wave group velocity, we have

$$C_n = \frac{1}{2}(\frac{g}{k_n})^{1/2}[1 + \frac{5}{2}(k_nA_n)^2 + 2\sum_{\ell=1}^{n-1}(\frac{k_n}{k_\ell})^{1/2}(k_\ell A_\ell)^2 + 4\sum_{\ell=n+1}^{N_\omega}(\frac{k_n}{k_\ell})^{3/2}(k_\ell A_\ell)^2]. \quad (5.2)$$

As (5.2) indicates, the second-order locked wave interactions generally increases the group velocity of each wave component in the wave field. In this example, the linear dispersion relation gives the minimum (maximum) group velocity of 4.62 m/s (13.61
m/s), while the second-order dispersion relation gives 5.15 m/s (13.93 m/s). As a result, the size of predictable zone becomes larger when second-order effects are accounted for, as shown in figure 5-3(c) where a quantitative comparison between the linear and second-order predictable zone is made. Hereafter we use $\mathcal{P}_S$ to denote the nonlinear predictable zone considering the second-order effects. We can see that in this example $\mathcal{P}_S$ estimates $\mathcal{P}_{NL}$ better than $\mathcal{P}_L$.

From the HOR reconstructed wave fields with different $(ka)_e$, and by looking at the converged $e(x,t)$ we estimate that $\mathcal{P}_{NL} - \mathcal{P}_L$ increases approximately linearly with increasing $(ka)_e$. This can be shown more clearly by using $\mathcal{P}_S$ to estimate $\mathcal{P}_{NL}$. Figure 5-4(a) plots the ratio of the volume of $\mathcal{P}_S$ and $\mathcal{P}_L$, denoted as $\mathcal{V}_{S}$ and $\mathcal{V}_{L}$ respectively, as a function of $(ka)_e$. It shows $\mathcal{V}_{S}/\mathcal{V}_{L}$ increases approximately linearly as $(ka)_e$ increases. We point out that for an irregular wave field with effective wave steepness of 0.19, the local wave steepness in the wave field can be as large as 1.0 in many places. In this case, $\mathcal{V}_{S}$ is greater than $\mathcal{V}_{L}$ about 20%. Figure 5-4(b) further shows the distribution of $e(x,t)$ at time $t/T_e=2.08$ (the midpoint of measurement duration) for the case of $(ka)_e=0.19$, when the reconstruction converges using HOR with $M=4$. At this time, $\mathcal{P}_S$ is larger than $\mathcal{P}_L$ for about 14%. This difference is most significant near the starting and ending time of the predictable zone, which leads to a total volume difference of 20%.

Based on these results, we see that for (very) small wave steepness $(ka)_e \rightarrow 0$, $\mathcal{P}_{NL} \rightarrow \mathcal{P}_L$. As wave steepness increases, $\mathcal{P}_{NL}$ contains and is greater than $\mathcal{P}_L$ ($\mathcal{P}_{NL} \supseteq \mathcal{P}_L$), with $\mathcal{P}_{NL} - \mathcal{P}_L$ increasing approximately linearly with increasing $(ka)_e$.

Note that although only wave elevation is provided as measurement, with reconstruction, we obtain other wave kinematics such as velocities. Figure 5-5 shows the comparison of reconstructed horizontal velocity profile at the measurement location under a wave crest using different nonlinear order in HOR versus the original data, for the case of $(ka)_e=0.11$. We see that, at this specific time, the deviation of linear and second-order reconstruction of velocity from the original data is significant, especially near the free surface, while third-order reconstruction agrees perfectly with the original data. This highlights again the importance of including higher-order effects in
Figure 5-4: (a) The ratio of the volume $V_S/V_L$ of second-order and linear predictable zone as a function of the effective wave steepness $(ka)_e$ (other spectrum parameters are kept the same in the generation of synthetic high-order $(M=5)$ HOS wave fields). (b) Distribution of $e(x, t)$ at $t/T_p = 2.08$ for wave field with $(ka)_e = 0.19$ when the reconstruction converges using HOR with $M=4$, where $P_L$ and $P_S$ indicate the linearized and nonlinear (considering second-order effects) predictable zone.

As further illustration, we use synthetic high-order $(M=5)$ HOS wave fields of different wave steepness (generated by the same JONSWAP spectrum as above) to validate HOR and examine the convergence of reconstruction. The location/duration of the single point elevation measurement $\mathcal{M}$ is the same as before. Table 5.1 shows the prediction error $\mathcal{E}^P$ as a function of $M$ in HOR for different wave steepness, where $P_S$ is used as an estimate for actual $P$. It further confirms that as the wave steepness increases, higher order of reconstruction is needed in order for the prediction error to reach the same level of reconstruction tolerance. Based on our tests, the reconstructed wave field always converges to the underlying wave field within the predictable zone for wave steepness up to the limit of HOS simulation.
Figure 5-5: Comparison of reconstructed versus original (synthetic) horizontal velocity along the depth under wave crest \( \left( t/T_p = 2.56 \right) \) at measurement location \( (x=0) \) for the case of \( (ka)_e = 0.11 \) (cf. figure 5-2): reconstruction using \( M=1 \) (---), \( M=2 \) (· · ·), \( M=3 \) (——) and original data (· · ·).

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Table 5.1: Prediction error \( \mathcal{E}^P \) as function of \( M \) in HOR for the reconstruction of the synthetic unidirectional high-order \( (M=5) \) HOS wave fields with different \( (ka)_e \) based on a single point measurement. \( L_w = 32, N = 512, \mathcal{E}^* = 0.01, \) and \( \mathcal{E} = \mathcal{E}^*/2 \) are used in HOR. The symbol "-" means \( \mathcal{E}^P(M) \simeq \mathcal{E}^P(M - 1) \).
5.1.2 Synthetic waves with multiple point measurements

We now consider reconstruction of synthetic unidirectional wave fields based on multiple point measurements. The wave fields with \((ka)_e=0.03\) and 0.12 are generated from a JONSWAP spectrum \(S(\omega)\) with same parameters as in §5.1.1 by HOS simulations with \(M=3\). Without loss of generality, we consider two point measurements at \(\xi_1/\lambda_p=0\) and \(\xi_2/\lambda_p=4.39\). The data for both measurements is taken in the duration of \(t/T_p \in [0, 25]\).

For the wave field with \((ka)_e=0.03\), we can again use the analytic reconstruction approach, which can be obtained in the context of linear wave theory. The detailed solution procedure is described in appendix 4.1.2. We use 512 uniform data points for both measurements. Figure 5-6 shows \(e(x, t)\) in \(x-t\), where \(\mathcal{P}_L\) is indicated. It confirms again that for very small wave steepness, \(\mathcal{P}_L\) estimates the actual predictable zone accurately, which corresponds to the dark blue region where \(e(x, t) \leq 0.01\). We see that outside \(\mathcal{P}_L\), \(e(x, t)\) is continuous but increases to \(O(1)\) within a distance/time of \(x/\lambda_p, t/T_p \approx O(1)\) from \(\mathcal{P}_L\).

HOR is used to reconstruct the wave field with \((ka)_e=0.12\). We use \(N=512\) and obtain \(L_\omega = 60\) and \(M = 3\) in order for \(\mathcal{E} \leq \mathcal{E}^*=0.01\). To illustrate the effect of \(L_\omega\) on the prediction, figure 5-7 shows the prediction error \(\mathcal{E}^P\) using fixed \(M = 3\) as a function of \(L_\omega\), where \(\mathcal{P}_S\) is used to estimate the actual \(\mathcal{P}\). As \(L_\omega\) increases, \(\mathcal{E}^P\) first decreases abruptly, and then after a turning point \(L_\omega^c \sim 60\), \(\mathcal{E}^P\) decreases slowly and converges. In this case, \(\mathcal{E}^P\) is about 1% for \(L_\omega=60\) which is the same level as the reconstruction tolerance, and \(L_\omega=60\) is selected as the optimal \(L_\omega\) by HOR.

Figure 5-8 displays \(e(x, t)\) in \(x-t\) for the nonlinear reconstruction. \(\mathcal{P}_S\) is also indicated to estimate the actual predictable zone, which corresponds to the blue region where \(e(x, t) \leq 0.01\). Outside \(\mathcal{P}_S\), \(e(x, t)\) increases to \(O(1)\) within a distance/time of \(x/\lambda_p, t/T_p \approx O(1)\) from \(\mathcal{P}_S\). The results in figures 5-6 and 5-8 both confirm the theoretical result that the predictable zone based on the combination of two point measurements (within a certain distance) is larger than the simple union of the predictable zones associated with each individual measurement.
Figure 5-6: $e(x, t)$ in the analytically-reconstructed wave field ($(ka)_e=0.03$) based on two point measurements (at $\xi_1/\lambda_p=0$ and $\xi_2/\lambda_p=4.39$ with time duration $t/T_p \in [0, 25]$). Parallelogram region enclosed by the white solid lines is the linear predictable zone $P_l$ based on the combination of two measurements. Parallelogram regions enclosed by dashed line and dot line are the linear predictable zones associated with each individual measurement.
Figure 5-7: The prediction error ($\mathcal{E}_P$) as a function of the number of free wave components ($L_\omega$) used in nonlinear ($M=3$) wave reconstruction based on two probes.
Figure 5-8: $e(x, t)$ in the numerically reconstructed nonlinear wave field ($(ka)_e=0.12$) based on two point measurements (at $\xi_1/\lambda_p=0$ and $\xi_2/\lambda_p=4.39$ with time duration $t/T_p \in [0, 25]$). Parallelogram region enclosed by solid lines is the nonlinear predictable zone (considering second-order group velocities) $P_S$ based on the combination of two measurements. Parallelogram regions by dashed lines are the predictable zones associated with each individual measurement.
Figure 5-9: Experimental set-up in the long wave tank at Texas A&M University: the glass-walled flume has a length of 36.1 m, a width of 0.91 m and a height of 1.22 m and is equipped with a permeable wave absorbing beach at downstream; wave generation is provided by a dry-back, hinged flap wavemaker capable of producing regular and irregular waves with period ranging from 0.25 s to 4.0 s and maximum height of 0.254 m. The water depth in the tank is \(h = 0.80\) m.

5.1.3 Comparison to two-dimensional wave tank experiments

So far, we have considered theoretical Stokes waves and synthetic wave fields. It is important to demonstrate HOR and verify the predictable zone theory for realistic physical wave field which contains measurement errors. Here we first consider two-dimensional experiments conducted in the narrow wave tank at Texas A&M University. Later in §5.2.3 we discuss a three-dimensional wave basin experiment. The measurement data are all provided by our collaborator at Texas A&M University through personal communication [47].

Setup of experiment

The experiment set-up is described schematically in figure 5-9, where \(x\) is the horizontal coordinate positive in the direction of wave propagation with \(x=0\) at the wavemaker and \(z\) positive upwards.

The irregular waves are generated from a JONSWAP spectrum with \(\gamma=1\) (Pierson-Moskowitz spectrum) and peak wavelength \(\lambda_p=1.25\) m. Specifically, two irregular wave trains with significant wave heights of \(H_s=0.04\) m and 0.09 m are generated...
to investigate the effects of nonlinearity. Resistant-type surface-piercing wave gauges are deployed to record the free-surface elevation at three locations \(x=7.0\) m, 8.0 m, and 9.0 m. A laser Doppler velocimeter (LDV) system is used to measure the wave kinematics at the location \(x=8.0\) m for seven vertical positions below the still water level (SWL) and two vertical positions above the SWL. The optic cable LDV measures the three velocity components simultaneously with high spatial resolution. As a non-intrusive measurement, the LDV measurement has been proved to be of high accuracy in other experiments (e.g. [97]; [1]). For both tests, the time series of wave elevation and velocity are recorded for about 250 s.

**Comparison of wave elevation**

The effective wave steepness \((ka)_{e}\) for the two irregular wave trains tested are 0.10 and 0.23 respectively. We choose partial records at \(x=8.0\) m which contain the steepest local waves to reconstruct these wave fields. Specifically, for the case of \((ka)_{e}=0.10\), the elevation record in the duration of \(112.24 \text{ s} \leq t \leq 118.20 \text{ s}\) is used as the given measurement. For the case of \((ka)_{e}=0.23\), the record in the duration of \(116.75 \text{ s} \leq t \leq 124.88 \text{ s}\) is used. In both cases, the records contain \(7 \sim 8\) dominant wave periods. In the wave reconstruction, we obtain \(L_{\omega}=22\), and \(M=3\) for \(\varepsilon < \varepsilon^*=0.01\). For the HOS simulation, we use \(N=512\). The computation domain in HOS contains about 50 dominant waves and about 500 shortest waves.

Figure 5-10 and 5-11 show the comparison of the measured and reconstructed wave elevation history at the measurement location (\(x=8.0\) m). The reconstructed wave fields are obtained by using HOR with \(M=1, 2,\) and 3. It is seen that the reconstructed wave records all agree perfectly with the measurement data.

**Comparison of wave kinematics**

Although the reconstructed wave fields with the use of linear, second-order, and third-order wave models all match the wave elevation record at the measurement location, the spatial wave profiles and detailed wave kinematics in these wave fields should differ owing to the effects of wave nonlinearity. We here investigate the importance
Figure 5-10: Comparison of reconstructed versus measured free-surface elevation history at the measurement location \((x=8.0 \text{ m})\) for irregular wave trains with \((ka)_c=0.1\). Plotted are the experimental record (---), HOR reconstruction with \(M=1\)(---), \(M=2\) (· · ·), and \(M=3\) (· · ·) based on HOS simulations (with \(N=512\)).

Figure 5-11: Comparison of reconstructed versus measured free-surface elevation history at the measurement location \((x=8.0 \text{ m})\) for irregular wave trains with \((ka)_c=0.23\). Plotted are the experimental record (---), HOR reconstruction with \(M=1\)(---), \(M=2\) (· · ·), and \(M=3\) (· · ·) based on HOS simulations (with \(N=512\)).
of the inclusion of wave nonlinearity in the reconstruction of wave kinematics.

Figure 5-12 and 5-13 show the comparison of the instantaneous horizontal velocity profile between the reconstructed and experimental data at the measurement location \((x=8.0 \text{ m})\) under a wave crest for two different wave steepness \((ka)_{e}=0.1\) and \((ka)_{e}=0.23\). We can see that for both cases, the third-order nonlinear reconstruction \((M=3)\) agrees well with the measurement, while the lower-order reconstructions largely overestimate the velocity especially in the region near the free surface. The fourth-order reconstructions are also shown in the figure, which are very close to the third-order results, confirming that third-order reconstructions are sufficient for these cases.

For the case of \((ka)_{e}=0.1\), the second-order reconstruction compares relatively well while the linear reconstruction over-predicts the velocity by about 40% near the free surface. For the case of \((ka)_{e}=0.23\), the second-order reconstruction over-predicts the velocity by about 25% and the linear reconstruction over-predicts by about 150% near the free surface. Note that since the experimental measurement at the free surface cannot be reliably obtained, the plots do not show the velocity profiles up to the exact free surface. For \((ka)_{e}=0.1\) (or 0.23), the crest wave elevation reaches 0.022 m (or 0.061 m) at time \(t=114.725 \text{ s} (120.075 \text{ s})\) (cf. figure 5-10 and 5-11). Note that the overestimation of the velocity by the low-order theories is most significant under the wave crest and is weakened at other time.

The velocity reconstruction is directly related to the determination of the wave dynamics that is of significance in practical applications. To illustrate the importance of including nonlinear effects in the reconstruction of wave dynamics, figure 5-14 shows the horizontal acceleration profiles along depth at the measurement location \((x=8.0 \text{ m})\) under the crest (at time \(t=120.075 \text{ s}\)) for the case of \((ka)_{e}=0.23\). We only compare the results from the three reconstructed wave fields \((M=1,2,3)\) as experimental data of acceleration is not available. We can see that the overestimation of the horizontal acceleration is even more significant by the linear and second-order reconstructions. Specifically, the linear (or second-order) reconstruction of the maximum acceleration (near the free surface) is about 7 (or 2.5) times that reconstructed by third-order
Figure 5-12: Comparison of reconstructed versus experimental horizontal velocity along the depth under wave crest at the measurement location (x=8.0 m): experimental measurement (▲), M=1 prediction (---), M=2 prediction (· · ·), M=3 prediction (-----) and M=4 prediction (· · ·) based on HOS simulation with N=512 spectral modes. The figure shows the horizontal velocity profile at time \( t=114.725 \) s and steepness \( (ka)_{c}=0.1 \).
Figure 5-13: Comparison of reconstructed versus experimental horizontal velocity along the depth under wave crest at the measurement location \( (x=8.0 \, \text{m}) \): experimental measurement \((\boldsymbol{\Delta})\), \( M=1 \) prediction \((-\cdot-\cdot-\cdot\cdot\cdot)\), \( M=2 \) prediction \((\cdot\cdot\cdot)\), \( M=3 \) prediction \((\ldots)\) and \( M=4 \) prediction \((-\cdot\cdot\cdot)\) based on HOS simulation with \( N=512 \) spectral modes. The figure shows the horizontal velocity profile at \( t=120.075 \, \text{s} \) and \((ka)_e=0.23\).
wave models. From the Morrison's formula, these would lead to a significant overestimation of the hydrodynamic wave force on an object by the linear and second-order reconstructions.

From above results, we see that the first- or second-order wave model may obtain a proper reconstruction of the wave elevation, which is provided as the measurement in this case, but we need to include higher-order nonlinear effects in the reconstruction in order to reconstruct well wave kinematics, such as velocity and acceleration, especially for steep waves. The reason that the linear reconstruction cannot properly reconstruct the wave kinematics is because linear reconstruction mistakenly treats all wave components in the wave field as free wave components. As we mentioned in §1.1, in the nonlinear wave field, there exist both free wave components and locked waves.
(bound waves and resonant waves). In the linear wave reconstruction, all components are treated as free waves. However, unlike free waves, wavenumbers and associated frequencies of the locked waves do not satisfy the dispersion relation. Thus, the resulting reconstructed wave kinematics is incorrect. The wave kinematics (particularly near the free surface) is overestimated since the wavenumbers of the locked waves are amplified when they are incorrectly treated as free waves. The second-order reconstruction improves the accuracy of the reconstructed wave kinematics as it takes into account the second-order locked waves. As wave steepness increases, however, the second-order reconstruction may not be sufficiently effective. In this case, higher-order reconstruction is needed.

**Forecast of the reconstructed wave field**

The predictable zone theory in Chapter 2 shows that the wave field at downstream (upstream) of the probe can be forecasted (hindcasted). To verify this theory, the wave elevations at both downstream and upstream of the measurement (used in wave reconstruction), $x=9.0$ m and $7.0$ m are also measured in the experiment. We compare these measurements with the predictions based on the single point measurement at $x=8.0$ m. Figure 5-15 shows the comparisons of the wave elevations at $x=8.0$ m, $9.0$ m and $7.0$ m for the case of $(ka)_e=0.23$.

We apply Fourier analysis on the measurement data and find that the wave energy is mainly contained in the frequency range $\omega \in [5, 10]$ rad/s. Note that for this physical experiment, the underlying wave field is unknown and thus the actual nonlinear predictable zone $P_{NL}$ cannot be obtained by comparing the converged reconstructed field with the underlying one. In this case, $P_L$ provides a conservative estimate for $P_{NL}$. From the linear predictable zone theory, we derive that at $x=8.0$ m, the predictable time duration is the same as the measurement duration ($t \in [116.75, 124.88]$s). At $x=9.0$ m, the wave motion can be predicted for $t \in [118.79, 125.90]$s, in which $t \in [124.88, 125.90]$s corresponds to forecast. Figure 5-15(b) shows that the prediction agrees well with the measurement for a duration a little longer than $[118.79, 125.90]$s, as $P_{NL}$ is a little larger than $P_L$. Outside the predictable
time duration, the error increases rapidly with evolution time. These confirm the theoretical linear predictable zone based on a single point measurement.

The linear predictable zone theory also indicates that at \( x = 7.0 \) m, the wave motion can be predicted for \( t \in [115.73, 122.84] \) s, in which \( t \in [115.73, 116.75] \) s corresponds to hindcast. The satisfactory comparison between the measurement and prediction for a duration a little larger than \([115.73, 122.84]\) s, shown in figure 5-15(c) further confirms the predictable zone theory.

5.2 Reconstruction and forecast of multidirectional waves

5.2.1 Synthetic waves with multiple point measurements

We consider reconstruction of a synthetic multidirectional nonlinear wave field based on multiple point measurements. The wave field is generated by the HOS simulation (with \( M = 3 \)) from a directional JONSWAP spectrum \( S(\omega, \theta) = S(\omega)D(\theta) \) with \( \gamma = 3.3, \omega_p = 1.57 \) rad/s, \( (ka)_c = 0.13 \), \( \Omega = [0.32, 1.6] \omega_p \) and directional spreading angle \( \theta \in [-\pi/6, \pi/6] \). The angular spreading function takes a form of \( D(\theta) = \cos(\pi \theta / (2 \Theta_0))^2 / \Theta_0 \), where \( \Theta_0 = \pi / 6 \). Seven point measurements are placed at fixed locations (see figure 5-16) with same measurement time duration \( t/T_p \in [0, 10] \). In the reconstruction, we use \( N = 1024 \times 1024 \) for HOS simulations and we obtain \( L_\omega = 40, L_\theta = 7 \) and \( M = 3 \) for \( \mathcal{E} < \mathcal{E}^* = 0.01 \). Figure 5-17 shows \( e(x, t) \) at two representative time \( t/T_p = 0 \) (reconstruction) and 11 (forecast). For simplicity, \( \mathcal{P}_L \) is also indicated as an estimate of \( \mathcal{P}_{NL} \). Excellent agreement between the reconstructed and the underlying wave field is obtained inside \( \mathcal{P}_L \), which validates the predictable zone theory developed in Chapter 2, while outside \( \mathcal{P}_L \), \( e(x, t) \) increases to \( O(1) \) within a distance of \( O(1) \lambda_p \) from \( \mathcal{P}_L \).

As an example to illustrate the effect of \( L_\theta \) on the prediction, figure 5-18 shows the change of prediction error \( \mathcal{E}^P \) with \( M = 3 \) as \( L_\theta \) increases from 1 to 7, where \( D \) is chosen as the predictable area at \( t/T_p = 0 \) shown in figure 5-16. It shows that \( \mathcal{E}^P \) decreases rapidly as we increase \( L_\theta \) as expected. In this example, there are only 7 measurement
Figure 5-15: Comparison of the wave elevation histories in the reconstructed wave field (with $(ka)_e=0.23$) with experimental measurements at locations of (a) $x=8.0$ m, (b) $x=9.0$ m, and (c) $x=7.0$ m. The plotted are the experimental measurements (—) and nonlinear $(M=3)$ wave reconstruction (—-). The dashdot line (—-—) indicates the predictable time duration at each location, predicted by the linear predictable zone theory. $[\tau_1, \tau_2]$ represents the measurement time duration at $x=8.0$ m, where $\tau_1=116.75$ s and $\tau_2=124.88$ s. In (a), $[\tau_1, \tau_2]$ is also the predictable time duration. In (b), wave forecast for $t \in [\tau_2, t_2]$ is obtained, and the prediction error $\varepsilon^D=10.29\%$, where $D$ corresponds to the predictable time duration $[t_1, t_2]$ at this location, with $t_1=118.79$ s and $t_2=125.90$ s. In (c), wave hindcast for $t \in [t_1, \tau_1]$ is obtained, and the prediction error $\varepsilon^D=10.95\%$, where $D$ corresponds to $[t_1, t_2]$ with $t_1=115.73$ s and $t_2=122.84$ s.
Figure 5-16: Distribution of the point-to-point prediction error $e(x,t)$ for the reconstructed multidirectional nonlinear wave field at time $t/T_p=0$. The wave reconstruction is based on seven probes located at $(x/A_p, y/A_p)=(0, 0)$, $(1.2, 0)$, $(1.2, 0.8)$, $(1.2, -0.8)$, $(2.4, 0)$, $(2.4, 1.6)$, $(2.4, -1.6)$ with same measurement time duration $t/T_p \in [0, 10]$, where $A_p=25$ m and $T_p=4$ s. The regions encircled by red lines are the theoretical predictable zones. The bullets denote the probe locations.

locations. Further increasing $L_\theta$ will cause non-uniqueness of the reconstruction and thus we finally choose $L_\theta=7$.

5.2.2 Synthetic waves with whole-area measurements

In practice, whole-area wave measurements such as remote sensing technology-based wave elevation measurements [76, 88] are also commonly used. Here we consider reconstruction of a synthetic multidirectional nonlinear wave field based on whole-area measurements. The wave field is generated from the same directional JONSWAP spectrum as in §5.2.1. We assume the wave elevation in a square region $A$: $\{15.4 \leq x/\lambda_p \leq 25.6, 15.4 \leq y/\lambda_p \leq 25.6\}$ with 256×256 uniform data points at time $t=0$ is given as the whole-area measurement.
Figure 5-17: Distribution of the point-to-point prediction error $e(x, t)$ for the reconstructed multidirectional nonlinear wave field at time $t/T_p=11$. The wave reconstruction is based on seven probes located at $(x/\lambda_p, y/\lambda_p)=(0, 0), (1.2, 0), (1.2, 0.8), (1.2, -0.8), (2.4, 0), (2.4, 1.6), (2.4, -1.6)$ with same measurement time duration $t/T_p \in [0, 10]$, where $\lambda_p=25$ m and $T_p=4$ s. The regions encircled by red lines are the theoretical predictable zones. The bullets denote the probe locations.
We first apply the analytic linear reconstruction approach based on the double Fourier transform to reconstruct the wave field. Figure 5-19 and 5-20 show $e(x, t)$ at time $t/T_p = 0$ (reconstruction) and $t/T_p = 3$ (forecast). At $t=0$, the original wave field in the measured area is recovered well. However, the wave field outside the measured area but within $\mathcal{P}_L$ is not properly recovered. We note that in the predictable zone theory, the predictable zone is determined based on the requirement that the complex wave amplitude $A(k, \theta)$ is completely determined from the given data. The analytic reconstruction approach used here does not ensure the requisite resolution of $A(k, \theta)$ in $\theta$ since the wave field is assumed to be doubly periodic in $x$ and $y$ directions. This results in larger errors in the region outside the measurement area (but within $\mathcal{P}_L$) at $t=0$. Similar behaviors are obtained in the forecasted wave field, as indicated by the distribution of $e(x, t)$ at $t/T_p = 3$ in figure 5-20.

As comparison, we then use HOR to reconstruct the nonlinear wave field. In the reconstruction, we use $N=1024\times1024$ for HOS simulations and obtain $L_\omega=40$, 

Figure 5-18: The change of prediction error $E^D$ with respect to $L_\theta$ for the reconstruction of multidirectional nonlinear wave field based on seven probes, where $\mathcal{D}$ represents the predictable area at $t/T_p=0$ shown in figure 5-16.
Figure 5-19: The point-to-point prediction error $\epsilon(x, t)$ between the analytically reconstructed linear wave field and the original linear wave field at $t/T_p=0$. The wave reconstruction is based on elevation data (in the region marked by dashlines) at $t=0$. The region encircled by solid lines is the theoretical predictable zone.

$L_\theta = 11$ and $M=3$ for $\mathcal{E} < \mathcal{E}^* =0.01$. Figure 5-21 and 5-22 show $\epsilon(x, t)$ at time $t/T_p = 0$ (reconstruction) and $t/T_p =3$ (forecast). Unlike the analytic linear approach, the numerical reconstruction method accurately predicts the nonlinear wave field inside the entire $\mathcal{P}_L$ at $t/T_p=0$ and $t/T_p=3$. These results verify the predictable zone theory and the efficacy of HOR for whole-area measurements.

### 5.2.3 Comparison to three-dimensional wave basin experiments

To further validate HOR for three-dimensional wave fields, we apply HOR to reconstruct a steep Bull’s Eye wave field that is created in a wave basin. The Bull’s Eye wave field is formed by a group of waves, coming from different directions and focusing at one spatial location. It is often used to study steep-wave interactions with structures in laboratory experiments [89]. The measurement data used here are from the
Figure 5-20: The point-to-point prediction error $e(x, t)$ between the analytically reconstructed linear wave field and the original linear wave field at $t/T_p=3$. The wave reconstruction is based on elevation data (in the region marked by dashlines) at $t=0$. The region encircled by solid lines is the theoretical predictable zone.
Figure 5-21: The point-to-point prediction error $e(x, t)$ between the reconstructed nonlinear wave field and the original nonlinear wave field at $t/T_p=0$. The wave reconstruction is based on elevation data (in the region marked by dashlines) at $t=0$. The region encircled by solid lines is the theoretical predictable zone.
Figure 5-22: The point-to-point prediction error $e(x,t)$ between the reconstructed nonlinear wave field and the original nonlinear wave field at $t/T_p=3$. The wave reconstruction is based on elevation data (in the region marked by dashlines) at $t=0$. The region encircled by solid lines is the theoretical predictable zone.
Figure 5-23: Layout of wave probes used in Bull's Eye wave experiment: 13 probes whose data are used for wave reconstruction (▲) and 4 probes whose data are used for wave reconstruction/forecast validation (○).
experiments conducted in the Offshore Technology Research Center (OTRC) wave basin [54]. In the experiment, time series of wave elevation at 17 locations around the wave-focusing center are measured. The layout of the measurement positions is sketched in figure 5-23. The specific positions of the measurements are given in table 5.2. The waves generated in the experiment have a peak period of 1.79 s (corresponding to a wavelength of 5.0 m) and a wave height of 0.211 m at the focusing point (corresponding to a wave steepness of 0.13). For detailed setup information of the experiments, see [54].

Measurements at 13 out of 17 locations are used for wave reconstruction. The other 4 measurements (at downstream) are used for comparisons to verify the efficacy of HOR. For wave-field reconstruction, we use wave records of same time duration $T=7.14s$ that is about 4 dominant wave periods and we set the record starting time as $t=0$. The estimated frequency and direction bands are $[\omega_a, \omega_b] = [2.650, 4.417]$ rad/s and $[\theta_a, \theta_b] = [-\pi/4, \pi/4]$. The optimized model parameters in HOR are: $L_\omega=12$, $L_\theta=11$ and $M=3$ for $E < E^*=0.01$. In the HOS simulation, we use $N=256 \times 256$. The linear predictable time range $[t_1, t_2]$ at each measurement location can be obtained according to the predictable zone theory in Chapter 2. They are shown in table 5.3. Note that $t < 0$ corresponds to hindcast while $t > T$ corresponds to forecast. It can be seen that at position $b_3$, the wave field can be forecasted up to $t=9.51$ s, which corresponds to a forecast of about 1.32 dominant wave period.

Figure 5-24 shows the comparison of the reconstructed wave elevation with the experimental data at representative measurement locations. Excellent agreement between the predictions and the experimental data in the linear predictable zones is

<table>
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<th>probe</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$b_1$</th>
<th>$b_2$</th>
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<td>2.286</td>
<td>2.032</td>
<td>2.413</td>
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</tr>
<tr>
<td>y (m)</td>
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<td>0.0</td>
<td>0.222</td>
<td>-0.222</td>
<td>-1.321</td>
<td>-1.321</td>
<td>-1.100</td>
<td>-1.544</td>
</tr>
</tbody>
</table>

Table 5.2: Coordinates of the probe locations in the Bull’s Eye wave experiment.
Figure 5-24: Comparisons of the reconstructed wave elevation (···) using HOR with $M=3$ with wave-basin measurement (——) at measurement locations: $r_1$, $f_2$, $l_3$, $c_4$, $b_3$. The linear predictable time range is indicated by —·—. The measurement duration $[0, T]$ in the 13 measurements used for reconstruction is indicated by the arrows.
Table 5.3: Predictable time range \([t_1, t_2]\) at each probe position determined by the predictable zone theory.

obtained for all measurements used and not used in wave-field reconstruction.

Finally we remark that for all the tests we perform, we find that the number of HOS evaluations \(N_{\text{eval}}\) is linearly proportional to \(L\) and \(M\). For this three-dimensional case with 13 point measurements each containing \(N_d \sim 4\) dominant wave periods, HOR uses \(L_{\omega}=12, L_{\theta}=11, M=3\) and \(N=256 \times 256\). The entire process requires \(O(1)\) hours using 16 processor cores on a Microway NumberSmasher computing cluster [67].

5.3 Optimal deployment of multiple point measurements

To validate the theoretical analysis of optimal deployment of multiple point measurements developed in Chapter 3, we consider a synthetic nonlinear multidirectional wave field generated by the HOS simulation (with \(M=3\)) from the same directional JONSWAP spectrum as in §5.2.1. As illustration, we assume that there are eight point measurements placed at fixed locations that provide wave elevation measurements in the time duration \(t/T_p \in [0, 4]\). According to the theoretical analysis in Chapter 3, we shall deploy as many measurements as possible in the central propagation direction (\(\theta=0\) in this case) with a uniform separation distance of \(C_{\text{min}}T\) where \(C_{\text{min}}\) is the minimum group velocity of the wave components in the wave field and \(T\) is the measurement time duration. In practice, for finite number \((L_{\omega}, L_{\theta})\) of wave components over \(\omega \in \Omega\) and \(\theta \in \Theta\), we can place a few probes at prescribed locations to avoid the situation that all measurements are simultaneously placed at the node.
points of standing waves formed by some wave components, as discussed in §4.2.3.

We consider two different deployment and compare the resultant prediction errors. In both cases, 3 fixed point measurements are placed at \((x/\lambda_p, y/\lambda_p) = (0, 0), (1.6, 0.8)\) and \((1.6, -0.8)\). In the optimal deployment shown in figure 5-25, the other 5 point measurements are located along the center propagation direction \(\theta = 0\) with the optimal separation distance, while in the random deployment shown in figure 5-26, the other 5 point measurements are located randomly.

We use HOR to reconstruct the nonlinear wave field and obtain \(e(x, t)\). In the reconstruction, we obtain \(L_\omega = 61, L_\theta = 7\) and \(M = 3\) for \(E < 0.01\), and we use \(N = 2048 \times 2048\) for HOS simulations. Figure 5-25 and 5-26 shows the distribution of \(e(x, t)\) at a sample time \(t/T_p = 2.5\) for the optimal and random deployment, respectively. For both cases, the original wave field is reconstructed well in the theoretical predictable zone, but the optimal deployment leads to a larger predictable zone compared to the random deployment.
Figure 5-25: The point-to-point prediction error $e(x,t)$ between the reconstructed and original nonlinear wave field at $t/T_p=2.5$ based on eight point measurements with the same measurement time duration $t/T_p \in [0,4]$. Measurement locations $(x/\lambda_p,y/\lambda_p)=(0,0), (1.6,0.8), (1.6,-0.8), (1.2,0), (2.4,0), (3.6,0), (4.8,0), (6.0,0)$. These locations are marked by bullets. The region encircled by red solid lines is the theoretical linear predictable zone.
Figure 5-26: The point-to-point prediction error $e(x, t)$ between the reconstructed and original nonlinear wave field at $t/T_p = 2.5$ based on eight point measurements with the same measurement time duration $t/T_p \in [0, 4]$. Measurement locations $(x/\lambda_p, y/\lambda_p) = (0, 0), (1.6, 0.8), (1.6, -0.8), (0.4, 0.4), (0.6, -0.8), (1.2, 0.8), (1.8, -1.2), (2.2, 0.6)$. These locations are marked by bullets. The region encircled by red solid lines is the theoretical linear predictable zone.
Chapter 6

Phenomenological wave breaking modeling

In the nonlinear wave reconstruction using HOR, in order to properly reconstruct and forecast steep wave field that may contain wave breaking events in the wave-field evolution, it is crucial for the nonlinear evolution engine (HOS) to have the ability of simulating wave breaking effects, so that the simulation can continue to post-breaking wave-field evolution. In this chapter, we develop a phenomenologically based wave breaking model that can be incorporated into the HOS simulation and is able to capture the main phenomenological effects of wave breaking, including detecting breaking onset at the right time, dissipating appropriate amount of energy and indicating the locations of breaking events. We develop the wave breaking model based on wave energetics as wave breaking is intrinsically caused by energy transfer in the wave system. We want the model to be self-adaptive to different wave-field conditions to simulate different strength of wave breaking automatically. The model is desired to simultaneously detect breaking onset and simulate breaking energy dissipation instead of based on data post-processing. In order for the model to be efficient for large-scale simulation, we develop the model using global spectral information of the wave field instead of local physical quantities.
6.1 Numerical simulation of nonlinear waves

We develop the wave breaking model based on direct simulation of wave-field evolution that contain wave breaking events, caused by different mechanisms, such as modulational instability and wave focusing. Based on the spectral information of simulated wave-field, a robust wave breaking model is incorporated into the simulation method to detect wave breaking onset and account for different strength of wave breaking energy dissipation automatically.

6.1.1 Direct simulation using a high-order spectrum method

In the context of potential flow, with proper normalizations such that both the gravitational acceleration and the water density are unity, we perform phase-resolved simulations of two and three-dimensional nonlinear deep water waves based on a high-order spectral (HOS) method [24]. HOS directly solves the field equation with the kinematic and dynamic boundary conditions on the free surface in the Zakharov form:

\[
\eta_t + \eta_x \cdot \Phi_x^S - (1 + \eta_x \cdot \eta_x)\Phi_z(x, \eta, t) = 0, \\
\Phi_t^S + \eta + \frac{1}{2} \Phi_x^S \cdot \Phi_x^S - \frac{1}{2} (1 + \eta_x \cdot \eta_x)\Phi_z^2(x, \eta, t) = -P_a,
\]

where \( x = (x, y) \) and \( z \) are the horizontal and vertical coordinates, \( \Phi(x, z, t) \) is the velocity potential and \( \Phi^S(x, t) \equiv \Phi(x, z = \eta(x, t), t) \) is the value of the potential on the free surface \( \eta(x, t) \).

The HOS method is capable of accounting for nonlinear wave interactions up to an arbitrary order \( M \) for a number \( N \) of wave modes. The computational effort is approximately linear in \( N \) and \( M \), with exponential convergence with \( M \) and \( N \) for waves up to \( \sim 80\% \) of Stokes limiting steepness. The capability of HOS method enables us to effectively model the nonlinear wave-wave interactions in a broadband spectrum including: (i) energy transfer due to resonant wave-wave interactions; (ii)
energy transfer due to side-band instability; and (iii) energy cascading to short waves due to non-resonant wave interactions.

The validity and efficacy of this method have been demonstrated extensively for nonlinear wave-wave interactions [24], and the method has been extended to include atmospheric forcing [25], variable finite depth [57], and effects of energy dissipation [117].

### 6.1.2 Initial wave group structures

We investigate three classes of initial wave group structures [23, 101], which correspond to different types of wave-wave interactions, including modulational instability and wave focusing, to study the phenomenon of wave breaking. The first class (hereafter referred to as class I) has a fundamental carrier wave with two small symmetric sidebands, with the initial carrier wave amplitude $a_0$ (or steepness $s_0=a_0 k_0$, where $k_0=1$) and the number of waves in one modulation length, $n$, as parameters. The initial wave group can be represented as:

$$\eta = \eta_0 + \eta_+ + \eta_- ,$$

with

$$\eta_0 = a_0 \cos(k_0 x) ,$$

is the primary wave, and

$$\eta_+ = \epsilon a_0 \cos\left(\frac{n+1}{n} k_0 x - \theta_I \right) ,$$
$$\eta_- = \epsilon a_0 \cos\left(\frac{n-1}{n} k_0 x - \theta_I \right) ,$$

are the two small symmetric sidebands, where $\epsilon=0.1$, $k_0=1$, and $n$ is the integer number of waves in the group. Following [23] (hereafter referred to as DP), we take $3 \leq n \leq 10$ and the phase angle $\theta_I$ was taken as $\pi/4$ as it provides the most rapid initial growth of the sideband modes.
The second class of wave groups (hereafter referred to as class II) has an initial bimodal spectrum of the form:

\[
\eta = a_0 \cos(k_0x) + a_0 \cos\left(\frac{n+1}{n}k_0x - \theta_{II}\right)
\]  

(6.7)

where \(k_0=1\), \(a_0\) and \(n\) are as defined as above. Following (Banner02), we take \(\theta_{II}\) as \(\pi/18\).

The third class of wave groups (hereafter referred to as class III) is characterized as a “chirped” wave packet and commonly implemented in wave tank experiments (e.g. [90], hereafter referred to as RM). It comprises carrier waves that coalesce rapidly due to their different phase velocities. The initial two-dimensional wave group can be represented as:

\[
\eta = \sum_{i=1}^{m} a_i \cos(k_i(x-x_b) + 2\pi f_i t_b),
\]  

(6.8)

where \(m\) is the total number of wave components and the \(i\)-th wave component has amplitude, wavenumber and frequency represented as \(a_i, k_i\) and \(f_i\) respectively. The \(f_i\) are uniformly distributed in a frequency band \([f_c - \Delta f/2, f_c + \Delta f/2]\) with central frequency \(f_c\). \(x_b\) and \(t_b\) are the designed wave focusing location and time. In this class of wave group, we further investigate two sub-classes. The first sub-class (referred to as class III-A) corresponds to const-amplitude spectrum same as RM, i.e., we set \(a_i = a_0=\text{const}\) for \(i=1,2,\ldots,m\) and use \(s = ma_0k_c\) as a steepness parameter. The second sub-class (referred to as class III-B) corresponds to const-steepness spectrum \([14, 52]\), i.e., we set \(a_i k_i=\text{const}\) for \(i=1,2,\ldots,m\) and use \(s = ma_i k_i\) as a steepness parameter.

### 6.2 Determining breaking onset and strength

#### 6.2.1 Energy transfer due to nonlinear wave-wave interaction

In an unforced wave system, before wave breaking happens, the total wave energy is conserved. But nonlinear resonant and non-resonant wave-wave interactions cause...
energy transfer between wave components. Any two finite waves propagating together will result in local fluctuations of the energy content of each component. It is found that the energy transfer in the wave system causes the wave instability [111] and triggers wave breaking. Thus in order to detect wave breaking, it is important to understand the energy transfer in the wave system.

In the HOS method, it is straightforward to obtain the wave energy spectrum $S(k, t)$ which is a function of wavenumber (or frequency) and time. For deep water waves, the energy spectrum for a specific wavenumber is conserved described by the energy balance equation [109]. But for a general dispersive wave system, wave energy spectrum may not be conserved while a more general concept, wave action [114, 42] $N(k, t) = S(k, t)/\omega(k)$, is conserved which is represented by the action balance equation [10]. As wave energy transfers between wave components, wave action of each wave component also fluctuates. In order to find a diagnosing mechanism to detect breaking onset which can be further applied to a general wave system, we are interested in analyzing the behavior of wave action.

Figure 6-1 to 6-3 show the evolution of wave action for different wave components in the three classes of wave groups. Specifically, for class I wave group structure, the fundamental carrier wave mainly loses energy before breaking happens (figure 6-1-(a)) while the side-bands mainly gain energy (figure 6-1-(b),(c)). For class II wave group structure, the two modes in the initial spectrum have a similar trend of losing energy although they do not lose energy simultaneously (figure 6-2-(a)). As comparison, a lower-wavenumber component (figure 6-2-(b)) and a higher-wavenumber component (figure 6-2-(c)) gain energy before breaking happens. For class III wave group structure, figure 6-3-(a) shows the evolution of wave action for the central frequency component in the initial wave packet. We can see that it also has a decreasing trend but with strong oscillation. Figure 6-3-(b) and (c) show the wave action evolution for a lower-wavenumber and higher-wavenumber components outside the initial wave packet respectively. They both have a increasing trend but the higher-wavenumber component has a stronger oscillation before breaking happens. From these figures we can see a common phenomenon that there is strong energy transfer between wave com-
ponents as the wave group evolves. Generally the peak wave component loses energy while lower- and higher- wavenumber components gain energy. Another characteristic is the energy of wave components, especially the high-wavenumber component, oscillates strongly before breaking happens.

6.2.2 Detection of breaking onset

Wave breaking happens when the transfer of energy becomes most significant, or the modulation is strongest [23]. In order to detect breaking onset, we need to quantify the overall strength of energy transfer in the wave system.

Due to wave-wave interaction, the wave action of each wave component changes dramatically with respect to time, as shown in figures 6-1, 6-2, 6-3. It is hard to measure the overall strength of energy transfer by observing only a few wave components. To reflect the overall effect of energy transfer between wave components, we look at the $j$-th ($j \geq 1$) moment $\phi_j(t; k_{\text{max}})$ of wave action defined as:

$$
\phi_j(t; k_{\text{max}}) \equiv \sum_{k=0}^{k_{\text{max}}} N(k, t) k^j \Delta k ,
$$

where $k_{\text{max}}$ is the maximum wavenumber that can be resolved in the simulation.

It is easy to show that for a short time duration $\Delta t$, if there is net energy transfer between wave components, the change of the $j$-th ($j \geq 1$) moment $\Delta \phi_j \equiv \phi_j(t+\Delta t) - \phi_j(t)$ is in general not zero. Specifically, if the overall effect of energy transfer is from low-wavenumber part to high-wavenumber part, then $\phi_j$ ($j \geq 1$) will increase and it will decrease if otherwise. Thus if the energy is transferring back and forth between wave components, we expect $\phi_j$ to oscillate. And the oscillation of $\phi_j$ can reflect the overall strength of energy transfer between wave components and further be used as diagnosing variables to detect wave breaking.

Specifically, we choose the second moment of wave action $\phi_2$ as the diagnosing variable in favor of nondimensionalizing (the time rate of change of $\phi_2$ is dimension-
Figure 6-1: Evolution of wave action for different wave components of class I wave group structure with $n=6$, $s_0=0.11$. 
Figure 6-2: Evolution of wave action for different wave components of class II wave group structure with \( n=5, s_0=0.07 \).
Figure 6-3: Evolution of wave action for different wave components of class III-A wave group structures with $m = 6$, $s = 0.04$. 
less). Hereafter $\phi_2$ is simply denoted as $\phi$, i.e.,

$$
\phi(t; k_{\text{max}}) = \sum_{k=0}^{k_{\text{max}}} N(k, t) k^2 \Delta k.
$$

(6.10)

To quantify the amount of oscillation/variation of $\phi$ as it evolves, we use the standard deviation $\sigma_\phi$ over a moving time window of its intrinsic oscillation period. Figures 6-4 to 6-9 show some examples of the evolution of $\phi$ and $\sigma_\phi$ for three classes of wave group structure. For each class of wave group structure, we show for a fixed $n$ (in class I, II) or $m$ (in class III), the corresponding marginal wave breaking and wave recurrence cases. It is shown that in the case of wave breaking, the moving average of $\sigma_\phi$ keeps increasing and suddenly becomes large when breaking happens, while in the case of wave recurrence, the moving average of $\sigma_\phi$ keeps increasing until recurrence happens and then it decreases. We denote the moving average of $\sigma_\phi$ as $\langle \sigma_\phi \rangle$. The change of $\langle \sigma_\phi \rangle$ is a good indicator of breaking onset, as breaking always happens when the time rate of change of $\langle \sigma_\phi \rangle$ is large enough.

Therefore, we use the time rate of change of $\langle \sigma_\phi \rangle$, defined as

$$
\chi(t) = \frac{d \langle \sigma_\phi \rangle}{dt},
$$

(6.11)

to detect wave breaking onset. Figures 6-4 to 6-9 also show the evolution of $\chi$ for wave recurrence and breaking for each class of wave group structure. We can see that, in wave recurrence, $\chi$ reaches its maximum when recurrence happens, while wave breaking, $\chi$ keeps increasing until wave breaks and the the maximum value is usually much larger than the recurrence maximum for any class of wave group structure. Thus it is reasonable to choose a threshold value $\chi_{\text{th}}$ which is larger than all recurrence maximum but less than all breaking maximum, such that $\chi(t) \geq \chi_{\text{th}}$ can be used to detect the breaking onset. Here we define the detected breaking onset time $t_b$ as:

$$
t_b = \min \{ t : \chi(t) \geq \chi_{\text{th}} \}.
$$

(6.12)

Later we will investigate the existence of such a common threshold value $\chi_{\text{th}}$ for any
class of wave group structure.

6.2.3 Determination of breaking energy dissipation

When wave breaking happens, it is observed that the energy dissipation is generally confined in the high frequency or wavenumber range of the wave spectrum [90, 32]. To account for energy dissipation due to breaking, we propose a breaking strength model which can be incorporated in the HOS method. The breaking strength model is a smooth low-pass filter applied in the wavenumber space for wave elevation/potential such that appropriate amount of energy in the high wavenumber range is filtered out at the time of breaking onset \( t = t_b \) to simulate the effect of wave breaking.

Specifically, we denote this low-pass filter as \( \Pi(k) \), and at time \( t = t_b \), we apply this filter in wavenumber space on wave elevation/potential spectra, both denoted as \( F(k) \) to obtain the filtered spectrum \( F_B(k) \), i.e.,

\[
F_B(k) = F(k) \cdot \Pi(k). \quad (6.13)
\]

Note that physically wave breaking is a transient phenomenon. Energy dissipation happens in a very short period. In the simulation, we simulate this transient process by applying the low-pass filter \( \Pi(k) \) for a short period of time \( t_{II} \). After time \( t_{II} \), the low-pass filter is turned off until the next breaking onset is detected. The determination of \( t_{II} \) is discussed later in §6.3.2.

For the low-pass filter \( \Pi(k) \), we expect it to be almost 1 for wavenumber \( k \) less than a critical wavenumber \( k_{cr} \), where \( k = |k| \), and almost 0 for \( k > k_{cr} \). A smooth form satisfying this requirement can be:

\[
\Pi(k) = \exp(-\left(\frac{k}{k_{cr}}\right)^n), \quad (6.14)
\]

where \( n \gg 1 \) is constant and in practice \( n=100 \) is sufficient. This filter ensembles the ideal low-pass filter \( \Pi(k)=1 \) for \( 0 \leq k \leq k_{cr} \) and 0 otherwise) except that it is infinitely differentiable at the turning point \( k=k_{cr} \).
Figure 6-4: The evolution of (a) $\phi$, (b) $\sigma_\phi$ (red line) and $\langle \sigma_\phi \rangle$ (green dashed line), (c) $\chi$ for Class I marginal breaking ($n=6$, $s_0=0.11$), where $\phi$ and $\sigma_\phi$ have the unit of (s).
Figure 6-5: The evolution of (a) $\phi$, (b) $\sigma_\phi$ (red line) and $\langle \sigma_\phi \rangle$ (green dashed line), (c) $\chi$ for Class I marginal recurrence ($n=6$, $s_0=0.10$), where $\phi$ and $\sigma_\phi$ have the unit of (s).
Figure 6-6: The evolution of (a) $\phi$, (b) $\sigma_\phi$ (red line) and $\langle \sigma_\phi \rangle$ (green dashed line), (c) $\chi$ for Class II marginal breaking ($n=5, s_0=0.07$), where $\phi$ and $\sigma_\phi$ have the unit of (s).
Figure 6-7: The evolution of (a) $\phi$, (b) $\sigma_\phi$ (red line) and $\langle \sigma_\phi \rangle$ (green dashed line), (c) $\chi$ for Class II marginal recurrence ($n=5$, $s_0=0.069$), where $\phi$ and $\sigma_\phi$ have the unit of (s).
Figure 6-8: The evolution of (a) $\phi$, (b) $\sigma_\phi$ (red line) and $\langle \sigma_\phi \rangle$ (green dashed line), (c) $\chi$ for Class III marginal breaking ($m=6$, $s=0.04$), where $\phi$ and $\sigma_\phi$ have the unit of (s).
Figure 6-9: The evolution of (a) $\phi$, (b) $\sigma_\phi$ (red line) and $(\sigma_\phi)$ (green dashed line), (c) $\chi$ for Class III-A marginal recurrence ($m=6$, $s=0.03$), where $\phi$ and $\sigma_\phi$ have the unit of $s$. 
Note that $k_{cr}$ determines the amount of energy dissipated by wave breaking. In general, the less $k_{cr}$, the more energy dissipated. We propose a method that can automatically determine $k_{cr}$ at the breaking time $t_b$ based on spectral information, so that we can simulate appropriate amount of energy dissipation for different types of wave breaking.

In the wave breaking onset detection process described above, we define $\phi(t; k_{max})$ as the second moment of wave action summing over the entire resolved wavenumber space $[0,k_{max}]$ and obtain $\chi(t)$ to detect wave breaking onset. Similarly, we define $\phi(t, k)$ as the second moment of wave action summing over partial wavenumber space $[0,k]$:

$$\phi(t, k) \equiv \sum_{\kappa=0}^{k} N(\kappa, t)\kappa^2 \Delta \kappa . \quad (6.15)$$

For each $k$, we can calculate the moving deviation $\sigma_{\phi}(t, k)$ and further obtain the time rate of change of $\langle \sigma_{\phi}(t, k) \rangle$ as:

$$\chi(t, k) \equiv \frac{d\langle \sigma_{\phi}(t, k) \rangle}{dt} . \quad (6.16)$$

It is observed that at the time of wave breaking ($t = t_b$) when $\chi$ first reaches $\chi_{th}$, there exists a critical wavenumber $k_{cr}$ such that

$$k_{cr} = \min\{k : \chi(k; t = t_b) \geq \chi_{th} \} . \quad (6.17)$$

And $k_{cr}$ decreases as the initial wave becomes steeper. This property is desired as smaller $k_{cr}$ corresponds to more energy dissipation. Thus it provides a potential way to simulate various strength of energy dissipation automatically.

To summarize, the wave breaking onset and strength model contains a common model parameter $\chi_{th}$, which determines the breaking onset time and automatically selects a critical wavenumber to apply the low-pass filter to simulate various energy dissipation strength. The value of $\chi_{th}$ should be larger than all the non-breaking values and its specific value can be calibrated by comparison with experiments.
6.3 Results

6.3.1 Existence of a common threshold $\chi_{th}$ for the breaking onset and strength model

In §6.2.2, we propose that wave breaking happens when $\chi(t) \geq \chi_{th}$. Now we investigate whether there exists such a common threshold value $\chi_{th}$. For each class of wave group structure tested, we change the number of $n$ (for class I and II) or $m$ (for class III) and for each $n(m)$, we investigate the marginal wave breaking and recurrence cases. We take the maximum of $\chi$ in all non-breaking situations, including all wave recurrence cases and wave breaking cases when breaking has not happened, as the lower bound of $\chi_{th}$. Table 1 shows the steepness parameter ($s_0$ for class I, II and $s$ for class III) and $\chi_{th}$ lower bound for three classes of wave group structures. Based on the ensemble of results for the three different classes of initial wave group structures in table 6.1, we propose that a common threshold $\chi_{th}$ for breaking onset should satisfy $\chi_{th} \geq 1.9 \times 10^{-5}$.

Its specific value is further calibrated by comparison with wave breaking experiments conducted by Rapp and Melville (1990) [90]. They investigated 2D wave breaking for chirped wave packets with different packet geometries and quantified wave breaking intensities based on the energy loss from wave breaking. They observed different breaking types including incipient, spilling and plunging breaking and measured the breaking intensities with respect to the initial wave steepness parameter $ak_c$. We use one set of their experiments (with packet parameters $f_c=0.88$ Hz, $\Delta f/f_c=0.73$, $x_b k_c=27.4$) to calibrate $\chi_{th}$ by comparing the energy loss data and obtain an optimal value of $\chi_{th}$ as $2.0 \times 10^{-5}$, corresponding to $R^2=0.99$ between the experimental and simulated breaking energy loss. Note that in the experiments, no breaking is observed for $ak_c < 0.25$. The energy loss for these cases is due to other factors such as friction with the wave tank, instead of breaking. Since we only compare the energy dissipation due to wave breaking, we modify the energy loss in the experimental data by subtracting the average energy loss of cases with $ak_c < 0.25$. In
order to better simulate their experiments in the wave tank which is actually three-
dimensional, we conduct 3D phase-resolved simulation in an area of 51.1m x 51.1m
(corresponding to 512 x 512 data points) with waves propagating in the +x direction.
In the calibration, it is found that the breaking energy dissipation is insensitive to
the low-pass filter applying time $t_n$. Without loss of generality, we set $t_n \sim T_p$ where
$T_p$ is the peak period in the wave group.

For the optimal value of $\chi_{th} = 2.0 \times 10^{-5}$, we further check with the simulation of
three classes of wave group structure. It is seen that for all the recurrence cases and
non-breaking time, $\chi$ remains below $\chi_{th}$, while whenever breaking happens, $\chi \geq \chi_{th}$.
This $\chi_{th}$ is just above the maximum of all $\chi_{th}$ lower bounds so that we can detect
wave breaking ahead of actual breaking time. We denote the detected breaking onset
using the proposed $\chi_{th}$ as $t_b$ and the actual breaking time as $\tilde{t}_b$. Table 6.1 also shows
the lead time between the detected and actual breaking time $t_{lead} \equiv \tilde{t}_b - t_b$. We can
see that the proposed threshold $\chi_{th}$ can well predict the breaking onset.

6.3.2 Validation of the breaking onset and strength model

To validate the proposed breaking onset and strength model with $\chi_{th} = 2.0 \times 10^{-5}$,
we simulate other sets of experiments conducted by Rapp and Melville (1990) [90]
(with packet parameters $f_c = 1.08$ Hz, 1.28 Hz) and calculate the wave breaking energy
dissipation. As discussed in §6.2.3, the breaking energy dissipation is determined
by the critical wavenumber $k_{cr}$ where the low-pass filter starts to take effect. Figure 6-10
shows the process of determining $k_{cr}$ for the case of $f_c = 1.08$Hz and initial $a_{kc} = 0.36$ as
an example. Figure 6-11 shows how $k_{cr}$ changes with the initial $a_{kc}$ in the wave group
for $f_c = 1.08$ Hz experiment sets. As $a_{kc}$ increases, the determined $k_{cr}$ decreases which
leads to more energy dissipation. Comparisons with the experimental data are shown
in figure 6-12 for $f_c = 1.08$Hz and figure 6-13 for $f_c = 1.28$Hz, both with $R^2 = 0.95$. In
figure 6-12 we also show the simulation results from Xiao et al. (2013) [118] where
a breaking model is proposed to be incorporated in HOS, but that model cannot
detect breaking onset and cannot simulate various types of breaking well. From
the comparisons we can see that our proposed breaking model can not only detect
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Table 6.1: Values of $Xth$ lower bound and $t_{lead}$ for different classes of wave group structures, where $T_p$ the period corresponding to the fundamental carrier wave for class I, the first component for class II and the central component for class III.
Figure 6-10: $\chi(k, t)$ as a function of $k$ at the time of wave breaking $t = t_b$ for RM measurement with $f_c=1.08$Hz and initial $ak_c=0.36$.

breaking onset well, but also simulate the breaking intensity well corresponding to different breaking types.

6.3.3 Verification of the breaking onset and strength model

To further verify the proposed breaking onset and strength model, we consider the 2D wave breaking experiments conducted by [52] (hereafter referred to as Kway) and [74] (hereafter referred to as Nepf).

[52] studied 2D deep-water wave breaking which is generated by wave focusing technique using wave packets of different spectrum. Specifically, they used wave pack-
Figure 6-11: The critical wavenumber $k_{cr}$ as a function of $ak_c$ for the set of RM measurement with $f_c=1.08$ Hz.
Figure 6-12: Simulated and measured wave breaking energy dissipation for sets of RM measurements with $f_c=1.08$ Hz, where black square represents measurement results, red circle represent simulation results in this work, and blue triangle represents simulation results of Xiao et al. (2013) [118].
Figure 6-13: Simulated and measured wave breaking energy dissipation for sets of RM measurements with $f_c=1.28$ Hz, where black square represents measurement results, red circle represent simulation results in this work.
ets with constant-steepness components, constant-amplitude components, and also components following the Pierson-Moskowitz distribution. For constant-steepness wave packet, they generated different intensities of wave breaking, ranging from incipient breaking to plunging breakers. For constant-amplitude and PM wave packets, only plunging breakers were generated. We follow their wave packet parameters (see Table 1 in Kway) and set the breaking time $t_b=10\text{s}$ ($t_b/T_c=8.33$) to conduct phase-resolved simulation. We conduct 3D phase-resolved simulation in an area of $51.1\text{m}\times51.1\text{m}$ (corresponding to $512\times512$ data points) with waves propagating in the $+x$ direction. Figure 6-14 and 6-15 show the simulation results for constant-steepness packet plunging breaker case. Specifically, figure 6-14 and 6-15 show the evolution of $\phi$ and $\chi$ respectively. We can see that around the designed breaking time ($\tau \equiv t/T_c=8.33$), the oscillation of $\phi$ suddenly becomes large so that the diagnosing variable $\chi$ increases abruptly. The breaking onset detection condition $\chi \geq \chi_{th}$ corresponds to a detected breaking time $\tau=7.80$ which is very close to the designed one. At the detected breaking time, the breaking strength model is applied and a critical wavenumber $k_{cr}$ is automatically selected as $k_{cr}/\Delta k=29$. The simulated energy dissipation $\Delta E/E_0=0.15$, compared well with experimental result. Table 6.2 shows the comparison with experimental results for four cases conducted in their experiments. We can see that the breaking onset model is able to detect breaking onset 1 to 2 typical periods ahead of the actual breaking time and the breaking strength model can simulate the breaking energy dissipation well compared to the experimental results.

[74] studied 2D and 3D deep water wave breaking also generated by wave focusing technique using wave packets with constant-steepness components. Here we
Figure 6-14: The evolution of $\phi$ (with unit (s)) from the simulation of Kway plunging breaker experiment with constant-steepness components.
Figure 6-15: The evolution of $\chi$ from the simulation of Kway plunging breaker experiment with constant-steepness components.
are specifically interested in their 2D wave breaking experiments as their 3D wave breaking is generated by spatially tapering the individual wave paddle movements in the wave basin which is nontrivial for the current numerical simulation method to simulate and it is beyond the scope of the present work. For the 2D wave breaking case, they generated a plunging breaker. We follow their wave packet parameters (32 wave components, $f_c=1.08\text{Hz}$, $\Delta f/f_c=0.73$, $a_k=0.50$, $y_b k_c=3.8$) to conduct the 3D phase-resolved simulation in an area of 51.1m×51.1m (corresponding to 512×512 data points) with unidirectional waves. We set the breaking time in simulation as $t_b=10s$ ($t_b/T_c=10.8$). With the breaking onset model, the detected breaking onset is at $\tau=t/T_c=9.18$ while the actual breaking happens at $\tau=9.56$. As for energy dissipation, they divided the wave energy spectrum into two frequency bands: the principle frequency band (0.7-1.5 Hz) which corresponds to the input signal and a second band (1.5-3.0 Hz). They observed energy increase (up to 1.4%) in the second band before breaking and energy loss (up to 2.6%) after breaking (see figure 6-16). As comparison, we also look at the energy change in the second frequency band in the simulation and find that the energy increases up to 1.1% before breaking is detected and energy loss is up to 2.3% after breaking is detected and the breaking strength model is applied which corresponds to a critical wavenumber $k_c/\Delta k=54$. The comparison is reasonable as we detect the breaking onset a little ahead of the actual breaking time so that the energy increase in the second band is not as much as in the experiment.
Figure 6-16: Evolution of the energy change in the second frequency band from Nepf 2D wave breaking measurement (black square) and simulation results (red circle), where $y$ is the wave propagation direction and the breaking location is $y_bk_c=3.8$, $E_0$ is the initial total wave energy.
Chapter 7

Quantification of wave breaking kinematics and dynamics

With the breaking model incorporated into the phase-resolved wave-field simulations, we can further quantify wave breaking energy dissipation. In this chapter, we discuss the quantification of wave breaking kinematics and dynamics.

7.1 Wave breaking kinematics and dynamics

In the open ocean, breaking waves cover a very wide range of scales, from very short gravity waves in which a moving convergent stagnation point is marked by a group of capillary ripples through intermediate scales (15 - 30 cm or so) where the breaking is unsteady and turbulent but only a few bubbles are produced, to actual whitecaps in which the breaking and the generation of turbulence is so vigorous that extensive patches of foam are generated [85].

At any instant, the fronts of the breaking waves define a distribution of isolated line or arc segments. It is suggested that the velocity \( c \) of the breaking front is a good measure of the scale of the breaking, since this is a well-defined quantity that might be measured from cine images of the sea surface [85].

For a specific breaking event, wave breaking kinematics include breaking front length \( L \) and the velocity \( c \) of the breaking front, and wave breaking dynamics in-
clude the energy and momentum loss due to wave breaking. To better quantify wave
breaking and connect wave breaking kinematics and dynamics, Phillips (1985) de-

defined a distribution $\Lambda(c)$ such that $\Lambda(c)dc$ represents the average total length per
unit surface area of breaking fronts that have velocities in the range $c$ to $c + dc$. It
can be further derived as function of breaking speed $c$. In practice, given $L_j$ and $c_j$
for a breaking event $j$, $\Lambda(c)$ is obtained by

$$
\Lambda(c) = \frac{1}{A\Delta c} \sum_j (L_j|c - \frac{\Delta c}{2} < c < c + \frac{\Delta c}{2}),
$$

(7.1)

where $A$ is the total sea surface area under consideration and $\Delta c$ is the speed increment
that can be resolved.

With $\Lambda(c)$ defined above, we can further deduce other wave breaking kinematics.
For example, the total length of breaking fronts per unit area is simply

$$
L = \int \Lambda(c)dc.
$$

(7.2)

In unit time, the fraction of sea-surface area traversed by breaking fronts with veloci-
ties between $c$ and $c + dc$ is $c\Lambda(c)dc$, so that the fraction of total surface area turned
over per unit time, the turnover rate, is

$$
R = \int c\Lambda(c)dc,
$$

(7.3)

which also expresses the total number of breaking waves of all scales passing a given
point per unit time. The distribution $c\Lambda(c)dc$ specifies the expected number per unit
time passing a fixed point with velocities in the interval $c$ to $c + dc$.

The distribution $\Lambda(c)$ can also be connected to the rate of energy loss per unit
length of front in the breaking events. Duncan (1981) [28] examined this question
through a series of laboratory experiments and showed that for deep-water continuing
active breakers, the breaking zone extends down the forward face of the wave over a
fixed fraction of its amplitude and that its shape is geometrically similar for waves of
different scales so that the cross-sectional area of the breaking zone is proportional to
the square of the local wavelength, \((c^2/g)^2\). The weight of the breaking zone per unit length of the front then exerts a tangential force per unit length proportional to \(c^4/g\) that acts on the incoming stream, whose speed is approximately \(c\). Consequently, the rate of energy loss per unit length of front is \(b(c^5/g)\), where \(b\) is a numerical constant estimated by Duncan from his experiments as approximately 0.06 [28, 85].

Thus the average rate of energy loss per unit area by breakers with speeds between \(c\) and \(c + dc\) is then

\[
\epsilon(c)dc = bg^{-1}c^5A(c)dc.
\]  

(7.4)

Based on the derivation of \(\epsilon(c)\) using wave spectra in the equilibrium range [85], Phillips (1985) derived that \(A(c) \propto c^{-7}\). Since \(\int A(c)cd\theta = \int A(c)dc\), we can further derive that \(A(c) \propto c^{-6}\).

### 7.2 Quantification of wave breaking energy dissipation

With the breaking model developed in chapter 6 that can be incorporated in HOS, besides the breaking onset and overall energy dissipation, we may further simulate the location and strength of breaking events. Specifically, given the same initial conditions, we conduct two simulations simultaneously. For one simulation, we do not incorporate the breaking model and the simulated wave elevation is denoted as \(\eta_0(x, t)\). For the other simulation, we incorporate the breaking model and the simulated wave elevation is denoted as \(\eta_B(x, t)\). Now we define the dissipation field \(\delta(x, t)\) as the absolute difference of \(\eta_0(x, t)\) and \(\eta_B(x, t)\), i.e.,

\[
\delta(x, t) = |\eta_0(x, t) - \eta_B(x, t)|.
\]  

(7.5)

Physically, \(\delta(x, t)\) is significantly nonzero where energy is dissipated due to wave breaking and the value of \(\delta(x, t)\) represents the strength of breaking events. Fig-
Figure 7-1: Dissipation field $\delta(x, t)$ based on phase-resolved three-dimensional wave-field simulation from directional JONSWAP spectrum with $\gamma=6$, $c=0.10$, $H_s=0.21$ m, $T_p=2s$ and directional range $\Theta=2\pi/9$.

Figure 7-1 shows an example of dissipation field calculated from phase-resolved three-dimensional wave-field simulation based on directional JONSWAP spectrum.

From the dissipation field $\delta(x, t)$, we can further quantify wave breaking by calculating the whitecap coverage rate $W_r$:

$$W_r = \frac{S_a}{S_{tot}},$$

where $S_a$ is the active wave breaking area and $S_{tot}$ is the total surface area. $S_a$ denotes surface area where dissipation is higher than a threshold value $\delta_c$. To determine $\delta_c$, we follow similar method as Kleiss and Melville (2011) deal with sea surface imagery for
whitecap kinematics [49]. Specifically, we calculate the probability density function of dissipation \( p(s) \), where \( s \equiv (\delta - \delta_{\text{min}})/(\delta_{\text{max}} - \delta_{\text{min}}) \), and get the complementary cumulative distribution \( G(s) = 1 - \int_0^s p(s)ds \). From the second derivative \( G''(s) \), we choose an point which corresponds to about 20% of the peak value to determine the threshold value \( s_c = (\delta_c - \delta_{\text{min}})/(\delta_{\text{max}} - \delta_{\text{min}}) \). Figures 7-2, 7-3 and 7-4 show the distribution of \( p(s) \), \( G(s) \) and \( G''(s) \) for the example in the figure 7-1.

Based on the dissipation field \( \delta(x, t) \) and the threshold value \( \delta_c \), we implement a depth-first-search (DFS) algorithm [16] to detect, label and track each breaking event \( j \). Then we are able to measure the size of each breaking event and calculate the breaking front length \( L_j \) following the contour method used by Kleiss and Melville.
Figure 7-3: Complementary cumulative distribution $G(s)$ for the example shown in figure 7-1, where $s \equiv (\delta - \delta_{\text{min}})/(\delta_{\text{max}} - \delta_{\text{min}})$. 
Figure 7-4: Second derivative $G''(s)$ of the complementary cumulative distribution for the example shown in figure 7-1, where $s \equiv (\delta - \delta_{\text{min}})/(\delta_{\text{max}} - \delta_{\text{min}})$. 
By tracking each breaking event in consecutive time instants, we can calculate the breaking velocity $c_j$. The information of $L_j$ and $c_j$ of all breaking events enables us to further obtain the Phillips statistics $\Lambda(c)$ discussed in §7.1, which is a useful quantity to connect wave breaking kinematics with dynamics.

As an example and to further validate the proposed breaking model, we generate synthetic two-dimensional wave field based on JONSWAP spectrum with a range of spectrum parameters and simulate its evolution to obtain the breaking dissipation field $\delta(x,t)$ and breaking statistics $\Lambda(c)$. Figure 7-5 shows the distribution of $\Lambda(c)$ for different peak enhancement factor $\gamma$ of JONSWAP spectrum. We can see that qualitatively the simulated $\Lambda(c)$ agrees well with theoretical result $\Lambda(c) \propto c^{-6}$, which to some extent further verifies the capability of the proposed breaking model to simulate the locations of wave breaking events.
Figure 7-5: Phillips statistics $\Lambda(c)$ obtained from phase-resolved wave-field simulation with the breaking onset and strength model based on JONSWAP spectrum of different peak enhancement factor $\gamma$ and other spectrum parameters $\epsilon=0.10$, $H_s=0.21\text{m}$, $T_p=2\text{s}$. In HOS simulation, we use $1024\times1024$ wave modes and nonlinear order $M=3$. 
Chapter 8

Conclusions and future work

8.1 Conclusions of thesis

In this thesis, we address the problem of phase-resolved reconstruction and forecast of nonlinear irregular wave field from limited measurements based on direct numerical simulations.

In Chapter 2, we investigate the phase-resolved predictability of irregular wave fields when information about the waves are known in specific space-times domains of the field. We define the theoretical linear predictable zone $P_L$ in space-time within which the wave field can be reconstructed based on this information. We obtain general closed-form expressions for $P_L$ using linearized wave theory for unidirectional waves based on single fixed or moving probes and general waves based on instantaneous whole-area measurements. For realistic ocean, we assume that there is a certain frequency and direction range, outside of which the wave energy is negligible.

This so-called predictable zone of the wave field is fully determined in terms of the maximum and minimum wave group velocities and direction spreading angle of the wave field, in addition to the spatial and temporal extents of the prescribed measurements. The theoretical approach for evaluating the predictable zone is generalized to the general situation of multidirectional wave fields with multiple fixed or moving probes or hybrid probe and whole-area measurements. Significantly, we find that the total predictable zone based on the combined effect of multiple data sources is a
superset of the simple union of the predictable zones associated with each individual
data source. Based on the understanding of the predictability of irregular wave fields,
in Chapter 3 we derive an optimal deployment of multiple probes in order to maxi-
mize the predictable zone in multidirectional wave fields. Remarkably, we find that
when $J$ ($J \gg 1$) probes are optimally deployed, the volume of the total predictable
zone can be $O(J^2)$ times larger than that of the union of the predictable zones of each
individual probe.

We remark that we do not consider (biased or unbiased) errors in the measure-
ments in deriving the predictable zone theory, since the theoretical solution of the
predictable zone is not affected by small values of these.

In Chapter 4, we develop the theoretical wave reconstruction approach for linear
waves based on simple measurement set-up which is obtained straightforwardly from
the predictable zone theory. We further develop an iterative high-order reconstruction
(HOR) method for nonlinear phase-resolved reconstruction of general two- and three-
dimensional irregular wave fields. For a given set of measurements in space-time
domain $\mathcal{M}$ of the wave field $\Phi$, HOR obtains a reconstructed field $\tilde{\Phi}$ which minimizes
the reconstruction error defined as the difference between $\Phi$ and $\tilde{\Phi}$ within $\mathcal{M}$. HOR
can, in practice, be implemented with any nonlinear evolution engine. In this work
we implement a high-order spectral (HOS) method [24], which accounts for nonlinear
wave interactions up to an arbitrary order $M$, with computational effort linearly
proportional to $M$ and the total number of spectral modes $N$ required.

We show that, under general conditions for $\Phi$ which can be represented by pertur-
bation series in wave steepness, HOR converges to a unique $\Phi$ with increasing numbers
$L$ of (free-wave mode) optimization parameters and nonlinear order $M$, which HOR
iteratively determines for a given reconstruction tolerance. In some space-time do-
main $\mathcal{P}_{NL}$ (beyond $\mathcal{M}$), the converged $\Phi$ approaches the underlying field $\tilde{\Phi}$ with error
bounded by the reconstruction tolerance. We find that $\mathcal{P}_{NL}$ always contains and gen-
erally extends beyond the linear predictable zone $\mathcal{P}_L$ developed in Chapter 2. The
importance of high-order nonlinear effects in reconstruction is shown by comparing
the reconstructed wave field using different order $M$ (compared to $\tilde{\Phi}$). It is found
that, even for relatively small wave nonlinearity, linear reconstruction is generally inadequate to predict underlying quantities such as velocities and pressure not in the measurements. The number of optimized free-wave components $L$ in HOR is generally much less than the number of modes $N$ required in HOS; while the number of HOS evaluations is typically linearly proportional to $L$ and $M$. With fast evaluation engine such as HOS using modern high-performance computing, HOR is practical for many realistic applications. These are illustrated in extensive tests in Chapter 4 and 5 including theoretical Stokes waves (with steepness up to the limit of HOS), synthetic uni- and multi-directional irregular wave fields generated by numerical simulations, and physical (nonlinear) wave fields in two- and three-dimensional wave tanks.

In all of the examples above, information of the underlying wave field $\Phi$ outside of the measurement domain $M$ is used to evaluate the predictive ability of the nonlinear HOR reconstruction and the domain $P_{NL}$ in which this prediction can be obtained. In actual application, $\Phi$ is generally not known outside of $M$, and $P_{NL}$ must be obtained independent of $\Phi$. In this case, $P_L$ (given in terms of $M$) provides a (conservative) estimate of $P_{NL}$. This highlights the importance of the linear predictable zone theory developed in Chapter 2.

In this work, we have considered general wave fields in deep water. HOR, implemented using HOS say, can be easily applied to more general problems such as wave fields on constant and/or variable finite depth, and internal waves, for which HOS has been extended [57, 2].

We also remark that, as in Chapter 2, we have not considered effects of noise and uncertainties in measurements. From (limited) Monte-Carlo numerical simulations [116] and from the present comparisons using physical wave tank data, the error in $\Phi$ is generally expected to be bounded in terms of the measurement error.

In Chapter 6, we develop a phenomenological wave breaking model to detect breaking onset and determine wave breaking energy dissipation, which can be incorporated into the nonlinear evolution engine of HOR, so that HOR can be able to reconstruct steep nonlinear wave fields that may contain breaking events. The breaking model is developed based on analysis of simulated two-dimensional wave breaking from three
different classes of initial wave group structures, which covers different wave-wave interaction mechanisms, including modulation instability and wave focusing. The developed wave breaking model is calibrated, validated and verified by different wave breaking measurements and excellent agreement is obtained between simulated wave breaking results and measured ones. The wave breaking model can be further used to predict the locations of breaking events, which is validated statistically by calculating the Phillips statistics in Chapter 7.

8.2 Future work

In this thesis, we do not consider the presence of wind as we focus on unforced wave evolution. We also do not consider imposed current or variable bottom topography for simplicity. In the future, to extend the capability of phase-resolved reconstruction and forecast of nonlinear irregular wave field to open ocean and coastal regions, it is helpful to consider the effects of wind, ambient current and varying bottom topography. The framework developed in this work is compatible with incorporating appropriate modelings of these effects.

8.2.1 Wind input

Effective modeling of wind input has been an extremely challenging task due to the complexity of the dynamics involved in the wind-wave interaction. A resonance theory of Phillips (1957)[82] and a shear instability theory of Miles (1957)[68] have been two important theories for the understanding of wave generation and evolution with the presence of wind and these two theories were soon combined[69, 70, 11] and they provided the basic formulation in the latest phase-averaged wave prediction models. For phase-resolved wave prediction, it is essential to appropriately model the effect of wind forcing in the numerical simulation. Lots of literature exists related to the wind forcing mechanism, see review articles by Donelan (1999)[26] and Young (1999)[120]. Common consideration of wind forcing mechanism includes energy transfer from wind to waves due to the form drag[103] or tangential stresses[104, 59, 79].
8.2.2 Ambient current

For the prediction of wave field in the open ocean or coastal regions, it is common to consider variable ambient current. The ambient current will have effects on both phase-resolved reconstruction and forecast of nonlinear irregular wave field based on wave measurements. The ambient current also influences the measured wave data such as the wave kinematics. For the determination of the predictable zone, the ambient current changes the group velocity and thus the predictable zone. For the forecast of wave field evolution, the ambient current can change wave characteristics[62], and cause wave refraction and reflection and thus changes the evolution mechanisms. Good review articles can be found from Peregrine (1976)[80] and Jonsson (1990)[46].

8.2.3 Varying bottom topography

For the application of phase-resolved reconstruction and forecast of wave field in near-shore regions, finite water depth and varying bottom topography need to be considered in the numerical simulations as they change the boundary conditions. The effect of varying bottoms, even for non-slowly varying bottom, can be directly considered in phase-resolved simulations[57] and thus easily to be incorporated in the developed framework in this thesis.
Appendix A

Second-order Stokes wave model

In the second-order Stokes wave model, the free surface elevation and velocity potential on the mean free surface can be expressed in the form:

\[ \eta^{(2)}(x, t) = \eta^{(1)}(x, t) + \eta^{(2)}(x, t) \quad \text{and} \quad \Phi^{(2)}(x, t) = \Phi^{(1)}(x, t) + \Phi^{(2)}(x, t), \]

where the first-order solution \( \eta^{(1)} \) and \( \Phi^{(1)} \) take the form of (4.25) and the bound wave solution, \( \eta^{(2)} \) and \( \Phi^{(2)} \), can be expressed as:

\[
\eta^{(2)}(x, t) = \sum_{p=1}^{N_T} A_p^2 [G_p^+ \cos(2\psi_p) + G_p^-] \\
+ \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_p A_q [H_{pq}^+ \cos(\psi_p + \psi_q) + H_{pq}^- \cos(\psi_p - \psi_q)], \tag{A.1}
\]

\[
\Phi^{(2)}(x, t) = \sum_{p=1}^{N_T} A_p^2 E_p^+ \sin(2\psi_p) \\
+ \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_p A_q [F_{pq}^+ \sin(\psi_p + \psi_q) + F_{pq}^- \sin(\psi_p - \psi_q)], \tag{A.2}
\]
where $N_T = L_\omega L_\theta$ is the total number of free wave components with $L_\omega$ and $L_\theta$ defined in §4.2.5. To simplify the expression, here we use a single subscript ($p$ or $q$) to represent a free wave component instead of using two ($mn$) as in §4.2.5. They can be connected by $p = (m - 1)L_\theta + n$, for $m = 1, 2, \ldots, L_\theta$ and $n = 1, 2, \ldots, L_\omega$. The coefficients are defined as following:

\[
G_p^+ = \frac{|k_p|}{4 \tanh(|k_p|h)} \left[ 2 + \frac{3}{\sinh^2(|k_p|h)} \right],
\]

(A.3)

\[
G_p^- = -\frac{|k_p|}{2 \sinh(2|k_p|h)},
\]

(A.4)

\[
H_{pq}^\pm = \frac{\omega_p^2 + \omega_q^2}{2g} \mp \frac{\omega_p \omega_q}{2g} \left[ 1 \mp \frac{g^2(k_p \cdot k_q)}{\omega_p^2 \omega_q^2} \right] \left\{ \frac{(\omega_p \pm \omega_q)^2}{\omega_p^2 \omega_q^2} - g|k_p \pm k_q| \tanh(|k_p \pm k_q|h) \right\} \pm \frac{\omega_p^3}{\sinh^2(|k_p|h)} \pm \frac{\omega_q^3}{\sinh^2(|k_q|h)}
\]

(A.5)

and

\[
E_p^+ = \frac{3\omega_p \cosh(2|k_p|h)}{8 \sinh^4(|k_p|h)},
\]

(A.6)

\[
F_{pq}^\pm = \frac{1}{(\omega_p \pm \omega_q)^2 - g|k_p \pm k_q| \tanh(|k_q \pm k_q|h)} \times \left\{ \mp \omega_p \omega_q (\omega_p \pm \omega_q) \left[ 1 \mp \frac{g^2(k_p \cdot k_q)}{\omega_p^2 \omega_q^2} \right] + \frac{1}{2} \left\{ \frac{\omega_p^3}{\sinh^2(|k_p|h)} \pm \frac{\omega_q^3}{\sinh^2(|k_q|h)} \right\} \right\}.
\]

(A.7)

Based on above formulas, we further have the second-order solution for the velocity
and acceleration field as:

\[
\mathbf{u}(\mathbf{x}, z, t) = \mathbf{u}^{(1)}(\mathbf{x}, z, t) + \mathbf{u}^{(2)}(\mathbf{x}, z, t),
\]

(A.8)

and

\[
\mathbf{a}(\mathbf{x}, z, t) = \mathbf{a}^{(1)}(\mathbf{x}, z, t) + \mathbf{a}^{(2)}(\mathbf{x}, z, t),
\]

(A.9)

where \(\mathbf{u}^{(1)}(\mathbf{x}, z, t)\) and \(\mathbf{a}^{(1)}(\mathbf{x}, z, t)\) are linear solutions as:

\[
\mathbf{u}^{(1)}(\mathbf{x}, z, t) = \sum_{p=1}^{N_T} g A_p k_{pz} \cos(\psi_p) f_1(k_p, z, h)
\]

(A.10)

and

\[
\mathbf{a}^{(1)}(\mathbf{x}, z, t) = \sum_{p=1}^{N_T} g A_p k_{py} \sin(\psi_p) f_1(k_p, z, h)
\]

(A.11)

where

\[
f_1(k_p, z, h) = \frac{\cosh(|k_p|(z + h))}{\cosh(|k_p|/h)}.
\]

(A.12)

and

\[
f_2(k_p, z, h) = \frac{\sinh(|k_p|(z + h))}{\cosh(|k_p|/h)}.
\]

(A.13)
And $u^{(2)}(x, z, t)$ and $a^{(2)}(x, z, t)$ are second-order interaction terms as:

$$u^{(2)}(x, z, t) = \sum_{p=1}^{N_T} 2k_{px}A_p^2E_p^+ \cos(2\psi_p)f_1(2k_p, z, h)$$

$$+ \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_pA_q[(k_{px} + k_{qx})F_{pq}^+ \cos(\psi_p + \psi_q)f_1(k_p + k_q, z, h)$$

$$+ (k_{px} - k_{qx})F_{pq}^- \cos(\psi_p - \psi_q)f_1(k_p - k_q, z, h)], \quad (A.14)$$

$$v^{(2)}(x, z, t) = \sum_{p=1}^{N_T} 2k_{py}A_p^2E_p^+ \cos(2\psi_p)f_1(2k_p, z, h)$$

$$+ \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_pA_q[(k_{py} + k_{qy})F_{pq}^+ \cos(\psi_p + \psi_q)f_1(k_p + k_q, z, h)$$

$$+ (k_{py} - k_{qy})F_{pq}^- \cos(\psi_p - \psi_q)f_1(k_p - k_q, z, h)], \quad (A.15)$$

$$w^{(2)}(x, z, t) = \sum_{p=1}^{N_T} 2|k_p|A_p^2E_p^+ \sin(2\psi_p)f_2(2k_p, z, h)$$

$$+ \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_pA_q[|k_p + k_q|F_{pq}^+ \sin(\psi_p + \psi_q)f_2(k_p + k_q, z, h)$$

$$+ |k_p - k_q|F_{pq}^- \sin(\psi_p - \psi_q)f_2(k_p - k_q, z, h)], \quad (A.16)$$

and

$$a^{(2)}_2(x, z, t) = \sum_{p=1}^{N_T} 4k_{px}\omega_pA_p^2E_p^+ \sin(2\psi_p)f_1(2k_p, z, h)$$

$$+ \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_pA_q[(k_{px} + k_{qx})(\omega_p + \omega_q)F_{pq}^+ \sin(\psi_p + \psi_q)f_1(k_p + k_q, z, h)$$

$$+ (k_{px} - k_{qx})(\omega_p - \omega_q)F_{pq}^- \sin(\psi_p - \psi_q)f_1(k_p - k_q, z, h)], \quad (A.17)$$
\[ a_{p}^{(2)}(x, z, t) = \sum_{p=1}^{N_T} 4k_{py}\omega_p A_p^2 E_p^+ \sin(2\psi_p)f_1(2k_p, z, h) \]
\[ + \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_p A_q (k_{py} + k_{qy})(\omega_p + \omega_q) F_{pq}^+ \sin(\psi_p + \psi_q)f_1(k_p + k_q, z, h) \]
\[ + (k_{py} - k_{qy})(\omega_p - \omega_q) F_{pq}^- \sin(\psi_p - \psi_q)f_1(k_p - k_q, z, h) \], \hspace{1cm} (A.18) \]

\[ a_{z}^{(2)}(x, z, t) = \sum_{p=1}^{N_T} -4k_{py}\omega_p A_p^2 E_p^+ \cos(2\psi_p)f_2(2k_p, z, h) \]
\[ + \sum_{p=1}^{N_T-1} \sum_{q=p+1}^{N_T} A_p A_q (k_{py} + k_{qy})(\omega_p + \omega_q) F_{pq}^+ \cos(\psi_p + \psi_q)f_2(k_p + k_q, z, h) \]
\[ + |k_p - k_q|(-\omega_p + \omega_q) F_{pq}^- \cos(\psi_p - \psi_q)f_2(k_p - k_q, z, h) \]. \hspace{1cm} (A.19) \]

Note that the above linear and second-order interaction terms are only valid for 
\( z \leq 0 \). For \( 0 < z < \eta \), we can apply Taylor expansion on \( z = 0 \) up to second-order to obtain the solution:

\[ u_{(2)}(x, z, t) = u_{(2)}(x, z = 0, t) + z \sum_{p=1}^{N_T} \frac{gA_p k_{px}\omega_p |k_p|}{\omega_p} \cos(\psi_p)f_2(k_p, z = 0, h) \]
\[ v_{(2)}(x, z, t) = v_{(2)}(x, z = 0, t) + z \sum_{p=1}^{N_T} \frac{gA_p k_{py}\omega_p |k_p|}{\omega_p} \cos(\psi_p)f_2(k_p, z = 0, h) \]
\[ w_{(2)}(x, z, t) = w_{(2)}(x, z = 0, t) + z \sum_{p=1}^{N_T} \frac{gA_p |k_p|^2}{\omega_p} \sin(\psi_p)f_1(k_p, z = 0, h) \], \hspace{1cm} (A.20) \]
\[ (A') = (y', 0 = z^d \xi | \mathcal{F}_{A'}) \cos (\xi | (y', 0 = z^d \xi | \mathcal{F}_{A'}) = \sum_{x \in \mathcal{F}_{A'}} z + (y', 0 = z^d \xi | \mathcal{F}_{A'}) = (y', 0 = z^d \xi | \mathcal{F}_{A'}) \]

\[ (A') = y', 0 = z^d \xi | \mathcal{F}_{A'} \sin \sum_{x \in \mathcal{F}_{A'}} z + (y', 0 = z^d \xi | \mathcal{F}_{A'}) = (y', 0 = z^d \xi | \mathcal{F}_{A'}) \]

\[ (A') = y', 0 = z^d \xi | \mathcal{F}_{A'} \tan \sum_{x \in \mathcal{F}_{A'}} z + (y', 0 = z^d \xi | \mathcal{F}_{A'}) = (y', 0 = z^d \xi | \mathcal{F}_{A'}) \]
Appendix B

High-order spectral (HOS) method

Dommermuth and Yue (1987) put forward a high-order spectral (HOS) method to study nonlinear gravity waves [24]. According to the HOS method, we consider the irrotational motion of a homogeneous, incompressible and inviscid fluid under a free surface in arbitrary depth, then the flow can be described by a velocity potential \( \Phi(\vec{x}, z, t) \) which satisfies Laplace’s equation within the fluid. Here \( \vec{x} = (x, y) \) is a vector in the horizontal plane. For simplicity, the time and mass units are chosen so that the gravitational acceleration and fluid density are unity. Following this method, we use the definition of surface potential:

\[
\Phi^S(x, t) = \Phi(x, \eta(x, t), t) \tag{B.1}
\]

where \( z = \eta(x, t) \) denotes the free surface, assumed to be continuous and single-valued. In terms of \( \Phi^S \), the kinematic and dynamic boundary conditions on the free surface are respectively:

\[
\eta_t + \nabla_x \Phi^S \cdot \nabla_x \eta - (1 + \nabla_x \eta \cdot \nabla_x \eta) \Phi^S_x(x, \eta, t) = 0 \tag{B.2}
\]

\[
\Phi^S_t + \eta + \frac{1}{2} \nabla_x \Phi^S \cdot \nabla_x \Phi^S - \frac{1}{2} (1 + \nabla_x \eta \cdot \nabla_x \eta) \Phi^2_x(x, \eta, t) = -P_a \tag{B.3}
\]
where $\nabla_x \equiv (\partial/\partial x, \partial/\partial y)$ denotes the horizontal gradient and $P_a$ is the atmospheric pressure. These two equations are the evolution equations for $\Phi^S$ and $\eta$.

Assuming weak nonlinearity, the velocity potential $\Phi$ is written as a perturbation series in wave steepness:

$$\Phi(x, z, t) = \sum_{m=1}^{M} \Phi^{(m)}(x, z, t),$$

(B.4)

where $M$ is the highest order of wave steepness. And the surface velocity potential can be written as a Taylor expansion at $z = 0$:

$$\Phi^S(x, t) \equiv \Phi(x, z = \eta, t) = \sum_{m=1}^{M} \sum_{l=0}^{M-m} \frac{\eta^l}{l!} \frac{\partial^l}{\partial z^l} \Phi^{(m)}(x, 0, t).$$

(B.5)

If $\Phi^S$ is given, $\Phi^{(m)}$'s on $z = 0$ can be obtained iteratively as:

$$\Phi^{(1)}(x, 0, t) = \Phi^S,$$

$$\Phi^{(m)}(x, 0, t) = -\sum_{l=1}^{m-1} \frac{\eta^l}{l!} \frac{\partial^l}{\partial z^l} \Phi^{(m-l)}(x, 0, t), \quad m = 2, 3, \ldots, M$$

(B.6)

Introducing the eigenmodes, $\Psi_n(x, z)$, which satisfy all but the free surface conditions, each $\Phi^{(m)}$ can be expressed as:

$$\Phi^{(m)}(x, z, t) = \sum_{n=1}^{N} \Phi_n^{(m)}(t) \Psi_n(x, z), \quad z \leq 0,$$

(B.7)

where $N$ is the number of eigenmodes needed. Assuming periodic condition in horizontal space, $\Psi_n$ has a form of:

$$\Psi_n = e^{ik_n x} + c.c. \quad \text{(B.8)}$$

for deep water, and

$$\Psi_n = \frac{\cosh(|k_n|(z + h))}{\cosh(|k_n|h) e^{ik_n x}} + c.c. \quad \text{(B.9)}$$

for constant finite water depth $h$. 230
Thus $\Phi_z(x, \eta, t)$ in (B.2) and (B.3) can be written as:

$$\Phi_z(x, \eta, t) = \sum_{m=1}^{M} \sum_{l=0}^{M-m} \sum_{n=1}^{N} \frac{\eta^l}{l!} \Phi_n^{(m)}(t) \frac{\partial^{l+1}}{\partial z^{l+1}} \Psi_n(x, 0) , \quad (B.10)$$

and (B.2) and (B.3) can be written as:

$$\eta_t + \nabla_x \Phi^S \cdot \nabla_x \eta - (1 + \nabla_x \eta \cdot \nabla_x \eta) \left[ \sum_{m=1}^{M} \sum_{l=0}^{M-m} \frac{\eta^l}{l!} \sum_{n=1}^{N} \Phi_n^{(m)}(t) \frac{\partial^{l+1}}{\partial z^{l+1}} \Psi_n(x, 0) \right] = 0 \,, \quad (B.11)$$

$$\Phi_t^S + \eta + \frac{1}{2} \nabla_x \Phi^S \cdot \nabla_x \Phi^S - \frac{1}{2} (1 + \nabla_x \eta \cdot \nabla_x \eta) \left[ \sum_{m=1}^{M} \sum_{l=0}^{M-m} \frac{\eta^l}{l!} \sum_{n=1}^{N} \Phi_n^{(m)}(t) \frac{\partial^{l+1}}{\partial z^{l+1}} \Psi_n(x, 0) \right]^2 = -P_a \,.
\quad (B.12)$$

Given the initial conditions $\eta(x, t = 0)$ and $\Phi^S(x, t = 0)$, with equation (B.6), the group of equations (B.11) and (B.12) can be solved using 4th-order Runge-Kutta method.

Note that the above equations only provide the solution for wave elevation and surface velocity potential. We can further derive the velocity and acceleration field under the surface in order to study the wave kinematics and dynamics. These can be obtained at the same time when we solve for elevation and surface velocity potential.

Specifically, for deep water and $z \leq 0$, from equations (B.4), (B.7) and (B.8), we have:

$$\Phi(x, z = 0, t) = \sum_{m=1}^{M} \sum_{n=1}^{N} \Phi_n^{(m)}(t) e^{i\kappa_n \cdot x} + c.c. \,.
\quad (B.13)$$

Therefore, after we obtain $\Phi^{(m)}(x, z = 0, t)$ using equation (B.6) at each time step, $\Phi_n^{(m)}(t)$ can be obtained by Fourier transform of $\Phi^{(m)}(x, z = 0, t)$. Then, the velocity field, $u(x, z, t) = \nabla \Phi(x, z, t)$, for $z \leq 0$ can be obtained as:
\[ u(x, z, t) = \sum_{m=1}^{M} \sum_{n=1}^{N} i k_n \Phi_n^{(m)}(t) \psi_n(x, z) + \text{c.c.} , \]
\[ v(x, z, t) = \sum_{m=1}^{M} \sum_{n=1}^{N} i k_n \Phi_n^{(m)}(t) \psi_n(x, z) + \text{c.c.} , \]
\[ w(x, z, t) = \sum_{m=1}^{M} \sum_{n=1}^{N} |k_n| \Phi_n^{(m)}(t) \frac{\sinh(|k_n|(z + h))}{\cosh(|k_n| h)} e^{ik_n x} + \text{c.c.} , \] (B.14)

For \( 0 < z \leq \eta \), similar to equation (B.5), we apply Taylor expansion at \( z = 0 \) and obtain:

\[ u(x, z, t) = \sum_{m=1}^{M} \sum_{l=0}^{M-m} \frac{z^l}{l!} \sum_{n=1}^{N} i k_n \Phi_n^{(m)}(t) \frac{\partial^l}{\partial z^l} \psi_n(x, 0) + \text{c.c.} , \]
\[ v(x, z, t) = \sum_{m=1}^{M} \sum_{l=0}^{M-m} \frac{z^l}{l!} \sum_{n=1}^{N} i k_n \Phi_n^{(m)}(t) \frac{\partial^l}{\partial z^l} \psi_n(x, 0) + \text{c.c.} , \]
\[ w(x, z, t) = \sum_{m=1}^{M} \sum_{l=0}^{M-m} \frac{z^l}{l!} \sum_{n=1}^{N} \Phi_n^{(m)}(t) \frac{\partial^{l+1}}{\partial z^{l+1}} \psi_n(x, 0) + \text{c.c.} . \] (B.15)

For the acceleration field, \( a(x, z, t) = (\nabla \Phi)_t(x, z, t) \). By introducing a variable \( \Lambda \), defined as:

\[ \Lambda(x, z, t) = \frac{\partial}{\partial t} \Phi(x, z, t) , \] (B.16)

and following the same procedure we did for \( \Phi \), we have:

\[ \Lambda^{(1)}(x, 0, t) = \Phi_t^S , \]
\[ \Lambda^{(m)}(x, 0, t) = - \sum_{l=1}^{m-1} \frac{\partial^l}{\partial z^l} \Lambda^{(m-l)}(x, 0, t) , \quad m = 2, 3, \ldots, M \] (B.17)

and

\[ \Lambda^{(m)}(x, z, t) = \sum_{n=1}^{N} \Lambda_n^{(m)}(t) \psi_n(x, z) , \quad z \leq 0 . \] (B.18)

Therefore, we can use the same formulas to calculate the acceleration field as those
to calculate the velocity field except we need to replace \( \Phi_n^{(m)}(t) \) with \( \Lambda_n^{(m)}(t) \).
Appendix C

Line-search algorithm and rank-two Hessian matrix update formulas in the quasi-Newton method

C.1 Line-search algorithm

In the Newton minimization method, the search vector $p_j$ of equation (4.38) is guaranteed to be a descent direction for the approximated quadratic form of the cost function (see equation (4.32)). However, this step may not sufficiently reduce the cost function and may not even decrease its value, indicating that $C(Y)$ is poorly modeled by a quadratic form in the vicinity of $Y_j$. One approach to alleviate this problem [20] is to adopt a line-search algorithm where one searches for an appropriate real positive step length $\nu_j$ along the search direction, $p_j$, which yields an acceptable next iterate, $Y_{j+1} = Y_j + \nu_j p_j$, that sufficiently decreases the cost function.

Essentially the line search problem is to determine

$$\nu_j = \arg\min_{\nu} \{ C(Y_j + \nu p_j) \} ,$$

(C.1)

which can be carried out by using any nonlinear minimization routine [87]. If the function evaluation is expensive, the full nonlinear determination of $\nu_j$ will be com-
putational expensive. It is therefore desirable to limit the number of such evaluations as much as possible. In this case we adopt the an algorithm whereby a step-length $\nu_j > 0$ is selected which reduces the cost function such that the average rate of decrease from $C(\Psi_j)$ to $C(\Psi_j + \nu_j p_j)$ is at least some prescribed fraction, $\mu$, of the initial rate of decrease at $\Psi_j$ along the direction $p_j$, i.e.,

$$C(\Psi_j + \nu_j p_j) \leq C(\Psi_j) + \mu \nu_j \delta C_{j+1},$$

where $0 < \mu < 1$ is a fractional number which, in practice, is set quite small (e.g. $10^{-4}$) so that hardly more than a decrease in function value is required. $\delta C_{j+1}$ is the rate of decrease of $C(\Psi)$ at $\Psi_j$ along the direction $p_j$ and is given by:

$$\delta C_{j+1} = \frac{\partial}{\partial \nu} C(\Psi_j + \nu p_j) \big|_{\nu=0} = g^T(\Psi_j) \cdot p_j.$$

The procedure we will adopt, is to first employ the full Newton search step and if $\nu_j=1$ fails to satisfy the criterion (C.2), then backtrack (i.e., reduce $\nu_j$) along the direction of the Newton step until an acceptable next iterate $\Psi_{j+1} = \Psi_j + \nu_j p_j$ is found.

If, at the $(j+1)$-th iteration, $\nu_j^{(m)}$ is the current step length that does not satisfy the condition (C.2), we compute the next backtracking step length, $\nu_j^{(m+1)}$, by searching for the minimum of the following function:

$$f(\nu) = C(\Psi_j + \nu p_j),$$

which we approximate by a quadratic expression as

$$f(\nu) \approx a + b\nu + c\nu^2,$$

where the real constants $a$, $b$, and $c$ are determined from the current information on the cost function $C(\Psi)$:

$$f(\nu = 0) = C(\Psi_j),$$

$$f(\nu = 0) = C(\Psi_j),$$

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\[ \frac{df}{d\nu}(\nu = 0) = \delta C_{j+1}, \quad (C.7) \]

and
\[ f(\nu = \nu_j^{(m)}) = C(\Upsilon_j + \nu_j^{(m)} p_j), \quad (C.8) \]

from which we obtain
\[ a = C(\Upsilon_j), \quad (C.9) \]

\[ b = \delta C_{j+1}, \quad (C.10) \]

and
\[ c = \frac{1}{[\nu_j^{(m)}]^2} \left[ C(\Upsilon_j + \nu_j^{(m)} p_j) - C(\Upsilon_j) - \nu_j^{(m)} \delta C_{j+1} \right]. \quad (C.11) \]

Thus, \( \nu_j^{(m+1)} \), which is the minimum of \( f(\nu) \), for \( m = 0, 1, 2, \ldots \) is given by:
\[ \nu_j^{(m+1)} = -\frac{b}{2c} = -\frac{[\nu_j^{(m)}]^2}{2} \frac{\delta C_{j+1}}{C(\Upsilon_j + \nu_j^{(m)} p_j) - C(\Upsilon_j) - \nu_j^{(m)} \delta C_{j+1}}, \quad (C.12) \]

from which it is clear that if \( C(\Upsilon_j + \nu_j^{(m)} p_j) < C(\Upsilon_j) \), then
\[ 0 < \nu_j^{(m+1)} < \frac{1}{2} \nu_j^{(m)} < \frac{1}{2^{m+1}}, \quad m = 0, 1, 2, \ldots \quad (C.13) \]

whereas if \( C(\Upsilon_j + \nu_j^{(m)} p_j) < C(\Upsilon_j) + \mu \nu_j^{(m)} \delta C_{j+1} \), then
\[ 0 < \nu_j^{(m+1)} < \frac{1}{2(1-\mu)} \nu_j^{(m)} < \frac{1}{[2(1-\mu)]^{m+1}}, \quad m = 0, 1, 2, \ldots \quad (C.14) \]

Thus, we start with \( \nu_j^{(0)} = 1 \) and proceed with the backtracking procedure of equation (C.12) until condition (C.2) is satisfied. In general, it is not desirable to decrease \( \nu_j^{(m+1)} \) too much since this may excessively slow down the iterative process, requiring many iterations to achieve very little progress towards the minimum. To prevent this slow down, we set \( \nu_j^{(m+1)} = 0.1 \nu_j^{(m)} \) if \( \nu_j^{(m+1)} < 0.1 \nu_j^{(m)} \) (but with \( \nu_j \) not to decrease below 0.1, i.e., \( \nu_{min}=0.1 \) to guard against too small a value of \( \nu \)) and then proceed
with the iteration.

To take advantage of the newly acquired information on the cost function beyond the first backtrack, one can replace the quadratic approximation of $f(\nu)$ of equation (C.8) by a cubic approximation. If $\nu_1$ and $\nu_2$ are two subsequent search steps, then according to the cubic approximation, the next search step is determined from:

$$
\nu = \frac{-b + \sqrt{b^2 - 3adC_{j+1}}}{3a},
$$

(C.15)

where $a$ and $b$ are given by:

$$
\begin{bmatrix}
a \\
b
\end{bmatrix} = \frac{1}{\nu_2 - \nu_1}
\begin{bmatrix}
1/\nu_2^2 & -1/\nu_1^2 \\
-\nu_1/\nu_2^2 & \nu_2/\nu_1^2
\end{bmatrix}
\begin{bmatrix}
f(\nu_2) - \nu_2\delta C_{j+1} - C(Y_j) \\
f(\nu_1) - \nu_1\delta C_{j+1} - C(Y_j)
\end{bmatrix}
$$

(C.16)

### C.2 Rank-two Hessian matrix update formulas

In the quasi-Newton method, there are also rank-two Hessian matrix update formulas in which $U_{j+1}$ is constructed from $U_j$ by adding a symmetric matrix of rank two.

The general form of a symmetric rank-two matrix update is given by:

$$
U_{j+1} = U_j + uu^T + \mu(uv^T + vu^T) + vv^T,
$$

(C.17)

where $u$ and $v$ are any two different vectors and $\mu$ is a scalar. The Hessian updates discussed in this section can all be derived from the following formula which satisfies the quasi-Newton condition (4.42):

$$
U_{j+1} = U_j + \frac{1}{s_j \cdot v}[(d_j + \nu_j g_j)v^T + v(d_j + \nu_j g_j)^T] - \frac{s_j^T \cdot (d_j + \nu_j g_j)}{(s_j \cdot v)^2}vv^T,
$$

(C.18)

for any arbitrary vector $v$ which is not orthogonal to $s_j$. Note that the rank-one matrix update formula (equation (4.47)) can be derived from the above equation by setting $v = d_j + \nu_j g_j$. 

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C.2.1 The Powell-Symmetric-Broyden (PSB) Update

In this update formula, the arbitrary vector $v$ is chosen to be $s_j$ to obtain:

$$U_{j+1} = U_j + \frac{1}{s_j^2} \left[ (d_j + \nu_j g_j) s_j^T + s_j (d_j + \nu_j g_j)^T \right] - \frac{s_j^T \cdot (d_j + \nu_j g_j)}{s_j^T} s_j s_j^T . \quad (C.19)$$

C.2.2 The Davidson-Fletcher-Powell (DFP) Update

In this update formula, the arbitrary vector $v$ is chosen to be $d_j$ to obtain:

$$U_{j+1} = U_j + \frac{\nu_j}{s_j^T \cdot g_j} g_j g_j^T + \frac{1}{s_j^T \cdot d_j} d_j d_j^T - \nu_j (s_j^T \cdot g_j) w_j w_j^T , \quad (C.20)$$

where

$$w_j \equiv \frac{1}{s_j^T} d_j - \frac{1}{s_j^T \cdot g_j} g_j . \quad (C.21)$$

Note that $w_j$ is orthogonal to $s_j$, i.e., $w_j^T \cdot s_j = 0$.

C.2.3 The Broyden-Fletcher-Goldfarb-Shanno (BFGS) Update

In above DFP update formula, we notice that $w_j$ is orthogonal to $s_j$. Consequently, any multiple of the rank-one matrix $w_j w_j^T$ can be added to $U_{j+1}$ without violating the quasi-Newton condition (4.42). This leads to the following update formula:

$$U_{j+1} = U_j + \frac{\nu_j}{s_j^T \cdot g_j} g_j g_j^T + \frac{1}{s_j^T \cdot d_j} d_j d_j^T - \mu \nu_j (s_j^T \cdot g_j) w_j w_j^T , \quad (C.22)$$

where $\mu$ is any scalar. In the case of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update, $\mu$ is set to zero, to obtain:

$$U_{j+1} = U_j + \frac{\nu_j}{s_j^T \cdot g_j} g_j g_j^T + \frac{1}{s_j^T \cdot d_j} d_j d_j^T . \quad (C.23)$$
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