New Applications in Revenue Management

by

Charles Mark Thraves Cortés-Monroy

B.S., University of Chile (2009)
M.S., University of Chile (2011)

Submitted to the Sloan School of Management in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Operations Research at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY June 2017

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Signature redacted

Author ........................................

Sloan School of Management

May 15, 2017

Signature redacted

Certified by ..................................

Georgia Perakis
William F. Pounds Professor of Management Sloan School of Management Thesis Supervisor

Signature redacted

Accepted by ........

Dimitris Bertsimas
Boeing Leaders for Global Operations Professor of Management Co-Director, Operations Research Center
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Abstract

Revenue Management (RM) is an area with important advances in theory and practice in the last thirty years. This thesis presents three different new applications in RM with a focus on: the firms’ perspective, the government’s perspective as a policy maker, and the consumers’ perspective (in terms of welfare).

In this thesis, we first present a two-part tariff pricing problem faced by a satellite data provider. We estimate unobserved data with parametric density functions in order to generate instances of the problem. We propose a mixed integer programming formulation for pricing. As the problem is hard to solve, we propose heuristics that make use of the MIP formulation together with intrinsic properties of the problem. Furthermore, we contrast this approach with a dynamic programming approach. Both methodologies outperform the current pricing strategy of the satellite provider, even assuming misspecifications in the assumptions made.

Subsequently, we study how the government can encourage green technology adoption through a rebate to consumers. We model this setting as a Stackelberg game where firms interact in a price-setting competing newsvendor problem where the government gives a rebate to consumers in the first stage. We show the trade-off between social welfare when the government decides an adoption target instead of a utilitarian objective. Then, we study the impact of competition and demand uncertainty on the three agents involved: firms, government, and consumers.

This thesis recognizes the need to measure consumers’ welfare for multiple items under demand uncertainty. As a result, this thesis builds on existing theory in order to incorporate demand uncertainty in Consumer Surplus. In many settings, produced quantities might not meet the realized demand at a given market price. This comes as an obstacle in the computation of consumer surplus. To address this, we define the concept of an allocation rule. In addition, we study the impact of uncertainty on consumers for different demand noise (additive and multiplicative) and for various allocation rules.

Thesis Supervisor: Georgia Perakis
Title: William F. Pounds Professor of Management
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Chapter 1

Introduction

Revenue Management (RM) has been established as one of the central pillars of OR/OM since the last third of the 20th century. In order to study this area, we also need to develop tools in the areas of: optimization, stochastic processes, game theory, machine learning, among others. In the recent years RM has captured the attention of numerous researchers and practitioners. In addition to practitioners who are the day-to-day users of these tools, institutions in charge of public policies must not ignore these decision making processes if they want to anticipate the agents’ actions. This thesis examines several different settings where RM plays an important role: from a data driven perspective (Chapter 2), a government policy perspective (Chapter 3), and the consumers’ perspective (Chapter 4).

1.1 Motivation

Despite the advancement in data acquisition, there is still a significant fraction of data that is not possible to capture. For example, customers in a market who decide not to purchase from a firm, then it is hard to estimate their willingness to pay. As a result, addressing unobserved information (often referred to as missing data) is an important pre-requisite for decision making processes. Chapter 2 studies this issue. This work is done in collaboration with one of the world-leading satellite data providers. Specifically, the company offers several data plans using a two-part tariff pricing structure.
Customers have a willingness to pay for consuming a certain amount of a service. Besides the modeling, data estimation, and optimization challenges; price experimentation is not a possibility in this case (unlike the retail setting). The reasons is that contracts between the satellite provider and its customers are set long in advance an in a long term basis.

Chapter 3 presents a problem in which the government wants to incentivize consumers to adopt a green technology through rebates. We model this setting as a Stackelberg game where firms compete in prices in order to maximize revenues given the government rebates and the demand function. Instead of looking into the utilitarian perspective of this problem (for example, the maximization of social welfare), we propose a model with a target adoption goal. We believe such a model is closer to reality\(^1\). As social welfare is not the government objective we use in our model, we analyze how suboptimal the social welfare becomes when the government sets a different objective such as expenditures but at the same time sets an adoption target of sales. Furthermore, we address another important question, how much spending will a utilitarian perspective give rise to versus a target of sales approach?

Unlike previous literature, this is the first work, to the best of our knowledge, that analyzes competition in this setting. Thus, the goal of this work is to understand the impact of competition relative to a monopoly. In particular, the focus of this thesis is on the agents involved, that is: the government, the firms, and the consumers. In practice, demand is not fully known, thus this thesis incorporates demand uncertainty. In particular, this thesis addresses the need to compute the consumer surplus for multiple items when demand is uncertain. To the best of our knowledge, there is no formal theory that captures the surplus of consumers when demand is uncertain and there are stock-outs. This has been the main motivation to develop a theory for consumer surplus under stochastic demand for multiple items. This is developed in Chapter 4 of this thesis.

In many operations management problems, consumer demand is highly uncertain by nature. The literature often models demand uncertainty using an additive and/or

\(^1\)This stems from private conversations with practitioners and public policy experts.
multiplicative noise. Then, given a market price (which could be a scalar for a single item or a vector for multiple items) and the demand realization, the surplus of consumers can be modeled as the utility experienced by a representative consumer\(^2\) minus the associated expenses. Thus, as is well known, a representative consumer utility function gives rise to the demand function. Nevertheless, as has been pointed out in the literature before, the demand function must satisfy several properties in order for the corresponding utility function to exist. This is the so-called integrability problem.

In addition to the previous nuances, the fact that demand is stochastic might lead to cases where the available units in the market are not enough to clear the realized demand for a given price (that is, give rise to stock-outs). Consequently, some consumers will enjoy the consumption (and therefore, they will gain the respective surplus), whereas others, not as lucky, will not take part in the transaction despite their willingness to pay being above the market price. Therefore, there is a need to define allocation rules among consumers. In this thesis we consider several allocation rules.

In Chapter 4, we study the impact of demand uncertainty on consumers. More precisely, would consumers be better off, on average, when demand is stochastic or when it is deterministic? In particular, this answer depends on the: (i) allocation rule, (ii) convexity/concavity of the demand curve, and (iii) the nature of the noise (additive or multiplicative). On one hand, the previous analysis can be performed considering the same market prices for both the deterministic and stochastic settings. On the other hand, the analysis can also be performed by including the firms' profit maximization problem\(^3\) and therefore, the prices used to compute the consumer surplus in the stochastic and the deterministic cases, will likely not be the same. Both of these cases are analyzed in this thesis.

\(^2\)The utility is measured in monetary units and is “compatible” with the expenditures incurred.

\(^3\)This problem corresponds to the price setting newsvendor problem, under competition if there is more than one good.
1.2 Thesis Contributions

Chapter 2 introduces a method to estimate parametric distribution functions for unobserved customers’ reservation prices given a continuous consumption level. We develop an algorithm that sequentially updates the density function of reservation prices and the consumption of unobserved customers. Then, we use this algorithm to generate missing data and complete the data of the problem. In order to determine how to price the plans a satellite provider offers, we formulate the optimization problem as a MIP. This problem is hard to solve, and as a result, we develop heuristics that make use of specific properties of the problem. We develop analytical and computational bounds for these heuristics that use different levels of information on the problem. Besides these heuristics, the problem is also formulated and solved as a dynamic programming. We perform a numerical study of the heuristics we introduce and compare their performance on realistic size instances with respect to the optimal prices (when optimal prices are possible to obtain) as well as the prices the company currently uses in practice. We find that the revenues attained by the heuristics we propose are always within 8% of the optimal solution. The revenues using the prices suggested by our heuristics give at least 7% extra revenue under different scenarios of unobserved consumers. Furthermore, we consider several instances under misspecification of the number of unobserved customers as well as the customers’ reservation prices. We show that it is preferable to underestimate the number of unobserved customers rather than overestimate it. This is due the fact that the latter case induces higher prices resulting in many lost sales and as a result hurting profits.

On the study of competition between firms carrying green technologies, in Chapter 3, one contribution lies in the study of the social welfare when the government’s objective is to set an adoption target of sales versus maximizing social welfare. More precisely, we provide bounds on how suboptimal the social welfare can be when the government focuses on achieving a sales target. We show that the suboptimality ratio is 3/4 in the case where there are no externalities. We show this bound is tight. We also provide a parametric bound as a function of externalities. Then, we show that
competition will always hurt firms while benefiting the government. Nevertheless, consumers will always benefit if the demand parameters are symmetric. In addition, we show that demand uncertainty favors consumers at the expense of the government, therefore, in a deterministic and symmetric setting, all the benefit of competition is absorbed by the government.

The main contribution of Chapter 4 lies in the development of a theory for Consumer Surplus when there are multiple items and demand is stochastic. We consider and analyze several allocation rules. We show how consumer surplus compares in a stochastic versus a deterministic demand setting under additive and multiplicative noise for the three most commonly used allocation rules: random allocation, highest willingness to pay, and lowest willingness to pay. In particular, we show that for a fixed market price, consumers are always worse off when the demand noise is multiplicative. However, if the noise is additive, the impact of uncertainty on consumers will depend on the convexity of demand and the allocation rule. Moreover, we study the case where the market price comes from a profit optimization problem of the firms.
Chapter 2

On a Variation of Two-part Tariff Pricing of Services: a Data Driven Approach

2.1 Introduction

2.1.1 Motivation

Two-part tariff is a pricing strategy widely used in the supply chain and pricing literature. In such strategy, the seller charges a fee for using a service and a price for each unit of the service consumed. In this work we focus on a variation of this pricing structure in two ways. First, the fee not only gives access to the service but also includes an amount of consumption. Second, the seller can offer multiple buying options under this pricing mechanism. The motivation to address this pricing strategy comes from the work done in collaboration with one of the world’s biggest satellite service providers. Indeed, the company’s revenues were over a billion dollars in 2015.

Due to an increasing demand for satellite data\(^1\), it is critical for the company to determine what prices to charge in the years to come. In addition, since there is almost no marginal cost for each unit of data downloaded, the pricing problem is

\(^1\)www.nsr.com/upload/research_reports/NSR_GASSD9_Brief.pdf
crucial for the company's finances. In the satellite context, the firm charges a fee (called *fixed price*) which includes a certain amount of units of data (called *bundle data*), and a *variable price* charged for those units consumed above the bundle data. Due to the high heterogeneity in customers' demand, the firm does not offer a single price but rather offers a set of multiple price contracts (also called *plans*). Under this setting, the company needs to decide for each plan the particular fixed and variable price. That is, having observed the demand from current customers, the firm would like to re-set the plans' prices in order to maximize its expected revenues. It is worth mentioning that plans' prices can not be changed on a frequent basis, since contracts with customers are established over a long time horizon. Thus, price experimentation is not a feasible option. Therefore, the pricing decisions must be made with extreme care, since these new prices will be responsible for the company's revenues in the years to come.

An important aspect of this problem relates to estimation. Furthermore, as is the case in many revenue management problems, there are different sources of unobserved (or missing) data related to this problem which must be incorporated in order to determine a new pricing strategy. Therefore, the challenge is not only on the price optimization problem itself, but also on addressing the issue of missing data. Missing data encompasses information that is not observed by the service provider: such as customers' willingness to pay, and demand of customers who are not subscribed to any of the current plans offered, yet they would subscribe if prices were lower. Once we integrate these missing data, we will be able to formulate the associated price optimization problem. We introduce algorithms with provable guarantees.

For the price optimization problem, the decision variables comprise: prices, customers' buying decisions on the plans offered, and the respective payments customers might incur. As we will see later, the size of the problem (which is mainly given by the number of customers and plans) as well as the binary nature of decision variables are some of the main limitations of the price optimization problem. In this chapter,

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2 Throughout the chapter, we will use the word *customers* to refer to *actual* and *potential customers*. In case we exclusively want to refer to those customers who are actually purchasing a plan, we will refer to them as *actual (or current, or observed) customers*. 

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we investigate how to overcome these limitations and solve realistic size instances that arise in practice.

Our goal in this chapter is to determine near optimal prices that attain revenues that outperform the revenues from the current plans set by the firm. In addition, it is important that the suggested prices are robust with respect to misspecifications on the data.

2.1.2 Contributions

As mentioned in the previous subsection, this chapter explores two main topics: estimation with missing data and price optimization for a variation of the two-part tariff problem. The proposed approach is explored in the context of a collaboration with one of the biggest satellite service providers in the world. It is worth mentioning that the methodology followed can be applied to other industries that face a similar problem structure, such as energy, amusement parks, and credit card clubs. In summary, the main contributions of the chapter are the following:

- **Estimation of Missing Data:** The development of a novel parametric approach for estimating missing data in a pricing problem when there are unobserved customers (that is, customers who currently do not purchase any plan), download demand from these customers, and willingness to pay for every type of customer.

- **MIP and Heuristics:** We propose a new model for the price optimization problem that relies on properties of the two-part tariff setting. Furthermore, we propose approximation methods that give provable “good” solutions. Proposing and analyzing such methods is important as they will allow us to solve realistic size instances for the satellite company. The original MIP formulation is hard to solve even for medium size instances.

- **Suboptimality Bounds:** We establish suboptimality bounds on the heuristics for different levels of information on the problem. As expected, the performance
of the bounds improves as the firm acquires more information on the customers' behavior. In practice, these bounds deliver a guarantee that the heuristic solution is at most 27\% from the optimal solution.

- **Revenue Improvement:** We establish that the company can improve its revenue by around 7\%. Yet, even if there are misspecifications on some of the assumptions, we show that we can still outperform the current prices and improve revenues. We notice that the proposed prices are consistently lower when there are more unobserved customers.

2.1.3 Literature review

Two-part tariffs were introduced more than a hundred years ago (see [12]). Numerous works have studied different aspects of this pricing strategy. Two-part tariffs were first adopted in the electricity industry in London at the end of the nineteenth century. This pricing strategy was introduced from the motivation to raise prices during peak-demand periods in which marginal costs were higher than regular demand periods. Yet, back on those days, it was not possible to charge a time dependent price. See [12] for more details on the origins of the applications of the two-part tariff pricing policy. In the next decades, other industries, such as telephone and gas, started to implement this pricing strategy. In a recent work, [4] state that this pricing mechanism has gained particular interest in industries that share some of the following characteristics: high usage heterogeneity, low marginal costs, and capacity constraints. Therefore, services such as credit card memberships, internet, and amusement parks have adopted this strategy to some extent\(^3\). One of the first works that captured the attention of researchers is by [17]. This work studied the case of two-part tariffs on Disneyland under a profit-maximization objective. The author concludes that two-part tariff pricing structure acts as a discriminatory pricing mechanism enhancing the park’s revenues. Similarly, [14] looks at the application of two-part tariffs on AT&T. [25] provides a more complete survey on different variations of the two-part tariff problem

\(^3\)\cite{4} call these industries as “access industries”, referring to industries in which consumers pay for the privilege to access a facility but do not acquire any right to, or use the facility itself.
under a profit-maximization framework. While some works have focused on the Pareto optimality of the problem, (see for example, [5] and [16]), others have focused on the welfare impact of two-part tariffs (see [13], and [20]). More recently, two-part tariff contracts have been studied in the supply chain context, see [19], [1], and [23]. Most of these works assume a model where the parameters are fully known and observable. This chapter differs in the following three aspects: (i) the access fee gives the right to consume a given amount of the service, (ii) the firm can offer multiple plan options of the service, and (iii) we study the problem from a data-driven perspective where a firm does not have any additional demand information other than the demand of its current customers. In particular, using a model based on not observable elements from the data indicates the presence of missing data.

Missing data has been widely studied during the last sixty years under many different settings. [8] introduced a framework to analytically compute a distribution for a survivor function of censored data. Several other works have also studied nonparametric likelihood estimation problems, see [24] and [11]. The EM algorithm was formally introduced ([2]) as a general way to compute a maximum likelihood estimate from incomplete data when the direct maximization problem is hard. Until today, this algorithm has been widely used in the literature to address missing data problems in parametric and nonparametric models. This method has been applied to estimate information that might not be observed by the firm such as: lost-sales, product substitution, and reservation prices. [26] used the EM algorithm to estimate the lost sales and substitutions from the observed sales transnational data using a discrete choice model, MNL. [10] developed a demand and substitution estimation model using the EM algorithm for assortment optimization. [27] used limited distribution information to optimize over problems without full information, while others have studied the value of full demand information (see for example, [6]). Other works have used Bayesian methods to estimate missing data (see [15]). In our case, the firm offers multiple “products” (plans), yet these products only differ in terms of (i) the bundle that is included in the fixed price and (ii) the fixed and variable prices. Therefore, a choice model (such as MNL) will not have the needed vector of attributes that char-
acterize each of the plans. Moreover, there is lack of price variability on the plans, since these have been set to the current prices for a long time. Our work assumes that a consumer's purchasing behavior is driven by reservation prices.

Works like [9] and [7], estimate reservation prices of customers from the attributes of the products using conjoint analysis. Unlike our case, as aforementioned, our products lack the needed attributes to apply this technique. [3] and [21] studied a pricing optimization problem for multiple items facing different customers segments characterized by a size and a reservation price vector. These papers formulate the firm's pricing problem as a MIP and develop heuristics and cuts to solve the problem. Similar to these works, we assume consumers choose the product that delivers higher (and positive) surplus. An alternative approach, besides random utility models and reservation prices, is the use of rankings over the products. [18] developed approximation algorithms for a pricing problem over items when the customer's purchase decision is based on ranking. In this chapter we do not consider a ranking approach since we are not addressing the assortment problem itself. Furthermore, ranking-based models often lead to an exponential number of parameters. [22] developed MIP formulations for a pricing problem under MNL demand. In this chapter, customers incur a fixed demand for the service and reservation price. As a result, customer purchasing decisions will be the outcome from their demand and reservation price, and the plans' prices.

2.1.4 Chapter Structure

The remainder of the chapter is structured as follows. Section 3.2 describes the model and assumptions. The methodology we introduce to address the missing data problem is given in Section 2.3. Section 2.4 describes in detail the price optimization problem, the developed heuristics, and suboptimality bounds. Section 2.5 describes the performance of the developed heuristics with respect to the optimal solution and current prices. Finally, Section 2.6 discusses our conclusions. Proofs and additional material are relegated to the Appendix.
2.2 Model

Consider a single satellite service provider who offers $S$ different plans. Each plan $s \in \{1, \ldots, S\}$ is defined by three variables: (i) a bundle of data, $B_s$ megabytes, (ii) a fixed price $f_s$, and (iii) a variable price $v_s$; in other words, a customer who purchases plan $s$ will pay the fixed price $f_s$ plus the variable price $v_s$ for each megabyte downloaded above the bundle data $B_s$. Wlog $B_1 < B_2 < \cdots < B_S$. This suggests that a customer who is subscribed to a plan $s \in \{1, \ldots, S\}$ and downloads $d$ megabytes in a month would incur a payment of

$$f_s + v_s \times \max\{d - B_s, 0\}.$$

The firm currently has a set of plans that it offers to its current customers. These can be described as the set of vectors $\{(f_s^0, v_s^0, B_s)\}_{s=1}^S$. We will refer to these plans as current plans. As mentioned in the previous section, the idea is to find the optimal values for the fixed and variable prices, while preserving the current number of plans and current bundle data for each plan. The reason we only focus on the prices are: first, changing the plans’ structures and prices lead to many degrees of freedom in the optimization problem, making harder its formulation and solving. In addition, changing only prices leads to better insights regarding the suggested new prices that should be charged. Second, changing the bundle data might, unlike prices, have additional costs for the company in the way the resources are allocated to meet customers’ actual demand under the current plans, thus the company we worked in collaboration with, preferred to only focus on the pricing problem.

The current plans give rise to two types of customers: (i) observed and (ii) unobserved customers. Observed customers are those who are actual clients, i.e., those who are subscribed to one of the current plans. Unobserved customers are potential customers who are not subscribed to any of the current plans, although they might if prices were lower. We denote by $n$ the number of observed customers, and $m$ the number of unobserved customers. As expected, the number of observed customers is known, but not the unobserved ones. Also we assume that the number of customers
does not vary from one month to another, neither do the customers’ downloads. This simplification will allow us to study the problem in a monthly horizon\(^4\). We will denote by \{1, \ldots, n + m\} the set of all customers (observed and unobserved), \{1, \ldots, n\} will denote the observed customers, and \{n + 1, \ldots, n + m\} the unobserved customers. Each customer \(i \in \{1, \ldots, n + m\}\) has a download value \(d_i\) and a reservation price \(r_i\); both quantities known by them. The reservation price is the maximum amount a consumer would pay for purchasing a plan. Hence, a consumer is represented by the pair of her download data and reservation price (see also the left panel of Figure 2-1). Customers are assumed to be rational, i.e., they will choose the cheapest plan according to their level of download; however, they will only purchase this plan if their reservation price is greater than or equal to the price of this cheapest plan. For ease of notation, let us call \(l_i\) the cheapest price customer \(i\) would face among the offered plans. That is,

\[
l_i = \min_{s \in \{1, \ldots, S\}} f_s + v_s \times \max\{d_i - B_s, 0\}. \tag{2.1}
\]

As a result, a customer \(i \in \{1, \ldots, n + m\}\) will purchase plan \(s\) if and only if \(r_i \geq l_i\) and \(l_i = f_s + v_s \times \max\{d_i - B_s, 0\}\). In case the customer is indifferent between more than one plan, we assume he chooses randomly among them. Customers subscribe to at most one plan. The right panel of Figure 2-1 depicts the same customers shown in the left panel, and two plans: \(s\) and \(t\). Customers who are willing to pay above the price of any of the plans will purchase the cheapest plan, i.e., the customer in the top left of Figure 2-1 will purchase plan \(s\), while the two customers in the top right will purchase plan \(t\). On the contrary, customers who are below all plans’ price curve are those whose reservation price is lower than any payment they would incur when purchasing the offered plans. As a result, these customers will not purchase any of them.

In setting the prices of the plans, there are important business rules that must be respected. First, the fixed price of the plans should be increasing in the plan

\(^4\)From the data, we notice that customers demand across different months do not differ significantly.
Figure 2-1: Left: Each customer is represented by a pair of reservation price and download. Right: Customers and plans' prices structure.

...
data is irrelevant. Naturally, \( c_s \in [0, B_s - B_{s-1}] \) for all \( s \in \{1, \ldots, S - 1\} \) (consider \( B_0 = 0 \)). Note that \( c_s \) does not play any significant role for the plan with smaller bundle data since this plan will be the cheapest for downloads in \([0, B_1]\). We consider \( c_s = (B_s - B - s - 1)/2 \) for \( s \in \{2, \ldots, S\} \) and \( c_1 = B_1 \).

![Figure 2-2](image-url)

Figure 2-2: Left: Case where the bundle data of plan \( t \) is attractive to some customers within a range of downloads. Right: Plan \( t \) bundle data at fixed price \( f_t \) is dominated by plan \( s \), therefore, plan \( t \)'s bundle data does not attract any consumers.

The firm’s problem is to find the optimal prices of the plans that maximize the expected revenue. Ideally, if we had full information on the pool of customers (their downloads and reservation prices), we would directly formulate the pricing optimization problem. Unfortunately, this is not the case. The left panel of Figure 2-3 shows a toy instance with full data knowledge, where all customers’ (observed and unobserved) downloads as well as their reservation prices are known. The right panel of Figure 2-3 shows the corresponding data that we actually observed from the instance depicted in the left panel of Figure 2-3. We can see that the only information that is captured by the seller is the payments and downloads of the observed customers. As a result, we can not go directly into the optimization problem of the firm without first addressing the missing data of the problem. The next section provides the details on the methodology we propose in order to overcome this issue.
Figure 2-3: Left: Full data scenario where the optimization problem is choosing the optimal fixed and variable prices, i.e. $\{f_s, v_s\}_{s=1}^4$. Right: The observed data.

2.3 Missing Data

As mentioned in Section 2.1.1, there are several sources of missing data which can be summarized as the following: (i) the number of unobserved customers ($m$), (ii) the download demand of unobserved customers ($d_i$ for $i \in \{n+1, \ldots, n+m\}$), and (iii) the reservation prices of all customers ($r_i$ for $i \in \{1, \ldots, n+m\}$).

2.3.1 Number of Unobserved Customers

The number of unobserved customers, those who are not purchasing any plan today yet they would if prices were lower, is unfortunately unidentifiable in the model. Therefore, we will consider different scenarios on the number of these. This will allow us to analyze how optimized prices and revenues will vary under the different cases. The values considered for these (unobserved) customers are $m \in \{n/3, 2n/3, n, 4n/3, 5n/3\}$, where $n$ is the number of observed customers. These scenarios were provided to us from the specific company we collaborated with.
2.3.2 Download of Unobserved Customers

With respect to the download values of the unobserved customers denote the pdf by $f_d$ and the cdf by $F_d$ of this probability distribution. In addition, consider the two underlying distributions that comprise this distribution: the download distributions of observed and unobserved customers. Call $f_d^{obs}$ and $f_d^{unobs}$ the pdf of the download distribution for observed and unobserved customers respectively, and similarly $F_d^{obs}$ and $F_d^{unobs}$ the respective cdfs. Given this, the density of the total download can be written as

$$f_d(x) = \frac{n}{n+m}f_d^{obs}(x) + \frac{m}{n+m}f_d^{unobs}(x).$$  \hspace{1cm} (2.2)

The idea of separating the download distribution for each customer type lies behind the fact that despite the fact we do not know the distribution of all customers’ downloads, we can at least say something about the distribution of downloads of the observed customers (since their download values is information we can observe). The left panel of Figure 2-4 shows the histogram of download values of the observed customers taken from the satellite company we collaborated with. It can be seen that the logarithm of the download values follows roughly a Gaussian distribution. Yet, a single Gaussian distribution clearly does not provide a good enough fit for the empirical data. Consequently, using a mixture of Gaussians considerably improves the fit, in particular for the right tail of the downloads, (see left panel of Figure 2-4). After investigating the log-likelihood, BIC, and AIC under different number of mixtures (see Table 2.1), we choose a mixture with five Gaussians since the marginal improvement of these statistics is quite significant up to five Gaussians. The selected fitted mixture of Gaussians over the empirical data of observed customers is depicted in the right panel of Figure 2-4. The cdf of this fitted distribution is denoted by $\hat{F}_d^{obs}$. Similarly, the estimated cdf for all download values is denoted by $\hat{F}_d$.

Section 2.3.4 describes the steps of the algorithm used to estimate the distributions, in particular $F_d$. In this procedure, we start by assuming that the distribution of downloads of all customers is the same as the distribution of downloads of ob-
Figure 2-4: Left: Histogram of download values of observed customers. Right: Histogram of densities of download values of observed customers and the fitted mixture with five gaussians.

<table>
<thead>
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<th>Number of Gaussians</th>
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<th>AIC</th>
</tr>
</thead>
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<tr>
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<td>132693.30</td>
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<tr>
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</tr>
<tr>
<td>8</td>
<td>-65750.25</td>
<td>131695.45</td>
<td>131546.49</td>
</tr>
</tbody>
</table>

Table 2.1: Log-likelihood, BIC, and AIC for the different number of Gaussians fitted on the empirical observable download values.

served customers. Then, as the algorithm iterates, the distribution is updated by looking at posterior probabilities. In other words, the cdf $F_d$ (and therefore $F_d^{\text{obs}}$) will be changing, while $F_d^{\text{obs}}$ remains unchanged. More details on this are given in Section 2.3.4 where the algorithm is described.
2.3.3 Reservation Prices

Customers' reservation prices are modeled through a parametric probability distribution. Naturally, this distribution should not be the same for different levels of downloads, since customers who download more might be willing to pay more. Consequently, we will assume that the distribution of customers' reservation price is conditioned on the download amount. The probability distribution function of the reservation prices is denoted by $\mathcal{F}_r$ (examples include normal, uniform, exponential, or other). The parameters of this distribution function are $\lambda \in \mathbb{R}^t$ and are given by a function $g$ of the download $d$ and a set of parameters to estimate $\theta \in \mathbb{R}^u$, i.e., $g : \mathbb{R}_+ \times \mathbb{R}^u \rightarrow \mathbb{R}^t$. In other words,

$$r|d \sim \mathcal{F}_r^{g(d,\theta)}.$$ 

The cdf of the reservation price under the function $g$, parameters $\theta$, and download value $d$; is denoted by $F_r^{g(d,\theta)}$. For example, $r|d \sim \mathcal{N}(\theta_1 + \theta_2 d, \theta_3 + \theta_4 d)$, where $\lambda = g(d, \theta) \in \mathbb{R}^2$ is the mean and standard deviation, and $\theta \in \mathbb{R}^4$ are the parameters that need to be estimated.

Once we consider a fixed distribution $\mathcal{F}_r$ and function $g$ (see Section 2.3.5 for details on this), the next step is to obtain the best estimation of the parameters $\theta$ that maximizes the observed likelihood. The details on the estimation of $\theta$ are described in the following section.

2.3.4 Estimation of the Parametric Model

In what follows, we provide the description of a procedure that allows us to estimate the parametric probability distribution functions. Recall that the two distributions we are trying to estimate are the download values of unobserved customers, and the reservation prices of customers, given their level of download.

In the following, we describe the details of each step of the estimation process.

0. Input: The algorithm needs the following inputs: the current plans’ bun-
dle data, the fixed and variable prices \(((f^0_s, v^0_s, B_s))_{s=1}^S\), the downloads of observed customers \(d_i\) for all \(i \in \{1, \ldots, n\}\), the (assumed) number of unobserved customers \(m\), the distribution type of reservation prices \(F_r\), a function \(g(d, \theta) : \mathbb{R}_+ \times \mathbb{R}^u \to \mathbb{R}^t\) that captures the parameters of the distribution of reservation prices, and a value for \(p \in (0,1]\) (the use of this parameter \(p\) will be explained in Step 5 of the algorithm).

**Output:** The estimated parameters of the customers’ reservation price distribution \(\hat{\theta}\), and the estimated unobserved customers’ download distribution \(\hat{F}_d^{unobs}\).

1. **Fit observed downloads:** Fit a mixture of Gaussians on the logarithm of the download values of observed customers, \(\hat{F}_d^{obs}\). As observed in Section 2.3.2, this distribution has a very good performance in fitting the observed customers’ downloads. Then, as a starting point, we will force the distribution of unobserved customers’ downloads to be the same as the observed ones, i.e. \(\hat{F}_d^{unobs} = \hat{F}_d^{obs}\), (thus \(\hat{F}_d = \hat{F}_d^{obs}\)).

2. **Sample unobserved downloads:** Sample the download values of the unobserved customers \(d_{n+1}, \ldots, d_{n+m}\) according to the distribution with cdf \(\hat{F}_d^{unobs}\).

3. **Compute minimum payment:** Compute the minimum payment that each customer would incur if she would purchase the cheapest plan given her level of download. In other words, compute \(l_i\) for all \(i \in \{1, \ldots, n+m\}\) according to Equation (2.1). The plans’ structure considered is the one of the current plans, that is, with the current bundle data, as well as current fixed and variable prices.

4. **Maximize log-likelihood:** Choose the parameter \(\theta\) that maximizes the log-likelihood. The log-likelihood consists of two terms: the first one is the sum of the logarithm of the probabilities that the observed customers purchase a plan (the cheapest one), and the second one is the logarithm of the probability that the unobserved customers do not purchase any of the offered plans. The
optimization problem that arises is

$$
\max_{\theta} \sum_{i=1}^{n} \ln \left(1 - F_{r}^{\theta(d_i, \theta)}(l_i)\right) + \sum_{i=n+1}^{n+m} \ln \left(F_{r}^{\theta(d_i, \theta)}(l_i)\right).
$$

(2.3)

Denote by \( \hat{\theta} \) the optimal solution obtained from the optimization problem (2.3).

At this point, the algorithm can be terminated since we have obtained an estimation of the distribution of the reservation prices, as well as of the downloads of unobserved customers that has been assumed to be equal to the distribution of downloads of observed customers. This last assumption is rather strong and perhaps unrealistic, because observed and unobserved customers may probably not have the same download pattern. In order to address this issue, we can iterate again over Steps 3 and 4 of the algorithm described above, by updating the values of the downloads of unobserved customers. More precisely, after Step 4, we can resample the downloads of unobserved customers and update the current estimations of the download distribution of all customers (including unobserved customers). In order to avoid degeneracy in the estimated downloads, we fixed a fraction \( p \) of the unobserved customers’ downloads at each iteration.

5. **Fix a portion of unobserved downloads:** Fix a fraction \( p \in (0, 1] \) of the (unfixed) unobserved customers’ downloads. For example, \( p = 1/3 \), means that at each iteration, a third of the unobserved customers’ downloads that have not yet been fixed will be fixed. Therefore, the number of unobserved download samples in Step 6 will be \( 2m/3 \) in the first iteration, \( 4m/9 \) in the second, \( 8m/27 \) in the third, and so on, until all values of unobserved customers are fixed, or until another stopping criterion is met (see Step 8).

6. **Resample unobserved downloads:** The objective in this step is to sample download values for unobserved customers (whose download values have not yet been fixed) while using the information from the estimated parameters \( \theta \) of the reservation price distribution in Step 4. To do this, we sample the a download according to the distribution of all downloads, and then we compute
the posterior probability that this customer will not purchase the corresponding
cheapest service at the sampled level of download.

More precisely, the prior distribution of the downloads of all customers ($\tilde{f}_d$)
is weighted by the probability that the customer does not purchase (i.e. is
an unobserved customer), thus unobserved customers’ downloads are sampled
proportional to

$$\tilde{f}_d^{\text{unobs}}(x) \propto F_r^{g(x, \tilde{\theta})}(l(x)) \times \tilde{f}_d(x),$$

where with some abuse of notation, $l(x)$ denotes the respective cheapest pay-
ment, given a download $x$ according to Equation (2.1).

7. Update download distribution: Since the unobserved customers’ downloads
have changed, we need to update the probability distribution of the customers’
download, i.e. $\bar{F}_d$. Accordingly, we fit a mixture of Gaussians to the whole set
of download values, including observed and unobserved customers’ downloads.

8. Stopping criteria: The algorithm stops when whether all unobserved cus-
tomers’ downloads have been fixed or there is no further improvement in the
log-likelihood value, otherwise go to Step 3.

A summary of the algorithm (which will be refer as Algorithm A.1.1 in what
follows) is presented in Appendix A.1. Once we run Algorithm A.1.1, we will have
an estimate of the downloads of customers ($\bar{F}_d$) and so unobserved customers too
($\tilde{F}_d^{\text{unobs}}$), and a probability distribution for the reservation prices for each level of
download, namely the cdf $F_r^{g(d, \tilde{\theta})}$. Thus, to generate the missing data, we proceed as
follows:

I Start with a number of unobserved customers $m$.

II Run Algorithm A.1.1.

III Sample the download values of unobserved customers $d_{n+1}, \ldots, d_{n+m}$ according to
the cdf $\tilde{F}_d^{\text{unobs}}$ obtained from the algorithm.
IV Sample the reservation prices: For every observed customer $i \in \{1, \ldots, n\}$ sample their reservation price $r_i$ as $\mathcal{F}_{r|r \geq l_i}^{q(d_i, \delta)}$ where $l_i$ is the minimum payment for consumer $i$ computed according to Equation (2.1). Similarly, for every unobserved customer $i \in \{n + 1, \ldots, n + m\}$ sample their reservation price $r_i$ as $\mathcal{F}_{r|r < c_i}^{q(d_i, \delta)}$.

Figure 2-5 shows an instance the sampled downloads and reservation prices. Customers below the price curve are those who are not observed with current plans, whereas customers above the price curve are the observed ones.

![Reservation prices vs download](image)

Figure 2-5: Sampled downloads and reservation prices.

### 2.3.5 Computations of the Estimation of Missing Data

In this subsection, we describe the different parametric models considered for the estimation of the distributions of missing data. On the one hand, we consider the distribution of the probability of the reservation price ($\mathcal{F}_r$), and on the other hand, we consider a functional form ($g$) where the parameters of the distribution are modeled as a function of the download ($d$) and the parameters ($\theta$) we need to estimate. Regarding the distribution of reservation prices, $\mathcal{F}_r$, we consider the Uniform, Exponential, and

---

6 For confidentiality reasons, we are not listing the prices in the y-axis.
Normal distributions. The reason we choose in our computations these distributions is because they have fairly simple but yet realistic distributions, endowed with nice properties, and hence make computations tractable.

Regarding the function of the distribution parameters, the following are the functions we considered

\[
g(d, \theta) \in \left\{ \begin{array}{l}
\theta_1 + \theta_2 d, \\
\theta_1 + \theta_2 d, \\

d^\theta, \\
\theta_2 (\theta_1 + d)^\theta_3, \\
\theta_3 (\theta_1 + \theta_2 d + d^2)^\theta_4
\end{array} \right\}.
\]

In the case of the Uniform distribution \(U[g_1(d, \theta), g_2(d, \theta)]\), the same functional form is considered for \(g_1\) and \(g_2\) with separate parameters \(\theta\) for each function. For the Exponential distribution, we consider the convention that the parameter \(\lambda\) of an exponential r.v. is such that the density function is given by \(f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\). That is, \(\lambda = g(d, \theta)\). For the Normal distribution, we initially considered two functions for the parameters, i.e. Normal\((g_1(d, \theta), g_2(d, \theta))\). However, when running Algorithm A.1.1, it resulted in high standard deviation values across all levels of downloads which is not particularly desirable\(^7\). In order to have more control on the behavior of the standard deviation of reservation prices for the normal distribution case, we consider a function \(g_2(d, \theta)\) with fixed values for \(\theta\). Due space limitations, we omit further details.

Despite the fact that the Exponential and Uniform distributions have closed form solutions for their corresponding cdfs, this is not the case for the Normal distribution\(^8\). Both distributions present key disadvantages in practice. The Uniform distribution has the drawback that it gives positive density on every part where there is some reservation price. Therefore, it results on support intervals for reservation prices where customers with low and high valuations are over-represented, while customers with medium valuations are underrepresented. In the case of the Exponential distribution, the main problem is that leads to extremely high standard deviations for

---

\(^7\)For example, having customers that download 1 Mb with a mean reservation price of $200 and a standard deviation of $10,000 might not be close to reality.

\(^8\)note that the log-likelihood expression in Equation (2.3) requires the cdf.
the reservation prices. As a result, for high levels of downloads, it will set overly high reservation prices. The Normal distribution turned out to be the one that has the best behavior among the three distributions considered. We run Algorithm A.1.1 with different parametric functions $g(d, \theta)$ for the mean according to Equation (2.4). Table 2.2 shows the average log-likelihood, over 10 random seeds, for different scenarios of unobserved customers and different parametric functions. It can be seen that the function that gives almost the best performance in terms of log-likelihood is $g(d, \theta) = \theta_2(\theta_1 + d)^{\theta_3}$, obtaining one of the highest average log-likelihood in all the different scenarios of unobserved customers. Also, it is worth noticing that parametric functions with a quadratic term in the download, do not add considerable value.

<table>
<thead>
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<th>$n \times 4/3$</th>
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</tr>
<tr>
<td>$\theta_2(\theta_1 + d)^{\theta_3}$</td>
<td>-44328</td>
<td>-61436</td>
<td>-72756</td>
<td>-81201</td>
<td>-88282</td>
</tr>
<tr>
<td>$\theta_3(\theta_1 + \theta_2 d + d^2)^{\theta_4}$</td>
<td>-23558</td>
<td>-35852</td>
<td>-44886</td>
<td>-52269</td>
<td>-58483</td>
</tr>
</tbody>
</table>

Table 2.2: Log-likelihood for different parametric functions of the mean reservation price after running Algorithm A.1.1. The reservation price follows a Normal distribution.

Figure 2-6 depicts the percentile of the reservation price distribution for each level of download after running Algorithm A.1.1. At the left of Figure 2-6 we show the case where $m = n/3$, and at the right where $m = n$. As is expected, the distribution of reservation prices, in the case where there are more unobserved customers, gives a lower mean. Note that the curve denoting the mean is lower in the plot of the right respect to the plot on the left. This behavior is expected, since the presence of more unobserved customers implies that there will be more people in the market who value the plans less.
Figure 2-6: Reservation price distribution for each level of download. At the left the number of unobserved customers is a third of the observed ones, while at the right the number of unobserved customers is assumed to be the same as the observed customers.

2.4 Optimization Model

At this point, we have introduced a methodology on how to obtain the necessary data for optimizing the prices of the various plans. In summary, the data needed for the pricing optimization problem includes: (i) customers' downloads, (ii) customers' reservation prices, (iii) the number of plans, (iv) the bundle data of each plan, and (v) the minimum range of megabytes over which the fixed price of a plan will be the cheapest option (this is given by $c_s$ for $s \in \{1, \ldots, S\}$, see Section 3.2). For ease of notation, let us call $N = n + m$ the total number of customers and denote $\alpha_{is}$ the amount of download of customer $i$ that exceeds the bundle data of plan $s$, namely

$$\alpha_{is} = \max\{d_i - B_s, 0\}$$

for each customer $i \in \{1, \ldots, N\}$ and plan $s \in \{1, \ldots, S\}$. The pricing optimization problem can be formulated as a MIP, where the decision variables correspond to the three following groups: plan prices, payments, and purchasing decision. In order to capture customers' purchase decision, we introduce binary variables. This will
translate in having \( N(S + 1) \) binary variables. Fortunately, despite the fact that customers can purchase any of the plans offered, they will decide between only two plans from the whole set of plans. This observation is presented more formally in the next proposition. Let us define \( \overline{s} : \mathbb{R}^+ \to \{1, \ldots, S\} \) such that \( \overline{s}(d) = \min\{s \in \{1, \ldots, S\} | d < B_s\} \). In other words, the function \( \overline{s}(d) \) corresponds to the lowest index plan with a bundle data greater than the download value \( d \). Equivalently,

\[
B_1 < \cdots < B_{\overline{s}(d)} - 1 \leq d < B_{\overline{s}(d)} < \cdots < B_S. \quad (2.5)
\]

**Proposition 1.** Given a set of plans \( \{1, \ldots, S\} \) such that \( B_1 < B_2 < \cdots < B_S \), a customer who downloads \( d \) megabytes will consider to buy only among the two plans: \( \overline{s}(d) - 1 \) and \( \overline{s}(d) \). If \( \overline{s}(d) = 1 \), the customer will only decide whether to buy plan \( s = 1 \).

Additionally, if there is a customer whose download \( d \) is such that \( d \in [B_s - c_s, B_s] \) for some plan \( s \in \{1, \ldots, S - 1\} \), then this customer will only consider to buy the respective plan \( s \). This happens to be the plan \( \overline{s}(d) \).

**Proof.** Proof. See Appendix A.2. \( \square \)

The first part of Proposition 1 suggests that a consumer will consider purchasing only between the two following options: (i) the plan with the biggest bundle data such that the bundle is below her level of download, or (ii) the plan with the smallest bundle data such that the bundle is above her level of download. In the first case, the customer will pay the fixed price plus the respective variable price times the downloaded megabytes above the bundle. In the second case, the customer will pay only the fixed price (since this plan’s bundle data is above the customer’s download). The second part of Proposition 1 suggests that there are some customers who would consider to purchase only the plan with bundle data just above the customer’s download. Let us define \( F \subseteq \{1, \ldots, N\} \) the set of customers under this category, i.e.,

\[
F = \{i \in \{1, \ldots, N\} | B_s - c_s \leq d_i \leq B_s, \text{ for } s = \overline{s}(d_i)\}, \quad (2.6)
\]

and denote it’s complement \( F^c = \{1, \ldots, N\} \setminus F \). In what follows we introduce the
decision variables. Let $f_s$ and $v_s$ represent the fixed and variable price respectively for every service $s \in \{1, \ldots, S\}$. For each customer $i \in \{1, \ldots, N\}$, the decision variable $z_i$ represents the payment incurred by customer $i$. Let $q_i$ is equal to 1 if the customer $i$ purchases any plan, and 0 otherwise. To keep track of the particular plan each customer purchases (if any), the decision variable $y_{is}$ is introduced for $s \in \{\bar{s}(d_i) - 1, \bar{s}(d_i)\}$ and $i \in \{1, \ldots, N\}$, so that $y_{is} = 1$, if customer $i$ purchases plan $s$, $y_{is} = 0$ otherwise. Note that according to the second part of Proposition 1 and the definition of the set $F$ (see Equation (2.6)), variables $y_{i, \bar{s}(d_i) - 1}$ can be forced to be zero for customers $i \in F$.

Before stating the MIP formulation, the following proposition states some valid inequalities induced by logical relationships among certain pairs of binary variables.

**Proposition 2.** Consider a set of plans $\{1, \ldots, S\}$ such that $B_1 < B_2 < \cdots < B_S$, and two customers $i, j \in \{1, \ldots, N\}$ where $r_i \geq r_j$. Then the following inequalities hold:

(a) If $d_i \leq d_j$, then $q_i \geq q_j$.

(b) If $\bar{s}(d_i) = \bar{s}(d_j)$ and $d_i \leq d_j$, then $y_{i, \bar{s}(d_i) - 1} \geq y_{j, \bar{s}(d_j) - 1}$.

(c) If $\bar{s}(d_i) = \bar{s}(d_j)$ and $d_i \geq d_j$, then $y_{i, \bar{s}(d_i)} \geq y_{j, \bar{s}(d_j)}$.

(d) If $\bar{s}(d_i) = \bar{s}(d_j)$ and $i, j \in F$, then $y_{i, \bar{s}(d_i)} \geq y_{j, \bar{s}(d_j)}$.

**Proof.** Proof. See Appendix A.3. □

In practice, instead of adding all the previous inequalities, we consider only a subset of these. Given $i, j \in \{1, \ldots, N\}$ where $r_i \geq r_j$, let $R = \{k \in \{1, \ldots, N\} | r_i > r_k > r_j\}$ and $D = \{k \in \{1, \ldots, N\} | d_i < d_k < d_j \text{ or } d_j < d_k < d_i\}$. Then, inequalities (a), (b), (c) from Proposition 2 are added only when $R \cap D = \emptyset$, and inequalities (d) of Proposition 2 when $R = \emptyset$. The purpose of adding these constraints only when there are no downloads and reservation prices between customers $i$ and $j$, is to avoid adding an extremely large number of constraints in the problem formulation. Let us denote $\mathcal{P}(r, d, B) \subseteq \mathbb{R}^{N+2N}$ the feasible set of variables $(q, y)$ generated by the constraints of Proposition 2. Using the results of Proposition 1 and 2, we can formulate the MIP as follows:
\[
\begin{align*}
\max_{f,v,z,q,y} & \sum_{i=1}^{N} z_i & \quad (2.7a) \\
\end{align*}
\]

\[
\begin{align*}
& z_i \leq f_s + v_s \alpha_{is} & \forall i \in F^c, \forall s \in \{\overline{s}(d_i) - 1, \overline{s}(d_i)\}, \\
& z_i \geq f_s + v_s \alpha_{is} - M(1 - y_is) & \forall i \in F^c, \forall s \in \{\overline{s}(d_i) - 1, \overline{s}(d_i)\}, \\
& z_i \leq f_{\overline{s}(d_i)} & \forall i \in F, \\
& z_i \geq f_{\overline{s}(d_i)} - M(1 - q_i) & \forall i \in F, \\
y_i,\overline{s}(d_i) - 1 + y_i,\overline{s}(d_i) = q_i & \forall i \in F^c, \\
z_i \leq r_i q_i & \forall i \in \{1, \ldots, N\}, \\
f_s \leq f_{s+1} & \forall s \in \{1, \ldots, S - 1\}, \\
v_s \geq v_{s+1} & \forall s \in \{1, \ldots, S - 1\}, \\
f_{s+1} \leq v_s(B_{s+1} - c_{s+1} - B_s) + f_s & \forall s \in \{1, \ldots, S - 2\}, \\
q_i, y_{is} \in \{0, 1\} & \forall i \in F^c, \forall s \in \{\overline{s}(d_i) - 1, \overline{s}(d_i)\}, \\
f_s, v_s, z_i \geq 0 & \forall s \in \{1, \ldots, S\}, i \in \{1, \ldots, N\}, \\
q, y \in \mathcal{P}(r, d, B). & \quad (2.7m)
\end{align*}
\]

The Objective Function (2.7a) represents the total revenue. Constraints (2.7b) enforces that a customer will not pay more than the corresponding payment of each plan. Constraint (2.7c) models the fact that if a customer purchases a particular plan, then she should pay at least that price. However, if the customer does not purchase any plan at all, then the constraint has no effect due to the big \(M\) constant introduced. The next constraint, (2.7f), imposes that if a customer purchases a particular plan, then the customer is purchasing \(q_i = 1\). Constraint (2.7g) has two goals: (i) if the customer does not purchase any plan, then her payment must be zero, and (ii) the customer payment can never exceed her reservation price. Constraints (2.7h) make sure that fixed prices are increasing in the plan indexes. Likewise, Constraint (2.7i) make sure that variable prices are decreasing in the plan indexes. Constraint (2.7j) enforces that plan \(s+1\) will be cheaper than plan \(s\) for the range of downloads in \([B_{s+1} - \)
Constraint (2.7m) represents the inequalities described in Proposition 2. The given formulation (without considering Constraint (2.7m)) has approximately $N(2S + 5) + 5S$ constraints, $2N + S$ continuous variables, and $3N$ binary variables. Constraint (2.7m) can eventually add $O(N^2)$ constraints. Nevertheless, in practice the number of these constraints are $O(N \ln(N))$. It is worth to point out that in this problem the number of customers is much bigger than the numbers of plans, i.e., $N \gg S$. It is worth to also note that the formulation can be easily extended to incorporate additional business rules. For instance, we can also introduce congestion and capacity constraints over the demand of particular geographical areas.

Propositions 1 and 2 help to significantly decrease the running time of the MIP and therefore allowing us to solve small problems with hundreds of customers. However, it is still is not enough to allow us more realistic size instances (like with thousands of customers). Therefore, in what follows we develop some heuristics that make use of this MIP formulation in order to solve bigger size instances. The details of the heuristics we introduce are given in the next subsection.

### 2.4.1 Heuristics

The first heuristic presented we refer to as Heuristic H0. It builds a feasible solution for the MIP. Then we present Heuristics H1 and H2(k) which are heuristics that improve the objective given a feasible solution.

**Heuristic H0**

Heuristic H0 consists of two steps, the first is to optimize only on the fixed prices as if variable prices would be set to infinity. This optimization problem can be solved much quicker than the full optimization problem. The reason is that customers are only considering to purchase a single plan. Then all the binary variables $y_{is}, \bar{a}(d_i) - 1$ can be set equal to 0. Additionally, the binary variables of plan-purchase ($y_{is}$) can be dropped since they will be equal to $q_i$. The formulation of the price optimization problem of the first step of Heuristic H0 is
The second step of Heuristic HO is to solve the original optimization problem (2.7) setting the fixed prices to what we obtained from the first step of Heuristic HO.

**Algorithm 2.4.1: Heuristic HO**

1. Solve Problem (2.8) to find a vector fixed prices;
2. Solve the original price optimization problem (Problem (2.7)) setting the fixed prices \( f \) obtained in Step 1, and fixing the variables \( z_i, q_i, \) and \( y_{is} \), for \( i \in F \);

Step 2 of Algorithm 2.4.1 is taking advantage of the structure of the problem by fixing not just the variable prices, but also the purchase decision variables of customers who are in set \( F \).

**Heuristic H1**

Heuristic H1 has a similar flavor as the previous Heuristic HO. The idea now is that once we have a feasible solution, we can iterate on solving Problem (2.7) fixing the variable prices \( (v) \), and then using the fixed prices obtained to solve Problem (2.7) fixing these variables \( (f) \). We iterate until the objective value stabilizes. The steps of the algorithm are described below.
Algorithm 2.4.2: Heuristic H1

1. Start with a feasible solution (indeed, we only need a vector of fixed and variable prices $f$ and $v$ that satisfy the respective Constraints (2.7h), (2.7i), (2.7j), and (2.7l));
2. Solve Problem (2.7) for the variable prices $v$, and fixing the fixed prices $f$ and the variables $z_i, q_i,$ and $y_{is}$ for $i \in F$;
3. Solve Problem (2.7) for the fixed prices $f$, and fixing the variable prices $v$ and the variables $z_i, q_i,$ and $y_{is}$ for $i \in F$;
4. If the change in the objective function is stabilized, go to Step 2; otherwise stop;

Heuristic H2(k)

So far, the previous heuristics have considered setting fixed and variable prices. In what follows, we consider fixing all the variables of certain plans to solve the optimization problem for only a subset of the plans simultaneously. Call $k \in \{1, \ldots, S\}$ the cardinality of the plans that will be simultaneously optimized. As in Heuristic H1, Heuristic H2(k) considers a feasible solution to start with, and then iterates over all combination of plans with cardinality $k$. Then we iterate again optimizing over all permutations. The algorithm is given below

Algorithm 2.4.3: Heuristic H2(k)

1. Start with a feasible solution;
2. For each set $T \subseteq \{1, \ldots, S\}$ such that $|T| = k$, solve Problem (2.7) fixing $f_s, v_s$ for all $s \notin T$, and set $z_i, y_{is}$ and $q_i$ for $i$ such that $\bar{s}(d_i) - 1, \bar{s}(d_i) \notin T$;
3. If the objective stabilizes, go to Step 2; otherwise stop;

Note that solving Heuristic H2(k) with $k = S$ is equivalent to solving Problem (2.7).

For practical purposes, $k \in \{1, 2, 3\}$ are the values used to solve Heuristic H2(k), since higher values of $k$ lead to long running times.

For Heuristic H0, we provide a performance guarantee on how suboptimal the heuristic’s solution is from the optimal solution. Let us call $z^{H0}$ the solution obtained by Heuristic H0, and $z^*$ the optimal value of Problem (2.7).
Proposition 3. With \( N \) customers (wlog assume \( r_1 \geq r_2 \geq \cdots \geq r_N \)) consider \( L \subseteq \{1, \ldots, N\} \) the subset of customers where we know their reservation prices \((1, N \in L)\). Consider \( \eta = \max_{j \in L} \{jr_j\} \). For each \( l \in \{1, \ldots, |L|\} \) denote \( o(l) \) the smallest \( l \)th element of the set \( L \). Then a bound on the suboptimality gap of Heuristic H0 is

\[
\frac{1}{\eta} + \sum_{l=1}^{|L|-1} \left( \min\left\{ 1, \frac{o(l+1)r_{o(l)}}{\eta} \right\} - \frac{o(l)r_{o(l)}}{\eta} + \ln \left( \max\left\{ 1, \frac{o(l+1)r_{o(l)}}{\eta} \right\} \right) \right) \leq \frac{z_{H0}}{z^*} \leq 1. \tag{2.9}
\]


The idea of the bound in Proposition 3 is that by making few assumptions over some customers’ reservation price, we can obtain a suboptimality gap for Heuristic H0. Let us call \( \rho = \frac{\eta}{r_{\hat{N}}} \). Note that if \( L = \{1, N\} \), the bound in Equation (2.9) reduces to \( \frac{1}{1+\ln(\min(\rho, N))} \). In particular, \( \frac{1}{1+\ln(\rho)} \) is a lower bound for Heuristic H0. This is independent of the number of customers. Figure 2-7 shows the bound on Equation (2.9) for different instances on reservation prices and different values of \( \rho \) with 1001 customers. Each series of Figure 2-7 considers a particular set of customers (in set \( L \)) for whom we assume their reservation prices are exactly the quantile between the highest and lower reservation prices according to the ranking of their reservation prices. For instance, in the series in Figure 2-7 where \( L = \{1, 501, 1001\} \), the reservation price of customer 501 (from highest to lowest reservation price) is set to be exactly the middle point between the highest and the lowest reservation price, i.e. \( r_{501} = \frac{r_1 + r_N}{2} \). It can be seen from Figure 2-7 that the bound improves significantly when assuming the median customer’s (501) reservation price is the average between the highest and lowest reservations. Yet, when making additional assumptions over customers’ reservation prices (series with \( L = \{1, 251, 501, 751, 1001\} \) and \( L = \{1, 101, 201, 301, 401, 501, 601, 701, 801, 901, 1001\} \)), the bound improves by a lower amount.

The previous bound does not use all of the information regarding the customers’ downloads. The following proposition states a bound using all customers’ reservation prices and download information. Let us define the sets of customer indexes \( N_s = \{i \in \{1, \ldots, N\}|B_{s-1} < d_i \leq B_s\} \), for all \( s \in \{1, \ldots, S\} \). Also consider \( \tau_s(j) = \{i \in N_s|r_i < \)
Figure 2-7: Lower bound on Heuristic H0 with $N = 1001$ for different degrees of information on customers’ reservation price. For each series, each customer $l$ in the respective set $L$ is assumed to have a reservation equal to the quantile according a uniform distribution on $[r_N, r_1]$, namely $r_l = \frac{|L|-l}{|L|-1} r_1 + \frac{l-1}{|L|-1} r_N$. The curve with $L = \{1, N\}$ corresponds to $\frac{1}{1 + \ln(\min\{N, \rho\})}$.

$$r_j, \frac{r_j}{d_j-B_{s-1}} \geq \min\{d_j, B_s-c_s, B_{s-1}\},$$

for $j \in N_s$ and $\kappa_s(j) = \{i \in N_s | r_i < \frac{r_j}{d_j-B_{s-1}} (B_s - c_s - B_{s-1})\}$, for $j \in \{i \in N_s | d_i \leq B_s - c_s\}$.

**Proposition 4.** In a setting with $N$ customers, reservation prices $r_i$ and download $d_i$ for each customer $i \in \{1, \ldots, N\}$. Consider $k_s = \max_{j \in N_s} \sum_{i \in N_s} r_j I\{r_i \geq r_j\}$, $\bar{k}_s = \max_{u \subseteq \{1, \ldots, s\}} k_u$, $\beta_s = \max_{j \in N_s} \sum_{i \in N_s} r_j I\{r_i \geq r_j\} + \sum_{i \in \kappa_s(j)} r_i$ and $\gamma_s = \max_{j \in N_s} \sum_{i \in N_s} \frac{r_j}{d_j-B_{s-1}} (B_s - c_s - B_{s-1}) I\{r_i \geq \frac{r_j}{d_j-B_{s-1}} (B_s-c_s-B_{s-1})\} + \sum_{i \in \kappa_s(j)} r_i$, for $s \in \{2, \ldots, S\}$; $\beta_1 = k_1$ and $\gamma_1 = 0$. Then a suboptimality bound on of Heuristic H0 is

$$\frac{\sum_{s \in \{1, \ldots, S\}} \bar{k}_s}{\sum_{s \in \{1, \ldots, S\}} \max\{\beta_s, \gamma_s\}} \leq \frac{z^{H0}}{z^*} \leq 1. \quad (2.10)$$

**Proof.** Proof. See Appendix A.5. □

The denominator of the left hand side of Equation (2.10) is an upper bound of the optimal solution and can be used with any feasible solution to compute an upper
bound on the suboptimality gap. Let us call it $z^{UB}$, i.e. 
$$z^{UB} = \sum_{s \in \{1, \ldots, S\}} \max \{\beta_s, \gamma_s\}.$$ 

### 2.4.2 Dynamic Programming

The pricing optimization problem can be modeled with dynamic programming, where each stage corresponds to a particular range of downloads between the bundle data of two plans. Therefore there are $S$ stages. In particular, stage 1 will be the interval of customers’ downloads between 0 and $B_1$, the second stage will be associated with customers’ download in the interval $[B_1, B_2)$, and so on; until stage $S$ which considers customers’ downloads’ between $[B_{S-1}, \infty)$. For ease of notation, denote $B_0 = 0$ and $B_S = \max_i \{d_i, i \in \{1, \ldots, N\}\}$. In each stage, the state $s = (s_f, s_v) \in \mathbb{R}^2$ will be given by the tuple consisting of the last fixed and variable priced used. The action $a = (a_f, a_v) \in \mathbb{R}^2$ on each stage will correspond to the fixed price of plan $t$, and the variable price of plan $t-1$. Note that the action in stage $t$ will be the exact state of the system in the next stage. Therefore, we discretize the actions space in order to discretize the states on each stage. To do this, for each stage $t$ we consider $\mathcal{R}_t$ a subset of customers’ reservation prices whose downloads are in $[B_{t-1}, B_t)$, such that none of the customers’ reservation prices are closer than some $\Delta > 0$ (we used different values of $\Delta$, and in the end utilize $\Delta = 10$ due its performance in objective and solving time). That is, instead of considering all the customers’ reservation prices in the interval $[B_{t-1}, B_t)$, we consider a subset of these. This is done to reduce the number of states of the problem. See Algorithm A.6.1 in Appendix A.6 for details on the algorithm that given a vector of reservation prices and $\Delta > 0$, returns a sub-array of values of the original array so that there are no values with a distance of $\Delta$. Regarding the variable prices, we consider a grid of variable prices from 0 to $\bar{v}$ moving with a step of $\delta_v = 0.1$.

The revenue function on stage $t$ given state $s$ and action $a$, $V_t(s, a)$ will be the revenue attained from those customers with downloads in the range $[B_{t-1}, B_t)$. Each of these customers’ potential payment is $l_i = \min\{a_f, s_f + (d_i - B_{t-1}) \times a_v\}$, and the payment is actually captured if this is less than or equal to the customer’s reservation price, namely $l_i \leq r_i$. Then, the whole dynamic program can be formulated as:
• Stages: \( t = 1, 2, \ldots, S \)

• States \( s = (s_f, s_v) \in A_{t-1} \), and \( A_0 = \{(0, \overline{v})\} \)

• Action \( a = (a_f, a_v) \in A_t := \{(a_f, a_v) \in \mathbb{R}^2|a_f \in \mathcal{R}_t, \exists i \in \mathbb{Z} \text{ such that } a_v = i\delta_v, 0 \leq a_v \leq \overline{v}\} \).

• Revenue Function:

\[
V_t(s, a) = \sum_{i \in \mathcal{N}_t, l_i(s, a, t) \leq r_i} l_i(s, a, t) + V_{t+1}^*(a)
\]

where \( l_i(s, a, t) = \min\{a_f, s_f + (d_i - B_{t-1}) \times a_v\} \).

• Revenue to Go:

\[
V_t^*(s) = \max_{a \in A_t} V_t(s, a)
\]

• Recursion \( s^{t+1} = a^t \)

• Border Condition \( V_{S+1}(s) = 0 \) for all \( s \)

Note that at stage \( t > 2 \), \( (s_f, s_v) = (f_{t-1}, v_{t-2}) \), for \( t = 2 \).

2.5 Results

In this section we illustrate the performance of the heuristics described in Section 2.4 over a set of instances. Each instance is generated as follows. First, consider the number of observed and unobserved customers \( (n \text{ and } m) \). Second, we sample without replacement the download of observed customers from the total downloads of these customers. Third, we sample downloads of unobserved customers and all customers’ reservation prices according to the distributions obtained after running Algorithm A.1.1.
The optimization methods used are: the MIP formulation (Problem (2.7)), Heuristic H0, Heuristics H1, Heuristic H2(k) with \( k \in \{1, 2, 3\} \), and the dynamic programming formulation. Since the latter heuristics, H1 and H2(k), need a feasible solution to start with, we start these heuristics from the solution obtained from Heuristic H0.

### 2.5.1 Heuristic performance

Table 2.3 shows the suboptimality gap attained by each heuristic. The instances considered are with fifty and one hundred observed customers. Unfortunately, it is not possible to compute the optimal solution for larger instances. It can be seen from Table 2.3 that the solution obtained from Heuristic H0 is always within 8% of the optimal solution. As expected, Heuristics H1 and H2(k) improve the solution attained in Heuristic H0. It can be seen that Heuristic H1 outperforms Heuristic H2(1) by a small percentage; however, Heuristic H2(2) outperforms Heuristic H1 by a considerable percentage. Regarding Heuristic H2(3), as expected, it performs better than Heuristic H2(2) but only by a very small percentage. It is also worth noticing that there is no need to use Heuristic H2(k) with values of \( k \) higher than 3, as the solution obtained from Heuristic H2(3) is very close to the optimal solution, as Table 2.3 shows (see Algorithm 2.4.3 for details on \( k \)).

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<td>5.97</td>
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Table 2.3: Percentage gap of the different heuristics with respect to the optimal solution.
For those instances for which the optimal solution can not be computed, we can compute an upper bound on the performance of the heuristics. Table 2.4 shows the percentage gap between the upper bound \( z^{UB} \) (given in the denominator of the left term of Equation (2.10)) and the solution attained from the different methods. Namely, each column of Table 2.4 shows \( \frac{(z^{UB} - z^{Heuristic})}{z^{UB}} \) for a particular heuristic over different instances. Note that this is an upper bound on the optimality gap, and so it is quite likely that the actual heuristic gap from the optimal solution is much lower. Indeed, it can be seen in the first three rows of Table 2.4 there is a gap of around 20% between the optimal solution an the upper bound. The running times of the heuristics for all the instances are given in Table A.1 in the Appendix.

<table>
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<th>( n )</th>
<th>( m )</th>
<th>( \text{Opt} )</th>
<th>( H0 )</th>
<th>( H1 )</th>
<th>( H2(1) )</th>
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<td>-</td>
<td>26.27</td>
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</tr>
</tbody>
</table>

Table 2.4: Percentage gap of the heuristics with respect to the upper bound \( z^{UB} \). \( \frac{(z^{UB} - z^{Heuristic})}{z^{UB}} \)

2.5.2 Performance of Heuristics versus Actual Prices

Heuristics vs Actual Prices In this subsection, we evaluate the performance of the heuristics with respect to the current prices the company charges. To do this, we
input the prices suggested by the heuristics and the current prices into new generated instances of full size. For example, consider Heuristic H2(1) solved over an instance with 1000 observed and 1333 unobserved customers (i.e. \( m = n \times 4/3 \)) obtaining prices \((f^{H2(1)}, v^{H2(1)})\). Then we generate a hundred scenarios of real size instances where the number of observed customers is the actual number of observed customers, while the unobserved customers are 4/3 of these. Then we compute the revenue attained from setting prices \((f^{H2(1)}, v^{H2(1)})\), and the revenue obtained from charging the current prices \((f^0, v^0)\) over each scenario. Denote \((\pi^H - \pi^0)/\pi^H\) as the average of the percentage revenues of Heuristic H with respect to the current prices, where \(H \in \{H0, H1, H2(k)\}\). Similarly, denote by \(\pi^H_n/\pi^H\) the average ratio between the revenues from observed customers with respect to the total revenues from Heuristic H.

![Figure 2-8](image-url)

Figure 2-8: Left: Average revenue difference between the heuristics and current prices over 100 real size instances. Right: Average percentage revenue from the observed customers over the revenues obtained from all customers over 100 real size instances.

The left panel of Figure 2-8 shows the average percentage difference of the revenues when applying the obtained prices from the various heuristics over a hundred generated real size instances as compared to the revenues obtained by setting the actual prices used in practice. As expected, considering more unobserved customers lead to a better performance of the heuristics. The reason behind this is that the
prices suggested by the heuristics will take into consideration these potential customers (unobserved ones) in order to gain some revenue from them. Note also that Heuristic H2(1) is the heuristic with better performance among all heuristics across all series.9

In addition, it is interesting to see what portion of the revenues comes from the observed customers. In other words, what percentage of the revenues obtained by the heuristic prices on the generated instances comes from the actual (observed) customers? To answer this question, we compute the average percentage of the revenues obtained from plugging the heuristic prices that come from the observed customers. The right panel of Figure 2-8 illustrates the average percentage of revenues that come from the observed customers in the generated set of real size instances for the different heuristics. This figure suggests that if the unobserved customers are a third of the observed ones, then the revenues obtained from the current (observed) customers will be between 87% to 92% of the total revenues. Similarly, if the number of unobserved customers is the same as the number of observed ones, the percentage lowers to 67% to 70%.

The next question we focus on is: how much we gain by solving bigger size instances? Figure 2-9 depicts the percentage difference of the revenues when applying the obtained prices from Heuristic H2(1) over a hundred generated real size instances as compared to the revenues obtained by setting the actual prices used in practice. We show the results for Heuristic H2(1) because among all heuristics this is the one able to solve bigger size instances (together with Heuristic H0). The horizontal axis of Figure 2-9 corresponds to the number of observed customers used to solve Heuristic H2(1). Each series in Figure 2-9 corresponds to a different number of unobserved customers. It can be seen that the bigger the instances considered in the heuristic, the better the heuristic prices will perform on a real size instance. Note that there is a significant improvement of the heuristic performance when the size of the instances solved increases from 100 to 500 observed customers, yet it can be seen that

9All the heuristics shown in both panels of Figure 2-8 are solved with the maximum number of customers these could handle.
the marginal improvement is much lower when solving with 1000 to 5000 unobserved customers.

![Graph showing revenue difference](image)

**Figure 2-9:** Average revenue difference attained over 100 instances of real size using prices obtain from Heuristic H2(1) versus the revenues attained on these instances by using the current prices.

**Heuristics vs Actual Prices under misspecifications on the data** Next, we analyze how robust the results are relative to misspecification on the number of unobserved customers, and the estimated parameters of the model. We consider the case in which the assumed number of unobserved customers (which is used to solve the heuristic) is not the actual number. In other words, consider Heuristic H2(1) is used to solve an instance of say \( n = 5000 \) observed customers and \( m = n \times 1/3 \) unobserved ones. Yet, it might happen that in reality \( m = n \times 4/3 \), thus the prices obtained from the heuristic are inputted on a real size instance with 4/3 unobserved customers with respect to the number of observed ones. The left panel of Figure 2-10 shows the average percentage revenue difference between the heuristic prices and current ones over a real size instances when miss assuming the number of unobserved customers. The horizontal axis of Figure 2-10 corresponds to the real number of unobserved customers, i.e., the number of unobserved customers used to sample the
real size instances where the heuristic and current prices are inputted to compute the corresponding revenues. Each series corresponds to the number of unobserved customers used to solve Heuristic H2(1). It can be seen that in most cases the heuristic prices still outperform the current ones, especially when the number of unobserved customers is 2/3 or more of the observed ones. It is interesting to observe how robust the revenue outcomes are with respect to the different assumptions over the unobserved customers. For example, assuming that \( m = n \times \frac{5}{3} \) can lead to revenue losses of 10% with respect to the current prices if \( m = n \times \frac{1}{3} \). Following the same logic, assuming \( m = n \times \frac{1}{3} \) might deliver a gain of 10% in revenues yet this gain could be much higher (~ 30%) if the number of unobserved customers assumed would be the correct one. According to the left panel of Figure 2-10, assuming \( m = n \times \frac{2}{3} \) (or \( m = n \)) appears to give rise to the most robust strategy, since in all cases the heuristic prices outperform the current ones.

Besides misspecification in the number of unobserved customers, we analyze the case when there are errors in the parameters estimated by the model. In order to see how robust the results are relative to potential errors in the estimated parameters, we perform the same analysis as before (left panel of Figure 2-10), but instead of considering instances generated by the exact estimated distribution parameters of reservation prices, we consider perturbations over these parameters. More precisely, originally, to generate a real size instance, reservation prices are assumed to follow a normal distribution where the mean is a function of the download parameters \( \tilde{\theta}_i(\hat{\theta}_i + d) \tilde{\theta}_i \), where \( \{\hat{\theta}_i\}_{i=1,2,3} \) are obtained from Algorithm A.1.1. Now, we will generate real size instances so that each parameter \( \tilde{\theta}_i \) will be uniformly distributed in a box around the estimated parameters \( \hat{\theta}_i \). More precisely, let

\[
\tilde{\theta}_i \sim U[1 - \Delta, 1 + \Delta] \times \hat{\theta}_i
\]

for \( i \in \{1, 2, 3\} \) be the parameters used to generate the reservation prices of the real size instances. Let us call \( \Delta \) noise deviation. The values considered for the noise deviation are \( \Delta \in \{0\%, 10\%, 20\%\} \). Each parameter \( \tilde{\theta}_i \) is generated independently of
the other parameters $\tilde{\theta}_j, j \neq i$.

As discussed before, we want to examine how these results change when the parameters of the reservation prices distribution are not exactly the same as the estimated ones (i.e., in addition to the misspecification in the number of unobserved customers, there might also be a misspecification on the estimated distribution parameters of reservation prices). The right panel of Figure 2-10 depicts the average percentage revenue differences over instances where the noise deviation is $\Delta = 20\%$. It can be seen that the performance is not as strong as the case with no noise deviation. Nevertheless, the heuristic prices outperform the current ones in all instances when unobserved customers are at least equal to the number of observed ones.

2.5.3 Optimal Prices

In what follows we also analyze how the prices (fixed and variable) obtained from the various heuristics differ when assuming a different number of unobserved customers. Figure 2-11 shows the ratio of the prices obtained from Heuristic H2(1) for instances with 5000 observed customers versus the current prices for each of the plans. It can be seen that the higher the number of unobserved customers we consider, the lower are the fixed and the variable prices obtained. The intuition is that for a given level of downloads, unobserved customers have on average lower reservation prices. As a result, introducing more of these customers will result in a decrease on prices in order to capture more revenue from this segment. This effect is more pronounced in fixed prices rather than variable prices (see Figure 2-11). Furthermore, as Figure 2-11 suggests, fixed and variable prices for the smallest bundle service should be increased, yet the opposite should be done for services with index between three and seven.

2.6 Conclusions

In this work, we have presented a framework that allows us to study a variation of the two-price tariff problem from a data perspective. The chapter focuses on two main aspects: first, the estimation of unobserved data from parametric probability
Figure 2-10: Average revenue difference on real size instances using Heuristic H2(1) (top) and Dynamic Programming (bottom) prices versus current ones when the number of unobserved customers used may differ from the real one. Left panel shows the case where there is no noise deviation on the reservation price distribution parameters, whereas the right panel shows the case where these parameters have a noise deviation of 20%.

Throughout the chapter, the problem is described for the particular case of this...
Figure 2-11: Left: Ratio of fixed prices obtained with Heuristic H2(1) run over instances with 5000 observed customers divided by the fixed prices of the current plans. Right: Ratio of variable prices obtained with Heuristic H2(1) run over instances with 5000 observed customers divided by the fixed prices of the current plans.

A satellite service provider; however, the methodology can be directly applied to settings that share a similar structure. Namely, downloads can be replaced by the particular unit of the service in consideration.

In this chapter, we also developed an algorithm to estimate the parameters of the probability distributions of the unobserved data: unobserved customers’ download, and all customers’ reservation prices. This algorithm iterates in a similar way as the EM-algorithm in order to maximize the likelihood. After experimenting with different probability distributions and parametric functions, we converge to the best parametric model. Furthermore, we used this model to generate scenarios of the unobserved data for the price optimization problem.

To formulate the pricing optimization problem as a MIP, we first develop useful properties that allow us to reduce its dimension and feasible space. Nonetheless, still the problem is very hard to solve on medium size instances. Consequently, we introduce several heuristics, which unlike the MIP formulation, allow us to solve bigger size instances with thousands of customers. One of these heuristics described, constructs a feasible solution, while the others improve this solution by iteratively.
solving the MIP formulation by fixing specific subsets of the decision variables. On the former heuristic we develop, suboptimality bounds that use partial and full knowledge on customers’ information. Over small instances (in which optimal revenues can be computed) the heuristics perform within 8% of the optimal solution. On larger instances, the heuristics’ performance is at least within 27% from the optimal solution. This is an upper bound to the suboptimality gap as the optimal solution can not be computed exactly.

The resulting prices obtained from the heuristics are inputted to real size instances in order to compute the gains/losses in revenue with respect to the current prices used by the company. The prices obtained from the heuristics outperform current prices in real size instances. This difference is more pronounced when more unobserved customers are present. More precisely, if the number of unobserved customers is a third (five thirds) of the observed ones, the increase in revenue will be 7% (31%) relative to the one attained from using the current prices. Since our assumed parameters might not match exactly the real ones, we perform an analysis of the heuristic prices respect to the current ones when misspecifying the number of unobserved customers as the distribution model parameters. We show that overestimating the number of unobserved customers might lead to revenue losses; however, underestimating unobserved customers turns to be a more conservative and robust strategy. Namely, heuristic prices consistently outperform the current prices when assuming a low value of unobserved customers.

In addition, we analyzed the case where in addition to misspecifications in the number of unobserved customers, there is noise in the distribution parameters estimated. In this case, we show that still heuristic prices can very well respect to current prices when underestimating the number of unobserved customers.

The prices our methods give rise to, are decreasing in terms of the number of unobserved customers. Following the logic that when more unobserved customers are present, then lower prices can take some revenue from these customers as their willingness to pay is lower.
Bibliography


Chapter 3

Competition and Externalities in Green Technology Adoption

3.1 Introduction

3.1.1 Motivation

Global warming has continuously increased during the last decades, bringing many undesired consequences to Earth and human life. Predictions suggest the consequences will continue to worsen over the years to come. One of the solutions to mitigate this problem is the adoption of green technologies. As a result, this prospect has captured the attention of public and private sectors. However, green products (such as electric vehicles and solar panels) remain usually unaffordable and many people continue using conventional ones. In order to overcome this issue and encourage green technology adoption, several governments started to offer subsidies (or tax rebates) to consumers so as to enhance the adoption of these technologies.

In the height of the economic recession, the US government passed the American Recovery and Reinvestment Act of 2009 which granted a tax credit for consumers who purchased an EV. Besides boosting the US economy, this particular tax incentive was aimed at fostering further research and economies of scale in the nascent electric vehicle industry. Indeed, sales of EVs in the US have effectively been increasing
throughout the past five years\(^1\).

Growing demand in these markets has attracted the interest of different car manufacturers. In December 2010, the all-electric car, Nissan Leaf, and the plug-in hybrid General Motors’ Chevy Volt were both introduced in the US market. After a slow first year, sales started to pick up and most major car companies are now in the process of launching their own versions of electric vehicles\(^2\). Even if we restrict to the highway capable vehicles (i.e., road cars with a top speed above 65 mph), one can count above 31 models of EVs available in the market (July 2016).

As we previously mentioned, General Motors and Nissan have recently introduced affordable electric vehicles in the US market. GM’s Chevy Volt was awarded the most fuel-efficient compact car with a gasoline engine sold in the US, as rated by the United States Environmental Protection Agency ([16]). However, the price tag of the Chevy Volt is still considered high for its category. The cumulative sales of the Chevy Volt in the US since it was launched in December 2010 until December 2014 amount to 73,357. It is likely that the $7,500 government subsidy offered to each buyer through a federal tax credit played a significant role in the sales volume. The manufacturer’s suggested retail price (MSRP) of GM’s Chevy Volt in April 2015 was $34,995 but the consumer was eligible for $7,500 tax rebates so that the effective price reduced to $27,495. The size of consumer subsidy has remained constant since launch in December 2010 until the end of 2014. In addition, most other EVs are eligible for the same $7,500 tax rebate, so that consumers can choose between the different vehicles while still receiving the same subsidy from the government.

In this chapter, we address the following questions: How does the recent competition in the EV industry affect consumer subsidies and green technology adoption? How should governments take into account competition effects while designing consumer subsidies for green technology adoption? Finally, how does competition affect the suppliers’ prices (MSRP), production quantities and consumers? Note that competition can either be symmetric (with perfectly substitutable products) or asymmet-

\(^1\)http://electricdrive.org/index.php?ht=d/sp/i/20952/pid/20952
\(^2\)http://energy.gov/articles/visualizing-electric-vehicle-sales
ric (with different products, such as family versus luxury cars). In this chapter, we consider both cases and study how a monopolistic setting differs from a competitive environment. By understanding the impact of competition in the green technology market, the government can then design more efficient subsidy programs.

Externalities are considered to be the environmental benefits from green technology adoption, measured as the monetary value of the reduction in CO₂ emissions of an EV relative to a regular car. We consider two types of markets: small and large externalities (defined formally in Section 3.4). We show that the analysis as well as some of the main insights are different depending on the level of externalities. Consequently, the impact of the competition is highly affected by the externality factors.

In this chapter, we introduce a model to study the effects of competition in green technology markets with substitutable products and uncertain demand, such as EVs. We show that externalities, demand uncertainty and suppliers’ asymmetry all play a key role in answering this question and we derive various insights regarding the impact of competition for green technology adoption.

3.1.2 Contributions

Given the recent growth of green technologies, supported by governmental subsidy programs, this chapter explores a timely problem in supply chain management. Understanding how the recent increasingly competition in this industry affects subsidy costs, as well as the economic surplus of suppliers and consumers, is an important part of designing sensible subsidy programs. The main contributions of this chapter are:

- Should the government maximize social welfare or minimize expenditures?

  We compare the outcomes when the government optimizes social welfare versus expenditures. We derive analytical tight bounds on the social welfare loss when the government minimizes expenditures, and investigate what is the most appropriate objective for the government. In particular, we show that maximizing social welfare can be very costly to the government. When externalities are
small, we show that the government can minimize expenditures and still attain a very good welfare level while significantly reducing the expenditures. Nevertheless, when externalities are large the social welfare loss may be significant.

- The interplay of demand uncertainty and suppliers asymmetry determines how the competition benefits redistribute among the government and the consumers. We show that the benefit from the presence of competition is shared among the different players. For example, in a market with small externalities, the suppliers are always worse-off in a competitive setting and the benefit is shared between the government and the consumers. Moreover, the exact sharing of this benefit depends on the interplay of demand uncertainty and suppliers asymmetry. We show that when the demand is deterministic and suppliers are identical, the entire benefit is absorbed by the government. We also determine that for asymmetric suppliers, the presence of competition does not benefit all the consumers. Finally, demand uncertainty favors consumers in terms of competition benefit.

- Competition always benefits the government and hurts suppliers, when externalities are small.
As expected, we show that when externalities are small, competition always hurts the suppliers. We also observe that competition always favors the government by allowing a reduction in expenditures. This result indicates that the government should encourage competition and incentivize new entrants to the electric vehicle industry.

- Competition always benefits consumers, when externalities are large.
By considering a market with large externalities, we show that the effects of competition may differ. In particular, it is not clear anymore that the government is better-off at the expense of the suppliers. In this case, competition always benefits consumers who enjoy (for a symmetric setting) higher available quantities and pay a lower effective price.
3.1.3 Literature review

Consumer subsidies for green technologies have been an active area of research for the last two decades (for recent works see, e.g., [13], [27], [9] and [30]). [30] analyzes the impact of subsidies on risk averse EV manufacturers in a newsvendor setting by looking at the trade-off between subsidy levels, risk aversion, demand uncertainty and performance. Similar to this chapter, other works incorporate the government as a player who decides a policy typically in the first stage, followed by the response of a firm. An example includes [13], where the authors consider a monopolistic supplier, and investigate the impact of demand uncertainty on subsidy policies for non-linear stochastic demands. The authors consider a cost minimization objective where the government offers consumer subsidies in order to achieve an adoption target level set by the government. Taking a slightly different perspective, several authors look into a social welfare maximization approach from the government. [27] compare, for the case of linear demand, different government intervention mechanisms and investigate under what conditions the system is coordinated in terms of welfare, prices and supply quantities. Our chapter considers a general framework where multiple competing manufacturers are present in the market that can include externalities. Indeed, motivated by recent developments in the EV industry, several car manufacturers started offering their own electric vehicle. Therefore, it seems appealing to study how competition affects the outcomes of the various players involved. In addition, we consider and compare both government objectives: maximizing the social welfare versus minimizing the government cost.

Another line of research related to this chapter aims to quantify the positive externalities generated by green technology products, such as solar panels and electric vehicles. In particular, several works have studied the monetary impact of the different pollutants in a dollar base, (see, e.g., [24]). Other works have studied the efficiency of green technologies by comparing the impact of these products versus the impact of regular ones. For instance, how much more efficient is an EV relative to a similar regular fuel vehicle? (See, e.g., [22]). Additional works consider not only the exter-
nalities produced by the reduced emissions from EVs, but also the different sources of energy used to charge the vehicles. For example, [2] studies the mix of energy sources in the US. [20] compare sources of energy in different countries, concluding that EVs actually reduce CO₂ emissions regardless of the sources of energy consumed. However, the more renewable the sources of energy are, the greater the benefit of EVs. [6] examine the social benefits of electric vehicle adoption in Sweden and report a pessimistic outlook for this technology in the context of net social welfare. [3] show that the adoption of electric vehicles has societal and environmental benefits, as long as the electricity grid is sufficiently clean. This paper assumes non-strategic industry players. In our chapter though, we incorporate the strategic response of the industry into the policy making decision.

Without considering demand uncertainty, there is a significant amount of empirical work in the economics literature on the effectiveness of subsidy policies for hybrid and electric vehicles. For example, [15] shows that there is a strong relationship between gasoline prices and hybrid adoption. [8] show that hybrid car rebates in Canada created a crowding out of other fuel efficient vehicles in the market. [18] argue that sales tax waivers are more effective than income tax credits for hybrid cars. The increase in hybrid car sales from 2000-2006 is mostly explained by social preferences and increasing gasoline prices. [1] show that the auto industry innovates more in clean technologies when fuel prices are high.

Our methodology is related to the newsvendor problem, an extensively studied problem in the literature (see, e.g., [26] and the references therein). Numerous extensions have been subsequently proposed. For example, [25] and [29] consider a price-setting newsvendor for a single supplier with additive and multiplicative noises. [11] also presents an overview of different facets of the newsvendor problem, studying the price-setting newsvendor for multiple substitutable products with a linear demand. The newsvendor under a competitive environment has been studied in [10] and [31]. In our setting, we model the problem as a Stackelberg game, where in the second stage, the suppliers are price-setting competing newsvendors responding to the subsidy announced by the government in the first stage.
The issue of how demand uncertainty creates a mismatch in supply and demand has been well studied in the literature. For example, [28] argues that consumers captured most of the incentives for the Toyota Prius, while the firm could not capture any of this surplus despite a binding production constraint. The author also shows that there was a shortage of vehicles manufactured to meet demand, when the Prius was launched. This reinforces our motivation for considering a newsvendor model in this context.

An additional stream of works on qualitative EV adoption have shown that most people primarily care about costs, range and performance of vehicles (e.g., [19]). In [23], the authors propose a way to forecast future sales of EVs in Germany and conclude that the two main barriers remain the price and the range. Indeed, most people do not consider the positive environmental effects of green technologies as their primary driver of the decision to purchase a green product: the features and price are more relevant, (see [5]). Consequently, consumer subsidies might play a key role in EV adoption, and hence, governments should design these subsidies with care. Studying the effects of competition in a model that includes externalities and demand uncertainty in this context is the main motivation of this paper.

Structure of the paper. In Section 3.2, we describe the model and assumptions we impose. In Section 3.3, we discuss the difference between two common government objectives: maximizing welfare versus minimizing expenditures. In Section 3.4, we study the effects of competition in markets with small and large externalities. In Section 3.5, we present some computational experiments using publicly available data from the electric vehicle industry in order to illustrate our model and insights. Finally, we present our conclusions in Section 4.6. Most of the proofs of the different propositions and theorems are relegated to the Appendix.

3.2 Model and Assumptions

Consider a green technology market for which the government designs a consumer subsidy in order to encourage adoption. In particular, we assume that the government
aims to achieve a target adoption level, denoted by $\Gamma$. The government offers a uniform subsidy (or rebate) $r$ directly to the end consumers. In the second stage, $n$ suppliers follow by deciding quantities $q_i$ and prices $p_i; \forall i = 1, 2, \ldots, n$ using a price-setting newsvendor model. In other words, the market is composed of $n$ substitutable green technology products (e.g., electric vehicles), where each product $i \in \{1, \ldots, n\}$ has a marginal production cost $c_i$. Consumer demand is modeled as an affine function with additive uncertainty as follows:

$$d = \bar{d} - B(p - r\epsilon) + \epsilon,$$

where $e \in \mathbb{R}^n$ is a vector of ones and $\epsilon \in \mathbb{R}^n$ is a random vector with components $\epsilon_i$. We denote by $F_i(\cdot)$ and $f_i(\cdot)$ the cumulative and density distribution functions respectively, for product $i$.

**Assumption 1.** We impose the following conditions on demand.

- $\epsilon_i$ are independent, with bounded support $[-A_i, A_i]$ and zero mean.

- The noise distributions have increasing failure rate (IFR), (i.e., $f_i(x)/(1-F_i(x))$ is a non-decreasing function).

- The noises satisfy $\bar{d} - Bc \geq A$ to ensure that demand is non-negative for any noise realization.

- The price elasticity matrix $B \in \mathbb{R}^{n \times n}$ is symmetric, strictly diagonal dominant and a Z-Matrix\(^3\).

The symmetry requirement of the matrix $B$ follows from the Slutsky condition (see, e.g., [17] and [21]), which essentially states that the demand function is derived from an underlying concave utility function of a representative consumer. The M-Matrix property is driven by the sign of the price elasticities in order to model a market with substitutable goods. More precisely, demand for a particular product is

\(^3\)A matrix $B \in \mathbb{R}^n$ is a Z-Matrix if all the off-diagonal elements are non-positive. Since $B$ is a Z-matrix and strictly diagonal dominant, then it is an M-matrix. A matrix $B$ is an M-matrix, if and only if $B_{ii} \geq 0 \forall i \in \{1, \ldots, n\}$ and $B_{ij} \leq 0 \forall i, j \in \{1, \ldots, n\}, i \neq j$.\[\text{\textcopyright 2022 AI2}\]}
a decreasing function of its own price and non-decreasing with respect to prices of competitive products. The diagonal dominance condition captures the fact that the net effect of a variation in the self price is more significant than the same variation in all the competitive prices. Finally, we impose the natural assumption that the target adoption level cannot be attained with zero rebate (otherwise, the problem is irrelevant).

Regarding the government objective, we consider two of the most common objective functions in the literature: minimizing government cost and maximizing social welfare. The Social Welfare (SW) is defined as the total system surplus, which includes the Firms’ Profits (Π), plus the Consumer Surplus (CS), minus Government Cost (GC) plus positive Externalities (EX):

\[ SW = Π + CS - GC + EX. \]  

(3.2)

Our goal is to study the effects of competition. In particular, we compare a monopolistic setting to a competitive environment (defined formally in Sections 3.2.1 and 3.2.2), both with \( n \) green technology products. In both cases, the total profits of the suppliers can be expressed as:

\[ Π = p' \min\{d, q\} - c'q, \]

where, \( p \) and \( q \) are the vectors of price and production quantities chosen by the suppliers, \( c \) is the vector of manufacturing costs and \( d \) is the uncertain demand vector.

Finally, we characterize the consumers by measuring the consumer surplus, that is a common metric to capture consumer satisfaction. More precisely, the consumer surplus is defined as the difference between the maximum price a consumer is willing to pay and the actual market price. We note that in our case, the market price is equal to the effective price paid by the consumers, i.e., \( \tilde{p} = p - r \). When demand is deterministic, we denote by \( d^{-1}(q) \) the effective price that will generate demand exactly equal to \( q \). The consumer surplus is given by: \( CS = \int_{\tilde{p}}^{\tilde{p}_{\text{max}}} d(w)dw \), where \( \tilde{p}_{\text{max}} \) corresponds to the value of the effective price that yields zero demand.
When demand is uncertain however, defining the consumer surplus is somewhat more subtle due to the possibility of a stock-out. Several papers on peak load pricing and capacity investments by a power utility under stochastic demand address partially this modeling issue (see [7], [14] and [4]). Nevertheless, the models developed in this literature are not applicable to the price-setting newsvendor. More specifically, in [4] the authors assume that the utility power facility has access to the willingness to pay of the customers so that it can decline the ones with the lowest valuations. This assumption is not justifiable in our setting where a “first-come-first-serve” logic with random arrivals is more suitable. In [27], the authors study a price-setting newsvendor model for public goods and consider the consumer surplus for linear additive stochastic demand. In [12], the authors extend the treatment of the consumer surplus for a general framework that includes non-linear demands for multiple products and consider various rationing capacity rules.

For general stochastic demand functions, the consumer surplus $CS(\epsilon)$ is defined for each realization of demand uncertainty $\epsilon$. If there was no supply constraint, considering the effective price and the realized demand, the total amount of potential consumer surplus is defined as: $\int_{\hat{p}_{sto}}^{p_{max}(\epsilon)} d(w,\epsilon)dw$. Since customers are assumed to arrive in a first-come-first-serve manner, irrespective of their willingness to pay, some proportion of these customers will not be served due to stock-outs. The proportion of served customers is given by the ratio of actual sales over potential demand: $\frac{\min\{d(\hat{p}_{sto},\epsilon),q_{sto}\}}{d(\hat{p}_{sto},\epsilon)}$. Therefore, the consumer surplus can be defined as the total available surplus times the proportion of that surplus that is actually served, i.e.,

$$CS(\epsilon) = \int_{\hat{p}_{sto}}^{p_{max}(\epsilon)} d(w,\epsilon)dw \cdot \frac{\min\{d(\hat{p}_{sto},\epsilon),q_{sto}\}}{d(\hat{p}_{sto},\epsilon)}.$$ (3.3)

We note that in this case, Consumer Surplus is a random variable that depends on the demand uncertainty through the noise $\epsilon$. Note that we are interested in computing the expected consumer surplus $\mathbb{E}_\epsilon[CS(\epsilon)]$. For stochastic demand, equation (3.3) has a similar interpretation as its deterministic counterpart. Nevertheless, we also incorporate the possibility that a consumer who wants to buy the product does not
find it available. For the linear demand function in equation (3.1), the expected consumer surplus is given by:

\[ CS = \frac{1}{2} \min\{d, q\}'B^{-1}d. \quad (3.4) \]

The social welfare also includes the externalities term denoted by \( EX \). Given that green technologies have a positive environmental impact, we consider that each unit sold of product \( i \) induces a positive externality factor \( k_i \) on the society (for example, due to reduction in emissions relative to conventional vehicles). Therefore, the total expected externalities are given by:

\[ EX = k' \min\{d, q\}, \quad (3.5) \]

where \( k \) is the vector with components \( k_i; \forall i = 1, 2, \ldots, n \).

The alternative government objective is to simply minimize the total expenditures, given by:

\[ GC = re' \min\{d, q\}. \quad (3.6) \]

We assume that the government is setting a rebate level \( r \) that is identical across all the different products. This assumption is supported by the EV market, where consumers are eligible for a tax rebate that amounts to $7,500 for the vast majority of the cars. In this paper, we study and compare both government objectives (3.2) and (3.6). Finally, we impose the following assumption on demand in order to ensure the concavity of the supplier’s problem.

**Assumption 2.** We impose the following condition on demand: \[ \frac{1}{f_i(-A_i) c_i} < e_i Be; \forall i \in \{1, \ldots, n\}, \text{ where } c_i > 0 \text{ and } A_i > 0. \]

Assumption 2 states that the price sensitivity matrix \( B \) has to be strictly diagonal dominant so that the self elasticities \( B_{ii} \) outweigh the sum of the cross elasticities by a factor of \( 1/[f_i(-A_i) c_i] \). This factor accounts for the demand uncertainty and ensures
that the effect of the noise does not overcome the diagonal dominance condition for substitutable goods. More precisely, the magnitude of the noise should not be too large, such that the density evaluated at the lowest noise realization is bounded away from zero. In addition, Assumption 2 is a sufficient condition to guarantee the concavity of the problem faced by the suppliers (in both the monopolistic and competitive settings). As a result, this ensures the existence and uniqueness of an equilibrium. Finally, we note that this assumption is easily satisfied for practical settings, as we will show in Section 3.5.

3.2.1 Monopolistic supplier

In this section, we consider a single firm that jointly manages all the different products. More precisely, the monopolist decides prices and production quantities for the \( n \) substitutable green technology products (e.g., \( n \) different versions of electric vehicles). In the alternative setting considered in the next section, we assume that each product is managed by a different supplier, leading to a competitive environment. Our goal is to compare the outcomes in both settings in order to study the effects of competition on the various players involved (government, suppliers and consumers). In the second stage, the firm sets its production quantities and prices for a given rebate level \( r \) set by the government, so as to maximize the total expected profit. We define the following function:

\[
\Psi_i(p_i) = \mathbb{E}\left[ \min\left\{ \epsilon_i, F_i^{-1}(1 - \frac{c_i}{p_i}) \right\} \right],
\]

which represents the negative of the expected shortages evaluated at the optimal newsvendor quantities. Note that since \( \epsilon \) has zero mean, \( \Psi_i(p_i) \leq 0 \). Note also that \( \Psi_i(p_i) = 0 \) when \( \epsilon = 0 \). The profit maximization problem can be written as:

\[
\max_{p, q} p' \mathbb{E}[\min\{d, q\}] - c'q.
\]
In order to keep the notation compact, we denote \( F^{-1}(1 - c/p) \) an \( n \) dimensional function with components \( F_i^{-1}(1 - c_i/p_i) \) \( \forall i \in \{1, \ldots, n\} \). We observe that the optimal production \( q^*(p) \) as a function of the price is given by:

\[
q^*(p) = \bar{d} - B(p - er) + F^{-1}(1 - c/p),
\]

from the first order condition of problem (3.8). We next characterize the optimal price response of the monopolist.

**Proposition 5.** Under Assumption 2, for a given rebate level \( r \), the unique solution of problem (3.8) is given by \( p \in \mathbb{R}^n \) that solves the following fixed point system of equations:

\[
p^N(r) = \frac{1}{2} B^{-1} \left[ \bar{d} + Ber + Bc + \Psi(p^N(r)) \right].
\]

**Proof.** Proof. See Appendix B.1. \( \square \)

Note that in order to find the optimal price for a given \( r \), one needs to solve a non-linear system of fixed point equations. In Appendix B.1, we show that there exists a unique solution to this system. Note that if Assumption 2 is not satisfied, the problem is not necessarily concave but is still numerically tractable (see [25]). After computing the optimal price \( p^N(r) \) from (3.10), one can derive the optimal quantity vector using (3.9). As expected, all the components of the optimal price vector are increasing with respect to \( r \), which is formally shown in Appendix B.2.

In the first stage, the government decides the uniform rebate \( r \) to be offered to consumers. The government aims to achieve a target adoption level \( \Gamma \) in expectation. We consider and study two different government objectives: minimizing expenditures and maximizing social welfare, defined in (3.6) and (3.2) respectively.

- **Minimizing Government Cost:** In this case, the government faces the following
optimization problem:

\[
\min_r \mathbb{E}[GC] \\
\text{s.t. } \mathbb{E}\left[e' \min\{q^N(r), d\}\right] \geq \Gamma.
\] (3.11)

Given the firm’s second stage response \( p^N(r) \) and \( q^N(r) \), the expected government cost is given by \( \mathbb{E}[GC] = re'\left\{ \bar{d} - B(p^N(r) - er) + \Psi(p^N(r)) \right\} \). We next show how to compute the optimal solution of problem (3.11).

**Proposition 6.** Under Assumption 2, problem (3.11) has a unique optimal solution. In addition, this optimal solution is such that the adoption constraint is exactly met.

**Proof.** Proof. See Appendix B.2. \(\square\)

Since the left hand side of the adoption constraint is monotonic with respect to \( r \), Proposition 6 implies that the optimal rebate can be computed using a binary search on \( r \) such that \( e'\left\{ \bar{d} - B(p^N(r) - er) + \Psi(p^N(r)) \right\} = \Gamma \). We next discuss the second alternative government objective.

**Maximize Social Welfare:** In this case, the government faces the following optimization problem:

\[
\max_r \mathbb{E}[SW] \\
\text{s.t. } \mathbb{E}\left[e' \min\{q^N(r), d\}\right] \geq \Gamma
\] (3.12)

As we previously discussed, the expected social welfare is given by:

\[
\mathbb{E}[SW] = p'\mathbb{E}\left[ \min\{d, q\} \right] - c'q - re'\mathbb{E}\left[ \min\{d, q\} \right] \\
+ k'\mathbb{E}\left[ \min\{d, q\} \right] + \frac{1}{2} \mathbb{E}\left[ \min\{d, q\} \right]'B^{-1}d.
\]

We note that while solving problem (3.12), it is not clear anymore that the optimal solution is obtained by the tightness of the adoption constraint, as it was the case for problem (3.11). In particular, if the target level \( \Gamma \) is large enough,
one can show that the adoption constraint is tight at optimality. Consequently, problems (3.11) and (3.12) yield the same optimal solution and thus are equivalent. However, if the target adoption is below a certain threshold value, the adoption constraint is not tight for problem (3.12) and the outcomes of both models will be different. One can characterize the threshold value depending on the parameters of the model. However, we will focus on the case where the target adoption is below the threshold in order to compare both government objectives (see Section 3.3).

3.2.2 Competing suppliers

In this section, we consider a competitive environment where each firm \( i \in \{1, \ldots, n\} \) is in charge of a single product. In particular, each supplier decides the price and production for its product by maximizing its own profit without knowing about the other supplier's decisions. We model this scenario by \( n \) suppliers competing in a price-setting newsvendor with cross demand elasticities, where each firm faces its own market (modeled as a price-demand stochastic function). As in the previous case, in the second stage, each firm decides upon \((p_i, q_i)\) so as to maximize its own profit. The optimization problem of each supplier can be formulated as:

\[
\max_{p_i, q_i} p_i \mathbb{E} \left[ \min\{d_i, q_i\} \right] - c_i q_i
\]

\[
\Leftrightarrow \max_{p_i} p_i \epsilon_i \left[ \bar{d} - B(p - er) + \Psi(p) \right] - c_i \epsilon_i \left[ \bar{d} - B(p - er) + F^{-1}(1 - \frac{\epsilon}{p}) \right] \quad (3.13)
\]

Note that each supplier \( i \) is affected by the decisions of the competitors through the demand \( d_i \) that explicitly depends on \( p_j, j \neq i \). As before, we have used the fact that the optimal production \( q^*(p) \) as a function of the price \( p \) is given by equation (3.9). We denote by \( D \) the matrix with diagonal elements of \( B \) and zero elsewhere, and \( X = (B + D)^{-1} \).

**Proposition 7.** Under Assumption 2, for a given rebate level \( r \), the unique solution of problem (3.13) is given by \( p \in \mathbb{R}^n \) that solves the following fixed point system of
\[ p^W(r) = X[\bar{d} + Ber + Dc + \Psi(p^W(r))]. \] (3.14)

Proof. Proof. See Appendix B.3. \qed

As in the previous case, after obtaining \( p^W(r) \), the vector of quantities can be computed by using equation (3.9). As before, one can see that the optimal prices are increasing in the rebate.

In the first stage, we consider the two different government objectives with the expected target adoption constraint (problems (3.11) and (3.12)), where the suppliers solve now (3.13) instead of (3.8). One can show similar results as in the monopolistic setting. In particular, under Assumption 2, the optimal solution when the government minimizes expenditures is such that the target adoption is exactly met (see Appendix B.10). In addition, a similar result on a threshold value holds for the social welfare maximization.

### 3.2.3 Competition versus decentralization

In the previous section, we considered the setting with \( n \) competing suppliers, where each supplier decides price and production for a single product. Nevertheless, the demand for each product is assumed to depend on all the \( n \) prices. In this paper, our goal is to study the impact of competition on the various players (government, suppliers and consumers) by comparing the outcomes driven by the monopolistic setting from Section 3.2.1 to the case of competing suppliers as described in Section 3.2.2. In other words, we compare the setting with a monopolist selling a single product versus the setting with \( n \) competing suppliers, each selling one product. We next show that under some reasonable assumptions, the monopolist setting (with a single product) is equivalent to the case of a single supplier selling the \( n \) products (for the symmetric case). Consequently, the price of decentralization is equivalent to the price of competition. In the remaining of the paper, we refer to the case with
competition to the setting with \( n \) suppliers each selling one product, and the case without competition to a monopolist selling a single product.

In order to show the equivalence between the impact of competition and the impact of decentralization, we consider symmetric suppliers and demand functions. We compare both settings by imposing the natural assumption that the total demand is equal in both cases. For the monopolistic setting, the demand is given by:

\[
d_M = \tilde{d}_M - b_M (p_M - r) + \epsilon_M,
\]

where \( p_M \) is the price set by the monopolist. In the case of \( n \) suppliers each selling one product, we have:

\[
d = \tilde{d} - B(p - e'r) + \epsilon,
\]

where \( p \) is the vector of \( n \) prices. Note that for the symmetric setting, all the elements of \( \tilde{d} \), \( p \) and \( \epsilon \) are identical (assuming that the uncertain shock affects the entire industry and splits equally among the suppliers). In addition, the matrix \( B \) is such that the diagonal elements are all \( B_{11} \), and the off-diagonal elements are denoted by \( \delta \). When both settings set the same price, we require that the total demands are equal. In particular, we impose the following condition.

**Assumption 3.** The demand parameters in both settings satisfy:

\[
\begin{align*}
\tilde{d}_M &= n\tilde{d}, \\
\epsilon_M &= n\epsilon, \\
b_M &= n[B_{11} + (n - 1)\delta].
\end{align*}
\]

Now that both settings are consistent in terms of facing the same total demand, one can show the following equivalence.

**Proposition 8.** Consider a symmetric setting with identical suppliers, manufacturing costs and demand functions, and assume that Assumption 3 is satisfied. Then, the
setting with \( n \) competing suppliers each selling one product is equivalent to the setting with a monopolist (i.e., a single supplier selling one product).


The result in Proposition 8 shows that the effect of decentralization is the same as the effect of competition for symmetric suppliers and demand functions. This also allows us to study the effect of competition for the case with asymmetric suppliers. (Note that the effect of competition for asymmetric suppliers is not clearly defined. Instead by looking at the effect of decentralization, one can make a meaningful comparison.) Consequently, in the remaining of the paper, we refer to the case with competition to the setting with \( n \) suppliers each selling one product; and the case without competition, to a monopolist selling \( n \) products.

3.3 Government Objectives: Comparisons

On the one hand, in many economics contexts, researchers consider that governments aim to maximize social welfare. On the other hand, various operational models were proposed where the government seeks to minimize expenditures. In this section, we are comparing the two objectives for a competitive market with green technology products. Note that maximizing social welfare takes explicitly into account the entire system utility and therefore, can be a desirable objective for the government. However, in many practical situations, policy makers are questioning the social welfare concept as it can be hard to measure and interpret. In addition, it is not clear that the government wants to incorporate the supplier surplus as the firms are already optimizing it. Similarly, the consumers are already offered rebates, so the government already takes care of the consumers. Another potential issue with maximizing social welfare can be the high cost to achieve this objective. Minimizing government expenditures appears to be a more realistic and practical approach for the decision makers, as it is easy to interpret and aims to save money for the government. However, one might wonder if it could result in a significant loss in terms of total welfare.

Recall that we have shown in the previous section, that when the target adoption
level \( \Gamma \) is larger than a certain threshold, both problems yield the same outcome at optimality, and hence are equivalent. Consequently, we focus on the case where the target adoption level is such that it induces different solutions for problems (3.11) and (3.12). In order to compare the outcomes under the two different objectives, we study how the government cost and social welfare compare. In other words, if the government decides to minimize expenditures, how far is the social welfare from the optimal value? Similarly, if the government decides to maximize social welfare, how far is the government cost from the optimal value? We address these two questions for both the monopolistic setting and the competitive environment, denoted by superscripts \( N \) and \( W \) respectively.

We denote by \( GC_{SW} \) and \( GC_{GC} \) the resulting government cost when the government maximizes social welfare and minimizes cost respectively. Similarly, we denote by \( SW_{SW} \) and \( SW_{GC} \) the resulting social welfare when the government maximizes social welfare and when the government minimizes cost respectively. Our goal is to characterize the ratios \( \frac{GC_{SW}}{GC_{GC}} \) and \( \frac{SW_{SW}}{SW_{GC}} \) in order to compare both government objectives.

We also denote by \( p_0^N \) the vector of prices in the monopolistic setting, when the rebate is set to zero \((r = 0)\) (and similarly \( p_0^W \) for the competitive environment). Let \( D_{\gamma} \) be the diagonal matrix with non negative entries \((\gamma_1^N, \ldots, \gamma_n^N)\), where \( \gamma_i^N = \frac{-k_i}{p_{0,i} - c_i} \) for the monopolistic case, and similarly define \( D_{\gamma}^W \) and \( \gamma_i^W = \frac{-k_i}{p_{0,i} - c_i} \) for the competition case. Recall that \( k_i \) represents the externality factor of product \( i \), from equation (3.5). Finally, we define \( \gamma^N = \min_{i \in \{1, \ldots, n\}} \{\gamma_i^N\} \) and \( \bar{\gamma}^N = \max_{i \in \{1, \ldots, n\}} \{\gamma_i^N\} \) (similarly \( \gamma^W \) and \( \bar{\gamma}^W \) for the case with competition). In other words, \( \gamma_i^N \) represents the externality factor of product \( i \) normalized by the profit margin at no rebate. We first consider a deterministic demand and derive closed form expressions for both ratios. We then discuss the case where demand is uncertain. The results for both settings are presented in the following Proposition.

**Proposition 9.** 1. Consider a monopolistic firm with deterministic demand. Then,
we have:
\[
\frac{3 + 2\gamma^N}{4 + 2\gamma^N + 2\gamma^N + (\gamma^N)^2} \leq \frac{SW_{GC}^N}{SW_{SW}^N} \leq 1
\]
\[
M \leq \frac{GC_{SW}^N}{GC_{GC}^N} \quad \forall M > 0
\]

Moreover, this bound is asymptotically tight.

2. Consider a competitive setting with deterministic demand. Then, we have:
\[
\frac{3 + 2\gamma^W}{(2 + \gamma^W)^2} \leq \frac{SW_{GC}^W}{SW_{SW}^W} \leq 1
\]
\[
M \leq \frac{GC_{SW}^W}{GC_{GC}^W} \quad \forall M > 0
\]

Moreover, this bound is asymptotically tight.

Proof. Proof. See Appendix B.5. 

Proposition 9 suggests that if the government focuses on maximizing social welfare, then the resulting expenditures may be arbitrarily large relative to the optimal government cost. Consequently, if the government has a given budget for its subsidy program, maximizing the social welfare may induce a solution that is not budget feasible, as the cost may be unbounded. This analysis potentially supports the fact that policy makers do not often seek to maximize social welfare. In addition, when the government minimizes expenditures, the resulting social welfare is not too far from optimal. For instance, when externalities are not present (i.e., \( k = 0 \)), the resulting social welfare obtained is at most 25% from the optimal value. Note that this theoretical guarantee holds for all instances under this class of demand models. For many practical settings, the ratio is very close to one when externalities are not significant (as we will show computationally in Section 3.5). In conclusion, for the case where externalities are small, the government should minimize expenditures. It will result in significant savings relative to maximizing social welfare, and still attain a near optimal welfare value.
For the case when $\gamma_N = \gamma'_N$ and $\gamma_W = \gamma'_W$, the lower bounds on the Social Welfare of Proposition 9 are the same expression as a function of $\gamma^N$ and $\gamma^W$ respectively. However, note that $\gamma^N \neq \gamma^W$. More precisely, we show that $\gamma^N \leq \gamma^W$ (see Lemma 5 (f) in Appendix B.11), and hence the lower bound in the competitive setting is lower (since $\frac{4(3+2\gamma)}{(2+\gamma)^2} = -\frac{2(\gamma+1)}{(\gamma+2)^2} \leq 0$).

Note that the previous analysis was conducted assuming a deterministic demand. Unfortunately, when demand is stochastic (by incorporating an additive noise), one cannot characterize the ratios in closed form anymore. However, through extensive numerical testing, we observed that the results of Proposition 9 are preserved when demand is stochastic. More specifically, we optimized over the parameters of the problem in order to find the minimal ratios (for both social welfare and government cost), using various noise distributions such as, uniform and truncated normal. The optimization always yields the worst case ratios for the case with a zero standard deviation, i.e., when demand is deterministic. Therefore, it suggests that the results of Proposition 9 are also valid when demand is stochastic, since the deterministic case yields the worst case. Figure 3-1 presents the values of the ratios in the monopolistic setting for a particular instance with no externalities ($k_1 = k_2 = 0$), as a function of the standard deviation. One can see that both ratios improve (get closer to one) as the standard deviation increases. In addition, for small standard deviations, minimizing government cost guarantees a near optimal social welfare (in this case, the social welfare ratio is between 0.8 and 0.98), while reducing the budget significantly.

As one can see from the results of Proposition 9, the social welfare ratio diverges relative to its optimal value, as externalities become more significant. In the limiting case, when the externality factor of one product approaches infinity, the lower bound on the social welfare ratio becomes arbitrarily close to zero. As a result, when externalities are large, the government should not minimize the expenditures anymore, as the loss in welfare may be very significant. Unfortunately, maximizing social welfare is still not a desirable option as it can be very costly. For this reason, we introduce an alternative government objective, which can be seen as an intermediate model between the two objective functions previously discussed. When the externality fac-
tors are large, one can actually expect that minimizing expenditures can be far from maximizing welfare. Indeed, minimizing expenditures does not account at all for the externalities. Motivated by this observation, we introduce the Intermediate Model (IM), that seeks to minimize the government expenditures minus externalities:

$$IM = (r - k)\min\{d, q\}.$$  \hspace{1cm} (3.19)

In other words, this model resembles maximizing social welfare without the firms’ profits and consumer surplus. Note that when the externalities are zero, this model actually coincides with minimizing expenditures. When externalities are strictly positive, one can see this objective as a modified cost, that accounts for the externalities. Since this objective accounts explicitly for externalities, one can expect that this objective is closer to the social welfare. In addition, the firms’ profits are actually already being maximized in the second stage and the consumers are offered subsidies from the government, so that these two terms (suppliers profit and consumer surplus) are somewhat already optimized.

Using the Intermediate Model we just discussed, the government faces the follow-
ing problem:

\[
\min_r \mathbb{E}[IM] \\
\text{s.t.} \mathbb{E}[e' \min\{q(r), d\}] \geq \Gamma 
\] (3.20)

Note that the Intermediate Model is equivalent to minimizing expenditures, when externalities are not very large, since one can show that the optimal solution is obtained when the adoption constraint is exactly met. However, if the externalities are large enough, it induces a higher rebate so that the adoption constraint is not tight at optimality. In order to differentiate between these two cases, we will refer as small externalities to the case when externalities are such that the optimal solution of problem (3.20) is tight. Similarly, large externalities refer to the case when the optimal solution is not tight.

As before, we characterize the ratios for the social welfare and the government cost relative to the optimal values. The results are presented in the following Proposition.

Proposition 10. 1. Consider a market with large externalities and assume a monopolistic firm with deterministic demand. Then, we have:

\[
1 - \frac{9(2 + \gamma^N)}{16(4 + 2\gamma^N + 2\gamma^N + (\gamma^N)^2)} \leq \frac{SW_{IM}^N}{SW_{SW}^N} \leq 1 \\
M < \frac{GC_{IM}^N}{GC_{GC}^N} \quad \forall M > 0 \\
16 \leq \frac{GC_{SW}^N}{GC_{IM}^N} \quad \text{for any instance with large externalities.}
\] (3.21)

Moreover, the bound is asymptotically tight.

2. Consider a competitive setting with deterministic demand. Then, we have:
\[
1 - \frac{9}{16} \frac{(\gamma^W + 2)(\gamma^W + 2) \left( \sqrt{3 + 2\gamma^W} - \sqrt{3 + 2\gamma^W} \right)^2}{\left( 3 + 2\gamma^W (\gamma^W + 1) - \sqrt{3 + 2\gamma^W (\gamma^W + 1)} \right)} \leq \frac{SW^W_{IM}}{SW^W_{SW}} \leq 1
\]

\[M < \frac{GC^W_{IM}}{GC^W_{GC}} \quad \forall M > 0\]  

(3.22)

\[4 \leq \frac{GC^W_{SW}}{GC^W_{IM}} \quad \text{for any instance with large externalities.}\]

**Proof.** Proof. See Appendix B.6. □

By comparing the ratios from Propositions 9 and 10, one can see that the IM model can be significantly better than the GC model in terms of social welfare. In particular, if the government minimizes expenditures, the welfare loss can be arbitrarily bad when externalities become large. In contrast, the ratio for the IM model achieves a constant guarantee that depends on the externality factors (with a worst case of 0.25). Note that when \( \gamma^N = \gamma^N \), the social welfare ratio is at most \( \frac{7}{16} = 0.4375 \), and similarly for the competition case. In addition, the worst case is obtained when \( \gamma^N = \gamma^W = 2 \) and \( \gamma^N = \gamma^W = 0 \), with a ratio of 0.25. In most practical instances, the resulting ratios are close to one so a good social welfare performance is obtained. Note that these bounds are clearly better relative to the case when the government minimizes expenditures.

In addition, by considering the IM model instead of maximizing social welfare, the government can potentially reduce the cost of the subsidy program significantly. As we previously mentioned, when the government maximizes social welfare, the expenditures can become very large and as a result, the subsidy program is very costly. The results of Proposition 10 show that by considering the IM model, the government can reduce its expenditures by a factor of at least 16 or 4 (in the monopolistic and competitive settings respectively). Consequently, when externalities are large, the best government objective is the IM model as it allows cost reduction while achieving a good social welfare performance.

Figure 3-2 presents the values of the ratios in the competitive environment for a particular instance with large externalities as a function of the standard deviation. One can see that both ratios improve (get closer to one) as the standard deviation
increases. Nevertheless, the government cost ratio gets closer to one much slower than the social welfare ratio. In addition, for small standard deviations, using the IM model guarantees near optimal social welfare (in this case, the social welfare ratio is between 0.74 and 0.77) while reducing the budget significantly (in this case, by a factor of at least 37).

Figure 3-2: Parameters: $\bar{d}_1 = 11.52, \bar{d}_2 = 5.59, B_{11} = 2.76, B_{12} = B_{21} = -0.07, B_{22} = 0.7, c_1 = 4.17, c_2 = 0.23, \epsilon_1, \epsilon_2 \sim U, \Gamma = 3, k_1 = 0.83$ and $k_2 = 8.92$.

### 3.4 Effects of competition

In this section, we examine the effects of competition by comparing the outcomes for the competitive environment relative to the monopolistic setting. Our main goal is to study how competition affects the prices, rebates, effective prices and production quantities. In addition, we are interested in quantifying the impact of competition on the various players involved (government, suppliers and consumers) as well as studying the role of externalities and demand uncertainty. Since we consider a general asymmetric competition setting, we study the competition effects on each different market/product. As we previously explained, there exist two different regimes depending on the magnitude of the externalities. Therefore, we divide the analysis into two different cases: small and large externalities. By small externalities, we refer to
problem (3.20) where the optimal solution is obtained when the adoption constraint is exactly met and all the government objectives yield the same optimal rebate. The regime with large externalities refers to the case in which the adoption constraint is not tight when solving problem (3.20). Indeed, when the externality factors are small (and below a certain threshold), the optimal solution of problem (3.20) is always driven by the tightness of the adoption constraint. However, when the externality factors become large, this is not the case anymore.

We first consider the general case which we call "asymmetric" markets or suppliers, and then the case when the suppliers are symmetric (namely, \( d_i = d_j \forall i, j, c_i = c_j \forall i, j, B_{ii} = B_{jj} \forall i, j, B_{ij} = B_{kl} \forall i, j, k, l \neq i, k \neq l \) and \( k_i = k_j \forall i, j \)).

3.4.1 Small externalities

As discussed, the analysis and results depend on the magnitude of the externality factors \( k_i \). In this section, we focus on the case where the externalities are relatively small.

Asymmetric case

We first consider the general case where the suppliers and the demand functions are asymmetric. Such asymmetries may arise due to different reasons, such as: (i) for some products, consumers may be more price sensitive than others (e.g., luxury versus cheap cars); (ii) cross elasticities may be different among different pairs of products (degree of substitution); (iii) the marginal costs may be different; (iv) differences in the market shares (e.g., large popular manufacturer versus small new entrant); and (v) differences in the externality factors (difference in gas emissions between different EVs). The following Proposition summarizes the comparisons of the optimal variables in the competitive environment relative to the monopolistic setting.\(^4\)

**Proposition 11.** Consider \( n \) asymmetric suppliers. Then, by comparing the monopolistic setting (denoted by the superscript \( N \)) to the competitive environment (denoted

\[^4\text{We define the relation operator } x \succeq_1 y (x \preceq_1 y), \text{ such that for any } x, y \in \mathbb{R}^n, \text{ there exists } k \in \{1, \ldots, n\} \text{ that satisfies } x_k \geq y_k (x_k \leq y_k).\]**
by the superscript $W$), we have:

\[
\begin{align*}
p^N & \geq p^W \\
p^N - r^N & \geq p^W - r^W \\
r^N & \geq r^W \\
q^N & \geq q^W \\
\Pi^N & \geq \Pi^W \\
GC^N & \geq GC^W
\end{align*}
\]

Proof. Proof. See Appendix B.7. \qed

In this case, the prices and rebates are smaller in the presence of competition. However, not necessarily all the effective prices paid by the consumers are smaller. Instead, we can only say that for at least one product this actually happens. This result differs from the classical insight that competition benefits consumers. In our problem, some of the consumers pay a higher price under the competitive setting. Similarly for production quantities, one can see that at least one product is under produced in the presence of competition. In addition, competition benefits the government by increasing the expected cost, at the expense of hurting the suppliers in terms of the expected profits. As expected, competition hurts the suppliers. The government, that is leading the game can take advantage of this effect by reducing the rebate offered to consumers while still meeting the desired target adoption level. The government can then take advantage of the competition at the expense of the suppliers. However, the effect on the consumers is not straightforward as they receive lower rebates but also pay a smaller price. We next study how the impact of competition is affected by the magnitude of demand uncertainty.

**Corollary 1.** As demand uncertainty increases:

- $p^N$ and $p^W$ remain the same.
- $(p^N - r^N) - (p^W - r^W)$ increases.
- $r^N - r^W$ decreases.

- $q^N - q^W$ increases.

- $\Pi^N - \Pi^W$ decreases.

- $GC^N - GC^W$ decreases.

- $CS^N - CS^W$ decreases.

The proof of Corollary 1 follows from the proof of Proposition 11 and is not reported for conciseness. Corollary 1 suggests that the magnitude of the demand uncertainty affects the different outcomes, and hence, it is important to consider a stochastic demand in our model and analysis. In addition, since the EV industry is a rather recent market, the absence of reliable data and the fact that consumers are still adapting to this new trend suggest that demand can be quite uncertain. We have shown that, when the demand is more uncertain: (i) it accentuates the competition effect on consumers (i.e., the consumers are “more better-off” in a competitive environment); (ii) it attenuates the competition effect on the government (i.e., the government is “less better-off” in a competitive environment); and (iii) it attenuates the competition effect on the suppliers (i.e., the suppliers are “less worse-off” in a competitive environment). In other words, when demand is more uncertain, the benefit of a competitive market for the government is mitigated. This follows from the fact that the suppliers reduce their production levels and as a result, the government needs to compensate by increasing the subsidies and bears the demand uncertainty risk. We further study the impact of demand uncertainty through a computational example in Figure 3-4.

As we previously discussed, the impact of competition on the consumers is not straightforward. In order to draw additional insights on the consumers, we study a simple scenario with deterministic demand and symmetric price elasticities. In this case, one can characterize which segments of consumers benefit from competition.

**Proposition 12.** In a deterministic setting with symmetric markets, the competition does not affect the effective prices, quantities, consumer surplus and social welfare. As
a result, competition only benefits the government at the expense of hurting the suppliers without affecting the consumers at all. In an asymmetric setting, competition benefits the consumer segment with:

- The highest marginal cost.
- The lowest market share.

In addition, the net change on effective prices due to the presence of competition is equal to zero, when the asymmetry is on costs and/or market share and not on price elasticities. In other words,

\[ \sum_{i=1}^{n}(p_i^N - r_i^N) = \sum_{i=1}^{n}(p_i^W - r_i^W). \]  

(3.23)

Proof. Proof. See Appendix B.8. \[ \square \]

Recall that as expected, the presence of competition induces lower prices. The government anticipates this effect and sets a lower rebate such that the effective price remains unchanged. When demand is deterministic, produced (or equivalently, sold) units also remain unchanged and hence, the expected adoption target is still exactly met. Since the effective price and production quantities are not affected by the presence of competition, consumers surplus and externalities will not be affected either. Interestingly, the social welfare factor remains unchanged too. Indeed, the social welfare stays the same as the welfare from competition is simply transferred from the suppliers to the government. In particular, the price increase is exactly compensated by the rebate reduction.

The second part of Proposition 12 suggests that competition benefits the segment with the highest marginal cost and/or with the lowest market share. In particular, in a competitive environment, each supplier decides price and production separately, leading to underproduction of the low cost product and overproduction of the high cost one. As a result, since the rebate is uniform across the products, the effective price is lower and the produced quantities are higher so that the consumers are clearly better-off.

\[ \text{We say that a segment benefits from the competition, if the effective price is lower and the produced quantities are higher so that the consumers are clearly better-off.} \]
price of the low cost product is higher, while the effective price of the high cost product is lower. Similarly, for a product with a large market share (captured in our model by the term $\bar{d}_i$), a competitive setting leads firms to overproduce the low market share product, and underproduce the high market share one. Consequently, the product with low market share benefits from the competition as the effective price decreases.

**Symmetric case**

In this section, we focus on the case where the parameters are symmetric across all the products.

Note that all the results of Proposition 11 still hold. In addition, since all the optimal variables are identical, one can see that the effective price and the production quantity are lower under the competitive environment. Finally, one can also show the following result on the expected consumer surplus.

**Corollary 2.** Consider $n$ symmetric suppliers and assume that $e' A \leq \Gamma$. Then, the expected consumer surplus follows the following relation:

$$CS^N \leq CS^W.$$  


The assumption $e' A \leq \Gamma$ ensures that the target adoption level set by the government is relevant and cannot be achieved by a large noise realization. Note that as the cross elasticity parameters $B_{ij}; i \neq j$ approach 0, the markets (and suppliers) become less dependent and as expected, the effect of competition vanishes. In particular, when $B_{ij} = 0; \forall i \neq j$ the monopolistic and competitive settings coincide.

We next consider varying the magnitude of the demand uncertainty, captured by the additive noises $\epsilon_i$. By reducing the demand uncertainty, one can see that the gap in the effective prices decreases. In other words, as $A_i$ goes to 0, $p^W - r^W$ approaches $p^N - r^N$, so that the effect of competition on the effective price diminishes as demand uncertainty decreases. However, the prices and the rebates are still larger for the
monopolistic setting. This implies that the effect of competition is totally absorbed by the government when the demand becomes deterministic. More specifically, we still obtain the reduction in the selling price $p$ induced by the competition among the suppliers but in this case, the government can decrease the rebates in a way so that the effective price $p - r$ remains unchanged. This is in contrast to classical insights about competition suggesting that consumers always benefit from the competition. This follows from the fact that the government aims to achieve a target adoption level. In a competitive environment, the government can anticipate the price reduction due to competition among the suppliers, and decrease the rebates so as to achieve the target. Since the sales depend explicitly on the effective price, the price reduction is exactly compensated by the rebate augmentation. As a result, the effective price paid by consumers is not affected by the competition (in a symmetric and deterministic setting). Interestingly, adding demand uncertainty modifies partially the outcomes. More precisely, the government and the consumers are now sharing the competition benefits at the expense of the suppliers. As before, competition decreases the price, the government reacts by reducing the rebate but in a way that the effective price decreases as well. As a result, the consumers are better-off in a competitive environment, when demand is symmetric and stochastic. As discussed, the government and the consumers share the competition benefit. Each player can extract some proportion of the benefits that depends on the magnitude of the demand uncertainty.

3.4.2 Large externalities

In this section, we consider the case where externalities are large. More precisely, by large we mean that the externalities are such that the constraint in problem (3.20) is not tight at optimality. As discussed before, for small externalities, the adoption constraint is always tight in the optimal solution. When the externalities are large enough (i.e., there exists a threshold value), the optimal solution of the government problem is such that the adoption constraint is not tight anymore. In this setting, the optimal rebate level is larger relative to the one that achieves the target adoption constraint with equality. This follows from the fact that the externalities generated
by an extra dollar of investment are profitable (i.e., larger than one dollar). Figure 3-3 illustrates the marginal government cost and externalities generated by an extra unit of rebate, in an environment without competition. Here, \( dGC/dr \) and \( dEX/dr \) denote the derivatives of the government cost and the expected externalities with respect to the rebate respectively. In this example, it is profitable for the government to increase the rebate beyond the value at which the constraint is binding. More precisely, the optimal rebate for the intermediate model is attained when the marginal cost equals the marginal externality factor (when the two curves cross in Figure 3-3).

As we discussed in Section 3.3, when externalities are large, the government should not aim to minimize expenditures. In particular, the welfare loss becomes unbounded as the externality factors grow. Therefore, we consider the intermediate model we introduced in (3.20), where the government minimizes the expenditures corrected by the externality factors. Recall that this objective allows the government to significantly reduce expenditures, while still achieving a near optimal welfare. We next present the results about the impact of competition on the different problem variables.

Figure 3-3: Parameters: \( \bar{d}_1 = 4, \bar{d}_2 = 3, B_{11} = B_{22} = 1, B_{12} = B_{21} = -0.1, c_1 = 2, c_2 = 2.2, k_1 = k_2 = 4, \epsilon_1, \epsilon_2 = 0 \) w.p. 1 and \( \Gamma = 2 \).

**Proposition 13.** Consider \( n \) asymmetric suppliers with deterministic demand. Then,
by comparing the monopolistic setting to the competitive environment, we have:

\[ p^N - r^N \geq p^W - r^W \]
\[ q^N \leq q^W \]
\[ e'q^N \leq e'q^W \]
\[ CS^N \leq CS^W \]

In the particular case with symmetric suppliers, we have:

\[ r^N = r^W \]
\[ p^N \geq p^W \]
\[ p^N - r^N \geq p^W - r^W \]
\[ q^N \leq q^W \]
\[ \Pi^N \geq \Pi^W \]
\[ GC^N \leq GC^W \]

Proof. See Appendix B.10. \qed

In this case, the rebate level is not necessarily smaller in the competitive environment. Note that under large externalities, competition will induce lower effective prices and larger production for some segments. Note that the results of Proposition 13 remain valid when demand is stochastic. In particular, all the inequalities remain the same except that in this case, for symmetric suppliers we have: \( r^N \geq r^W \).

When the externalities are large, Proposition 13 states that in the symmetric case, prices and effective prices decrease in the presence of competition. Furthermore, quantities produced increase under competition. A lower effective price leads to a higher demand and as a result, the firms increase production. Note that this effect did not occur in the case with small externalities. Indeed, since the rebate was set to exactly meet the target adoption level, the firms could produce less units in the presence of competition, even though the effective price was lower. Another
interesting result, is that in this case, the total firms' profits and government costs are not necessarily lower in a competitive environment (with asymmetric suppliers), which again, was not the case under small externalities. Actually, for symmetric suppliers the government cost is even higher in the presence of competition.

We next compare the impact of competition on markets with small and large externalities. For the case of symmetric suppliers, one can see that competition always hurts the suppliers in terms of expected profits (no matter how small or large the externalities are). However, the effects on the government and the consumers differ depending on the magnitude of the externalities. On the one hand, when externalities are small, the government always benefits from competition in terms of expected cost. On the other hand, when externalities are large, the competition has the opposite effect on the government cost (for symmetric suppliers). This difference is driven by the fact that the target adoption level is exactly met a optimality, for small externalities. As a result, the government can take advantage of the competing suppliers by decreasing the rebates, and hence reducing the expenditures. However, recall that for large externalities, the adoption constraint is not tight anymore. As a result, the competition induces larger production quantities and demand, so that the government has to subsidize additional units and hence the overall expenditures increase. This is explained by the fact that the government perceives a significant environmental benefit (through the large externalities) from any dollar invested in consumer subsidies and as a result, aims to let competition increase the production quantities.

By studying the expected consumer surplus, one can see that under small externalities, the presence of competition does not always benefit consumers (for asymmetric suppliers). Indeed, since the adoption constraint is exactly met, the production quantities in the presence of competition will be decreased. Consequently, this affects the consumers that dispose of less available supply and may reduce the expected consumer surplus. However, when externalities are large, the adoption constraint is not tight and therefore, the competing suppliers will produce more so that it always benefits consumers.
For asymmetric suppliers, one can note that the impact of competition on the price depends on the magnitude of the externalities. In particular, for small externalities, all the prices are lower in the competitive environment but when externalities are large, this is not necessarily the case.

### 3.5 Computational experiments

In this section, we calibrate our model with publicly available data and test our insights computationally. Several car manufacturers have launched EVs in the last four years, yet many of them lack of sufficient historical sales data. Therefore, we have decided to consider only the two major EV models for which richer data is available: Chevy Volt and Nissan Leaf\(^6\). We assume the marginal cost of producing an EV is equal to 90\% of the average manufacturer’s suggested retail price (MSRP) over the years 2011, 2012 and 2013. Note that the average margin for car dealers in 2015 was 13\%\(^7\). We estimate the price elasticities and the intercept terms of the demand (i.e., the matrix \(B\) and the vector \(d\) respectively), by performing a least squares estimation of the sales for the two EVs over the 3 year period. Since the data set is small relative to the number of parameters, we assume that the self elasticities are the same (i.e., \(B_{11} = B_{22}\)). Note that this also helps avoid over fitting. The adoption target \(\Gamma\) is set to 45 (thousand), which corresponds to the amount of EVs sold in 2013 by the two manufacturers. We obtain the following estimates: \(d_1 = 110.21, d_2 = 73.43, B_{11} = B_{22} = 3.14, B_{12} = B_{21} = -0.81, c_1 = 30.73, c_2 = 28.8\). Finally, we assume that the additive noises \(\epsilon_1\) and \(\epsilon_2\) are uniformly distributed with support equal to \(\pm 10\%\) of their corresponding market shares \(\bar{d}_1\) and \(\bar{d}_2\).

The positive externalities of an EV correspond to the reduction in CO\(_2\) emissions throughout their lifetime, converted to US dollars. Following the analysis in [2], the emission rate per unit of energy amounts to 755 [Kg CO\(_2\) × MWh\(^{-1}\)]. In addition, the efficiency of an EV is of the order of 0.155 [KWh × Km\(^{-1}\)], whereas the average

\(^6\)http://insideevs.com/monthly-plug-in-sales-scorecard/
lifetime vehicle mileage is about 152,137 [Miles]. Consequently, the total emissions of an EV amounts to
755 [Kg CO$_2$ x MWh$^{-1}$] $\times$ 0.155 [KWh x Km$^{-1}$] $\times$ 152,137 [Miles] $\times$ 1.61 [Km x Miles$^{-1}$] = 28.66 [Ton CO$_2$]. In the case of a regular gasoline vehicle, the net emission factor can be taken to be 10.6 [CO$_2$ x gal$^{-1}$]. Then, considering an average consumption rate of 24.4 [Miles x gal$^{-1}$], the lifetime emissions generated by a regular vehicle amount to 10.6 [CO$_2$ x gal$^{-1}$] $\times$ 152,137 [Miles] / (20.4 [Miles x gal$^{-1}$]) = 78.9 [Ton CO$_2$]. As a result, an estimate for an EV gas emission reduction is 78.9 − 28.6 = 50.3 [Ton CO$_2$]. In order to convert this number to US dollars, we use the value assigned by the United States Environmental Protection Agency (EPA) to a Ton of CO$_2$. We consider the average value for the years 2014 – 2033, which is 52.0 \([\$ x (Ton CO$_2$)$^{-1}$] (in US dollars of 2014). Therefore, the monetary positive externality of an EV is equal to 50.3 [Ton CO$_2$] $\times$ 52.0 \([\$ x (Ton CO$_2$)$^{-1}$] = $2,612. Therefore, in our model $k_1 = k_2 = k = $2,612.

3.5.1 Demand Uncertainty

First, we test the sensitivity of our results with respect to the magnitude of the demand uncertainty. In other words, we are interested in studying how the impact of competition is affected by demand uncertainty. We assume that the additive noises for each product $i$ is uniformly distributed, i.e., $U[-A_i, A_i]$. We modify the magnitude of the demand uncertainty by varying $A_i$ for each product, ranging from 0 (i.e., a deterministic setting) to 30% of the market share (i.e., a fairly volatile market). Figure 3-4 shows the ratios of the government cost, the profits of the suppliers, and the consumer surplus for different values of $A_i$. The left plot is under small externalities (in this case, $k = 2.61$) while the right plot assumes large externalities (in this case, $k = 35$).

From the left plot, one can see that when $A_i = 0$ (i.e., a deterministic setting), competition benefits the government (by reducing the expected cost) and hurts the

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suppliers in terms of profits. As demand uncertainty increases, the benefit of the government is transferred to the consumers. Note that for low demand uncertainty, competition does not always benefit consumers. Nevertheless, incorporating demand uncertainty favors the consumers that can extract some benefit from the presence of the competition, at the expense of the government who needs to pay a larger cost. From the right plot, one can see that when externalities are large, consumers always benefit from competition (see Proposition 13). Surprisingly, increasing demand uncertainty for large externalities leads to a lower benefit for consumers in the competitive environment.

Figure 3-4: Parameters: $\bar{d}_1 = 110.21$, $\bar{d}_2 = 73.43$, $B_{11} = B_{22} = 3.14$, $B_{12} = B_{21} = -0.81$, $c_1 = 30.73$, $c_2 = 28.8$, $\epsilon_1 \sim \bar{d}_1 \times U[-A_1, A_1]$, $\epsilon_2 \sim \bar{d}_2 \times U[-A_2, A_2]$, $\Gamma = 45$. $k = 2.61$ in the left plot, and $k = 35$ in the right plot.

3.5.2 Varying the target adoption

We next vary the value of the target adoption level $\Gamma$ set by the government. In Figure 3-5, we plot the ratios of the government cost, the profits of the suppliers, and the consumer surplus for both small and large externalities as a function of $\Gamma$. One can see that the value of $\Gamma$ does not have any qualitative impact on the main results and insights of the paper. When externalities are large, the optimal solutions do not depend on $\Gamma$ at all, so that the impact of competition remains constant. For small externalities, the results are slightly affected by the value of $\Gamma$. In particular, varying
\(\Gamma\) from 35 to 50 (i.e., a 40% increase) does not change the consumer surplus ratio and has an effect of less than 4.5% on the other two ratios. This suggests that no matter what is the value of the desired adoption target set by the government, the insights regarding the impact of competition remain the same.

Figure 3-5: Parameters: \(\bar{d}_1 = 110.21\), \(\bar{d}_2 = 73.43\), \(B_{11} = B_{22} = 3.14\), \(B_{12} = B_{21} = -0.81\), \(c_1 = 30.73\), \(c_2 = 28.8\), \(\epsilon_1 \sim \bar{d}_1 \times U[-0.1, 0.1]\), \(\epsilon_2 \sim \bar{d}_2 \times U[-0.1, 0.1]\). \(k = 2.61\) in the left plot, and \(k = 35\) in the right plot.

### 3.5.3 Substitution

Next, we analyze the effects of varying the cross-elasticity term \(B_{12} = B_{21}\), that captures the degree of competition among the firms. Note that if \(B_{12} = 0\) (i.e., independent suppliers), then the monopolist and competitive settings coincide, as expected. In Figure 3-6, we plot the ratios of the government cost, the profits of the suppliers, and the consumer surplus for different values of demand substitution captured by \(|B_{12}|/B_{11}\).

When externalities are small (left plot of Figure 3-6), increasing the degree of substitution reduces the government cost and the firms' profits. As expected, the more intense the competition is, the better it is for the government and the worse for the suppliers. Note that the consumers perceive a very small benefit in this case. This follows from the fact that the government can take advantage by counterbalancing the reduction in prices by decreasing the rebates. As a result, consumers do not
benefit much of the competition. In the case of large externalities, (see right plot of Figure 3-6), the suppliers are still hurt from the competition. However, the effect of competition on the government and the consumers is quite different as before. In this case, since the target adoption constraint is not tight, the competition induces larger total production (see Proposition 13). Consequently, the competition increases the government cost and the consumer surplus. Figure 3-6 shows how this effect scales with the degree of substitution. One can see that the impact of competition on the different players (government, suppliers and consumers) highly depends on the magnitude of the externalities.

![Graph showing the relationship between competition and externalities](image)

Figure 3-6: Parameters: $d_1 = 110.21, d_2 = 73.43, B_{11} = B_{22} = 3.14, c_1 = 30.73, c_2 = 28.8, \epsilon_1 \sim \bar{d}_1 \times U[-0.1, 0.1], \epsilon_2 \sim \bar{d}_2 \times U[-0.1, 0.1], \Gamma = 45. k = 2.61$ in the left plot, and $k = 30$ in the right plot.

### 3.5.4 Externalities

As discussed, the impact of competition on the different players highly depends on the magnitude of the externalities. Next, we study the effect of varying the level of externalities on the government cost, the profits of the suppliers, and the consumer surplus by considering a competitive setting. We vary the level of externalities $k$ by changing the price of carbon. As we previously mentioned, the average value for the years 2014 – 2033 is 52.0 \$/\text{(Ton CO}_2\text{)}^{-1}]. The plot in Figure 3-7 examines the
effect on the various players if the price of carbon were to increase in the future. Alternatively, this is equivalent to the case in which technological progress will allow a significant reduction of the CO₂ emissions for EVs.

When externalities are small (and below a certain threshold), one can see that the price of carbon does not affect either of the quantities of interest. However, when externalities become large (in this case, when the price of carbon is greater than 520), all three quantities increase with respect to the price of carbon. Consequently, increasing the price of carbon hurts the government but benefits both the suppliers and the consumers. Recall that the government incorporates the environmental benefit of the externalities directly in its objective. As a result, it becomes advantageous for the government to increase rebates. In particular, for any subsidized dollar, the government perceives a great return due to the externalities and to the high price of carbon. Ultimately, it costs more to the government, but benefits the other parties. This analysis suggests that for markets with large externalities, the government should be careful about incorporating externalities in its objective, as the cost can become very high, as we show in Section 3.3.

![Figure 3-7: Parameters: $d_1 = 110.21$, $d_2 = 73.43$, $B_{11} = B_{22} = 3.14$, $B_{12} = B_{21} = -0.81$, $c_1 = 30.73$, $c_2 = 28.8$, $\epsilon_1 \sim d_1 \times U[-0.1, 0.1]$, $\epsilon_2 \sim d_2 \times U[-0.1, 0.1]$, $\Gamma = 45$. The value of the x-axis is the price of a Ton of CO₂.](image)
3.6 Conclusions

In this chapter we introduce and study a model for competition in green technology adoption. In particular, as it became prevalent in the EV industry, the government offers consumer subsidies with the goal to reach a target adoption level. Multiple competing suppliers then decide price and production so as to maximize expected profits. We compare the outcomes of a competitive environment relative to a monopolistic setting.

We first show that when the government minimizes expenditures, the resulting social welfare remains close to optimal, especially when externality factors are low. As a result, when externalities are not significant, it seems better for the government to minimize the cost of the subsidy program as it provides good social welfare, while keeping a reasonable budget. For markets with large externalities, we first show that minimizing expenditures can yield a very low welfare relative to optimal. We then introduce an intermediate model where the government minimizes expenditures corrected by externality factors, and we derive tight bounds that characterize the worst case guarantee on the social welfare.

Second, we investigate the impact of competition on the government, the suppliers and the consumers. When the externalities are low, we show that competition induces lower prices and rebates, but not necessarily lower effective prices. For symmetric suppliers, we show that both the effective prices and the production quantities are lower in a competitive environment. As a result, competition hurts the suppliers (that perceive a lower expected profit), benefits the government (that can decrease the rebates) and benefits the consumers (that enjoy a larger expected surplus). More precisely, the benefit of competition is shared between the government and the consumers, as a function of the demand uncertainty. When demand is deterministic, the government absorbs the entire benefit, and the consumers are not affected by the competition at all. When demand becomes more uncertain, the consumers manage to extract some of the competition benefit and share it with the government.

For markets with large externalities, the impact of competition differs. It becomes
optimal for the government to offer subsidies such that the expected sales exceed the target adoption level in order to take advantage of the high environmental benefits. We show that it is not clear anymore that the rebates are lower in a competitive environment and as a result, the government can be hurt by competition. In addition, the consumers will now always benefit from the presence of the competition.

In conclusion, the effect of competition on the different agents depends on the interplay of suppliers’ asymmetry, externality factors and demand uncertainty. In most cases, competition tends to hurt the suppliers as expected. However, the impact on the government and the consumers appears to be more subtle, as we illustrate in this paper.

Bibliography


Chapter 4

A Unifying Framework for Consumer Surplus under Demand Uncertainty

4.1 Introduction

Operations Management (OM) has focused its efforts on a wide variety of problems. Many of these problems involve consumers in different contexts. In these efforts, the models developed have evolved in complexity in order to capture the different intricacies of reality. One particular intricacy that frequently arises, is the uncertainty in demand. In order to capture this underlying uncertainty, the introduction of a stochastic noise in demand functions has been a common practice in several OM models. However, stochastic demand makes welfare analysis more complicated as has been stated in works such as [15]. Mainly, including a stochastic term in the demand function might not allow the existence of an underlying representative consumer utility function. As a result, utility analysis may not be possible.

Besides these concerns, stochasticity might bring scarcity into the picture leading to other nuances. Given a price, the demand function represents the maximum number of units that will be consumed, or at least that are desired to be consumed in the market. This demand entails multiple consumers where each of them is endorsed with a particular maximum willingness to pay for the item. Then, when demand exceeds the produced items for a given market price, the portion of customers willing
to purchase the item can not be fully served given the scarcity in place\textsuperscript{1}. Therefore, the resulting consumer surplus will depend on which of those consumers might end up buying the item. In addition to the single item case, we consider the multiple item case. This setting has additional subtleties that are not present in the single item case.

Previous works that have studied the impact of demand uncertainty have mainly focused on the quantities, prices, and firms' profits. An example of this is the famous newsvendor problem that comes up with the optimal production quantity before demand realizes. Similarly, in the price-setting newsvendor problem, the seller has also control over the price. This allows him to choose the best range of demand realizations in terms of maximizing his expected profits. For instance, works such as [17] and [14] have explored the optimal prices and profits earned by firms in the stochastic setting vis-à-vis its deterministic counterparts for additive and multiplicative demand noises in the single item case. However, these papers do not address the effects of uncertainty over the consumers. This chapter attempts to answer these questions.

The contributions of this work are: (i) Define in a systematic and mathematical way the concept of allocation rules when demand is stochastic. We consider the most "common" allocation rules, and use them to define Consumer Surplus for single as well as multiple items. (ii) Study the impact of demand uncertainty compared to when demand is deterministic for fixed market prices and quantities, and see how this impact depends on the nature of demand noise, the convexity/concavity of demand, and the allocation rule. (iii) Extend the latter analysis for the case where firms also have control and decide their optimal prices and quantities.

4.1.1 Literature Review

The consumer surplus, in the way we know it today, namely the difference between consumers' maximum willingness to pay minus the price paid, was first introduced

\textsuperscript{1}Note that the context here is such that quantities and prices are fixed before demand realizes, similar to the newsvendor problem.
by Jules Dupuit\(^2\) in 1844 as *utilité relative*. Later on, Marshall relabeled this as “consumers’ surplus” in his work, *the Principles of Economics* in 1890, denoting it as the left triangle of the (inverse) demand function.

Since then, hundreds of academic papers were published on the topic. Some works develop analytical frameworks (e.g., [27] and [23]), whereas some others focus on applications. In particular, the concept of consumer surplus was studied in a large variety of contexts. Examples include online auctions ([2]), commodity markets ([22]), public transportation ([10]), and government regulations ([16]), just to name a few. Recently, several works study consumer surplus from an empirical perspective. In [13], the authors use firm-level data to show that information technology has increased productivity and also created substantial value for consumers. In [4], it is shown that increased product variety (through electronic markets such as Amazon) can be a significant source of consumer surplus gain. The authors demonstrate that the increased product variety of online bookstores enhanced consumer welfare by $731$ million to $1.03$ billion in the year 2000. Even more recently, in [7], the authors measure the impact of the transportation network company Uber on consumers, and convey that for each dollar spent by consumers, about $1.60$ of consumer surplus is generated. In addition, they conclude that: “Back-of-the-envelope calculations suggest that the overall consumer surplus generated by the UberX service in the United States in 2015 was $6.8$ billion”.

In most previous works, the framework developed to study and calculate consumer surplus focuses on assuming that the demand curve is a deterministic function of the price. In the operations management community, many applications involve stochastic demand. Examples include supply chain management, revenue management, queuing systems and service operations. If one is interested in measuring consumer satisfaction in such an application, we need an appropriate definition of consumer surplus. Addressing this question is one of the main focuses of this chapter.

Several papers on peak load pricing and capacity investments by a power utility

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\(^2\)Before Dupuit, Jean-Baptiste Say had the notion that the utility of a transaction was equal to the good’s price.
under stochastic demand address partially this modeling issue (see [5], [8] and [3]).

Nevertheless, the models developed in this literature are not applicable to settings where consumers arrive in a random fashion and not according to their valuations. More specifically, in [3] the authors assume that the utility power facility has access to the willingness to pay of the customers so that it can decline the ones with the lowest valuations. This assumption is not justifiable in a setting where a “first-come-first-serve” logic with random arrivals is more suitable (e.g., in the retail industry).

In [18], the authors study a price setting newsvendor model for public goods and consider the Consumer Surplus for linear additive stochastic demand. In [6], the authors study the impact of demand uncertainty on consumer subsidies for green technology adoption and propose a definition similar to the one we present in this paper. However, this chapter is the first to provide a rigorous treatment of extending the definition of consumer surplus under uncertain demand for multiple items (see Section 4.3), as well as for different rationing capacity rules.

The second contribution of this chapter is to use the consumer surplus extension we introduce in order to measure the impact of demand uncertainty on consumers. Studying the impact of demand uncertainty (or stock-outs) was extensively studied in the operations management literature. Examples include supply chain (e.g., [12]), capacity investment (e.g., [11]) and subsidies for green technology adoption (e.g., [6]).

**Structure of the chapter.** In Section 4.2, we first start our analysis with the single item model. We extend the treatment for multiple items in Section 4.3. We then use the extension of consumer surplus to study and quantify the impact of demand uncertainty in Section 4.4. Section 4.5 analyze the impact of uncertainty on the optimal prices and quantities of the firm(s). Finally, the conclusions are summarized in Section 4.6. Most of the proofs of the Theorems and Propositions are relegated to the Appendix.
4.2 Single Item

Consider the case of a supplier selling a single item, who faces a stochastic demand curve \( d(p, \epsilon) \). In particular, we explore two different noise natures: (i) additive\(^3\), i.e., \( d(p, \epsilon) = d(p) + \epsilon \), where \( \epsilon \) is a random variable with zero mean; and (ii) multiplicative, i.e., \( d(p, \epsilon) = d(p) \cdot \epsilon \), where \( \epsilon \) is a positive random variable with mean equal to 1.

The function \( d(p) \) is called the *nominal demand function* and corresponds to the setting with deterministic demand, i.e., no demand uncertainty. Namely, \( \epsilon = 0 \) with probability 1 in the additive noise case, and \( \epsilon = 1 \) with probability 1 for multiplicative noise. We assume that \( d(p) \) is non-decreasing and invertible. We refer to \( d^{-1}(q) \) as the *nominal inverse demand*. Note that \( d^{-1}(q, \epsilon) \) is defined so that \( d(d^{-1}(q, \epsilon), \epsilon) = q \). The goal is to study the Consumer Surplus for a general stochastic demand function. First, let us briefly recall the Consumer Surplus definition when demand is deterministic.

### 4.2.1 Deterministic Demand

Consumer Surplus is an economic measure of consumer net welfare when consuming a product given the expenditures incurred. This can be computed as the area between the market price \( p \), and the inverse demand curve. Thus, for a given market price and quantity pair \( (p, q = d(p)) \), the Consumer Surplus \( CS_{det} \) can be computed as:

\[
CS_{det} = \int_0^{d(p)} (d^{-1}(w) - p) \, dw
\]  

(4.1)

Equivalently, Consumer Surplus can also be computed by integrating over the price space as

\[
CS_{det} = \int_p^{p_{\text{max}}} d(z) \, dz,
\]  

(4.2)

where \( p_{\text{max}} \) corresponds to \( d^{-1}(0) \), i.e. the inverse of the demand function at zero, or \( +\infty \) in the case it is not defined (e.g. \( d(p) = 1/p \)); usually called “the null price”.

\(^3\)With some abuse of notation, we denote \( d(p, \epsilon) \) the stochastic demand function, and \( d(p) \) its respective deterministic part; thus if the argument is only the price, we will refer to the latter one, whereas if the argument is a price and a random variable (\( \epsilon \)), then we will refer to the former one.
In the remainder of the chapter, we focus on the Consumer Surplus expression when the integration is taken over the quantity space, i.e. expression (4.1). Note that since \( d(z) \geq 0 \) and \( p \leq d^{-1}(0) \), one can see that \( CS_{det} \) is non-increasing with respect to the market price \( p \). Consumer Surplus represents the surplus induced by consumers who are willing to pay more than the market price.

**4.2.2 Stochastic Demand**

When demand is stochastic, defining the Consumer Surplus is somewhat more subtle due to the possibility of stock-outs. More precisely, if the demand \( d(p, \epsilon) \) at price \( p \), for a particular realization \( \epsilon \), happens to be greater than the available units \( q \) (i.e. \( d(p, \epsilon) - q > 0 \)), some consumers who are willing to make a purchase will not be able to do so due to lack of supply. This stock-out event clearly affects consumers' satisfaction, and therefore, should naturally be accounted in the Consumer Surplus definition.

Note that for stochastic demand functions, the Consumer Surplus \( CS(\epsilon) \) will be a function of the noise realization \( \epsilon \). A trivial upper bound of the Consumer Surplus can be considered as the case in which there is infinite supply. Then the Consumer Surplus will simply be \( \int_0^{d(p, \epsilon)} \left( d^{-1}(w, \epsilon) - p \right) dw \). In reality, as we previously mentioned, we have to account for situations where demand exceeds supply. In particular, the actual Consumer Surplus will be a fraction of this upper bound (that does not account for limited supply), based on the fraction of customers who are actually served. The way of formalizing this precise fraction of served consumers depends on rationing capacity rules that we discuss next.

**Rationing Capacity Rules**

One can consider several rules of rationing the available supply. In many operational systems, suppliers don’t have access to customer valuations and simply assign capacity to the first customers who show up to the store. Examples include car dealers, fashion, electronic items and online shopping. In this case, the customers’ arrival times are
independent of their valuations for the product and therefore, we label this rule as a R (Random allocation). In other words, potential consumers who are interested in purchasing the product, have the same chances to be served. Other two alternative common rules are H (Highest willingness to pay) and L (Lowest willingness to pay). As their names indicate, the seller can allocate the available supply to the consumers with the highest (respectively, lowest) willingness to pay and discard the consumers with lowest (respectively, highest) valuations. Note that these two allocation rules are the best (respectively, the worst) in terms of the total consumers’ surplus. Besides these three allocation rule examples, the seller can consider other allocation rules. More precisely, an allocation rule \( A \) can be any allocation of the available capacity \( q \) at price \( p \) to the customers, for a given demand realization \( \epsilon \). Mathematically, an allocation \( A \) is defined as a function \( A : [0, d(p, \epsilon)] \to [0, 1] \) so that for any price \( p \), quantity \( q \) and noise realization \( \epsilon \), we have:

\[
\int_0^{d(p, \epsilon)} A(w) dw = \min\{d(p, \epsilon), q\}. \tag{4.3}
\]

Then, for any \( w \in [0, d(p, \epsilon)] \), \( A(w) \) can be interpreted as the chance that an infinitesimal consumer get the item. Equation (4.3) captures the fact that the total supplied units under any allocation rule \( A \), should be equal to the volume of sales (i.e., the minimum between demand and quantity). Then, it is not hard to see that, given a price \( p \), a supply \( q \), and a demand realization \( d(p, \epsilon) \), the allocation rules functions \( A^R, A^H, A^L \); for the rules R, H, and L are \( A^R(w) = \min\left\{1, \frac{q}{d(p, \epsilon)}\right\}, A^H(w) = 1_{\{w \leq q\}}, \) and \( A^L(w) = 1_{\{w > d(p, \epsilon) - q\}} \), respectively. Consequently, given an allocation rule, the Consumers’ Surplus can be simply defined as the sum of the surpluses of consumers willing to make a purchase, weighted by the corresponding chance that each of these consumers get the item. Although this definition can be made for any allocation rule, most of our analysis will focus on the three most popular rules aforementioned.
In this section, we provide the Consumer Surplus definition under stochastic demand for a general demand function $d(p, \epsilon)$. We first report the expressions for the three allocation rules aforementioned and conclude by considering any allocation rule $A$. In addition, we provide a graphical interpretation by illustrating the definitions for the case of linear demand with additive noise. A similar methodology and graphical intuitions can be found in [26] for the total welfare (consumer and supplier surpluses).

We start with the H rule. In this case, the Consumer Surplus for any given noise realization $\epsilon$ and any pair $(p, q)$ is given by:

$$
CS^H(\epsilon) = \int_0^{d(p, \epsilon)} (d^{-1}(w, \epsilon) - p) \mathbb{1}_{\{w \leq q\}} dw = \int_0^{\min\{d(p, \epsilon), q\}} (d^{-1}(w, \epsilon) - p) dw.
$$

This rule is the best that can be achieved for the pool of customers, as customers with high valuations usually want the product the most, and are served with the highest priority. In the case where demand exceeds the units produced, i.e., $d(p, \epsilon) > q$, the $d(p, \epsilon) - q$ customers with the lowest valuations are not served and so the Consumer Surplus for those consumers is zero. See the left panel of Figure 4-1.

Next, we consider the L rule. In this case, the Consumer Surplus for any given noise realization $\epsilon$ and any pair $(p, q)$ is given by:

$$
CS^L(\epsilon) = \int_0^{d(p, \epsilon)} (d^{-1}(w, \epsilon) - p) \mathbb{1}_{\{w \geq d(p, \epsilon) - q\}} dw = \int_0^{d(p, \epsilon)} (d^{-1}(w, \epsilon) - p) dw.
$$

This rule is the worst that can be achieved for customers, as the customers with high valuations want the product the most and are served with the lowest priority. This can correspond to the case where low valuation customers are more deal seekers and arrive first to the store. When demand exceeds supply, i.e., $d(p, \epsilon) > q$, the $d(p, \epsilon) - q$ customers with the highest valuations are not served. See the center panel of Figure 4-1.

Next, we consider the random allocation rule. In this case, the Consumer Surplus
for any given noise realization $\epsilon$ and any pair $(p, q)$ is given by:

$$CS^R(\epsilon) = \int_0^{d(p,\epsilon)} \left( d^{-1}(w, \epsilon) - p \right) \min \left\{ 1, \frac{q}{d(p,\epsilon)} \right\} dw. \quad (4.6)$$

In this case, customers arrive in a first-come-first-served manner, irrespective of their willingness to pay. Under certain demand realizations, some proportion of these customers will not be served due to stock-outs. The proportion of served customers under one of these demand realizations is given by the ratio of actual sales over potential demand, i.e., $\min \left\{ 1, \frac{q}{d(p,\epsilon)} \right\}$. Therefore, the Consumer Surplus can be defined as the total available surplus times the proportion of served customers. In this case, the Consumer Surplus can be depicted as the area in gray in between the inverse demand curve realization and the price, see the right panel of Figure 4-1. Equivalently, when demand exceeds supply, each infinitesimal consumer has a surplus which is weighted by $q/d(p,\epsilon)$.

![Figure 4-1](image.png)

Figure 4-1: Left: High willingness to pay allocation rule. Center: Low willingness to pay. Right: Random allocation rule. The shaded region represents the Consumer Surplus where demand exceeds supply.

Finally, we consider any allocation rule $A$. In this case, the Consumer Surplus for any given noise realization $\epsilon$ and any pair $(p, q)$ is given by:

$$CS^A(\epsilon) = \int_0^{d(p,\epsilon)} \left( d^{-1}(w, \epsilon) - p \right) A(w) dw. \quad (4.7)$$

Note that all the definitions above coincide with Equation (4.1) when the noise $\epsilon$
"vanishes" (i.e., $\epsilon = 0$ for additive and $\epsilon = 1$ for multiplicative) and $q = \mathbb{E}[d(p, \epsilon)]$. In addition, the reason to pick the integral along the quantity space on the definition of the Consumer Surplus in Equation (4.7), is that it allows to weight the surplus of each demand by the corresponding allocation. For situations where one is interested in quantifying the impact of demand uncertainty on consumers (see Section 4.4), one can compare $CS_{det}$ to the expected Consumer Surplus $\mathbb{E}[CS^A(\epsilon)]$ for the allocation rule $A$. Finally, note that for any given rule, the following property holds.

**Observation 1.** For any allocation rule $A$, any price $p$, production quantity $q$ and given noise realization $\epsilon$, we have:

$$CS^L(\epsilon) \leq CS^A(\epsilon) \leq CS^H(\epsilon).$$

(4.8)

The proof of Observation 1 can be found in the Appendix C. Consequently, we have that $\mathbb{E}[CS^L(\epsilon)] \leq \mathbb{E}[CS^A(\epsilon)] \leq \mathbb{E}[CS^H(\epsilon)]$. As a result, by studying the H and L rules, we cover both the worst and best case scenarios for the consumers.

### 4.2.3 Utility Perspective

In this section, we recall an alternative way to derive the Consumer Surplus formula through the representative consumer utility function. In many cases, the consumer’s utility from consumption is directly compared with the expenses incurred in this consumption. Therefore, consumption takes place until the marginal utility equals the price. In this setting, there is an underlying assumption that the utility function is measured in monetary units, the utility of each unit of wealth is equal to one, and the income of the representative consumer is high enough relative to the expenditures$^4$. See for examples works as [25], [9] and [19]. The representative consumer has

$^4$The first claim, that the utility is measured in monetary units is direct, otherwise it could no be directly compared to the expenditures. The second point, relates to the fact that the consumer does not also values the consumption of products, but also has a constant valuation for capital. Indeed, if this would not be the case, then the consumer would simply purchase as much quantity of the item until his marginal utility of consumption is equal to zero (instead to equating marginal utility to the price). The last point, is that we are assuming that the individual income is high enough so there is no budget constraints that would cap the individual’s optimal consumption.
a utility function denoted by $u(v)$ and seeks to maximize her surplus by deciding her consumption level $v$ for a given price $p$. Thus the net surplus of consuming $v$ at price $p$ is

$$CS_{det} = u(v) - pv.$$  \hspace{1cm} (4.9)

Then, as is well known, consumers demand for a given price $p$ will correspond to the solution of maximizing the expression (4.9), which give rise to the demand function $d(p)$. Note that taking FOC on Expression (4.9) with respect to $v$ leads to $u'(v) = p$, thus the derivative of the utility function is indeed the inverse demand function. Calling $u'(v) = d^{-1}(v)$, note $u(v) = \int_0^v d^{-1}(w)dw$. Then, Consumer Surplus at a given price $p$ is given by $CS_{det} = u(v) - pv = u(d(p)) - pd(p) = \int_0^{d(p)} d^{-1}(w)dw - pd(p)$ which coincides with Equation (4.1).

When demand is stochastic, we consider a representative consumer endowed with a utility function that depends on the consumed $v$ and a random variable $\epsilon$, namely $u(v, \epsilon)$. Assume that $u(v, \epsilon)$ is strictly concave and differentiable almost everywhere with respect to $v$. Assume also that $u(0, \epsilon) = 0$ for all $\epsilon$, and $\frac{\partial}{\partial v} u(v, \epsilon) \bigg|_{v=0} > 0$ for all $\epsilon^5$. As in the deterministic case, the maximization of the consumers surplus, $u(v, \epsilon) - pv$, gives the demand for the given price $p$ and noise realization $\epsilon$.

Recall that in the stochastic setting, some consumers may not be served due to the possibility of a stock-out. As before, among the $d(p, \epsilon)$ customers that are demanding the item, only $\min\{d(p, \epsilon), q\}$ will be served, depending on the allocation rule $\mathcal{A}$. The Consumer Surplus can then be written as follows:

$$CS^\mathcal{A}(\epsilon) = \int_0^{d(p, \epsilon)} \left( \frac{\partial}{\partial x} u(x, \epsilon) - p \right) \mathcal{A}(x) dx$$

$$= \int_0^{d(p, \epsilon)} \frac{\partial}{\partial x} u(x, \epsilon) \mathcal{A}(x) dx - p \min\{d(p, \epsilon), q\}. \hspace{1cm} (4.10)$$

Here, $\frac{\partial u}{\partial x}$ represents the marginal utility of each infinitesimal consumer. The first term

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5The above set of assumptions is similar as the deterministic case, and very common in the literature. In particular, it ensures the existence and uniqueness of an optimal solution to the optimization problem faced by the representative consumer.
in Equation (4.10) represents the total utility of consumers weighted by the probability that each infinitesimal consumer receives the item, over all consumers who demand the product. The second term corresponds to the cost of consuming \( \min\{d(p, \epsilon), q\} \) items at price \( p \). Note that in the deterministic case with additive noise, we have \( q = d(p, 0) \) so that the first term of Equation (4.10) reduces to \( \int_0^q \frac{\partial}{\partial x} u(x, 0) dx = u(q, 0) \).

In what follows, we derive the CS expressions under the three allocation rules: Random (R), Highest willingness to pay (H), and Lowest willingness to pay (L). We use the general expression from Equation (4.10) and substitute the corresponding allocation rule in each case.

First, for the H rule, we have:

\[
CS^H(\epsilon) = \int_0^{d(p, \epsilon)} \frac{\partial}{\partial x} u(x, \epsilon) \mathbb{I}_{x \geq q} dx - p \min\{d(p, \epsilon), q\}
= \int_0^{\min\{d(p, \epsilon), q\}} \frac{\partial}{\partial x} u(x, \epsilon) dx - p \min\{d(p, \epsilon), q\}
= u\left( \min\{d(p, \epsilon), q\}, \epsilon \right) - p \min\{d(p, \epsilon), q\}.
\]

Second, for the L rule, we have:

\[
CS^L(\epsilon) = \int_0^{d(p, \epsilon)} \frac{\partial}{\partial x} u(x, \epsilon) \mathbb{I}_{x < d(p, \epsilon) - q} dx - p \min\{d(p, \epsilon), q\}
= \int_{\max\{d(p, \epsilon) - q, 0\}}^{d(p, \epsilon)} \frac{\partial}{\partial x} u(x, \epsilon) dx - p \min\{d(p, \epsilon), q\}
= u\left( d(p, \epsilon), \epsilon \right) - u\left( \max\{d(p, \epsilon) - q, 0\}, \epsilon \right) - p \min\{d(p, \epsilon), q\}.
\]

Third, for the random allocation rule, we have:

\[
CS^R(\epsilon) = \int_0^{d(p, \epsilon)} \frac{\partial}{\partial x} u(x, \epsilon) \min\left\{1, \frac{q}{d(p, \epsilon)}\right\} dx - p \min\{d(p, \epsilon), q\}
= u\left( d(p, \epsilon), \epsilon \right) \min\left\{1, \frac{q}{d(p, \epsilon)}\right\} - p \min\{d(p, \epsilon), q\}
= \left( u\left( d(p, \epsilon), \epsilon \right) - p d(p, \epsilon) \right) \min\left\{1, \frac{q}{d(p, \epsilon)}\right\}.
\]

Note that for the random allocation rule, one can compute the Consumer Surplus for any noise realization \( \epsilon \) as if demand were equal to sales, but then we need to
account for stock-outs by randomly serving a portion of the consumers. Consequently, the consumers that will be served derive a utility of $u(d(p, e), c)$ and pay a price of $pd(p, e)$. However, only the fraction $\min\left\{1, \frac{q}{d(p, e)}\right\}$ is served. Observe that no matter the allocation rule, the second term related to the cost paid by consumers is the same and equal to $p \min\{d(p, e), q\}$. As expected, the impact of the allocation rule is expressed in the first term that corresponds to the utility perceived by the consumers.

Another interesting fact is how demand functions can be derived from solving the optimization problem (4.19) and establish the connection between utility and demand functions. In particular, we next briefly depict the underlying utility functions that give rise to linear demand functions under the additive and multiplicative noise. Consider that the utility function is $u(v, e) = \frac{a + e}{b}v - \frac{1}{2b}v^2$. Then solving the optimization problem (4.19) leads to the demand function $d(p, e) = a - bp + e$. Note this corresponds to the stochastic demand with additive noise. So if the seller prices and quantities are $(p, q)$, and a random allocation rule, consumers' surplus will be

$$CSR(\epsilon) = \min\left\{1, \frac{q}{d(p, e)}\right\}\left(\frac{a + \epsilon}{b}d(p, e) - \frac{(d(p, e))^2}{2b}\right) - p \min\{d(p, e), q\}$$

$$= \min\{d(p, e), q\}\frac{1}{2b}d(p, e)$$

Consider now the alternative example where the utility function is $u(d, e) = \frac{a}{b}v - \frac{1}{2be}v^2$. Then solving the optimization problem (4.19) gives the demand function $d(p, e) = (a - bp)e$. Note this corresponds to the stochastic demand with multiplicative noise. So if the seller prices and quantities are $(p, q)$, and a random allocation rule, consumers' surplus will be

$$CSR(\epsilon) = \min\left\{1, \frac{q}{d(p, e)}\right\}\left(\frac{a}{b}d(p, e) - \frac{(d(p, e))^2}{2be}\right) - p \min\{d(p, e), q\}$$

$$= \min\{d(p, e), q\}\frac{1}{2be}d(p, e).$$
4.3 Multiple Items

In this section, we consider a setting with \( n \geq 2 \) items. The demand vector (composed of the \( n \) different demand functions), \( d(p, e) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), is expressed as a function of the price vector \( p \in \mathbb{R}^n \), and a random variable vector \( e \in \mathbb{R}^n \) with support \( \Omega \subset \mathbb{R}^n \).

Assume that \( d(p, e) \) is continuous in \( p \) and \( e \), and differentiable almost everywhere with respect to \( p \). As in Section 4.2, we will later consider the following two cases: (i) additive noise\(^6\): \( d(p, e) = d(p) + e \), and (ii) multiplicative noise: \( d(p, e) = D_x d(p) \),

where we use the notation \( D_x \) to refer to the diagonal matrix with the elements of the vector \( x \) in its diagonal. As stated by [15], the Slutsky condition on demand can only be met if the realizations of random variables \( e_i \) are identically for all items, i.e. \( D_x = \varepsilon_1 I \) where \( I \in \mathbb{R}^{nxn} \) is the identity matrix\(^7\). Assume \( \mathbb{E}[\varepsilon_i] = 0 \) and \( \mathbb{E}[\varepsilon_i] = 1 \) for all \( i \in \{1, \ldots, n\} \) in the additive and multiplicative cases respectively. We also assume that the demand of each item is decreasing in its own price, and non-decreasing in the other prices. In other words, \( \frac{\partial d_i}{\partial p_i} < 0 \) for all \( i \in \{1, \ldots, n\} \), and \( \frac{\partial d_i}{\partial p_j} \geq 0 \) for all \( i, j \in \{1, \ldots, n\}, i \neq j \). This is a common assumption that captures the fact that the items are substitutable goods (e.g., two competing brands of the same product category). In addition, we assume that the price change of a particular item has a stronger effect on its own demand when compared to the sum of the price changes of the other items, i.e., \( \sum_{j \neq i} \left| \frac{\partial d_i}{\partial p_j} \right| < \left| \frac{\partial d_i}{\partial p_i} \right| \) for all \( j \in \{1, \ldots, n\} \). The latter condition is called the \textit{strict diagonal dominance}, and is a common assumption in the literature (see, e.g., [1]). The above assumptions imply that the negative of the demand Jacobian (in terms of the derivatives with respect to prices) is a non-singular \( M \)-matrix, and hence the Jacobian of the inverse demand function is non-positive with a strictly negative diagonal. In addition, the demand Jacobian is not zero, so that the inverse demand function is well defined.

\(^6\)As in Section 4.2, when the function \( d() \) is written with a price and a noise argument will denote the stochastic demand function, whereas if it only has a price as argument then will correspond to the deterministic demand function.

\(^7\)Throughout the paper, when referring to the multiple items case under multiplicative noise, we will mean \( d(p, e) = D_x d(p) \). Yet if the reader would like to impose the Slutsky condition, then she can simply replace the \( n \) random variables by a single one, say \( \varepsilon_1 \), so that \( d(p, \varepsilon) = \varepsilon_1 d(p) \).
4.3.1 Deterministic Demand

In the deterministic setting, the demand vector is a function of the price vector only, i.e., \( d(p) : \mathbb{R}^n \to \mathbb{R}_+^n \). In this case, the market price vector \( p \) and quantities \( q \) are such that \( d(p) = q \). Note that for a given \( q \), the inverse demand function \( d^{-1}(q) \) represents the maximal price vector for which consumers will demand \( q \) items, i.e., the maximum willingness to pay of consumers. This interpretation is important as we will compute the Consumer Surplus by integrating consumers’ willingness to pay (i.e., the inverse demand function) over the quantity space. Alternatively, one can integrate over the price space, as in [24] where the author computes the Consumer Surplus for multiple items under deterministic demand. In this case, the Consumer Surplus is expressed as the path integral over the sum of the demand functions of each item. As a result, the value of the integral may be path dependent if the Slutsky conditions\(^8\) are not satisfied. In this paper, however, we will compute the Consumer Surplus as the path integral over the inverse demand function (as opposed to the demand function). We propose to integrate the inverse demand (over the quantity space) instead of the demand function (over the price space), as this allows us to extend the definition of the Consumer Surplus for cases where there is a mismatch between demand and supply. As we discuss in Section 4.3.2, when demand is stochastic, the produced units can sometimes be lower than demand. As in the single item case, we address this issue by introducing an \( n \)-dimensional allocation rule that allocates the available supply to consumers (see Section 4.3.2 for more details). This motivates us to compute the Consumer Surplus as the path integral over the inverse demand function, as follows:

\[
CS = \int_C (d^{-1}(r) - p) \cdot dr, \tag{4.11}
\]

where \( C \) is a path that goes from \( 0 \in \mathbb{R}^n \) to \( q = d(p) \) defined by the function \( r : [a, b] \to \bigotimes_{i=1}^n [0, q_i] \) that is continuous and differentiable almost everywhere. Note that the expression in Equation (4.11) is uniquely determined (i.e., path independent), if

\(^8\) Slutsky conditions are satisfied if the demand cross derivatives are equal, i.e., \( \frac{\partial d_i}{\partial p_j} = \frac{\partial d_j}{\partial p_i} \) for all \( i \neq j \).
the cross derivatives of the inverse demand function (or demand function) are equal, see [24]. Otherwise, the expression in Equation (4.11) would depend on the path of integration. Note that in the single item case, there is not such an issue, since the path moves along a unique direction over a segment.

So far, we (have logically) assumed that demand and supply are matching, i.e., \( d(p) = q \). The more general setting when supply is not necessarily matching demand \( (d(p) \neq q) \), can be addressed with the methodology introduced in Section 4.3.2.

### 4.3.2 Stochastic Demand

Consider the price and quantity vectors \( p \) and \( q \), where does not need to hold that \( q = \mathbb{E}[d(p, \epsilon)] \). As discussed, when demand is stochastic, there might be cases where production is lower than the units demanded for each item. Our goal is to define the Consumer Surplus for multiple items while accounting for the potential stock-outs. As in the single item case, we first present the expressions of the Consumer Surplus for the R, H and L allocations rules, and then generalize to a general allocation rule.

Using the Random allocation, for each item \( i \in \{1, \ldots, n\} \), consumers receive the item with probability \( \min\{1, \frac{q_i}{d_i(p, \epsilon)}\} \). Consequently, each component of the inverse demand function has to be weighted by this factor when integrating over the path. More precisely, the expression for the Consumer Surplus for any noise realization \( \epsilon \) can be written as:

\[
CS^R(\epsilon) = \int_{C^\epsilon} \sum_{i=1}^{n} \left( d_i^{-1}(r^\epsilon, \epsilon) - p_i \right) \min\left\{1, \frac{q_i}{d_i(p, \epsilon)}\right\} dr_i^\epsilon. \quad (4.12)
\]

Here, the path \( C^\epsilon \) goes from \( 0 \) to \( d(p, \epsilon) \), and \( r^\epsilon \) corresponds to the parametric function of this path for a given \( \epsilon \). Observe that different noise realizations can lead to different integration paths. For consistency, we would like to define the Consumer Surplus over one particular path that follows the same structure for the different noise realizations. More precisely, for two different noise realizations \( \epsilon^1 \) and \( \epsilon^2 \), we want to require that the Consumer Surplus is computed over the "same" path. However, using the exact same path is not possible as the two noise realizations induce two different end points.
Therefore, we next define the notion of equivalent paths to refer to those paths that follow the same pattern under different noise realizations. Before introducing this definition, we consider the two-dimensional case with the path \( r^\varepsilon : [a, b] \rightarrow \mathbb{R}^2 \) defined as:

\[
\begin{align*}
    r^\varepsilon(w) &= \begin{cases}
        \left( \frac{wd_1(p, \varepsilon)}{d_1(p, \varepsilon)} \right) & \text{if } w \in [0, 1] \\
        \left( \frac{w-1}{d_2(p, \varepsilon)} \right) & \text{if } w \in (1, 2].
    \end{cases}
\end{align*}
\tag{4.13}
\]

In this case, the path will move sequentially in each dimension: first from the origin \((0, 0)^T\) to \((d_1(p, \varepsilon), 0)^T\), and then to \((d_1(p, \varepsilon), d_2(p, \varepsilon))^T\). As a result, the path follows the same pattern for different noise realizations.

**Definition 1.** Consider a demand function \( d(p, \varepsilon) \), a market price vector \( p \), two noise realizations \( \varepsilon_1, \varepsilon_2 \in \Omega \), and their respective paths \( C_{\varepsilon_1}, C_{\varepsilon_2} \). Two parametric functions \( r_{\varepsilon_1} \) and \( r_{\varepsilon_2} \) with the same domain \([a, b]\) \( \subset \mathbb{R} \) for the respective paths are equivalent if:

\[
\frac{r_{\varepsilon_1}^i(w)}{d_i(p, \varepsilon_1)} = \frac{r_{\varepsilon_2}^i(w)}{d_i(p, \varepsilon_2)} \quad \forall i \in \{1, \ldots, n\}, \forall w \in [a, b].
\tag{4.14}
\]

The idea behind Equation (4.14) is that for each \( w \) in the domain of the parametric function, the proportion between the distance covered by the path in each dimension \( i \in \{1, \ldots, n\} \) over the total demand for the item is equal for two different noises. Therefore, two equivalent paths follow the same pattern for different noise realizations, and provide a consistent way to define the Consumer Surplus when demand is stochastic. In particular, it is natural to consider an equivalent set of paths \( r^\varepsilon \) in order to compute the expectation over \( \varepsilon \) of the Consumer Surplus in (4.12).

From now on, whenever the notation \( r^\varepsilon \) is used for the parametric function of a path, we will assume that all the paths are equivalent, unless stated otherwise.

---

\(^9\)For simplicity, we consider that both parametric functions have the same domain. Yet, the definition of equivalent paths can be extended to the more general setting without this assumption.
Note that unlike the deterministic case, Equation (4.12) is not path independent even if the cross derivatives of the demand function are equal. It actually turns out that most of the time, the expression is path dependent. We next derive the expression for the Consumer Surplus for the H and L rules for any given path and any noise realization.

For the H rule, each item \( i \in \{1, \ldots, n\} \) will be allocated to the \( q_i \) consumers with the highest valuation for item \( i \). Since the inverse demand functions are strictly decreasing in their respective quantities and non-increasing in the other quantities (i.e., \( \frac{\partial d_i}{\partial q_i} < 0 \) and \( \frac{\partial d_i}{\partial q_j} \leq 0 \) for all \( i \neq j \in \{1, \ldots, n\} \)), items will be allocated to those consumers whenever the integration path \( r_i^\epsilon \) is less than or equal to the units available, \( q_i \), i.e., \( r_i^\epsilon \leq q_i \). However, the exact set of these \( q_i \) consumers will vary depending on the integration path. The Consumer Surplus expression can be written as:

\[
CS^H(\epsilon) = \int_{C^\epsilon} \sum_{i=1}^{n} (d_i^{-1}(r, \epsilon) - p_i) 1_{\{r_i^\epsilon \leq q_i\}} dr_i^\epsilon.
\]

For the L rule, the items will be allocated to the consumers with the lowest valuation (among the ones that are willing to pay at least the market price). In this case, the Consumer Surplus expression can be expressed as follows:

\[
CS^L(\epsilon) = \int_{C^\epsilon} \sum_{i=1}^{n} (d_i^{-1}(r, \epsilon) - p_i) 1_{\{r_i^\epsilon \geq d_i(p, \epsilon) - q_i\}} dr_i^\epsilon.
\]

As in Section 4.2, one can define a general allocation rule and compute the Consumer Surplus under this general allocation. An \( n \)-dimensional allocation \( A \) for multiple items is defined as a function \( A : \mathcal{O}_{j=1}^{n}[0, d_j(p, \epsilon)] \to [0, 1]^n \) so that for every price vector \( p \), quantity vector \( q \), noise realization \( \epsilon \), and path \( C^\epsilon \) defined by the function \( r^\epsilon \), we have:

\[
\int_{C^\epsilon} D_{A(r^\epsilon)} dr^\epsilon = \min \{q, d(p, \epsilon)\}.
\]  

Equation (4.15) implies that the total units allocated for each item is equal to the minimum between demand and supply for this item (the minimum is applied component-
wise). As a result, the Consumer Surplus under any allocation rule $\mathcal{A}$ is given by:

$$\text{CS}^\mathcal{A}(\epsilon) = \int_{C^\epsilon} \sum_{i=1}^{n} (d_i^{-1}(r^\epsilon, \epsilon) - p_i) A_i(r^\epsilon) dr_i^\epsilon.$$  \hspace{1cm} (4.16)

Note that we define a single allocation function from $\mathbb{R}^n$ to $\mathbb{R}^n$, instead of considering a separate one-dimensional function for each item. This allows us to capture potential dependencies among the allocations of the different items. In addition, one can have different allocation rules for different items. Finally, the expected Consumer Surplus can be obtained by taking the expectation over the random vector $\epsilon$:

$$\mathbb{E}[\text{CS}^\mathcal{A}(\epsilon)] = \int_{\mathbb{R}^n} \text{CS}^\mathcal{A}(\epsilon) dF(\epsilon).$$  \hspace{1cm} (4.17)

As in the single item case, the consumer surplus under the different allocations rules follow the following ordering.

**Observation 2.** For any allocation rule $\mathcal{A}$, any price $P$, production quantity $Q$ a given noise realization $\epsilon$, and a path $C^\epsilon$ defined by a parametric function $r^\epsilon$, we have:

$$\text{CS}^L(\epsilon) \leq \text{CS}^\mathcal{A}(\epsilon) \leq \text{CS}^H(\epsilon).$$  \hspace{1cm} (4.18)

The proof of Observation 2 can be found in the Appendix. Consequently, we have:

$$\mathbb{E}[\text{CS}^L(\epsilon)] \leq \mathbb{E}[\text{CS}^\mathcal{A}(\epsilon)] \leq \mathbb{E}[\text{CS}^H(\epsilon)].$$

As we can see, the consumer surplus for multiple items when demand is stochastic depends on the noise realization, the allocation rule, and also on the path of integration. When supply exceeds demand, there is no stock-out and we are back to the deterministic case. Consequently, the consumer surplus is path independent (assuming that the cross derivatives are symmetric). When demand exceeds supply, some consumers might not be served and in this case, the path of integration affects the value of the consumer surplus. As we explained before, the allocation rule dictates the order in which consumers are served for a particular product. For example, the $H$ rule will serve first the consumers with the highest valuation for the item. When we have multiple items, the seller also needs to decide the order in which he/she...
allocates the different items. The path of integration captures the cross allocation strategy of the seller. Since we have shortages for certain products, some consumers might not be served and this incurs a loss in the consumer surplus. We observe that this loss depends on the order in which the seller allocates the different items. When looking at the inverse demand curves for multiple items, each curve corresponds to the willingness to pay of a different product. Under our assumptions, the willingness to pay for product $i$ is non-increasing with respect to both $q_i$ and each $q_j$, $j \neq i$. As a result, the order in which items are allocated affects the consumer surplus.

Illustrative Example

We next provide a graphical illustration of the Consumer Surplus under the three different allocation rules for different paths of integration. Consider a linear demand function with an additive noise of the form $d(p, \epsilon) = d - Bp + \epsilon$ satisfying the conditions mentioned at the beginning of Section 4.3, and denote $p^0$ and $q^0$ the given price and quantities. Consider the three following paths:

- **Path 1:** This path follows a straight line from 0 to $d(p^0, \epsilon)$ and is called the “canonical path”. Therefore, it can be parametrized as:

  $$r(w) = wd(p^0, \epsilon) \text{ for } w \in [0, 1].$$

- **Path 2:** This path goes first from 0 to $\begin{pmatrix} d_1(p^0, \epsilon) \\ 0 \end{pmatrix}$, and then from $\begin{pmatrix} d_1(p^0, \epsilon) \\ 0 \end{pmatrix}$ to $d(p^0, \epsilon)$. Therefore, it can be parametrized as:

  $$r(w) = \begin{cases} \begin{pmatrix} wd_1(p^0, \epsilon) \\ 0 \end{pmatrix} & \text{if } w \in [0, 1], \\ \begin{pmatrix} d_1(p^0, \epsilon) \\ (w-1)d_2(p^0, \epsilon) \end{pmatrix} & \text{if } w \in (1, 2]. \end{cases}$$

\[10\] It suffices that $d > 0$ and $B$ is a symmetric, strictly diagonally dominant, and $Z$-matrix.
Table 4.1: For each allocation rule and integration path, the three numbers in each box are: (i) the Consumer Surplus, (ii) the Consumer Surplus first term, i.e., the one that comes from \( \int_a^b (d_1^{-1} (r^e(w), \epsilon) - p_1) A_1 (r^e(w)) \frac{dr_1^e(w)}{dw} \, dw \), and (iii) the Consumer Surplus second term, where \( A_i(r^e(w)) \) is the allocation rule. Parameters: \( \overline{d}_1 = 1 \), \( \overline{d}_2 = 7 \), \( B_{11} = B_{22} = 1 \), \( B_{12} = B_{21} = -0.25 \).

<table>
<thead>
<tr>
<th>Rule</th>
<th>Path 1</th>
<th>Path 2</th>
<th>Path 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random</td>
<td>56.400</td>
<td>55.600</td>
<td>57.200</td>
</tr>
<tr>
<td></td>
<td>31.200</td>
<td>36.400</td>
<td>26.000</td>
</tr>
<tr>
<td></td>
<td>25.200</td>
<td>19.200</td>
<td>31.200</td>
</tr>
<tr>
<td>High</td>
<td>60.560</td>
<td>59.067</td>
<td>60.667</td>
</tr>
<tr>
<td></td>
<td>35.360</td>
<td>39.867</td>
<td>29.467</td>
</tr>
<tr>
<td></td>
<td>25.200</td>
<td>19.200</td>
<td>31.200</td>
</tr>
<tr>
<td>Low</td>
<td>52.240</td>
<td>52.133</td>
<td>53.733</td>
</tr>
<tr>
<td></td>
<td>27.040</td>
<td>32.933</td>
<td>22.533</td>
</tr>
<tr>
<td></td>
<td>25.200</td>
<td>19.200</td>
<td>31.200</td>
</tr>
</tbody>
</table>

- **Path 3:** This path goes first from 0 to \( \begin{pmatrix} 0 \\ d_2(p^0, \epsilon) \end{pmatrix} \), and then from \( \begin{pmatrix} 0 \\ d_2(p^0, \epsilon) \end{pmatrix} \) to \( d(p^0, \epsilon) \). Thus, it can be parametrized as:

\[
\begin{align*}
r(w) &= \begin{cases}
0 & \text{if } w \in [0,1], \\
wd_2(p^0, \epsilon) & \text{if } w \in (1,2].
\end{cases}
\end{align*}
\]

As we previously mentioned, different paths of integration represent the cross allocation strategy of the seller. This implies that in path 2, the seller first allocates all the \( q_1^0 \) items, and then subsequently allocates all the \( q_2^0 \) items (path 3 follows the opposite order). In path 1, the seller allocates all the items simultaneously.

Figure 4-2 depicts the Consumer Surplus in terms of the integrated areas along different allocation rules and integration paths for the case of a linear demand function (with an additive noise) with two items. In each case, the Consumer Surplus corresponds to the area between the inverse demand function and the price along the considered path. We compute the Consumer Surpluses values for a particular instance in Table 4.1.
Figure 4-2: Consumer Surplus under different allocation rules and integration paths. 
**Left:** Inverse demand function in the first component. **Right:** Inverse demand function in the second component. **Top:** R rule. **Middle:** H rule. **Bottom:** L rule.
In the next section, we present an alternative approach to derive the Consumer Surplus based on the consumer utility.

4.3.3 Utility Perspective

Very often, the Consumer Surplus is derived from the solution of a utility maximization problem faced by an aggregate representative consumer. This approach provides a more intuitive interpretation on the expressions from Section 4.3.2. In addition, we study the impact of demand uncertainty on the utility maximization problem. Consider a representative consumer endowed with a utility function \( u(v, \epsilon) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) that captures the utility derived from the goods in a monetary terms. We assume that \( u(v, \epsilon) \) is strictly concave and differentiable almost everywhere in \( v \). We also assume that \( u(0, \epsilon) = 0 \) and \( \frac{\partial}{\partial v_i} u(v, \epsilon) \bigg|_{v=0} > 0 \), for all \( \epsilon \in \Omega \). Note that the representative consumer knows the noise realization \( \epsilon \), whereas the firm knows only the utility function \( u \) and the probability distribution of \( \epsilon \). In addition, consumer’s rationality is bounded in the sense that one cannot foresee the optimization problem solved by the firm, i.e., anticipate the market prices and quantities. The optimization problem faced by the representative consumer for a given realization of \( \epsilon \) is given by:

\[
\max_{v \geq 0} \; u(v, \epsilon) - p^T v. \tag{4.19}
\]

Note that the assumptions we imposed on the utility function ensure that there exists a unique optimal solution to problem (4.19). Note also that this optimal solution depends on the price \( p \) and on the noise realization \( \epsilon \). Therefore, the optimal demanded quantities will be an (explicit or implicit) function \( d(p, \epsilon) \) obtained by solving problem (4.19) for each price vector. By using the first order condition, we obtain:

\[
p = \nabla_v u(v, \epsilon) \equiv d^{-1}(v, \epsilon). \tag{4.20}
\]

As expected, the inverse demand function \( d^{-1}(v, \epsilon) \) can be defined as the gradient of the utility function in the first \( n \) variables (i.e., the vector \( v \)). Consider now a price
and a quantity vector \((p^0, q^0)\). Note that the consumer surplus cannot be computed as
\[ u(d(p^0, \epsilon)) - (p^0)^T d(p^0, \epsilon), \]
where \(d(p^0, \epsilon)\) is the demand function obtained from solving problem (4.19)). In particular, one has to account for the scenarios where the produced quantities are lower relative to demand. Since the aggregate consumer is a representation of an infinite number of infinitesimal consumers that go along some path from \(0\) to the units demanded \(d(q^0, \epsilon)\), the gradient of the utility function represents the valuations of each infinitesimal consumer at each point. In the deterministic case, no stock-outs occur (since \(q^0 = d(p^0)\)) and the Consumer Surplus is precisely
\[ u(d(p^0)) - (p^0)^T d(p^0) = u(q^0) - (p^0)^T q^0. \]
Alternatively, this can be expressed as a path independent\(^{11}\) integral:
\[
CS = \int_C \left( \nabla u(r) - p^0 \right) \cdot dr = \int_a^b \frac{du(r(w))}{dw} dw - p^0 \cdot q^0 = u(q^0) - u(0) - (p^0)^T q^0
\]
\[= u(q^0) - (p^0)^T q^0. \tag{4.21} \]

We next consider the case when demand is stochastic using an allocation rule \(A\) as defined in equation (4.15). In this case, the Consumer Surplus expression can be written as:
\[
CS^A(\epsilon) = \int_C \left( \nabla u(r, \epsilon) - p^0 \right)^T D_{A(r^*)} dr^\epsilon
\]
\[= \int_C \left( \nabla u(r^\epsilon, \epsilon) \right)^T D_{A(r^*)} dr^\epsilon - (p^0)^T \min\{q^0, d(p^0, \epsilon)\}, \tag{4.22} \]
where \(D_{A(r^*)}\) is an \(n \times n\) matrix with the allocation functions in its diagonal. Note that the first term in equation (4.22) represents the utility derived from the consumers, and the second term corresponds to the price paid for the total number of allocated items. Note that equation (4.22) is the same as equation (4.16), so that both approaches are consistent.

\(^{11}\)The line integral of a function that can be expressed as a gradient is path-independent (Gradient Theorem).
More on the Path Interpretation

Let us give a glimpse to the single item case and give an alternative interpretation of the utility maximization problem of the single consumer when the resulting outcome might have scarcity, then I will go to the multiple items case to proceed with a similar alternative interpretation of the utility maximization problem which will bring a new light on the path interpretation.

In the single item case, we have that the single representative consumer solves

$$\max_{v \geq 0} u(v, \epsilon) - pv$$

for a given $p \in \mathbb{R}$ and $\epsilon \in \Omega$, leading to a demand $d = d(p, \epsilon)$. Consider for a second that there is no scarcity, then

$$u(d, \epsilon) = \int_0^\infty du(w, \epsilon) \frac{d}{dw} \mathbb{1}_{\{du(w, \epsilon) \leq p\}} dw = \int_0^d du(w, \epsilon) dw = \lim_{m \to +\infty} \sum_{i=1}^m \frac{du\left(\frac{i}{m}d, \epsilon\right)}{dw} \frac{1}{m} d$$

In the argument of the limit of the last expression, it can be interpreted as if the utility attained corresponds to the sum of the utilities of $m$ different fractional customers (fractional since each customer consumption is $d/m$) whose utility is the derivative of the utility function evaluated at the item consumed. Thus, each fractional consumer purchases (or has the desire of purchasing) as long as her utility is above the price.

Now, if there is scarcity, namely $q < d(p, \epsilon)$, then not all of these $m$ customers will get the item with probability one, therefore under an allocation rule $A$ (see section 4.2.2 for the definition of allocation rule), the attained utility of the $m$ consumers will be

$$\sum_{i=1}^m \frac{du\left(\frac{i}{m}d, \epsilon\right)}{dw} \frac{1}{m} d A\left(\frac{i}{m}d\right) \approx \int_0^d du(w, \epsilon) \frac{d}{dw} A(w) dw$$

Note that the allocation rule is not present in the utility maximization problem,
thus it is considered to be exogenous to consumers. Also, we divided the demand \((d)\) in \(m\) customers evenly spaced in the segment \([0, d]\), however this can also be done in a non-homogeneous way.

In summary, the utility maximization problem can be seen as the decision of purchase multiple customers if whether to purchase or not at a given price, where each of this customer might or not get the item depending on the availability of the item upon the demand and the allocation rule.

Let us look now to the multiple item case. The single representative consumer problem is

\[
\max_{\nu} u(\nu, \epsilon) - p^T \nu
\]

for a given \(p \in \mathbb{R}^n\) and \(\epsilon \in \Omega\), leading to a demand \(d = d(p, \epsilon)\). Consider a path \(C\) from \(0\) to \(d\) described by the parametric function \(r: [a, b] \to \mathbb{R}^n\), and let us assume for a second that there is no scarcity, then

\[
u(d, \epsilon) = \int_{C} \nabla_{\nu} u(r, \epsilon) \cdot dr
= \int_{a}^{b} \nabla_{\nu} u(r(w), \epsilon) \cdot \frac{dr(w)}{dw} dw
= \lim_{m \to +\infty} \sum_{i=1}^{m} \nabla_{\nu} u(r(a + \frac{i(b-a)}{m}), \epsilon) \cdot \frac{dr(w)}{dw} \bigg|_{w=a+\frac{i(b-a)}{m}} \frac{(b-a)}{m}
\]

(note that we know the latter integral is path independent). Again we see that we are summing up the utilities of \(m\) consumers along the path \(C\) weighted by the respective (fractional) amount they consume \((\frac{dr(w)}{dw} \bigg|_{w=a+\frac{i(b-a)}{m}} \frac{(b-a)}{m})\). The path can be seen as the combination over which we represent the single representative consumer as different combinations of multiple fractional (infiniteesimal) consumers with a utility for the items who are given the items according to the path considered. So in a 2 items example, consider the “path 1” that goes in straight line from \(0\) to \(d\), and “path 2” (see one page later approx.) the one in which we go in straight line from \(0\) to \((d_1, 0)^T\) and then goes to \(d\). Consider \(m\) consumers evenly spaced along the respective paths. In “path 1”, we will have that each consumer \(i\) valuation for each item \(j\) is \(\nabla_{\nu_j} u(\frac{i}{m} d, \epsilon),\)
while the number of items purchased is \( \frac{dr_j(w)}{dw} \bigg|_{w=\frac{1}{m}} = \frac{d_j}{m} \). In the case of "path 2", there is a first segment that consumes item 1 and then another that consumer item 2, despite this might suggest that consumers' consumption of items are independent, this representation of consumption under this path (as in any other path) emulates the single representative consumer. So for the \( i \)-th consumer \( i \in \{1, \ldots, m\} \), we have that her valuation for item 1 is \( \nabla_{v_1} u(\begin{pmatrix} d_1 \\ 0 \end{pmatrix}^T, \epsilon) \) while the number of items purchased is \( \frac{dr_1(w)}{dw} \bigg|_{w=\frac{2i}{m}} = \frac{d_1}{m/2} \) and their valuation for item 2 is \( \nabla_{v_2} u(\begin{pmatrix} 0 \\ d_2 \end{pmatrix}^T, \epsilon) \) but they get 0. For the \( i \)-th consumer \( i \in \{\frac{m}{2}, \ldots, m\} \), we have that her valuation for item 1 is \( \nabla_{v_1} u(\begin{pmatrix} d_1 \\ 0 \end{pmatrix}^T, \epsilon) \) but get 0, and for item 2 is \( \nabla_{v_2} u(\begin{pmatrix} 0 \\ d_2 \end{pmatrix}^T, \epsilon) \) while the number of items purchased is \( \frac{dr_2(w)}{dw} \bigg|_{w=\frac{2i}{m}} = \frac{d_2}{m/2} \) and 0 for item 2.

If in addition, we consider that there is scarcity, \( q_i < d_i(p, \epsilon) \) for some \( i \), and an allocation rule \( A \), the utility of consumers will be given by

\[
 u(d, \epsilon) = \lim_{n \to \infty} \sum_{i=1}^{m} \nabla_{v} u \left( r(a + \begin{pmatrix} b-a \end{pmatrix}^T/m, \epsilon) \frac{dr(w)}{dw} \bigg|_{w=a+\frac{b-a}{m}m} \right) \frac{(b-a)}{m}
\]

The latter expression is path dependent, thus, this representation from a single to multiple consumers will depend on the path considered.

In summary, the single representative consumer can be also seen under a set of multiple infinitesimal consumers that receive the item along a considered path. When there is no scarcity, the welfare will independent of the path (as long as the Jacobian of the demand function satisfies the conditions). In the presence of scarcity, the resulting utility will be dependent on the path considered.

Having defined the Consumer Surplus for a general allocation rule under stochastic demand, we next compare the expected consumer surplus to a deterministic average demand case. We show that the impact of demand uncertainty on consumers depends on several factors such as the convexity properties of the demand function, the noise structure and the allocation rule.
4.4 Impact of Demand Uncertainty on Consumers

In this section, we compare the Consumer Surplus in the deterministic setting relative to the case when demand is stochastic. We consider both additive and multiplicative noises and the three different allocation rules discussed in Section 4.2.2. We show that the impact of demand uncertainty may depend on the convexity properties of the nominal demand function $d(p)$, as well as the noise nature and the allocation rule. In this section, the analysis is done considering the same price in the deterministic and stochastic settings.

4.4.1 Single Item

When demand is deterministic, the supplier matches supply and demand at price $p^0$, i.e., $q^0 = d(p^0)$. When demand is stochastic, we assume that the price is still equal to the same value of $p^0$ but the quantity $q^{sto}$ can be different than $d(p^0)$. For instance, the quantity could be the optimal newsvendor ordering (see, e.g., [21] and [20]), yet in the following we are not necessarily assuming that it is this amount.

We first consider the case of a multiplicative noise: $d(p, \epsilon) = \epsilon d(p)$. The results about the impact of demand uncertainty on consumers are summarized in the following Proposition.

**Proposition 1.** Consider a stochastic demand function with a multiplicative noise and any given capacity allocation rule $A$. Then:

$$\mathbb{E}[CS^A(\epsilon)] \leq CS_{det}. \quad (4.23)$$

The proof can be found in the Appendix.

Note that the result holds for any value of $q^{sto}$ and any noise distribution. Observe also that the result applies to any allocation rule. Therefore, no matter how sophisticated the supplier is in terms of allocating the available quantities, the consumers are always hurt when demand is uncertain relative to the deterministic setting, (for a multiplicative noise). In this case, positive shock demand scenarios (i.e., $\epsilon > 1$) will
induce more consumers, but their valuation will not increased as much, specially for those customers who value the item the most. In addition, the maximal valuation, \( d^{-1}(0) \), remains the same regardless of the noise realization. This can be seen when plotting the inverse demand curve for different noise realizations, where the inverse demand curve will “pivot” on the point \((0, d^{-1}(0))\). Note that this does not happen in the additive noise case.

We next consider the case of an additive noise, i.e., \( d(p, \epsilon) = d(p) + \epsilon \). In this case, the impact of demand uncertainty on consumers depends on three different factors: (i) the convexity properties of the nominal demand function \( d(p) \); (ii) the allocation rule considered and (iii) the relation between \( d(p^0) \) and \( q^{sto} \). The results are summarized in the following proposition.

**Proposition 2.** Consider a stochastic demand function with an additive noise and the three allocation rules, H, L and R. Then, we have:

- **H rule:** If \( q^{sto} \geq d(p^0) \) and \( d^{-1}(\cdot) \) is convex,

  \[
  \mathbb{E}[CS^H(\epsilon)] \geq CS_{det}. \tag{4.24}
  \]

- **L rule:** If \( q^{sto} < d(p^0) \),

  \[
  \mathbb{E}[CS^L(\epsilon)] \leq CS_{det}. \tag{4.25}
  \]

- **R rule:** If \( q^{sto} \leq d(p^0) \) and \( d^{-1}(\cdot) \) is concave,

  \[
  \mathbb{E}[CS^R(\epsilon)] \leq CS_{det}. \tag{4.26}
  \]

The proof can be found in the Appendix. Note that the conditions in each case are only sufficient. In other words, if one of the conditions is not satisfied, we can find examples in which the inequality can be in either direction. It follows that the impact of demand uncertainty on consumers (under an additive noise) is driven by (i) the allocation rule, and (ii) the demand convexity/concavity. An interesting special
case to consider is when demand is linear and \( q^{sto} = d(p^0) \). In this case, the impact of demand uncertainty on consumers depends crucially on the ability of the supplier to identify and discriminate consumers. In particular, under the H rule (best scenario for the consumers), consumers are better-off when demand is uncertain, whereas this conclusion is reversed under the R or the L rules. Regarding the convexity/concavity of demand, a convex demand (and thus a convex inverse demand) will favor the consumers. This follows from the fact that positive demand shocks will result in new customers with a much higher valuation relative to the deterministic setting (see the left panel of Figure 4-3). However, for a concave demand (and hence concave inverse demand), positive demand shocks will introduce new consumers with valuations only slightly higher relative to the deterministic setting (see the left panel of Figure 4-3).

![Figure 4-3: Left: Convex inverse demand curve. Right: Concave inverse demand curve. The shaded region represents the additional Consumer Surplus from the new customers due to the positive demand shock.](image)

In addition, if \( q^{sto} \) is thought in terms of the optimal newsvendor quantity, the condition \( q^{sto} \leq d(p^0) \) translates to \( F^{-1}_\epsilon(F_{\epsilon}(\frac{p^0-c}{p^0})) \leq 0 \) for the case of an additive noise, where \( F_\epsilon \) is the CDF of the noise \( \epsilon \) and \( c \) is the (constant) marginal production cost. In other words, this condition is satisfied for products with low profit margins. More precisely, if the noise \( \epsilon \) is symmetric, the condition translates to \( c \leq p^0 \leq 2c \).

One can also consider an alternative setting where the supplier produces \( q^0 \) units in both the deterministic and stochastic cases so that \( q^0 \) does not necessarily equates \( d(p^0) \). In this case, we can extend most of the results presented in this section.
(the results are not presented due to space limitations). Nevertheless, it seems very reasonable to assume that when demand is deterministic \( q^0 = d(p^0) \) as the supplier can tailor production to exactly match the demand.

### 4.4.2 Multiple Items

In this section, we extend the analysis and results for multiple items. In particular, we are interested in using the expressions of the expected Consumer Surplus from Section 4.3 in order to study the impact of demand uncertainty on consumers. If the different items are independent (i.e., no cross-item effects), one can extend the results from the single item case by simply adding the expected Consumer Surplus for each item separately. We are interested in the case where the demand for the \( n \) items are dependent on the others prices, so that the price of item \( i \) affects the demand of item \( j \neq i \).

We consider the setting with \( n \) items with the price vector \( p^0 \). When demand is deterministic, the level of production quantities is set exactly to match the nominal demand, i.e., \( q^0 = d(p^0) \). When demand is stochastic, the supplier produces quantities \( q^{sto} \) which similarly as in the single item case, these not necessarily equate to \( d(p^0) \). As before, we consider both cases of multiplicative and additive noises.

**Proposition 3.** Consider a stochastic demand function with a multiplicative noise and any given allocation rule \( A \), and integration path \( C \). Then:

\[
\mathbb{E}[CS^A(\epsilon)] \leq CS_{det}. \tag{4.27}
\]

**Proof.** Proof. See Appendix C. \( \square \)

**Proposition 4.** Consider \( n \) items and a stochastic demand function with an additive noise and the \( R \) allocation rule. Then, we have:

- **R rule:** If \( q^{sto} \leq d(p^0) \) and \( d_i^{-1}(\cdot) \) is concave for all \( i \in \{1, \ldots, n\} \),

\[
\mathbb{E}[CS^R(\epsilon)] \leq CS_{det} \tag{4.28}
\]
for any given path $C_e$.

Proof. Proof. See Appendix C. ■

Note that for the H rule, the Consumer Surplus inequality from single items does not hold anymore necessarily. Thus, we have found counterexamples in which $\mathbb{E}[CS^H(e)] < CS_{det}$ even under linear demand, symmetric items, canonical path (the so called “Path 1”), independent and symmetric noises. Take $n = 3$ symmetric case where $\bar{d} = (7 7 7)^T$, $B_{ii} = 1$, $B_{ij} = -0.4$, and $p^0 = (1 1 1)^T$ with noise plus minus $\pm 1$ w.p. 0.5 each independently. Then $\mathbb{E}[CS^H(e)] = 346.539 < 346.8 = CS_{det}$. Yet, $\mathbb{E}[CS^H(e)] \geq CS_{det}$ happens to hold at least for the symmetric case when $n = 2$ under this settings.

In regard of the L rule, due to Observation 2, we have that $\mathbb{E}[CS^L(e)] \leq CS_{det}$ when the inverse nominal demands are concave and $q^{sto} \leq d(p^0)$. After extensive numerical computations, we believe the inequality remains to be true when dropping the concavity assumption, yet we have not been able to formally prove it.

4.5 Revenue Maximization Problem

In the previous sections, in particular Section 4.4, prices are considered exogenous, that is, the price vector used for the computation of the consumer surplus was the same for deterministic as for stochastic demand cases. Yet, in reality market prices come from a revenue maximization problem that firms are solving. Consequently, the equilibrium prices attained in both settings of study (deterministic and stochastic) might possibly differ, and likewise the Consumer Surplus. In order to capture the endogenous nature of prices, this section incorporate the firms pricing and production strategies in the firm' revenue maximization problem. Following the same order of analysis of the previous sections, the analysis of single items will be perform first under additive and multiplicative demand noises; following with the multiple items case.
4.5.1 Single Item

For the deterministic case, the firm faces a single value demand function \( d(p) \). Assume that the marginal production cost is \( c > 0 \). Note that in this case firms will produce exactly the demand obtained at the chosen price, i.e., \( q = d(p) \). The firm’s profit maximization problem is

\[
\max_p (p - c)d(p).
\]

(4.29)

Call \((p^d, q^d)\) the tuple such that \( p^d \) is the maximizer of Problem (4.29), and \( q^d = d(p^d) \). Then, the Consumer Surplus can be simply computed by using the expression (4.1).

For the stochastic demand case, consider a firm that faces a demand function \( d(p, \epsilon) \) with the same cost structure as in the deterministic case. The optimization problem of the firm can be expressed as

\[
\max_{p, q} p \mathbb{E} \left[ \min \{d(p, \epsilon), q\} \right] - cq,
\]

(4.30)

which indeed is the well known price-setting newsvendor problem. Denote \((p^s, q^s)\) the optimal solution of Problem (4.30). Before stating the relation between the Consumer Surpluses, let us mention a known result upon the optimal prices of Problems (4.29) and (4.30) for additive and multiplicative demand noises (see [14] and [17]).

**Lemma 1.** If the noise is additive, \( p^d \geq p^s \). Yet, if demand noise is multiplicative, \( p^d \leq p^s \).

The next proposition, states the Consumer Surplus relation between the deterministic with respect to the stochastic setting in a multiplicative noise setting when the optimal prices and quantities correspond to the optimal ones decided by the firm in each of the cases.

**Proposition 5.** Under a stochastic demand function with a multiplicative noise
and any given capacity allocation rule $A$, it holds that:

$$E[CS^A(\epsilon)] \leq CS_{det}. \quad (4.31)$$

The proof can be found in the Appendix C.

Similar as in the result shown in Proposition 1, consumers are always better off in the deterministic case not just for same prices and quantities set to the nominal demand, but also for the optimal prices and quantities chosen by the firm in the respective deterministic and stochastic situations. The next proposition states the results for the additive demand noise case:

**Proposition 6.** Consider a stochastic demand function with an additive noise and the $H$ allocation rule such that $q^s \geq d(p^s)$ and $d^{-1}(\cdot)$ is convex, then it holds that:

$$E[CS^H(\epsilon)] \geq CS_{det}. \quad (4.32)$$

The proof can be found in the Appendix C. For the cases when the allocation is R or L the result is not as clear as in the analysis of Section (4.4). Indeed, for the R rule, consider the instance with linear demand $d(p, \epsilon) = 5 - p + \epsilon$, $c = 2$, and $\epsilon \sim U[-0.5, 0.5]$, then turns out that $E[CS^R(\epsilon)] = 1.1368 > 1.1250 = CS_{det}$ in the optimal prices and quantities for each case (where, being consistent with the assumptions made in Proposition 2, it holds that $q^s < d(p^s, 0)$). For the L rule, we have counterexamples where consumers are better off in the stochastic case, yet these are such that the optimal firm revenue is not unimodal with respect to the price (quantities set to the optimal newsvendor). Yet, we believe that if the revenues of the firm are unimodal with respect to the price, then it must be that consumers are always better off in the deterministic setting.
4.6 Conclusions

Stochastic demand functions have an impact not only in firms’ profits, as has been vastly studied in most OM models, but also on the resulting welfare of consumers. As observed, demand uncertainty might lead to stock-out situations where not all consumers will enjoy the consumption of it. It is in this context that allocation rules are an appropriate tool to describe the rationing of the available supply on the demand, and therefore compute the corresponding Consumer Surplus. Also, we observed the connection of the Consumer Surplus with utility functions used in OM that directly combined utility and expenditures.

Then, we studied the impact of uncertainty on consumers under two setting: (i) same prices for deterministic and stochastic cases yet not necessarily optimal, and (ii) optimal prices and quantities for deterministic and stochastic settings. It was shown that the under a multiplicative noise, consumers are always better on average in the deterministic setting, whereas the answer in the additive noise case will depend on the allocation rule and the convexity of demand. In particular, stochasticity has a positive impact on consumers the more convex is the demand, and when the allocation favor those consumers with highest valuations. The aforementioned result holds for single as multiple items cases. Yet in the additive scenario, precise sufficient conditions are given for the three allocation rules worked with, yet for multiple items sufficient conditions are only given for the random allocation rule. Indeed, for the highest allocation rule we found a counterexample with respect to the inequality that holds for the single item case. In regard of the lowest willingness to pay (also under additive noise and multiple items), we believe that consumers are always better on average in the deterministic case, yet we were not able to prove it. For the second analysis, (ii), we show that given the firm optimal decisions in prices and quantities, consumers are still always better (in expectation) of in the deterministic case for multiplicative noise. Ideally, we want to look at the results of (ii) also for additive demand case and for multiple items.
Bibliography


Chapter 5

Conclusions

This thesis studies different and important topics in Revenue Management. In particular, we study a two-part tariff pricing problem, government rebate policies in green technology markets, and Consumer Surplus under uncertainty.

Often in revenue management problems, data on important variables might not be observable. Thus, addressing the issue of missing data is a key and practical problem to consider. Furthermore, pricing problems can be very large in scale and therefore, hard to solve computationally in practice. Hence, the importance of developing solution methods that provide near optimal solutions, while outperforming current practices is very important. Such methods have the potential to convince practitioners to implement these solutions.

In terms of green technology rebate policies, we conclude that the social welfare loss is not that high when the government uses as its main goal a target adoption of sales. On the contrary, a utilitarian approach might lead to much higher government expenditures. Furthermore, this thesis concludes that the presence of competition reduces firms’ profits at the expense of benefiting the government and the consumers.

Finally, Consumer Surplus is very important to model and study when the demand function is stochastic. As demand for multiple items might not match exactly the realized demand, allocations rules are important to use by the modeler in order to determine how items will be allocated to those consumers willing to make a purchase. We show that the average impact on consumers’ surplus depends on the nature of
the noise, the demand shape, and the allocation rule. A multiplicative noise tends to hurt consumers in the stochastic versus the deterministic setting. Yet this might not necessarily be the case when the noise is additive. In the latter case, consumers benefit by uncertainty more as demand becomes “more convex”, and the allocation rule gives preference to consumers with higher valuations.
Appendix A

Appendix of Chapter 2
A.1 Algorithm A.1.1

Algorithm A.1.1: Algorithm for fitting probability distribution of customers’ download and reservation prices

**Input**: Current plans parameters \{\( (f_{s_0}^0, v_{s_0}^0, B_s) \}_{s=1}^S \), downloads of observed customers \( d_i \) for all \( i \in \{1, \ldots, n\} \), the number of unobserved customers \( m \), the distribution of reservation prices \( F_r \) and a function \( g(d, \theta) : \mathbb{R}_+ \times \mathbb{R}^u \to \mathbb{R}_+ \), a value for \( p \in (0, 1] \).

**Output**: Parameters of customers’ reservation price distribution \( \hat{\theta} \), and unobserved customers’ download distribution \( \hat{F}_d^{\text{unobs}} \).

1. Fit a mixture of Gaussians over the observed customers’ download \( \hat{F}_d^{\text{obs}} \), and set \( \hat{F}_d^{\text{unobs}} \triangleq \hat{F}_d^{\text{obs}} \).
2. Sample download values for unobserved customers \( d_{n+1}, \ldots, d_{n+m} \) according to \( \hat{F}_d^{\text{unobs}} \).
3. Compute the minimum payment \( l_i \) for all \( i \in \{1, \ldots, n+m\} \) according to Equation (2.1);
4. Solve \( \hat{\theta} = \arg \max_{\theta} \sum_{i=1}^n \ln \left(1 - F_r^{g(d_i, \theta)}(l_i)\right) + \sum_{i=n+1}^{n+m} \ln \left(F_r^{g(d_i, \theta)}(l_i)\right); \)
5. Sample downloads, for unobserved customers whose downloads have not been fixed, according to the posterior probability distribution, as \( \tilde{f}_d^{\text{unobs}}(x) \sim F_r^{g(x, \theta)}(l(x)) \times \hat{f}_d(x); \)
6. Fix a fraction of \( p \) of the unobserved customers’ downloads that have not been fixed before;
7. Fit distribution \( \hat{F}_d \) to the download values of all customers ;
8. Stop if convergence criterion is satisfied. Otherwise go to Step 3.

A.2 Proof of Proposition 1

To make notation more succinct, we call \( \bar{s} = \bar{s}(d) \). The proof of the first claim of Proposition 1 is by contradiction dividing it into two cases: in the first case a customer prefers a plan with index lower than \( \bar{s} - 1 \), and in the second case the customer prefers a plan with index greater than \( \bar{s} \).
By contradiction, assume that customers prefer plan $t = \tilde{s} - 2$. Then this implies that $f_t + v_t(d - B_t) > f_{\tilde{s} - 1} + v_{\tilde{s} - 1}(d - B_{\tilde{s} - 1})$. However note that

$$f_t + v_t(d - B_t) \geq f_{\tilde{s} - 1} + v_t(d - B_{\tilde{s} - 1})$$

$$\geq f_{\tilde{s} - 1} + v_{\tilde{s} - 1}(d - B_{\tilde{s} - 1})$$

The first inequality follows from Constraint (2.7j) which implies that $f_{\tilde{s} - 1} \leq v_t(B_{\tilde{s} - 1} - c_{\tilde{s} - 1} - B_t) + f_t \leq v_t(B_{\tilde{s} - 1} - B_t) + f_t$. The second inequality follows from Constraint (2.7i). Note that the term $d - B_{\tilde{s} - 1}$ is non-negative by definition of $\tilde{s}$. This is a contradiction.

If a customer prefers a plan $t < \tilde{s} - 2$, the logic follows similarly by induction.

Now consider the case that the customer prefers a plan $t > \tilde{s}$. Then, this would imply that $f_t < f_{\tilde{s}}$, but this contradicts Constraint (2.7h). As a result, the customer will only be willing to purchase between plans $\tilde{s} - 1$ and $\tilde{s}$.

For the second part of Proposition 1, assume that a customer $i$ is such that $d \in [B_s - c_s, B_s]$. First, note that by definition of $\tilde{s}$, this plan must be $\tilde{s}$, (see Equation (2.5) and note that $c_s \in [0, B_s]$). Using the result of the first part of Proposition 1, we know that consumer $i$ will only be willing to purchase plans $\tilde{s} - 1$ or $\tilde{s}$. Then note that the payment under plan $\tilde{s} - 1$ can be bounded above as follows:

$$f_{\tilde{s} - 1} + v_{\tilde{s} - 1}(d - B_{\tilde{s} - 1}) \geq f_{\tilde{s} - 1} + v_{\tilde{s} - 1}(B_s - c_s - B_{\tilde{s} - 1})$$

$$\geq f_{\tilde{s}}$$

The first inequality uses the fact that $d \in [B_s - c_s, B_s]$. The second inequality follows from Constraint (2.7h).

### A.3 Proof of Proposition 2

We will show the first part of the Proposition 2, the rest follows using the same logic and we will omit for the sake of brevity. Consider two customers $i, j$ such that $r_i \geq r_j$ and $d_i \leq d_j$. If customer $j$ buys plan $s$, then if customer $i$ purchases plans $s$, she would
incur on a payment lower than customer $i$ (since $d_i < d_j$). In addition, customer $i$ reservation price is higher than customer $j$, then it follows that she would be willing to purchase plan $s$ if not another plan that is cheaper. Therefore, customer $i$ ends up purchasing a plan.

### A.4 Proof of Proposition 3

Before proving Proposition 3, consider the following inequality

$$\sum_{i=1}^{j} \min \left\{ a, \frac{b}{i} \right\} \leq b \left( \min \left\{ 1, \frac{ja}{b} \right\} - \frac{a(l-1)}{b} + \ln \left( \max \left\{ 1, \frac{a_j}{b} \right\} \right) \right)$$

(A.1)

This holds because

$$\sum_{i=1}^{j} \min \left\{ a, \frac{b}{i} \right\} \leq \int_{l-1}^{\min \{j, k\}} a \, dx + \int_{\min \{j, k\}}^{j} \frac{b}{x} \, dx \leq \min \{ b, ja \} - a(l-1) + \ln \left( \max \{ b, aj \} \right)$$

$$= b \left( \min \left\{ 1, \frac{ja}{b} \right\} - \frac{a(l-1)}{b} + \ln \left( \max \left\{ 1, \frac{a_j}{b} \right\} \right) \right)$$

Let us focus first on the first optimization problem solved in Heuristic H0, (i.e. Problem (2.8)). Consider that there are only two plans 1, and 2, so that $0 < B_1 < B_2 = \infty$ (the second plan is the unlimited data plan). Furthermore, consider that all consumer downloads are greater than or equal than $B_1$. Note that the optimal fixed prices of Problem (2.8), will be $f_2 = r_i$ when $i$ is one of the customers, otherwise $f_2$ could be increase by some small amount and get higher revenue. Let us now consider that the optimal price is $r_k$, then this will imply that $kr_k \geq jr_j \forall j \in \{1, \ldots, N\}$. Then for each $l \in \{1, \ldots, |L| - 1\}$, it holds that for every $i \in \{ o(l) + 1, \ldots, o(l+1) \}$, $r_i \leq r_{o(l)}$, and $ir_i \leq kr_k$. Then it follows that

$$r_i \leq \min \left\{ r_{o(l)}, \frac{kr_k}{i} \right\}.$$  

(A.2)
We have

\[
\frac{z^{H0}}{z^*} \geq \frac{kr_k}{\sum_{i=1}^{N} r_i} \\
\geq \frac{kr_k}{r_1 + \sum_{l=1}^{[L]-1} \sum_{i:o(l)+1} \min \{r_o(l) \cdot \frac{kr_k}{kr_k} \}} \\
\geq \frac{r_1}{\eta} + \sum_{l=1}^{[L]-1} \left( \min \left\{ 1, \frac{o(l+1)r_o(l)}{kr_k} \right\} - \frac{o(l)r_o(l)}{kr_k} + \ln \left( \max \left\{ 1, \frac{o(l+1)r_o(l)}{kr_k} \right\} \right) \right) \\
\] (A.3)

The first inequality follows from the fact that \( z^{H0} = kr_k \), and that \( z^* \) can be at most the sum of all reservation prices. The second inequality is due to Equation (A.2). The third inequality follows from (A.1). The fourth inequality follows from the fact that the expression in (A.3) is increasing in \( kr_k \). Indeed, taking the derivative of the expression in (A.3) with respect to \( \frac{1}{kr_k} \) we have that:

\[
\frac{d}{d \left( \frac{1}{kr_k} \right)} \left( \frac{r_1}{kr_k} + \sum_{l=1}^{[L]-1} \left( \min \left\{ 1, \frac{o(l+1)r_o(l)}{kr_k} \right\} - \frac{o(l)r_o(l)}{kr_k} + \ln \left( \max \left\{ 1, \frac{o(l+1)r_o(l)}{kr_k} \right\} \right) \right) \right) \\
= -r_1 + \sum_{l=1}^{[L]-1} \left( r_o(l) o(l) - \frac{1}{\{1 \leq o(l+1)r_o(l) \}} \frac{kr_k}{kr_k} - \frac{1}{\{1 > o(l+1)r_o(l) \}} \frac{o(l+1)r_o(l)}{kr_k} \right) \\
\left( \frac{r_1}{kr_k} + \sum_{l=1}^{[L]-1} \left( 1 - \frac{o(l)r_o(l)}{kr_k} + \ln \left( \frac{o(l+1)r_o(l)}{kr_k} \right) \right) \right)^2 \] (A.4)

Each term in the numerator sum of (A.4) is non-negative, since on the one hand (if \( 1 \leq \frac{o(l+1)r_o(l)}{kr_k} \)) then \( o(l) r_o(l) \leq kr_k \) for all \( l \in \{1, \ldots, [L] - 1\} \), and on the other hand if \( (1 \leq \frac{o(l+1)r_o(l)}{kr_k} \)) then \( o(l) \leq o(l+1) \) for all \( l \in \{1, \ldots, [L] - 1\} \). Then the derivative of the expression in (A.3) with respect to \( kr_k \) is greater or equal to zero.
A.5 Proof of Proposition 4

For the lower bound of Heuristic H0, for each service \( s \in \{1, \ldots, S\} \), consider \( j^*_s = \arg \max_{j \in N_s} r_j \sum_{i \in N_s} r_j 1_{\{r_i > r_j\}} \) and the fixed prices \( f_s = \max_{i \in \{1, \ldots, s\}} r_{j^*_s} \). Note that these fixed prices are increasing. This give rise to a feasible solution for Problem (2.8). Thus, it gives a lower bound for Heuristic H0.

Now let us show that the denominator of the left hand side of Equation (2.10) is an upper bound to the optimal solution of Problem (2.7). Denote by \( f, v \) the optimal prices of this problem. In what follows, consider a fixed plan \( s \in \{2, \ldots, S-1\} \); for the sake of brevity we will omit the cases when \( s \in \{1, S\} \) as these are simpler than the former ones and the same arguments apply. Define the sets \( \Omega^v_s(f_{s-1}, v_{s-1}, f_s) = \{ i \in N_s | d_i \leq B_{s-1} + \frac{f_s-f_{s-1}}{v_{s-1}}, r_i \geq f_{s-1} + v_{s-1}(d_i - B_{s-1}) \}, \Omega^f_s(f_{s-1}, v_{s-1}, f_s) = \{ i \in N_s | d_i \leq B_{s-1} + \frac{f_s-f_{s-1}}{v_{s-1}}, r_i \geq f_s \}, \Omega^v_s(f_{s-1}, v_{s-1}, f_s) = \{ i \in N_s | d_i > B_{s-1} + \frac{f_s-f_{s-1}}{v_{s-1}}, r_i = f_s \}, \Omega^f_s(f_{s-1}, v_{s-1}, f_s) = \{ i \in N_s | d_i > B_{s-1} + \frac{f_s-f_{s-1}}{v_{s-1}}, r_i = f_s \}. \) The sets \( \Omega^v_s (\Omega^f_s) \) represent the customers whose download is in between the bundle data of plans \( s-1 \) and \( s \) and who will prefer to pay the variable price of service \( s-1 \) (fixed price of service \( s \)). The sets \( \Omega^v_s (\Omega^f_s) \) represent those customers in the sets \( \Omega^v_s (\Omega^f_s) \) whose reservation price is exactly equal to the price they pay. See Figure A-1.

Figure A-1: Customers who are in the set \( \Omega^v_s(f_{s-1}, v_{s-1}, f_s) \) are those in the left shaded region, while customers in the set \( \Omega^f_s(f_{s-1}, v_{s-1}, f_s) \) are in the right shaded region.
The revenues captured by the optimal prices from the customers in the set \( N_s \) can be expressed as

\[
\sum_{i \in \Omega_s^v(f_{s-1}, v_{s-1}, f_s)} f_{s-1} + v_{s-1}(d_i - B_{s-1}) + \sum_{i \in \Omega_s^l(f_{s-1}, v_{s-1}, f_s)} f_s. \tag{A.5}
\]

Consider \( \delta^* = \min \{ \delta \geq 0 | \Omega_s^v(f_{s-1} + \delta, v_{s-1}, f_s + \delta) \cup \Omega_s^l(f_{s-1} + \delta, v_{s-1}, f_s + \delta) \neq \emptyset \} \). Then, Equation (A.5) can be upper bounded by

\[
\sum_{i \in \Omega_s^v(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)} f_{s-1} + \delta^* + v_{s-1}(d_i - B_{s-1}) + \sum_{i \in \Omega_s^l(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)} (f_s + \delta^*), \tag{A.6}
\]

since the set of indexes in the sums of Equation (A.5) and Equation (A.6) are the same, and the arguments in both sums of Equation (A.6) are greater than those of Equation (A.5) by \( \delta^* \geq 0 \). The reason why the set of indexes in the sums are the same, is because \( \Omega_s^v(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) = \Omega_s^v(f_{s-1}, v_{s-1}, f_s) \). Clearly \( \Omega_s^v(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \subseteq \Omega_s^v(f_{s-1}, v_{s-1}, f_s) \). Let us show that \( \Omega_s^l(f_{s-1}, v_{s-1}, f_s) \subseteq \Omega_s^v(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \).

By contradiction there is a \( k \in \Omega_s^v(f_{s-1}, v_{s-1}, f_s) \) and \( k \notin \Omega_s^v(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \), then \( f_{s-1} + v_{s-1}(d_k - B_{s-1}) \leq r_k < f_{s-1} + \delta^* + v_{s-1}(d_k - B_{s-1}) \), or equivalently \( 0 \leq r_k - (f_{s-1} + v_{s-1}(d_k - B_{s-1})) \). Call \( \delta^{**} = r_k - (f_{s-1} + v_{s-1}(d_k - B_{s-1})) \), then \( d_k \leq B_{s-1} + \frac{f_{s-1} - f_{s-1}}{v_{s-1}} \) and \( r_k = f_{s-1} + \delta^{**} + v_{s-1}(d_k - B_{s-1}) \). So \( \Omega_s^v(f_{s-1} + \delta^{**}, v_{s-1}, f_s + \delta^{**}) \neq \emptyset \) and \( 0 \leq \delta^{**} < \delta^* \), which leads to a contradiction. Then it holds that \( \Omega_s^v(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) = \Omega_s^l(f_{s-1}, v_{s-1}, f_s) \). A similar argument can be used to show that \( \Omega_s^l(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) = \Omega_s^v(f_{s-1}, v_{s-1}, f_s) \).

Let us define the following set

\[
\Psi_s(\theta, f_{s-1}, v_{s-1}, f_s) = \left[ \Omega_s^v(0, \theta, (B_s - c_s - B_{s-1})) \cup \Omega_s^l(0, \theta, (B_s - c_s - B_{s-1})) \right] \cap \left[ \Omega_s^v(f_{s-1}, v_{s-1}, f_s) \cup \Omega_s^l(f_{s-1}, v_{s-1}, f_s) \right].
\]

Consider \( \theta^* = \min \{ \theta \geq 0 | \Psi(\theta, f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \neq \emptyset \} \). Recall that revenues obtained from the optimal prices \( f, v \) of customers in set \( N_s \) can be upper bounded by Equation (A.6). Then the following inequalities hold:
\[
\sum_{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)} f_{s-1} + \delta^* + v_{s-1}(d_i - B_{s-1}) + \sum_{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)} (f_s + \delta^*) \\
= \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i < \theta^*(B_s-c_s-B_{s-1})\}} f_{s-1} + \delta^* + v_{s-1}(d_i - B_{s-1}) \\
+ \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i \geq \theta^*(B_s-c_s-B_{s-1})\}} f_{s-1} + \delta^* + v_{s-1}(d_i - B_{s-1}) \\
+ \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i \geq \theta^*(B_s-c_s-B_{s-1})\}} f_s + \delta^* \\
+ \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i < \theta^*(B_s-c_s-B_{s-1})\}} f_s + \delta^* \\
\leq \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i < \theta^*(B_s-c_s-B_{s-1})\}} r_i \\
+ \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i \geq \theta^*(B_s-c_s-B_{s-1})\}} r_i \\
+ \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i \geq \theta^*(B_s-c_s-B_{s-1})\}} \theta^*(B_s - c_s - B_{s-1}) \\
= \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i < \theta^*(B_s-c_s-B_{s-1})\}} r_i \\
+ \sum_{\{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i \geq \theta^*(B_s-c_s-B_{s-1})\}} \theta^*(B_s - c_s - B_{s-1}) \\
\leq \sum_{\{i \in \Omega^*_s(0,\theta^*,\theta^*(B_s-c_s-B_{s-1}))\mid r_i < \theta^*(B_s-c_s-B_{s-1})\}} r_i + \sum_{\{i \in \Omega^*_s(0,\theta^*,\theta^*(B_s-c_s-B_{s-1}))\mid r_i \geq \theta^*(B_s-c_s-B_{s-1})\}} \theta^*(B_s - c_s - B_{s-1})}
\]

The first inequality is due to the two following facts: (i) \( r_i \geq f_{s-1}+\delta^*+v_{s-1}(d_i-B_{s-1}) \), for \( i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*) \), and \( r_i \geq f_s + \delta^* \) for \( i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*) \); (ii) due to Lemma 2 (this Lemma is shown at the end of this proof), we have that \( \theta^*(B_s - c_s - B_{s-1}) \geq f_s + \delta^* \), for \( i \in \Omega^l(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*) \), while \( \theta^*(B_s - c_s - B_{s-1}) \geq f_s + \delta^* \geq f_{s-1}+\delta^*+v_{s-1}(d_i-B_{s-1}) \), for \( i \in \Omega^v(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*) \). The second inequality is due to the fact that \( \{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i < \theta^*(B_s-c_s-B_{s-1})\} \), \( d_i < B_s - c_s \} \in \{i \in \Omega^*_s(0,\theta^*,\theta^*(B_s-c_s-B_{s-1}))\mid r_i < \theta^*(B_s-c_s-B_{s-1})\} \}. 

To illustrate this, consider by contradiction that there exists some \( h \in \{i \in \Omega^*_s(f_{s-1}+\delta^*,v_{s-1},f_s+\delta^*)\mid r_i < \theta^*(B_s-c_s-B_{s-1})\} \), \( d_i < B_s - c_s \} \), but \( h \notin \{i \in \Omega^*_s(0,\theta^*,\theta^*(B_s-c_s-B_{s-1}))\mid r_i < \theta^*(B_s-c_s-B_{s-1})\} \}. 

Then, it holds that \( d_h \leq B_s - c_s \). Since \( h \notin \{i \in \Omega^*_s(0,\theta^*,\theta^*(B_s-c_s-B_{s-1}))\mid r_i < \theta^*(B_s-c_s-B_{s-1})\} \}, it
follows that \( r_h < \theta^*(d_h - B_{s-1}) \). Consider \( \theta^{**} \equiv \frac{r_h}{d_h - B_{s-1}} \), thus \( 0 \leq \theta^{**} < \theta^* \). Note that \( h \in \Psi(\theta^{**}, f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \), since \( d_h \leq B_s - c_s \) and \( r_h = \theta^{**}(d_h - B_{s-1}) \), which implies that \( h \in \overline{\Omega}^\theta_s(0, \theta^{**}, \theta^*(B_s - c_s - B_{s-1})) \). Putting this together with the fact that \( h \in \Omega^\theta_s(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \cup \Omega^\delta_s(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \), leads to \( h \in \Psi(\theta^{**}, f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \). Nevertheless, this contradicts the definition of \( \theta^* \), since \( \theta^{**} < \theta^* \) and \( \theta^{**} \geq \theta^* \).

Next we consider two possible cases: (i) \( \overline{\Omega}^\theta_s(0, \theta^*, \theta^*(B_s - c_s - B_{s-1})) \cap \varnothing \), (ii) \( \overline{\Omega}^\delta_s(0, \theta^*, \theta^*(B_s - c_s - B_{s-1})) \cap \varnothing \), (or both).

(i) Consider \( k \in \overline{\Omega}^\theta_s(0, \theta^*, \theta^*(B_s - c_s - B_{s-1})) \), then \( r_k = \theta^*(d_k - B_{s-1}) \), equivalently \( \theta^* = \frac{r_k}{d_k - B_{s-1}} \). Then \( \{ i \in \Omega^\theta_s(0, \theta^*, \theta^*(B_s - c_s - B_{s-1})) \mid r_i < \theta^*(B_s - c_s - B_{s-1}) \} = \{ i \in N_s \mid d_i \leq B_s - c_s \land r_i \leq \theta^*(d_i - B_{s-1}) \} \).

(ii) Consider \( k \in \overline{\Omega}^\delta_s(0, \theta^*, \theta^*(B_s - c_s - B_{s-1})) \), then \( r_k = \theta^*(B_s - c_s - B_{s-1}) \). Equivalently \( \theta^* = \frac{r_k}{d_k - B_{s-1}} \). Then \( \{ i \in \Omega^\delta_s(0, \theta^*, \theta^*(B_s - c_s - B_{s-1})) \mid r_i < \theta^*(B_s - c_s - B_{s-1}) \} = \{ i \in N_s \mid d_i \leq B_s - c_s \land r_i \leq \theta^*(d_i - B_{s-1}) \} \).
In the second to last equality we are using the fact that \( d_k > B_s - c_s \). Therefore we have that

\[
\sum_{\{ i \in \Omega(0, \theta^* (B_s - c_s - B_{s-1})) \mid r_i < \theta^* (B_s - c_s - B_{s-1}) \}} r_i + \sum_{\{ i \in N \mid r_i \geq \theta^* (B_s - c_s - B_{s-1}) \}} r_k = \sum_{i \in \Upsilon(k)} r_i + \sum_{\{ i \in N \mid r_i \geq r_k \}} r_k \leq \beta_s \leq \max\{ \beta_s, \gamma_s \}.
\]

In summary, we have shown that for \( s \in \{2, \ldots, S - 1\} \)

\[
\sum_{i \in \Omega(f_{s-1}, v_{s-1}, f_s)} f_{s-1} + v_{s-1}(d_i - B_{s-1}) + \sum_{i \in \Omega(f_{s-1}, v_{s-1}, f_s)} f_s \leq \max\{ \beta_s, \gamma_s \}.
\]

Note that for \( s \in \{1, S\} \), the proof follows using the similar arguments. Thus

\[
\sum_{s \in \{1, \ldots, S\}} \sum_{i \in \Omega(f_{s-1}, v_{s-1}, f_s)} f_{s-1} + v_{s-1}(d_i - B_{s-1}) + \sum_{i \in \Omega(f_{s-1}, v_{s-1}, f_s)} f_s \leq \sum_{s \in \{1, \ldots, S\}} \max\{ \beta_s, \gamma_s \}.
\]

**Lemma 2.** \( \theta^* (B_s - c_s - B_{s-1}) \geq f_s + \delta^* \)

**Proof.** Proof. Consider \( k \in \Psi_s(\theta^*, f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*) \), then let us analyze the four possible cases:

(i) \( k \in \Omega(f_{s-1}, v_{s-1}, f_s) \) and \( k \in \Omega(f_{s-1} + \delta^*, v_{s-1} + \delta^*, f_s + \delta^*) \).

In this case

\[
\theta^* (B_s - c_s - B_{s-1}) = \frac{r_k}{d_k - B_{s-1}} (B_s - c_s - B_{s-1}) \geq \left( \frac{f_{s-1} + \delta^*}{d_k - B_{s-1}} + v_{s-1} \right) (B_s - c_s - B_{s-1}) \geq f_{s-1} + \delta^* + v_{s-1}(B_s - c_s - B_{s-1}) = f_s + \delta^*
\]

The first equality follows from the fact that \( \theta^* = \frac{r_k}{d_k - B_{s-1}} \) since \( k \in \Omega(f_{s-1}, v_{s-1}, f_s + \delta^*) \). The first inequality holds because \( r_k \geq f_{s-1} + \delta^* + v_{s-1}(d_k - B_{s-1}) \).
(due to $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$). The second inequality follows from the fact that $d_k \leq B_s - c_s$.

(ii) $k \in \Omega_s^0(0, \theta^*, \theta^*(B_s - c_s - B_{s+1}))$ and $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$.

In this case

$$\theta^*(B_s - c_s - B_{s-1}) = \frac{r_k}{d_k - B_{s-1}}(B_s - c_s - B_{s-1}) \geq r_k \geq f_s + \delta^*$$

The first inequality holds because $d_k \leq B_s - c_s$. The second inequality follows directly from the fact that $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$.

(iii) $k \in \Omega_s^0(0, \theta^*, \theta^*(B_s - c_s - B_{s+1}))$ and $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$.

This case can not happen since it would imply that $d_k > B_s - c_s$ and $d_k \leq B_{s-1} + \frac{L_s - f_{s-1}}{v_{s-1}} \leq B_s - c_s$. Note that the first inequality is due to $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$ while the second follows from Constraint (2.7j).

(iv) $k \in \Omega_s^0(0, \theta^*, \theta^*(B_s - c_s - B_{s+1}))$ and $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$.

In this case

$$\theta^*(B_s - c_s - B_{s-1}) = r_k \geq f_s + \delta^*$$

The first equality follows from the fact that $\theta^* = \frac{r_k}{B_s - c_s - B_{s-1}}$ since $k \in \Omega_s^0(0, \theta^*, \theta^*(B_s - c_s - B_{s+1}))$. The first inequality follows directly from the fact that $k \in \Omega_s^0(f_{s-1} + \delta^*, v_{s-1}, f_s + \delta^*)$. 

$\square$
A.6 Sub-array Algorithm used in Dynamic Programming

Algorithm A.6.1: Algorithm for selecting a sample of customers’ reservation prices.

**Input**: Array of reservation prices in increasing order, \( r \). The minimum difference to allow between reservation prices, \( \Delta \).

**Output**: An array \( q \) of the reservation prices so that none of them are closer than \( \Delta \).

1. \( q \leftarrow \[ \] \); /* Initialize the output array as empty */
2. \( l \leftarrow -\infty \); /* Last value added in the output array */
3. for \( i \leftarrow 1 \) to \( \text{Length}(r) \) do /* Loop over the elements in the input array */
   4. if \( r[i] - l \geq \Delta \) then /* Is the new value at least \( \Delta \) from the last value added? */
   5. \( q \leftarrow [q; r[i]] \); /* Add the new value in the output array */
   6. \( l \leftarrow r[i] \); /* Update the last value added */
4. return \( q \)
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<th>$m$</th>
<th>Opt</th>
<th>H0</th>
<th>H1</th>
<th>H2(1)</th>
<th>H2(2)</th>
<th>H2(3)</th>
<th>DP</th>
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<td>9</td>
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<td>1</td>
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<td>0</td>
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<td>7</td>
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<td>0</td>
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<tr>
<td></td>
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<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>10</td>
<td>23</td>
<td>55</td>
</tr>
</tbody>
</table>

| 100 | 33  | 7   | 7   | 6   | 7     | 16    | 51    | 37  |
|     | 67  | 8   | 0   | 3   | 0     | 11    | 59    | 43  |
|     | 100 | 12  | 0   | 10  | 0     | 11    | 40    | 69  |
|     | 133 | 24  | 1   | 38  | 1     | 24    | 142   | 83  |
|     | 167 | 129 | 1   | 24  | 1     | 85    | 229   | 116 |

| 500 | 167 | 4364 | 24 | 320 | 43 | 988 | 2154 | 1228 |
|     | 333 | 20259 | 72 | 1416 | 71 | 1702 | 16184 | 847 |
|     | 500 | 119865 | 98 | 11737 | 253 | 4554 | 21639 | 2259 |
|     | 667 | - | 56 | 4658 | 302 | 5411 | 14027 | 1333 |
|     | 833 | - | 232 | 9555 | 323 | 11880 | 93474 | 2709 |

| 1000 | 333 | - | 390 | 5723 | 362 | 8932 | 129918 | 2552 |
|      | 667 | - | 312 | 74649 | 743 | 30377 | 244210 | 3428 |
|      | 1000 | - | 670 | 54250 | 12445 | 73255 | 125909 | 4003 |
|      | 1333 | - | 2212 | 145932 | 30702 | 142258 | 387919 | 3417 |
|      | 1667 | - | 857 | 361175 | 14878 | 258955 | 327276 | 8274 |

| 5000 | 1667 | - | 21101 | - | 347874 | - | - | 34696 |
|      | 3333 | - | 30859 | - | 53019 | - | - | 45417 |
|      | 5000 | - | 16177 | - | 101539 | - | - | 60853 |
|      | 6667 | - | 54249 | - | 122044 | - | - | 67552 |
|      | 8333 | - | 60152 | - | 1153472 | - | - | 48775 |

| 10000 | 3333 | - | - | - | - | - | - | 132421 |
|      | 6667 | - | - | - | - | - | - | 151172 |
|      | 10000 | - | - | - | - | - | - | 198616 |
|      | 13333 | - | - | - | - | - | - | 216111 |
|      | 16667 | - | - | - | - | - | - | 233630 |

Table A.1: Running time in seconds.
Appendix B

Appendix of Chapter 3

B.1 Proof of Proposition 5

Note that we have:

\[ \nabla_p \pi(p) = \tilde{d} - B(p - er) + \Psi(p) - B(p - c), \]

\[ \nabla^2_p \pi(p) = -2B + J_p \Psi(p), \]

where \([J_p \Psi(p)]_{ij} = \frac{c^2}{f_i(F_i^{-1}(1 - \frac{x}{\rho_i}))p_i^2} \mathbb{1}_{i=j} \). For existence, note that from \( \nabla_p \pi = 0 \), we obtain \( p = \frac{1}{2}B^{-1}\Psi(p) + \frac{1}{2}B^{-1}(\tilde{d} + Ber + Bc) \). Starting with \( p^{(0)} = c \), then evaluating, we have \( p^{(1)} = \frac{1}{2}B^{-1}\Psi(c) + \frac{1}{2}B^{-1}(\tilde{d} + Ber + Bc) = \frac{1}{2}B^{-1}(\tilde{d} - Bc - A + er) + c \geq c \), so that \( p^{(1)} \geq p^{(0)} \). Assume that \( p^{(i)} \geq p^{(i-1)} \). We have: \( p^{(i+1)} = \frac{1}{2}B^{-1}\Psi(p^{(i-1)}) + \frac{1}{2}B^{-1}(\tilde{d} + Ber + Bc) + \frac{1}{2}B^{-1}(\Psi(p^{(i)}) - \Psi(p^{(i-1)})) \geq p^{(i)} \). Note that the sequence is bounded since \( p^{(i)} < \frac{1}{2}B^{-1}(\tilde{d} + Ber + Bc) \forall i \) and therefore, it converges. To show uniqueness, a sufficient condition is that \( \nabla^2_p \pi \) is negative definite, which is implied if \( -\nabla^2_p \pi \) is a strictly diagonally dominant (SDD) M-Matrix, i.e., if \( \forall i, \forall p_i \geq c_i \frac{c^2}{2p_i^2 f_i(F_i^{-1}(1 - \frac{x}{\rho_i}))} \leq c_i'Be \). Since \( f_i \) is IFR (i.e., \( \frac{d}{dx} \left( \frac{f_i(x)}{1-F_i(x)} \right) \geq 0 \)), this is satisfied if \( \frac{1}{f_i(A_i)} \leq 2c_i'Be \). Note that the latter follows directly from Assumption 2. As a result, we conclude that there exists a unique optimal solution. 

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B.2 Proof of Proposition 6

Note that \( E[GC(r)] = r E[e' \min\{q, d\}] = r \frac{1}{2} e'(d + Ber - Bc + \Psi(p(r))) \). We next show that \( E[e' \min\{q, d\}] \) is non-decreasing in \( r \), and so is \( E[GC(r)] \). We have:

\[
\frac{d}{dr} E[e' \min\{q, d\}] = \frac{1}{2} e' Be + \frac{1}{2} e' J_p \Psi(p) \nabla_r p(r).
\]

If \( \nabla_r p(r) \geq 0 \), then the result follows. Taking the derivative with respect to \( r \) of \( \nabla_p \pi = 0 \), we obtain:

\[
0 = \nabla_r \nabla_p \pi(p(r), r) = \nabla_r \left( \Psi(p) + \tilde{d} - B(p - er) - B(p - c) \right) \\
= J_p \Psi(p) \nabla_r p(r) - 2B \nabla_r p(r) + Be = \nabla_p^2 \pi(p(r)) \nabla_r p(r) + Be.
\]

Note that \( -\nabla_p^2 \pi \) is a non-singular M-Matrix, since it is a Z-Matrix with positive diagonal elements and SDD. Therefore, \( (-\nabla_p^2 \pi)^{-1} \geq 0 \). So \( \nabla_r p(r) = (-\nabla_p^2 \pi)^{-1} Be \geq 0 \) and this concludes the proof.

B.3 Proof of Proposition 7

Equivalently to the \( n \) optimization problems that each firm faces in equation (3.13), one can reduce this system to a single optimization problem with \( p \in \mathbb{R}^n \) as decision variables and \( \pi^W(p) = \pi(p) + \frac{1}{2} (p - c)'(B - D)(p - c) \) as the objective function. Therefore:

\[
\nabla_p \pi(p)^W = \tilde{d} - B(p - er) + \Psi(p) - D(p - c), \\
\nabla_p^2 \pi(p)^W = -B - D + J_p \Psi(p).
\]

For existence, note that from \( \nabla_p \pi^W = 0 \), we have \( p = X \Psi(p) + X (\tilde{d} + Ber + Bc) \). Starting with \( p^{(0)} = c \), then evaluating, we obtain \( p^{(1)} = X \Psi(c) + X (\tilde{d} + Ber + Bc) = X (\tilde{d} - Bc - A + er) + c \geq c \), so that \( p^{(1)} \geq p^{(0)} \). Assume that \( p^{(i)} \geq p^{(i-1)} \). We have:

\[
p^{(i+1)} = X \Psi(p^{(i-1)}) + X (\tilde{d} + Ber + Bc) + X (\Psi(p^{(i)}) - \Psi(p^{(i-1)})) \geq p^{(i)}.
\]

Note the sequence
is bounded, since \( p(i) < X(d+Ber+Be) \forall i \) and therefore, it converges. For uniqueness, a sufficient condition is that \( \nabla^2 \pi^W \) is negative definite, which is implied if \(-\nabla^2 \pi^W\) is an SDD M-Matrix, i.e., if \( \forall i, \forall p_i \geq c_i \frac{e_i^2}{p_i f_i (F_i^{-1}(1-p_i))} \leq e_i'(B+D)e \). Since \( f_i \) is IFR (i.e., \( \frac{d}{dx} \left( \frac{f_i(x)}{1-F_i(x)} \right) \geq 0 \)), this is satisfied if \( \frac{1}{f_i(A_i)} < c_i e_i'(B+D)e \). Note that the latter follows directly from Assumption 2. As a result, we conclude that there exists a unique optimal solution.

B.4 Proof of Proposition 8

The proof goes as follows. First, write the optimal equations for the government and the supplier in both cases: (i) a monopolist that sells a single product, and (ii) a monopolist that sells \( n \) symmetric products. Then, by using Assumption 3, one can formally show that the optimal equations coincide for both cases. Consequently, the optimal decision variables \( p, q \) and \( r \) are the same and hence, both settings are equivalent (the details are omitted for conciseness).

B.5 Proof of Proposition 9

1. Let us call \( q_0^N \) the vector of quantities produced when there is no rebate in the setting without competition. It can be seen that if \( e'q_0^N < 2e'q_0^N + e'B_k \leq \Gamma \), both optimization problems (3.11) and (3.12) yield the same outcome, since the adoption constraint is tight at optimality in both problems. We next consider the case where \( e'q_0^N < \Gamma < 2e'q_0^N + e'B_k \) for which the solution of problem (3.12) is not tight. The government costs ratio is given by:

\[
\frac{GC_{SW}^N}{GC_{GC}^N} = \frac{4(e'q_0^N + e'B_k)(2e'q_0^N + e'B_k)}{2e'B_k} \times \left( \frac{2(\Gamma - \frac{e'q_0^N}{2})^2 - \frac{e'q_0^N e'q_0^N}{2}}{e'B_k} \right)^{-1} = \frac{4(e'q_0^N + e'B_k)(2e'q_0^N + e'B_k)}{4(\Gamma - \frac{e'q_0^N}{2})^2 - e'q_0^N e'q_0^N}.
\]
Then making $\Gamma \downarrow e'q_0^N$, the denominator goes to zero, whereas the numerator stays bounded away from zero, since $e'q_0^N < 2e'q_0^N + e'Bk \Rightarrow e'q_0^N + e'Bk > 0.$ Then, we obtain:

$$M < \frac{GC_{SW}^N}{GC_{GC}^N} \quad \forall M > 0.$$ 

For ease of notation, we use $\bar{\gamma} = \gamma^N$ and $\gamma = \gamma^N$. We next show that:

$$\frac{3 + 2\gamma}{4 + 2\gamma + 2\bar{\gamma} + \bar{\gamma}^2} \leq \frac{SW_{GC}^N}{SW_{SW}^N} \leq 1.$$
The second inequality follows from the fact that the government maximizes social welfare. In order to show the first inequality, one can see that:

\[
\frac{SW_{GC}^N}{SW_{SW}^N} = \frac{\frac{3}{2}q_0'N' - 1q_0'N + k'q_0'N + \frac{(1-\gamma)q_0'N + 2e'q_0'N + e'Bk - 1)}{2e'B_k}}{\frac{3}{2}q_0'N' - 1q_0'N + k'q_0'N + \frac{(e'_q^N + e'Bk)^2}{2e'B_k}}
\]

\[
\geq \frac{\frac{3}{2}q_0'N' - 1q_0'N + k'q_0'N + \frac{(e'_q^N + e'Bk)^2}{2e'B_k}}{\frac{3}{2}q_0'N' - 1q_0'N + k'q_0'N + \frac{(1+\gamma)^2(e'_q^N)^2}{2e'B_k}}
\]

\[
\geq \frac{(\frac{3}{2} + \gamma)q_0'N' - 1q_0'N + \frac{(1+\gamma)^2(e'_q^N)^2}{2e'B_k}}{(3 + 2\gamma)q_0'N' - 1q_0'N e'Be + (1 + \gamma)^2(e'_q^N)^2}
\]

\[
= 1 - \frac{(1+\gamma)^2(e'_q^N)^2}{(3 + 2\gamma)q_0'N' - 1q_0'N e'Be + (1 + \gamma)^2(e'_q^N)^2}
\]

\[
= 1 - \frac{(1 + \gamma)^2(e'_q^N)^2}{(3 + 2\gamma)(q_0'N' - 1q_0'N e'Be - e'_q^N e'_q^N) + (4 + 2\gamma + 2\gamma + \gamma^2)(e'_q^N)^2}
\]

\[
\geq 1 - \frac{(1 + \gamma)^2(e'_q^N)^2}{(4 + 2\gamma + 2\gamma + \gamma^2)(e'_q^N)^2}
\]

\[
= 1 - \frac{(1 + \gamma)^2}{4 + 2\gamma + 2\gamma + \gamma^2}
\]

The first inequality follows from \(e'_q^N \leq \Gamma \leq 2e'_q^N + e'Bk\) and the second inequality from \(D, Be \leq \gamma Be \Rightarrow B^{-1}D, Be \leq \gamma e \Rightarrow q_0'N' - 1q_0'N e'Be \leq \gamma q_0'N' e'.\) The third inequality is obtained from \(\gamma q_0'N' - 1q_0'N \leq k'q_0'N\) and \(\frac{d}{dx} \left(\frac{1+\gamma}{1+\alpha} \right) = \frac{a}{(1+\alpha)x} \geq 0\) if \(a \geq 0.\)

Finally, the forth inequality follows from Lemma 6.

However, one can show that if \(k = \gamma(p_0^N - c) = \gamma B^{-1}q_0^N,\) for \(\gamma \in \mathbb{R}^+\), the gap is
tight. The social welfare ratio can be written as:

\[
\frac{SW_{GC}^N}{SW_{SW}^N} = \frac{\frac{3}{2}q_0^N' B^{-1}q_0^N + k'q_0^N + \frac{(1-e'q_0^N)(3e'q_0^N-\Gamma)}{2e'Be}}{\frac{3}{2}q_0^N' B^{-1}q_0^N + k'q_0^N + \frac{(e'q_0^N+e'Bk)^2}{2e'Be}} \\
\geq \frac{\frac{3}{2}q_0^N' B^{-1}q_0^N + k'q_0^N}{\frac{3}{2}q_0^N' B^{-1}q_0^N + k'q_0^N + \frac{(1+\gamma)^2(e'q_0^N)^2}{2e'Be}} (\Gamma \geq e'q_0^N) \\
= \frac{\frac{3}{2}q_0^N' B^{-1}q_0^N + k'q_0^N}{\frac{1}{2}q_0^N' B^{-1}q_0^N + \frac{(1+\gamma)^2(e'q_0^N)^2}{2e'Be}} (e'Be = \gamma e'q_0^N) \\
= \frac{(3+2\gamma)q_0^N' B^{-1}q_0^N e'Be}{(3+2\gamma)(q_0^N' B^{-1}q_0^N e'Be - e'q_0^N e'q_0^N) + (1+\gamma)^2(e'q_0^N)^2} \\
= \frac{(3+2\gamma)(q_0^N' B^{-1}q_0^N e'Be - e'q_0^N e'q_0^N) + (2+\gamma)^2(e'q_0^N)^2}{(3+2\gamma)(q_0^N' B^{-1}q_0^N e'Be - e'q_0^N e'q_0^N) + (2+\gamma)^2(e'q_0^N)^2} - (1+\gamma)^2(e'q_0^N)^2 \\
= 1 - \frac{(1+\gamma)^2(e'q_0^N)^2}{(3+2\gamma)(q_0^N' B^{-1}q_0^N e'Be - e'q_0^N e'q_0^N) + (2+\gamma)^2(e'q_0^N)^2} \\
\geq 1 - \frac{(1+\gamma)^2(e'q_0^N)^2}{(2+\gamma)^2(e'q_0^N)^2} (q_0^N' B^{-1}q_0^N e'Be - e'q_0^N e'q_0^N \geq 0) \\
= 1 - \left(\frac{1+\gamma}{2+\gamma}\right)^2 = \frac{3+2\gamma}{(2+\gamma)^2}.
\]

Note that \(1 - \left(\frac{1+\gamma}{2+\gamma}\right)^2 \in \left[\frac{5}{9}, \frac{3}{4}\right]\), when \(\gamma \in [0,1]\). In addition, when \(\gamma > 0\), we have:

\[
\frac{SW_{GC}^N}{SW_{SW}^N} \to 0 \text{ as } \gamma \to \infty.
\]

2. Define \(Y = (B^{-1} + D^{-1})^{-1}\), and denote by \(q_0^W\) the vector of quantities produced when there is no rebate for the setting with competition. Note that \(q_0^W = YB^{-1}(\bar{D} - Bc)\). One can see that if \(e'q_0^W < e'q_0^W + \frac{e'YB^{-1}(\bar{D} - Bc - q_0^W + Bk)e'Ye}{e'YB^{-1}Ye} e'Ye < \Gamma\), both optimization problems, the analogous of (3.11) and (3.12) in the presence of competition, yield the same outcomes since the adoption constraint is tight at optimality. We next consider the case where \(e'q_0^W < \Gamma < e'q_0^W + \frac{e'YB^{-1}(\bar{D} - Bc - q_0^W + Bk)e'Ye}{e'YB^{-1}Ye} e'Ye\) for which the solution of problem (3.12) is not tight.
The government cost ratio is given by:

\[
\frac{GC^w_{SW}}{GC^w_{GC}} = \frac{e'YB^{-1}(\bar{d} - Bc - q_0^W + Bk)}{e'YB^{-1}Ye} \times \left( \frac{e'q_0^W + e'YB^{-1}(\bar{d} - Bc - q_0^W + Bk)e'Ye}{e'YB^{-1}Ye} \right) \frac{e'Ye}{\Gamma(1 - e'q_0^W)}.
\]

If \( \Gamma \downarrow e'q_0^W \), then the denominator goes to zero. Note that the left term of the numerator is bounded away from zero, if the optimal rebate level when maximizing \( SW \) is positive.

Again, for ease of notation, we use \( \bar{\gamma} = \bar{\gamma}^N \), and \( \gamma = \bar{\gamma}^N \). We next show that:

\[
\frac{3 + 2\bar{\gamma}}{(2 + \gamma)^2} \leq \frac{SW^W_{GC}}{SW^W_{SW}} \leq 1.
\]

The second inequality follows from the fact that the government maximizes social welfare. To show the first inequality, we can write the ratio of social welfare as:

\[
\frac{SW^W_{GC}}{SW^W_{SW}} = (\bar{d} - Bc - \frac{1}{2}q_0^W + Bk)'B^{-1}q_0^W + \frac{(e'YB^{-1}(\bar{d} - Bc - q_0^W + Bk))^2}{2e'YB^{-1}Ye}.
\]

The first inequality is due to the fact that \( \Gamma > e'q_0^W \). In the fourth inequality, we use the fact that \( \bar{d} - Bc = BY^{-1}q_0^W = B(B^{-1} + D^{-1})q_0^W = (I + BD^{-1})q_0^W \) (since \( q_0^W = YB^{-1}(\bar{d} - Bc) \)), and the fact that by definition of \( \gamma_i \) for all \( i \in \{1, \ldots, n\}, k = D_\gamma (p_0^W - c) = D_\gamma D^{-1}q_0^W \). This last equality, \( p_0 - c = D^{-1}q_0^W \), follows from the first order conditions for all the suppliers, which can be stated as \( \bar{d} - Bp - D(p - c) = 0 \) (in the case when \( r = 0 \)). We denote \( a = D^{-1/2}Ye \),
and \( b = D^{-1/2}q_0^W \). Note that \( a, b \geq 0 \), as \( a = D^{-1/2}Ye = D^{-1/2}YB^{-1}Be \geq 0 \) since \( D^{-1/2} \geq 0, YB^{-1} \geq 0 \) by Lemma 5 (b), and \( Be \geq 0 \) by Lemma 5 (c); while \( b = D^{-1/2}q_0^W = D^{-1/2}YB^{-1}(\tilde{d} - Bc) \geq 0 \). Therefore:

\[
\frac{SW_{GW}^W}{SW_{SW}^W} \geq 1 - \frac{(a'(I + D\gamma)b)^2}{b'(2I + 2D\gamma + D^{1/2}B^{-1}D^{1/2})ba'D^{1/2}B^{-1}D^{1/2}a + (a'(I + D\gamma)b)^2}
\]

\[
\geq 1 - \frac{(a'(I + D\gamma)b)^2}{b'(3I + 2D\gamma)ba'a + (a'(I + D\gamma)b)^2}
\]

\[
\geq 1 - \frac{(a'(I + D\gamma)b)^2}{(a'(3I + 2D\gamma)^{1/2}b)^2 + (a'(I + D\gamma)b)^2}
\]

\[
= \frac{(a'(3I + 2D\gamma)^{1/2}b)^2}{(a'(3I + 2D\gamma)^{1/2}b)^2 + (a'(I + D\gamma)b)^2}
\]

\[
\geq \left( \frac{a'(3I + 2D\gamma)^{1/2}b}{a'(2I + D\gamma)b} \right)^2 \geq \frac{3 + 2\gamma}{(2 + \gamma)^2}.
\]

The second inequality follows from \( D^{1/2}B^{-1}D^{1/2} \geq I \), as this is equivalent to \( B^{-1} \geq D^{-1} \) which follows from Lemma 5 (a). The third inequality follows from Lemma 6. The fourth inequality is by using Lemma 3. For the last inequality, let \( x_i = a_ib_i(\geq 0) \) for all \( i \in \{1, \ldots, n\} \). We want to show that:

\[
\frac{\sum_{i=1}^{n} x_i \sqrt{3 + 2\gamma_i}}{\sum_{i=1}^{n} x_i (2 + \gamma_i)} \geq \frac{\sqrt{3 + 2\gamma}}{2 + \gamma} \iff (2 + \gamma) \sum_{i=1}^{n} x_i \sqrt{3 + 2\gamma_i} \geq (\sqrt{3 + 2\gamma}) \sum_{i=1}^{n} x_i (2 + \gamma_i)
\]

\[
\iff (2 + \gamma) \sqrt{3 + 2\gamma_i} \geq (\sqrt{3 + 2\gamma})(2 + \gamma_i) \quad \forall i
\]

\[
\iff \frac{\sqrt{3 + 2\gamma_i}}{2 + \gamma_i} \geq \frac{3 + 2\gamma}{2 + \gamma} \quad \forall i \in \{1, \ldots, n\}.
\]

The last statement holds, since \( \frac{d}{dx_i} \left( \frac{\sqrt{3 + 2\gamma_i}}{2 + \gamma_i} \right) = \frac{1 + \gamma_i}{(2 + \gamma_i)^2 \sqrt{3 + 2\gamma_i}} \leq 0 \). As a result, we obtain:

\[
\left( \frac{a'(3I + 2D\gamma)^{1/2}b}{a'(2I + D\gamma)b} \right)^2 = \left( \frac{\sum_{i=1}^{n} x_i \sqrt{3 + 2\gamma_i}}{\sum_{i=1}^{n} x_i (2 + \gamma_i)} \right)^2 \geq \left( \frac{\sqrt{3 + 2\gamma}}{2 + \gamma} \right)^2 = \frac{3 + 2\gamma}{(2 + \gamma)^2},
\]

which concludes the proof. \( \blacksquare \)

**Lemma 3.** Assume \( y, z \in \mathbb{R}^n \) are non negative vectors and \( D \in \mathbb{R}^{n \times n} \) such that \( D \) is a diagonal positive matrix. Then, \( (z'(3I + 2D)^{1/2}y)^2 + (z'(I + D)y)^2 \leq \)
Proof. Proof. We have:

\[
(z'(3I + 2D)^{1/2}y)^2 + (z'(I + D)y)^2 = \\
\sum_{i=1}^{n} \sum_{j=1}^{n} z_i y_i y_j (1 + D_{ii})(1 + D_{jj}) + \sum_{i=1}^{n} \sum_{j=1}^{n} z_i y_i y_j \sqrt{(3 + 2D_{ii})(3 + 2D_{jj})} \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i y_i y_j (1 + D_{ii} + D_{jj} + D_{ii}D_{jj} + \sqrt{9 + 6D_{ii} + 6D_{jj} + 4D_{ii}D_{jj}}) \\
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} z_i y_i y_j (1 + D_{ii} + D_{jj} + D_{ii}D_{jj} + \sqrt{(D_{ii} + D_{jj} + 3)^2}) \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i y_i y_j (4 + 2D_{ii} + 2D_{jj} + D_{ii}D_{jj}) \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i y_i y_j (2 + D_{ii})(2 + D_{jj}) = \left[ z'(2I + D)y \right]^2. \\
\]

\(\Box\)

B.6 Proof of Proposition 10

1. If \(\frac{2e'q_0^N + k'B}{4} < \Gamma\), then the constraint of problem (3.20) is tight, and we are back to the small externalities case so that Proposition 9 applies. We next consider the case where \(e'q_0 < \Gamma \leq \frac{2e'q_0^N + k'B}{4}\) (i.e., large externalities and the adoption constraint is not tight). Note that \(2e'q_0 < k'B\). Observe that the region of \(\Gamma\) is such that \(e'q_0 < \Gamma \leq \frac{2e'q_0^N + k'B}{4} < 2e'q_0^N + k'B\), then the optimal solution of problem (3.12) is such that the adoption constraint is not tight.

The government cost ratio is given by:

\[
\frac{GC_{IM}^N}{GC_{GC}^N} = \frac{(k'B - 2e'q_0^N)(k'B e + 2e'q_0^N)}{8e'Be} \times \frac{e'Be}{2\Gamma(\Gamma - e'q_0^N)} \\
= \frac{(k'B - 2e'q_0^N)(k'B e + 2e'q_0^N)}{16\Gamma(\Gamma - e'q_0^N)},
\]

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where the expression approaches $+\infty$ when $\Gamma \downarrow e'tq^N$, since $k'B\gamma - 2e'tq^N = (k'B + 2e'tq^N) - 4e'tq^N > 4\Gamma - 4\Gamma = 0$. Therefore, we have:

$$M < \frac{GC'_{IM}^N}{GC'_{GC}^N} \quad \forall M > 0.$$

We next show the inequality for the social welfare ratio. For ease of notation, we use $\bar{\gamma} = \gamma^N$, and $\gamma = \gamma^N$. In the general case, we have: $k \leq D_\gamma B^{-1}q^N_0$. Note that $\bar{\gamma} \geq 2$, as otherwise, if $\bar{\gamma} < 2$, we have $k'Be = e'BD_\gamma B^{-1}q^N_0 \leq \bar{\gamma}e'BB^{-1}q^N_0 = \bar{\gamma}e'q^N_0 < 2e'q^N_0$ which is a contradiction. The social welfare ratio is given by:

$$SW'_{IM}^N \frac{SW'_{IM}^N}{SW'_{SW}^N} = \left( \frac{3}{2} q^N_0 B^{-1}q^N_0 + k'q^N_0 + \frac{(k'B - 2e'tq^N_0)(7k'B + 10e'tq^N_0)}{32e'B} \right) \times \left( \frac{3}{2} q^N_0 B^{-1}q^N_0 + k'q^N_0 + \frac{(k'B + e'tq^N_0)^2}{2e'B} \right)^{-1}$$

$$= 1 - \frac{(e'tq^N_0 + k'B)^2}{2e'B} - \frac{(k'B - 2e'tq^N_0)(7k'B + 10e'tq^N_0)}{32e'B}$$

$$= 1 - \frac{1}{16} (e'tq^N_0 + k'B)^2 - \frac{(k'B - 2e'tq^N_0)(7k'B + 10e'tq^N_0)}{48q^N_0 B^{-1}q^N_0 e'B + 32k'q^N_0 B^{-1}q^N_0 e'B + 16(k'B + e'tq^N_0)^2}$$

$$= 1 - \frac{9}{16} \frac{(2e'tq^N_0 + k'B)^2}{2e'B}$$

$$\geq 1 - \frac{9}{16} \frac{(2e'tq^N_0 + k'B)^2}{2e'tq^N_0 B^{-1}q^N_0 e'B + 2q^N_0 B^{-1}q^N_0 e'B + (k'B + e'tq^N_0)^2}$$

$$\geq 1 - \frac{9}{16} \frac{(2e'tq^N_0 + k'B)^2}{2q^N_0 B^{-1}q^N_0 e'B - 2\bar{\gamma}(e'tq^N_0)^2 + (k'B + 2e'tq^N_0)^2}$$

$$\geq 1 - \frac{9}{16} \frac{(2e'tq^N_0 + \bar{\gamma}e'tq^N_0)^2}{2q^N_0 B^{-1}q^N_0 e'B - 2\bar{\gamma}(e'tq^N_0)^2 + (\bar{\gamma}e'tq^N_0 + 2e'tq^N_0)^2}$$

$$= 1 - \frac{9(2 + \bar{\gamma})^2}{16(4 + 2\bar{\gamma} + \bar{\gamma}^2)}.$$

The first inequality follows from $k'q^N_0 \geq \gamma_0 B^{-1}q^N_0$, the second inequality follows from Lemma 6, and the third inequality is implied by the fact that $k'B \gamma \leq \bar{\gamma} e'tq^N_0$. Finally, the fourth inequality follows from $\frac{d}{dx} \frac{x}{a + x} \geq 0$ for $a \geq 0$, and the fact that $k'B \gamma \leq \bar{\gamma} e'tq^N_0$. Note that $\frac{9(2 + \bar{\gamma})^2}{16(4 + 2\bar{\gamma} + \bar{\gamma}^2)}$ is increasing in $\bar{\gamma}$ and $\gamma$. Note also that $\bar{\gamma} \geq 2$ and $\gamma \geq 0$, so that the worst case is obtained when $\bar{\gamma} = 2$ and $\gamma = 0$, leading
to a ratio of 0.25.

Finally, we show the last inequality for the government cost ratio:

$$\frac{GC_{SW}^N}{GC_{IM}^N} = \frac{2(e'q_0 + e'Bk)(2e'q_0 + e'Bk)}{e'Bk} \times \frac{8e'Be}{(k'Be - 2e'q_0)(k'Be + 2e'q_0)} = 16 \frac{k'Be + e'q_0}{k'Be - 2e'q_0} \geq 16.$$

2. We next consider the competitive environment. The lower bound for the social welfare ratio can be shown in a similar fashion as before, and is not reported due for conciseness. We next show the bounds for the government cost ratio.

Comparing the costs for the IM and GC models, we obtain:

$$\frac{GC_{IM}^W}{GC_{GC}^W} = \frac{(k'Ye - e'q_0^W)(k'Ye + e'q_0^W)}{4e'Ye} \times \frac{e'Ye}{\Gamma(\Gamma - e'q_0^W)} = \frac{(k'Ye)^2 - (e'q_0^W)^2}{4\Gamma(\Gamma - e'q_0^W)}.$$

Note that the above expression approaches $+\infty$ when $\Gamma \downarrow e'q_0^W$, as $k'Ye - e'q_0^W > 0$. Therefore:

$$M < \frac{GC_{SW}^W}{GC_{GC}^W} \quad \forall M > 0.$$

We next show the second inequality for the government cost ratio (using IM relative to SW):

$$\frac{GC_{SW}^W}{GC_{IM}^W} \geq 4.$$

It is sufficient to show that $\frac{r_{SW}^W}{r_{IM}^W} \geq 2$ and $\frac{e'q_{SW}^W}{e'q_{IM}^W} \geq 2$. We have:

$$\frac{r_{SW}^W}{r_{IM}^W} = \frac{e'Ye - e'q_0^W}{e'Ye - e'q_0^W} \geq 2.$$
The inequality follows from $eY e \leq eY B^{-1} Y e$ by using Lemma 5 (e), and the fact that $Y B^{-1} \geq 0$ by using Lemma 5 (b). The total production quantities ratio can be written as:

$$\frac{e'q_W}{e'q_M} = \left( e'q_W + e'Y B^{-1}(\tilde{d} - Bc - q_0^W + Bk) \frac{e'Y e}{e'Y B^{-1} Y e} \right) \times \frac{1}{e'q_0^W + \frac{e'Y e - e'q_W}{2}}$$

$$\geq 2 \frac{e'q_0^W + e'Y k + e'Y B^{-1}(\tilde{d} - Bc - q_0^W)}{e'q_0^W + e'Y k}$$

$$= 2 \frac{e'q_0^W + e'Y k + e'Y B^{-1}(Y^{-1} - B^{-1})(\tilde{d} - Bc)}{e'q_0^W + e'Y k} \geq 2.$$

The first inequality follows from $eY e \leq eY B^{-1} Y e$ by using Lemma 5 (e). The second inequality follows from $e'Y B^{-1} Y (Y^{-1} - B^{-1})(\tilde{d} - Bc) \geq 0$, using that: $Ye \geq 0$ by Lemma 5 (d), $B^{-1} Y \geq 0$ by Lemma 5 (b), and $Y^{-1} \geq B^{-1}$ as shown in the proof of Lemma 5 (e).

### B.7 Proof of Proposition 11

Before presenting the proof, we define $z_i = F_i^{-1}\left(1 - \frac{p_i}{p_N}\right)$ for $i \in \{1, \ldots, n\}$, and the function $\Theta: \mathbb{R}^n \to \mathbb{R}^n$ such that $\Theta_i(z) = \int_z^{A_i} (\epsilon_i - z) f_i(\epsilon_i) \, d\epsilon_i$. The proof is divided in the following steps:

1. We first show that at least one of the prices for the monopolist setting is larger, i.e., $p_N \geq p^W$. For each setting (with and without competition), one can write a system of $n + 1$ optimal equations in the $n + 1$ decision variables ($p$ and $r$). The first $n$ equations are obtained from the first order conditions on the prices, and are given by (for more details, see Appendix B.1):

   **Competition:** $\nabla \pi(p) = \tilde{d} - B(p - er) - \Theta(z) - D(p - c)$,

   **Non-Competition:** $\nabla \pi(p) = \tilde{d} - B(p - er) - \Theta(z) - B(p - c)$. 

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The last equation is obtained by optimizing the government problem and is the same in both settings (using the tightness of the adoption constraint):

\[ e' \left( \tilde{d} - B(p - r) - \Theta(z) \right) = \Gamma. \quad (B.1) \]

By summing up the first \( n \) equations (from the \( n \) different products) and subtracting equation (B.1) from this sum, we obtain:

\[
\text{Competition: } \sum_{i=1}^{n} B_{ii}(p_{i}^w - c_i) = \Gamma, \quad (B.2) \\
\text{Non-Competition: } \sum_{i=1}^{n} \left( B_{ii} - \sum_{j \neq i} B_{ij} \right)(p_{i}^N - c_i) = \Gamma. \quad (B.3)
\]

By comparing the above two equations and using the assumption that the matrix \( B \) is strictly diagonal dominant, one can see that at least one of the Non-Competition prices has to be larger, i.e., \( p^N > p^W \).

2. We next show that all the prices under competition are larger, i.e., \( p_i^N \geq p_i^W \) \( \forall i = 1, \ldots, n \). We present the proof for the case with \( n = 2 \) in order to simplify the illustration. One can extend the same argument in an iterative fashion for \( n > 2 \). From the previous step, we know that at least for one product (without loss of generality, product 1), we have \( p_1^N \geq p_1^W \). We assume by contradiction that \( p_2^N < p_2^W \). We then look at the optimality equation for product 2 in both settings. By comparing the two equations, one can see that we need to require \( r^N < r^W \) (the details are omitted for conciseness). Therefore, we have: \( p_1^N - r^N > p_1^W - r^W \). In addition, by still looking at the same equations, one can see that: \( p_2^N - r^N > p_2^W - r^W \). We next look at the last equation (the tightness of the adoption constraint). Recall that this equation is the same for both settings. For \( n = 2 \), we have:

\[
\tilde{d}_1 + \tilde{d}_2 - B_{11}(p_1 - r) - B_{22}(p_2 - r) - B_{12}(p_2 - r) - B_{21}(p_1 - r) - \Theta_1(z_1) - \Theta_2(z_2) = \Gamma.
\]
We know that: \( p_N^2 - r_N > p_W^2 - r_W, r_N < r_W \) and \( p_N^1 > p_W^1 \). In addition, since \( p_N^2 < p_W^2 \), we also have \( \Theta_2(z_N^2) > \Theta_2(z_W^2) \). By using Assumption 2 and the strict diagonal dominance of the matrix \( B \), we obtain a contradiction so that the adoption constraint cannot be achieved in both settings. Therefore, we conclude that \( p_N^1 > p_W^1 \) and \( p_N^2 > p_W^2 \), or more generally \( p_i^N > p_i^W \) for \( i = 1, 2, \ldots, n \).

3. We next show that at least one of the effective prices for the monopolist setting is larger, i.e., \( p_N - r_N \geq 1 \) \( p_W - r_W \). From the previous step, we have \( p_i^N \geq p_i^W \) \( i = 1, 2, \ldots, n \) and hence, \( \Theta_i(z_i^N) \leq \Theta_i(z_i^W) \) \( i = 1, 2, \ldots, n \). In order to satisfy the adoption constraint in both settings, we must have \( p_N - r_N \geq 1 \) \( p_W - r_W \).

4. We next show the inequality for the rebates, i.e., \( r_N \geq r_W \). By again looking at the adoption target constraint and using the fact that \( p_i^N \geq p_i^W \) \( i = 1, 2, \ldots, n \), one can see that in order to satisfy the constraint in both settings, we need to have \( r_N \geq r_W \).

5. We next show the inequality for the production quantities, i.e., \( q_N \geq q_W \). Note that the sum of all the quantities is given by: \( e'q = e'(\bar{d} - B(p - er) + z) = \Gamma + e'(\Theta(z) + z) \). We next show that \( e'q \) is non-increasing in each component of the vector \( z \). We have:

\[
\frac{d (e'q)}{dz_i} = \int_{z_i}^{A_i} (\epsilon_i - z_i) f(\epsilon_i) d\epsilon_i + \int_{-A_i}^{z_i} \int_{z_i}^{A_i} z_i f(\epsilon_i) d\epsilon_i + \int_{z_i}^{A_i} z_i f(\epsilon_i) d\epsilon_i = F_i(z_i) \geq 0.
\]

Recall that we have shown that \( p_N \geq p_W \) and as a result, \( z_N \geq z_W \). Therefore, we have \( e'q_N \geq e'q_W \) and consequently, at least one component of \( q_N \) must be larger.

6. We next show the inequality for the profits: \( \Pi_N \geq \Pi_W \). Consider \( \Pi_N(p, z, r) \) and \( \Pi_W(p, z, r) \) evaluated at a given value of the rebate \( r \). We have:

\[
\Pi_N(p_N^N, z_N^N, r_N^N) = \max_{p, z} \Pi_N(p, z, r_N^N) \geq \max_{p, z} \Pi_N(p, z, r_W^N) \geq \Pi_N(p_W^N, z_W^N, r_W^N).
\]

The first equality comes from the definition of \( p_N^N \) and \( z_N^N \). Then, the first
inequality follows from the fact that \( r^N \geq r^W \) (shown in step 4). Indeed, we have:
\[
\Pi^N \left( p + e(r^N - r^W), z, r^N \right) = \left( p + e(r^N - r^W) \right) \left( \tilde{d} - B(p - \epsilon r^W) \right) - c' \left( z + \tilde{d} - B(p - \epsilon r^W) \right) \geq p' \left( \tilde{d} - B(p - \epsilon r^W) \right) - c' \left( z + \tilde{d} - B(p - \epsilon r^W) \right) = \Pi^N \left( p, z, r^W \right).
\]
In other words, one can increase the price \( p \) by the difference in rebates \( r^N - r^W \) and increase the profit. Finally, the second inequality follows from the feasibility of \( p^W \) and \( z^W \).

7. Finally, we show the inequality for the government cost, i.e., \( GC^N \geq GC^W \). We have:
\[
GC^N = r^N e' \mathbb{E} \left[ \min\{d, q^N\} \right] = r^N \Gamma \geq r^W \Gamma = r^W e' \mathbb{E} \left[ \min\{d, q^W\} \right] = GC^W,
\]
since we have shown in step 4 that \( r^N \geq r^W \).

### B.8 Proof of Proposition 12

After solving problem (3.11) for the cases with and without competition, we obtain:
\[
p^N - r^N = B^{-1}(\tilde{d} - q^N_0) - \frac{e'(\Gamma - e'q^N_0)}{e'Be} \quad \text{and} \quad p^W - r^W = B^{-1}(\tilde{d} - q^W_0) + \frac{e'(\Gamma - e'q^W_0)}{e'Ye} B^{-1} Ye.
\]
Therefore, we obtain:
\[
p^N - r^N - (p^W - r^W) = B^{-1}(\tilde{d} - q^N_0) - \frac{e'(\Gamma - e'q^N_0)}{e'Be} \quad \text{and} \quad B^{-1}(\tilde{d} - q^W_0) + \frac{e'(\Gamma - e'q^W_0)}{e'Ye} B^{-1} Ye.
\]

If the elasticity terms are symmetric (i.e., \( B_{ij} = B_{lk} \) \( i \neq j, l \neq k \) and \( B_{ii} = B_{jj} \)), we have \( B^{-1} Ye = \mu e \) for some \( \mu \geq 0 \), and then \( \frac{1}{e'Y} B^{-1} Ye = \frac{\mu}{\mu e'Be} e = \frac{1}{e'Be} e \). Therefore:
\[
p^N - r^N - (p^W - r^W) = B^{-1}(q^W_0 - q^N_0) - \frac{e'(q^W_0 - q^N_0)}{e'Be} e
\]
\[
= B^{-1}(q^W_0 - q^N_0) - \frac{e'BB^{-1}(q^W_0 - q^N_0)}{e'Be} e
\]
\[
= (I - \frac{ee'B}{e'Be}) B^{-1}(q^W_0 - q^N_0)
\]
\[
= (I - \frac{ee'B}{e'Be})(B^{-1} B^{-1} - \frac{B^{-1}}{2})(\tilde{d} - Bc)
\]
\[
= (I - \frac{ee'B}{e'Be}) \left[ (B + BD^{-1}B)^{-1} - \frac{B^{-1}}{2} \right] (\tilde{d} - Bc).
\]

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Since all the elasticities are symmetric, we denote by $\alpha = B_{ii}$ and $\beta = B_{ij} \ i \neq j$. Then, $B^{-1}$ has only two different terms: $\frac{\alpha + (n-2)\beta}{\alpha + (n-2)\beta - 1}$ and $\frac{\alpha - (n-2)\beta}{\alpha - (n-2)\beta - 1}$ in its diagonal and off-diagonal respectively. Let $a = [(B + BD^{-1}B)^{-1} - \frac{B^{-1}}{2}]_{ii}$ and $b = [(B + BD^{-1}B)^{-1} - \frac{B^{-1}}{2}]_{ij}$. One can obtain: $b - a = \frac{-\beta}{2(\alpha - \beta)(\alpha - \beta)} \geq 0$

Note that $(I - \frac{1}{n}ee')(B + BD^{-1}B)^{-1} - \frac{B^{-1}}{2}$ is a symmetric matrix with $-\frac{1}{n}(b - a)$ and $\frac{1}{n}(b - a)$ in its diagonal and off-diagonal respectively. Note also that each row sums to zero. We have:

$$p^N - r^N - (p^W - r^W) = (I - \frac{1}{n}ee')[(B + BD^{-1}B)^{-1} - \frac{B^{-1}}{2}](\bar{d} - Bc)$$

$$= -(b - a)[(\bar{d} - Bc) - e\frac{e'(\bar{d} - Bc)}{n}]$$

$$= -(b - a)[(\bar{d} - (\alpha - \beta)c - \beta e'ce) - e\frac{e'(\bar{d} - (\alpha - \beta)c - \beta e'ce)}{n}]$$

$$= -(b - a)[(\bar{d} - (\alpha - \beta)c) - e\frac{e'(\bar{d} - (\alpha - \beta)c)}{n}]$$

Observe that the average (or sum) of all the changes in effective prices is zero. If $\bar{d}$ and $c$ are symmetric, we have $p^N - r^N - (p^W - r^W) = 0$ so that $q^N = q^W$. As a result, this leads to the same consumer surplus and social welfare. If there are asymmetries in $\bar{d}$ or $c$, one can see that $p^N_i - r^N_i - (p^W_i - r^W_i)$ is positive in the product with the lowest $\bar{d}_i - (\alpha - \beta)c_i$, and negative for the segment with the highest one. Consequently, if there are asymmetries only in $\bar{d}$, the product with the lowest $\bar{d}_i$ will have $p^N_i - r^N_i > p^W_i - r^W_i$, and the opposite for the one with the highest $\bar{d}_i$. A similar result applies for asymmetries in $c$, but with the signs reversed. Finally, note that we have:

$$e'(p^N - r^N - (p^W - r^W)) = -(b - a)(e'(\bar{d} - (\alpha - \beta)c) - e\frac{e'(\bar{d} - (\alpha - \beta)c)}{n}) = 0.$$  

B.9 Proof of Corollary 2

We show the inequality for the expected consumer surplus in the symmetric case, i.e., $CS^N \leq CS^W$. First, we define $z_i = F_i^{-1}\left(1 - \frac{\alpha}{p_i}\right)$ for $i \in \{1, \ldots, n\}$, and the functions
\( \Theta, \Xi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \Theta_i(z) = \int_{z}^{A_i} (\epsilon_i - z) f_i(\epsilon_i) \, d\epsilon_i \), and \( \Xi_i(z) = \int_{-A_i}^{z} (\epsilon_i - z) f_i(\epsilon_i) \, d\epsilon_i \). Then, the consumer surplus can be written as follows:

\[
CS(p, z) = \frac{1}{2} \left( \tilde{d} - B(p - er) \right)' B^{-1} \left( \tilde{d} - B(p - er) - \Theta(z) \right) + \frac{1}{2} e' D_{B^{-1}} \Xi(z) \\
= \frac{1}{2} \left( \tilde{d} - B(p - er) - \Theta(z) \right)' B^{-1} \left( \tilde{d} - B(p - er) - \Theta(z) \right) + \frac{1}{2} e' D_{B^{-1}} \Xi(z) \\
+ \frac{1}{2} \Theta(z)' B^{-1} (\tilde{d} - B(p - er) - \Theta(z)) \\
= \frac{\Gamma^2 e' B^{-1} e}{2n^2} + \frac{1}{2} \left( e' D_{B^{-1}} \Xi(z) + \frac{\Gamma}{n} e' B^{-1} \Theta(z) \right).
\]

The last equality follows from the symmetry of the suppliers so that each firm sells \( \Gamma/n \), in expectation. Therefore: \( E[\min(d, q)] = \tilde{d} - B(p - er) - \Theta(z) = \frac{\Gamma}{n} e \). We define the function \( \Omega : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \Omega(z) = \sum_{i=1}^{n} e_i \int_{z}^{A_i} \epsilon_i f_i(\epsilon_i) \, d\epsilon_i \). We next compute the gradient of the expected consumer surplus with respect to \( z \):

\[
\nabla_z CS = \frac{1}{2} D_{B^{-1}} (\Omega(z) + D_{\tilde{F}(z)} z) - \frac{1}{2n} \Gamma B^{-1} \tilde{F}(z) \\
\leq \frac{1}{2} D_{B^{-1}} (\Omega(z) + D_{\tilde{F}(z)} z) - \frac{1}{2n} e' D_{B^{-1}} D_{\tilde{F}(z)} \\
= \frac{n}{2} B_{i1}^{-1} e \int_{z_1}^{A_1} \left( \epsilon - \frac{\Gamma}{n} \right) f_1(\epsilon) \, d\epsilon \leq \frac{n}{2} B_{i1}^{-1} e \int_{z_1}^{A_1} \left( A_1 - \frac{\Gamma}{n} \right) f_1(\epsilon) \, d\epsilon \leq 0_{n \times 1}.
\]

The first inequality follows from \( B^{-1} \geq 0 \), and the second equality is from the problem symmetry. The second inequality uses the fact that \( \epsilon \) is bounded by \( A_1 \) and the third inequality follows from the assumption \( e'A \leq \Gamma \). Finally, since the optimal prices satisfy \( \tilde{F}(z_i) p_i = c \), \( z \) is a non-decreasing function of \( p \). Therefore, \( p^N \geq p^W \) implies that \( z^N \geq z^W \) and this concludes the proof.

### B.10 Proof of Proposition 13

We start by presenting the proof for the case of asymmetric suppliers.

- \( q^N \leq q^W \)
The total production quantities are given by:

\[
e'q^N = \frac{k'Be + c'(d - Bc)}{4} \leq \frac{2k'Ye + 2e'YB^{-1}(d - Bc)}{4} = e'q^W.
\]

The inequality follows from Lemma 5 (b) and (d). Therefore, \( q^N \leq q^W \).

- \( p^N - r^N \geq p^W - r^W \)

We have shown that \( e'q^N \leq e'q^W \) and hence, \( e' (p^N - er^N) \geq e' (p^W - er^W) \) (since \( Be > 0 \)). Then \( p^N - r^N \geq p^W - r^W \).

- \( CS^N \leq CS^W \)

For simplicity, we consider the case when \( B \) is symmetric so that \( B_{ii} = B_{jj} \) and \( B_{ij} = B_{kl} \) for all \( i \neq j, k \neq l \) (products with the same self and cross elasticities).

We have:

\[
CS^N = \left( \frac{q_0^N + Be k'Be - 2e'q_0^N}{2e'Be} \right)^\prime B^{-1} \left( \frac{q_0^N + Be k'Be - 2e'q_0^N}{2e'Be} \right) \\
= \left( \frac{q_0^N + Be k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye} \right)^\prime B^{-1} \left( \frac{q_0^N + Be k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye} \right) \\
\leq \left( 2YB^{-1}q_0^N + Ye \frac{k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye} \right)^\prime B^{-1} \\
\times \left( 2YB^{-1}q_0^N + Ye \frac{k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye} \right) = CS^W.
\]

The first equality follows from the symmetry, and the inequality uses Lemma 5 (b) and (d).

We next consider symmetric suppliers.

- \( r^N = r^W \)

We have:

\[
r^N = \frac{k'Be - 2e'q_0^N}{2e'B} = k'Be - 2 \beta e' q_0^N = k'Ye - 2e'YB^{-1}q_0^N = r^W.
\]

The second equality follows from multiplying the numerator and the denominator by any scalar \( \beta \neq 0 \). The third equality is obtained by using the fact that
the suppliers are symmetric: $YB^{-1}e = \beta e$ for some $\beta > 0$.

- $p^N \geq p^W$

We have:

$$p^N = B^{-1}(\bar{d} - q_0^N) + \frac{k'Be - 2e'q_0^N}{4e'Be}e = B^{-1}(\bar{d} - q_0^N) + \frac{k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye}e$$

$$\geq B^{-1}(\bar{d} - 2YB^{-1}q_0^N) + \frac{k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye}e$$

$$\geq B^{-1}(\bar{d} - 2YB^{-1}q_0^N) + \frac{k'Ye - 2e'DYB^{-1}q_0^N}{2e'Ye}(I - B^{-1}Y)e = p^W.$$

The second equality uses the fact that $r^N = r^W$, the first inequality uses Lemma 5 (b), and the second inequality uses the fact that $I - B^{-1}Y \leq 1/2$ which follows from Lemma 5 (b).

- $p^N - r^N \geq p^W - r^W$ and $q^N \leq q^W$ follow directly from the asymmetric case shown above.

- $\Pi^N \geq \Pi^W$

The profit in the monopolistic setting is given by:

$$\Pi^N = \left( q_0^N + \frac{Be k'Be - 2e'q_0^N}{2} \right)' B^{-1} \left( q_0^N + \frac{Be k'Be - 2e'q_0^N}{2} \right).$$

The profit in the competitive environment is given by:

$$\Pi^W = \left( 2YB^{-1}q_0^N + Ye \frac{k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye} \right)' D^{-1}$$

$$\times \left( 2YB^{-1}q_0^N + Ye \frac{k'Ye - 2e'YB^{-1}q_0^N}{2e'Ye} \right)$$

$$\geq \left( 2YB^{-1}q_0^N + Ye \frac{k'Be - 2e'q_0^N}{2e'Be} \right)' D^{-1} \left( 2YB^{-1}q_0^N + Ye \frac{k'Be - 2e'q_0^N}{2e'Be} \right)$$

$$= \left( q_0^N + \frac{Be k'Be - 2e'q_0^N}{2e'Be} \right)' 4B^{-1}YD^{-1}YB^{-1} \left( q_0^N + \frac{Be k'Be - 2e'q_0^N}{2e'Be} \right).$$
We need to show that $4B^{-1}YD^{-1}YB^{-1} \leq B^{-1}$. We have:

$$4B^{-1}YD^{-1}YB^{-1} = 4B^{-1}YD^{-1}DD^{-1}YB^{-1} \leq B^{-1}DB^{-1} \leq B^{-1}.$$ 

The first inequality follows from $YD^{-1} \leq I/2$, since $YD^{-1} + YB^{-1} = I$, and using Lemma 5 (b). The second inequality follows from Lemma 5 (a). Therefore, $\Pi^N \geq \Pi^W$.

- $GC^N \leq GC^W$

Since $r^N = r^W$ and $q^N \leq q^W$, we have: $GC^N = q^N r^N \leq GC^W = q^W r^W$.

### B.11 Additional Lemmas

**Lemma 4.** *(Inverse of M-matrices)* Let $A, B \in \mathbb{R}^{n \times n}$ be M-Matrices so that $A \geq B$, then $A^{-1} \leq B^{-1}$.

*Proof.* Proof. See *Matrix Analysis 2012, Horn and Johnson.*

**Lemma 5.** Let $B \in \mathbb{R}^{n \times n}$ be a symmetric strictly diagonally dominant M-Matrix, $D \in \mathbb{R}^{n \times n}$ its diagonal, and $Y = (B^{-1} + D^{-1})^{-1}$ then:

(a) $B^{-1}D \geq I$.

(b) $YB^{-1} \geq I/2$.

(c) $Be > 0$.

(d) $Ye \geq Be/2$ ($> 0$).

(e) $eYe \leq eYB^{-1}Ye$.

(f) For given $d, B, c, k$ and cdf distributions $F_i$ for each $\epsilon_i$, $i \in \{1, \ldots, n\}$, $\gamma^N \leq \gamma^W$.

*Proof.* Proof.

(a) $B^{-1}D \geq I$. It follows from Lemma 4 since $B$ and $D$ are M-matrices such that $B \leq D$. 

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(b) \( YB^{-1} \geq I/2 \). Note that \( YB^{-1} = (B^{-1} + D^{-1})^{-1} = (B(B^{-1} + D^{-1}))^{-1} = (I + BD^{-1})^{-1} \). By using Lemma 4, it suffices to show that \( I + BD^{-1} \leq 2I \). The latter holds since \( BD^{-1} \leq I \) by using Lemma 5 (a).

(c) \( Be > 0 \). Note that for each \( i \in \{1, \ldots, n\} \), \( [Be]_i = \sum_{j=1}^n B_{ij} = B_{ii} + \sum_{j=1, i \neq j}^n B_{ij} \geq B_{ii} - \sum_{j=1, i \neq j}^n |B_{ij}| > 0 \) since \( B \) is strictly diagonal dominant.

(d) \( Ye \geq Be/2 \ (> 0) \). By Lemma 5 (b), we have \( YB^{-1} \geq I/2 \). Since \( Be > 0 \), we also have \( YB^{-1}Be \geq (I/2)Be \), and then \( Ye \geq Be/2 \). In addition, \( Be/2 > 0 \) follows from part (c).

(e) \( eYe \leq eYB^{-1}Ye \). Note that \( eYe = eYY^{-1}Ye \). Since \( Ye > 0 \) by using Lemma 5 (d), we only need to show that \( Y^{-1} \geq B^{-1} \). Since \( D^{-1} > 0 \), we have \( Y^{-1} = B^{-1} + D^{-1} > B^{-1} \), and then \( eYe \leq eYB^{-1}Ye \).

(f) For given \( \bar{d}, B, c, k \) and cdf distributions \( F_i \) for each \( \epsilon_i; i \in \{1, \ldots, n\} \), \( \gamma^N \leq \gamma^W \).
Recall that: \( \gamma^N = \frac{k}{p^N_{i} - c_i} \) and \( \gamma^W = \frac{k}{p^W_{i} - c_i} \), where \( p^N \) and \( p^W \) are the equilibrium prices when \( r = 0 \). Then, showing \( \gamma^N \leq \gamma^W \) is equivalent to show that \( p^N \geq p^W \).

From the first order conditions of the suppliers' problems (assuming \( r = 0 \)), we have: \( p^N = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p^N)] \) and \( p^W = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p^W)] - \frac{B^{-1}}{2} (D - B)(p^W - c) \) for the monopolist and competitive settings respectively. Denote the function \( g : \mathbb{R}^n \to \mathbb{R}^n \) such that \( g(p) = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p)] \). Consider \( p^{(0)} = p^W \), and \( p^{(i)} = g(p^{(i-1)}) \) for \( i \geq 1 \). Note that \( p^{(1)} = g(p^{(0)}) = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p^{(0)})] = p^{(0)} + \frac{B^{-1}}{2} (D - B)(p^{(0)} - c) \geq p^{(0)} \). In order to show that \( p^{(i)} \geq p^{(i-1)} \) for all \( i \geq 2 \), consider a given \( i \geq 2 \), and assume by induction that \( p^{(i-1)} \geq p^{(i-2)} \). Note that \( p^{(i)} = g(p^{(i-1)}) = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p^{(i-1)})] = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p^{(0)})] + \sum_{j=1}^{i-1} \frac{B^{-1}}{2} [\Psi(p^{(j)}) - \Psi(p^{(j-1)})] = p^{(i-1)} + \sum_{j=1}^{i-1} \frac{B^{-1}}{2} [\Psi(p^{(i-1)}) - \Psi(p^{(i-2)})] \geq p^{(i-1)} \). The last inequality follows from \( B^{-1} \geq 0, J_p \Psi(p) \geq 0, \) and \( p^{(i-1)} \geq p^{(i-2)} \). We then conclude that \( p^{(i)} \geq p^{(i-1)} \), for all \( i \geq 1 \). In addition, the sequence is bounded above by \( \frac{B^{-1}}{2} (\bar{d} - Bc) \), as \( p^{(i)} = \frac{B^{-1}}{2} [\bar{d} + Bc + \Psi(p^{(i-1)})] \leq \frac{B^{-1}}{2} (\bar{d} - Bc) \) since \( \Psi(p) \leq 0 \). As a result, the sequence \( p^{(i)} \) converges, as \( i \to \infty \). Moreover, it converges to \( p^N \) (since \( p^N = g(p^N) \)). Therefore, \( p^N \geq p^W \), and \( \gamma^N \leq \gamma^W \).
Lemma 6. Let $y, z \in \mathbb{R}^n$ be non negative, and $C \in \mathbb{R}^{n \times n}$ be a symmetric positive semidefinite matrix. Then, $z'Czy'C^{-1}y - (y'z)^2 \geq 0$.

Proof. Proof. Let $x = \sqrt{z'Cz} C^{1/2} (\frac{zy'}{z'Cz} - I) y$.

\[
v'v = z'Cy' \left( \frac{zz'C}{z'Cz} - I \right) C^{-1} \left( \frac{zz'C}{z'Cz} - I \right) y = z'Cy' \left( \frac{zz'C}{z'Cz} - I \right) \left( \frac{zz'C}{z'Cz} - C^{-1} \right) y
\]

\[
= z'Cy' \left( \frac{zz'C}{z'Cz} - \frac{zz'C}{z'Cz} + C^{-1} \right) y
\]

\[
= z'Cy' \left( C^{-1}y - \frac{zz'C}{z'Cz} \right) y = z'Cy'C^{-1}y - y'zz'y.
\]

Note that the positive semi definitiveness ensures a unique square root of $C$. Nevertheless, this assumption can be relaxed and the Lemma still holds. □
Appendix C

Appendix of Chapter 4

Proof of Observation 1

If the quantity produced exceeds the demand, i.e., \( q > d(p, \varepsilon) \) then we have \( CS^H(\varepsilon) = CSA(\varepsilon) \). We next consider the case where \( q \leq d(p, \varepsilon) \). We have:

\[
CS^H(\varepsilon) - CSA(\varepsilon)
= \int_0^{d(p, \varepsilon)} 1_{\{w \leq q\}} (d^{-1}(w, \varepsilon) - p) \, dw - \int_0^{d(p, \varepsilon)} A(w)(d^{-1}(w, \varepsilon) - p) \, dw
= \int_0^{d(p, \varepsilon)} (1 - A(w))(d^{-1}(w, \varepsilon) - p) \, dw + \int_0^{d(p, \varepsilon)} A(w)(d^{-1}(w, \varepsilon) - p) \, dw
\geq \int_0^{q} (1 - A(w))(d^{-1}(q, \varepsilon) - p) \, dw + \int_q^{d(p, \varepsilon)} A(w)(d^{-1}(q, \varepsilon) - p) \, dw
= (d^{-1}(q, \varepsilon) - p) \left( q - \int_0^{d(p, \varepsilon)} A(w) \, dw \right)
= (d^{-1}(q, \varepsilon) - p) \left( q - d(p, \varepsilon) \frac{q}{d(p, \varepsilon)} \right)
= 0
\]

We next consider any allocation rule \( A \). If the quantity produced exceeds the demand, i.e., \( q > d(p, \varepsilon) \) then we have \( CSA(\varepsilon) = CS^L(\varepsilon) \). We next consider the case
where \( q \leq d(p, \varepsilon) \):

\[
CS^A(\varepsilon) - CS^L(\varepsilon)
= \int_0^{d(p, \varepsilon)} A(w)((d^{-1}(w, \varepsilon) - p)\, dw - \int_0^{d(p, \varepsilon)} \mathbb{1}_{\{w \geq d(p, \varepsilon) - q\}}(d^{-1}(w, \varepsilon) - p)\, dw
= \int_0^{d(p, \varepsilon)} (A(w) - \mathbb{1}_{\{w \geq d(p, \varepsilon) - q\}})(d^{-1}(w, \varepsilon) - p)\, dw
\geq \int_0^{d(p, \varepsilon) - q} A(w)(d^{-1}(q, \varepsilon) - p)\, dw + \int_{d(p, \varepsilon) - q}^{d(p, \varepsilon)} (A(w) - 1)(d^{-1}(q, \varepsilon) - p)\, dw
= (d^{-1}(q, \varepsilon) - p)\left(\int_0^{d(p, \varepsilon)} A(w)\, dw - q\right)
= (d^{-1}(q, \varepsilon) - p)\left(d(p, \varepsilon)\frac{q}{d(p, \varepsilon)} - 1\right)
= 0
\]

**Proof of Observation 2**

Consider a price vector \( p \), a quantity vector \( q \), a noise realization \( \varepsilon \in \Omega \), a path \( C^\varepsilon \) defined by a parametric function \( r^\varepsilon \) and any allocation rule \( A \). For simplicity assume the path is non-decreasing on each component. For each \( i \in \{1, \ldots, n\} \), partition of path \( C^\varepsilon \) into the two following paths: \( C^\varepsilon_{1,i} = \{ y \in \mathbb{R}^n | y = r^\varepsilon(w), r^\varepsilon_i(w) \leq q_i, w \in [a, b] \} \) and \( C^\varepsilon_{2,i} = \{ y \in \mathbb{R}^n | y = r^\varepsilon(w), r^\varepsilon_i(w) > q_i, w \in [a, b] \} \). Given an \( \varepsilon \), consider \( \tilde{q}^i = \arg\min_{x \in C^\varepsilon_{1,i}} \{ d^{-1}_i(x, \varepsilon) \} \) (if the solution is not a singleton, select any among these
solutions).

\[
\begin{align*}
\text{CS}^H(\epsilon) - \text{CS}^A(\epsilon) \\
= \int_{C^*} \sum_{i=1}^{n} \left( d_i^{-1}(r^\epsilon, \epsilon) - p_i \right) \mathbb{I}_{\{r^\epsilon < q_i\}} \, dr_i^\epsilon - \int_{C^*} \sum_{i=1}^{n} \left( d_i^{-1}(r^\epsilon, \epsilon) - p_i \right) \mathbb{A}_i(r^\epsilon) \, dr_i^\epsilon \\
= \sum_{i=1}^{n} \int_{C^*_{1,i}} \left( d_i^{-1}(r^\epsilon, \epsilon) - p_i \right) \left( 1 - \mathbb{A}_i(r^\epsilon) \right) \, dr_i^\epsilon - \sum_{i=1}^{n} \int_{C^*_{2,i}} \left( d_i^{-1}(r^\epsilon, \epsilon) - p_i \right) \mathbb{A}_i(r^\epsilon) \, dr_i^\epsilon \\
\geq \sum_{i=1}^{n} \int_{C^*_{1,i}} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \left( 1 - \mathbb{A}_i(r^\epsilon) \right) \, dr_i^\epsilon - \sum_{i=1}^{n} \int_{C^*_{2,i}} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \mathbb{A}_i(r^\epsilon) \, dr_i^\epsilon \\
= \sum_{i=1}^{n} \int_{C^*_{1,i}} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \, dr_i^\epsilon - \sum_{i=1}^{n} \int_{C^*_{2,i}} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \mathbb{A}_i(r^\epsilon) \, dr_i^\epsilon \\
= \sum_{i=1}^{n} \int_{C^*} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \mathbb{I}_{\{r^\epsilon < q_i\}} \, dr_i^\epsilon - \sum_{i=1}^{n} \int_{C^*} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \mathbb{A}_i(r^\epsilon) \, dr_i^\epsilon \\
= \sum_{i=1}^{n} \min\{q_i, d_i(p, \epsilon)\} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) - \sum_{i=1}^{n} \min\{q_i, d_i(p, \epsilon)\} \left( d_i^{-1}(\tilde{q}_i^1, \epsilon) - p_i \right) \\
= 0
\end{align*}
\]

The inequality is due to two facts: first, \(d_i^{-1}(x, \epsilon) \geq d_i^{-1}(\tilde{q}_i, \epsilon)\) for \(x \in C_{1,i}\) by definition of \(\tilde{q}_i\). Second, \(d_i^{-1}(\tilde{q}_i, \epsilon) \geq d_i^{-1}(x, \epsilon)\) for \(x \in C_{2,i}\) since all \(x \geq \tilde{q}_i\) for all \(x \in C_{2,i}\) (due to the non-decreasing of the parametric function), and the fact that the inverse demand function is non-increasing in each of the quantities (because the Jacobian in the quantities of the inverse demand function is non-positive).

To show now that \(CS^A(\epsilon) \geq CS^L(\epsilon)\), consider a similar setting as before until just before defining the path partitions. For each \(i \in \{1, \ldots, n\}\), consider the partition of path \(C^\epsilon\) into the two following paths: \(C_{3,i}^\epsilon = \{y \in \mathbb{R}^n | y = r^\epsilon(w), r_i^\epsilon(w) < d_i(p, \epsilon) - q_i, w \in [a, b]\}\) and \(C_{4,i}^\epsilon = \{y \in \mathbb{R}^n | y = r^\epsilon(w), r_i^\epsilon(w) \geq d_i(p, \epsilon) - q_i, w \in [a, b]\}\). Consider \(\tilde{q}_i = \arg\max_{x \in C_{3,i}^\epsilon} \{d_i^{-1}(x, \epsilon)\}\) (if the solution is not a singleton, select any among these
The inequality is due to two facts: first, \( d_i^{-1}(x, \epsilon) \leq d_i^{-1}(q^i, \epsilon) \) for \( x \in C_{4,i}^\epsilon \) by definition of \( q^i \). Second, \( d_i^{-1}(q^i, \epsilon) < d_i^{-1}(x, \epsilon) \) for \( x \in C_{3,i}^\epsilon \) since all \( x \leq q^i \) for all \( x \in C_{3,i}^\epsilon \) (due to the non-decreasing of the parametric function), and the fact that the inverse demand function is non-increasing in each of the quantities (because the Jacobian in the quantities of the inverse demand function is non-positive).
Proof of Proposition 1

We show the result for the H rule. Then, by using Observation 1, we conclude that the result holds for any allocation rule. We have:

\[
\mathbb{E}[CS^H(\epsilon)] = \mathbb{E} \left[ \int_0^{d(p^0,\epsilon)} (d^{-1}(w,\epsilon) - p^0) \mathbb{1}_{\{w \leq \epsilon\}} \, dw \right]
\]

\[
= \mathbb{E} \left[ \int_0^{ed(p^0)} \left( d^{-1} \left( \frac{w}{\epsilon} \right) - p^0 \right) \mathbb{1}_{\{w \leq \epsilon\}} \, dw \right]
\]

\[
= \mathbb{E} \left[ \int_0^{d(p^0)} (d^{-1}(v) - p^0) \mathbb{1}_{\{w \leq \epsilon\}} \, dv \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^{d(p^0)} (d^{-1}(v) - p^0) \epsilon \, dv \right]
\]

\[
= \int_0^{d(p^0)} (d^{-1}(v) - p^0) \, dv
\]

\[
= CS_{det},
\]

where the last equality follows from \( \mathbb{E}[\epsilon] = 1 \).

Proof of Proposition 2

- H rule

We have:

\[
\mathbb{E}[CS^H(\epsilon)]
\]

\[
= \mathbb{E} \left[ \int_0^{d(p^0)+\epsilon} \mathbb{1}_{\{w \leq p^0\}} (d^{-1}(w - \epsilon) - p^0) \, dw \right]
\]

\[
\geq \mathbb{E} \left[ \int_0^{d(p^0)+\epsilon} \mathbb{1}_{\{w \leq d(p^0)\}} (d^{-1}(w - \epsilon) - p^0) \, dw \right]
\]

\[
= \mathbb{E} \left[ \mathbb{1}_{\{\epsilon \leq 0\}} \int_0^{d(p^0)+\epsilon} (d^{-1}(w - \epsilon) - p^0) \, dw + \mathbb{1}_{\{\epsilon > 0\}} \int_0^{d(p^0)} (d^{-1}(w - \epsilon) - p^0) \, dw \right]
\]

We denote by \( A(\epsilon) \) the argument of the expectation in the last equation. We next show that \( A(\epsilon) \) is convex. The first and second derivatives of the first term
of $A(\epsilon)$ are given by:

\[
\frac{d}{d\epsilon} \int_0^{d(p^0)+\epsilon} (d^{-1}(w-\epsilon) - p^0) \, dw = \frac{d^{-1}(d(p^0)) - p^0}{dx} - \int_0^{d(p^0)+\epsilon} \frac{dd^{-1}(w-\epsilon)}{dx} \, dw
\]

\[
= - \int_{-\epsilon}^{d(p^0)} \frac{dd^{-1}(x)}{dx} \, dx \quad (x = w - \epsilon)
\]

\[
= d^{-1}(-\epsilon) - p^0
\]

\[
\frac{d^2}{d\epsilon^2} \int_0^{d(p^0)+\epsilon} (d^{-1}(w-\epsilon) - p^0) \, dw = -\frac{dd^{-1}(-\epsilon)}{dx} \geq 0
\]

Regarding the second term of $A(\epsilon)$, we have:

\[
\frac{d}{d\epsilon} \int_0^{d(p^0)} (d^{-1}(w-\epsilon) - p^0) \, dw = -\int_0^{d(p^0)-\epsilon} \frac{dd^{-1}(w-\epsilon)}{dx} \, dw
\]

\[
= - \int_{-\epsilon}^{d(p^0)-\epsilon} \frac{dd^{-1}(x)}{dx} \, dx \quad (x = w - \epsilon)
\]

\[
= -d^{-1}(d(p^0) - \epsilon) + d^{-1}(-\epsilon)
\]

\[
\frac{d^2}{d\epsilon^2} \int_0^{d(p^0)} (d^{-1}(w-\epsilon) - p^0) \, dw = \frac{d^{-1}(d(p^0) - \epsilon)}{dx} - \frac{d^{-1}(-\epsilon)}{dx} \geq 0
\]

Note that the last inequality follows from the convexity of $d(p)$. As a result, $A(\epsilon)$ is convex for both $\epsilon \leq 0$ and $\epsilon > 0$. In addition, the derivatives of the two terms of $A(\epsilon)$ coincide when $\epsilon = 0$ and therefore, $A(\epsilon)$ is convex. One can now use Jensen Inequality as follows:

\[
\mathbb{E}[CS^H(\epsilon)] = \mathbb{E} \left[ \int_0^{d(p^0)+\epsilon} \mathbb{I}_{(w \leq \phi^t \omega)}(d^{-1}(w-\epsilon) - p^0) \, dw \right]
\]

\[
\geq \int_0^{d(p^0)} (d^{-1}(w) - p^0) \, dw = CS_{det}.
\]

- L rule
\[ CSL(\epsilon) = \int_0^{d(p^0, \epsilon)} \left( d^{-1}(w, \epsilon) - p^0 \right) I_{\{ w \geq d(p^0, \epsilon - q^{sto}) \}} \, dw \]

\[ = \int_0^{d(p^0, \epsilon)} \left( d^{-1}(w - \epsilon) - p^0 \right) I_{\{ w \geq d(p^0, \epsilon) \}} \, dw \]

\[ \leq \int_0^{d(p^0) + \epsilon} \left( d^{-1}(w - \epsilon) - p^0 \right) I_{\{ w \geq \epsilon \}} \, dw \]

\[ = \int_{\max(0, \epsilon)}^{d(p^0) + \epsilon} \left( d^{-1}(w - \epsilon) - p^0 \right) \, dw \]

\[ = \int_{\max(-\epsilon, 0)}^{d(p^0)} \left( d^{-1}(v) - p^0 \right) \, dv \]

\[ \leq \int_0^{d(p^0)} \left( d^{-1}(v) - p^0 \right) \, dv \]

\[ = CSL_{\text{det}} \]

Therefore, for any \( \epsilon \) we have: \( CSL(\epsilon) \leq CSL_{\text{det}} \) (as long as \( q^{sto} \leq d(p^0) \)). Taking expectation over \( \epsilon \), we obtain: \( \mathbb{E}[CSL] \leq CSL_{\text{det}} \).

**Random rule**

We first state and prove the following Lemma.

**Lemma 7.** Consider \( a > 0 \) and \( f(\cdot) \) a non-negative concave function. The function \( h(x) = \frac{1}{a + x} \int_0^{a + x} f(w - x) \, dw \) is also concave (for \( x > -a \)).

**Proof.** We compute the first and second derivatives of \( h(x) \):

\[ \frac{dh(x)}{dx} = \frac{f(a)}{a + x} - \frac{1}{(a + x)^2} \int_0^{a + x} f(w - x) \, dw - \frac{1}{a + x} \int_0^{a + x} f'(w - x) \, dw \]

\[ = \frac{f(a)}{a + x} - \frac{1}{(a + x)^2} \int_0^{a + x} f(w - x) \, dw - \frac{f(a)}{a + x} + \frac{1}{a + x} f(-x) \]

\[ = -\frac{1}{(a + x)^2} \int_0^{a + x} f(w - x) \, dw + \frac{f(-x)}{a + x} \]
\[
\frac{d^2 h(x)}{dx^2} = -\frac{f(a)}{(a + x)^2} + \frac{2}{(a + x)^3} \int_0^{a+x} f(w - x)dw \\
+ \frac{1}{(a + x)^2} \int_0^{a+x} f'(w - x)dw - \frac{f'(-x)}{a + x} - \frac{f(-x)}{(a + x)^2} \\
\leq \frac{2}{(a + x)^3} \int_0^{a+x} f(w - x)dw - \frac{2f(-x)}{(a + x)^2} - \frac{f'(-x)}{a + x} \\
= 0 
\]

The inequality is due the fact that \((f(y) \leq f(x) + f'(x)(y - x))\). Note that alternatively, one can show that the function \(h(x) = \frac{1}{2} \int_0^x f(z)dz\) is concave for \(x > 0\). □

Using the random allocation rule, we have:

\[
E[CS^R(\epsilon)] = E \left[ \int_0^{d(p^0,\epsilon)} \min\left\{1, \frac{q_{sto}}{d(p^0,\epsilon)} \right\} (d^{-1}(w, \epsilon) - p^0) dw \right] \\
= E \left[ \int_0^{d(p^0)+\epsilon} \min\left\{1, \frac{q_{sto}}{d(p^0)+\epsilon} \right\} (d^{-1}(w - \epsilon) - p^0) dw \right] \\
\leq E \left[ \int_0^{d(p^0)+\epsilon} \min\left\{1, \frac{d(p^0)}{d(p^0)+\epsilon} \right\} (d^{-1}(w - \epsilon) - p^0) dw \right] \\
\leq E \left[ \int_0^{d(P)} \frac{d(p^0)}{d(p^0)+\epsilon} (d^{-1}(w - \epsilon) - p^0) dw \right] \\
\leq \int_0^{d(P)} \frac{d(p^0)}{d(p^0)} (d^{-1}(w) - P) dw \\
= CS_{det}
\]

The first inequality follows from \(q_{sto} \leq d(p^0)\) and the second inequality is obtained by dropping the minimum. The third inequality is a consequence of Lemma 7 noting that \(d^{-1}(x) - p^0\) is a non-negative concave function, i.e., we use
Lemma 7 with \( f(y) = d^{-1}(y) - p^0 \), \( a = d(p^0) \) and \( x = \epsilon \). Then, the last inequality follows from Jensen inequality.

**Proof of Proposition 3**

For multiplicative noise we will show that \( \mathbb{E}[CS^H(\epsilon)] \leq CS_{det} \) for any given path \( \mathcal{C} \).

Consider the equivalent paths \( \mathcal{C}^\epsilon \) and the parametric functions \( r^\epsilon : [a, b] \to \mathbb{R}^n \), which we approximate by the equivalent paths \( \mathcal{C}^\epsilon,m \) with the parametric functions \( r^\epsilon,m : [a, b] \to \mathbb{R}^n \) with \( m \) line segments so that \( r^\epsilon,m(w) = \frac{w_{j+1} - w_j}{w_{j+1} - w_j} r^\epsilon(w_j) + \frac{w_j - w_{j+1}}{w_{j+1} - w_j} r^\epsilon(w_{j+1}) \) for \( w \in (w_j, w_{j+1}] \) for \( j \in \{0, \ldots, m-1\} \) where \( w_j = a + \frac{b-a}{m} j \) for \( j \in \{0, \ldots, m\} \). Note that in the multiplicative demand case, it holds that \( r^\epsilon_i(w) = \epsilon_i r_i(w) \) and \( \frac{dr^\epsilon_i(w)}{dw} = \epsilon_i \frac{dr_i(w)}{dw} \), and similarly for the parametric functions of the approximation paths \( r^\epsilon,m_i(w) = \epsilon_i r^m_i(w) \) and \( \frac{dr^\epsilon,m_i(w)}{dw} = \epsilon_i \frac{dr^m_i(w)}{dw} \).

\[
\mathbb{E}[CS^H_{\mathcal{C}}(\epsilon)] = \mathbb{E}\left[ \int_{\mathcal{C}^\epsilon} d^{-1}(r^\epsilon, \epsilon) D_{1_{r^\epsilon, \epsilon, q_0}} dr^\epsilon \right]
\leq \mathbb{E}\left[ \lim_{m \to \infty} \int_{\mathcal{C}^\epsilon,m} d^{-1}(r^\epsilon,m, \epsilon) D_{1_{r^\epsilon,m, \epsilon, q_0}} dr^\epsilon,m \right]
\leq \mathbb{E}\left[ \lim_{m \to \infty} \sum_{j=0}^{m-1} \int_{w_j}^{w_{j+1}} d^{-1}(r^\epsilon,m(w), \epsilon) D_{1_{r^\epsilon,m(w), \epsilon, q_0}} \frac{dr^\epsilon,m(w)}{dw} dw \right]
= \mathbb{E}\left[ \lim_{m \to \infty} \sum_{j=0}^{m-1} \int_{w_j}^{w_{j+1}} d^{-1}(D_1 D_{r^\epsilon,m(w)}) D_{1_{r^\epsilon,m(w), \epsilon, q_0}} \frac{dr^\epsilon,m(w)}{dw} dw \right]
= \mathbb{E}\left[ \lim_{m \to \infty} \sum_{j=0}^{m-1} \int_{w_j}^{w_{j+1}} d^{-1}(r^m(w)) D_{1_{r^\epsilon,m(w), \epsilon, q_0}} \frac{dr^m(w)}{dw} dw \right]
= \mathbb{E}\left[ \lim_{m \to \infty} \sum_{j=0}^{m-1} \int_{w_j}^{w_{j+1}} d^{-1}(r^m(w)) \frac{dr^m(w)}{dw} dw \right]
= \int_a^b d^{-1}(r(w)) \frac{dr(w)}{dw} dw
= CS_{det}
\]
Proof of Proposition 4

- Random allocation rule

This is the proof for the general path case

\[ \mathbb{E}[CS^R(\epsilon)] = \mathbb{E} \left[ \int_{c_\epsilon} \sum_{i=1}^{n} (d_i^{-1}(r^\epsilon, \epsilon) - p_i^0) \min \left\{ 1, \frac{q_i^{sto}}{d_i(p^0, \epsilon)} \right\} dr_i^\epsilon \right] \]
\[ = \mathbb{E} \left[ \int_{c_\epsilon} \sum_{i=1}^{n} (d_i^{-1}(r^\epsilon - \epsilon) - p_i^0) \min \left\{ 1, \frac{q_i^{sto}}{d_i(p^0) + \epsilon} \right\} dr_i^\epsilon \right] \]
\[ \leq \mathbb{E} \left[ \int_{c_\epsilon} \sum_{i=1}^{n} (d_i^{-1}(r^\epsilon - \epsilon) - p_i^0) \min \left\{ 1, \frac{d_i(p^0)}{d_i(p^0) + \epsilon} \right\} dr_i^\epsilon \right] \]
\[ = \mathbb{E} \left[ \int_{a}^{b} \sum_{i=1}^{n} (d_i^{-1}(r^\epsilon(w) - \epsilon) - p_i^0) \min \left\{ 1, \frac{d_i(p^0) + \epsilon}{d_i(p^0)} \right\} \frac{dr_i^\epsilon(w)}{dw} dw \right] \]
\[ = \mathbb{E} \left[ \int_{a}^{b} \sum_{i=1}^{n} (d_i^{-1}(r(w) - (I - D^{-1}d(p^0)D_r(w))\epsilon) - p_i^0) \min \left\{ 1, \frac{d_i(p^0) + \epsilon}{d_i(p^0)} \right\} \frac{dr_i(w)}{dw} dw \right] \]
\[ \leq \mathbb{E} \left[ \int_{a}^{b} \sum_{i=1}^{n} (d_i^{-1}(r(w) - (I - D^{-1}d(p^0)D_r(w))\epsilon) - p_i^0) \frac{dr_i(w)}{dw} dw \right] \]
\[ = \int_{a}^{b} \sum_{i=1}^{n} \mathbb{E} \left[ (d_i^{-1}(r(w)) - (I - D^{-1}d(p^0)D_r(w))\epsilon) - p_i^0 \right] \frac{dr_i(w)}{dw} dw \]
\[ \leq \int_{a}^{b} \sum_{i=1}^{n} (d_i^{-1}(r(w)) - p_i^0) \frac{dr_i(w)}{dw} dw \]
\[ = CS_{det} \]

The second equality is due to the relation between the demand function \(d(q, \epsilon)\) and the nominal demand function \(d(q)\) when the noise is additive. The first inequality is due to the fact that we are assuming \(q^{sto} \leq d(p^0, \epsilon)\). The third equality is the integration over the parameter of the parametric path. The fourth
equality is using the replacing $r^e$ and $dr^e$ by $r$ and $dr$. The last inequality is
due to the fact of Jensen inequality, thus the only thing left to show is that
\[ d_i^{-1}(r(w) - (I - D^{-1}_d(p^o)D_{r(w)})\epsilon) \]
is concave in $\epsilon$ for all $i \in \{1, \ldots, n\}$. Indeed

\[ \nabla^2 d_i^{-1}(r(w) - (I - D^{-1}_d(p^o)D_{r(w)})\epsilon) = (I - D^{-1}_d(p^o)D_{r(w)})\nabla^2 d_i^{-1}(r(w) - (I - D^{-1}_d(p^o)D_{r(w)})\epsilon)(I - D^{-1}_d(p^o)D_{r(w)}) \]

The latter is negative semi-definite since $\nabla^2 d_i^{-1}$ is negative semi-definite.

**Proof of Proposition 5**

Consider for the stochastic case the H rule (which is the best case). Recall
$(p^s, q^s)$ are the optimal prices and quantities from solving Problem (4.30), and
$p^d$ is the optimal solution of solving Problem (4.29) then,

\[
\mathbb{E}[CSH(\epsilon)] = \mathbb{E}\left[\int_0^{d(p^s)} (d^{-1}(w, \epsilon) - p^s) 1_{\{w \leq q^s\}} dw\right]
\]
\[
= \mathbb{E}\left[\int_0^{d(p^s)} (d^{-1}(\frac{w}{\epsilon}) - p^s) 1_{\{w \leq q^s\}} dw\right]
\]
\[
= \mathbb{E}\left[\int_0^{d(p^s)} (d^{-1}(v) - p^s) 1_{\{v \leq q^s\}} \epsilon dv\right]
\]
\[
\leq \mathbb{E}\left[\int_0^{d(p^d)} (d^{-1}(v) - p^s) \epsilon dv\right]
\]
\[
= \int_0^{d(p^d)} (d^{-1}(v) - p^s) dv
\]
\[
\leq \int_0^{d(p^d)} (d^{-1}(v) - p^d) dv
\]
\[
= CS_{det},
\]

where the second to last equality follows from $\mathbb{E}[\epsilon] = 1$, and the last inequality
follows from the fact that $p^d \leq p^s$ for multiplicative demand noise (see Lemma 1),
plus the fact that the Consumer Surplus in the deterministic case is decreasing.
on the price. The last follows since

$$\frac{d}{dp} \left( \int_0^{d(p)} (d^{-1}(w) - p) \, dw \right) = \int_0^{d(p)} -1 \, dv = -d(p) \leq 0$$

**Proof of Proposition 6**

$$E[C^{SH}(\epsilon)] = E \left[ \int_0^{d(p^*, \epsilon)} (d^{-1}(w, \epsilon) - p^*) 1_{\{w \leq q^*\}} \, dw \right]$$

$$\geq E \left[ \int_0^{d(p^d, \epsilon)} (d^{-1}(w, \epsilon) - p^d) 1_{\{w \leq q^*\}} \, dw \right]$$

$$\geq E \left[ \int_0^{d(p^d)} (d^{-1}(w - \epsilon) - p^d) 1_{\{w \leq d(p^d)\}} \, dw \right]$$

$$\geq \int_0^{d(p^d)} (d^{-1}(w) - p^d) \, dw$$

$$= CS_{det}.$$ 

The first inequality is because the expression inside the expectation is decreasing on $p^\ast$, indeed

$$\frac{d}{dp^\ast} \int_0^{d(p^*, \epsilon)} (d^{-1}(w, \epsilon) - p^*) 1_{\{w \leq q^*\}} \, dw$$

$$= \frac{\partial d(p^*, \epsilon)}{\partial p^\ast} (d^{-1}(d(p^*, \epsilon), \epsilon) - p^*) 1_{\{w \leq q^*\}} + \int_0^{d(p^*, \epsilon)} -1 1_{\{w \leq q^*\}} \, dw$$

$$= - \min\{ d(p^*, \epsilon), q^* \}$$

$$\leq 0.$$ 

In respect to the other inequalities used in the original expressions aforementioned, the second inequality is due the fact that $q^* \geq d(p^\ast)$ and the fact that $d(p^d) \geq d(p^d)$ since $p^\ast \leq p^d$ by Lemma 1. The third inequality is due Proposition 2 for the H rule.