PION PRODUCTION IN PION-NUCLEON COLLISIONS

IN THE CHEW-LOW-WICK FORMALISM

by

LEONARD SIDNEY RODBERG
A.B., Johns Hopkins University
(1954)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
August, 1956
Physics
Thesis
1956
Pion Production in Pion-Nucleon Collisions in the Chew-Low-Wick Formalism

Leonard S. Rodberg

Submitted to the Department of Physics on August 20, 1956, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

The production of a single p-wave pion in pion-nucleon collisions has been studied using the Chew-Low-Wick formalism. This theory is a low-energy, static-source theory. The purposes of this study are: to determine the ability of the theory to handle high-energy processes; to determine the effects of high-energy processes upon the low-energy predictions of the theory; and to obtain information on the behavior of the production cross-section.

An equation has been derived which, in the one-intermediate-meson approximation, gives the matrix element for meson production in terms of the matrix element for scattering. This equation exhibits a crossing symmetry under a time reversal of one of the outgoing mesons.

The angular momentum and isotopic spin dependences have been extracted. This allows one to take advantage of the conservation of the total angular momentum and total isotopic spin, and to use the symmetry of the theory between angular momentum and isotopic spin. The resulting equations have the form of twenty coupled integral equations; they are uncoupled by neglecting integrals containing non-resonant denominators. Solutions have been obtained by including scattering in only the \( T = J = \frac{3}{2} \) state.

The resulting production cross-section for \( \pi^- \) incident on protons shows a peak for an incident energy of 700 Mev. For energies in this region each outgoing meson can attain the 33-resonant energy relative to the nucleon. The peak is the result of this "two-particle resonance".

The one-meson approximation does not maintain the requirements of unitarity. It is found that the production cross-sections violate unitarity in the region of 500 Mev. This is attributable to the inadequacies of the one-meson approximation and to the fact that high-energy scattering effects are important. The equation for the production matrix element shows that the high-energy scattering amplitudes act to damp the production cross-section.

The solutions for meson production have been used to examine the two-meson corrections to the equation for the scattering matrix element. These corrections are small relative to the scattering amplitude for the 33-state, but large relative to the amplitudes for the 11- and 13-states.

The equation for the matrix element describing the photo-production of two p-wave pions has been derived. All quantities involved have been expressed in terms of multipoles. The structure of the equation is identical with the equation for meson production, leading to the belief that double photoproduction will show a peak for incident photon energies of about 800 Mev.

Thesis Supervisor: Herman Feshbach
Title: Professor of Physics
## TABLE OF CONTENTS

I. Introduction 1
   A. Chew-Low Theory 1
   B. Purposes of This Study 6

II. Derivation of Equations 9
   A. T-Matrix 9
   B. Crossing Symmetry 13
   C. One-Meson Approximation 15

III. Separation of Angular Momentum-Isotopic Spin Dependences 22
   A. Introduction 22
   B. Expansion of States 24
   C. Elimination of Magnetic Quantum Numbers 26

IV. Approximate Solution 37
   A. 33-Resonance Approximation 37
   B. Solutions 43

V. Results 52
   A. Representation of the Scattering Amplitude 52
   B. Production in the 33-State 54
   C. Contribution of the "Crossed" Terms 55
   D. Production Cross-Sections 57
   E. Discussion of Results 61
   F. The One-Meson Approximation 65

VI. Photoproduction 70
   A. Hamiltonian 70
   B. Equation for Photoproduction Matrix Element 71
   C. Commutator Term 73
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D. Multipole Expansion</td>
<td>76</td>
</tr>
<tr>
<td>E. Isotopic-Spin Reduction</td>
<td>80</td>
</tr>
<tr>
<td>F. Reduction of Inhomogeneous Terms</td>
<td>81</td>
</tr>
<tr>
<td>G. Discussion</td>
<td>84</td>
</tr>
<tr>
<td>VII. Summary and Conclusion</td>
<td>87</td>
</tr>
<tr>
<td>Appendix A. Representation of Meson-Nucleon States</td>
<td>90</td>
</tr>
<tr>
<td>Appendix B. Results of Angular Momentum-Isotopic Spin Reduction</td>
<td>93</td>
</tr>
<tr>
<td>Appendix C. Numerical Coefficients</td>
<td>96</td>
</tr>
<tr>
<td>Appendix D. Gauge-Invariance for the Static-Source Theory</td>
<td>98</td>
</tr>
<tr>
<td>Figure 1 Pion Production Cross-Sections</td>
<td>101</td>
</tr>
<tr>
<td>Figure 2 Experimental Cross-Sections</td>
<td>102</td>
</tr>
<tr>
<td>Table 1 Total Cross-Sections for Production of a Single Pion in Pion-Nucleon Collisions</td>
<td>103</td>
</tr>
<tr>
<td>Table 2 Corrections to the One-Meson Approximation</td>
<td>104</td>
</tr>
<tr>
<td>Bibliography</td>
<td>105</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>108</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

During the past few years a promising new approach to the problem of pion-nucleon interactions has been made by Chew and Low.¹,²

The theory is constructed to give a correct representation of low-energy processes. Our purpose in studying a high-energy process, pion production in pion-nucleon collisions, is threefold: to determine the ability of the theory to handle high-energy processes; to determine the effect of these high-energy processes upon the low-energy predictions of the theory; and to gain some indication as to the expected experimental behavior of the production cross-section.

We shall present a brief summary of the Chew-Low approach to meson-nucleon scattering. In the light of this survey we shall discuss more fully the aims of this study.

A. Chew-Low Theory

The theory is an attempt to eliminate the convergence problems which have plagued perturbation approaches to meson-nucleon interactions. It deals only with matrix elements describing complete physical processes and involves only the renormalized coupling constant. This coupling constant is much smaller than the unrenormalized coupling constant of pseudoscalar meson theory. The expansion which is used in the theory is in terms of the number of real mesons in an intermediate state, not in powers of the coupling constant. For these reasons it is hoped that the convergence of the series will be quite rapid. One of the purposes of this study is to examine this problem.
The theory makes use of the linear charge-symmetric Yukawa theory in the limit of no nucleon recoil. The nucleon is represented by a fixed charge density; the interaction is taken to be the non-relativistic limit of a relativistic interaction. Then one expects that the theory will be applicable chiefly to low-energy processes.

The following Hamiltonian is used:

\[ H = H_0 + H_1 \]  \hspace{1cm} (1.1)

\[ H_0 = \sum_k a_k^+ a_k \omega_k \]  \hspace{1cm} (1.2)

\[ H_1 = \sqrt{4 \pi} f^0 \gamma \int \rho(x) \sigma \nabla \phi(x) \, d^3x = \sum_k (a_k v_k + a_k^+ v_k^+) \]  \hspace{1cm} (1.3)

Throughout we use the units \( \hbar = c = \mu = 1 \). The notation is:

- The terms "pion" and "meson" are used interchangeably.
- \( k \) denotes the momentum and charge (isotopic spin) of the meson, and gives a complete description of its state.
- \( \rho \) is the meson mass.
- \( \omega_k = (k^2 + \rho^2)^{\frac{1}{2}} \) is the meson energy.
- \( a_k \) and \( a_k^+ \) are the meson annihilation and creation operators, respectively, and satisfy the commutation relations

\[ [a_p, a_q] = [a_p^+, a_q^+] = 0 \]

\[ [a_p, a_q^+] = \delta_{pq} \]

\( f^0 \) is the unrenormalized unrationalized coupling constant.

\( \gamma \) is the nucleon isotopic spin operator, and is a vector in isotopic spin space.
\( \rho(x) \) is the source function which represents phenomenologically the nucleon density and provides a finite cut-off.

\( \sigma \) is the nucleon spin operator.

\( \phi(x) \) is the meson field operator which we expand as

\[
\phi(x) = \sum_k \frac{1}{k \sqrt{2\omega_k}} \left\{ a_k e^{ik \cdot x} + a_k^* e^{-ik \cdot x} \right\} \tag{1.4}
\]

It is a vector in isotopic space. The interaction includes a scalar product in isotopic space

\[
\gamma \cdot \phi = \sum_{\alpha=1}^{3} \gamma_\alpha \phi_\alpha .
\]

We shall use "box normalization", but shall not explicitly include the normalization constant since it eventually cancels out of all formulas.

We should note that the theory does not include a meson-meson interaction, and takes no account of the "strange" particles which are expected to be important only at high energies. Since the nucleon is fixed, the pion is pseudoscalar, and the theory is linear, only p-wave pions interact with the nucleon.

We shall call \( V_k \) the "interaction Hamiltonian"; it is given by

\[
V_k = \sqrt{4\pi} \int \gamma_k \cdot \sigma \, \nu(k) \, \frac{1}{\sqrt{2\omega_k}} \tag{1.5}
\]

where

\[
\nu(k) = \int e^{ik \cdot x} \, \rho(x) \, ds \tag{1.6}
\]

For numerical calculations we shall take \( \nu(k) \), the cut-off function, to be a square cut-off:

\[
\nu(k) = \begin{cases} 
1 & k < k_{\text{max}} \\
0 & k > k_{\text{max}} 
\end{cases} \tag{1.7}
\]
Chew and Low have found that the value of $k_{\text{max}}$ is of the order of the nucleon rest mass.

The approach uses the non-relativistic limit of an equation developed by Low\textsuperscript{2} for the relativistic case. For meson-nucleon scattering the equation which Chew and Low\textsuperscript{1} have discussed can be written

$$\langle q; T; p \rangle = - \sum_n \left\{ \frac{(q; T^*; n)(n; T; p)}{\alpha_n - \alpha_p - i \varepsilon} + \frac{(p; T^*; n)(n; T; q)}{\alpha_n + \alpha_p} \right\}$$

(1.8)

Here $\langle n; T; p \rangle = \langle n; V_p; 0 \rangle$ is the T-matrix element describing the transition from a state with a physical nucleon and a meson $p$ to a state containing a physical nucleon and $n$ mesons. We suppress the spin and charge indices describing the nucleon.

By the term "physical nucleon" we mean a nucleon in complete interaction with the pion field. It is the lowest eigenstate of the total Hamiltonian and in general satisfies the equation

$$H\langle 0 \rangle = E_0 \langle 0 \rangle$$

Since the energy of a physical nucleon serves only to determine a reference level, we set it equal to zero: $H\langle 0 \rangle = 0$.

The states $\langle n \rangle$ are eigenstates of the total Hamiltonian and satisfy $H\langle n \rangle = \omega_n \langle n \rangle$.

The notation and the meaning of the quantities involved here will be more fully discussed in the subsequent development. However, the important point here is that $\langle n; T; p \rangle$ is closely related to the T-matrix element describing an actual physical process. It differs from it only in being defined for all values of $\alpha_n$ and $\alpha_p$. It is equal to the physical T-matrix element when energy is
conserved, that is, for $\omega_p = \omega_n$. Then, except for a single term, only physically-observable processes occur in this equation.

The one exception is the zero meson term containing

$$(0| T | p) = (0| V_p | 0).$$

This is a numerical multiple of

$$(0| \sigma_p \gamma_p | 0).$$

Since $|0\rangle$ is a physical interacting state we cannot evaluate this quantity without knowing the detailed properties of the physical nucleon. However, we shall show in Chap. 3 that this quantity is simply a numerical, state-independent, multiple of the matrix element $(u| \sigma_p \gamma_p | u)$ where $|u\rangle$ is a "bare", non-interacting, nucleon. This can be evaluated using the commutation relations of $\sigma$ and $\gamma$. Then we write

$$f^0 (0| \sigma_p \gamma_p | 0) = f (u| \sigma_p \gamma_p | u)$$

(1.9)

where $f$ is defined as the renormalized coupling constant. Comparison of the solution of the scattering equation with experiment and a study of low-energy meson photoproduction shows that $f^2 \approx 0.08$.

Chew and Low$^2$ have solved (1.8) by making the one-meson approximation. This involves neglecting all matrix elements relating to states $|n\rangle$ containing more than one real meson (Remember that each state contains all of the virtual processes which are discussed in ordinary perturbation theory, so that this is not equivalent to an expansion in powers of the coupling constant). Their solution can be written in the form

$$\lambda_{TJ} \frac{P_T}{\alpha_p} \cot \delta_{TJ} = 1 - \omega_p r_{TJ}(\omega_p).$$

Here $T$ and $J$ denote the isotopic spin and angular momentum of the scattering state, $\lambda_{TJ}$ is a constant proportional to the renormalized coupling constant, and $\delta_{TJ}$ is the scattering phase shift for the TJ-state. For constant $r_{TJ}$ this has the form of an effective-
range approximation similar to that used in low-energy nucleon-nucleon scattering (except that here we are dealing with $p$-waves so that this contains $p^8\cot\delta$ rather than $p\cot\delta$). Experimentally such a straight-line behavior is found for the 33-state.

The one-meson approximation gives an energy-dependence to $r_{33}(\omega_p)$ which is not observed. It is then of interest to determine whether the contributions of high-energy processes will affect this energy dependence.

The great success of the theory has been to show that a resonance in the 33-state is a prediction of the theory which is independent of any approximations. The known low-energy scattering data (for incident meson energies less than 400 Mev) can be interpreted in terms of such a resonance. The symmetry of the theory can be used to show that only $r_{33}(0)$ is positive, and thus that only $\delta_{33}$ can go through $90^\circ$ (in the low-energy region).

The general background of the meson theory involved here is discussed in Ref. 3. The derivation and solution of (1.8) is given in Ref. 1.

B. Purposes of This Study

The chief purpose of our study is to investigate the internal properties of the theory. The two-meson term of the scattering equation (1.8) involves the amplitude for production of a single pion in pion-nucleon collisions. It is therefore of interest to examine this amplitude in order to find the corrections to the Chew-Low solution of the scattering equation. This should also serve as a test of the convergence of the expansion in (1.8).
In addition we can hope to gain information about the physical process of pion production and its contribution to observed high-energy cross-sections. There are some features of the total pion-nucleon cross-sections, in particular a peak for an incident energy of about 900 Mev, which one might attempt to explain by this process. Of course there are large reservation in applying this theory to high-energy processes. As we have discussed, the interaction is the non-relativistic limit of a correct relativistic interaction and nucleon recoil is neglected. However, this is the only theory which we now have which gives correct predictions for low-energy processes, and we can expect it, at the very least, to give an indication of the general behavior of high-energy processes.

The available information on the pion-production process is not sufficient to be able to compare the predictions of this theory with experiment. However, we can use general conservation laws such as the unitarity of the scattering matrix as a check on the solutions we obtain.

We shall derive an equation for the meson-production amplitude similar to equation (1.8) for the meson-scattering amplitude. We shall use the same methods as used by Wick\textsuperscript{3} to derive (1.8). The result has the same form as (1.8) and can be solved by methods quite similar to those used by Chew and Low. The equation which we shall derive gives the meson-production amplitude in terms of the meson-scattering amplitude. We shall use the experimental information on this scattering amplitude, but shall take the viewpoint that only known low-energy phenomena should be included.
Thus our result will be the meson-production amplitude as determined by the known low-energy scattering data. A comparison with experiment and with known conservation laws will then serve as a test of the assumption that the high-energy scattering amplitudes will not affect the low-energy solutions. Such a test is important because this theory contains the implicit assumption that low-energy processes are determined solely by low-energy effects. In other words, a cut-off is inserted to disguise our ignorance of high-energy effects, with the hope that the results will be relatively independent of the cut-off. If the high-energy effects are important, the validity of such a cut-off is questionable.
II. DERIVATION OF EQUATIONS

In this chapter we shall derive an equation for the T-matrix element describing the production of one pion in a pion-nucleon collision. This proceeds as with the derivation of the T-matrix element for the scattering problem and uses the same methods. The resulting equation exhibits a "crossing" symmetry under a transformation of one of the pions. This will be demonstrated and discussed. We then proceed to make the one-meson approximation on our production equation, and obtain a form which can be interpreted in terms of Feynman diagrams. There are strong reasons, however, for not relying on these diagrams, and these are pointed out out.

A. T-Matrix

We wish to derive an equation for the matrix element describing the production of a pion in pion-nucleon collisions. To do this we use the formalism developed by Wick; it permits a rapid derivation of results which require an involved discussion if ordinary methods are used, but it is restricted to the no-recoil theory.

We begin with the scattering matrix as defined by Gell-Mann and Goldberger:

$$\mathcal{S}_{pq,k} = (pq-|k+\rangle$$ (2.1)

Here $|k+\rangle$ is a state containing a physical nucleon and one real meson described by the variable $k$. (We suppress the nucleon indices describing spin and charge.) This state contains only outgoing waves. It is a stationary, time-independent state,
but the restriction to outgoing (or incoming) waves can be included by adding a positive (or negative) imaginary part to the energy. The state \( (pq^-) \) contains a physical nucleon and two real mesons, \( p \) and \( q \), and contains only incoming scattered waves.

From this form for the \( S \)-matrix we want to compute the \( T \)-matrix, which is directly related to the production cross-section by

\[
\sigma(pq,k) = \frac{2\pi}{v_k} |T_{pq,k}|^2 \rho(E) \tag{2.2}
\]

Here \( v_k \) is the velocity of the incident meson, and \( \rho(E) \) is the density of final states of the system.

We first represent the initial state by

\[
|k+\rangle = a_k^+|0\rangle - \frac{1}{H - \omega_k - i\varepsilon} v_k|0\rangle \tag{2.3}
\]

\( i\varepsilon \) is added to insure outgoing scattered waves. \( |0\rangle \) is the physical nucleon, the ground state of the total Hamiltonian. This form can be obtained from the requirement that the state \( |k+\rangle \) be an eigenstate of the total Hamiltonian with an energy \( \omega_k \) (with the nucleon self-energy normalized to zero), that it contain only outgoing scattered waves, and that asymptotically it contain a free meson and a physical nucleon (see Appendix A).

Then

\[
S_{pq,k} = (pq-|a_k^+|0\rangle - (pq-| \frac{1}{H - \omega_k - i\varepsilon} v_k|0\rangle \tag{2.4}
\]

We now write

\[
a_k|pq-\rangle = \delta_{pk}|q-\rangle + \delta_{qk}|p-\rangle - \frac{1}{H - \omega_p - \omega_q + \omega_k + i\varepsilon} v_k^+|pq-\rangle \tag{2.5}
\]
which is obtained by imposing requirements on this function similar to those described above (Appendix A). Using this form, and the orthogonality of these states,

\[ S_{pq,k} = -(pq-!) \frac{1}{H - \omega_p - \omega_q - \omega_k - i\xi} |10\rangle 
- (pq-!) \frac{1}{H - \omega_k - i\xi} V_k |10\rangle = 
= \left( \frac{1}{\omega_p + \omega_q + \frac{1}{2}} - \frac{1}{\omega_p + \omega_q + \omega_k + i\xi} \right) (pq-!) V_k |10\rangle = 
= -2\pi i \delta(\omega_p + \omega_q - \omega_k) (pq-!) V_k |10\rangle \quad (2.6) 
\]

from which

\[ T_{pq,k} = (pq-!) V_k |10\rangle \quad (2.7) \]

We have here used

\[ \left[H_1, a_k^+\right] = - \left[H_1, a_k\right] = -V_k. \]

We will now expand this form for the T-matrix, and thus obtain an equation which we may use to determine its dependence on the momentum and charge of the particles involved.

We will write

\[ |pq-\rangle = a_p^+|q-\rangle - \frac{1}{H - \omega_p - \omega_q + i\xi} V_p|q-\rangle \quad (2.8) \]

(Appendix A)

This form is apparently not symmetric between the two outgoing mesons, but a further expansion of the states involved shows that this symmetry is maintained. We use this form because the resulting equation is the most convenient for further calculations. It should be noted that this form is not unique; indeed, an equivalent symmetric form is

\[ |pq-\rangle = a_p^+a_q^+|0\rangle - \frac{1}{H - \omega_p - \omega_q + i\xi} (V_p a_q^+ + V_q a_p^+) |0\rangle \quad (2.9) \]
The use of this form leads to an equation which, while it is symmetric in the two mesons, is much more complicated and difficult to handle.

Now, using (2.7) and (2.8),

\[ T_{pq,k} = (q-|a_p V_k|0) - (q-|V_p^+ \frac{1}{H - \omega_p - \omega_q - i\varepsilon} V_k^+|0) = \]

\[ = (q-|V_k^+ \frac{1}{H + \omega_p} V_p^+|0) - \]

\[ - (q-|V_p^+ \frac{1}{H - \omega_p - \omega_q - i\varepsilon} V_k^+|0) - \]

\[ - (q-|V_k^+ [V_k, a_p]|0) \] (2.10)

Our theory is linear in the meson field, so that the last term is zero.

For comparison, the equation which is obtained using the symmetric form (2.9) is

\[ T_{pq,k} = (0|V_k^+ \frac{1}{H + \omega_p + \omega_q} \left\{ V_p^+ \frac{1}{H + \omega_q} V_q^+ + V_q^+ \frac{1}{H + \omega_p} V_p^+ \right\}|0) - \]

\[ - (q-|V_p^+ \frac{1}{H - \omega_p - \omega_q - i\varepsilon} V_k^+|0) - \]

\[ - (p-|V_q^+ \frac{1}{H - \omega_p - \omega_q - i\varepsilon} V_k^+|0) - \]

\[ - (0| \left\{ V_q^+ \frac{1}{H - \omega_q - i\varepsilon} V_p^+ + V_p^+ \frac{1}{H - \omega_p - i\varepsilon} V_q^+ \right\} \times \]

\[ \times \frac{1}{H - \omega_p - \omega_q - i\varepsilon} V_k^+|0) \] (2.10S)

It is indeed symmetric, but is not nearly as compact as (2.10). Since (2.10) contains many virtual processes implicitly in the physical state (q-|), while (2.10S) contains the operators for these processes explicitly written out, one has the feeling that
any approximation to (2.10S) will be less accurate than a similar approximation of (2.10). This is borne out by actual calculations using the approximation scheme discussed below. As an example, one finds that (2.10S) contains an approximation to the Chew-Low equation (1.8) for the scattering amplitude, while (2.10), in a comparable term, contains the scattering amplitude itself. For this, and other reasons, we prefer to use (2.10), although, as we shall see, this has disadvantages.

B. Crossing Symmetry

We here note a feature of this equation which follows from the Bose statistics of the pions. This is the "crossing" symmetry discussed by Gell-Mann and Goldberger\(^5\) and in Ref. 1. The manner in which this symmetry is operative here can be seen by a simple extension of the method of Feldman and Matthews\(^6\), using the relativistic treatment. But it is perhaps more instructive to examine it in this formalism.

Consider the T-matrix describing an inverse process, in which two mesons are incident on a nucleon and one meson emerges. Let us use \(\times \) to denote reversal of the direction of the momentum and the sign of the charge. This T-matrix is given by

\[
T_{q,-p} = (q-|V_k|^p) = (q-|V_k a_p^+|0) - (q-|V_k H - \omega_p - i\xi V_p|0) = \\
\times_p, q (0|V_k|^0) - (q-|V_p H - \omega_q + \omega_p - i\xi V_p|0) + \\
+ (q-|V_k H - \omega_p - i\xi V_p|0) + (q-|V_k a_p^+|0) (2.11)
\]

where we have used (2.3) and (A.14).
Making the energy-dependence explicit, and recognizing that
\[ V_p = \left[ H_1, a_p^+ \right] = - \left[ H_1, a_p \right] = V_p^+ , \]
we have
\[ T_{pq,k}(\omega_p, \omega_q, \omega_k) = T_{q,-p k}(-\omega_p, \omega_q, \omega_k) - \delta_{p,q} (0|V_k|0) \] (2.12)

Instead of destroying a meson of given momentum and charge, we create a meson of opposite momentum and charge. Field-theoretically, the sign of the energy is associated with the creation or destruction of a particle. This corresponds to the fact that we are now destroying meson \(-p\) rather than creating \(p\). We should note that this equation relates two \(T\)-matrices differing in the momentum and charge of only one of the mesons. In addition it is important to remember that the nucleons are unchanged by this transformation; the same nucleon is incident in both cases.

The effective change from \(i\mathcal{E}\) to \(-i\mathcal{E}\) represents the change from incoming to outgoing waves (the direction of the particles in the physical process should not be confused with the direction of the waves used in defining the \(S\)-matrix). In a sense, we have performed a time-reversal on one of the mesons, leaving all other particles unchanged.

The delta-function term corresponds to the possibility that one of the incident mesons does not interact with the nucleon while the other one is absorbed. Naturally, on the "energy shell", where energy is conserved and \(\omega_p + \omega_q = \omega_k\), this term does not contribute.
C. One-Meson Approximation

Let us now expand the inverse operator in the complete set of states containing ingoing waves and any number of real mesons. We will use only ingoing waves from now on and will suppress the minus sign. Since these states are eigenstates of the total Hamiltonian, i.e. \( H|\psi_n\rangle = E_n|\psi_n\rangle \), we have

\[
\frac{1}{H - E} = \sum_n \frac{|\psi_n\rangle \langle \psi_n|}{E_n - E} \tag{2.13}
\]

where the sum is over all numbers of mesons and all momenta and charges.

Using this, we can expand the equation for \( T_{pq,k} \):

\[
T_{pq,k} = \sum_n \frac{(q|V_k|n)(n|V_p^+|0)}{\alpha_n + \alpha_p} - \sum_n \frac{(q|V_p^+|n)(n|V_k|0)}{\alpha_n - \alpha_p - \alpha_q - i\epsilon} \tag{2.14}
\]

We are now going to make the "one-meson approximation", that is, we are going to drop all terms containing more than one intermediate meson. Before doing this, we should remember that this equation is still inherently symmetric between the two outgoing mesons. When we make the one-meson approximation we destroy this symmetry. The interaction of meson \( q \) with the nucleon is still treated exactly, since it is described by an exact eigenstate of the total Hamiltonian. However, the parts of the equation which propagate and scatter meson \( p \) are being treated only approximately. We should therefore symmetrize our final result so as to restore the necessary symmetry between these particles.

An exact solution would give \( T_{pq,k} = T_{qp,k} \). Because of our approximations this is no longer the case, and we should
$\frac{1}{2}(T_{pq,k} + T_{qp,k})$. In finding the total production cross-section this is not necessary since we sum over all meson variables.

The only justification for making the one-meson approximation is that it permits a solution to be found. The usual argument, that the terms which are dropped do not have vanishing denominators, does clearly not hold here since it is just the two-meson term which has such a pole. The error incurred by this approximation can only be estimated by finding the matrix element for the next higher inelastic process; as stated previously, this is one of the purposes of this study. In our case the two-meson term involves the matrix element for double-meson production, which we may assume is small compared to that for single production (at least in the intermediate-energy region).

Before writing our approximate equation, it is better to examine each of the terms involved since several transformations must be performed.

a) One of the one-meson terms contains the matrix element $(q-!V_k!s-)$. Proceeding as with (2.11),

\[
(q-!V_k!s-) = (q-!V_k a_s^+!0) - (q-!V_k \frac{1}{H - \omega_q + \omega_s - i\varepsilon} V_s !0) =
\]

\[
= \delta_{s,q} (0!V_k !0) - (q-!V_k \frac{1}{H - \omega_q + \omega_s - i\varepsilon} V_s !0) - (q-!V_k !0) + (q-!V_k , a_s^+ !0) =
\]

\[
= \delta_{s,q} (0!V_k !0) - (q-!V_k ^+ \frac{1}{H - \omega_q + \omega_s - i\varepsilon} V_s !0) - (-!V_k \frac{1}{H - \omega_q + \omega_s - i\varepsilon} V_s !0) - (q-!V_k , a_s^+ !0) =
\]

\[
= \delta_{s,q} (0!V_k !0) - (-q!V_k !0)
\]

(2.15)
Here, as in the crossing theorem, \(-s\) implies opposite charge, momentum, and energy. In fact, this quantity differs from \(T_{q,sk}\) only in having incoming waves in the initial state.

The first of the one-meson terms thus becomes

\[
- \sum_{s} \frac{(q|V_{k}|s)(s|V_{p}^{+}|0)}{\omega_{s} + \omega_{p}} = \]

\[
= - \frac{(0|V_{k}|0)(q|V_{p}^{+}|0)}{\omega_{q} + \omega_{p}} + \sum_{s} \frac{(-s|V_{k}|0)(s|V_{p}^{+}|0)}{\omega_{s} + \omega_{p}} \]

(2.16)

b) By a similar manipulation the second term becomes

\[
\frac{(0|V_{p}^{+}|0)(q|V_{k}|0)}{\omega_{p}} + \sum_{s} \frac{(-s|V_{p}^{+}|0)(s|V_{k}|0)}{\omega_{s} - \omega_{p} - \omega_{q} - i\varepsilon} \]

(2.17)

Since

\[V_{p}^{+} = V_{-p},\]

(2.18)

this transformation has changed a coefficient which is not directly related to the meson-production amplitude into a delta-function term and a term which bears a direct relation to this amplitude. This result is interesting also since we have obtained from a "one-meson" term a part which looks like a "zero-meson" term. The zero-meson terms which are obtained directly from the expansion are

\[
- \frac{(q|V_{k}|0)(0|V_{p}^{+}|0)}{\omega_{p}} + \frac{(q|V_{p}^{+}|0)(0|V_{k}|0)}{\omega_{p} + \omega_{q}} \]

(2.19)

We see that these terms are related to the above terms by an exchange of the order of the matrix elements, and a change in sign. If we were to use the language of ordinary perturbation theory (which we shall do henceforth but with careful reservations),
we would say that these terms represent the "crossed" and "uncrossed" terms. They would represent processes which are related through performing a time reversal on some of the boson fields (just as in our crossing theorem stated above); such processes would be interpreted through Feynman diagrams as, for instance,

\[ \begin{array}{c}
\text{\textbackslash \textbackslash} \\
\text{\textbackslash \textbackslash} \\
\text{\textbackslash \textbackslash} \\
\text{\textbackslash \textbackslash} \\
\end{array} + \begin{array}{c}
\text{\textbackslash \textback\textbackslash} \\
\text{\textback\textbackslash} \\
\text{\textback\textbackslash} \\
\text{\textback\textbackslash} \\
\end{array} \]

Then the result of the above transformation has been to bring these parts of the equation into a form in which this important symmetry is apparent. We see in addition that it is dangerous to apply the language of ordinary perturbation theory in an expansion such as this which involves eigenstates of the total Hamiltonian. A term which we have called a "one-meson" term actually contained a part analogous to the usual "Born approximation", and a term containing the meson-production amplitude—a "two-meson" amplitude.

Taking this result as a clue, we should examine the "two-meson" terms to see whether they contain terms which should properly be included in a consistent "one-meson" approximation. It is found that this is indeed the case, through relations such as

\[
(q|V_k|rs) = \delta_{rq} (0|V_k|s) - (q|V_r \frac{1}{H - \omega_q + \omega_r - i\epsilon} V_k|s) - (q|V_k \frac{1}{H - \omega_p - \omega_s + i\epsilon} V_r|s) \tag{2.20}
\]

The last two terms of this result are related to the matrix element for production of two pions, and we shall not consider
them further. The delta-function term, however, reduces this part from a "two-meson" term to a "one-meson" term:

\[ - \sum_s \frac{(0|V_k^s|s)(q|V_p^+|0)}{\omega_s + \omega_q + \omega_p} \]  \hspace{1cm} (2.21)

A similar term is obtained from the other two-meson term:

\[ - \sum_s \frac{(0|V_p^+|s)(q|V_k^s|0)}{\omega_s - \omega_p - i\epsilon} \]  \hspace{1cm} (2.22)

Combining all of these results, we obtain the equation with which we will be dealing for the remainder of this paper:

\[
T_{pq,k} = (pq|V_k^s|0) = \frac{(0|V_p^+|0)(q|V_k^s|0) - (q|V_k^s|0)(0|V_p^+|0)}{\omega_p} + \\
+ \frac{(q|V_p^+|0)(0|V_k^s|0) - (0|V_k^s|0)(q|V_p^+|0)}{\omega_p + \omega_q} \\
- \sum_s \left\{ \frac{(0|V_p^+|s)(q|V_k^s|0)}{\omega_s - \omega_p - i\epsilon} - \frac{(-sq|V_k^s|0)(s|V_p^+|0)}{\omega_s + \omega_p} \\
- \frac{(-sq|V_p^+|0)(s|V_k^s|0)}{\omega_s - \omega_p - \omega_q - i\epsilon} + \frac{(0|V_k^s|s)(q|V_p^+|0)}{\omega_s + \omega_p + \omega_q} \right\} \hspace{1cm} (2.23)
\]

It should be remembered that there is implied in all of these terms a sum over the spin and charge states of the intermediate nucleon.

Observe that a solution to this equation would give the meson-production amplitude in terms of the meson-scattering amplitude.

We see also that there is a symmetry, related to the crossing symmetry, between the various terms of this equation. This was one reason for using form (2.8) rather than (2.9).
It is tempting to relate the terms of this equation to the Feynman diagrams used in a perturbation theory in which the expansion is in powers of the coupling constant. At first sight this can be done quite readily, since the energy denominators are the appropriate ones. However, Feynman diagrams have their greatest utility in the fact that a unique set of diagrams for a given process can be written down, and from this the matrix element can be obtained unambiguously. In our case, in which we are not expanding in powers of the coupling constant and are not using non-interacting ("interaction representation") states, neither of these statements is true. As we have stated, more than one equation for the two-meson amplitude can be obtained by using different representations for the final two-meson state. We could associate with each of these equations a set of diagrams, but the sets would be different. Of course, if we were to expand our physical states in terms of the "bare" nucleon, the "bare" nucleon plus one virtual meson, etc., that is, write out explicitly all the processes involved in the nucleon self-energy and in vertex renormalization, we would find that the two sets of diagrams were equivalent. But in doing so we would be losing the essential feature of the Chew-Low approach, that it deals only with physical quantities, and would be returning to conventional, and unsuccessful, perturbation theory.

A second objection to the use of such diagrams is that some of the matrix elements occurring in this equation are clearly not those which one would intuitively write down. For instance, the
second one-meson term (with the $\frac{1}{m_s + m_p}$ denominator) contains the "crossed" relative of the matrix element we would ordinarily include.

For this reason we conclude that a representation in terms of diagrams is not applicable for this theory. They may be useful for gaining a familiarity with the individual terms of the equation, and they provide a "language" for discussing them, but beyond this one must proceed with care.

After these words of caution, we proceed to reduce this equation to a form in which we may attempt to solve it.
III. SEPARATION OF ANGULAR MOMENTUM-ISOTOPIC SPIN DEPENDENCE

We now proceed to reduce the approximate equation we have obtained to a form which can be solved. To do this we introduce the angular momentum and isotopic spin eigenstates for the initial and final states. We can then extract all dependence on magnetic quantum numbers using the techniques of Racah\(^8\). After introducing the renormalized coupling constant, and using the symmetry of this theory between angular momentum and isotopic spin, we present the resulting equation. In this final form it gives the matrix element as a function of only the isotopic spin and angular momentum of the initial state, "partial" angular momenta and isotopic spins required to specify the final state, and the energies of the outgoing mesons.

A. Introduction

In the previous chapter we have specified our states by the magnitude and direction of the linear momentum, the direction of the nucleon spin, and the charges of the various particles (all contained in the meson variables \(p, q, \) and \(k\) while the nucleon indices are suppressed). These are not the most convenient variables to use, however.

The total angular momentum and the total isotopic spin commute with the Hamiltonian for the system. Then these variables are constants of the motion; a system which is initially in a state characterized by these variables (and the corresponding "magnetic" quantum numbers) will remain in such a state. In other words, the scattering or T-matrix will be diagonal.
There will then be a great simplification since we can represent the T-matrix by an amplitude for scattering or production in a state characterized by $J$, the total angular momentum; $m_J$, the total angular momentum magnetic quantum number; $T$, the total isotopic spin; and $m_T$, the total isotopic spin "magnetic" quantum number.

We can simplify things still further, however. The Hamiltonian is constructed to be a scalar under rotations in both coordinate and isotopic space. In particular the T-matrix, which can be written as the matrix element of the Hamiltonian, is invariant under such a rotation. Therefore, the scattering or production amplitudes are independent of the magnetic quantum numbers, and can be described completely in terms of the total angular momentum, total isotopic spin, and the magnitude of the momentum (or the energy). We will see below that this statement is not quite true when more than two particles exist in a given state.

As a result, the number of quantities necessary to completely describe a process is greatly reduced. In addition, many processes can be described by relatively few amplitudes, so that valuable relationships may be found between them. From a broader point of view, this is just another example of the methods which physics can use to show the underlying relationships, and the basic unity, of physical phenomena.
B. Expansion of States

Let us now proceed to expand our states in eigenstates of angular momentum and isotopic spin. In doing this we will of course have to make the nucleon indices explicit. However, in the interests of keeping the notation within reason, we will include both the quantum number and its projection along the "z-axis" in a single index.

We first expand the meson-momentum states in eigenstates of orbital angular momentum:

\[ |k\rangle = \sum_{\ell_k} Y_{\ell_k}^{\ell_k}(\hat{r}) |\ell_k\rangle \]  

(3.1)

\[ (pq| = \sum_{\ell_p \ell_q} Y_{\ell_p}^{p}(\hat{p}) Y_{\ell_q}^{q}(\hat{q}) |\ell_p\ell_q\rangle \]  

(3.2)

where \( Y_{\ell_k}^{\ell_k}(\hat{r}) \) is the normalized spherical harmonic of order \( \ell_k \).

\(|\ell_k\rangle\) and \(|\ell_p\ell_q\rangle\) are characterized by the orbital angular momentum of the mesons and the spin indices of the nucleons. We specify the initial nucleon by \( n_1 \) and the final nucleon by \( n_2 \), where these contain both spin and charge variables. We can then expand these states in eigenstates of total angular momentum:

\[ |\ell_kn_1\rangle = \sum_{J_1} (J_1|\ell_kn_1\rangle |\ell_kJ_1\rangle \]  

(3.3)

The expansion coefficients are the familiar Clebsch-Gordan coefficients; in the notation of Ref. 7, making all indices explicit,

\[ (J_1|\ell_kn_1\rangle = (\ell_k \frac{1}{2} m_k m_1 \; \ell_k \frac{1}{2} J_1 m_{J_1}) \]  

(3.4)

There is an additional complication in the final state. Two angular momenta are required to specify the state of three
particles. Any two particles can be coupled together to form what we shall call a "partial" angular momentum; then this momentum is coupled to the third particle to form the total angular momentum. We choose to couple the final nucleon to meson \( p \) to form partial momentum \( j \). The reason for this particular choice will be given later. Then we write the final state as

\[
(l_p l_q n s | j, J_s) (J_l q | J_s) (l_p l_q J_s)
\]

(3.5)

where, in the notation of Ref. 7,

\[
(l_p n s | j) = (l_p \frac{1}{2} m_p m_s ; l_p \frac{1}{2} j m_j)
\]

\[
(J_l q | J_s) = (j l q m_j m_q ; j l q J_s m_J)
\]

(3.6)

It is convenient here to represent the T-matrix element as

\[
T_{pq,k} = (pq | V_k | 0) = (pq | T | k)
\]

(3.7)

Then, using the above expansions this becomes

\[
(pq | T | k) = \sum_{J_1, J_2, J_3} \gamma_{(\tilde{r})} \gamma_{(\tilde{q})} \gamma_{(\tilde{k})} (l_p n s | j) (J_l q | J_s) x
\]

\[
J_1, J_2, J_3
\]

\[
x (J_1 | l_k n_1) (l_p l_q J_s | T | l_k J_1)
\]

(3.8)

However, as we stated above, the T-matrix is diagonal in the total angular momentum:

\[
(J_s | T | J_1) = (J_1 | T | J_1) \delta_{J_1, J_s} \delta_{m J_1, m J_s}
\]

(3.9)

In addition the T-matrix element is independent of magnetic quantum numbers.
Then, writing
\[
(\ell_p \ell_q \ell_j; T | \ell_k J) = T_j^T, \tag{3.10}
\]
(We shall not specify the angular momenta since we know that in this theory only \(\ell_p = \ell_q = \ell_k = 1\) contribute.) we have our final result:
\[
(pq; T | k) = \sum_{\ell_k, \ell_p, \ell_q} Y_{\ell_p} (\hat{\vec{p}}) Y_{\ell_q} (\hat{\vec{q}}) Y_{\ell_k}^+(k) (\ell_p n_s; j) x_{j,j} x (j_l \ell_q; J) (J_l \ell_n k) T_j^j \tag{3.11}
\]
The same process must be repeated for the isotopic spin, but it proceeds in exactly the same way. Throughout this paper we shall carry out our transformations on the angular momentum, with the understanding that the same process must be performed on the isotopic spin. In our particular case this parallel can be extended from the purely geometrical factors we are considering here to the dynamical properties of the theory. This is because the interaction Hamiltonian (1.3) is symmetric between the nucleon spin and isotopic spin.

C. Elimination of Magnetic Quantum Numbers

We want to eliminate all of the expansion coefficients, specifying the initial and final states, from the equation for the T-matrix. We will then obtain an equation containing only dynamical quantities, the geometrical factors being represented by numerical matrices. To do this we must make use of the techniques of Racah algebra of tensor operators.

In the first place, it should be realized that the quantity \(T_j^j\) can be thought of as the "reduced" matrix element of the
scalar operator $T$; i.e., all dependence on magnetic quantum numbers has been removed by means of Clebsch-Gordan coefficients.

Also, when using the Racah techniques, tensor operators must be defined so that

$$O^v_u = (-1)^u o^{-v}_{-u}$$  \hspace{1cm} (3.12)

where $O^v_u$ is a tensor of degree $v$ with component $u$. Scalar products must be defined as

$$N \cdot O = \sum_u (-1)^u N^v_u O^v_{-u}$$  \hspace{1cm} (3.13)

We must now introduce the special interaction Hamiltonian of this theory. We previously saw that in terms of momentum eigenstates it is

$$V_k = \sqrt{\frac{4\pi}{r^6}} \gamma_k ik \cdot \sigma \text{ v(k) } \frac{1}{\sqrt{2m_k}}$$  \hspace{1cm} (1.5)

We must now write it in terms of angular momentum eigenstates and properly-defined tensor operators. Actually, of course, these are related through the fact that an irreducible tensor operator of degree $v$ transforms under rotations as a spherical harmonic of degree $v$, and the spherical harmonics are the angular momentum eigenstates. We then perform both of these requirements at the same time by writing the components of $\sigma$ (and $\gamma$) as

$$\sigma_+ = -\frac{1}{\sqrt{2}} (\sigma_1 + i \sigma_3)$$

$$\sigma_0 = \sigma_3$$

$$\sigma_- = \frac{1}{\sqrt{2}} (\sigma_1 - i \sigma_3)$$  \hspace{1cm} (3.14)
and by writing the three components of \( k \) as

\[
\left( \frac{4\pi}{3} \right)^{\frac{1}{2}} k \left\{ Y_{l,1}(\vec{k}), Y_{l,0}(\vec{k}), Y_{l,-1}(\vec{k}) \right\} \tag{3.15}
\]

The possibility of doing this shows that this interaction scatters only \( p \)-waves--\( \ell = 1 \).

Then in this representation the interaction Hamiltonian becomes

\[
V_k = 4\pi i f^e \gamma_k \sigma_k \nu(k) \frac{k}{\sqrt{\delta_{0_k}}} Y_k^+(\vec{k}) \tag{3.16}
\]

where

\[
Y_k^+(\vec{k}) = Y_{l,m_k}^+(\vec{k}) = (-1)^m_k Y_{l,-m_k}(\vec{k}) \tag{3.17}
\]

We can remove the spherical harmonics by multiplying by

\[
Y_j^+(\vec{p}) Y_q^+(\vec{q}) Y_k^+(\vec{k})
\]

and integrating over the angles of \( \vec{p}, \vec{q}, \) and \( \vec{k} \). This has the effect of removing the sums over orbital angular momenta (which of course had only one term anyway, the \( p \)-wave term) and magnetic quantum numbers.

Using the previous expansions of the states we can remove all of the geometrical factors from the meson-production amplitudes and the meson-scattering amplitudes. The latter are

\[
(p|V_k|0) = (p|T|k) = \sum_{f_p, f_k, j} Y_{f_p}^+(\vec{p}) Y_{f_k}^+(\vec{k}) (J_p n_s; J)(J_k n_1; J) T_J \tag{3.18}
\]

(with, of course, a similar expansion in terms of isotopic spin)
From the unitarity of the S-matrix it follows that the T-matrix satisfies the equation

$$ T^+ - T = 2\pi i \sum_n \delta(\omega_n - \omega_p) T^T $$

(3.19)

If, following Chew-Low\(^1\), we assume that only one-meson states need be counted in the matrix product on the right (this is rigorously true for energies below the threshold for meson production), we find that

$$ T^J = - \frac{2\pi^2}{p \omega_p} e^{i\delta_j} \sin \delta_j $$

(3.20)

where \( \delta_j \) is the familiar scattering phase shift.

It will prove convenient to extract some factors from this and define a new function by

$$ T^J = - \frac{(4\pi)^2}{p \omega_p} \frac{v(k) v(p)}{\sqrt{\omega_k \omega_p}} S_j $$

(3.21)

("S" for "single"-meson process)

It is also convenient to used (3.11) and define

$$ T^J = \sqrt{2} \frac{v(p) v(q) v(k)}{\sqrt{\omega_k \omega_p \omega_q}} D_j $$

(3.22)

("D" for "double"-meson process)

We are now in a position to remove all geometrical factors from our equation (2.23). We insert our expansions for the meson-production amplitude, the meson-scattering amplitude, and the interaction Hamiltonian. We then multiply by the Clebsch-Gordan coefficients in the expansion of the meson-production amplitude (3.11):

$$ (l \frac{1}{2} m_p \frac{1}{2} m_n ; l \frac{1}{2} j m_j)(j l \frac{1}{2} m_j \frac{1}{2} m_q ; j l \frac{1}{2} j m_j) \times $$

$$ x (l \frac{1}{2} m_k \frac{1}{2} n_1 ; l \frac{1}{2} j m_j) $$
and sum over all magnetic quantum numbers. We shall not carry this out in detail since it is a straightforward application of standard procedures for these reductions, and reveals no new physics. However, there are several points which are unique to our application:

a) The Born approximation (zero-meson) term involves the matrix element of the interaction Hamiltonian between physical nucleon states. The operators involved are $\sigma$ and $\tau$, so the matrix element is, for instance,

$$ (n' | \sigma_{k} \tau_{k} | n) $$  \hspace{1cm} (3.23)

Since we have defined $\sigma$ and $\tau$ to transform as vectors (irreducible tensors of the first rank), we can immediately remove the dependence on magnetic quantum numbers. We have

$$ (0 | \sigma_{k} \tau_{k} | 0) \equiv \left( \frac{1}{2} m' \frac{1}{2} t' \right) \left( \sigma_{m_{k}} \tau_{t_{k}} \frac{1}{2} m \frac{1}{2} t \right) = $$

$$ = -(-1)^{m} + t \frac{1}{2} \left( \frac{1}{2} \frac{1}{2} - m' \frac{1}{2} \frac{1}{2} l - m_{k} \right) x $$

$$ x \left( \frac{1}{2} \frac{1}{2} - t' \frac{1}{2} l - t_{k} \right) (0 | \sigma \tau | 0) $$ \hspace{1cm} (3.24)

$(0 | \sigma \tau | 0)$ is the "reduced" matrix element and contains all the properties of the physical nucleon. The physical nucleon differs from the bare nucleon in its ever-present interaction with the meson field. We now follow Chew-Low and interpret the effect of this field to be a renormalization of the meson-nucleon coupling constant. Thus

$$ (0 | \sigma \tau | 0) = \frac{f}{f_{o}} (u | \sigma \tau | u) = 6 \frac{f}{f_{o}} $$ \hspace{1cm} (3.25)
where \( |u\rangle \) is the bare nucleon, \( f \) is the renormalized coupling constant, and \( (u|\sigma T|u) \) has been evaluated using (3.24) for particular values of the magnetic quantum numbers.

Then we have extracted the physical content from the matrix element (in fact, all the physical content which the spirit of this theory permits us to extract), and can proceed with the quantum number sums. It should be noticed that we have here demonstrated the oft-mentioned fact that the renormalization of the coupling constant is independent of magnetic quantum numbers.

b) Because of the symmetry between and in the Hamiltonian, the theory is completely symmetric between angular momentum and isotopic spin. Then, as has been often stated, we need be concerned with only one. We can then perform the summing over just the angular momentum quantum numbers.

The meson-production amplitude is specified by two angular momenta, the total momentum \( J \) and the partial momentum \( j \). For our case, in which only p-wave mesons interact, \( j \) and \( J \) can assume the values \( \frac{1}{2} \) and \( \frac{3}{2} \). Ordinarily the total angular momentum could have the value \( \frac{5}{2} \) since we have two mesons in the final state. But in our problem there is only one meson in the initial state and therefore a maximum angular momentum of \( \frac{3}{2} \) for the system.

We will then obtain four meson-production amplitudes which we label \( D_{2j}^{2J} \): \( D_{1}^{1}, D_{1}^{3}, D_{2}^{1}, D_{2}^{3} \). We obtain equations for each of these. However, the equations are coupled so that each equation contains all four amplitudes. The result can be conveniently
represented by considering $D'$ to be a four-component vector. The coupling between the equations is then represented by $4 \times 4$ matrices. Our result thus far is then a set of four coupled equations--integral equations, as we shall see.

We have chosen a particular coupling scheme in order to obtain the partial momentum $j$. This choice was perfectly arbitrary and was chosen merely for reasons of convenience which will become apparent later. Any other coupling scheme can be obtained from this by a unitary transformation. In the language we are using here this transformation is a rotation in a four-dimensional vector space. The elements of the unitary rotation matrix are the Racah coefficients $W^{(abcd;ef)}$. Since these coefficients are real, this matrix is also orthogonal. As an application of these properties, Dyson has used the transformation between different coupling scheme to show, within the framework of the Low equation, the relation between the unitarity statement and the dispersion relations.

c) If we now include isotopic spin, each component of our previous 4-component vector is itself a 4-component vector in the isotopic-spin representation. The isotopic spin can thus be included by writing the meson-production amplitude in the "product space", with 16 dimensions. The production amplitude becomes a 16-component vector; the matrices become $16 \times 16$ matrices. We therefore have 16 coupled equations.

However, the symmetry of this theory enables us to reduce this number somewhat. Let us first establish our notation: our
meson-production amplitude, including both angular momentum and isotopic spin will be written $D_{TJ}^{T_{12}}$, i.e. $D_{11}^{11}$, etc. (We follow the usual convention and use $2T$ and $2J$.)

The following relations follow immediately from the symmetry between angular momentum and isotopic spin (analogous to the identity of $\delta_{31}$ and $\delta_{13}$ for the scattering problem):

$$
\begin{align*}
D_{13}^{11} &= D_{21}^{11} \\
D_{11}^{13} &= D_{11}^{31} \\
D_{13}^{13} &= D_{21}^{21} \\
D_{31}^{13} &= D_{13}^{31} \\
D_{13}^{33} &= D_{13}^{33} \\
D_{31}^{33} &= D_{31}^{31}
\end{align*}
$$

(3.26)

The use of this symmetry therefore allows us to reduce the dimensionality of our "product space" from 16 to 10.

d) Two of the one-meson terms involve the meson-production amplitude, but with one meson "crossed"—negative energy, etc. This quantity differs from the crossed matrix element in having incoming rather than outgoing waves.

We note here the expansion and the notation we shall use for this quantity:

$$
T_{-s_1,q,-p} = (-s_1!V_p^+!0) = \sum y_s^+(\vec{s}) y_q(\vec{q}) y_p(\vec{p}) (pns_1!) x
$$

$$
x (j_1!j) (J_1!sn') T_j^J(-)
$$

(3.27)

We have here "cross-coupled" the particles; that is, we have coupled a meson in the final state to the nucleon in the initial state, and vice versa. It is not immediately obvious that this is legitimate—that is, that angular-momentum conservation still has a meaning in such a situation. But its correctness can be
checked by transforming the Clebsch-Gordan coefficients\textsuperscript{9} into the scheme of (3.11) where we know that angular momentum is conserved.

Now the reduction of (2.23) consists in performing sums over the magnetic quantum numbers of products of Clebsch-Gordan coefficients. These sums can be expressed in a form which is independent of the magnetic quantum numbers by means of Racah coefficients\textsuperscript{8,9}. Then the elements of the zero-meson term, and of the coupling matrices in the one-meson terms, are products of Racah coefficients. The final result of this reduction can be written

\[
D = \frac{A}{\omega_p} + \frac{B}{\omega_p + \omega_q} + \frac{2}{\pi} \int d\omega_s s^3 v^s(p) \left\{ \frac{\kappa D}{\omega_s - \omega_p - i\epsilon} - \frac{\mathcal{L} F}{\omega_s + \omega_p} \right. \\
- \frac{\mathcal{M} F}{\omega_s - \omega_p - \omega_q - i\epsilon} + \frac{\mathcal{N} D}{\omega_s + \omega_p + \omega_q} \left. \right\}
\]

where \( F \) is defined by

\[
T_j^J(\cdot) = i \left( \frac{4\pi}{i} \right)^3 \sqrt{\mathcal{E}} p q s v(p) v(q) v(s) F_j^J
\]

(3.28)

The detailed results are explained in Appendix B. The general form is as follows: \( D \) is a 10-component vector. \( A \) and \( B \) are also 10-component vectors which contain the single-meson scattering amplitudes \( S_{TJ}(\omega_q) \). Otherwise they are energy-independent numbers. \( \kappa, \mathcal{L}, \mathcal{M}, \) and \( \mathcal{N} \) are 10 x 10 matrices which also contain \( S_{TJ}(\omega_s) \). This is their only energy dependence.

These equations have the form of 10 coupled integral equations. However, the situation is somewhat worse than this since they contain \( F \) as well as \( D \). \( F \) satisfies the same equation as \( D \), but with the sign of \( \omega_p \) reversed. We thus really have 20 coupled equations.
It is appropriate here to say something about the energy-dependencies of the scattering and production amplitudes. For the scattering case, it turns out that $S_{TJ}$ is a function of the final meson energy only. This result arises because the energy dependence of the interaction Hamiltonian is factorable. For this reason the scattering amplitudes off and on the energy shell are closely related.

The equation for $S_{TJ}$ is, in the one-meson approximation,

$$S_1(z) = \frac{\lambda_1}{z} + \frac{2}{\pi} \int d\omega_s s^s v^s(s) \left| \frac{S_1(\omega_s)}{\omega_s - z} \right|^2$$

$$+ \sum_j A_{1j} \frac{2}{\pi} \int d\omega_s s^s v^s(s) \left| \frac{S_1(\omega_s)}{\omega_s + z} \right|^2$$

where

$$\lambda_1 = \begin{pmatrix} 11 \\ 33 \end{pmatrix} ; \quad \lambda_1 = \frac{2}{3} \begin{pmatrix} -4 \\ -1/2 \end{pmatrix} ; \quad A_{1j} = \frac{1}{9} \begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}$$

The function $S_{TJ}(z)$ is defined by this equation for all $z$. It approaches the physical scattering amplitude for $z \rightarrow \omega_p + i\epsilon$; taking the imaginary part of this equation gives the one-meson approximation to the unitarity statement (3.20):

$$\lim_{z \rightarrow \omega_p + i\epsilon} S_1(z) = \frac{1}{2} \frac{e^{\frac{i}{2} \delta_1(p)}}{p^s v^s(p)} \frac{\sin \delta_1(p)}{p^s v^s(p)}$$

We see that $S_{TJ}$ depends only on $z$, or $\omega_p$, as stated above.

The same situation holds in our case also. The vectors $A$ and $B$ are functions of $\omega_q$. Then the energy dependence of $D$ is $D(\omega_p, \omega_q)$. Similarly we can write $F$ as $F(-\omega_p, \omega_q)$. In terms of the variable $z$, these are $D(z, \omega_q)$ and $F(-z^+, \omega_q)$. 
With these energy dependences made explicit, (3.28) is

\[ D(z, \omega_q) = \frac{A(\omega_q)}{z} + \frac{B(\omega_q)}{z + \omega_q} + \frac{2}{\pi} \int d\omega_s s^a v^a(s) \left\{ \begin{array}{c} \frac{\mathcal{K}(\omega_s) D(\omega_s, \omega_q)}{\omega_s - z} \\ \mathcal{L}(\omega_s) F(-\omega_s, \omega_q) \\ \mathcal{M}(\omega_s) F(-\omega_s, \omega_q) \\ \frac{\mathcal{N}(\omega_s) D(\omega_s, \omega_q)}{\omega_s + \omega_q + z} \end{array} \right\} \]

(3.28)

We see that in our case also the amplitude is a function of only the final meson energies; \( \omega_k \) does not enter into the equations (This is another way in which Feynman techniques are not applicable—the denominator \( \frac{1}{\omega_s - \omega_p - \omega_q} \) would be written \( \frac{1}{\omega_s - \omega_k} \), using ordinary methods).

We now proceed to the approximations which are needed to uncouple and solve these equations.
IV. APPROXIMATE SOLUTION

The equations we have obtained consist of a set of 20 coupled integral equations. In order to uncouple these equations we must make further approximations. On the basis of known low-energy data we neglect all scattering except in the 33-state. If we also neglect the terms with non-resonant denominators, the equations become uncoupled. We examine the results of our choice of a scheme to couple the momenta of the outgoing mesons and nucleon. This leads to an understanding of the structure of the matrices obtained in the angular-momentum reduction. We then proceed to solve the equations making use of the Chew-Low equation for the scattering amplitude, and of the techniques of analytic function theory. One interesting result is that the inclusion of the 33-scattering amplitude and the neglect of the others makes the production amplitudes for the $T = J = \frac{3}{2}$ state quite different from those of the other states.

A. 33-Resonance Approximation

The present static-source model is a low-energy theory, and assumes that high-energy effects do not make significant contributions to low-energy processes. We then want to include only known low-energy scattering data in finding our solution to the meson-production problem. Our results will be those contributions to the production process which arise from these low-energy effects. We base this viewpoint not only on the limitations of the
theory but also on the lack of experimental data on high-energy scattering.

The first and most important approximation will be to neglect all meson scattering except in the 33-state \((T = J = \frac{3}{2})\). Experimentally\(^{12}\) it is known that this state has a resonance for an incident kinetic energy (in the laboratory system) of about 165 Mev. The phase shifts for the other states are only known reliably for very low energies, where they are not masked by the resonance. For these energies it is known that they are small, of the order of \(\frac{1}{10}\) the 33-phase shift. Then we can feel quite confident of this approximation for low energies.

There does appear to be theoretical evidence\(^{16,17}\) that the neglect of the high-energy behavior of the 11 and 13 phase shifts is not consistent with the present theory. Our results can serve as another, and very different, check of the theory.

The experimental information on high-energy pion-nucleon interactions might also cast doubts about the validity of this approximation. The total cross-sections are not small, and in fact the total cross-section in the \(T = \frac{1}{2}\) state shows a peak around 1 Bev. However, this peak is too large to be attributable to the scattering states we have neglected\(^{13,14,15}\).

We will then consider our approximation to be consistent with the spirit of the theory and with the presently-available experimental information.

Our approximation therefore consists in neglecting \(S_{11}(\omega)\) and \(S_{13}(\omega)\) for all value of \(\omega\) (remember that \(S_{13} = S_{31}\)). Keeping only \(S_{33}(\omega)\), the vectors \(A(\omega_q)\) and \(B(\omega_q)\), and the matrices
Denoting $S_{33} = S_3$, we can write these quantities as

$$A(\omega_q) = S_3(\omega_q) A \quad B(\omega_q) = S_3(\omega_q) B$$

$$\kappa(\omega_s) = S_3^+(\omega_s) \kappa \quad \mathcal{L}(\omega_s) = S_3(\omega_s) \mathcal{L} \quad \mathcal{M}(\omega_s) = S_3(\omega_s) \mathcal{M} \quad \mathcal{N}(\omega_s) = S_3^+(\omega_s) \mathcal{N}$$

where $A$ and $B$ are constant numerical vectors, and $\kappa$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are constant numerical matrices.

Using these substitutions we can write our equation as

$$D(\omega_p, \omega_q) = \frac{S_3(\omega_q) A}{\omega_p} + \frac{S_3(\omega_q) B}{\omega_p + \omega_q} + \frac{2}{\pi} \int d\omega_s \frac{s^s v^s(s)}{\omega_s - \omega_p - i\varepsilon} \left\{ \frac{S_3^+(\omega_s) \kappa D(\omega_s, \omega_q)}{\omega_s - \omega_p} - \frac{S_3^+(\omega_s) \mathcal{L} F(-\omega_s, \omega_q)}{\omega_s + \omega_p} - \frac{S_3^+(\omega_s) \mathcal{M} F(-\omega_s, \omega_q)}{\omega_s - \omega_q - \omega_p} + \frac{S_3^+(\omega_s) \mathcal{N} D(\omega_s, \omega_q)}{\omega_s + \omega_q + \omega_p} \right\}$$

(4.2)

We see that the dependence of $D(\omega_p, \omega_q)$ on $\omega_q$ is such that we can immediately write

$$D(\omega_p, \omega_q) = S_3(\omega_q) D'(\omega_p, \omega_q)$$

(4.3)

Notice that $D'(\omega_p, \omega_q)$ does depend on $\omega_q$, so that $D(\omega_p, \omega_q)$ does not have the energy dependence of a meson which is simply scattered.

In the interests of notational economy we will drop the "prime" from $D'$, remembering that in our final answer it must be multiplied by $S_3(\omega_q)$. 

$\kappa(\omega_s), \mathcal{L}(\omega_s), \mathcal{M}(\omega_s), \mathcal{N}(\omega_s)$ become simply multiples of $S_{33}$. 

...
Then the equation we must solve is

\[
D(\omega_p, \omega_q) = \frac{A}{\omega_p} + \frac{B}{\omega_p + \omega_q} + \\
+ \frac{2}{\pi} \int d\omega_s \, s^3 \, v^s(s) \left\{ \frac{S_3^+(\omega_s) \mathcal{X} D(\omega_s, \omega_q)}{\omega_s - \omega_p - i\epsilon} - \\
- \frac{S_3(\omega_s) \mathcal{F}(-\omega_s, \omega_q)}{\omega_s - \omega_p} - \frac{S_3(\omega_s) \mathcal{M} F(-\omega_s, \omega_q)}{\omega_s - \omega_q - \omega_p - i\epsilon} + \\
+ \frac{S_3^+(\omega_s) \mathcal{N} D(\omega_s, \omega_q)}{\omega_s + \omega_q + \omega_p} \right\}
\]

(4.4)

The numerical quantities are given in Appendix C.

The structure of the matrices in the one-meson term is now of interest. To illustrate the reasons for their particular structure we shall resort to the use of diagrams—remembering all the while the reservations noted earlier concerning their use.

The first term, with denominator \( \frac{1}{\omega_s - \omega_p - i\epsilon} \), will be represented as

\[
\text{Circle 1 represents the matrix element } D; \text{ circle 2 represents } S. \text{ We see now why the matrix } \mathcal{X} \text{ is diagonal. The meson incident on circle 1 is the incident meson of our problem, } k; \text{ therefore, the total angular momentum and isotopic spin involved in this term are the same as those for the total problem. Meson } p \text{ is scattered in circle 2. Since angular momentum and isotopic spin are conserved by such a scattering, meson } s \text{ is characterized by}
\]
the quantities $t$ and $j$. These are what we previously called the partial momenta, with which meson $p$ is coupled to final nucleon. Then the element $D$ in this term is the same as the element $D$ on the left side of (4.4); thus the matrix $K$ is diagonal.

With our assumption about the size of the various scattering matrix elements, this term is non-zero only if meson $s$, and therefore meson $p$, is coupled to the nucleon in a $t = j = \frac{3}{2}$ state. Then this term only enters in the equations for the quantities $D_{3\bar{3}}^{TJ}(\omega_p, \omega_q)$.

The third term, with denominator $\frac{1}{\omega_s - \omega_q - \omega_p - i\varepsilon}$ can be represented as

Circle 1 represented a scattering of the incident meson $k$. Since, under our assumptions, this can only occur in the $3\bar{3}$-state, this term only contributes for $T = J = \frac{3}{2}$; i.e., it enters only in the equations for $D_{3\bar{3}}^{TJ}(\omega_p, \omega_q)$.

The remaining terms can be discussed similarly, but we shall not do so since our next approximation will be to neglect them. We shall call these terms the "crossed" terms, by analogy with the meson-scattering case, since they have non-resonant denominators (no poles for positive energies).

The basis for neglecting them cannot be simply that they have no poles. The resonant denominators are treated as a delta-function plus a principal-value integral through the identity
\[
\frac{1}{\omega_s - \omega - i\epsilon} = P \frac{1}{\omega_s - \omega} + \pi i \delta(\omega_s - \omega) \quad (4.5)
\]

The delta-function picks out only one point of the integrand, and the principal value takes a "symmetrical average" so that the singularity is cancelled out. Thus the integrals containing resonant denominators need not be exceptionally large. They may very well have strong energy dependences, however. On the other hand, the crossed terms will have a weak energy dependence, roughly like \(\frac{1}{\omega}\). In addition it may happen that the numerical matrices act to reduce their contribution (in certain cases this happens in the scattering case; see below). We will later check their contribution by iteration. It might be hoped that this would be sufficient because of their weak energy dependence.

In our approximation of neglecting \(S_{11}\) and \(S_{13}\), the equation for \(S_3\) becomes

\[
S_3(\omega_p) = \frac{\lambda_3}{\omega_p} + \frac{2}{\pi} \int d\omega_s \, s^3 \, v^2(s) \left\{ \frac{1}{\omega_s - \omega_p - i\epsilon} + \frac{\frac{1}{2}}{\omega_s + \omega_p} \right\} |S_3(\omega_s)|^2
\]

where \(\lambda_3 = \frac{2}{3} f^2 = 0.0533\).

We now propose to neglect the crossed term here also. If this is done, this equation can be solved exactly. However, this will not be our prime concern here since we are going to use the experimental phase shifts rather than the Chew-Low solution. A check of the size of the crossed term, by inserting the experimental phase shifts (see next chapter) shows that the crossed integral is of the same order of magnitude, or somewhat smaller than, the uncrossed integral. However the factor of \(\frac{1}{\eta}\), from the
crossing matrix, results in our making only a 10% error by dropping this term.

Then we take as the equation for the scattering amplitude

\[ S_3(z) = \frac{\lambda_3}{z} + \frac{2}{\pi} \int d\omega_s \ s^a \ v^a(s) \ \frac{|S_3(\omega_s)|^2}{\omega_s - z} \]  

(4.7)

where we write \( z = \omega_p + i \epsilon \).

B. Solutions

We will first discuss the production amplitude for the cases when \( T \) and \( J \) are not \( \frac{3}{2} \). In these cases the term involving the matrix \( M \) does not enter, and the equations (4.4) become (writing out the vectors in column form)

\[
\begin{pmatrix}
D^{11}_{11}(z,\omega_q) \\
D^{11}_{13}(z,\omega_q) \\
D^{11}_{33}(z,\omega_q)
\end{pmatrix} = \frac{f}{81} \begin{pmatrix}
\frac{1}{z} \\
\frac{1}{8\sqrt{2}} \\
4
\end{pmatrix} + \frac{f}{81} \begin{pmatrix}
\frac{1}{z + \omega_q} \\
-4\sqrt{2} \\
40
\end{pmatrix}
\]

\[ + \frac{2}{\pi} \int d\omega_s \ s^a \ v^a(s) \ \frac{S_2^+(\omega_s)}{\omega_s - z} \begin{pmatrix}
0 \\
0 \\
D^{11}_{33}(\omega_s,\omega_q)
\end{pmatrix} \]  

(4.8)

with a similar set of equations for \( D^{13}_{tj} \), using the inhomogeneous terms given in Appendix C.

We see that the terms for which \( t \) and \( j \) are not \( \frac{3}{2} \) are given by the Born approximation terms (times \( S_3(\omega_q) \), of course).
$D_{11}^{33}$ and $D_{13}^{33}$ are given by linear, uncoupled, integral equations of the form

$$D(z, \omega_q) = \frac{A}{z} + \frac{B}{z + \omega_q} + \frac{2}{\pi} \int d\omega_S s^* v^2(s) \frac{S_+^+(\omega_S) D(\omega_S, \omega_q)}{\omega_S - z} \quad (4.9)$$

Since the equation is linear in $D$, we can let $D$ be the sum of two terms, each arising from one of the inhomogeneous terms.

If we refer to equation (4.7) for the meson-scattering amplitude $S_3(z)$, we see that the contribution of the first term is simply

$$\frac{A}{\lambda_3} S_3(z).$$

To find the contribution of the second term, we observe that, as in the scattering case, the function $D$ defined by this equation is an analytic function of $z$ except on the real axis. It has simple poles on the real axis arising from the inhomogeneous terms, and a line of poles--or a branch cut--from 1 to $\omega$. Since $D$ is analytic except for simple poles, and is finite at infinity (going as $\frac{1}{z}$ for $z \to \infty$), it is specified entirely by its residues at the poles.

Then let us represent the contribution of the second term as $g(z, \omega_q) S_3(z)$ where $g(z, \omega_q)$ will be a function which is real on the axis. We want this function to have a pole at $z = -\omega_q$ with a residue $B$; but we want no pole at $z = 0$. Since $S_3(z)$ has a pole at $z = 0$, $g(z, \omega_q)$ must be proportional to $z$.

These conditions will be satisfied if we let

$$g(z, \omega_q) = \frac{-Bz}{\omega_q (z + \omega_q)} S_3(-\omega_q)$$

The line of poles from $z = 1$ to $\omega$ is contributed by $S_3(z)$ which, from (4.7), has these poles.
Now we notice that $S_3(z)$ as defined by (4.7) does not satisfy the one-meson crossing theorem\(^1,\)\(^5\). This theorem relates the vector $S(z)$ to $S(-z)$ by a matrix transformation. Since it is not satisfied in our approximation, the question arises as to the meaning of $S_3(-\omega_q)$ which occurs in $g(z, \omega_q)$. This question does not arise, of course, if one simply takes (4.7) as the defining equation for $S_3(z)$ and ignores all previous knowledge about a crossing theorem. From that equation, $S_3(-z)$ would be defined simply by changing the sign of $z$ on the right-hand side. This is confirmed by taking the limit of our solution as $z \to \infty$, and comparing this with the limit of the right-hand side of (4.9) as $z \to \infty$. This shows unambiguously that we must indeed obtain $S_3(-z)$ in this way.

Then the solutions we obtain for $D^{11}_{33}$ and $D^{13}_{33}$ have the form:

$$D^{11}_{33}(\omega_p, \omega_q) = \left\{ \begin{array}{c} \frac{A_{33}}{\lambda_3} - \frac{B_{33}}{\omega_q (\omega_q + \omega_p)} S_3(-\omega_q) \\ \omega_p \end{array} \right\} S_3(\omega_p) \quad (4.10)$$

We turn now to the solutions for $T = J = \frac{3}{2}$. These involve $F(-\omega_p, \omega_q)$ which we must now find. The equation for this quantity can be found simply by changing $\omega_p + i\epsilon$ to $-\omega_p + i\epsilon$ in (2.23). It will be remembered that in (3.27) we defined the angular momentum coupling (of the pions to the nucleon) for this matrix element to be different from that which we used for $D(\omega_p, \omega_q)$ in (3.11). We must take (2.23), the equation before angular momentum reduction, introduce the new coupling scheme, and perform the sums. The resulting equation has the same form as (4.4) except for the change in sign of $\omega_p$.

It will turn out that $D(\omega_p, \omega_q)$ is small compared to $F(-\omega_p, \omega_q)$ so that we can neglect the terms corresponding to those containing
the $\mathcal{K}$ and $\mathcal{N}$ matrices. In addition, unless $\omega_q$ is large, the denominator $\frac{1}{\omega_s - \omega_q + \omega_p}$ will not have a pole. Unfortunately, the only positive statement we can make relating terms with and without poles is that the energy variation of the terms without poles is smaller than that of the terms containing poles. However, the only practical way to obtain a solution is to ignore the terms which one does not think will be important, and then perform an iteration to test their size. We will take this viewpoint.

A further justification for neglecting these terms is that our only use of $F(-\omega_p, \omega_q)$ will be in the integral of the $\mathcal{M}$-term of (4.4). The result should be to some extent independent of the detailed structure of $F(-\omega_p, \omega_q)$.

We then ignore all integral terms except that with the denominator $\frac{1}{\omega_s - \omega_p + i\epsilon}$. Doing this we obtain (with the same approximation as before concerning the scattering amplitudes):

$$F(-\omega_p, \omega_q) = \frac{B}{\omega_p} - \frac{A}{\omega_q - \omega_p} + \frac{2}{\pi} \int d\omega_s \, s^s v^s(s) \frac{S^s(\omega_s) \mathcal{M} F(-\omega_s, \omega_q)}{\omega_s - \omega_p + i\epsilon}$$

(4.11)

Here $\mathcal{M}$ is the same matrix as that occurring in (4.4) and defined in (C.3). Therefore the integral term enters only in the equations for $F_{tj}^{33}(-\omega_p, \omega_q)$. However, these are just the terms we need to solve (4.4).

The vectors $A$ and $B$ are the same as those in (4.4) and defined by (C.1). It is interesting that they occur in this equation, but with their positions reversed compared with (4.4). This can be given some intuitive meaning in terms of diagrams. $F$ is closely related to the matrix element for the inverse process as given by (2.11). If we then consider the diagrams which we would use to
represent the Born terms in the equation for $D$, and reverse the direction of meson $p$, we see that the diagrams are exchanged. The coefficients will be the same if the coupling scheme is appropriately chosen.

We again use the technique of adjusting the poles and the residues of the function $F(-z, \omega_q)$, and obtain the solutions

$$F_{tj}^{33}(-\omega_p, \omega_q) = \left\{ \frac{B_{t1}^{33}}{\lambda_3} + \frac{A_{t1}^{33}}{S_{3}^{+}(\omega_q)} \frac{\omega_p}{\omega_q - \omega_p} \right\} S_{3}^{+}(\omega_p) \quad (4.12)$$

We now insert these solutions in the integrals occurring in (4.4). Using (4.7), the equation for $S_3(z)$, these integrals can be expressed in terms of $S_3(z)$. The resulting set of equations is

$$D_{tj}^{33}(z, \omega_q) = \left\{ \frac{B_{t1}^{33}}{\lambda_3} + \frac{A_{t1}^{33}}{S_{3}^{+}(\omega_q)} \frac{z + \omega_q}{z \omega_q} \right\} S_{3}(z + \omega_q) +$$

$$+ \frac{2}{\pi} \int d\omega_s s^3 v^3(s) \frac{S_{3}^{+}(\omega_s) D_{tj}^{33}(\omega_s, \omega_q) \delta_{t,2} \delta_{j,2}}{\omega_s - z} \quad (4.13)$$

where we have written the elements of the matrix $\mathcal{K}$ explicitly.

Thus the integral containing $F(-\omega_p, \omega_q)$ has cancelled the Born approximation terms. Of course, this cancellation is only apparent since the poles at $z = 0$ and $z = -\omega_q$ are still present in the inhomogeneous term. If we now treat these terms as the inhomogeneous parts of an integral equation, we will find that this cancellation has indeed made a tremendous difference. The behavior at the poles of the Born terms is unchanged, but these poles are not in the physically-interesting region. What is
changed is the behavior of the inhomogeneous term in the physical region. This is now controlled by the behavior of the meson-scattering amplitude.

Several points require discussion before we can accept this result. The F-term is obviously important. If we had originally thought it small, we would have solved the D-equation without it, and then added on its contribution. If we do this, we find that the terms which have acted to cancel the Born terms in the above equation perform this same task here. The cancellation is not as complete, about 40%, but in such a direction and of such a magnitude as to indicate that such a perturbation treatment is not sufficient. As our above result shows, this is indeed the case.

We have here followed the steps: a) neglected D in the equation for F; b) solved for F; c) inserted F in the equation for D. We might try the reverse procedure: a) neglect F in the equation for D; b) solve for D; c) insert D in the equation for F (in some of the terms which we have neglected in writing (4.11).

If we do this, we find that the Born terms of the F-equation are cancelled. As we stated above, however, their poles are still in the remaining terms. In this case one of the poles is in the physical region. This, plus the position of the poles in the integral terms, has the effect of maintaining the dominance of the original Born terms—even though they now appear in disguised form. Therefore, if we now solve for F, we find a solution which is not very different from the one obtained by neglecting D.
When we "complete the cycle", and insert this solution for $F$ into the D-equation, we obtain the same result.

A last point might be the question of the dependence of the result in (4.13) on the form of $F$. This is difficult to test since other forms result in integrals which cannot be evaluated so simply; and a numerical integral over an integrand which contains at least two poles is not easy to do reliably. Some examples seem to indicate that the pole at $\omega_q = \omega_p$, the pole in the integral at $\omega_p$, and the pole in the integral at $\omega_p + \omega_q$, combine to dominate the behavior of the F-integral. The final result seems to be not greatly disturbed by changes in $F$, as long as the poles are unchanged.

We can interpret this cancellation by remembering that this formulation of field theory "buries" many virtual processes within each term. In this case it turns out that the F-integral contains within it the processes represented by the Born terms.

The question naturally arises whether the inclusion of the other scattering states would not produce similar cancellations in other terms—since terms similar to the F-integral would appear in the other equations if it were not for our neglect of these states. It would seem that this would be the case, since the cancellation did not depend on the sign of $\lambda_3$ (which determines the low-energy behavior of the scattering, and in particular, the resonance). However, the inclusion of these terms would take us far afield from our 33-resonance model. In addition, it would not be consistent with our approach, which is to use only known
low-energy effects. We will later view our results as a test of these assumptions.

Returning to (4.13), this can be solved by the same techniques as before. Here the inhomogeneous term contains a line of poles, but the technique is essentially unchanged. The result for $D_{33}^{33}$ is ($D_{11}^{33}$ and $D_{13}^{33}$ are given by the inhomogeneous terms alone)

$$D_{33}^{33}(\omega_p, \omega_q) = \left\{ \frac{B_{33}^{33}}{\lambda_3} + \frac{A_{33}^{33}}{S_{+}^{3}(\omega_q)} \right\} \left( S_3(\omega_p + \omega_q) \right) - \omega_p S_3(\omega_p) C_3(\omega_p, \omega_q)$$

$$+ \frac{A_{33}^{33}}{S_{+}^{3}(\omega_q)} S_3(\omega_p) C_3(0, \omega_q)$$

(4.14)

where

$$C_3(\omega_p, \omega_q) = \frac{2}{\pi} \int d\omega_s s^a v^a(s) \frac{S_3(\omega_s + \omega_q)}{\omega_s(\omega_p + i\varepsilon - \omega_q)}$$

We now collect our results, and remember (4.3), in which we have extracted $S_3(\omega_q)$. Our solutions are presented on the next page.
\[ D_{11}^{11}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) \left\{ \frac{32}{\omega_p} - \frac{64}{\omega_p + \omega_q} \right\} \]
\[ D_{13}^{11}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) \left\{ \frac{8\sqrt{2}}{\omega_p} - \frac{4\sqrt{2}}{\omega_p + \omega_q} \right\} \]
\[ D_{33}^{11}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) S_3(\omega_p) \left\{ \frac{4}{\lambda_3} - \frac{40}{\omega_q} \frac{\omega_p}{\omega_p + \omega_q} \right\} \]
\[ D_{11}^{13}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) \left\{ \frac{-20}{\omega_p} + \frac{4}{\omega_p + \omega_q} \right\} \]
\[ D_{13}^{13}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) \left\{ \frac{-8\sqrt{5}}{\omega_p} + \frac{4\sqrt{5}}{\omega_p + \omega_q} \right\} \]
\[ D_{31}^{13}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) \left\{ \frac{-5\sqrt{2}}{\omega_p} + \frac{7\sqrt{2}}{\omega_p + \omega_q} \right\} \]
\[ D_{33}^{13}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) S_3(\omega_p) \left\{ \frac{-2\sqrt{10}}{\lambda_3} - \frac{7\sqrt{10}}{\omega_q} \frac{\omega_p}{\omega_p + \omega_q} \right\} \]
\[ D_{11}^{33}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) S_3(\omega_p + \omega_q) \left\{ \frac{2}{\lambda_3} - \frac{28}{S_3^+(\omega_q)} \frac{\omega_p + \omega_q}{\omega_p \omega_q} \right\} \]
\[ D_{13}^{33}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) S_3(\omega_p + \omega_q) \left\{ \frac{2\sqrt{5}}{\lambda_3} - \frac{5\sqrt{5}}{S_3^+(\omega_q)} \frac{\omega_p + \omega_q}{\omega_p \omega_q} \right\} \]
\[ D_{33}^{33}(\omega_p, \omega_q) = \frac{r}{81} S_3(\omega_q) \left\{ \frac{10}{\lambda_3} + \frac{10}{S_3^+(\omega_q)} \frac{\omega_p + \omega_q}{\omega_p \omega_q} \right\} \]
\[ x \left( S_3(\omega_p + \omega_q) - \omega_p S_3(\omega_p) C_3(\omega_p, \omega_q) + 10 \frac{S_3(\omega_p)}{S_3^+(\omega_q)} C_3(0, \omega_q) \right) \]

(4.15)
V. RESULTS

In this chapter we use our solutions to obtain the cross-sections for production of a single pion in pion-nucleon collisions. Our solutions involve the scattering amplitude for the 33-state; we use the experimental value of the phase shift to determine this quantity. We then find that production in the 33-state is greatly reduced compared to the other states. The contributions of the crossed terms are evaluated and found to be small. We then discuss kinematic corrections to the no-recoil approximation; these affect chiefly the density of final states. The production cross-sections for $\pi^+ + p$ and $\pi^- + p$ are evaluated. The former shows a peak for an incident energy of about 700 Mev. However, the results violate the limits placed on the cross-section by the requirements of unitarity; the implications of this for the present theory are discussed. We then evaluate the two-meson corrections to the Chew-Low equation for the scattering amplitude, and find that these are large. Finally we discuss a test of the approximations we have used for the production problem.

A. Representation of the Scattering Amplitude

We now have the meson-production amplitudes expressed in terms of the scattering amplitude for the 33-state. For numerical evaluation we require a representation of this amplitude as a function of energy. We shall use the experimental phase shift; the motivation for this is the hope that our solutions may perhaps be relations between the production and scattering amplitudes...
which are to some extent independent of the approximation represented by (4.7)—the one-meson approximation with the crossing term neglected. The most recent\textsuperscript{12} evaluation of the phase shift gives
\[
\tan \delta_{33}(p) = \frac{.25 p^3}{1 + .79 p^3 \left( \frac{1.25 - 1}{1.95 - \omega_p} \right)}
\] (5.1)
(We use throughout the units \( \mu = 1 \).)
This form has been used since it corresponds to the expected low-energy behavior of a \( p \)-wave phase shift, and has a resonance behavior. We remember that
\[
S_3(\omega_p) = \frac{1}{2} \frac{1}{p^3} \frac{\delta_{33}(p)}{\sin \delta_{33}(p)} \sin \delta_{33}(p).
\] (5.2)

The form (5.1) has been used to fit the experimental data for incident kinetic energies up to 300 Mev. However, in some of the integrals we must use, and to examine our process at high energies, we require the phase shift up to energies of the order of 1 Bev. We shall assume that the \( 33 \)-phase shift remains fairly constant and real at high energies; we do this most conveniently simply by using (5.1) for all energies. A numerical solution\textsuperscript{17} of the Chew-Low equation for the scattering amplitude indicates another \( 33 \)-resonance for \( \omega_p = 5 \), but such a resonance is not observed experimentally. We shall continue to follow the policy of including only known low-energy data. Assuming the high-energy phase shift constant (reaching a maximum of about \( 155^\circ \)) has the additional advantage of making \( S_3(\omega_p) \) quite small at high energies. We are going to assume a square cut-off at \( \omega_p = 6 \), simply for convenience. Then the factor of \( \frac{1}{p^3} \) makes the high-energy
contributions of $S_3(\omega_p)$ quite negligible. We assume it real on
the hypothesis that production is small (the basis of the one-
meson approximation in the scattering problem).

B. Production in the 33-State

We first examine the production amplitudes for $T = J = \frac{3}{2}$. With
the above assumptions about the scattering amplitude, they
are all small compared to those for other states. To obtain an
order of magnitude: the ratio of $D_{ij}^{33}$ to $D_{i3}^{TJ}$ ($T$ and $J$ not $\frac{3}{2}$) is
of the order of

$$\left| \frac{S_3(\omega_p + \omega_q)}{S_3(\omega_p)} \right|.$$ 

For $\omega_p = \omega_q = 1.5$ (incident energy of about 400 Mev) this is $\frac{1}{10}$.

$S_3(\omega_p + \omega_q)$ is always above resonance since $\omega_p + \omega_q \geq 2$;
this plus the $\frac{\lambda_p}{\rho}$ behavior cause it to be much smaller, for all
real energies, than $S_3(\omega_p)$. Therefore the integral containing
the $M$ matrix and $F(-\omega_p, \omega_q)$ has had a large damping effect on
these terms. Of course, this damping effect does not hold for
all values of the energy; in the non-physical region, for
$\omega_p + \omega_q < 2$, these terms are enhanced. But in the region of
physical production these terms turn out to be practically negli-
gible (Even in the Born approximation, neglecting all integral
terms, production in the 33-state is smaller by a factor of
about 3 compared to that of the other states; this is just be-
cause of the size of the coefficients which result from the
angular-momentum-isotopic-spin reduction.)
We have evaluated the integral (4.14) for $c_3(m, m)$ and $p_q$. We have found that the contribution of this term to $D_{33}^{33}$ is somewhat smaller than that of $S_{33}(\omega_p + \omega_q)$. Therefore the conclusion is the same for all $T = J = \frac{3}{2}$ amplitudes: under the present assumptions they are negligible.

C. Contribution of the "Crossed" Terms

We must examine the error which has been caused by the neglect of the terms in (4.4) which contain no poles. We shall do this by iteration, using the solutions (4.15).

In order to do this we require a representation of $S_3(-\omega)$ which appears in the solutions for $D_{33}^{33}$ and $D_{33}^{13}$. This function is defined by

$$S_3(-\omega) = \frac{-\lambda_3}{\omega} + \frac{2}{\pi} \int d\omega_8 s^3 v^2(s) \frac{|S_3(\omega_8)|}{\omega_8 + \omega}$$

Using the phase shift as given in (5.1), we can evaluate this numerically. It is found that the contribution of the integral is small compared to that of the inhomogeneous term and has nearly the same energy dependence. It can thus be represented (within 10%) by

$$S_3(-\omega) \approx \frac{-0.20}{\omega} \quad (5.3)$$

We can now use this to obtain the crossed terms. These involve the matrices $\mathbf{L}$ and $\mathbf{N}$; when these are evaluated and all except the $33$-scattering state neglected, it is found that the integrals involve only the production amplitudes for which $t = j = \frac{3}{2}$ (see (B.8)).
The results can be described as follows: the crossed terms have a weak energy dependence, as expected, varying by about a factor of 3 over the energy range $1 < \omega < 6$. Their contribution to $D_{33}^{11}$ and $D_{33}^{13}$ is less than 5% of the solution given in (4.15). The contribution to $D_{33}^{33}$ is about 20%, chiefly because our original solution is so small. The contributions to the other terms vary from 5% to 30%.

Thus for the $t = j = \frac{3}{2}$ amplitudes the crossed terms are quite small. These are the terms for which we have solved an integral equation; we thus expect our solutions to remain valid even in the presence of the crossed terms.

The crossed terms make relatively large contributions to the other amplitudes, chiefly because these amplitudes are smaller than the resonating $t = j = \frac{3}{2}$ amplitudes. However, these other amplitudes have the same general energy dependence as the crossed terms. This, plus the fact that they are small, leads us to believe that we will not make a large error if we neglect the crossed terms entirely.

These results have been obtained for low energies. Since our solutions decrease fairly rapidly for large energies, while the crossed terms do not, we expect their contribution to be larger at higher energies. However, the energies involved are those of the final mesons. These are small even for large incident-meson energies, so that our conclusions should hold over all ranges of the total energy which we shall consider.
D. Production Cross-Sections

The general expression for the production cross-section is

$$\sigma'(pq,k) = \frac{2\pi}{v_k} \left| \langle pq \mid T \mid k \rangle \right|^2 \rho(E)$$  \hspace{1cm} (5.4)

To apply this we use the expansion (3.11) in terms of spherical harmonics and Clebsch-Gordan coefficients, and the representation (3.22) to remove some of the energy dependences. The double sums over angular momenta and isotopic spin can be reduced by methods similar to those of Ref. 9. The result can be expressed entirely in terms of real functions and the real spherical harmonics

$$Y_L^J(p) Y_{L'}^J(q).$$

We shall first consider the effects of recoil on the cross-section (except that we consider no change in the meson theory arising from these effects). In the absence of nucleon motion,

$$v_k = \frac{k}{v_k},$$  \hspace{1cm} (5.5)

$$\rho(E) = \frac{p \cdot q \, \Phi_p \, \Phi_q \, d\Omega_p \, d\Omega_q}{(2\pi)^6}$$

We will now treat the nucleon non-relativistically; for incident kinetic energies up to 1200 Mev this introduces an error of less than 5%.

The no-recoil approximation assumes that the momentum of the nucleon is unchanged by the interaction with the meson. This is rigorously true for the kinetic energy of the nucleon in the center-of-mass system for a scattering process. For a meson-production process the kinetic energy of the nucleon will be
reduced since there are additional particles to carry momentum and energy. Therefore there is more energy available to the mesons in the final state than the no-recoil approximation would suggest. Such an effect is important since at low energies the cross-section is a sensitive function of the available volume in phase space.

If we denote the momentum of the nucleon by \( \vec{p} \), the mass of the nucleon by \( M \), the total center-of-mass energy by \( E_0 \), and the incident kinetic energy by \( T \), we have

\[
\vec{p} + \vec{q} + \vec{P} = 0
\]

\[
E_0 = \sqrt{(M + 1)^2 + 2MT}
\tag{5.6}
\]

and

\[
E_0 = M + \frac{\left( \frac{\vec{p}}{2M} + \frac{\vec{q}}{2M} \right)^2}{2M} + \omega_p + \omega_q
\]

To find the total cross-section for production we use (5.5) and integrate over the energy of meson \( p \) (or meson \( q \); for the total cross-section we can use either). We will take the minimum value of this energy to be \( \omega_p = 1 \). Using expression (5.6) we find that the maximum value of this energy (for a given incident energy) occurs when \( q \approx \frac{p}{M} \). Using this we find

\[
K = 1 - M' + \sqrt{(M' - 1)^2 + 2M'(E_0 - M)}
\tag{5.7}
\]

where \( K = (\omega_p)_{\text{max}} + 1 \) is the effective available energy for the final mesons, and \( M' = M \left( 1 - \frac{1}{M} \right)^{-1} \).

In the absence of recoil we would have \( K = E_0 - M - \frac{k^2}{2M} \)

where the last term is the kinetic energy of the nucleon in the center of mass system.
Another effect of recoil is to change the density of states. If we include recoil non-relativistically, we find

\[
\rho(E) = \frac{1}{(2\pi)^6} \frac{p \cdot q \omega_p \omega_q d\omega_p d\Omega_p d\Omega_q}{1 + \frac{\omega_q}{Mq} \left( q + p \cos \theta \right)}
\]

(5.8)

where \( \theta \) is the angle between \( \mathbf{p} \) and \( \mathbf{q} \). We cannot expand the denominator in powers of \( \frac{1}{M} \) since \( q \to 0 \) and the expansion would diverge. We will include its effect by letting

\[
\frac{q + p \cos \theta}{q} \simeq 2 \quad \text{and} \quad 2 \alpha_q \simeq \alpha_k.
\]

There is a similar correction to the incident velocity, which in the center of mass is written

\[
v = v_k + v_N = k \left( \frac{1}{\alpha_k} + \frac{1}{\alpha_N} \right).
\]

(5.9)

We approximate this by

\[
k \frac{\omega_k}{\alpha_k} \left( 1 + \frac{\omega_k}{M} \right)
\]

and, combining this with the above correction to the density of states, obtain a total correction factor of

\[
\frac{1}{(1 + \frac{\omega_k}{M})^2}
\]

(5.10)

Since we are not including nucleon recoil in the meson theory, there is no correlation between the directions of the outgoing mesons (provided we also neglect the angular dependence of the density of states).
Including these corrections, and performing the Clebsch-Gordan reductions mentioned above, the total production cross-section is

\[ \sigma_2(k) = 96 \frac{1}{(1 + \frac{m_k}{M})^2} \sum_{T,t,J,j} (2J + 1) \frac{1}{(\frac{m_1}{M})^2} \times \]

\[ \times \int_{\Omega} p^s q^s |A(T,t,j,j)_{t'}|^2 \]  

\[ \text{in units of } (\frac{\hbar}{f^c})^2 = 20 \text{ mb}. \]

Here

\[ (\frac{m_1}{M})^2 = (1 \frac{1}{2} m_k m_1 \Gamma_1 m_\Gamma)^2 \]

projects out the fraction of the incident state which is in the isotopic spin state T.

In order to evaluate the integrals involved here, we have used convenient approximations for \( S_3(\alpha_p) \) which permit the integrals to be performed analytically. These give the experimental results as represented by (5.1) within 10% over a large energy range and are chosen to be particularly accurate in the regions where the quantities concerned are large. They are

\[ \text{Re } S_3(\alpha_p) \approx \frac{0.21}{p^s} \frac{(1 + 0.79 p^s)(1.95 - \alpha_p)}{(\alpha_p - 1.8)^s + \frac{1}{16}} \]

\[ \text{Im } S_3(\alpha_p) \approx \frac{0.0050}{p^s} \frac{p}{(\alpha_p - 1.8)^s + \frac{1}{16}} \]  

(5.12)

We have also approximated the phase space factor by

\[ p^s = \frac{K - 1}{K} \alpha_p (\alpha_p^s - 1) \]  

(5.13)

in order to be able to perform the integrations.
The results for pions on protons are given in Fig. 1 and Table 1. In Fig. 2 we give the experimentally observed total cross-sections (including all processes) for pions on protons, and a curve representing a compilation of the available data on inelastic processes for $\pi^- + p$.

E. Discussion of Results

The striking feature of these results, of course, is the magnitude of the predicted cross-section. It follows from unitarity—the conservation of matter—that the total inelastic cross-section for a state of total angular momentum $J$ is limited by

$$\sigma_{\text{in}} \leq \frac{1}{2} \pi (2J + 1) \lambda^2 \quad (5.13)$$

In our problem we have $J = \frac{1}{2}$ and $\frac{3}{2}$, so that the total inelastic cross-section must be less than $3\pi \lambda^2$. We have drawn this curve on Fig. 1. We see that the predicted production cross-section for $\pi^- + p$ crosses the "unitarity barrier" for $T \approx 500$ Mev. This indicates that the one-meson approximation, with its implication that multi-meson states can be neglected in the unitarity statement (3.19), breaks down for these energies. It should be noticed that the violation of unitarity occurs in the region of the threshold for production of three mesons. It is thus not surprising that the one-meson approximation is not valid.

We might also interpret this to mean that our neglect of the $1\bar{l}$- and $1\bar{3}$-scattering amplitudes was unjustified. We might take a suggestion from the behavior of the production amplitude for the $3\bar{3}$-state which is strongly damped by the $3\bar{3}$-scattering state. It
is plausible that the inclusion of the other scattering states will have this effect on the corresponding production amplitudes.

We can use the general requirement of unitarity to tell us that this will indeed be the case. This requirement implies that for a given inelastic (production) cross-section there is a minimum value which the scattering cross-section may take. Thus a large production cross-section automatically implies a large scattering cross-section. Therefore our large production cross-section implies that the high-energy effects of the scattering states which we have neglected are important.

We might consider here the structure of the solutions if we had not dropped these scattering states. The inhomogeneous terms would contain a linear combination of the $S_{TJ}(\omega_q)$. Thus instead of the factorization of $S_{33}(\omega_q)$ given in (4.3), we would separate out these linear combinations. However, since low meson energies are involved here, $S_{33}(\omega_q)$ will still be dominant and the inclusion of the other states in these terms will not have a large effect.

The inclusion of these scattering states would mean that the integral terms containing both the $\kappa$ and the $M$ matrices would now be present in all terms. The presence of the $\kappa$ term does not have a large effect since it means, effectively, that we replace the Born approximation terms by $S_{11}(\omega_p)$ and $S_{13}(\omega_p)$. However, these quantities can be represented fairly accurately by their Born approximations so that again our solutions would not be greatly changed.
The terms containing the $M$ matrix act as they did on the 33-amplitudes. They result in the solutions being proportional to $S_{11}(\omega_p + \omega_q)$ and $S_{13}(\omega_p + \omega_q)$ and to integrals of these quantities. Thus these terms bring in the high-energy behavior which then controls the condition of unitarity and causes a damping of the production amplitude.

Although it is apparent that we cannot believe our results for high energies, we believe that the feature of a two-particle "resonance" at an incident energy of about 700 Mev would also result from a more correct treatment. This peak results from the enhanced scattering which results when each outgoing meson has an energy corresponding to the single-particle scattering resonance. This behavior has been the basis for related models which have been suggested\textsuperscript{15} to explain the observed peak in the total $\pi^- + p$ cross-section at about 850 Mev. The peak due to such single-particle resonances is expected to be broad because of the wide range of incident energies over which the particles can attain the resonance energy relative to the nucleon (with different angles between the outgoing mesons the nucleon can absorb different amounts of recoil). In fact, our result in Fig. 1 shows a peak with a half-width of the order of 150 Mev, as compared to the low-energy scattering resonance, with a width of about 50 Mev. The position of the peak is determined by the single-particle resonance energy and by the phase space and recoil factors.

It is interesting that the cross-section for $\pi^+ + p$, as shown in Fig. 1, has no pronounced peak. It is known experimentally\textsuperscript{18} that the total cross-section for $\pi^- + p$ shows a maximum
at about 850 Mev, while the first sign of a peak in the $\pi^+ + p$ cross-section is at 1.35 Bev. Our results appear to be in agreement with these observations. The absence of a "resonance" in the $\pi^+ + p$ cross-section arises from two causes: a) The main contribution to the resonance for $\pi^- + p$ is from the $1l$-production state; the angular-momentum-isotopic-spin "numerics" are such as to produce a cancellation in the $t = j = \frac{3}{2}$ part of the $l3$-production state. (It is this part which contains the product of two scattering amplitudes and thus gives rise to the two-particle resonance). Since the $1l$-state does not appear for $\pi^+ + p$ (for which $T = \frac{3}{2}$), the resonance contribution is greatly reduced; b) The previously-discussed damping of the $33$-state makes its contribution negligible. We believe that the first condition will appear in a more complete solution since it is solely a matter of kinematics (momentum coupling); the second will almost certainly be modified in a complete solution (in fact, a similar feature will probably appear in other states).

At high energies the terms contributing to the peak have become small, and the controlling factors are the Born terms and the volume of phase space. These combine to give the rise in the cross-section after the resonance.

It is interesting to observe that the experimental inelastic cross-section is greater than $3\pi \lambda^2$ for incident energies greater than 600 Mev. Thus we certainly can not expect to obtain a correct description of meson production processes in a theory such as ours which gives contributions only for $J = \frac{1}{2}$ and $\frac{3}{2}$. 
One can calculate the differential cross-sections for particular processes, such as $\pi^- + p \rightarrow \pi^+ + \pi^- + n$. This is the only process (below the threshold for production of two mesons) which results in a $\pi^+$ from an initial state of $\pi^- + p$. Therefore, observation of the $\pi^+$ can be used as an unambiguous indication that a meson-production process has taken place.

However, we do not feel that our solutions (4.15) are sufficiently accurate to warrant the presentation here of the differential cross-sections as "predictions" of the theory. In fact, this calculation can serve as a test of some of our approximations, as will be discussed in the next section.

F. The One-Meson Approximation

The Chew-Low equation for the scattering amplitude\(^1\) contains a sum over all numbers of intermediate mesons. The solution obtained by Chew and Low assumed the one-meson approximation. We can now use our solutions for the two-meson amplitude to test this approximation.

The two-meson term in this equation is

\[
\sum_{r,s} \left\{ \frac{(p| T^+ |rs) (rs| T |k)}{\omega_r + \omega_s - \omega_p - i\varepsilon} + \frac{(k| T^+ |rs) (rs| T |p)}{\omega_r + \omega_s + \omega_p} \right\} (5.14)
\]

in the notation which we introduced in Chap. III. We can now use the expansions introduced earlier to express this term in angular-momentum-isotopic-spin eigenstates. We then obtain the following correction to Eq. (3.29) for $S_1(z)$:

\[
C_1 = \sum_{t,j} \frac{4}{r^3} \int \frac{d\omega_r}{\pi} \frac{d\omega_s}{\pi} r^3 s^3 \left\{ \frac{|D_{tj}^1|^2}{\omega_r + \omega_s - \omega_p - i\varepsilon} + \sum_m A_{1m} \frac{|D_{tj}^m|^2}{\omega_r + \omega_s + \omega_p} \right\} (5.15)
\]
$A_{im}$ is the crossing matrix which also appears in the one-meson term of (3.29). The same matrix appears in each term since the crossing symmetry represents the effect of the time-reversal operation on the Boson components of the system. It is a function only of the total angular momentum and total isotopic spin, and is independent of the internal coordinates used to describe a particular intermediate state.

These integrals are most conveniently evaluated by using the variable $\omega = \omega_r + \omega_s$, in which case the integrals take the form

$$\int \frac{d\omega_r}{\omega_r + \omega_s} \frac{r^s s^s}{z} \frac{|D_{tj}(\omega_r,\omega_s)|^2}{|D_{tj}(\omega_r,\omega)|^2} = \int \frac{d\omega}{\omega + z} \left[ \int \frac{d\omega_r}{\omega_r + \omega_s} r^s s^s |D_{tj}(\omega_r,\omega)|^2 \right]$$

The integral in the bracket is closely related to the integral involved in the computation of the total cross-section.

The results are presented in Table 2, where they are compared with the corresponding scattering amplitudes.

We might remark first that they illustrate the damping effect of the additional term in the equation for $D_{tj}^{33}$. The correction term $\text{Im} C_1$ is directly proportional to the total production cross-section in the $i$-state (except for the incident-velocity-factor in the equation for the cross-section). We see that while the cross-section for the $33$-state is comparable with the other states at low energies, it is extremely small relative to them at higher energies.

As far as the imaginary parts of these amplitudes are concerned, it appears that the scattering and production processes
act in opposite directions. In the case of the 11-and 13-states, the scattering is small while the production is large. Thus the two-meson contributions are large. For the 33-state, the scattering is large and the production is small, so that the two-meson corrections are relatively small.

The real parts are fairly large, and are independent of energy. In addition, they contribute at energies below the meson-production threshold. It might be noticed that the real part is composed of both the direct and the crossed term. The latter contributes about one-third of the total and thus, as we have stated before, cannot be treated as negligible.

It would appear from these results that the one-meson approximation is not justified.

One other point merits discussion. While the two-meson corrections are large, they are not as large as the calculated total cross-sections lead us to expect. This can be understood as follows: The one meson terms have the form

$$\int \frac{d\omega_s}{s^3} \frac{|S_i|^2}{\omega_s + z}$$

The total scattering cross-section in the i-state is proportional to $k^4 |S_i|^2$, so that this integral is proportional to

$$\int \frac{\sigma_i}{s(\omega_s + z)}.$$ 

Similarly, the production cross-section has the form

$$\sigma_s \sim \sum_{t,j} k \int \frac{d\omega_p}{p^3} q^3 |D_{ij}^1|^2$$
so that, using (5.16), the two-meson terms take the form

\[
\int \frac{d\omega}{s} \frac{\sigma}{s(s_0 + z)}
\]

We expect that the scattering cross-section will be large for low energies, and the production cross-section large for high energies. Therefore the contribution of the production cross-section relative to the scattering cross-section is reduced by the denominator, which is proportional to the square of the energy.

In order to test the one-meson approximation for the meson-production equation, we have available an entirely different method.

We have used (2.8), a representation which is not symmetric in the two outgoing mesons. It can be shown that it is equal to (2.9), which is symmetric between mesons \( p \) and \( q \). Thus the \( T \)-matrix must be symmetric between the two mesons.

We can now consider a process such as \( \pi^- + p \rightarrow \pi^+ + \pi^- + n \) and can compute various differential or total cross-sections for this process. In doing so we can let meson \( p \) be the positive meson, or we can use meson \( q \). The results should be identical.

We have calculated the total cross-section for this process and find the following:

<table>
<thead>
<tr>
<th>( T ) (Mev)</th>
<th>( \sigma(\pi^+\pi^-;\pi^-) ) with ( p = \pi^+ ) (mb)</th>
<th>( \sigma(\pi^+\pi^-;\pi^-) ) with ( q = \pi^+ ) (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>275</td>
<td>0.0865</td>
<td>0.139</td>
</tr>
<tr>
<td>405</td>
<td>0.917</td>
<td>2.45</td>
</tr>
<tr>
<td>540</td>
<td>2.217</td>
<td>11.1</td>
</tr>
<tr>
<td>685</td>
<td>2.783</td>
<td>10.9</td>
</tr>
<tr>
<td>860</td>
<td>3.317</td>
<td>6.05</td>
</tr>
</tbody>
</table>
These results differ by a factor of about three. The difference can be attributed, in particular, to the dominance of $D_{13}^{11}$; this causes the cross-section as calculated with $q = \pi^+$ to be larger (since then $p = \pi^-$ and is always coupled to the final neutron in a $t = \frac{3}{2}^-$-state) and to have a resonance. It is this amplitude which causes the total cross-section for all $\pi^- + p$-production processes to have a resonance, and to be so large; if it did not have such a large contribution: a) these two partial cross-sections would not differ so greatly, and b) the total cross-section would be smaller and thus would not cross the "unitarity barrier" at such a low energy.

Now equation (2.10) for the T-matrix is symmetric between mesons $p$ and $q$ since we have used a final state which can be shown to be symmetric. The asymmetry arises when we make the one-meson approximation. Thus we can use the asymmetry which is introduced as a qualitative measure of the error due to the one-meson approximation. From the above table it would appear that this error is a large one. Unfortunately, this is not an unambiguous test since it appears (from considerations such as those of the previous paragraph) that an asymmetry is also induced by our neglect of the 11 and 13 scattering states.
VI. PHOTOPRODUCTION

We consider in this chapter the photoproduction of two p-wave pions. A gauge-invariant Hamiltonian is presented which has the familiar form for a static theory when the radiation gauge is used. We then derive an equation for the matrix element describing the photoproduction of two pions. A multipole expansion of the initial state and of the matrix element is derived. The isotopic-spin reduction of this equation is discussed. We write each term in the form of a multipole expansion so that the linear momentum, polarization, and charge variables may be removed; the equation may then, to any approximation, be reduced to multipole, angular momentum, and isotopic spin variables. The available experimental evidence is examined, and the possibility of obtaining meaningful results from this theory is discussed.

A. Hamiltonian

The Hamiltonian which we shall use consists of the original Hamiltonian (1.1) plus the additional terms

\[ H(A) = - \int d^3x \, J \cdot A = - \int d^3x \, (J_\text{M} + J_\text{int}) \cdot A \]  \hspace{1cm} (6.1)

where

\[ J_\text{M} = i e (\varphi_+ \nabla \varphi_- - \varphi_- \nabla \varphi_+) = -e (\varphi_1 \nabla \varphi_2 - \varphi_2 \nabla \varphi_1) \]  \hspace{1cm} (6.2)

\[ J_\text{int} = -1 \sqrt{4\pi} e \varphi(x) \sigma \cdot t \cdot \gamma \cdot \varphi \]  \hspace{1cm} (6.3)

Here \( t_3 \) is the meson charge-identification operator, i.e.,

\[ t_3 \varphi_+ = \varphi_+ ; \quad t_3 \varphi_- = -\varphi_- ; \quad t_3 \varphi_0 = 0 \]  \hspace{1cm} (6.4)
The form for the meson current $j_M$ follows from the usual gauge-invariant method of introducing the electromagnetic field:

$$\nabla \rightarrow \nabla - ie t_3 \vec{A} \tag{6.5}$$

The form of the so-called "interaction current" $j_{\text{int}}$ follows from a gauge-invariant method of introducing the electromagnetic field into the interaction Hamiltonian $H_I$ (see Appendix D). There is some ambiguity in this, but we have obtained a gauge-invariant form which reduces to (6.3) provided we use the usual radiation gauge:

$$\phi = 0 ; \quad \nabla \cdot \vec{A} = 0 \tag{6.6}$$

We will use this gauge in later developments.

B. Equation for Photoproduction Matrix Element

To first order in the electromagnetic field the matrix element for photoproduction of two mesons is

$$M_{pq,k} = (pq; H_k |0) \tag{6.7}$$

where

$$H_k = H(\vec{A}_k) \quad \text{and} \quad \vec{A}_k = \frac{\vec{p}}{\sqrt{2k}} e^{i\vec{k} \cdot \vec{x}} \tag{6.8}$$

We now use expansion (2.8) for the final two-pion state and obtain

$$M_{pq,k} = -(q; \left[ H_k , a_p \right] |0) - $$

$$- (q; H_k \frac{1}{\omega_p} V^+_p |0) - (q; V^+_p \frac{1}{\omega_p} \frac{1}{\omega_q} H_k |0) \tag{6.9}$$
We can expand the inverse operators using (2.3):

\[ M_{pq,k} = -(q! [H_k, a_p] !0) - \]

\[ - \sum_n \left\{ \frac{(q!H_k!n)(n!V_p^+!0) + (q!V_p^+!n)(n!H_k!0)}{\omega_n + \omega_p} \right\} \]

\[ - \sum_n \left\{ \frac{(q!H_k!n)(n!V_p^+!0)}{\omega_n - \omega_p - \omega_q - i\varepsilon} \right\} \]

We can obtain the one-meson approximation to this result, expressed in terms of physical scattering and photoproduction matrices, using identities (2.15) and (2.20). The result is

\[ M_{pq,k} = H_{pq,k}^c + \frac{(0!V_p^+!0)(q!H_k!0) - (q!H_k!0)(0!V_p^+!0)}{\omega_p} + \]

\[ + \frac{(q!V_p^+!0)(0!H_k!0) - (0!H_k!0)(q!V_p^+!0)}{\omega_p + \omega_q} \]

\[ - \sum_s \left\{ \left( \frac{(0!V_p^+!s)(s!H_k!0) - (s!H_k!0)(s!V_p^+!0)}{\omega_s - \omega_p - \omega_q - i\varepsilon} \right) + \frac{(-s!V_p^+!0)(s!H_k!0) + (0!H_k!0)(s!V_p^+!0)}{\omega_s + \omega_p + \omega_q} \right\} \]

where we denote

\[ H_{pq,k}^c = -(q! [H_k, a_p] !0) \]

We see that the zero-meson terms involve the meson-scattering amplitude and the single-meson photoproduction amplitude. The one-meson terms require knowledge of the meson-production amplitude. With this knowledge, this equation is an integral equation for the two-meson photoproduction amplitude.

We might discuss briefly the multipole character of this result, of which more will be said later. The initial state consists of a nucleon (and a photon); the final state, of two
p-wave pions and a nucleon. Then there is no change in parity of the pion-nucleon system, and we obtain non-zero results only from odd magnetic multipoles and even electric multipoles.

The initial state has angular momentum \( \frac{1}{2} \); the final state can have angular momentum \( \frac{1}{2}, \frac{3}{2}, \) and \( \frac{5}{2} \). Thus we would expect to obtain contributions from magnetic dipole, electric quadrupole, and magnetic octupole radiation. The last does not enter, however. The photon can interact directly with a pair of pions through \( \vec{p} \cdot \vec{A}_k \), but since these are p-wave pions it can give a maximum of 2 units of angular momentum (the nucleon does not enter into this interaction directly). The term \( \vec{j}_{\text{int}} \cdot \vec{A}_k \) is linear in the pion field; since it does contain the nucleon spin, it can involve the transfer of 2 units of angular momentum, but no more.

We are then concerned with only magnetic dipole and electric quadrupole terms (as in single-meson photoproduction).

C. Commutator Term

Let us examine the term containing the commutator of \( H_k \).

We first notice that the term containing the interaction current does not enter into our problem. It involves only electric dipole radiation\(^{23}\) and leads to the production of s-wave mesons only (as long as we use a spherical source and assume that in the limit it approaches a point source).

We need only be concerned with \(- (q! [H_M, a_p] \mid 0)\)

where

\[
H_M = e \int d^3x \left( \phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1 \right) \cdot \vec{A}_k
\]

(6.13)
The result for the commutator can be written

\[
- \left[ \hat{H}_M, a_{p,i} \right] = \frac{2ie \hat{e} \cdot \hat{p}}{\sqrt{8k\omega \omega_{k-p}}} \epsilon_{31j} \left\{ a_{p-k,j} + a_{k-p,j}^+ \right\}
\]  

(6.14)

Here \( \epsilon_{31j} \) is the antisymmetric tensor: \( \epsilon_{312} = -\epsilon_{321} = 1 \) and the indices \( i \) and \( j \) refer to the charges of these pions (for convenience we use Hermitian variables in isotopic-spin space).

The matrix elements which we need are

\[
(q|a_{p-k,j}|0) = -(q|H + \omega_{p-k} v^+_{p-k}|j|0) = -\frac{(q|V_{k-p,j}|0)}{\omega_q + \omega_{p-k}}
\]

(6.15)

\[
(q|a_{k-p,j}|0) = (q|H - \omega_{k-p} + i\epsilon v_{k-p}|j|0) = \frac{(q|V_{k-p,j}|0)}{\omega_q - \omega_{k-p} + i\epsilon}
\]

We then find our result

\[
H_{pq,k}^c = \frac{2ie \hat{e} \cdot \hat{p}}{\sqrt{8k\omega \omega_{k-p}}} \epsilon_{31j} \frac{2\omega_{k-p}}{\omega_q - \omega_{k-p}} (q|V_{k-p,j}|0)
\]

(6.16)

This term is thus directly proportional to the meson-scattering amplitude. For the case in which there are one p-wave and one s-wave pion in the final state, this term (along with the interaction-current term) gives the total photoproduction amplitude.

Let us extract the factorable momentum dependences, as we have done before. That is, since

\[
v_{pq} = 1 \frac{\sqrt{4\pi}}{f \, \gamma_1} \hat{p} \cdot \hat{e} \, v(p) \frac{1}{\sqrt{2\omega_p}}
\]

we can write

\[
(q|V_{k-p,j}|0) = \frac{v(|k-p|)}{\sqrt{2\omega_{k-p}}} (\hat{k} - \hat{p}) \cdot \hat{R}_j(q)
\]

(6.17)
Therefore
\[ H_{pq,k}^C = -ie \frac{\epsilon_{311} v(|k-p|)}{\sqrt{4k\omega p}} \frac{(\hat{e} \cdot \hat{p}) (\vec{k} - \vec{p})}{k\omega_p - k \cdot p} \cdot \vec{R}_j(q) \] (6.18)

We can picture this term physically as the production of mesons p and k-p in the meson cloud, with p being emitted directly, and k-p scattering on the nucleon into meson q. In a diagram

\[ \text{Diagram} \]

Since only p-wave pions are scattered by the meson-nucleon interaction, meson q is a p-wave. This term gives all angular-momentum waves for meson p since it is emitted directly from the pion cloud. We can separate the p-wave part with the result

\[ H_{pq,k}^C = -ie \frac{\epsilon_{311} v(|k-p|)}{\sqrt{4k\omega p}} \left\{ G_M(k,p) \cdot \vec{\sigma}_M + G_Q(k,p) \cdot \vec{\sigma}_Q \right\} \cdot \vec{R}_j(q) \] (6.19)

\[ \vec{\sigma}_M, \vec{\sigma}_Q \] is the projection operator for magnetic dipole photoproduction; \[ \vec{\sigma}_Q \], for electric quadrupole

\[ \vec{\sigma}_M = (\hat{p} \cdot \hat{e}) \hat{k} - (\hat{p} \cdot \hat{k}) \hat{e} \]

\[ \vec{\sigma}_Q = (\hat{p} \cdot \hat{e}) \hat{k} + (\hat{p} \cdot \hat{k}) \hat{e} \]

and

\[ G_M(k,p) = \frac{3}{4\pi k} \left\{ \frac{\omega_p}{\omega_p + p} \log \frac{\omega_p + p}{\omega_p - p} \right\} \] (6.20)

\[ G_Q(k,p) = \frac{1}{4\pi k} \left\{ -\frac{1}{2} \omega_p^2 - 1 - \frac{\omega_p^2}{4} \log \frac{\omega_p + p}{\omega_p - p} \right\} + G_M(k,p) \]
D. Multipole Expansion

We now wish to remove the dependence on magnetic quantum numbers (both angular momentum and isotopic spin) and thus make explicit the dependence on multipole order. Since the equation is linear, and since the multipoles of order \( L \) transform as irreducible tensors of rank \( L \), the equations for the various multipole amplitudes will be uncoupled.

The first step is to express the electromagnetic field in terms of multipoles. We must keep the direction of the incident photon arbitrary instead of choosing it along the \( z \)-axis. In order to produce a separation of the various multipoles, we must rotate the coordinate system or, equivalently, rotate the direction of \( k \); in this way their tensor character is exhibited (This was the motivation for our keeping the incident meson direction arbitrary in the meson-production case, also.).

Let us first examine a plane-wave photon with circular polarization moving along the \( z \)-axis. This can be expanded\(^{25} \) as

\[
\vec{A}_p^0 = \vec{u}_p e^{ikz} = \vec{u}_p \sum_{\ell} \frac{i^\ell}{\ell!} \sqrt{4\pi (2\ell + 1)} J_\ell (kr) Y_{\ell 0}(\theta, \phi) \tag{6.21}
\]

where \( \vec{u}_p = \frac{1}{\sqrt{2}} (\vec{u}_1 + iP \vec{u}_2) \) with \( P = \pm 1 \) and \( \vec{u}_1 \times \vec{u}_2 = \frac{\vec{k}}{k} \). The circular polarization is along the \( k \)-axis since the field must be transverse (the radiation gauge). We can express \( \vec{u}_p \) in terms of properly-defined tensors by

\[
\vec{u}_p = -P \vec{\chi}_p \quad \text{where} \quad \vec{\chi}_\pm = \pm \frac{1}{\sqrt{2}} (\vec{e}_x \pm i\vec{e}_y) \tag{6.22}
\]

If we now define the "vector spherical harmonic"\(^{25} \) by

\[
\vec{y}_{L,M} (\phi, \theta) = \sum_{m,m'} (\ell \ell 1 m m' | \ell J M) Y_{\ell m} (\phi, \theta) \vec{\chi}_{m'}, \tag{6.23}
\]
we see that we can write this wave as

\[ \hat{A}_p^0 = -P \sum_{L,P} \frac{1}{2} \sqrt{4\pi(2L+1)} \, j_L(kr) \, (L \, 1 \, 0 \, P \mid L \, 1 \, L \, P) \, \hat{Y}_{L \, L \, 1 \, P}(\theta, \phi) \]  

(5.24)

For a given value of \( L \), \( \ell \) can take the values \( L+1 \), \( L \), and \( L-1 \).

We now wish to define multipole fields by the requirements that they transform under rotations as irreducible tensors of a given rank, and that they have a well-defined parity under reflections. To do this we note that the vector spherical harmonic \( \hat{Y}_{L \, \ell \, 1 \, P} \) is a tensor of rank \( L \) with parity \((-1)^\ell \). Then we define the electric and magnetic multipole fields\(^{26}\) by

\[ \hat{A}_{LP}(m) = -P \sqrt{\frac{2}{\pi}} \, j_L(kr) \, (L \, 1 \, 0 \, P \mid L \, 1 \, L \, P) \, \hat{Y}_{L \, L \, 1 \, P}(\theta, \phi) \]  

\[ \hat{A}_{LP}(e) = -\sqrt{\frac{4 \pi}{2L+3}} \, j_{L+1}(kr) \, (L+1 \, 1 \, 0 \, P \mid L+1 \, 1 \, L \, P) \, \hat{Y}_{L \, L \, 1 \, P}(\theta, \phi) \]  

(6.25)

Using these, we can write the plane wave as

\[ \hat{A}_p^0 = \pi \sum_{L=1}^{\infty} i^L \sqrt{2L+1} \left\{ \hat{A}_{LP}(m) + i \, P \, \hat{A}_{LP}(e) \right\} \]  

(6.26)

We note again that the direction of the circular polarization is that of the propagation vector, in line with the requirement that \( \nabla \cdot \vec{A} = 0 \). We must now perform a rotation so as to obtain an arbitrary direction. In doing so this relation between the direction of propagation and the direction of polarization must be maintained. It is this requirement—in other words, that we deal with a vector field with a gauge condition imposed—which makes this case differ from the incident pseudoscalar meson considered earlier.
Now any tensor of rank $L$ is transformed under rotations through

$$\psi'_{JM'} = \sum_M \mathcal{D}^J_{MM'}(\alpha', \beta', \gamma) \psi_{JM}$$  \hspace{1cm} (6.27)

where $\mathcal{D}^J_{MM'}(\alpha', \beta', \gamma)$ is the irreducible representation of dimension $(2J+1)$ of the rotation group. $\alpha$, $\beta$, and $\gamma$ are the Euler angles. Since the rotation in general changes the axis of quantization, the new tensor (or eigenfunction) is a linear combination of the old tensor components with the same $J$ (since $J^2$ is invariant under rotations) but different $M$. The representations of the group satisfy the Clebsch-Gordan relation

$$\mathcal{D}^{J_1}_{M_1M_1'}(\alpha, \beta, \gamma) \mathcal{D}^{J_2}_{M_2M_2'}(\alpha', \beta', \gamma) = \sum_{J,M} (J_1J_2M_1M_2; J_1J_2J M) \times$$

$$\times (J_1J_2M_1M_2'; J_1J_2J M) \mathcal{D}^J_{MM'}(\alpha', \beta', \gamma)$$  \hspace{1cm} (6.28)

In general they are hypergeometric functions; in practice only one of them is needed. It can be found easily from the addition theorem for spherical harmonics as follows:

$$Y_{LM}(\theta', \phi') = \sum_m \mathcal{D}^L_{mM}(\alpha', \beta', \gamma) Y_{LM}(\theta, \phi)$$

$$Y_{LO}(\theta', \phi') = \sqrt{\frac{4\pi}{2L+1}} \sum_m Y_{LM}^+(\beta', \alpha') Y_{LM}(\theta, \phi)$$

so that

$$\mathcal{D}^L_{MO}(\alpha, \beta, \gamma) = \frac{\sqrt{4\pi}}{2L+1} Y_{LM}^+(\beta, \alpha')$$  \hspace{1cm} (6.29)

One other property is needed:

$$\mathcal{D}^J_{MM'} = (-1)^{M-M'} \mathcal{D}^{-J,-M,-M'}$$  \hspace{1cm} (6.30)

This follows from the fact that $\mathcal{D}$ is a unitary transformation.
With these facts, and the knowledge that the multipole fields transform as irreducible tensors, we see that the plane wave representing propagation in any direction can be written

$$ A_P = \pi \sum_{L=1}^{\infty} \sum_{M}^{\pm} \frac{1}{L^{2L+1}} D_{LM}(\hat{k}) \left\{ A_{LM}^{(m)} + \frac{1}{P} A_{LM}^{(e)} \right\} \tag{6.31} $$

The last step is to convert this back into the form of a plane-polarized wave. If we fix the direction of $\hat{k}$, we can specify the axis $\hat{u}_1$. We measure the angle of the polarization vector $\hat{e}$ with respect to this axis, so that

$$ \hat{e} = \hat{u}_1 \cos \phi + \hat{u}_2 \sin \phi \tag{6.32} $$

Thus we find our final result

$$ A_k = \hat{e} e^{i \hat{k} \cdot \hat{x}} = \frac{1}{\sqrt{2}} \sum_{P} e^{-iP \phi} \hat{A}_P = $$

$$ = \frac{\pi}{\sqrt{2}} \sum_{L,M,P} e^{-iP \phi} \frac{1}{L^{2L+1}} D_{LM}(\hat{k}) \left\{ A_{LM}^{(m)} + \frac{1}{P} A_{LM}^{(e)} \right\} \tag{6.33} $$

We can now proceed in exact analogy to the meson-production case. We define an $M$-matrix by

$$ M_{pq,k} = (pq \mid M \mid k_{\text{en}1}) \tag{6.34} $$

where $M$ is a scalar operator, write the initial photon state as

$$ \kett{k}\ell = \sum_{L,P,M,1} \hat{1}L^{2L+1} e^{-iP \phi} D_{LM}(\hat{k}) \kett{LM} \tag{6.35} $$

(i = m or e) and obtain the expansion

$$ M_{pq,k} = (pq \mid M \mid k_{\text{en}1}) = \sum_{P,L,J,J,1} \hat{1}L^{2L+1} e^{-iP \phi} D_{LM}(\hat{k}) x \ltimes Y_{P}^{(\hat{\phi})} Y_{q}^{(\hat{\phi})} (pq;J) (Jq;J)(n_{1};L,J) (L;P,J;M;L,J) \tag{6.36} $$
We see that this has exactly the same form as the meson case (3.8) with $Y_{\ell k}^+(\tilde{\kappa})$ for the incident meson replaced by $\frac{1}{\sqrt{2\ell+1}} e^{-i\mathbf{P} \cdot \mathbf{L}} \mathcal{O}_{\text{MP}}^L(\tilde{\kappa})$ for the incident photon (the constants are just for convenience). The equation can be reduced exactly as before. The final mesons are treated the same; the initial state is "projected out" by multiplying by $e^{i\mathbf{P}' \cdot \mathbf{L}'} \mathcal{O}_{\text{MP}'}^L(\tilde{\kappa})$ and integrating over $d\phi$ and $d\Omega_\kappa$.

The result contains the same matrices as before, except that the incident state can now have any angular momentum (actually, for our case, only $\ell = 1$) instead of just $\ell = 1$.

E. Isotopic-Spin Reduction

The isotopic-spin reduction must also be treated in a somewhat different manner from before. The Hamiltonian is no longer a scalar in isotopic space, but transforms as a mixed scalar and vector. The unperturbed Hamiltonian (1.1) is a scalar. The Hamiltonian containing the electromagnetic field $H(A)$ is proportional to the operator $t_3$, and thus transforms as a vector.

In general the matrix elements of a tensor operator are given by $^8$

$$
(TM; 0^V_u |T'M') = (-1)^T + T' + \nu \frac{1}{\sqrt{2T+1}} x
$$

$$
x (\nu T' u M' ; \nu T M) (T; ; 0^V ; ; T') (6.37)
$$

where $(T; ; 0^V ; ; T')$ is the "reduced matrix element" and is independent of magnetic quantum numbers. We also have the selection rule that the three angular momenta must satisfy a "triangular inequality" which can be expressed by the vector relation $T + T' + \nu = 0$. 
Since our initial state (one nucleon) has $T = \frac{1}{2}$, this means that $\Delta T = 0, \pm 1$ (our operator is a vector, a tensor of rank 1). For our case we have

$$(TM; H^T \vec{r} \frac{1}{2} m_1) = (-1)^T \frac{1}{\sqrt{2T+1}} (1 \frac{1}{2} 0 m_1 \frac{1}{2} T M) H^T \quad (6.38)$$

One sees that this has the same form as the corresponding quantity for meson scattering, with a neutral meson incident. We can then also use our previous methods here, except that now we cannot sum over the states of the initial meson, but must instead sum over the total isotopic-spin "magnetic" quantum number. This causes a difference of only a constant factor, so that the result of the isotopic reduction is identical to that for meson-production.

Then for magnetic dipole we would obtain the same matrices as before; for electric quadrupole they would be the same as those for an incident d-wave pion. Here we do not have the symmetry between spin and isotopic spin to reduce the number of matrix elements, so we would get two sets of $16 \times 16$ matrices.

F. Reduction of Inhomogeneous Terms

There are two remaining details before one would be in a position to completely reduce the equations. The removal of the magnetic quantum number dependence is of course not enough; we must also determine from the physical theory the "reduced matrix elements". The reduced matrix elements corresponding to $(pq; H_k \mid 0)$ and $(p; H_k \mid 0)$ are just the multipole photoproduction amplitudes which are determined from their respective integral equations (in principle).
We must, however, determine from first principles the quantity \( \langle 0; H_k | 0 \rangle \). For this we take some suggestions from the approach of Chew and Low\(^{23} \). We first notice that this quantity, and therefore the term with the \( \frac{1}{a_P} \) denominator, contributes only to the magnetic dipole amplitude. Now the expectation value of the Hamiltonian \(- \vec{J} \cdot \vec{A}_k\) is directly proportional to \(- \vec{\mu}_V \cdot \vec{H}\), where \( \vec{\mu}_V \) is the isotopic-vector part of the magnetic moment. We do not have exactly this because there is an implied sum over the intermediate nucleon spin and charge. However, the form of the matrix element depends only on its multipole and tensor character, and not on the magnetic quantum number values. Then we can write this quantity as

\[
\langle 0; H_k | 0 \rangle = - \frac{\mathbf{\epsilon}}{2} \mathbf{k} \cdot \mathbf{\epsilon} \cdot \frac{\mathbf{\mu}_P - \mathbf{\mu}_N}{2} F(k^2) \left( \mathbf{\epsilon} \cdot \mathbf{1} \right) (0; \sigma | 0) \quad (6.39)
\]

where we have noted that \( \vec{H}_k = \nabla \times \vec{A}_k = \mathbf{1} \mathbf{k} \times \mathbf{\epsilon} \mathbf{e}^{i \mathbf{k} \cdot \mathbf{x}} \) and have used the fact that \( \mathbf{\mu}_V = \frac{\mathbf{\mu}_P - \mathbf{\mu}_N}{2} \). In addition we have used the definition of the renormalized coupling constant given in (3.25). \( F(k^2) \) is a form factor which gives the momentum dependence (or, equivalently, the space dependence) of the magnetic moment.

In order to make use of the general tensor methods we have set up, we must expand this result in irreducible tensors. We can do this using (6.22) and (6.32) to represent the polarization vector. The result is

\[
\mathbf{k} \times \mathbf{\epsilon} = \frac{\mathbf{1} \mathbf{k}}{\sqrt{2}} \sum_{M_P} e^{-i \mathbf{P} \cdot \mathbf{\phi}} \mathcal{O} \frac{1}{M_P} \mathcal{F}(\mathbf{k}) \hat{\lambda}^M \quad (6.40)
\]

where \( \hat{\lambda}^M \) is the unit tensor of rank 1 defined in (6.22).
Then our matrix element is

\[
\langle 0 \mid H_k \mid 0 \rangle = \frac{\hbar}{i} \frac{\mathcal{D}_B - \mathcal{D}_N}{4} \frac{\mathcal{F}(k^2)}{\sqrt{k}} \sum_{\mathcal{M}_\mathcal{P}} \gamma_3 \sigma_\mathcal{M}_\mathcal{P} \langle 0 \mid e^{-i\mathcal{P} \phi} \mathcal{D}_\mathcal{M}_\mathcal{P} \rangle (\mathcal{R})
\]

which can be reduced exactly as we did in (3.24), and has the same form as the general expansion (6.36) (when we finally extract the nucleon indices).

Lastly we must recast expression (6.19) into this same form so as to be able to use orthogonality and the Clebsch-Gordan relation (6.28).

The magnetic dipole part has already been done, since we can write

\[
\vec{\sigma}_\mathcal{M} \cdot \vec{\sigma} = (\vec{p} \cdot \vec{e}) (\vec{k} \cdot \vec{\sigma}) - (\vec{p} \cdot \vec{k}) (\vec{e} \cdot \vec{\sigma}) = (\vec{\sigma} \times \vec{p}) \cdot (\vec{k} \times \vec{e})
\]

Using the definition of the vector spherical harmonics given in (6.23), and using result (6.40), we find

\[
\vec{p} \times \mathcal{K}_\mathcal{M} = i \frac{e^{i\mathcal{P} \phi}}{\sqrt{2}} Y_{111,\mathcal{M}} (\vec{p})
\]

and

\[
\vec{\sigma}_\mathcal{M} \cdot \vec{\sigma} = -\frac{4\pi}{3} k \frac{e^{-i\mathcal{P} \phi}}{\mathcal{D}_\mathcal{M}_\mathcal{P}} \sum_{\mathcal{M},\mathcal{P},\mathcal{m},\mathcal{m}'} \mathcal{D}_{\mathcal{M}_\mathcal{P}} (\mathcal{R}) (1 \ 1 \ m \ m' \ | 1 \ 1 \ 1 \ M) x Y_{1m} (\vec{p}) \sigma_{m'}
\]

It is thus in a form suitable for extraction of all the quantum numbers. (6.43) illustrates, incidentally, that this quantity is an irreducible tensor of rank 1, since the vector spherical harmonic $Y_{111,\mathcal{M}}$ has been so defined.
Similarly we find
\[
\overrightarrow{Q} \cdot \overrightarrow{\sigma} = - \sqrt{\frac{4\pi}{3}} k P \sum_{M,P,m,m'} P e^{-iP} D^{2}_{MP}(k) (l_1 l_2 M) \times
x Y_{lm}(\hat{\phi}) \sigma_{m'}
\]
(6.45)

We have thus made explicit the multipole dependence of each term in the complete equation (6.10) for the photoproduction of two p-wave pions. We have expressed it in a form in which the geometric dependence of each term can be easily removed by the same methods as used for meson-production. We shall not proceed further with the reduction since no attempt will be made to solve the equation.

G. Discussion

Recent experimental evidence\textsuperscript{28} for incident photon energies up to 600 Mev indicate that there is a preference for one of the outgoing pions to be in an s-state. We have seen that the meson current leads to an outgoing meson containing all angular momenta, while the interaction current leads only to s-waves. A theory\textsuperscript{24} for this process similar to ours shows that the interaction current gives the largest s-wave contribution; that the p-wave pion emerges primarily from the 33-resonant state; and that because of this the latter carries most of the energy so that the s-wave pion emerges with little energy. We would then expect the production of two p-wave pions to become important only at high energies. However, we have seen in the meson-production case that the one-meson approximation is invalid for high energies.
The evidence on single-meson photoproduction shows a resonance which has been interpreted in terms of a resonance in the 33-state of the final meson-nucleon system. In the present theory this is directly related to the scattering resonance in that state. In fact a large part of the photoproduction amplitude turns out to be directly proportional to the scattering amplitude.

Now the equation for our photoproduction amplitude is very similar to that for the meson-production amplitude. If we believe that the peak in the meson-production amplitude is a feature of a complete solution, we would imply that this behavior would also be present in the double-photoproduction amplitude. These resonances arise from the enhanced rescattering of the outgoing mesons and thus should be, to a certain extent, independent of the incident state (for comparable energies). Although these resonances will tend to make the double p-wave production amplitude large, they will not necessarily overcompensate for the "angular momentum barrier" which favors the production of one s-wave and one p-wave pion. This is because the angular momentum eigenstates do not favor the presence of both outgoing pions in a 33-state relative to the nucleon.

One is forced to the conclusion that the results predicted by this equation will be very difficult to check (as opposed to that predicting one s-wave and one p-wave). The effect will only be large for high energies, where the static theory is not reliable and where any approximation is suspect (the result for the s- and p-wave production can be expressed exactly). We can
only use it to give an indication of the general behavior of the cross-section: that it will tend to show a peak for incident photon energies of the order of 800 Mev.
VII. SUMMARY AND CONCLUSION

We have considered the problem of the production of a single pion in pion-nucleon collisions using the Chew-Low-Wick formalism. In order to obtain a solution in closed form it has been necessary to include only the zero- and one-meson terms in an expansion over intermediate states. In this approximation the resulting equations are linear integral equations for the production amplitudes.

These equations have been solved by neglecting the terms which have non-resonant denominators and by using analytic function theory. We have been able to use the latter because the nature of the interaction for this theory allows the initial meson energy to be factored. The result is then an equation in only one (complex) variable.

Our equation gives the meson production amplitude in terms of the meson scattering amplitude. To represent the latter we have taken the viewpoint of including only know low-energy experimental data. Following this approach, we have neglected all scattering except in the \( T = J = \frac{3}{2} \) state. One would hope that this approximation would be good since for moderate incident energies the outgoing mesons will have low energies. It is these outgoing mesons whose behavior is described by single-particle scattering amplitudes in the solutions.

The total production cross-sections which we have obtained using our solutions are quite large. By making the one-meson approximation for the scattering problem we have implied that
production is small. The one-meson approximation for the production problem assumes that the amplitude for production of two mesons is small.

We have used our solutions to evaluate the corrections to the one-meson approximation for the scattering equation. These are found to be small compared to the \(33\)-scattering amplitude, but large compared to the \(11\)-and \(13\)-scattering amplitudes. It appears that in regions where there is no scattering resonance, and for high energies, the two-meson corrections are not negligible.

Unfortunately, the inclusion of any terms beyond the one-meson terms makes solution of the equations very difficult. The two-meson term of the scattering equation involves the amplitudes for production of one meson. It thus couples the twenty equations for these amplitudes to the three equations for the scattering amplitudes. The two-meson terms of the single-production equation involve the amplitudes for production of two mesons. It thus appears to be a very complex problem to obtain a self-consistent solution.

It appears that one must really obtain such a self-consistent solution, however. The scattering and production amplitudes must satisfy the requirements of unitarity. The large production amplitudes which we have obtained are not consistent with the negligible scattering amplitudes we have assumed for the \(11\) and \(13\) states. Then a complete solution must include these states. Since the scattering in these states is known to be small for
low energies, the production amplitudes will be comparable with the scattering amplitudes in such a complete solution, and the production amplitudes cannot be treated as perturbations on the scattering solutions.

This situation has a further unpleasant consequence. The equations show that the high-energy scattering amplitudes serve to damp the production amplitudes and maintain unitarity. We thus find, in agreement with other theoretical approaches, that the high-energy behavior of these scattering amplitudes is important. This is in disagreement with the philosophy of a cut-off theory, which assumes that high-energy effects are not important and thus that the conclusions of the theory are independent of the cut-off. One should notice that the conclusion that the results of this theory are cut-off dependent is itself cut-off independent; that is, this conclusion depends only on the structure of the equation, which is the same in the non-relativistic and the relativistic case.

Another property of the Low equation which we have observed is that many virtual processes are contained in each term. One cannot be certain that a term which is neglected would not completely cancel, or make important contributions to, other so-called "lower-order" terms. One is led to doubt, from this and the above considerations, whether, even in this renormalized theory, higher-order terms can be neglected.

We are forced to the conclusion that we still do not have an approximation scheme which is capable of handling both low- and high-energy processes.
Appendix A

REPRESENTATION OF MESON-NUCLEON STATES

We wish to derive representations of the states which are involved in our problem and in the equation for the relevant T-matrix. The method of Wick\(^3\) will be used.

Consider a state \(|k^+\rangle\) containing only outgoing scattered waves. It is an eigenstate of the total Hamiltonian and satisfies the equation

\[
H |k^+\rangle = \omega_k |k^+\rangle \quad (A.1)
\]

where we have set the nucleon self-energy equal to zero. Let us write the state

\[
|k^+\rangle = a_k^+ |0\rangle + \chi^{(+)} \quad (A.2)
\]

This form is chosen since the first term represents asymptotically the incident state of a free meson of momentum \(k\) and a physical nucleon. This can be shown as follows (This is done in Ref. 3, but we reproduce the derivation here and extend it since there are several points which are not made clear there):

Consider a free meson which we describe by the wave packet

\[
g(x,t) = \sum_k c_k e^{i(kx - \omega_k t)} \quad (A.3)
\]

which we construct so that, for \(t \leq 0\), it and its gradient are zero in the region in which \(\rho(x)\), the nucleon density, is non-zero. Now let us write the incident, time-independent state as \(a_k^+ |0\rangle\) and form the wave packet

\[
\tilde{\phi}(t) = \sum_k c_k e^{-i\omega_k t} a_k^+ |0\rangle \quad (A.4)
\]
Since 
\[ [H, a_k^+] = \omega_k a_k^+ \quad \text{and} \quad [H_1, a_k^+] = V_k, \]  
(A.5)  
\[ H a_k^{+\dagger} |0\rangle = \omega_k a_k^{+\dagger} |0\rangle + V_k |0\rangle. \]  
(A.6)

We then find that 
\[ i \frac{d\bar{\phi}}{dt} - H \bar{\phi} = \sum_k c_k V_k |0\rangle. \]  
(A.7)

This can be written in the following form, using the interaction Hamiltonian of this theory (although the result is clearly more general):
\[ i \frac{d\bar{\phi}}{dt} - H \bar{\phi} = \sqrt{4\pi} \int d^3x \phi(x) \gamma \sqrt{\sum_k c_k e^{ikx}} |0\rangle. \]  
(A.8)

Because of the restrictions we have placed on the wave packet, the right hand side is zero; thus \( \bar{\phi} \) satisfies the Schrödinger equation when the meson does not overlap the nucleon, and we have the correct incident wave.

Using the commutation relations (A.5), and representation (A.2),
\[ (H - \omega_k) \chi^{(+)\dagger} = -(H - \omega_k) a_k^{+\dagger} |0\rangle = -V_k |0\rangle \]  
(A.9)

and
\[ \chi^{(+)\dagger} = -\frac{1}{H - \omega_k - i\epsilon} V_k |0\rangle \]  
(A.10)

where we insert the \( i\epsilon \) to obtain outgoing scattered waves.

Thus we obtain
\[ |k+\rangle = a_k^{+\dagger} |0\rangle - \frac{1}{H - \omega_k - i\epsilon} V_k |0\rangle \]  
(A.11)

In order to obtain the equation for the T-matrix element, we need a representation of the quantity \( a_k |n-\rangle \) where \( |n-\rangle \) is a state containing a physical nucleon and \( n \) mesons. We obtain
the desired representation as follows:

\[ H a_k |n-\rangle = (\omega_n - \omega_k) a_k |n-\rangle - V_k^+ |n-\rangle \]  
(A.12)

\[ a_k |n-\rangle = \delta_{k,n} |n-k-\rangle - \frac{1}{H - \omega_n + \omega_k + i\epsilon} V_k^+ |n-\rangle \]  
(A.13)

The notation \( \delta_{k,n} |n-k-\rangle \) means a state obtained by removing a meson \( k \) from the state containing \( n \) mesons—-if this state originally contained a meson in the state \( k \). We see from (A.12) that this is the desired form for the homogeneous solution since \( |n-k-\rangle \) is an eigenstate of \( H \) with the eigenvalue \( \omega_n - \omega_k \).

As examples of this result, we have

\[ a_k |p-\rangle = \delta_{k,p} |0\rangle - \frac{1}{H - \omega_p + \omega_k + i\epsilon} V_k^+ |p-\rangle \]  
(A.14)

\[ a_k |q-\rangle = \delta_{k,q} |p-\rangle + \delta_{k,q} |p-\rangle - \frac{1}{H - \omega_p - \omega_q + \omega_k + i\epsilon} V_k^+ |q-\rangle \]  
(A.15)

We now want a representation of the final two-meson state. Let us write this state as

\[ |pq-\rangle = a_p^+ |q-\rangle + \psi(-) \]  
(A.16)

We can form a wave packet as before to show that the first term contains two free mesons asymptotically, but this is not necessary. We proceed with the algebra as before:

\[ (H - \omega_p - \omega_q) |pq-\rangle = 0 \]

\[ (H - \omega_p - \omega_q) \psi(-) = - (H - \omega_p - \omega_q) a_p^+ |q-\rangle \]

\[ |pq-\rangle = a_p^+ |q-\rangle - \frac{1}{H - \omega_p - \omega_q + i\epsilon} V_p |q-\rangle \]  
(A.17)
Appendix B

RESULTS OF ANGULAR MOMENTUM-ISOTOPIC SPIN REDUCTIONS

We give here the results of summing over all magnetic quantum numbers. These are given in terms of the Racah coefficients\(^8,9\) which have been tabulated\(^11\).

We have previously written the resulting equation as

\[
D(\omega_p, \omega_q) = \frac{A(\omega_q)}{\omega_p} + \frac{B(\omega_q)}{\omega_p + \omega_q} + \frac{2}{\pi} \int d\omega_s s^3 \psi^2(s) \left\{ \frac{\mathcal{K}(\omega_s) D(\omega_s, \omega_q)}{\omega_s - \omega_p - i\epsilon} + \right.
\]

\[
- \frac{\mathcal{L}(\omega_s) F(-\omega_s, \omega_q)}{\omega_s + \omega_p} - \frac{\mathcal{M}(\omega_s) F(-\omega_s, \omega_q)}{\omega_s - \omega_p - \omega_q - i\epsilon} +
\]

\[
+ \frac{\mathcal{N}(\omega_s) D(\omega_s, \omega_q)}{\omega_s + \omega_q + \omega_p} \right\} \tag{3.28}
\]

To obtain this we first change the summation to an integral:

\[
\sum_s \rightarrow \frac{1}{(2\pi)^3} \int d\omega_s \, d\Omega_s \, \omega_s \, s \tag{B.1}
\]

We use the expansions of the T-matrices (3.11) and (3.18), multiply by the appropriate spherical harmonics, and integrate over the angles of p, q, k, and s. We then multiply by the Clebsch-Gordan coefficients occurring in (3.11) and perform the sums. Lastly we extract all of the factorable energy dependences using (3.16), (3.21), and (3.22). We have also treated the matrix elements of \(\sigma\) and \(\tau\) as shown in (3.24) and (3.25).

The results, as written in (3.28), can be described as follows: \(D(\omega_p, \omega_q)\) is a 10-component vector with components \(D_{tj}^{Tj}(\omega_p, \omega_q)\).
A(\omega_q) is a 10-component vector with components given by

\[ A_{tj}^{TJ}(\omega_q) = \frac{1}{4} \sum_{T', J'} b_t^{T}(T') b_{T'}^{J}(J') S_{T, J'}(\omega_q) \]

\[ + (-1)^{T+j} \frac{1}{2} f \delta_{t, \frac{1}{2}} \delta_{J, \frac{1}{2}} S_{TJ}(\omega_q) \]  

(B.2)

where

\[ a_t^{T}(T') = (-1)^{t+T'} \sqrt{2t'+1} (2T'+1) W(T', T'_2; \eta) W(T'_2, T'; \eta) \]

and similarly for \( a_j^{J}(J') \). All of the terms have this structure: the product of identical quantities, summed over appropriate momenta. This is, of course, because of the symmetry between angular momentum and isotopic spin.

B(\omega_q) is a 10-component vector with components

\[ B_{tj}^{TJ}(\omega_q) = -\frac{1}{4} \sum_{T', J'} c_t^{T}(T') c_{T'}^{J}(J') S_{T, J'}(\omega_q) \delta_{T, \frac{1}{2}} \delta_{J, \frac{1}{2}} \]

\[ - f \sum_{T', J'} c_t^{T}(T') c_{T'}^{J}(J') S_{T, J'}(\omega_q) \]  

(B.3)

where

\[ b_t^{T}(T') = (-1)^{t+T'} \sqrt{2t'+1} (2T'+1) W(T', \frac{11}{2}; \eta) \]

\[ c_t^{T}(T') = (-1)^{t+T'} \sqrt{2t'+1} (2T'+1) \sum_{T''} (-1)^{T''} (2T''+1) x \]

\[ W(T'', T'_2; \eta) W(T', \frac{11}{2}; \eta) W(T'' T'_2; \eta) \]

\( \kappa(\omega_s) \) is a 10 x 10 diagonal matrix (This particular coupling scheme was chosen in order to make this matrix diagonal). Let us write an element of the vector \( \kappa D \) as

\[ \sum_{T', J', t, t', J'} \kappa_{TJ, T'_j, T'_j; \omega_s}^{T, T'J'} D_{t, t'}^{T', J'}(\omega_s, \omega_q) \]  

(B.4)
In this notation an element of the matrix \( \mathcal{K} \) is

\[
\mathcal{K}(T_j^t, T'_j^t; \omega_s) = S_{tj}^+(\omega_s) \delta_{T,T'} \delta_{t,t'} \delta_{j,j'} \delta_{j,j'}
\]  

(B.5)

Using the same notation the remaining matrices are:

\[
\mathcal{L}(T_j^t, T'_j^t; \omega_s) = -d(T_T^t) d(J_J^t) S_{T,J'}^+(\omega_s)
\]

where

\[
d(T_T^t) = (-1)^{T+t} (2T+1) \sqrt{2t+1} (-1)^{'T'+t'} (2T'+1) \sqrt{2t'+1} \times \sum_{t''} (-1)^{t''} (2t''+1) W(t''T_T^t;ll) \times W(t''T_T^t;ll) W(T'T_J^t;ll) \]  

(B.6)

\[
\mathcal{M}(T_j^t, T'_j^t; \omega_s) = S_{TJ}(\omega_s) \delta_{T,T'} \delta_{j,j'} \delta_{t,t'} \delta_{j,j'}
\]  

(B.7)

\[
\mathcal{N}(T_j^t, T'_j^t; \omega_s) = -g(T_T^t) g(J_J^t) S_{tj}^+(\omega_s)
\]

where

\[
g(T_T^t) = (-1)^{T+t+T'+t'} \sqrt{2t+1} (2T'+1) \sqrt{2t'+1} \times \sum_{T''} (-1)^{T''} (2T''+1) W(T'T_T''^t;ll) \times W(T'T_T''^t;ll) \]  

(B.8)
Appendix C

NUMERICAL COEFFICIENTS

We here give the results of evaluating numerically the coefficients given in Appendix B. The coefficients given here are those which are needed when we assume that \( S_{11} = S_{13} = 0 \). Thus they are the quantities defined in (4.1) and used in (4.4) and (4.8), etc.

The vectors \( A \) and \( B \), when written in column form, are:

\[
\begin{pmatrix}
A_{11}^{11} \\
A_{13}^{11} \\
A_{33}^{11} \\
A_{13}^{13} \\
A_{31}^{13} \\
A_{33}^{13} \\
A_{11}^{33} \\
A_{13}^{33} \\
A_{33}^{33}
\end{pmatrix}
= \frac{\mathbf{f}}{81}
\begin{pmatrix}
32 \\
8\sqrt{2} \\
4 \\
-8\sqrt{5} \\
-5\sqrt{2} \\
-2\sqrt{10} \\
-28 \\
5\sqrt{5} \\
10
\end{pmatrix}
\]

\[
\begin{pmatrix}
B_{11}^{11} \\
B_{13}^{11} \\
B_{33}^{11} \\
B_{13}^{13} \\
B_{31}^{13} \\
B_{33}^{13} \\
B_{11}^{33} \\
B_{13}^{33} \\
B_{33}^{33}
\end{pmatrix}
= \frac{\mathbf{f}}{81}
\begin{pmatrix}
-64 \\
-4\sqrt{2} \\
40 \\
4\sqrt{5} \\
7\sqrt{2} \\
7\sqrt{10} \\
4 \\
4 \\
10
\end{pmatrix}
\]
The matrix \( \mathbf{K} \) is diagonal and, with the approximation of including only the 33-scattering state, contains only three non-zero elements. It can then be written

\[
\mathbf{K}(T_{t,j',t',j'}) = \mathbf{\delta}_{T,T'} \mathbf{\delta}_{t,t'} \mathbf{\delta}_{J,J'} \mathbf{\delta}_{J,\frac{3}{2}} \mathbf{\delta}_{J,\frac{3}{2}}
\]  

Thus the only non-zero elements are those for which \( t = j = \frac{3}{2} \).

The matrix \( \mathbf{M} \) has the same structure, except that it is non-zero only for states in which the total momenta \( T \) and \( J \) are \( \frac{3}{2} \):

\[
\mathbf{M}(T_{t,j',t',j'}) = \mathbf{\delta}_{T,T'} \mathbf{\delta}_{t,t'} \mathbf{\delta}_{J,J'} \mathbf{\delta}_{J,\frac{3}{2}} \mathbf{\delta}_{J,\frac{3}{2}}
\]
Appendix D

GAUGE INVARIANCE FOR THE STATIC-SOURCE THEORY

In the static-source theory the meaning of gauge invariance is ambiguous because of the phenomenological manner of treating the nucleons. The usual, local-theory, requirement is that the theory be invariant under the simultaneous replacements:

\[ \psi \rightarrow e^{i(\frac{1+T_3}{2})G} \psi \]
\[ \varphi \rightarrow e^{it_3G} \varphi \]
\[ \hat{A} \rightarrow \hat{A} + \nabla G \]  \hspace{1cm} (D.1)

Having replaced the nucleons by a density function \( \rho(x) \), we can no longer do this. The usual means of obtaining gauge invariance is to include the electromagnetic field only in the combination \( \nabla - i \epsilon \hat{A} \). Then, from another viewpoint, we are treating the nucleons as fixed and have no kinetic-energy operator available in which to make this replacement.

One means of solving this problem is that of including a path integral\(^{20,21}\) in the interaction Hamiltonian. However, this has the unpleasant feature that it is clearly not unique (the path is arbitrary); since the path integral can give non-zero contributions, this is undesirable. In addition, and more important, such an approach adds an element to the theory which does not appear in a local theory; this is undesirable if we wish to view theory as a limit of a true local theory.
The requirement of gauge invariance can be formulated as

$$H_1(\tilde{A} + \nabla G) = e^{iD} H_1(\tilde{A}) e^{-iD}$$  \hspace{1cm} (D.2)

where

$$D = \int d^3x \left\{ \rho_M(x) + \left( \frac{1 + \gamma_3}{2} \right) \delta(x) \right\} G(x)$$

$$\rho_M(x) = ie \left( \frac{\partial \varphi}{\partial t} \varphi + - \frac{\partial \varphi}{\partial t} \varphi_+ \right) .$$

Let us insert the electromagnetic field into the interaction Hamiltonian in the usual way, but also provide an additional function of the field:

$$H_1(\tilde{A}) = \sqrt{4\pi} \int d^3x \varphi(x) e^{i f(\tilde{A})} \gamma \tilde{\sigma} \cdot (\nabla - i e \gamma_3 \tilde{A}) \varphi$$  \hspace{1cm} (D.3)

Noting that

$$e^{iD} \gamma^\pm e^{-iD} = \gamma^\pm e^{\pm i e G(0)}$$  \hspace{1cm} (D.4)

$$e^{iD} \varphi^\pm(x) e^{-iD} = e^{\pm i e G(x)} \varphi^\pm(x)$$

and writing

$$\gamma \cdot \varphi = - \gamma_+ \varphi_- - \gamma_- \varphi_+ + \gamma_0 \varphi_0 ,$$

we have

$$e^{iD} H_1(\tilde{A}) e^{-iD} = \sqrt{4\pi} \int d^3x \varphi(x) e^{-ie f(\tilde{A})} \varphi$$

$$\times e^{ie \left[ G(0) - G(x) \right]} \left\{ \gamma_+ \tilde{\sigma} \cdot [\nabla - ie (\tilde{A} + \nabla G)] \varphi_- \gamma_+ \tilde{\sigma} \cdot [\nabla + ie (\tilde{A} + \nabla G)] \varphi_+ \right\}$$  \hspace{1cm} (D.5)

Therefore we must have

$$e^{-ie f(\tilde{A})} e^{ie \left[ G(0) - G(x) \right]} = e^{-ie f(\tilde{A} + \nabla G)}$$  \hspace{1cm} (D.6)
Now \[ G(0) - G(x) = \int d^3x' \ G(x') \left[ \delta(x') - \delta(x' - x) \right] = \]
\[ = \int d^3x' \ G(x') \nabla^2 \varphi(x';0,x) = \]
\[ = -\int d^3x \ \nabla G(x') \cdot \nabla \varphi(x';0,x) \]  
(\{D.7\})

(suggested by some work in Ref. 22)

where \[ \varphi(x';0,x) = \frac{1}{4\pi} \left[ \frac{1}{|x'|} - \frac{1}{|x' - x|} \right] . \]

We have assumed that \( G(x') \) vanishes as \( x' \to \infty \). This is not required on physical grounds but is a reasonable condition to place on the gauge function. It seems that one cannot really avoid the lack of uniqueness in the gauge prescription, but this does seem to be an innocuous place for this indefiniteness to appear.

Assuming this condition, we obtain gauge invariance if we let
\[ I(\tilde{A}) = \int d^3x' \ \tilde{A}(x') \cdot \nabla \varphi(x';0,x) = -\int d^3x' \ \nabla \cdot \tilde{A}(x') \varphi(x';0,x) \]
(\{D.8\})

Then \( \tilde{A} \) appears only in the form \( \nabla \cdot \tilde{A} \), and this term will not enter into the matrix elements if we use the radiation gauge
\[ \phi = 0 \ ; \ \nabla \cdot \tilde{A} = 0 \]

This gauge is generally used in applications to avoid the complications of the longitudinal field.
PION PRODUCTION CROSS-SECTIONS

\[ \sigma^2 (\pi^- + p) \]

\[ \sigma^2 (\pi^+ + p) \]

Incident Kinetic Energy (Mev)

0 200 300 400 500 600 700 800 900 1000 1100

\[ \sigma^2 \text{ (mb)} \]

0 8 16 21 32 48
EXPERIMENTAL CROSS-SECTIONS
Total and Inelastic

Incident Kinetic Energy (MeV)
Table 1

TOTAL CROSS-SECTIONS FOR PRODUCTION OF A SINGLE PION IN PION-NUCLEON COLLISIONS

<table>
<thead>
<tr>
<th>Incident Kinetic Energy (Mev)</th>
<th>$\sigma(\pi^- + p)$ (mb)</th>
<th>$\sigma(\pi^+ + p)$ (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>0.0725</td>
<td>0.0216</td>
</tr>
<tr>
<td>275</td>
<td>0.560</td>
<td>0.130</td>
</tr>
<tr>
<td>335</td>
<td>3.28</td>
<td>0.702</td>
</tr>
<tr>
<td>400</td>
<td>7.71</td>
<td>1.62</td>
</tr>
<tr>
<td>455</td>
<td>15.9</td>
<td>3.12</td>
</tr>
<tr>
<td>540</td>
<td>33.4</td>
<td>6.56</td>
</tr>
<tr>
<td>685</td>
<td>37.1</td>
<td>9.08</td>
</tr>
<tr>
<td>860</td>
<td>29.7</td>
<td>10.5</td>
</tr>
<tr>
<td>1030</td>
<td>29.9</td>
<td>12.9</td>
</tr>
<tr>
<td>1250</td>
<td>33.6</td>
<td>16.0</td>
</tr>
<tr>
<td>1500</td>
<td>38.2</td>
<td>19.4</td>
</tr>
</tbody>
</table>
CORRECTIONS TO THE ONE-MESON APPROXIMATION

We here denote $c_1$ as the two-meson correction to the equation for the scattering amplitude $S_1(\omega)$.

The experimental phase shifts given by Anderson are

$$\tan \delta_{11} = -0.021 \, p^3$$
$$\tan \delta_{13} = 0.0125 \, p^3$$
$$\tan \delta_{33} = \frac{0.25 \, p^3}{1 + .79 \, p^2} \left( \frac{1.95 - 1}{1.95 - \omega_p} \right)$$

Using these, $S_{11}(\omega_p)$ and $S_{13}(\omega_p)$ are constant.

On the next page we tabulate the corrections to the one-meson approximation. These have been calculated using our solutions for the meson-production amplitude.

These corrections to the equation for the scattering amplitude are compared with the scattering amplitudes computed using the experimental phase shifts given above.
<table>
<thead>
<tr>
<th>$\omega_p$</th>
<th>Re $S_{11}$</th>
<th>Re $C_{11}$</th>
<th>Im $S_{11}$</th>
<th>Im $C_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>-0.010</td>
<td>0.0081</td>
<td>0.00022</td>
<td>0</td>
</tr>
<tr>
<td>1.75</td>
<td>-0.010</td>
<td>0.0107</td>
<td>0.00022</td>
<td>0</td>
</tr>
<tr>
<td>2.25</td>
<td>-0.010</td>
<td>0.0159</td>
<td>0.00022</td>
<td>0.000103</td>
</tr>
<tr>
<td>3.25</td>
<td>-0.010</td>
<td>0.0425</td>
<td>0.00022</td>
<td>0.0239</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\omega_p$</th>
<th>Re $S_{13}$</th>
<th>Re $C_{13}$</th>
<th>Im $S_{13}$</th>
<th>Im $C_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>0.0062</td>
<td>0.00969</td>
<td>0.00078</td>
<td>0</td>
</tr>
<tr>
<td>1.75</td>
<td>0.0062</td>
<td>0.0106</td>
<td>0.00078</td>
<td>0</td>
</tr>
<tr>
<td>2.25</td>
<td>0.0062</td>
<td>0.0123</td>
<td>0.00078</td>
<td>0.000041</td>
</tr>
<tr>
<td>3.25</td>
<td>0.0062</td>
<td>0.0202</td>
<td>0.00078</td>
<td>0.00463</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\omega_p$</th>
<th>Re $S_{33}$</th>
<th>Re $C_{33}$</th>
<th>Im $S_{33}$</th>
<th>Im $C_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>0.115</td>
<td>0.00483</td>
<td>0.012</td>
<td>0</td>
</tr>
<tr>
<td>1.75</td>
<td>0.0775</td>
<td>0.00449</td>
<td>0.105</td>
<td>0</td>
</tr>
<tr>
<td>2.25</td>
<td>-0.0275</td>
<td>0.00419</td>
<td>0.0420</td>
<td>0.00038</td>
</tr>
<tr>
<td>3.25</td>
<td>-0.0080</td>
<td>0.00370</td>
<td>0.0054</td>
<td>0.00073</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY

Theory and Formalism

2. F. E. Low, Phys. Rev. 27, 1392 (1955)

Crossing Theorem


Angular Momentum Reduction and Tensor Algebra Methods

10. F. J. Dyson, Phys. Rev. 100, 344 (1955)
11. Table of the Racah Coefficients ORNL-1679

Scattering Phase Shifts

12. H. L. Anderson, PRC, 1956

High-Energy Resonance in T = \frac{1}{2} State

13. F. J. Dyson, Phys. Rev. 92, 1037 (1955)
15. B. T. Feld, PRC, 1956
Analysis of Chew-Low Theory


17. G. Salzman, PRC, 1956

Experimental Data

18. R. P. Shutt, PRC, 1956

19. J. Crussard, W. D. Walker, and M. Koshiba,
   Phys. Rev. 94, 736 (1954)
   M. Blau and M. Caulton, Phys. Rev. 96, 150 (1954)
   A. V. Crewe, U. E. Kruse, and H. D. Taft,

Gauge Invariance

20. R. G. Sachs, Phys. Rev. 74, 433 (1948)

21. R. H. Capps and W. G. Holladay,
   Phys. Rev. 92, 931 (1955)


Static-Source Theory of Photoproduction


24. R. E. Cutkosky and F. Zachariasen, Phys Rev (to be published)

Multipole Fields


27. G. Goertzel, Phys. Rev. 70, 897 (1946)
Photoproduction--Experiments and Analysis


ACKNOWLEDGEMENTS

It is a pleasure to thank Professor Feshbach for his stimulation, advice, and encouragement, and to express my gratitude for his continued personal interest.

I wish to thank my parents for their enthusiasm and support during the formative years.

I want to thank my wife for typing this thesis. More important, I want to express in some measure my indebtedness to her for offering the kind of support and encouragement which only a wife can give.
BIOGRAPHICAL NOTE

Leonard Sidney Rodberg was born in Baltimore, Maryland, on December 14, 1932. He was educated in the Baltimore public schools and graduated from the Baltimore Polytechnic Institute in February, 1951. He then entered Lehigh University and remained there, in the Department of Engineering Physics, until June, 1953. In September, 1953, he entered Johns Hopkins University and remained there until June, 1954, when he received a Bachelor of Arts Degree. In September, 1954, he entered M.I.T.

While at Johns Hopkins University he held a Junior Instructorship. He has held a National Science Foundation Fellowship during the academic years 1954-56.

He is a member of Phi Eta Sigma, Tau Beta Pi, Sigma Xi, and the American Physical Society.