Tax-Efficient Asset Management Via Loss Harvesting

by

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Submitted to the Department of Electrical Engineering and Computer Science
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Abstract

In this thesis, we study loss-harvesting—an investment strategy that realizes capital losses immediately but defers realizing capital gains as long as possible. We begin by describing a computational framework for studying the properties of loss-harvesting empirically. The main advantage of our framework is flexibility. In particular, our framework is independent of any particular choice of a source for stock return time series. After combining the framework with the Capital Asset Pricing Model as a source for simulated stock returns data, we perform a thorough sensitivity analysis and study the performance of loss-harvesting under various conditions of the financial market. By combining the framework with historical stock return time series from the S&P 500 Index, we study the performance of loss-harvesting from a different and more practical, point of view. Through this empirical exploration, we identify three new findings about loss-harvesting: (1) introducing a transaction cost rate of 1% reduces alpha by about 50% after taxes; (2) introducing regular cash contributions reduces alpha after taxes; and (3) under specific market conditions, a simple passive buy-and-hold investment strategy outperforms loss-harvesting.

Thesis Supervisor: Andrew W. Lo
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Increasing the number of stocks above the base value of 500 stocks does not result in a significant change of either the before-tax or the after-tax alpha; with a number of stocks fewer than the base value, however, tax alpha decreases. Importantly, increasing the number of stocks results in a higher variance of both before-tax and after-tax alpha as indicated by the 25-th and 75-th percentiles in the left panels.

Higher stock turnover leads to lower tax alpha both before and after accounting for taxes, because stock turnover requires recognizing capital gains. This dynamic is in the opposite direction of loss-harvesting’s goal, which is to realize capital losses as soon as possible, and defer capital gains as long as possible.

Higher tax rate brackets lead to higher tax alpha both before and after taxes, because larger tax rates result in more substantial tax credits from realized capital losses.

Increasing the cash contribution rate leads to higher tax alpha before taxes, and importantly, lower tax alpha after taxes. This empirical result is significant because: (1) it pinpoints the qualitative difference in reporting alpha of loss-harvesting before versus after accounting for taxes; (2) it is crucial due to the practical implications of the performance of loss-harvesting in the presence of regular cash contributions.
Increasing the cash withdrawal rate (i.e. more negative) leads to lower alpha before taxes, and higher alpha after taxes. In general, the magnitude of both effects is rather small.  

Higher monthly market premium rate leads to lower loss-harvesting alpha before taxes, because loss-harvesting opportunities reduce under a higher market premium rate. Interestingly, the empirical results for loss-harvesting after accounting for taxes imply that loss-harvesting does best under moderate (about 9% annual) market premium rate.  

Increasing the systematic market risk rate leads to higher loss-harvesting alpha both before and after accounting for taxes, because the number of loss-harvesting opportunities grows with increasing the systematic volatility in the market.  

Increasing the idiosyncratic stock risk rate leads to higher loss-harvesting alpha both before and after accounting for taxes, because the number of loss-harvesting opportunities grows with increasing the specific volatility in the market.  

Increasing the dividend-yield rates leads to higher loss-harvesting alpha both before and after accounting for taxes.  

Similarly to the loss-harvesting empirical results with the Capital Asset Pricing Model as a source of simulated stock return rates, increasing the monthly cash contribution rate leads to higher loss-harvesting alpha before taxes, and lower loss-harvesting alpha after taxes on historical stock market data from the S&P 500 Index.  

Increasing the cash withdrawal rate leads to higher loss-harvesting alpha on historical stock market return data from the S&P 500 Index both before and after taxes.  

Increasing the tax rate leads to higher loss-harvesting alpha on historical stock market return data from the S&P 500 Index both before and after taxes.
A heat map of after-liquidation tax alpha in basis points with the Capital Asset Pricing Model as a source of simulated stock return rates.

The after-liquidation tax alpha varies smoothly in adjacent years for every tax rate.

A heat map of after-liquidation tax alpha in basis points on historical market data from the S&P 500 Index. The after-liquidation tax alpha does not vary smoothly in time for any tax rate. In particular, there is a significant drop in after-liquidation tax alpha in 2002 and in 2008.

A heat map of after-liquidation tax alpha in basis points with the Capital Asset Pricing Model as a source of simulated stock return rates with an artificial sharp price shock in 2002. The after-liquidation tax alpha drops significantly in 2002, similarly to Figure 4-14. Note that the colorbar scales are different.

Increasing the transaction cost rate $r_t$ leads to lower loss-harvesting alpha both before and after accounting for taxes. Moreover, note that transaction cost rate of 1% reduces the alpha of loss-harvesting after taxes by about 50%; thus, transaction costs are vital for the use of loss-harvesting in practice.
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Chapter 1

Introduction

The excess return of a portfolio with respect to a benchmark portfolio is called alpha. Everything else equal, investors are interested in portfolios with larger alpha. In practice, a significant part of the investment decision process is focused on security selection, whose goal, in general, is to maximize portfolio returns subject to specific constraints (e.g. maximizing risk-adjusted returns).

Once a portfolio generates capital gains, however, investors are required to pay taxes to realize those capital gains. Jeffrey and Arnott (1993) find that taxes often are the single largest inefficiency of a portfolio. Unfortunately, the majority of portfolio management research has been focused on investing that is free of taxes. Since taxes may affect portfolio returns considerably, it is not surprising that the academic literature has recently turned attention to investment strategies that make use of tax specifics to improve the investment performance.

Tax-aware investment. Tax-aware investment, as discussed by Horan and Adler (2010), seeks to increase the returns of a portfolio of securities by guiding the investment management through tax-conscious decisions. We note that tax-aware investment is of interest to all participants in the financial market:

- individual investors that need to pay taxes on realized capital gains;
- financial services firms that strive to stay ahead of their competitors;
- security exchanges that are an integral part of the market when it comes to
transaction costs;

- governments that may be interested in reshaping the tax system due to potentially having smaller revenue as a result of the existence of tax-aware investment strategies.

**Loss-harvesting** One particular investment strategy that makes use of tax specifics to guide the investment process is loss-harvesting (Stein and Narasimhan (1999)). *Loss-harvesting* seeks to realize capital losses immediately, and seeks to delay realizing capital gains for as long as possible. Once investors realizes capital losses, they are eligible to claim tax credits on those losses; later, those tax credits could either (1) be reinvested in the portfolio; or (2) be used to offset taxes on capital gains.

To implement a *loss-harvesting* strategy, an investor needs fine control over their portfolio—at the level of an individual security. Such financial services have long been available to the wealthiest investors. However, the majority of investors have their assets in ETFs and mutual funds that do not support such granular control over a portfolio. The major impediment for tax-aware investing reaching the masses earlier has been technological: implementing a *tax-efficient* asset management strategy requires significant hardware and software investment. Since computing is now prevalent, it is likely that a major portion of the capitals managed in ETFs and mutual funds will move to tax-efficient investing.
Chapter 2

Background

2.1 Literature Review

The majority of the academic literature on portfolio management has been developed under the assumption that taxes do not exist. Garland (1987) argues that this has been so because taxes obscure the already involved investment management theories. However, taxes are real, and moreover, they often are the largest expense that investors face in practice (Garland (1987); Jeffrey and Arnott (1993)). This observation has served as a motivation for the recent growth of the research effort in portfolio management in the presence of taxes.

Investing in practice Shoven, Dickson, and Clemens (2000) argue that the mutual fund industry has developed significantly as a result of the shift from households investing in the stock market to indirect ownership of equities through mutual funds. This change, however, has lead to questioning the tax-efficiency of mutual funds where the actions of one shareholder may affect the taxes of other shareholders, because once a mutual fund realizes capital gains, all of its shareholders may need to pay taxes. Longmeier and Wotherspoon (2006) perform a thorough analysis of the impact of taxes on mutual fund and stock index returns, and find that a significant portion of the mutual funds and the stock indices underperform with regards to returns after taking taxes into account. More recently, companies, such as Wealthfront (2015), have started offering investment services for personal accounts.
that incorporate tax-efficient wealth management practices.

**Loss-harvesting** Going back to the academic literature, Constantinides (1984) finds that the optimal stock trading strategy in the presence of taxes, is to realize capital losses immediately and to defer capital gains for as long as possible; notably, this result was derived under the assumption that there are no transaction costs. This trading idea is the core of loss-harvesting, which is described by Stein and Narasimhan (1999).

Subsequently, Berkin and Ye (2001) find that loss-harvesting reduces taxes in comparison to a simple buy-and-hold strategy; after performing sensitivity analysis, the authors find that loss-harvesting yields returns that increase as the tax rate bracket of an investor increases. In 2003, Berkin and Ye argue that loss-harvesting should be combined with HIFO accounting when it comes to realizing losses, and perform a thorough robustness analysis that describes the performance of loss-harvesting under various market and investment conditions. Importantly, they assume that there are no transaction costs.

Stein, Vadlamudi, and Bouchey (2008) find that loss-harvesting may be improved under the setting of maximizing returns subject to minimizing risk exposure; for that, they argue in favor of realizing capital gains earlier under specific conditions.

### 2.2 Our Contribution

We contribute to the research on loss-harvesting by providing a framework for evaluating said strategy empirically by building off of the research of Berkin and Ye (2003). One major advantage of this framework is its independence of any particular source of market return data. This flexibility enables using the framework in combination with different sources of market return data.

By combining the framework with the Capital Asset Pricing Model as a source for stock return data, we carry out an extensive sensitivity analysis of the various free parameters in the model, which proves useful in understanding loss-harvesting under various market and investment conditions.
Moreover, by combining the framework with historical market return data from the S&P 500 Index, we carry out a comparative sensitivity analysis, which helps in understanding the performance of loss-harvesting under market conditions that are not exactly modelled via the Capital Asset Pricing Model.

Finally, by using our framework, we obtain three new results:

- Introducing 1% transaction costs result in reduction of the loss-harvesting alpha after taxes by 50%. This finding is significant, because it means that efficient transaction costs are vital for the performance of loss-harvesting.

- Introducing regular cash contributions reduces the alpha of loss-harvesting after taxes. This finding contradicts previous results by Berkin and Ye (2003). Such a result is quite significant, because it poses a question as to whether loss-harvesting can be efficiently combined with regular cash contributions, which is of real practical interest, as described in the white paper by Wealthfront (2015).

- Depending on the market conditions, the alpha of loss-harvesting after taxes could be negative, i.e. it is possible that loss-harvesting has worse returns than a simple buy-and-hold strategy. Such a finding is important, because it means that it may be possible to improve loss-harvesting by identifying conditions under which it is better to temporarily halt the strategy.

### 2.3 Thesis Outline

We begin in Chapter 3 with describing the computational framework that we are going to use to study loss-harvesting empirically. Our empirical studies are contained in Chapter 4. In particular, in Section 4.1 we perform sensitivity analysis over the free parameters in the loss-harvesting trading strategy with simulated stock return data from the Capital Asset Pricing Model, and we find that regular cash contribution reduce the alpha of loss-harvesting. Then, in Section 4.2 we perform similar sensitivity analysis with historical stock return data from the S&P 500 Index from 1990 to 2014, and we observe loss-harvesting alpha patterns that are not present in the study over
simulated data. We attempt to bridge the gap between the empirical results with different source of stock returns data in Section 4.3 by identifying a time-invariant modeling limitation in our use of the Capital Asset Pricing Model. Then, in Section 4.4, we study loss-harvesting in the presence of transaction costs, and find that 1% of transaction costs reduce the alpha of loss-harvesting by about 50%. Finally, we conclude in Chapter 5.
Chapter 3

Methods

In this Chapter, we lay out the computational framework that we are going to use for the empirical evaluation of a loss-harvesting investment strategy.

In Section 3.1, we describe a framework for evaluating tax alpha that is independent of any particular source of stock return rates and stock dividend rates. This framework computes the tax alpha of a tax-aware strategy that uses loss-harvesting by maintaining two portfolios: (1) a base buy-and-hold portfolio that does not utilize loss-harvesting, and (2) a tax-efficient one that does utilize loss-harvesting; tax alpha is defined as the difference of the returns between the two portfolios.

Then, in Section 3.2, we extend the framework in Section 3.1 by introducing fees on all transactions.

Finally, in Section 3.3 we describe two possible sources of stock return rates and stock dividend yield rates that we are going to use for evaluation in the next chapter: (1) the well-known Capital Asset Pricing Model; and (2) historical market data from the S&P 500 Index.

3.1 Framework

Before presenting the main core of the framework by Algorithm 7 in Subsection 3.1.7, we start by describing some preliminary procedures the serve as building blocks.
3.1.1 Definitions

Firstly, we start with several definitions:

- The total number of different stocks under consideration, denoted by $N_S \in \mathbb{N}$, is a constant.

- The portfolio is managed for a total of $N_T \in \mathbb{N}$ number of discrete periods, where $N_T$ is a constant.

- The tax rate, denoted by $\tau_r \in [0, 1]$, is a constant.

- A shares matrix $A \in \mathbb{R}^{N_S \times N_T}$ is a matrix such that $A_{s,t}$ is the amount of shares of stock $s \in \{1, \ldots, N_S\}$ purchased in period $t \in \{1, \ldots, N_T\}$.

- A cost matrix $C \in \mathbb{R}^{N_S \times N_T}$ is a matrix such that $C_{s,t}$ is the cost of one share of stock $s \in \{1, \ldots, N_S\}$ purchased in period $t \in \{1, \ldots, N_T\}$.

- A portfolio is given by a pair $P = (A, C)$ of a shares matrix $A$, and a cost matrix $C$.

- A price vector $\bar{p} \in \mathbb{R}^S$ is a column vector of prices such that $\bar{p}_s$ is the price of one share of stock $s \in \{1, \ldots, S\}$.

- A stock index set $S \subseteq \{1, \ldots, N_S\}$ is an element from the powerset of all available stocks $\{1, \ldots, N_S\}$.

3.1.2 Portfolio Values

We consider two different ways to quantify the value of a portfolio $P$. The first one is called the before-tax value of a portfolio, and corresponds to the cash-equivalent of the portfolio. The second one is the after-tax value of the portfolio $P$ if we were to close the entire portfolio by liquidating all positions, and paying the taxes due on the realized capital gains or collecting tax credit on the realized capital losses.
Both of those valuations are outlined by Algorithm 1. GetPortfolioValue computes the before-tax value $V^\text{before-tax}_P$ and the after-tax value $V^\text{after-tax}_P$ of a portfolio $P = (A, C)$, where both portfolio values are computed with respect to a price vector $\vec{p}$, and a stock index set $S$. The purpose of the stock index set parameter $S$ is to enable computing the portfolio value on a subset of the stocks in the portfolio $P$. For example, if $S = \{1, \ldots, N_S\}$, the portfolio value is computed with respect to all constituent stocks in $P$. On the other hand, if $S = \{s\}$, the portfolio value is computed with respect to the single stock $s \in [1, N_S]$. Also, GetPortfolioValue computes the amount of tax due on the net capital gains (or tax credit on the net capital losses) with respect to the price vector $\vec{p}$ and the stock index set $S$, if we liquidated all positions in the stock index set $S$ from $P$.

Algorithm 1: GetPortfolioValue($P = (A, C), S, \vec{p}, r_\tau$)

**Input**: portfolio $P$, stock index set $S$, price vector $\vec{p}$, tax rate $r_\tau$

**Output**: before-tax value $V^\text{before-tax}_P$, after-tax value $V^\text{after-tax}_P$, amount of tax $\tau_P$ if we liquidated all of $P$.

1. Let $V^\text{before-tax}_P := \sum_{s \in S} \sum_{t=1}^{N_T} \vec{p}_s A_{s,t}$ be the before-tax portfolio value of $P$ with respect to the stock index set $S$.
2. Let $G_P := \sum_{s \in S} \sum_{t=1}^{N_T} (\vec{p}_s - C_{s,t}) A_{s,t}$ be the amount of capital gains (or capital losses when negative) on the portfolio $P$ with respect to the stock index set $S$.
3. Let $\tau_P := -r_\tau G_P$ be the amount of tax due on the capital gains $G_P$. Note that when $G_P < 0$, there is a net capital loss on the portfolio $P$ with respect to the stock index set $S$, and $\tau_P > 0$ is the amount of tax credit due on that loss.
4. Let $V^\text{after-tax}_P := V^\text{before-tax}_P + \tau_P$ be the after-tax portfolio value, obtained by offsetting the before-tax portfolio value with the amount of tax $\tau_P$ due (which is negative if there is a tax due, or positive if there a is tax credit due to losses) if we liquidated the entire portfolio $P$. Note that when there a is net capital loss on $P$ with respect to the stock index set $S$, $V^\text{after-tax}_P > V^\text{before-tax}_P$.

5. return $V^\text{before-tax}_P, V^\text{after-tax}_P, \tau_P$.

### 3.1.3 Index Turnover

The investment strategies under consideration implicitly maintain an index of stocks. It is reasonable to assume that some of the stocks may no longer be available for
purchasing at certain times due to liquidation or bankrupt events.

We are going to allow for this flexibility by replacing a stock currently in a given portfolio with a new substitute stock that has not been part of the portfolio previously. More formally, all shares of a stock \( s \in \{1, \ldots, N_S\} \) are going to be sold, and the obtained cash equivalent is going to be used to purchase shares of a stock that has not previously been included in the portfolio. Algorithm 2, \text{ReplaceStock}, describes this procedure precisely.

Algorithm 2: \text{ReplaceStock}(P = (A, C), \bar{p}, r_\tau, t^*, s, p_s)

\begin{align*}
\text{Input} : & \text{ portfolio } P, \text{ price vector } \bar{p}, \text{ tax rate } r_\tau, \text{ current time period } t^*, \text{ stock } s \in \{1, \ldots, N_S\}, \text{ price } p_s \text{ of one share of replacement stock } s. \\
\text{Output:} & \text{ portfolio } P' \text{ obtained after replacing stock } s, \text{ tax } r_\tau \in \mathbb{R} \text{ due on realized capital gains (or tax credit if losses) from liquidating shares of stock } s \text{ from portfolio } P.
\end{align*}

1. Let \( V_s \) and \( r_\tau \) respectively be the before-tax portfolio value, and amount of tax with respect to the stock index set \{s\} consisting of stock \( s \), as per \text{GetPortfolioValue}(P, \{s\}, \bar{p}, r_\tau). \text{Note that when } r_\tau > 0, \text{ we have obtained tax credit, which could be reinvested in the portfolio. On the other hand, when } r_\tau < 0, \text{ we have to pay the amount } -r_\tau \text{ by possibly liquidating shares further. None of those scenarios are considered as part of ReplaceStock.}

2. \text{for each period } t \in \{1, \ldots, N_T\} \text{ do}
3. \quad | \text{ Sell all } A_{s,t} \text{ shares of stock } s \text{ purchased in period } t: A_{s,t} \leftarrow 0.
4. \text{end}
5. Set the cost of the replacement stock: \( C_{s,t^*} \leftarrow p_s. \)
6. Spend the amount \( V_s \) on shares from the replacement stock: \( A_{s,t^*} \leftarrow V_s/p_s. \)
7. The portfolio obtained after replacing stock \( s \) is \( P' = (A, C). \)
8. return \( P', r_\tau \)

3.1.4 Harvesting Losses

Harvesting losses is the core of the loss-harvesting strategy. This process identifies all shares that are at a loss with respect to the price that they were bought at, and sells them all to realize capital losses. After that, we can claim a tax credit on the realized capital losses, and subsequently, repurchase the shares of stocks that we sold.

Note that, in practice, such a strategy would not be allowed to immediately collect a tax credit due to the \textit{wash sale rule}. The framework that we describe does
not follow the wash sale rule. However, one way to handle this limitation would be to purchase shares of a stock that has similar characteristics but is considered “substantially different” according to the wash sale rule. Algorithm 3, **HarvestLosses**, describes this procedure formally.

**Algorithm 3: HarvestLosses** ($P = (A, C), \bar{p}, r_\tau, t^*$)

**Input**: portfolio $P$, price vector $\bar{p}$, tax rate $r_\tau$, current time period $t^*$

**Output**: portfolio $P'$ obtained after harvesting losses from $P$, amount of tax credit $\tau(P)$ due to loss-harvesting.

1. Let $H(P) := \{(s, t) \in \{1, \ldots, N_S\} \times \{1, \ldots, t^* - 1\} : C_{s,t} > \bar{p}_s\}$ be the set of pairs of a stock $s \in \{1, \ldots, N_S\}$ and a period $t \in \{1, \ldots, N_T\}$ such that the $A_{s,t}$ shares were purchased at a cost $C_{s,t}$ that is strictly higher than the price $\bar{p}_s$.
2. Let $L(P) := \sum_{(s,t) \in H(P)} (C_{s,t} - \bar{p}_s) A_{s,t}$ be the amount of capital losses due to liquidating all shares according $H(P)$.
3. Let $\tau(P)_{\text{loss-harvesting}} := r_\tau L(P)$ be the amount of tax credit due to loss-harvesting.
4. for each stock $s \in \{1, \ldots, N_S\}$ do
5.   Let $T := \{t : (s, t) \in H(P)\}$ be the set of periods for which we have shares of stock $s$ to harvest.
6.   Let $a_s := \sum_{t \in T} A_{s,t}$ be the total shares of stock $s$ that we are going to harvest.
7.   for period $t \in T$ do
8.     Sell all $A_{s,t}$ shares purchased in period $t$: $A_{s,t} \leftarrow 0$.
9.   end
10. Buy $a_s$ shares of stock $s$ for the current period $t^*$: $A_{s,t^*} \leftarrow A_{s,t^*} + a_s$.
11. end
12. $P' = (A, C)$ is the portfolio obtained after harvesting losses.
13. return $P', \tau(P)_{\text{loss-harvesting}}$.

### 3.1.5 Cash Contributions

In addition to managing a portfolio via loss-harvesting strategy, we are interested in the effects of periodic cash contributions and withdrawals. Algorithm 4: **ContributeCash** describes precisely the process of contributing cash to a portfolio. In particular, we spend $c > 0$ amount of cash contribution to purchase shares in proportion to the fraction of portfolio value invested in each stock $s \in \{1, \ldots, N_S\}$.
Algorithm 4: \texttt{ContributeCash}(P = (A, C), \overrightarrow{p}, t^*, c)

\textbf{Input}: portfolio \(P\), price vector \(\overrightarrow{p}\), current time period \(t^*\), cash contribution amount \(c \in \mathbb{R}_{\geq 0}\).

\textbf{Output}: portfolio \(P'\) obtained after investing \(c\) amount of cash in portfolio \(P\).

1. Let \(\overrightarrow{v}_s := \sum_{t=1}^{t^*} \overrightarrow{p}_s A_{s,t}\) be the amount of portfolio value invested in stock \(s \in \{1, \ldots, N_S\}\).
2. Let \(\overrightarrow{w}_s := \frac{\overrightarrow{v}_s}{(\sum_{s'=1}^{N_S} \overrightarrow{v}_{s'})}\) be the fraction of the portfolio value invested in stock \(s \in \{1, \ldots, N_S\}\).
3. Let \(\overrightarrow{a}_s := \frac{(c \overrightarrow{w}_s)}{\overrightarrow{v}_s}\) be the amount of shares of stock \(s\) to be purchased.

4. \textbf{for each stock} \(s \in \{1, \ldots, N_S\}\) \textbf{do}
5. \quad Buy \(\overrightarrow{a}_s\) shares of stock \(s\) in period \(t^*\): \(A_{s,t^*} \leftarrow A_{s,t^*} + \overrightarrow{a}_s\).
6. \textbf{end}

7. The portfolio obtained after investing \(c\) amount of cash is given by \(P' = (A, C)\).
8. \textbf{return} \(P'\)

3.1.6 Cash Withdrawals

To withdraw cash from a portfolio, we need to close existing portfolio positions. We are going to consider two different accounting methods for closing positions.

**HIFO Accounting**  The first method for withdrawing cash that we consider is called Highest In, First Out (HIFO) Accounting, and its precise description is given by Algorithm 5: \texttt{WithdrawCashHIFO}.

HIFO accounting liquidates the shares of a given stock \(s\) in order of non-increasing basis-cost; that is, if there are shares of a stock \(s \in \{1, \ldots, N_S\}\) with different basis-costs, HIFO accounting first liquidates the shares with higher basis-cost. Since we do not want to change the portfolio composition significantly, similarly to contributing cash to a portfolio, we withdraw cash from each stock \(s\) in proportion to the portfolio value invested in stock \(s\).

One important observation of cash withdrawal is the following: after we liquidate positions as a result of withdrawing cash, we may need to withdraw further cash to pay taxes on realized capital gains. We handle this scenario in lines 14-19 of Algorithm 5 by withdrawing more cash if we have incurred taxes due on realized capital gains, or contributing cash-equivalent from tax credits due to realizing capital losses.
Algorithm 5: WithdrawCashHIFO\((P = (A, C), \bar{p}, r_\tau, c)\)

**Input**: portfolio \(P\), price vector \(\bar{p}\), tax rate \(r_\tau\), cash amount \(c \in \mathbb{R}_{\geq 0}\) to be withdrawn.

**Output**: portfolio \(P'\) obtained after withdrawing \(c\) amount of cash from \(P\).

1. Let \(\bar{v}_s := \sum_{t=1}^{N_T} \bar{p}_s A_{s,t}\) be the portfolio value invested in stock \(s \in \{1, \ldots, N_S\}\).
2. Let \(\bar{w}_s := \bar{v}_s / (\sum_{s'=1}^{N_S} \bar{v}_{s'})\) be the fraction of the portfolio value invested in stock \(s\).
3. Let \(\bar{a}_s := (c \bar{w}_s) / \bar{v}_s\) be the amount of shares of stock \(s\) to be liquidated. *Note that, we liquidate a fraction of the shares of each stock \(s\) in proportion to portfolio value invested in the stock \(s \in \{1, \ldots, N_S\}\).*
4. Initialize \(\tau := 0\) to be the extra amount of cash that needs to be withdrawn to cover the realized capital gains (if any) due to liquidating positions from the portfolio \(P\).
5. for each stock \(s \in \{1, \ldots, N_S\}\) do
6.     while \(\bar{a}_s > 0\) do
7.         Let \(T := \{t \in \{1, \ldots, N_T\} : A_{s,t} > 0\}\) be the set of periods \(t\) such that portfolio \(P\) has positive number of shares from stock \(s\) purchased in period \(t \in \{1, \ldots, N_T\}\), i.e. \(A_{s,t} > 0\).
8.         Let \(t' := \text{arg max}_{t \in T} C_{s,t}\) be the period with the shares from stock \(s\) that were purchased at the highest cost among all periods \(t \in T\).
9.         Let \(a := \min(\bar{a}_s, A_{s,t'})\) be the shares of stock \(s\) purchased in period \(t'\) that we are going to sell.
10.        Sell \(a\) shares of stock \(s\) purchased in period \(t'\): \(A_{s,t'} \leftarrow A_{s,t'} - a\).
11.        Update \(\bar{a}_s \leftarrow \bar{a}_s - a\).
12.        Update \(\tau \leftarrow \tau + r_\tau a (\bar{p}_s - C_{s,t'})\) by accounting for the tax due on the realized capital gains on selling \(a\) shares of stock \(s\) purchased in period \(t'\).
13.     end
14. end
15. if \(\tau > \epsilon\) then
16.     Note that to withdraw cash from the portfolio, we have sold out of positions.
17.     As a result, we may need to withdraw further cash to cover the tax due on the realized capital gains.
18.     return WithdrawCashHIFO\((A, C), \bar{p}, r_\tau, \tau\)
19. end
20. else
21.     return \((A, C)\)
22. end
AVCO Accounting Another method for withdrawing cash from a portfolio is Average Cost Accounting, and its precise description is given by Algorithm 6, WithdrawCashAVCO.

Algorithm 6: WithdrawCashAVCO(P = (A, C), p̄, t*, rτ, c)

Input: portfolio P, price vector p̄, current period t*, tax rate rτ, cash amount c ∈ ℝ≥0 to be withdrawn.

Output: portfolio P′ obtained after withdrawing c amount of cash from P.

1 Let \( \bar{v}_s := \sum_{t=1}^{N_T} \bar{p}_s A_{s,t} \) be the portfolio value invested in stock \( s \in \{1, \ldots, N_S\} \).
2 Let \( \bar{w}_s := \bar{v}_s / (\sum_{s'=1}^{N_S} \bar{v}_{s'}) \) be the fraction of the portfolio value invested in stock \( s \).
3 Let \( a_s := (c\bar{w}_s) / \bar{v}_s \) be the amount of shares of stock \( s \) to be liquidated. Note that, we liquidate a fraction of the shares of each stock \( s \) in proportion to portfolio value invested in the stock \( s \in \{1, \ldots, N_S\} \).
4 Initialize \( \tau := 0 \) to be the extra amount of cash that needs to be withdrawn to cover the realized capital gains due to liquidating positions from the portfolio P.
5 for each stock \( s \in \{1, \ldots, N_S\} \) do
6   Let \( n_s := \sum_{t=1}^{N_T} A_{s,t} \) be the total shares of stock \( s \) in the portfolio P.
7   for each period \( t \in \{1, \ldots, N_T\} \) do
8     Let \( a := \bar{a}_s (A_{s,t} / n_s) \) be the shares of stock \( s \) purchased in period \( t \) to be liquidated.
9     Sell \( a \) shares of stock \( s \) purchased in period \( t \): \( A_{s,t} \leftarrow A_{s,t} - a \).
10    Update \( \tau \leftarrow \tau + r\tau a (\bar{p}_s - C_{s,t}) \) by accounting for the tax due on the realized capital gains (or losses) on selling \( a \) shares of stock \( s \) purchased in period \( t \).
11 end
12 end
13 Let \( P' = (A, C) \) be the portfolio obtained after withdrawing cash from P.
14 if \( \tau < 0 \) then
15   return ContributeCash(P′, p̄, t*, −τ)
16 end
17 else if \( \tau \geq \epsilon \) then
18   return WithdrawCashAVCO(P′, p̄, t*, rτ, τ)
19 end
20 else
21   return P′
22 end

The main difference between HIFO and AVCO accounting is that AVCO accounting liquidates shares of a stock \( s \) in proportion to the value invested in stock \( s \) from all
lots of stock $s$, instead of liquidating shares of stock $s$ in order of decreasing basis-cost as HIFO accounting does.

### 3.1.7 Framework Outline

Having described all of the preliminary building blocks, we give the outline of the framework in Algorithm 7: \textbf{GetReturns}. We start by initializing two identical portfolios, one of which is going to follow a simple buy-and-hold strategy, whereas the other follows the tax-efficient loss-harvesting strategy. For every time period we repeat the following steps:

1. Obtain the amount of cash contribution ($c > 0$) or cash withdrawal ($c < 0$) for the current time period.

2. Obtain dividend yield rates for all stocks.

3. Obtain the cash-equivalent of all dividends after accounting for taxes, and add the respective cash-equivalent amount the cash contribution amount, i.e. reinvest the dividends in the portfolio.

4. Update the current prices of all stocks according to the stock return rates for the current time period.

5. Perform any stock index turnover.

6. Harvest losses from the tax-efficient portfolio.

7. Contribute or withdraw any outstanding cash equivalent.

8. Compute before-tax and after-tax return rates for the current time period.

By using the returns for every period, we can compute the tax alpha for every time period as the difference of the returns between the tax-efficient loss-harvesting strategy and the simple base strategy. We are going to use this framework for computing tax alpha in the next chapter to quantify the performance of loss-harvesting with respect to a base buy-and-hold strategy.
Algorithm 7: GetReturns

Input: returns \( \bar{r}_s^{(t)} \) for each stock \( s \) at each period \( t \), before-tax dividend yield rates \( \bar{d}_s^{(t)} \), before-tax cash contributions \( c^{(t)} \in \mathbb{R} \) for \( t \in \{1, \ldots, N_T\} \), tax rate \( r_t \).

Output: before-liquidation and after-liquidation returns of both the tax-efficient portfolio (with loss-harvesting), and the base portfolio that is not tax-efficient.

1 Create two initially-identical portfolios \( P^{(0)} = (A, C) \) and \( P_0^{(0)} = (A, C) \) that hold equal number of shares from each stock \( s \in \{1, \ldots, N_S\} \) purchased at period 0. Note that \( P^{(t)}_\ast \) will be the tax-efficient portfolio at period \( t \), and \( P^{(t)} \) will be the base portfolio, which is not tax-efficient, at period \( t \).

2 for each period \( t \in \{1, \ldots, N_T\} \) do

3 Let \( P^{(t)} := P^{(t-1)} \) and \( P^{(t)}_\ast := P^{(t-1)}_\ast \), the initial portfolios in the beginning of period \( t \) equal the portfolios at the end of period \( t - 1 \).

4 Let \( c_s := c^{(t)}_s \) be the amount of cash contribution if positive or cash withdrawal if negative at period \( t \) for the tax-efficient portfolio. Analogously, \( c := c^{(t)} \).

5 Let \( \bar{d}^{(t)}_{\text{after-tax}} := (1 - r_t) \bar{d}^{(t)}_{\text{before-tax}} \) be a vector with the after-tax dividend yield rates for all stocks.

6 Let \( y := \sum_{s=1}^{N_S} (\bar{p}_s \bar{d}_s^{(t)}_{\text{after-tax}} (\sum_{t=1}^{N_T} A_{s,t})) \) be the total dividend yield cash-equivalent from portfolio \( P^{(t)} \). Analogously, let \( y_s \) be the total dividend yield cash-equivalent from \( P^{(t)}_\ast \) in the same period \( t \).

7 Consider \( y_s \) and \( y \) as part of the cash contributions, i.e. update \( c_s \leftarrow c_s + y_s \) and \( c \leftarrow c + y \).

8 Let \( \bar{p}^{(t)} := (1 + r^{(t)} - \bar{d}^{(t)}_{\text{before-tax}}) \bar{p}^{(t-1)} \) be the price vector for period \( t \).

9 for each stock \( s \in \{1, \ldots, N_S\} \) leaving the stock index do

10 Apply ReplaceStock to update portfolio \( P^{(t)} \) with the replacement stock \( s \), and update \( c \leftarrow c - \tau_s(P^{(t)}) \) because of the tax due on the realized capital gains after liquidating shares of stock \( s \) \( P^{(t)} \). Analogously, apply ReplaceStock for portfolio \( P^{(t)}_\ast \), and update \( c_s \leftarrow c_s - \tau_s(P^{(t)}_\ast) \).

11 end

12 Apply HarvestLosses to update the tax-efficient portfolio \( P^{(t)}_\ast \) by taking loss-harvesting opportunities, and update \( c_s \leftarrow c_s + \tau(P^{(t)}_\ast) \) loss-harvesting.

13 Invest the amount of cash \( c > 0 \) in portfolio \( P^{(t)} \) via ContributeCash, or withdraw \(-c \) amount of cash via WithdrawCashAVCO when \( c < 0 \). Analogously with \( c^s \) with the different of using HIFO accounting, i.e. applying WithdrawCashHIFO when \( c^s < 0 \) instead of WithdrawCashAVCO.

14 Compute before-tax and after-tax returns for both portfolios \( P^{(t)} \) and \( P^{(t)}_\ast \) via GetPortfolioValue.
3.2 Introducing Transaction Costs

The framework described in the previous Section has one major limitation—it has the implicit assumption that there are transaction fees. In practice, high transaction costs are likely to reduce tax alpha, and it is important to study to what degree (if at all) loss harvesting is profitable once transaction costs are considered.

Due to the outlined limitation, we are going to introduce changes to the framework described above by adding a new parameter—a fixed transaction cost rate $r_t$. For example, a transaction cost rate of 1% means that we have to pay a 1$ of transaction costs on buying or selling 100$ worth of stock shares. More formally, we define $r_t$ to be transaction fees in a percentage of the value of every transaction.

We modify the framework to include transaction costs with the following three changes:

1. The after-liquidation value of a portfolio is adjusted with respect to the transaction cost rate $r_t$.

2. The loss harvesting criterion is changed to take transaction costs into account, i.e. we harvest losses only when there is still positive tax credit after transaction fees are accounted for. Note that this change leads to fewer loss harvesting opportunities as transaction costs increase because to loss harvest a particular stock share, its basis-cost should be sufficiently higher than its current price so that resulting tax credit can offset the transaction fee.

3. Cash contributions and cash withdrawals are also subject to transaction costs. We implement this change by proportionally decreasing/increasing the cash contribution/withdrawal according to the transaction rate $t_r$.

Finally, Appendix A contains formal description of all necessary changes to the framework.
3.3 Sources of Stock and Dividend Yield Return Rates

In this Section, we are going to describe two different sources of stock return rates and dividend yield return rates: one based off of the Capital Asset Pricing Model, and the other—historical market data from the S&P 500 Index.

3.3.1 Capital Asset Pricing Model

We use the Capital Asset Pricing Model as a stochastic source of stock return rates and dividend yield return rates.

We describe the generative process for stock return rates in Algorithm 8, which takes four parameters: monthly risk-free rate \( r_f \), expected market return rate \( \mu_{rm} \), market risk rate \( \sigma_{rm} \), and stock-specific risk rate \( \sigma_s \). These four hyperparameters govern the distribution of possible stock return rates.

In addition to the source of stock return rates, the framework outlined in the previous section also requires a source of dividend yield return rates. For simplicity, we assume that the dividend yield return rates are the same for each stock \( s \in \{1, \ldots, N_S\} \), and each period \( t \in \{1, \ldots, N_T\} \), i.e., \( d_{st}^{(t)} \) before-tax = \( r_d \in [0, 1] \).

Algorithm 8: SampleStockReturnRates(\( r_f, \mu_{rm}, \sigma_{rm}, \sigma_s \))

**Input**: monthly risk-free rate \( r_f \), monthly expected market return rate \( \mu_{rm} \), monthly market risk rate \( \sigma_{rm} \), monthly stock-specific risk rate \( \sigma_s \).

**Output**: monthly stock return rates \( r_s^{(t)} \).

1. for each stock \( s \in \{1, \ldots, N_S\} \) do
2.   Let \( \beta_s \sim \mathcal{N}(\mu, \sigma^2, a, b) \), the beta for stock \( s \in \{1, \ldots, N_S\} \), be sampled from a truncated normal distribution with parameters \( \mu = 1, \sigma = 0.3, a = -1, b = 3 \).
3. end
4. for each period \( t \in \{1, \ldots, N_T\} \) do
5.   for each stock \( s \in \{1, \ldots, N_S\} \) do
6.     Let \( r_s^{(t)} \sim \mathcal{N}((1 - \beta_s)r_f + \beta_s\mu_{rm}, \beta_s^2\sigma_{rm}^2 + \sigma^2) \) be the monthly return rate of stock \( s \) in period \( t \).
7.   end
8. end

By combining a sample of stock and dividend yield return rates from Algorithm 8
with the framework outlined in the previous section, we can obtain the loss-harvesting
tax alpha for one particular sample of stock and dividend yield return rates. We call
this combination a single realization, because we are simulating the loss-harvesting
strategy against a base strategy for a particular set of stock return rates and dividend
yield return rates in order to obtain the loss-harvesting tax alpha return. To obtain a
distribution of tax alpha returns, we are going to obtain multiple realizations. Overall,
this methodology draws similarity with the Monte Carlo methodology due to the
stochastic nature of the stock return rates of the Capital Asset Pricing Model.

3.3.2 Historical Data from S&P 500 Index

The source of stock return rates and dividend yield return rates given in the previous
section is stochastic in nature, and its usefulness in practice is limited by the modeling
assumptions of the Capital Asset Pricing Model. Thus, that source of stock and
dividend yield return rates provides one particular formulation under which we can
evaluate the tax alpha return of a loss-harvesting investment strategy.

To have a more thorough understanding of the tax alpha return of loss-harvesting
investment, we evaluate the strategies on historical market data. In particular, we
are going to use stock and dividend yield return rates coming from historical S&P 500
Index data. To that end, we use the Center for Research in Security Prices (CRSP)
dataset from Wharton Research Data Services (WRDS). In particular, we obtain stock
return rates and dividend yield return rates from two columns of monthly stock data:
(1) “Holding Period Return”; and (2) “Holding Period Return Without Dividends”,
where (2) is directly in the form of \( \tilde{r}_s^{(t)} \) for every \( s \in \{1, \ldots, N_S\} \) and \( t \in \{1, \ldots, N_T\} \)
as in the previous section, and \( \tilde{d}_s^{(t)} \), before-tax is obtained as the difference between
(1) and (2).

There is one significant difference between using historical market data and using
the Capital Asset Pricing Model as a source of stock return rates—since the loss-
harvesting strategy is completely deterministic, and the historical stock returns are
also non-random, the framework outlined in the previous section produces a single
loss-harvesting tax alpha return rate i.e. there is no stochastic element, in contrast
with the Capital Asset Pricing Model source of stock return rates.
Chapter 4

Results

In this Chapter, we are going to apply the framework from Chapter 3 and present an empirical study of loss-harvesting.

In Section 4.1, we are going to look at the performance of loss-harvesting under various settings of the Capital Asset Pricing Model. In particular, we are going to apply tax alpha sensitivity analysis on the free parameters of the Capital Asset Pricing Model. This analysis is fundamental to understanding how the different parameters of the model (that correspond to different market conditions) affect the tax alpha performance of loss-harvesting.

In Section 4.2, we are going to look at the performance of loss-harvesting on historical market data from the S&P 500 Index.

In Section 4.3, we will compare the tax alpha performance with historical market data and the tax alpha performance under the Capital Asset Pricing Model. In particular, we are going to observe patterns of the tax alpha performance with historical data that are not present in the tax alpha performance under the Capital Asset Pricing Model.

Finally, in Section 4.4, we are going to introduce transaction costs in the computational framework from Chapter 3 and study how their presence affect the tax alpha performance.
4.1 Capital Asset Pricing Model

In this Section, we are going to apply the framework from Chapter 3 and perform sensitivity analysis over the parameters of the Capital Asset Pricing Model. Table 4.1 describes the parameters that we are going to study, their units, base values and respective values that we are going to use in the sensitivity analysis. In the following subsections, we are going to present empirical results from the loss-harvesting strategy with keeping and varying on parameter at a time with values from the Parameters column of the table.

4.1.1 Number of Stocks

Figure 4-1 shows the empirical results from the sensitivity analysis over the number of stocks managed in both the base buy-and-hold portfolio and the loss-harvesting portfolio. We observe that in comparison to the base value of 500 stocks, a higher number of stocks does not significantly increase either the before-liquidation or the after-liquidation tax alpha, whereas decreasing the number of stocks in the portfolio leads to a reduced tax alpha.

The economic intuition is that if there is a smaller number of stocks, the number of loss-harvesting opportunities decreases, and thus the tax alpha is lower. On the
Figure 4-1: Increasing the number of stocks above the base value of 500 stocks does not result in a significant change of either the before-tax or the after-tax alpha; with a number of stocks fewer than the base value, however, tax alpha decreases. Importantly, increasing the number of stocks results in a higher variance of both before-tax and after-tax alpha as indicated by the 25-th and 75-th percentiles in the left panels.

On the other hand, from the panels, we observe that significantly increasing the number of stocks does not lead to a significantly different alpha. One possible explanation is that even if we increase the number of stocks in both portfolios, we would still hold the same initial portfolio value, and thus we would invest proportionally less cash equivalent in each stock.

### 4.1.2 Stock Turnover

We expect that increasing the monthly stock turnover leads to both lower before-liquidation and lower after-liquidation tax alpha because higher turnover requires realizing more capital gains, and consequently paying taxes on those realized capital gains. Note that these dynamics do not align well with the loss-harvesting strategy,
which heavily favors recognizing capital losses that lead to a tax credit.

The empirical results in the panels of Figure 4-2 agree with the above expectations. We conclude that the loss-harvesting strategy fares better in market conditions with a low stock turnover.

Figure 4-2: Higher stock turnover leads to lower tax alpha both before and after accounting for taxes, because stock turnover requires recognizing capital gains. This dynamic is in the opposite direction of loss-harvesting’s goal, which is to realize capital losses as soon as possible, and defer capital gains as long as possible.

4.1.3 Tax Rate

We expect that higher tax rate brackets lead to both higher before-liquidation and after-liquidation tax alpha, because larger tax rates result in more substantial tax credits due to recognized capital losses. This intuition matches the empirical results in panels of Figure 4-3. We conclude that investors in a higher tax rate bracket can obtain larger tax alpha returns from the loss-harvesting strategy.
Figure 4-3: Higher tax rate brackets lead to higher tax alpha both before and after taxes, because larger tax rates result in more substantial tax credits from realized capital losses.

### 4.1.4 Cash Contribution Rate

We expect that increasing the cash contribution rate leads to both higher before-liquidation and after-liquidation tax alpha, because the additional capital invested every month would result in more future loss-harvesting opportunities.

However, by looking at the plots in Figure 4-4, we see that the empirical results do not match our expectation. In particular, the intuition is matched only for before-liquidation tax alpha, but not for the after-liquidation tax alpha.

This finding is one of the most significant empirical results that we have obtained due to several reasons: (1) this result defies basic economic intuition; (2) this result exemplifies the discrepancy between reporting before-liquidation and after-liquidation tax alpha for a loss-harvesting strategy.

We can explain this finding in the following way—cash contributions do lead to additional loss-harvesting opportunities in the future, but they also increase the
value of the portfolio. Since we have defined tax alpha according to the value of the portfolio, it is not that surprising that higher monthly cash contribution rates lead to lower after-liquidation tax alpha. On the other hand, we observe that the magnitude of the effect for after-liquidation tax alpha is not very significant—from no monthly cash contributions to cash contributions of 5% of initial portfolio value we lose about five basis points of tax alpha.

Figure 4-4: Increasing the cash contribution rate leads to higher tax alpha before taxes, and importantly, lower tax alpha after taxes. This empirical result is significant because: (1) it pinpoints the qualitative difference in reporting alpha of loss-harvesting before versus after accounting for taxes; (2) it is crucial due to the practical implications of the performance of loss-harvesting in the presence of regular cash contributions.

4.1.5 Cash Withdrawal Rate

In light of the sensitivity analysis over the cash contribution rate in the last Subsection, we are interested in how the cash withdrawal rate affects before-liquidation and after-liquidation tax alpha. For consistency with the cash contribution rate, the cash
withdrawal rate is defined as a percentage of the initial portfolio value. The largest monthly cash withdrawal rate that we consider is 0.1% of initial portfolio value, which corresponds to withdrawing 36% of the initial portfolio value over the course of the 25-year horizon.

Based on the empirical results from the previous Subsection, two main forces govern the effect of the monthly cash withdrawal rate on after-liquidation tax alpha: (1) higher monthly cash withdrawal rate leads to realizing more capital gains, and subsequently paying taxes on them, which is expected to decrease the after-liquidation tax alpha; (2) higher monthly cash withdrawal rate reduces the base value of the portfolio, which leads to higher after-liquidation tax alpha. Since those two forces are in opposite directions, it is not clear which one is stronger a priori.

Figure 4-5: Increasing the cash withdrawal rate (i.e. more negative) leads to lower alpha before taxes, and higher alpha after taxes. In general, the magnitude of both effects is rather small.

Figure 4-5 shows the empirical results from the sensitivity analysis over the cash withdrawal rate. We observe that before-liquidation tax alpha decreases as the cash
withdrawal rate increases, whereas for after-liquidation tax alpha, we see the opposite effect with the important note that the magnitude of the effect is insignificant (i.e. less than two basis points).

4.1.6 Market Premium Rate

Increasing the market premium rate should, intuitively, lead to lower loss-harvesting tax alpha (both before and after taxes), because the market does better on average the higher the market premium is, since higher market permium rates lead to reduced number of loss-harvesting opportunities.

The empirical results match our intuition only before accounting for taxes, as shown in the top panels of Figure 4-6. However, our intuition above does not explain the empirical results for after-liquidation tax alpha where we observe that the highest average after-liquidation tax alpha is achieved with a moderate value for the market premium rate. We conclude that the loss-harvesting strategy favors moderate market premium rates.

4.1.7 Systematic Market Risk Rate

We expect that increasing the systematic market risk rate leads to both higher before-liquidation and higher after-liquidation tax alpha, because the number of loss-harvesting opportunities increases when the volatility of the market is greater. This intuition matches the empirical results from the sensitivity analysis over the systematic market risk rate in the panels of Figure 4-7.

4.1.8 Stock-Specific Risk Rate

Similarly to the results in the previous Subsection, we expect that higher stock-specific risk rate leads to both higher before-liquidation and after-liquidation tax alpha. The sensitivity analysis empirical results in the panels of Figure 4-8 match this intuition.
Figure 4-6: Higher monthly market premium rate leads to lower loss-harvesting alpha before taxes, because loss-harvesting opportunities reduce under a higher market premium rate. Interestingly, the empirical results for loss-harvesting after accounting for taxes imply that loss-harvesting does best under moderate (about 9% annual) market premium rate.

4.1.9 Dividend-Yield Rate

When it comes to loss-harvesting alpha under varying monthly dividend-yield rates, there are two main opposing forces: (1) paying dividends requires closing positions, i.e. realizing capital gains, and subsequently paying taxes on them, which decreases the portfolio return; (2) reinvesting the cash from the dividends back in the portfolio, which leads to more loss-harvesting opportunities in the future. We expect that, in general, higher monthly dividend-yield rate, leads to lower loss-harvesting tax alpha.

The empirical results from the sensitivity analysis over the monthly dividend-yield rate parameter are given in the panels of Figure 4-9. We observe that increasing the monthly dividend-yield rate leads to both higher before-liquidation tax alpha, and higher after-liquidation tax alpha, which provides evidence that (2) is the stronger of the two forces. In light of the results of the sensitivity analysis for the cash
Figure 4-7: Increasing the systematic market risk rate leads to higher loss-harvesting alpha both before and after accounting for taxes, because the number of loss-harvesting opportunities grows with increasing the systematic volatility in the market.

withdrawal rate, paying dividends in (1) decreases the value of the portfolio, which is one additional reason why tax alpha may increase as a result of higher dividend-yield rate.
Figure 4-8: Increasing the idiosyncratic stock risk rate leads to higher loss-harvesting alpha both before and after accounting for taxes, because the number of loss-harvesting opportunities grows with increasing the specific volatility in the market.
Figure 4-9: Increasing the dividend-yield rates leads to higher loss-harvesting alpha both before and after accounting for taxes.
4.2 Historical Data from S&P 500 Index

In Section 4.1, we studied the tax alpha performance of loss-harvesting under the Capital Asset Pricing Model as a source of simulated stock return rates.

In this Section, we apply the framework from Chapter 3 to historical market data from the S&P 500 Index. As a consequence of using historical market data, there are only two parameters to vary: (1) the tax rate, and (2) the monthly cash contribution/withdrawal rate.

We are going to apply sensitivity analysis over the two parameters in the following three Subsections, which will help us in understanding the tax alpha performance of loss-harvesting in a setting that is less dependent on the modeling assumptions.

Note that the base setup the parameter values are chosen as follows: (1) 35% tax rate, and (2) no cash contribution or withdrawal.

4.2.1 Cash Contribution Rate

Recall that in Section 4.1, we observed that before-liquidation tax alpha decreases over time, because as portfolio ages, the loss-harvesting opportunities decrease. Intuitively, cash contributions are expected to increase the before-liquidation tax alpha because contributing capital can be thought equivalently in terms of investing in young portfolios that tend to have more loss-harvesting opportunities. That intuition matched the empirical results in Section 4.1 from the Capital Asset Pricing Model as a source of market data.

The before-liquidation tax alpha performance with the S&P 500 Index as a source of market data is given in the top two panels of Figure 4-10. From the top left panel, we observe that increasing the monthly cash contribution rate, which is given as a percentage of initial portfolio value, leads to higher before-liquidation tax alpha. From the top right panel, we observe most of the before-liquidation tax alpha is generated when the portfolio is young. Those results match the results from the previous Section.

In Section 4.1, we observed that increasing the cash contribution rate leads to a
Figure 4-10: Similarly to the loss-harvesting empirical results with the Capital Asset Pricing Model as a source of simulated stock return rates, increasing the monthly cash contribution rate leads to higher loss-harvesting alpha before taxes, and lower loss-harvesting alpha after taxes on historical stock market data from the S&P 500 Index.

lower after-liquidation tax alpha, which matches the results in the bottom panels of Figure 4-10. In general, the magnitude of the effect is not particularly large, because we lose 17 basis points of after-liquidation tax alpha by going from a portfolio with no contributions to a portfolio with monthly cash contribution rate of 5% of the initial portfolio value. Note that a monthly cash contribution rate of 5% corresponds to investing cash equivalent to the initial portfolio value every 20 months, which is rather extreme.

Also, we observe that the after-liquidation tax alpha generated during the last 5 years of the horizon of 25 years is highest; this in sharp contrast with the empirical results from the previous Section, and signifies that after-liquidation tax alpha in this setup is generated quite differently. We are going to explore this further in Section 4.3.
4.2.2 Cash Withdrawal Rate

The before-liquidation tax alpha performance with the S&P 500 Index as a source of market data with respect to different amounts of cash withdrawal rates is given in the top two panels of Figure 4-11. We observe that higher cash withdrawal rate results in both higher before-liquidation and higher after-liquidation tax alpha.

Similarly to the analysis for cash contribution rate, we observe that the most before-liquidation tax alpha is generated when the portfolio is young, whereas the most after-liquidation tax alpha is generated when the portfolio is old.

Figure 4-11: Increasing the cash withdrawal rate leads to higher loss-harvesting alpha on historical stock market return data from the S&P 500 Index both before and after taxes.

The results mimic those from Section 4.1. Note that the magnitude of the effect is quite small.

1Note that cash withdrawal rate is defined to be negative, and thus highest cash withdrawal rate corresponds to smaller values on the x-axis.
4.2.3 Tax Rate

Figure 4-12 shows the sensitivity analysis of tax rate on the loss-harvesting alpha. From the panels on the left, it becomes clear that the tax rate is the most important parameter when it comes to tax alpha performance over historical market data from the S&P 500 Index. Moreover, there is evidence a linear relationship between tax rate and tax alpha performance. In particular, 10% of additional tax rate adds about 25 additional basis points of both before-liquidation and after-liquidation tax alpha. Moreover, the effect seems to be increasing at higher tax rates, which provides some evidence for some higher-order terms within the relationship.

Similarly to the cash contribution and withdrawal rate analyses, we observe that the most before-liquidation tax alpha is generated when the portfolio is young, whereas the most after-liquidation tax alpha is generated when the portfolio is old.

Figure 4-12: Increasing the tax rate leads to higher loss-harvesting alpha on historical stock market return data from the S&P 500 Index both before and after taxes.
4.3 Comparison

In the previous sections, we observed that the after-liquidation tax alpha differs significantly between the two setups with different sources of market return data (CAPM vs historical S&P Index data).

In particular, in Section 4.1 we observed that after-liquidation tax alpha over market data from the Capital Asset Pricing Model is highest in the middle 5 years of the 25-year horizon, whereas in Section 4.2 we observed that after-liquidation tax alpha over historical market data from the S&P 500 Index is highest in the last 5 years of the 25-year horizon.

By looking at Figure 4-13 and Figure 4-14, we observe a sharp contrast in the after-liquidation tax alpha between the two different sources of market data with respect to time. The first heat map shows that the after-liquidation tax alpha varies smoothly in adjacent years for every tax rate.
Figure 4-14: A heat map of after-liquidation tax alpha in basis points on historical market data from the S&P 500 Index. The after-liquidation tax alpha does not vary smoothly in time for any tax rate. In particular, there is a significant drop in after-liquidation tax alpha in 2002 and in 2008.

Market data from CAPM varies smoothly in time. On the other hand, the second heat map shows that the after-liquidation tax alpha with historical market data from the S&P 500 Index does not change smoothly. We observe that in 2002 and 2008 the tax alpha becomes negative in the latter case but not in the former. This pattern is likely a result of the market conditions in 2002 and 2008.

Note that our Capital Asset Pricing Model implementation is time-invariant. We hypothesize that the negative after-liquidation tax alpha is a result of a major market price shock.

One way to check if this is, potentially, a plausible explanation is to introduce an artificial significant downward price shock in the framework in Section 4.1 and verify if we observe a similar negative after-liquidation tax alpha. The empirical results of this experiment are shown in Figure 4-15. In particular, there is a sharp price shock introduced in 2002. We observe results similar to those from the experiment with historical market data from the S&P 500 Index: there is negative after-liquidation
tax alpha in 2002, and after that tax alpha recovers. Therefore, we conclude that
alpha generated by the loss-harvesting strategy is likely to dramatically depend on the
particular market conditions, and as we see by the performance on historical data, it
is possible for loss-harvesting to perform worse than a simple buy-and-hold strategy.

4.4 Transaction Costs

In this Section, we perform sensitivity analysis over the transaction rate $r_t$, which
was introduced in Section 3.2. In particular, we combine the extended framework
from Section 3.2 with historical market data from S&P 500 Index.

In practice, we expect that increasing the transaction cost rate leads to lower
before-liquidation and lower after-liquidation tax alpha because some portion of the
tax credit due to loss-harvesting will be used for covering transaction fees. Finding
the transaction cost rate above which loss-harvesting is not profitable is important,
because such knowledge can be used by an investor to make an informed decision as
to whether loss-harvesting is profitable given the transaction costs they will need to

Figure 4-15: A heat map of after-liquidation tax alpha in basis points with the Capital
Asset Pricing Model as a source of simulated stock return rates with an artificial
sharp price shock in 2002. The after-liquidation tax alpha drops significantly in 2002,
similarly to Figure 4-14. Note that the colorbar scales are different.
Figure 4-16: Increasing the transaction cost rate $r_t$ leads to lower loss-harvesting alpha both before and after accounting for taxes. Moreover, note that transaction cost rate of 1% reduces the alpha of loss-harvesting after taxes by about 50%; thus, transaction costs are vital for the use of loss-harvesting in practice.

The empirical results from the sensitivity analysis experiment are shown in Figure 4-16. We see that our expectations of reduced alpha with increasing transaction costs match the experimental results. Moreover, it is important to note the values on the $x$-axis—we observe that even with a exceedingly high transaction cost rate of 10%, loss-harvesting still returns positive before-liquidation and positive after-liquidation tax alpha. There are several reasons why loss-harvesting is profitable even after high transaction cost rates are included in the framework:

1. Both the loss-harvested portfolio and the base one are subject to the same transaction cost rate, which means high transaction cost rates affects both portfolios.

2. Recall that the loss-harvesting criterion was modified so that shares are sold at a loss only when the tax credit is sufficient to cover the transaction fee.
Thus, as the transaction cost rate $r_t$ increases, loss-harvesting initiates fewer and fewer transactions. From a different perspective, as the transaction cost rate $r_t$ increases, the tax-efficient strategy becomes similar to the base investment strategy that does not perform any loss-harvesting.

From the above observation, we conclude that loss-harvesting is profitable even after accounting for transaction costs. However, in practice, the magnitude of alpha matters a lot. We note that introducing transaction cost rate of 1% reduces the alpha of loss-harvesting after taxes by 50%, which means that transaction costs play a vital role in the performance of loss-harvesting in practice.
Chapter 5

Conclusion

First, we started off with the research by Berkin and Ye (2003) on the loss-harvesting trading strategy, and outlined a computational framework for studying said strategy that is independent of any particular source of stock returns data. By using the framework in combination with the Capital Asset Pricing Model as a source of simulated stock returns, we obtained our first notable result. In particular, we provide evidence that introducing regular monthly cash contributions with the idea of refreshing the portfolio and the hope to boost returns, results in reduced alpha after accounting for taxes. This finding is extremely significant at a time when the interest in tax-efficient investing has increased to the point that there are companies, such as Wealthfront, that offer financial services that incorporate loss-harvesting. There are few notable differences between the loss-harvesting that we have studied, and Wealthfront’s implementation of the strategy: (1) we harvest losses on a monthly basis, whereas Wealthfront looks for such opportunities on a daily basis, and (2) their criterion for loss-harvesting is not identical to ours. Those differences could be sufficient for their claim that regular cash contributions do boost loss-harvesting returns to hold. Further research in this direction is necessary to reconcile those contrasting views, and the extent to which cash contributions can increase the returns of loss-harvesting after taxes.

Second, we applied our framework in combination with historical stock returns data from the S&P 500 Index, and performed similar sensitivity analysis on tax
rates, and both cash contribution and withdrawal rates. Analogously to the study with the simulated stock returns data, we find that higher tax rates increase loss-harvesting alpha after taxes. Moreover, higher cash contribution rates reduce alpha after taxes. Interestingly, we observe that the loss-harvesting alpha per year varies a lot in comparison to the first study. In particular, the loss-harvesting strategy yields negative alpha both in 2002 and 2008, which means that the simple buy-and-hold strategy performs better than the loss-harvesting strategy in some market conditions. Such a finding implies that it may be possible to guide the loss-harvesting strategy to make better trading decisions in the presence of signals about the general market conditions. This observation lead us to the experiment of artificially introducing a sharp price shock in the simulated stock returns data. Then, we obtained similar patterns to those with the historical stock returns data, and concluded that the use of the time-invariant Capital Asset Pricing Model to study the loss-harvesting strategy in a simulated setting has severe limitations.

Third, we extend our computational framework to include transaction costs, and obtain our third key result. In particular, we find that 1% transaction costs reduce the alpha of loss-harvesting by 50%. This finding is significant, because it points out the importance that transaction costs play in the performance of loss-harvesting strategy.

Finally, we identify directions for future research that may lead to improving returns of the loss-harvesting strategy: (1) determining market conditions that result in loss-harvesting losses, and (2) optimizing the loss-harvesting performance in the presence of transaction costs. We hope to explore those directions in future work.
Appendix A

Transaction Costs

In this appendix, we formally describe the changes that we apply to the framework from Chapter 3 to incorporate transaction costs.

For consistency, we denote the fixed transaction cost rate by $r_t$, which is unitless, and is given as a percentage of the total transaction value. For example, a transaction cost rate of 1% corresponds to paying a $1\$ of transaction costs on buying or selling $100\$ worth of stock shares.

A.1 Portfolio Value

The first change that we introduce is to modify the definition of the after-tax portfolio value. Recall that the after-tax portfolio value is computed by liquidating all portfolio positions, and paying the taxes on the portfolio gains. However, if we were to liquidate the entire portfolio, we have to include the transaction costs that such a liquidation event would incur. For that, we need to discount the after-tax portfolio value proportionally to the transaction cost rate. We use Algorithm 1 \texttt{GetPortfolioValue} as a template, and introduce the necessary changes in Algorithm 9 in red color.
Algorithm 9: GetPortfolioValue2($P = (A, C), S, \bar{p}, r_\tau, r_t$)

**Input:** portfolio $P$, stock index set $S$, price vector $\bar{p}$, tax rate $r_\tau$, transaction cost rate $r_t$

**Output:** before-tax value $V_{P}^{\text{before-tax}}$, after-tax value $V_{P}^{\text{after-tax}}$, amount of tax $\tau_P$ if we liquidated all of $P$.

1. Let $V_{P}^{\text{before-tax}} := \sum_{s \in S} \sum_{t=1}^{N_t} \bar{p}_s A_{s,t}$ be the before-tax portfolio value of $P$ with respect to the stock index set $S$.

2. Let $G_P := \sum_{s \in S} \sum_{t=1}^{N_t} (\bar{p}_s - C_{s,t}) A_{s,t}$ be the amount of capital gains (or capital losses when negative) on the portfolio $P$ with respect to the stock index set $S$.

3. Let $\tau_P := -r_\tau G_P$ be the amount of tax due on the capital gains $G_P$. *Note that when $G_P < 0$, there is a net capital loss on the portfolio $P$ with respect to the stock index set $S$, and $\tau_P > 0$ is the amount of tax credit due on that loss.*

4. Let $V_{P}^{\text{after-tax}} := (1 - r_t) V_{P}^{\text{before-tax}} + \tau_P$ be the after-tax portfolio value, obtained by offsetting the before-tax portfolio value with the amount of tax $\tau_P$ due (which is negative if there is a tax due, or positive if there is a tax credit due to losses) if we liquidated the entire portfolio $P$. *Note that when there a is net capital loss on $P$ with respect to the stock index set $S$, $V_{P}^{\text{after-tax}} > V_{P}^{\text{before-tax}}$.

5. *Return* $V_{P}^{\text{before-tax}}, V_{P}^{\text{after-tax}}, \tau_P$.

**A.2 Harvesting Losses**

The second change that we introduce is to modify the loss-harvesting criterion. In particular, we harvest losses only when there is sufficient tax credit to offset the transaction cost of liquidating the shares to be loss-harvested. Note that this change leads to fewer loss harvesting opportunities as transaction costs increase, because to loss harvest a particular stock share, its basis-cost should be sufficiently higher than its current price to offset the larger transaction fee.

We use Algorithm 3 HarvestLosses as a template, and introduce the necessary changes in red color in Algorithm 10. Importantly, we update the loss-harvesting criterion so that it accounts for two transactions: one transaction for liquidating the share due to loss harvesting, and one additional transaction for buying back the share, which is an important part of maintaining the loss-harvested portfolio; those two types of transactions are highlighted in blue color.
Algorithm 10: HarvestLosses2\((P = (A, C), \bar{p}, r_\tau, t^*, r_t)\)

**Input**: portfolio \(P\), price vector \(\bar{p}\), tax rate \(r_\tau\), current time period \(t^*\), transaction cost rate \(r_t\)

**Output**: portfolio \(P'\) obtained after harvesting losses from \(P\), amount of tax credit \(\tau(P)\)loss-harvesting gathered by loss-harvesting

1. Let \(H(P) := \{(s, t) \in \{1, \ldots, N_S\} \times \{1, \ldots, t^* - 1\} : (C_{s,t} - \bar{p}_s)t_\tau > 2r_tC_{s,t}\}\) be the set of pairs of a stock \(s \in \{1, \ldots, N_S\}\) and a period \(t \in \{1, \ldots, N_T\}\) such that the \(A_{s,t}\) shares were purchased at a cost \(C_{s,t}\) that is strictly higher than the price \(\bar{p}_s\).
2. Let \(L(P) := \sum_{(s,t) \in H(P)}(C_{s,t} - \bar{p}_s)A_{s,t}\) be the amount of capital losses due to liquidating all shares according \(H(P)\).
3. Let \(\tau(P)\)loss-harvesting := \(r_\tau L(P)\) be the amount of tax credit due to loss-harvesting.
4. **for each stock** \(s \in \{1, \ldots, N_S\} \) **do**
5.  
6.  
7.  
8.  
9. **end**
10. **for period** \(t \in T\) **do**
11.  
12.  
13. **end**

14. \(P' = (A, C)\) is the portfolio obtained after harvesting losses.

15. **return** \(P', \tau(P)\)loss-harvesting

A.3 Contributing And Withdrawing Cash

Note that cash contributions and withdrawals lead to modifying the portfolio constituents, and we need to account for the necessary transaction costs.

One way to implement this for cash contributions is to modify the cash contribution amount from \(c\) to \(c(1 - r_t)\) so that transaction costs are accounted for implicitly. This change requires the addition of a single line to Algorithm 4 ContributeCash just before line 1. On the other hand, to implement this change for both types of accounting rules (WithdrawCashAVCO and WithdrawCashHIFO), we modify the cash withdraw amount from \(c\) to \(c(1 + r_t)\). Note that difference in the sign between the two modifications. Whereas a positive transaction cost rate decrease the amount of cash contributed to a portfolio, it increase the amount of cash-equivalent that needs
to be withdrawn from a portfolio to for the necessary transaction fees.

A.4 Summary

The above three modifications are sufficient to extend the framework in Chapter 3 to include transaction costs. It is worth mentioning that this extension is in line of the original framework when $r_t = 0$, i.e. the extended framework agrees with the original one when the transaction cost rate is $r_t = 0$. 
Bibliography


