FLUID OSCILLATIONS IN CYLINDRICAL SHELLS

by

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ABSTRACT

The problem of sloshing of liquids in cylindrical shells with flexible walls is considered. Theoretical considerations predict that the effect of the wall flexibility is to reduce the natural frequencies of oscillation below those that would occur if the walls of the tank were assumed rigid. This effect is quite small in the model used in the experimental investigation and at present can be said to be unverified.

A brief discussion of the mathematical solutions and the approximations made show that the analysis as such is limited to experimental type procedures but that the approximations made are quite compatible with full scale configurations.

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Furthermore, the author is indebted to his thesis supervisor, Prof. Marten Landahl and his colleague Prof. Erik Mollo-Christensen for their valuable discussions on the theory included in this work and to Profs. Holt Ashley and James Mar for supplying needed references. The idea of performing experiments on models of fuel tank configurations is the suggestion of Prof. Paul Sandorff although in a slightly different form. All of the men mentioned in the above paragraph are on the faculty of the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology.
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\( \omega \)  circular frequency
\( \rho_f \)  mass density of fluid
\( \rho_w \)  mass density of shell wall
\( \phi, \psi \)  Velocity potential
\( \nabla^2 \)  Laplacian Operator
\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2} \]
\( \nabla^4 \)  Modified Double Laplacian
\[ \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2}{a^2} \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{1}{a^2} \frac{\partial^4}{\partial \theta^4} \]
OBJECT

The object of this paper is to estimate the effect that non rigid shell walls have on the determination of the sloshing frequencies of liquid in a cylindrical vessel. Since the fluid oscillations are governed by Laplace's equation with a free surface condition, the eigen-frequencies are determined by the boundary condition at the outer radius of the cylinder. In this paper, the equations for a thin cylindrical shell shall be considered to apply in this case. The assumption of a specific form of solution shall result in an eigenvalue problem, the form of which is slightly different from that of the case where a rigid wall is assumed, but which approaches the results of the rigid wall case in the proper limits.

An experimental investigation shall be also attempted to devise procedures for measuring the predicted effect, and the results of preliminary investigations outlined, along with some suggestions for future work.
CHAPTER I

INTRODUCTION

The problem of the so called "free oscillations" of fluid in vessels is quite well known, and has been investigated extensively for the past century and a half. Recent work has been submitted by Bauer (1) for complete sets of solutions for sloshing in cylindrical sector and ring tanks, and Miles (2) who considered primary tank bending for known deflection modes. Nelson (3) considered sloshing in tanks being drained of fluid and the bending sloshing coupling in deriving the equations of motion for missiles in flight.

While each of these three authors have considered the effect of non-rigid tank structure on the sloshing, they all have assumed that a Green's function for the beam bending of the structure was known. In this paper the shell deflections will be considered on a scale where they are assumed to deform in a membrane like manner, and the deflections thereby determined.

With the advent of large liquid fueled boosters for missile and space applications, the structural design of these configurations is bound to be as efficient as possible, resulting in thin skinned pressurized cylinders
for the most part. Since the effective thickness of the cylinder wall is usually much less than the radius of the tank, the question immediately arises as to the error introduced in the calculations by considering the wall to be rigid. If one considers the equations for the deflections of thin cylindrical shells along with Laplace's equation, which governs the fluid oscillations, coupled together by matching the relative velocity of the fluid and the wall, and by applying the sloshing force at the wall as an inhomogeneous term in the shell equations, a new eigen value problem arises that includes the rigid wall case as a special example.

Care must be taken in the formal procedures towards solution however, because the shell equations increase the order of the system by a factor of eight after separating variables, and the question of proper end conditions on the shell deflections must be considered along with the completeness of the solutions.

The procedure to be used may be thought of as a perturbation on the solution of the rigid wall problem. In this investigation, the boundary conditions satisfied by the chosen solution shall be presented rather than finding a general solution that satisfies arbitrary boundary conditions, a far more complex problem in practice if not in theory.
CHAPTER II

PRESENTATION OF THE EQUATIONS OF MOTION FOR
THE FLUID SHELL SYSTEM

Consider an inviscid, incompressible fluid of density \( \rho_f \) undergoing small oscillations in a cylindrical vessel of radius \( a \) and length \( L \). If the fluid motion is irrotational initially, it remains so and the equation of conservation of mass can be written in terms of a velocity potential as

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad (2-1)
\]

The associated velocity perturbations are

\[
\nu_r = \frac{\partial \Phi}{\partial r} \quad (2-2a)
\]
\[
\nu_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \quad (2-2b)
\]
\[
\nu_x = \frac{\partial \Phi}{\partial x} \quad (2-2c)
\]

where \( r \) is the radial coordinate \( (0 \leq r \leq a) \) and \( \theta \) the angular coordinate in a cylindrical polar coordinate system centered on the tank axis. Select \( x = 0 \) to be in the mean position of the free fluid surface and \( x = L \) to be at the base of the tank. Although the cylinder is assumed to be full of fluid initially, there
need be no fear of liquid spilling over the top in view of the linear approximations to be made.

There are three boundary conditions associated with this potential. The first condition states (i) that the fluid velocity normal to the base of the tank vanish

\[ \frac{\partial \Phi}{\partial x} = 0 \quad \text{at} \quad x = L \quad (2-3a) \]

(the assumption of a rigid tank base perpendicular to the tank axis is an unrealistic one in view of the methods of construction used in missile bottoms, but since the phenomena of interest are associated with the tank walls, it is useful for the purposes of analysis as a model).

The free surface condition (ii) arises from the fact that there cannot exist a pressure differential at the fluid surface and gives a relation between the pressure rate of change on the surface and the rate of change of potential energy directly from Kelvin's equation. In linearized form it is expressed as

\[ \frac{\partial^2 \Phi}{\partial t^2} - g \frac{\partial \Phi}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad (2-3b) \]

If \( u, v, \) and \( w \) are defined as the axial, tangential, and radial displacements of the shell, the condition of zero relative velocity (iii) at the wall then reads in linear form
\[ \frac{\partial \Phi}{\partial r} = \frac{\partial w}{\partial t} \quad \text{at} \quad r = a \]  

(2-3c)

It should be remarked that although Laplace's equation is time independent, the boundary conditions are not. Equation (2-3b) can be satisfied if the potential varies sinusoidally in time, and solutions of the following form are permissible

\[ \Phi(x, r, \theta, t) = \Phi(x, r, \theta) e^{i\omega t} \]  

(2-4)

The solutions for \( \Phi \) are of course obtained by separation of variables.

The equations for the deflections of cylindrical shells have been derived by various investigators using energy methods. Fung, Sechler, and Kaplan (4) present these in a suitable form and they are recorded below.

\[ \frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2a} \frac{\partial^3 u}{\partial x \partial \theta^2} + \frac{1+\nu}{2a} \frac{\partial^2 w}{\partial x^2} - \frac{\omega^2}{2} \frac{\partial^2 w}{\partial x^2} - \frac{1}{\varepsilon} \rho_w \frac{\partial^2 u}{\partial t^2} = 0 \]  

(2-5a)

\[ \frac{1+\nu}{2a} \frac{\partial^3 u}{\partial x \partial \theta^2} + \frac{1-\nu}{2a} \frac{\partial^2 u}{\partial x^2} - \frac{1}{2a} \frac{\partial^2 w}{\partial x^2} + \frac{h^2}{12a^2} \left[ \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial \theta^4} \right] + \frac{h^2}{12a^2} \left[ \frac{1}{(1-\nu)} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right] = 0 \]  

(2-5b)

\[ 2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial \theta} - \frac{w}{a} - \frac{a^2}{12} \nabla^4 w - \frac{1}{12} \left[ \frac{2-\nu}{a} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right] + \frac{2-\nu}{\varepsilon} \rho_w \frac{\partial^2 w}{\partial t^2} \]  

(2-5c)

\[ = \frac{c_l(1-\nu)}{\varepsilon} \left\{ \rho_{tf} \frac{\partial \Phi}{\partial t} (x, a, \theta) - \rho_f g \right\} \]
where
\[ \nabla_1^4 = \frac{\partial^4}{\partial x^4} + \frac{2}{a^4} \frac{\partial}{\partial x} \frac{\partial^4}{\partial \theta^4} + \frac{1}{a^4} \frac{\partial^4}{\partial \theta^4} \] (2-5d)

In (2-5), \( E \) is Young's modulus, \( \nu \) is Poisson's ratio, \( \rho_w \) is the density of the material in the shell wall, and \( p \) is the internal pressure.

The inhomogeneous term is the driving force associated with the fluid pressure of sloshing and with the static pressure due to the fluid depth of \( x \). The term proportional to \( x \) can be subtracted out as a steady state solution, and only the oscillatory term kept.

These equations could be written in another form (called the Donnell form after their initiator) and are so derived by Yu (5). This form is characterized by the separation of the dependent variables \( u, v, \) and \( w \) such that \( w \) can be solved directly from the driving pressure and \( u, v \) determined from \( w \) by two separate equations. While the use of the equations of this form may at first seem to be handier, the final form of their presentation is somewhat cumbersome and the physical interpretation of the terms is lost in the mathematical manipulations. If the full equations are to be solved for a specific configuration, the use of these relations is recommended.

However, for the purposes of this analysis, where approximations based on the physical situation are to be made,
this generality serves little use and these equations will not be used.

The equations as presented are of the eighth order and of constant coefficients, so exponential solutions in \( x \) can be expected. However for an analysis in general terms, an eighth order algebraic equation is of little use unless one wants to get specific solutions for a given configuration with a machine.

It is fairly obvious that some sort of approximations must be made to make equations (2-5) amenable to solutions. In the next section, the various dimensionless parameters of the problem shall be considered, seeking always to justify approximations that will simplify the equations to such a degree that they are capable of being handled.
CHAPTER III

IMPORTANT DIMENSIONLESS GROUPINGS

Sandorff (7) discussed the important dimensionless parameters necessary for the modeling of full scale fluid tank configurations. In this paper he lists many parameters of interest, perhaps the most important of which, for this discussion is

\[
\frac{\beta_f g l^2}{\lambda} = \frac{\beta_f g a^2}{Eh} \quad \text{(for cylinders)} \quad (3-1)
\]

This parameter represents the similarity relation between the structural elastic characteristics of the tank wall and the surging input force of the fluid. The same parameter will be shown to result in the present analysis.

Another grouping of interest is the ratio of the mass per unit length of the tank wall to the fluid mass per unit length. The following relationship is usually true for thin shell

\[
\frac{\rho wh}{\rho_f a} \ll 1 \quad (3-2)
\]

In view of this order of magnitude relation, it is not unreasonable to assume that terms involving this ratio are negligible, or as Miles put it in reference (2), that
the fluid can be assumed to possess all of the kinetic energy, while the shell stores the potential energy.

The effect of the steady pressure can be simply stated in a grouping that is recognizable as the hoop strain in the shell wall, which is usually small

\[
\frac{pa}{Eh} \ll 1
\]

(3-3)

The terms that have this ratio as a coefficient can be neglected to first order and if further refinement is desired, and expansion in powers of \( \frac{pa}{Eh} \) can be used.

The other small ratio in the problem is the shell wall thickness over the cylinder radius. Terms of order

\[
\frac{h^2}{12 a^2} \ll 1
\]

(3-4)

shall be neglected consistently. This is equivalent to the statement that there is no variation in radial stress throughout the thickness of the shell.

While these approximations may gloss over some of the critical details in a stress analysis, it makes the determination of the shifts in natural frequencies amenable to analysis. It must be stressed that these approximations are an extension of those used in the derivation of (2-5). Of course the suitability of the method depends upon the problem at hand, and for a given configuration a machine solution is desirable for this analysis will only predict gross trends for preliminary design considerations.
CHAPTER IV

SOLUTIONS OF THE EQUATIONS OF MOTION

Equation (2-1) in cylindrical polar coordinates is solved by separation of variables. Since the Θ = 0 axis is of little importance because of the symmetry of the cylinder, and for solutions non-singular on the axis, a solution to (2-1) can be written directly as

\[ \Phi(x,r,\theta,t) = \cosn\theta \left[ \tilde{A}_n \cosh q_n + \tilde{B}_n \sinh q_n \right] J_n(qr) e^{i\omega t} \]  

(4-1)

where \( J_n \) is Bessel's function of the first kind and order \( n \), and \( q \) is some modal parameter. The real part of the function is assumed.

Substituting (4-1) into (2-3a,b) yields, after some manipulation

\[ \varphi(x,r,\theta) = \tilde{A}_n \cosn \theta \frac{\cosh q(x-L)}{\cosh q L} J_n(qr) \]  

(4-2a)

where \( \omega \) is now determined by

\[ \omega^2 = q g \tanh q L \]  

(4-2b)

If by separation of variables, the functions \( u, v, \) and \( w \) are written as

\[ u = \hat{u}(x) \cosn\theta e^{i\omega t} \]  

(4-3a)
\[ u = \hat{u}(x) \sin n\theta e^{i\omega t} \]  
\[ w = \hat{w}(x) \cos n\theta e^{i\omega t} \]  

then the boundary condition (2-3c) can be written operationally as

\[ \hat{w}(x) \cos n\theta = \frac{1}{(i\omega)} \frac{\partial \phi}{\partial r}(x, a, \theta) \]  

(4-2c)

It should be stated at this time that \( q \) need not be real for these solutions to exist. In fact, for \( q \) purely imaginary

\[ \Phi'(x, r, \theta, t) = \cos n\theta \left[ A_n' \cos kr + B_n' \sin kr \right] I_n(kr)e^{i\omega t} \]  

(4-4a)

where

\[ k = iq \]  

(4-4b)

is also a solution with trigonometric instead of hyperbolic variation in \( x \) and the modified Bessel function \( I_n \) instead of \( J_n \). In this instance

\[ \omega^2 = g kr + \tan kL \]  

(4-4c)

again is real so that undamped solutions still are possible.

Since \( q \) is to be selected such that the resultant solutions satisfy the equations and the boundary conditions the possibility arises that \( q \) may not be real and the natural frequencies may have imaginary parts. To solve
the full problem with arbitrarily homogeneous boundary conditions, complex \( q \) must in fact be considered so that a complete set of solutions are available for an eigenfunction expansion. This point shall be considered again in the next section.

If for the moment, the mathematical rigor of the following procedure is overlooked, and formally the following substitution is made

\[
\hat{u}(x) = \bar{u} \sinh q(x-L)
\]

\[
\hat{v}(x) = \bar{v} \cosh q(x-L)
\]

\[
\hat{w}(x) = \bar{w} \cosh q(x-L)
\]

where \( \bar{u} \), \( \bar{v} \), and \( \bar{w} \) are constant, one finds that these constants are proportional to \( \frac{\partial \Phi}{\partial t}(x,a) \). If the approximations of Chapter III are inserted into (2-5), three equations for \( \bar{u} \), \( \bar{v} \), and \( \bar{w} \) result with an inhomogeneous term that is some constant times \( \frac{\partial \Phi}{\partial t}(x,a) \).

After solving the resultant set of algebraic equations, a single equation for \( q \) results after some manipulation. Formally the equation for \( \hat{w} \) is

\[
\hat{w} = -\frac{\alpha^2 \beta}{E_t} \left[ 1 - \frac{n^2}{\bar{b}^2 a^2} \right] \frac{\partial \Phi}{\partial t}(x,a,\theta,t)
\]

(4-5)

By applying equations (2-3c) to (4-5), one finds that the equation for \( q \) becomes
\[ \frac{\partial \Phi}{\partial \nu} (x,a) = \frac{-i\omega}{Eh} \alpha^2 \left[ 1 - \frac{n^2}{\xi^2} \right]^2 \frac{\partial \Phi}{\partial t} (x,a) \]

or upon insertion of (4-2a)

\[ q \frac{J_n'(q\alpha)}{J_n(q\alpha)} = \frac{\omega^2 \alpha^2 \beta}{Eh} \left[ 1 - \frac{n^2}{\xi^2} \right]^2 \]

Substituting for \( \omega^2 \) from (4-2b) and changing notation slightly

\[ \frac{J_n'(\xi)}{J_n(\xi)} = \alpha_0 \left[ 1 - \frac{n^2}{\xi^2} \right]^2 \tan h \xi A \]  \hspace{1cm} (4-6)

where

\[ \xi = q \alpha \]  \hspace{1cm} (4-7a)

\[ \alpha_0 = \frac{\alpha^2 \beta \gamma}{Eh} \]  \hspace{1cm} (4-7b)

\[ A = \frac{L}{a} \]  \hspace{1cm} (4-7c)

The transcendental equation (4-6) can be solved by plotting \( \frac{J_n'(\xi)}{J_n(\xi)} \) and \( \alpha_0 \left[ 1 - \frac{n^2}{\xi^2} \right]^2 \tan h \xi A \) on the same scale vs. \( \xi \) and selecting the intersecting points. Sample curves for the cases \( n = 0 \) and \( n = 1 \) are shown on figures (2) and (3).

Since in general (4-4) is also permissible as a proper eigenfunction of the equations of motion, a similar procedure can be followed for \( k \) and the result obtained.
\[
\frac{I_n'(x)}{I_n(x)} = \alpha_0 \left[ 1 + \frac{n^2}{x^2} \right]^2 \tan \frac{xA}{x} \tag{4-8a}
\]

where
\[
x = \kappa a \tag{4-8b}
\]

These curves can also be plotted and the result for \( n = 0 \) is given on figure 4. Since the problem of interest is to obtain corrections on the known solutions, equation (4-8) will not be discussed further.

Figure 3 represents the case of greatest interest, for not only will this mode contain most of the sloshing energy, but the forces exerted on the shell by the sloshing liquid have a resultant component when integrated around the circumference of the tank. Since the parameter \( \alpha_0 \) is usually small compared to unity, the solutions for \( \xi \) are not far from the zeroes of \( J_1' \) and one can expand in a Taylor's series about these points. If \( \xi_i^{(p)} \) is defined as the \( p^{th} \) zero of \( J_1 \), then
\[
\xi_i^{(p)} \simeq \xi_i^{(p)} + \left[ \frac{d}{dz} \left( \frac{J_1'(z)}{J_1(z)} \right) \right]^{-1} \left[ 1 - \frac{1}{\xi_i^{(p)}} \right]^2 \tan \frac{\xi_i^{(p)} A}{x} + \ldots \tag{4-9}
\]

Since
\[
\frac{d}{dz} \left( \frac{J_n'(z)}{J_n(z)} \right) \bigg|_{z=\xi_n^{(p)}} = \left( \frac{n^2}{\xi_n^{(p)}} - 1 \right)
\]

the solution can be written as
\[ \Xi_1^{(p)} \approx \xi_1^{(p)} - \left[ 1 - \frac{1}{\xi_1^{(p)}} \right] \alpha_0 \tanh \xi_1^{(p)} A + \ldots \] (4-10)

However, as may be seen from figure 3, another root arises that did not occur in the rigid wall case.

If the function \( J_1 \) is expanded for small \( \Xi \) as follows

\[
\begin{align*}
J_1(z) & \sim \frac{z}{2} + \ldots \\
J_1'(z) & \sim \frac{1}{2} + \ldots \\
+ \tanh z & \sim z + \ldots
\end{align*}
\]

a solution is

\[ \Xi^2 \sim \frac{A \alpha_0}{1 + 2A \alpha_0} \]

For \( A \alpha_0 \) small enough to make this approximation valid

\[ \Xi_1^{(o)} \sim \sqrt{A \alpha_0} \] (4-11)

The frequency associated with this mode is

\[ \omega_1^{(o)} \sim \frac{g}{\alpha} \sqrt{A \alpha_0} + \tanh \sqrt{\alpha_0 A^3} \]

or by again expanding the \( \tanh \)

\[ \omega_1^{(o)} \sim \sqrt{\frac{g \alpha_0}{\alpha}} A \] (4-12)

This represents a very low frequency oscillation

for normal ranges of \( \alpha_0 \). Written in dimensional terms
it becomes

$$\omega_1^{(p)} \approx \sqrt{\frac{g^2 \rho_a L}{E_h}} \frac{1}{a}$$

This solution can be interpreted as the transition of the trivial root $J_n'(0) = 0$ for $n > 1$, but $J_1'(0) = \frac{1}{2}$ so in the case $n = 1$, it represents a degenerate case of this type. This usually represents an oscillation so slow that it is of no great importance. The other frequencies are

$$\omega_1^{(p)} = \sqrt{\frac{g \varepsilon_1^{(p)}}{\alpha}} + a \pi h \varepsilon_1^{(p)} A \left( 1 - \frac{\alpha_0}{2} \left( 1 - \frac{1}{\varepsilon_1^{(p)}} \right) + a \pi h \varepsilon_1^{(p)} A + \ldots \right) \quad (4-13)$$

for a reduction in resonant frequency of

$$\frac{\alpha_0}{2} \left( 1 - \frac{1}{\varepsilon_1^{(p)}} \right) + a \pi h \varepsilon_1^{(p)} A \times 100 \%$$

It should be noted that while for model size configurations, this effect is quite small, but can grow to significant proportions in a full size vehicle. If one recalls that the inclusion of fluid viscosity, which introduces some damping, also produces a reduction in resonant frequency, and the two effects are additive.

Since most methods of construction of large liquid fueled boosters do not employ uniform cylinders as the prime structural members but use various stringers and stiffeners for added strength, this analysis is valid for the unsupported panels between these members. For example, if there are six longitudinal members much more
resistant to deflections than the skin, then the solution for $J_3$ will be affected by this effect, and deflections of the wall become important for $n \geq 3$. 
CHAPTER V

MATHEMATICAL ASPECTS OF THE SOLUTIONS

As was mentioned in deriving equation (4-6), the mathematics used up to this point is far from rigorous and a whole spectrum of fine mathematical points have been quietly ignored. If one regards the operational method of setting derivatives of the functions u, v, and w equal to powers of some parameter q, this has been shown to be analogous to a Laplace Transform, except that this problem is defined in a finite interval, which is not of the best structure for the Laplace Transform. A properly set problem in this instance would require an integral transform of the form

\[ F(p) = \int_0^L e^{-px} f(x) \, dx \]  

(5-1)

with the appropriate inversion formula. A partial integration of a derivative will imply a transform of the form

\[ \int_0^L e^{-px} f'(x) \, dx = e^{-pL} f(L) - f(0) + pF(p) \]  

(5-2)

which clearly was not done in the preceding analysis.

Essentially the procedure that was followed was to apply the transform for a tank that was extended beyond the limits of the fluid and assume that the fluid forces could be continued so that the functions would
have no discontinuities at the end points. This method of course does not yield a proper solution to the problem, for arbitrary boundary conditions on the shell deflections are not satisfied. Formally a solution can be found that will satisfy arbitrary boundary conditions explicitly, but this is much less convenient for imaginary values of $q$ will definitely have to be considered. The way this apparent lack of rigor can be circumvented is to investigate the end conditions that the solution derived actually satisfies. If these do not correspond to a physically realizable or practical situation, the solution as it stands will be of little use. The end conditions that are satisfied are

\[
W(0) = K_n \cos n\theta e^{i\omega t} \quad (5-2a)
\]

\[
W(L) = \frac{K_n \cos n\theta}{\cosh qL} e^{i\omega t} \quad (5-2b)
\]

for $K_n$ an arbitrary constant related to $A_n$ and similar conditions for $u$ and $v$. In particular, $u(L) = 0$, permitting the assumption of a fixed end at $x = L$. The deflections at the ends of the cylinder walls for $n = 1$ are similar to those produced if the cylinder was restrained at the base and the base translated in the $\theta = 0$ direction with amplitude $K_n / \cosh qL$. These conditions are remarkably similar to those that would exist in an experiment in which the tank base was fixed and translated laterally in a
direction parallel to the \( \theta = 0 \) axis sinusoidally in time. Thus the solution found does seem to represent some type of physical reality in the case \( n = 1 \) and is a proper solution for these end conditions. However the results of equation (4-6) are valid only for these end conditions and can be expected only to show trends in the shifting of the natural frequencies for other end restraints.

A further word should be mentioned about the completeness of the problem. Solutions (4-2) and the eigen condition (4-6) can be used to satisfy an initial value problem in the fluid while (4-4) and (4-8) need be used for the general initial value problem in the shell. Both are required for an entirely general initial value problem in the fluid and shell, and can be accomplished by a standard eigenfunction expansion, but the details are rather complicated for \( J_n \) satisfies an orthogonality condition while \( I_n \) does not, the same holding true for cosines but not for hyperbolic functions. However it can be shown that the expansion in general can be carried out, in a non-trivial manner, to solve the initial value problem in general.
CHAPTER VI

EXPERIMENTAL PROCEDURE

An attempt was made to measure the effect predicted in the preceding sections. A cylindrical tank was constructed out of three sheets of 26x6x.002 inch high strength stainless steel shim stock sheets. The method of fabrication was to employ an epoxy adhesive to glue three seams down the sides of the cylinder, allowing sufficient overlap for adequate strength. The sheets were formed on a 5.0 inch diameter tube and the finished tank is nominally 5.0 inches in diameter. A base plate was turned out of steel and was adhered to the inside of the cylinder. The base had taps in the bottom for the admission of water, the test fluid, and air to pressurize the cylinder. The top was formed from 1/8 inch plexiglas and glued to a 5.0 inch outside diameter ring. These two rings at top and bottom were sufficient to give the model its cylindrical shape. A small pressure tap was located in the top for the measurement of the internal pressure.

Nine strain gages were attached to the cylinder walls while it was still on the forming tube. These were each located in one arm of a four arm bridge and
the gage bridge combination was excited by a 5 volt 3000 cps signal from a carrier amplifier, which amplified and then rectified the output. The output was measured on a recording oscillograph. (see diagram 1)

The most satisfactory method of exciting the primary sloshing mode in the tank was found to be a trapeze type arrangement constructed as follows. The tank base was suspended by two bars from a horizontal rod. (see photograph) A magnetic shaker applied an excitation force to the base. The deflection of the shaker was so small that the base could be said to be undergoing purely horizontal oscillations instead of moving in an arc. The output of a low frequency oscillator was amplified and the magnet driven by the power amplifier.

An accelerometer was attached directly to the base of the tank to measure the driving force on the base and this was recorded along with the gage output. This method of measurement compensated for any resonances in the supporting structure or in the trapeze itself, by comparing the output of the gage with that of the accelerometer. Phase shifts between the input of the accelerometer and the strain gage were observed visually at the lowest resonance.

Since the cylinder was designed to hold at least
ten pounds per square inch differential pressure, the entire structure was sealed off, the thin walls of the shell preventing less permanent means of closure. At first, forcing was attempted on an instrument vibration test table in the basement Aerelastic and Structural Dynamics Research Laboratory of the Massachusetts Institute of Technology. This method of forcing was found to be unsatisfactory because of the poor wave form that resulted due to the backlash in the gears and torsion in the shafts that drove the oscillating platform. Many undesirable harmonics resulted and the accelerations were so harsh that several small cracks were observed in the seams in the tanks' walls. Attempts were made to patch these cracks from the outside with epoxy glue, but the leaks continued. The patches were much stiffer than the shell wall but were located far from any strain gages so the effect was estimated to be small compared to the stiffness of the structure as a whole. These failures were attributed to the questionable properties of the epoxy adhesive used under a condition of alternating stress far below its static strength (fatigue).

Another difficulty was noted in the loss of sensitivity of the strain gages. At the start of the experiment five of the nine gages gave an output large enough to be measured on the oscillograph, but at the commencement
of data gathering, only one gave a readable output. This of course limited the conclusions that could be drawn about the distribution of strains in the tank.

Strictly speaking the measurements that were made were of the strain in the circumferential direction \( \omega^\circ \), not \( w \). These are proportional but there is a phase shift in time. These details are taken care of in the presentation of results.

The table below lists the equipment used in the experimental investigation.


Power amplifier - Bogen 100 watts, Model HO 125

Magnetic shaker - 50 pound magnet, property of the Aeroelastic and Structural Research Lab.

Strain gages - Baldwin Type A-7

Accelerometer - Statham Lab. \( \pm 20 \) g, Serial No. 193

Carrier Amplifier - Consolidated Engineering Type 1-118

Recording oscillograph - Heiland #2, Model 712C

Strain gage data was measured on a 40 \( \mu \)amp galvanometer. Accelerometer data was measured on a 100 \( \mu \)amp galvanometer.
CHAPTER VII
EXPERIMENTAL RESULTS

For the tank model used in the experimental investigation, the following properties were observed:

\[ a = 2.5 \ (\pm 0.1) \text{ inches} \]
\[ L = 25.0 \ (\pm 0.1) \text{ inches} \]
\[ h = 0.002 \text{ inches} \]
\[ B = 29,000,000 \text{ psi} \]
\[ g = 32.17 \text{ ft/sec/sec} \]
\[ \rho_f = 0.0012 \text{ slug/cu. inch} \]

Since

\[ \alpha_0 = \frac{a^2 \rho_f g}{Eh} = 1.86 \times 10^{-4} \]

the effect of the non-rigid walls on the natural frequencies should be negligible, and the only perturbing effect from the known solution should result from the fluid viscosity.

The first resonant frequency predicted by rigid wall theory that has the lowest frequency occurs at what has been called the primary lateral mode

\[ f_i^{(p)} = \frac{\omega_i^{(p)}}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{\alpha_i^{(p)} g}{a} + \tanh \alpha_i \frac{L}{a}} \]

\[ f_i = 2.66 \text{ cps} \]

for this configuration. This result is in excellent agreement with that observed experimentally.
Figure 5 shows a plot of the ratio of double amplitude of the strain gage to that of the accelerometer as determined by the data from the oscillograph, samples of which are shown on figures 6 and 7. Both accelerometer and strain gage readings were corrected by subtracting the amplitude of the zero trace before the ratio was taken. This tended to correct somewhat for random effects in the data, but some scatter is still noticeable. The most troublesome noises are caused by the first resonances in the shell wall, if the inertia of the shell is considered, and has a resonant frequency at around 40 cps or approximately fifteen times greater than the first sloshing frequency. This is in agreement with approximation (3-2). Prediction of the frequencies of these oscillation is tractable only with machine analysis or by some approximate theory, such as Reissner's "shallow shell" theory, which is valid only for higher m. However an order of magnitude estimate reveals that this resonance can be expected in that frequency range.

As can be observed, the inputs are not very accurate sinusoids. The fault here is in the power amplifier, it being an audio amplifier and introducing appreciable distortion below 5-10 cps. It is for this reason that the results on phase shift show
such a large degree of scatter. Plotted against an arbitrary reference, there seemed to be a $40^\circ$ phase lead at 2.0 cps and a $145^\circ$ lag at 3.0 cps. The determination of relative phase was much poorer than that of the amplitude, and the curves show considerable scatter from that drawn. Most of the data seemed to fall in the $-40/50^\circ$ range for frequencies between 2.5 and 2.7 cps. However, the scale on the traces was such that a 0.05 difference in the starting and end points of the part of the trace which was chosen as reference, could cause an error of $10^\circ$ to $15^\circ$ in that frequency range, so that portion of figure 5 should be taken as only approximate until more refined measurements are possible.

In spite of all of the scatter, a rather sharp resonance was seen to occur in the region of 2.65 cps which is what the theory predicts.

The measurements of higher harmonics did not yield any satisfactory information, and the strain gage was losing sensitivity with every test. On the whole, the few tests that were made at frequencies in the range of the other sloshing modes showed no consistent trend.

No attempt was made to measure the effect predicted in equation (4-12), for the period of this oscillation
is about 20 seconds for the configuration that was
used in the experiment.

It is deeply regretted that further measurements
were not able to be made. The tank was leaking quite
freely by the time it was decided to stop taking data.
The leaks were due to the 40 cps resonance that weakened
the seams, for one of the three seams was located in a
region of highly concentrated stress.
CHAPTER VIII

CONCLUSIONS

Theoretical considerations predict that a fluid oscillating in a thin cylindrical shell will have resonant frequencies lower than those corresponding frequencies in a tank with rigid walls. The magnitude of this effect is proportional to the parameter $\alpha$. The experimental investigation did not verify this theory.

The results of the experiment were significant however in that they agreed with the theoretical prediction that the effect should be immeasurable in this model. Thus it can be stated that the mechanism postulated is not ruled out by the experimental results and that more investigation must take place before the theory can be said to be verified.

The model used in the investigation had walls that were definitely non-rigid if one considers the plate stiffness of the walls as the criterion. The walls of the cylinder were observed to bulge out as the tank was filled with water, and could easily be pushed in one half inch or more with light pressure.
from the finger. However, since the mechanism postulated is proportional not to the plate stiffness but to the parameter $\alpha_o$, the model used can be said to be essentially rigid.

It is estimated that for extremely thin shells with small modulus and relatively large radius, this effect could become important. Future investigations along this line could be made on a series of shells with rubber walls, in which case the parameter $\alpha_o$ could be made large enough to cause an appreciable shift in the resonant frequency. It is hoped that this work will be done in the near future
FIGURE 1: Coordinate System
\[ \frac{J_1'(\xi)}{J_1(\xi)} \] and
\[ \alpha_0 \tanh \xi A \frac{1}{(1 - \frac{1}{\xi})^2} \]
\[ \frac{I'(x)}{I(x)} \text{ and } \alpha \tan \delta A \]

\[ n = 0 \]

\text{FIGURE 4}
FIGURE 5: Experimental Results
40 cps. tank natural frequency from background

Strain gage

120 cps noise - rectified A.C. pickup

Accelerometer

1 sec
paper speed 3.1 ips

/Strain gage

45°

Accelerometer

2.01 cps

1 sec
3.28 ips

Data Sample: 1

FIGURE 6
Data Sample: 2

FIGURE 7
Diagram 1: Schematic Layout of Experimental Apparatus
FUEL TANK MODEL
REFERENCES


