MEMBRANE-FLEXURAL FAILURE MODES OF SQUARE
HORIZONTALLY RESTRAINED R/C SLABS

by

AMNON JACOBSON

B.Sc., Technion, Israel Institute of Technology
(1954)

M.Sc., Technion, Israel Institute of Technology
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Signature of Author ________________________________
Department of Civil Engineering

Certified by ________________________________
Thesis Supervisor

Accepted by ________________________________
Chairman, Departmental Committee on Graduate Students
TO

RIVKA, ANNATE AND MICHAL
ABSTRACT

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AMNON JACOBSON

Submitted to the Department of Civil Engineering on 1965, in partial fulfillment of the requirement for the degree of Doctor of Science.

A theoretical insight into the compressive membrane-flexural failure modes is developed. The analysis is based on data obtained from experimental investigation. The domain of behavior is defined by yield criteria, and the behavior is followed by a "stress path". The idealized linear-elastic stage is solved including a non-linear geometry effect for thin slab, and shear stress for thick slab. For the complete perfectly plastic stage a rigid plastic solution, together with modified rigid plastic solution that account for elastic inplane strains are considered. An elasto-plastic approximate solution is used for the "spreading mechanism" stage. The idealized case is correlated to restraint conditions in actual practice.

This method of solution leads to prediction of the ultimate loads and deflections for the various slabs considered. In particular, distinction is made between under-reinforced thin slab associated with geometrical instability and thick slab, associated with concrete strength. For the thick slab, the 3 dimensional state of stress leads to "stress hardening" effect that requires new definition for over-reinforcement. The effect of reinforcement ratio on the failure mode is analyzed. Elastic release is introduced on the boundary and insight into the behavior of internal panel in two way continuous slab is developed.

Thesis Supervisor:.................................Myle J. Holley

Title:.........................................Professor of Civil Engineering
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LIST OF SYMBOLS

$x, y, z$  Rectangular coordinates.

$L = a$  Span of the square slab.

$y_0$  Distance from the neutral axis to the center of the reinforcement.

$y_i$  Distance from neutral axis to the bottom of the slab.

$d_i = h$  Total depth of the slab.

$d$  Depth of the center of reinforcement from the compression face of the concrete.

$\bar{h}$  Total effective depth including elastic equivalent for the reinforcement.

$k$  Ratio of distance between compression face to neutral axis and $d$.

$t$  $k$ for unrestrained slab, if two dimensional yield criteria is considered. (rect. block)

$t_1, t_L$  $k$ for restrained and unrestrained slab respectively, if 3 dimensional yield criteria is considered. (rect. block)

$t_{bal}$  $k$ for balanced cross section. (rect. block)

$A_s$  Area of reinforcement.

$r$  Percentage of reinforcement.

$p$  Uniformly distributed transverse load.

$\Delta p$  The reduction in transverse load due to appearance of yield region.

$P_{ult}$  Ultimate load of restrained slab at small deflection.
\( p_m \)  Maximum capacity of slab for rigid-plastic solution.

\( p_u \)  Ultimate load for unrestrained slab.

\( u_x, u_y \)  Horizontal displacements at the reference axis in x and y directions respectively.

\( \omega \)  Vertical deflection.

\( \omega_c \)  Central deflection.

\( \omega_{oc} \)  The particular part of the solution for the central deflection.

\( (\omega_{oc})_{ult} \)  Deflection of ultimate load for restrained slab.

\( \varepsilon_c \)  Total strain at the extreme fiber of the concrete.

\( \varepsilon_{cy} \)  Idealized concrete yield strain for linear-elastic perfectly plastic stress strain relations.

\( \varepsilon_u \)  Ultimate strain of the concrete.

\( \varepsilon_s \)  Total strain of the reinforcement.

\( \varepsilon_y \)  Yield strain of the reinforcement.

\( \gamma \)  Total curvature strain.

\( (\gamma)_c \)  Elastic quantity.

\( (\gamma)_p \)  Plastic quantity.

\( M_x, M_y \)  Bending moments per unit length of sections of slabs perpendicular to x and y axes, respectively.

\( M_{xy} \)  Twisting moment.

\( M_o \)  The ultimate moment for unrestrained slab. (Johanson moment).
\( V_x, V_y \)  Shearing forces parallel to z axis per unit length of sections of slab perpendicular to x and y directions, respectively.

\( P_x, P_y \)  Normal forces per unit length of sections of a slab perpendicular to x and y directions, respectively.

\( P_{xy} \)  Membrane shearing force.

\( T_0 \)  Yield load for the reinforcement.

\( ( \ )_{i-1} \)  Quantity with respect to the reinforcement level.

\( a \)  Generalized stress coordinate.

\( q \)  Generalized strain coordinate.

\( \nu \)  Poission ratio.

\( E_c, E_s \)  Moduli of elasticity of concrete and reinforcement respectively.

\( I \)  Elastic moment of inertia.

\( D = EI \)  Slab stiffness.

\( f_c, f_s \)  Stresses in the concrete and the reinforcement, respectively.

\( f_c' = 0.85 f_c' \)  Strength of concrete.

\( f_y \)  Yield stress of reinforcement.

\( \overline{V} \)  Shear stress in the concrete.

\( L(, \xi, \tau, g) \)  Functions as defined in the text.

\( F, \psi \)  Stress functions as defined in the text.

\( i, j, n, m, p, q, s \)  Integers as defined in the text.
\[ A_n, B_n, C_n, D_{pq}, \]
\[ E_{mn}, F_{mn}, G_{mn}, K_n \]
\[ H_n, \gamma_{pq}, \phi_n, \lambda \]

Constants as defined for each case in the text.

\[ \alpha \]
Parameter for rigid plastic solution of thin slab.

\[ \beta, \gamma \]
Parameters for rigid plastic solution of thick slab.

\[ \mu \]
Modification factor for rigid plastic solution of thin slab.

\[ \mathcal{S} \]
Modification factor of rigid plastic solution of thick slab.

\[ \eta = \frac{f_{32}}{f_c''} \]
Modification factor for 3 dimensional yield criteria.

\[ (\quad)_L \]
Quantities for unrestrained slab.

\[ (\quad)_m \]
Quantities for maximum capacity for rigid-plastic solution.

\[ m_1, m_2 \]
Shape factors to account for modification of stress distribution of axial stress for membrane force and moment calculation respectively.

\[ X_0 \]
Boundary coordinate of spreading rigid plastic region into an elastic region.

\[ U_e \]
Elastic energy.

\[ U_p \]
Plastic work.

\[ B \]
Bending rigidity of horizontal elastic support.
\[ \xi = \frac{h}{k, n} \]

Correction factor for elastic supports.

Geometrical parameters for the dimensions of the elastic support.

Elastic stiffness of the restraining cells used in the experiments.
1

CHAPTER 1

INTRODUCTION

1.1 GENERAL

This thesis is related to a series of tests on square, simply supported, uniformly loaded, horizontally restrained R/C slabs (Fig. 1.1.1). The main objective of these tests was to advance the understanding of the effect of restraint against horizontal movement on the behavior of R/C slabs. New tools were invented and used here only as far as the above mentioned aim was concerned. It is recognized, however, that there is much more to be done in the basic application of plastic theory to R/C slabs. e.g., more generalized yield criteria to account for mixed stress resultants that can be carried by the concrete are not yet known.

1.2 THE MODES OF FAILURE OF HORIZONTALLY RESTRAINED R/C SLABS

The analysis presented is based upon actual experimental observations of the deflected shapes and crack patterns and measurements of the horizontal restraining force at the boundary and central deflection (Fig. 1.2.1-4). By following the "stress path" of a slab, the true deflection field at each mode is estimated. The properties of the material at each stage are used together with the equilibrium equations to explain the behavior.

The following stages are suggested:
**Fig. 1.1.1**: Simply supported uniformly loaded horizontally restrained R/C slab.

**Fig. 1.2.1**: Typical experimental load-deflection curve for restrained R/C slab.
(1) "Uncracked".
   a) Elastic. \( \varepsilon_c \leq \varepsilon_u ; \varepsilon_s < 0 \)
   b) Some cracking at the center of the slab, where
      \[ \varepsilon_c \leq \varepsilon_u ; 0 \leq \varepsilon_s < \varepsilon_y \]

   In this stage either no cracks appear at the bottom of the slab, or only a few hair cracks may be present at the center of the slab.

(2) "Spreading mechanism". (Fig. 1.2.2).

   This stage consists of a plastic central region surrounded by elastic region. This stage is limited by:
   a) Thin slab: "geometrical instability", where extremal value of axial compression is obtained due to non linear geometric effects.
   b) Thick slab: complete yielding (zero resistance) associated with extremal value of moment capacity due to concrete strength.

(3) "Mechanism". \( \varepsilon_c > \varepsilon_u ; \varepsilon_s > \varepsilon_y \)

   In this stage the carrying capacity of the slab is reduced as the deflection increases. The limit of this stage is given by \( \frac{M}{M_o} = 1 \).

(4) "Tensile membrane."

   Initially all the points on the yield surface (Fig. 2.3.2) move to satisfy the coordinates \( \frac{M}{M_o} = 1 ; \frac{R}{R_o} = 0 \).

   This stage is limited by:
   a) Tensile failure of the reinforcement.
   b) Compression failure at the boundary, thus no more support is provided to the tensile membrane.
FIG. 1.2.2: "SPREADING MECHANISM"

FIG. 1.2.3: OBSERVED CRACK PATTERNS AT THE BOTTOM OF THE TESTED SLABS.

FIG. 1.2.4: TYPICAL EXPERIMENTAL LOAD—RESTRAINING FORCE RELATIONS.
The measured axial forces at the boundary confirm the various modes described here as indicated on Fig. 1.2.4.

All possible simplifications conforming with the actual mode of behavior were made. The errors introduced are limited to the corresponding mode, and do not propagate to the next one.

The tensile membrane stage is not considered in this research, since it is believed that this mode of behavior is fairly well understood in comparison to the other modes. Thus, stage 4 is not considered in the following analysis.

1.3 PREVIOUS INVESTIGATIONS

Recently, it was found experimentally by Ockelston, Powell and Thomas, that the ultimate load of a horizontally restrained R/C slab is much higher than predicted by conventional yield line theory. It was indicated that the reason is compression membrane action at small deflections.

More specifically, a rigid plastic solution for a beam was suggested by Powell. The solution was based upon the kinematic relations and equilibrium conditions. However, such a solution is sufficient only when the deflections are large enough, since then the compressive action is small. The actual ultimate load occurs at small deflections, where elasto-plastic relations have a major effect on the compression action.

A rigid plastic solution for thin circular slabs was suggested by Wood. The solution is based upon equipotential
yield criteria and kinematic relations of the deflection field. Again, the solution has the same character as Powell's solution. However, it was recognized by Wood that the solution should be modified to take into account deformations due to compression membrane action. No explanation was given for the instability phenomena associated with the transition from elasto-plastic to rigid-plastic behavior. Thus the ultimate load can not be predicted.

Park expanded the rigid-plastic approach to solve for different cases of rectangular slabs with various boundary conditions. A rigid-plastic strip approximation was used, and the analysis results were checked against tests. However, an empirical value was assigned for the deflection at ultimate load, since the solution is based upon the same principles as the former solutions. Thus, the effect of structural parameters such as span/depth and reinforcement ratios can not be investigated.

Tensile membrane solutions for circular slabs were suggested by Wood. The solutions are based on "spreading" of the tensile membrane region into the rigid-plastic cone deflection field. Park expanded these solutions for rectangular slabs.

1.4 OBJECTIVE

The objective of this research is to develop insight into the elastic and elasto-plastic behavior of horizontally restrained R/C slabs which will lead to prediction of the ultimate load, the effect of span/depth and reinforcement
ratios and to more realistic plastic analysis for the re-
strained slab. In particular, the following points are of
interest.

(1) The effect of span/depth and reinforcement ratios on
the load deflection characteristics, taking into account
elastic, elasto-plastic and plastic modes of behavior.

(2) An understanding of the compressive membrane action
as a function of the deflected shape, based on no-tension
theory. Effects of span/depth and reinforcement ratios, dif-
ferent degrees of horizontal restraint against lateral move-
ment, and different rotational restraints.

(3) Development of procedures for prediction of the
ultimate load and deflection by analytical methods, for
various span/depth and reinforcement ratios.

It is anticipated that the tools obtained will be useful
in the development of design procedures and more advanced re-
search.

1.5 SCOPE

In order to achieve the objective, the fundamental
problem of simply supported horizontally restrained slabs is
solved. Any practical boundary condition can be calculated
on the basis of this solution.

The following topics are considered in the analysis:

(1) Formulation of the general problem and yield criteria
so that the solution obtained can be applied to various span/
derth depth ratios, reinforcement ratios, and boundary conditions.
a) Development of yield criteria considering not only tensile failure (Wood) but also compression failure, and how they are affected by different reinforcement ratios. For thick slabs a three dimensional yield criteria is considered.

b) Introduction of the "stress path" as a tool for the solution of the elasto-plastic problem if the behavior is assumed to be elastic-perfectly plastic.

(2) Using large deflection relationships, the compressive membrane action is calculated. By also considering stress path, a clear distinction between the behavior of thin and thick slabs is established. Within each group, the effect of reinforcement ratio is checked.

a) For the thin under-reinforced slabs, the ultimate load corresponds to "geometrical instability" associated with extremal value of the compression reaction at the boundary, and therefore with the loss of the stiffness.

b) For the thick slabs, the ultimate load is a function of the concrete strength, the reinforcement ratio and the associated deflection field. Revision of the criteria for balanced section for thick slab is suggested.

(3) Using the yield criteria, the stress path and the elastic solution for the compressive membrane action, a solution for "spreading mechanism" is developed, where the elastic solution is modified by additional plastic solution.
for spreading plastic zone at the center of the slab.

(4) Rigid-plastic solution based on yield line theory, is developed in order to find an upper bound solution for the problem, including:

a) Rigid-plastic deflection field and equilibrium conditions that are compatible with 2 dimensional yield criteria for thin slabs, and 3 dimensional for thick slabs.

b) Modification of the rigid-plastic solution to include the effect of axial strain.

c) Application of the rigid plastic solution for the complete mechanism stage, where actual yield line develops. The solution is applied as a continuation of the elasto-plastic stage, thus, the initial axial and curvature strains are included.

(5) The ideal case of a simply supported restrained slab is correlated to restraint conditions in actual practice. The aim of this correlation is to develop an insight into the behavior of continuous slabs, as well as to specify the flexibility of the restraint used in the experimental phase of the research.

The analysis is based on experimental investigations, consisting of static uniformly distributed loads applied to square R/C panels. The tests were carried out on 3 span/depth ratios: 20; 10; 5. For each separate ratio the percentage of reinforcement was varied to include: 0; 0.5; 1; 2; 3 percent.
of reinforcement. The horizontal restraint was applied by loading cells. The restraining force, and the central deflection were both measured as a function of the transverse load. A brief description of the test procedures and test results, as far as they related to the present analysis is given in appendix II.

The analysis was checked against the experimental data for each stage. In order not to repeat the same ideas, however, numerical results are compared for only two percentages of reinforcement at each thickness: \( r = 1\% \) and \( r = 3\% \). All the rest of the data covering the whole range of behavior stands in good agreement to the analysis.

1.6 THE NATURE OF THE PROBLEM

The tensile strength of a brittle material such as concrete is at present only a small fraction \( \left( \frac{1}{8} \div \frac{1}{10} \right) \) of its compressive strength. In order to increase the moment capacity of a concrete member, reinforcement of material with a higher tensile strength is added.

Restraint against horizontal expansion of slabs provides higher moment capacity by a self induced pre-stressing of the R/C cross section. The nature of the behavior of the restrained slab together with the effect of the various parameters is discussed in this section by using a simple physical model of the equilibrium conditions of a slab element.

Consider a quarter of the square slab, bounded by the diagonals (see also Fig. 3.1.1). After deformation, the re-
FIG. 1.6.1: EQUILIBRIUM OF SLAB ELEMENT "A".

NOTE: "TENSILE (PLASTIC) FAILURE": \[ M_R = M_{pl}(\varepsilon_y, \varepsilon_u) \]

"COMPRESSION (ULTIMATE) FAILURE": \[ M_R = M_{ult}(\varepsilon_s, \varepsilon_y, \varepsilon_u) \]
sultants of the acting forces along the diagonals \((P, m_r)\) are acting with an arm \((\omega)\) from the undeformed position. The shear forces as well as the mixed membrane forces and moments, are zero along the diagonals of the square slab under a uniformly distributed load. The acting resultants are schematically shown in Fig. (1.6.1).

Due to the displacement boundary conditions, a compressive force is induced by the requirement that the horizontal displacement is zero at the reinforcement level along the boundary of the slab. This compression increases proportional to the rotation of the slab along the boundary, and decreases due to elastic shortening of the slab under axial compression and the geometrical "shortening" of the span due to the deflection ("non-linear geometric" effect). For a discussion of the horizontal displacements and their effect on the axial force see Appendix I.

The slab element is under equilibrium if the acting moment \((M_a)\) is equal to the resisting moment \((M_r)\). The acting moment is produced by the forces acting on the surface and the boundaries, i.e. by the total horizontal restraining force \(P\) acting with some average deflection arm \(w\), and the transverse load \(\lambda\) and its reaction. The resisting moment is a property of the material. For R/C materials, the ultimate resisting moment increases as a function of the axial compression, under tensile failures.

The ultimate capacity of the slab can be attained under the following conditions:
The change of the acting moment with an additional increment of the deflection is larger than the change of the resisting moment (instability $\delta m_a > \delta m_R$) under tensile failure.

The acting moment is larger than the resisting ultimate moment (ultimate strength $M_a > M_{ult}$).

In the simplified model considered, the instability condition is satisfied when $\frac{dP}{d\omega} = 0$. This can occur due to:

a) A deflection $w$, for which the change of the horizontal movement at the restraining point is zero, i.e. the expansion due to the rotation is equal to the shortening due to the axial compression and the "non-linear geometric" effect (geometrical instability). This condition occurs for under-reinforced thin slabs (span-depth ratio of 20).

b) Crushing of the concrete due to stresses that are higher than the compressive strength of the concrete. The result is that additional deflection does not cause additional increment of the axial compression. This condition occurs for medium and thick slabs (span-depth ratio which is less than 10), with low reinforcement ratio, since these slabs are associated with large increases of the capacity due to the restraint.
(2) Where the ultimate strength governs, the resisting moment of the R/C cross section is limited by the so-called "compression failure", where concrete is crushed prior to the complete yielding of the reinforcement. Here, no instability results due to initial crushing:

a) A complete "plastic hinge" has not been developed yet.

b) Significant "stress hardening" effect that results increase of $M_R$.

At ultimate load, the acting moment is equal to the resisting moment. If the compression failure is imposed because of the restraint, then with some deflection the restraining force is reduced. Finally, a full plastic hinge is developed and $M_R$ reduces rapidly.

This condition occurs for medium and thick under-reinforced slabs ($\frac{b}{d} < 10$) with sufficient reinforcement. Thus, the ultimate deflection of thick slabs ($\frac{b}{d} = 5$) with high reinforced ratios ($r = 3\%$) is considerably larger than the thick slabs with low reinforcement ratios. The reason is that the ultimate load under compression failure of the former is associated with crushing of the concrete together with a significant change of geometry, while the ultimate load under tensile failure of the latter is associated with crushing alone, resulting in instability.

The resisting moment is proportional to the concrete strength. For thick slabs ($\frac{b}{d} = 5$) the applied load is of the same order of magnitude as the inplane stresses. The
result is that the strength of the concrete increases due to the reduction in the "deviator stress". For thick restrained slabs, the effect is much higher ($\sim 1.6 \, M_R$) than for under-reinforced unrestrained slabs ($\sim 4.2 \, M_R$). The reason is that much more of the concrete area is active when the neutral axis is lowered by the compression force.

Due to this "stress hardening" effect, thick slabs exhibit greater capacity than is predicted for over-reinforced slabs, and a ductile type of failure rather than a brittle one. Higher reinforcement ratios can be used, since "over-reinforcement" is a function of the applied load.

For thick slabs ($\frac{L}{d} = 5$) the effect of the shear deformation on the bending is to reduce the effect of the restraint prior to ultimate load. This is caused mainly due to the fact that the deformation is not linearly proportional to the distance from the neutral axis.

Restraint is similar to reinforcement in that it increases the strength of R/C slabs. Restraint differs from reinforcement in that it is also a sensitive function of the deflected shape and magnitude. Thus, addition of a restraint at the bottom of the slab is not merely over reinforcement of the slab, since with increased deflections the restraining effect is reduced. For that reason the location of the restraint is of major effect. At the bottom of the slab the effect is not only to increase the "prestressing" effect, but also to preserve the effect with increasing deflections. If the restraint will be located near the neutral axis, the restraint effect
will be relatively small (see Appendix I). In continuous R/C slabs with sufficient top and bottom reinforcement above the supports, such that the bottom reinforcement does not yield once plastic rotation starts above the support, the natural location of the restraining action is at the bottom reinforcement. This fact was confirmed by a series of tests on large panels, continuous above the supports with bottom reinforcement only. (slabs #42 - #49). These slabs gave the same results as the corresponding simply supported slabs with the restraint located at the reinforcement level (see Appendix II). It is suggested that analysis for continuous slabs will be associated with this location, while secondary effects such as the elongation of the neutral axis due to cracking of the concrete prior to the yielding will be neglected (see Appendix I).

In practice, elastic members exist at the boundary of a single panel. This condition occurs in continuous slabs as well as in the experimental phase of this research. The relative stiffness for horizontal movement of a slab and its boundary members is a major factor in the resistance of the slab. For the internal panel of a two way slab, the restraining effect is lost if the panel is surrounded by a slab strip which is capable of acting as horizontal beam, however with a width less than half of the span of the panel. The restraining effect is increased with the width of the strip, and reaches a maximum value when the width is approximately two times the span of the panel.

The same restraining cells were used in the experimental
setup (see Appendix II) for all the slabs tested. Thus, for thin slabs \( \frac{L}{d_i} = 20 \) with 1\% reinforcement the restraining cells provided 82\% of the axial membrane compression for infinity stiff restraint, while for thick slabs \( \frac{L}{d_i} = 5 \) with 3\% reinforcement, the restraining cells produced 49\% of this maximum value.
CHAPTER 2

THE YIELD CRITERIA AND THE STRESS PATH

2.1 THE PLASTIC POTENTIAL FUNCTION

In generalized variable terms, for elastic perfectly-plastic material. (Fig. 2.1.1)

\[ q_i = E_{ij} Q_j + (q_i)_{pl}. \quad i=1, \ldots, n \]  

(2.1.1)

A functional representation of the yield stress is given by

\[ f(Q_1, \ldots, Q_n) = 0 \]  

(2.1.2)

If \( f < 0 \), elastic conditions are maintained.

The plastic strain \( (q_i)_{pl} \), which is orthogonal to the yield surface, is given by the gradient

\[ (q_i)_{pl} = \lambda \frac{\partial f}{\partial Q_i} \]  

(2.1.3)

\( \lambda \) is indeterminate constant.

\[ \begin{align*}
\text{IF} & \quad \lambda > 0, & f &= 0 \\
\lambda &= 0 & f &< 0
\end{align*} \]

2.2 YIELD CRITERIA FOR CONCRETE (BRITTLE MATERIAL)

The two dimensional yield criteria for brittle material
**FIG. 2.1.1:** IDEALIZED ELASTIC-PERFECTLY PLASTIC STRESS-STRAIN RELATIONS.

**FIG. 2.2.1:** TWO DIMENSIONAL YIELD CRITERIA FOR BRITTLE MATERIAL.

**FIG. 2.2.2:** THREE DIMENSIONAL YIELD CRITERIA FOR BRITTLE MATERIAL.
was given by Griffith, when maximum stress was considered at the tip of a crack in a brittle material. If the crack is randomly oriented in the material, the strength criteria is given for the plane stress case in Fig. (2.2.1) for the principal directions $i = 1, 2$.

The mathematical representation of the yield criteria is given by:

$$\max \left[ \frac{f_{c_i}}{f_c} ; \frac{f_{c_j}}{f_c} \right] \leq 1$$  \hspace{1cm} (2.2.1)$$

Since no-tension theory for R/C is used, the tensile strength of the concrete is neglected.

By an application of the same procedure to three dimensional state of stress, the yield criteria is as shown in Fig. (2.2.2); for the diagonal plane ($f_{c_1} = f_{c_2} ; f_{c_3}$). If the cone shape surface is simplified into parallel faces (see also Fig. (6.1.1)), one gets:

$$\max \left[ \frac{f_{c_{\text{max}}} - f_{c_{\text{min}}}}{f_c} \right] \leq 1$$ \hspace{1cm} (2.2.2)$$

The uniaxial stress-strain relations of the concrete are idealized for the linear elastic-perfectly plastic case (Fig. 2.2.3a).

For $\varepsilon_c < \varepsilon_u$, a linear elastic perfectly plastic idealization is used, when the strength is given by $f_c$. 
Fig. 2.23: Uniaxial stress-strain relations for the materials.
For \( \varepsilon_c \gg \varepsilon_u \), idealization into rectangular stress block with strength \( f_c'' = 0.85 f_c \) is used to get compatible neutral axis for stress and strain computations of a rectangular block.

Linear elastic-perfectly plastic strain relations are assumed for the reinforcement steel. Since behavior of slabs at large deflections (membrane action) is not considered in the analysis, the maximum strain \( \varepsilon_{sm} \) is less than the strain necessary for initiation of strain hardening in the steel.

For R/C (a composite material) with concrete and steel strains that are larger than the yield strains, \( (\varepsilon_c \gg \varepsilon_u, \varepsilon_s > \varepsilon_y) \), the stress distribution in a cross section is given in Fig. (2.2.4).

The shear and vertical stress distributions are based on elastic-cracked cross section. For a thick and heavily reinforced R/C slab, this distribution is highly uncertain.

If the order of magnitude of stress component is considered, it is known that:

For thin slabs \( \left( \frac{b}{d_i} = 20 \right) \) \( f_{c_{1,2}} \gg \gamma_m \gg f_{c_3} \).

For thick slabs \( \left( \frac{b}{d_i} = 5 \right) \) \( f_{c_{1,2}} = 0 (f_{c_3}) \).

Thus, for thin slabs the effect of shear and vertical stress is neglected, while these effects are considered for the thick slabs.

In the following analysis, it is assumed that conditions
\[ V_0 = \frac{V}{d(1-\frac{1}{2}k)} \]

**b. IDEALIZED SHEAR STRESS**

\[ f_{c_3} = \frac{1}{2} \left[ 1 - \frac{(d-\frac{1}{2}k)^2}{2kd^2(1-\frac{1}{2}k)} \right] \]

\[ \frac{f_{c_3}}{f_c} = \frac{1}{2} \left[ 1 - \frac{1}{1-\frac{1}{2}k} \left( \frac{k^2}{2} - \frac{d(1-k)-\frac{1}{2}}{d} \right) \right] \]

**c. IDEALIZED VERTICAL STRESS**

**FIG. 2.24: IDEALIZED STRESS DISTRIBUTION FOR B/C CROSS SECTION**

\[ \varepsilon_c \geq \varepsilon_u, \quad \varepsilon_s \geq \varepsilon_y \quad (f = 1) \]

\[ \varepsilon_s = \frac{f_s}{E_s} \]

\[ f_c'' = 0.85kd \]

**FIG. 2.31: STRESS-STRAIN DISTRIBUTION UNDER "COMPRESSION FAILURE"**

\[ \varepsilon_c = \varepsilon_u, \quad \varepsilon_s < \varepsilon_y \quad (f < 1) \]
are set forth so that shear failure is prevented during any mode of behavior. Precautions were taken in the experimental program, where shear failures were prevented by proper support and reinforcement details.

For the slabs considered, the direction of the orthogonal reinforcement does not coincide with the principal directions of stresses along the yield line. However, it is assumed that the mixed stress resultants \((M_{xy}, P_{xy})\) are carried by the concrete without any effect on the uniaxial stress relations at yielding. This assumption is equivalent to assuming a stepped yield line (Fig. 3.3.1), where the steps are not in the direction of the principal stresses. On each step only the yield moment and force are acting, while the mixed stress resultants are equal to zero. More basic research is needed in order to find the generalized yield criteria for R/C cross sections, which consider also mixed stress resultant, to replace the present "square yield criteria". Then the analysis could be extended to any system of coordinates.

2.3 **YIELD FUNCTION FOR THIN SLAB**

If the center of reinforcement is chosen as the reference axis, the stress resultants are given by:

\[
P_i = \int f_{c_i} \, dA \quad (\text{positive if compression})
\]

\[
(M_i)_{x,y} = \int f_{c_i} \, \xi \, dA
\]

\[
\quad i = 1, 2
\]
The uniaxial perfectly plastic stress resultants are given by:

\[ T_o = A_s f_y \]

\[ M_o = A_s f_y \alpha \left( 1 - \frac{1}{2} \frac{f_y}{f_c^*} \right) \]

The strength criteria given by eq. (2.2.1), leads to a yield function for the thin slab, formulated in a dimensionless functional representation as in eq. (2.1.2)

\[ \max \left[ f_1 (Q_{11}, Q_{21}), f_2 (Q_{12}, Q_{22}) \right] = 1 \]

where

\[ Q_{11} = \frac{(M_i)^{r-1}}{M_o} \]

\[ Q_{22} = \frac{P_i}{T_o} \]

Substituting the corresponding stresses, the yield function reduces to:

\[ f_1 = Q_{11} - \alpha Q_{22} + (1 - \alpha) Q_{22}^2 = 1 \]

\[ \text{where} \quad \alpha = \frac{1 - \frac{r}{t}}{1 - \frac{1}{2} t} \]

\[ t = r \left( \frac{f_y}{f_c^*} \right) \]

The yield condition is given as a limiting border for the elastic domain in Fig. (2.3.2). Due to the proportionality between stress and strain, each point in this domain
represents a unique axial-curvature strain value. Each point on the yield condition curve defines a unique location of the neutral axis, while the magnitude of the strain is indeterminate if the strain is non-zero, and is given by the gradient of the equipotential yield line (eq. 2.1.3). The location of the neutral axis $k$ is given for some limiting cases on the yield function:

$$k = 1; \text{ when } ( q_{2i} )_{pl} = 0 , \text{ from eq. (2.1.3)}$$

$$\frac{df_i}{dA_{2i}} = 0$$

using eq. (2.3.1)

$$Q_{2i} = \alpha \frac{\alpha}{2(1-\alpha)}$$

$$Q_{i_i} = \frac{\alpha^2}{4(1-\alpha)}$$

$k = t$; $Q_{i_i} = 1$

$k = 0$; $Q_{i_i} = 0$, $Q_{2i} = -1$

The yield function represents the limiting value for the elastic domain only if the so-called "tensile failure" is considered. However, capacity of the concrete may be limited by "compression failure". Thus, interaction relations are also developed along the compression limit curve which is not an equipotential line, since elastic work is done for any interval on this line, by stretching the un-yielded reinforcement.

For $\varepsilon_c = \varepsilon_u$, $\varepsilon_c < \varepsilon_y$ assuming a rectangular stress
block as in Fig. (2.3.1) (Whitney), the following relations are developed:

For the case \( k \leq 1 \)

\[
Q_{ii} = \left[ \frac{1}{2} Q_{zi} - 1 + \left( \frac{1}{4} Q_{zi}^2 - Q_{zi} + \frac{1.85 \alpha - 2.7}{\alpha - 1} \right)^{1/2} \right] \left\{ 2 - \alpha - \left(1 - \alpha \right) \left[ \frac{1}{2} Q_{zi} - 1 + \left( \frac{1}{4} Q_{zi}^2 - Q_{zi} + \frac{1.85 \alpha - 2.7}{\alpha - 1} \right)^{1/2} \right] \right\}
\]  

(2.3.2)

For the case \( k = 2 \) \( \frac{d_i}{a} = 1.2 \)

\[Q_{ii} = 0.96 \left[ 1 + \frac{\alpha^2}{4(1 - \alpha)} \right]
\]

\[(Q_{zi})_{\text{max}} = \left[ 1 + \frac{d_i}{a} \frac{2 - \alpha}{2(1 - \alpha)} \right]
\]  

(2.3.3)

Each point in the elastic domain represents a unique state of stress and strain. Some characteristic elastic lines are considered.

For the elastic limit: \( \epsilon_c = \epsilon_{cy} , \epsilon_s = |\epsilon_y| \)

For \( k \leq 1 \), triangular stress block considered.

\[
Q_{ii} = \left[ Q_{zi} - 1 + \left( Q_{zi}^2 - 2Q_{zi} + \frac{2\alpha - 3}{\alpha - 1} \right)^{1/2} \right] \left\{ \frac{2 - \alpha}{2} - \frac{1 - \alpha}{3} \left[ Q_{zi} - 1 + \left( Q_{zi}^2 - 2Q_{zi} + \frac{2\alpha - 3}{\alpha - 1} \right)^{1/2} \right] \right\}
\]  

(2.3.4)

For \( 1.2 < k < \infty \)

\[
Q_{ii} = 2.5 \alpha - 0.87 Q_{zi} (1 - \alpha) - 3.04 \frac{1 - \alpha}{12 \left( \frac{1 - \alpha}{2 - \alpha} \right)^2 (1.35 \alpha - 1.72)}
\]  

(2.3.5)

The graphical representation of the elastic-perfectly plastic domain is given in Fig. 2.3.2., scaled for \( d_i = \frac{3}{4} \), \( r = 1\% \), \( \alpha = 0.94 \). For values other than this, the
FIG. 2.3.2: ELASTO-PLASTIC DOMAIN FOR R/C SLAB.
(SCALED FOR $d = 3/4''$, $f = 1\%$, $\alpha = 0.94$).
absolute values are different, as well as the relative relation between compression and tensile failures. Later, where thick slabs are considered (Ch. 6), a modified definition of balanced cross section as a function of axial compression and the 3-dimensional state of stress is suggested.

2.4 THE STRESS PATH

For a given loading, a unique point is determined in the elasto-plastic domain.

The stress path shows the history of the stress resultants at a given point in the structure.

If the rigid-plastic solution is considered, the stress path follows along the yield condition curve \( f = 1 \) only. For a restrained slab structure, maximum elastic strains are assumed to develop for infinitely small deflections in order to develop a full plastic hinge with equipotential characteristics. Thus, the initial conditions are given by:

\[
k = 1 \quad \begin{bmatrix} \frac{P}{I_s} & \frac{M_{x-1}}{M_{e}} \end{bmatrix} \Rightarrow \text{Max}
\]

Since elastic strains are assumed to be zero, the strain of the reinforcement level \( \varepsilon_{sp} \) is also assumed to be zero. With increasing deflection the neutral axis moves upward toward the compression face, while the stress path is moving downward, along the yield conditions curve. (Fig. 2.3.2)

For the elasto-plastic solution, the stress path is included inside the elastic domain if elastic strains are
maintained. At the point of entry into the yield condition curve, continuity of strains must be obtained, thus

\[ k_{\text{elastic}} = k_{\text{plastic}} \]

Movement along the yield condition curve is associated with additional strains, thus:

\[ \varepsilon_s = \varepsilon_y + \varepsilon_{sp} \]

The change of strain while moving along the yield condition curve corresponds to the sum of the change in elastic strain and the additional plastic strain.

Consider the stress conditions of all the points along the diagonal of a square slab (Fig. 2.4.1). The various modes discussed in section 1.2 correspond to different strain states. If the yield condition for the slab is given by the stress coordinates \( Q_1 \) and \( Q_2 \), the "stress path" of each mode, which represents the state of stress under a given load \( n \), is described in Sec. 1.2.

For stage (1) ("elastic") \( f < 1 \) everywhere

For stage (2) ("spreading mechanism")

\[ \begin{align*}
&\text{near center slab:} \quad f = 1 \\
&\text{near the boundary:} \quad f < 1 
\end{align*} \]

For stage (3) & (4) \( f = 1 \) everywhere

The stress path for \( x = y = 0 \) and \( x = y = \frac{a}{4} \) (Fig. 3.1.1) is given for each case studied (see also Fig. 4.4.2, 3, 6.2.1 & 4). It is useful for understanding of:

1. The various parameters that influence the behavior of the slab.
2. Elasto-plastic solution for the slab.
3. The diagnosis of the true mode of failure.
FIG. 2.4.1: STRESS RESULTANTS ALONG THE DIAGONAL OF A SQUARE SLAB FOR VARIOUS MODES OF FAILURE.
CHAPTER 3

UPPER BOUND SOLUTION

3.1 RIGID-PLASTIC FORCE DISPLACEMENT RELATIONS

Principles of limit analysis are applied when the region $f<1$ disappears or becomes insufficient to restrain the plastic region from motion.

The kinematic field assumed is rigid plastic yielding along the diagonals. (Fig. 3.1.1)

The admissible kinematic field must satisfy $y=0$:

$$\begin{align*}
    y=0 & \quad \omega = -\overline{\omega}_0 \left(1 - 2 \frac{|x|}{L}\right) \\
    U_0 < x \leq \frac{L}{2} & \quad \epsilon_x = 0 \\
    x = \frac{L}{2} & \quad U = 0 \\
    x = 0 & \quad U = U_0
\end{align*}$$

From strain-displacement relations

$$\epsilon_x = U_{,x} + \frac{1}{2} \omega_{,x}^2 = 0$$

After integration and using the boundary conditions, one gets:

$$U = \frac{\overline{\omega}_0}{L} \left(1 - 2 \frac{|x|}{L}\right)$$

Since for $x=0$

$$K_{x=0} = \frac{\overline{\omega}_0 2 \sqrt{2}}{L} \quad \epsilon_x = \frac{2 U}{\sqrt{2}}$$
**Figure 3.1.1**: Kinematic field for restrained slab.

**Figure 3.2.1**: Equilibrium of a free rigid part.
Then, along the diagonals \(|x| = |y|\)

\[
\varepsilon_{|x|=|y|} = \frac{2}{\sqrt{2}} \frac{\overline{a}^2}{L} (1 - 2 \frac{|x|}{L})
\]

(3.1.4a)

\[
\gamma_{|x|=|y|} = \gamma_{x=0} = \text{CONST.}
\]

(3.1.4b)

From geometrical relations:

\[
d(1 - K_{|x|=|y|}) = \left(\frac{\varepsilon}{\gamma}\right)_{|x|=|y|}
\]

Using eq. (3.1.4), get

\[
K_{|x|=|y|} = \left| - \frac{\overline{a}^2}{2a} (1 - 2 \frac{|x|}{L}) \right|
\]

(3.1.5)

From eq. (2.1.3)

\[
\frac{\varepsilon}{\gamma} = \frac{\delta F}{\delta P} = \frac{\alpha - \frac{2(1-\alpha)}{1 - \alpha}}{\frac{1}{\alpha^2}}
\]

(3.1.6)

and from eq. (3.1.4)

\[
Q_2 = \frac{\alpha}{2(1-\alpha)} + \frac{\overline{a}^2}{4a} \frac{2-\alpha}{1-\alpha}
\]

(3.1.7)

substitute in eq. (2.3.1)

\[
Q_1 = 1 + \frac{\alpha^2}{4(1-\alpha)} - \frac{\omega^2 (2-\alpha)^2}{d^2 16(1-\alpha)}
\]

3.2 RIGID-PLASTIC LOAD-DEFLECTION RELATIONS

Consider equilibrium of the rigid part. (Fig. 3.2.1)
If rotations are small:

$$\omega_y = \tan \phi \approx \sin \phi \approx \phi$$

$$\cos \phi \approx 1$$

The total equilibrium equations are:

$$\Sigma Z = 0$$

$$\frac{R L^2}{4} = \int_{-\frac{L}{2}}^{\frac{L}{2}} V\left(\frac{s}{L}\right) dy$$

$$\Sigma X = 0$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} P\left(\frac{s}{L}\right) dy + \int_{-\frac{L}{2}}^{\frac{L}{2}} V\left(\frac{s}{L}\right) \omega_x dy = \frac{R L^2}{4} \omega_x + \frac{2}{\sqrt{2}} \int_0^{\frac{L}{\sqrt{2}}} P(s) ds$$

$$\Sigma M\left(\frac{L}{2}\right) = 0$$

$$\frac{2}{\sqrt{2}} \int_0^{\frac{L}{\sqrt{2}}} M(s) ds = \frac{R L^2}{4} \frac{L}{6} + \frac{2}{\sqrt{2}} \int_0^{\frac{L}{\sqrt{2}}} P(s) \omega(s) ds$$

After integration, get in dimensionless form.

$$\frac{P}{L T_o} = \frac{\alpha}{2(1 - \alpha)} - \frac{2 - \alpha}{8(1 - \alpha)} \frac{\bar{\omega}_o}{d}$$

$$\frac{R}{R_o} = 1 + \frac{\alpha^2}{4(1 - \alpha)} - \frac{\alpha(2 - \alpha)}{4(1 - \alpha)} \frac{\bar{\omega}_o}{d} + \frac{1}{16} \frac{(2 - \alpha)^2}{1 - \alpha} \left( \frac{\bar{\omega}_o}{d} \right)^2$$

when

$$\bar{\omega}_o = \frac{24 M_o}{L^2}$$
In Fig. (3.2.2) the variation of the axial force as a function of the central deflection is given.

It is seen that the stress field obtained for the compressive axial force is far from being the true one. A modified plastic solution is needed.

Comparison of \( f_L = W(\tilde{\omega}_3) \) with the experimental results for \( r = 1\% \) and \( r = 3\% \) is shown in Fig. (3.3.2)

3.3 MODIFIED RIGID PLASTIC SOLUTION

The rigid-plastic solution appears to give reasonable load-deflection relations for loads that are higher than the ultimate load. For smaller loads, the rigid plastic idealization leads to the imaginary state

\[
\bar{\omega}_3 = 0 \quad \frac{M_{1,r}}{M_o} \rightarrow \text{max} \quad \psi_p = 0 \\
\frac{P}{T_o} \rightarrow \text{max} \quad \varepsilon_{ep} = 0
\]

Actually, the yield is not concentrated along a line, and elastic change of curvature due to bending as well as elastic inplane strains due to axial force have a major effect. The result is that the conditions along the yield line are "softened".

The effect of elastic inplane strains is taken into account in this section by considering the additional strain caused by the strains of the rigid-plastic regions. The effect of change in curvature is taken into account in section
FIG. 3.2.2: VARIATION OF THE AXIAL FORCE AS A FUNCTION OF THE CENTRAL DEFLECTION

FIG. 3.3.1: "STEPPEPD" YIELD LINE
5.2, by using the modified-plastic solution as a continuation of the elasto-plastic solution.

If eq. (3.1.2) is modified, to account for elastic shortening due to axial compression \( P \) and if a strip approximation is assumed (i.e. \( P_{xy} = 0 \) everywhere), one gets for point \( X_i \) along the diagonal \( x=y \), Fig. (3.3.1).

\[
X_i < x \leq \frac{L}{2} \\
\varepsilon_x' = u_x' + \frac{1}{2} \gamma_{xx} - \frac{P}{Eh} = 0
\]  

\[
0 \leq x \leq X_i \\
\varepsilon_x'' = u_x'' - \frac{P}{Eh} = 0
\]  

The boundary conditions are given by:

\[
X = \frac{L}{2} \quad u' = 0
\]

\[
X = X_i \quad u' = u_o'
\]

\[
X = 0 \quad u'' = 0
\]

Solving for \( u' \) AND \( u'' \), get

\[
u' = \frac{\omega_o^2}{L} \left( 1 - 2 \frac{x}{L} \right) + \frac{PL}{2Eh} \left( 1 - 2 \frac{x}{L} \right)
\]  

\[
u'' = \frac{P}{Eh} x
\]
Using (3.1.5)

\[
\frac{\Delta}{\bar{t}} = \frac{w''(1-x) - 2(1-x)w}{2 \bar{w}_o} = \frac{\bar{w}_o}{2} \left(1 - \frac{2|x|}{L}\right) + \frac{PL^2}{4Eh \bar{w}_o} = \\
= \frac{M_0}{T_o} \left[ \alpha - \frac{2(1-\alpha)}{T_o} P \right]
\]

\[Q_1 = \frac{P}{T_o} = \frac{1}{\bar{w}_o} M \left[ \frac{\alpha \bar{w}_o}{2(1-\alpha)} - \frac{\bar{w}_o}{4(1-\alpha)} \right] \left(1 - \frac{2|x|}{L}\right)
\]

(3.3.5)

where

\[M = \frac{2 - \alpha}{1 - \alpha} \frac{T_o L^2}{8 \alpha E h}
\]

(3.3.6)

Using eq. (2.3.1), get

\[Q_2 = \frac{M_{x+1}}{M_0} = \\
= \frac{1}{4(1-\alpha)} \frac{\bar{w}_o^2 + 2M \bar{w}_o}{(\bar{w}_o + M)^2} - \frac{\alpha(2-\alpha)}{4(1-\alpha)} \frac{\bar{w}_o}{d} \frac{\bar{w}_o M}{(\bar{w}_o + M)^2} - \frac{(2-\alpha)^2}{16(1-\alpha)} \left(\frac{\bar{w}_o}{d}\right)^2 \frac{\bar{w}_o^2}{(\bar{w}_o + M)^2}
\]

(3.3.7)

Using equilibrium conditions as in section 3.2, one
gets for the average axial compression along the boundary:
\[
\frac{P}{T_o L} = \frac{\alpha}{2(1-\alpha)} \frac{\ddot{u}_s}{\dot{\omega}_s + \mu} - \frac{2-\alpha}{8d(1-\alpha)} \frac{\ddot{u}_s^2}{\dot{\omega}_s + \mu}
\]  

(3.3.8)

and from moment equilibrium

\[
\frac{m}{T_o^2} = 1 + \frac{\alpha^2}{4(1-\alpha)} \frac{\ddot{u}_s^2 + 2\mu \ddot{\omega}_s}{(\dot{\omega}_s + \mu)^2} - \frac{\alpha(2-\alpha)}{8(1-\alpha)} \frac{\ddot{u}_s}{d} \frac{2\ddot{u}_s + 3\ddot{\omega}_s \mu}{(\ddot{\omega}_s + \mu)^2} + \frac{(2-\alpha)^2}{48(1-\alpha)} \left( \frac{\ddot{\omega}_s}{d} \right)^2 \frac{3\ddot{u}_s^2 + 4\ddot{\omega}_s^2 \mu}{(\ddot{\omega}_s + \mu)^2}
\]  

(3.3.9)

For \( \mu \to 0 \) eq. (3.3.9) reduces to eq. (3.2.5)

For the cases studied ( \( r=1\% \) and \( r=3\% \)), the important parameters are represented in Table (3.3.1)

<table>
<thead>
<tr>
<th>r</th>
<th>0.01</th>
<th>0.03</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>0.138</td>
<td>0.414</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.725</td>
<td>0.74</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.125</td>
<td>0.110</td>
</tr>
</tbody>
</table>

**TABLE 3.3.1:** PARAMETERS FOR RIGID PLASTIC SOLUTIONS, \( d_1 = 3/4' \).
The load-deflection relations are compared with the experimental data and the rigid-plastic solution in Fig. (3.3.2).
FIG. 3.3.2: LOAD DEFLECTION RELATIONS—UPPER BOUND. \( d_1 = 3/4'' \)
4.1 DEFINITION AND METHOD OF SOLUTION

The definition of instability in the present problem corresponds to sudden loss of stiffness. Due to geometrical changes, the axial compression imposed by the boundary conditions reaches a maximum and then decreases. The result is that the stiffness of the reinforced concrete cross section, which is proportional to the axial compression, is lost.

The yield-line solution satisfies closely the load-deflection relations for the second equilibrium condition. Since the ultimate load is reached for loads that are much higher than the Johanson load, for loads close to the ultimate load a full plastic yielding has just been developed along the yield line. This, however, is not the limiting capacity of the slab, since this plastic moment is a function of the axial compression.

Consider equilibrium of the region limited by the diagonals (Fig. 3.2.1). The equilibrium condition may be described as

\[ M_{\text{resisting}} = M_{\text{acting}} \]

For \( \lambda \approx \lambda_{ur} \)

\[ M_{\text{resisting}} = M_{\text{plastic}} \]
From eq. (2.3.1)

$$M_{PLASTIC} = \int M_{pl} \, ds = \sum_{m} A_{m} P^{m} \quad m \geq 0$$

Where $P$ is the total axial compression along the "stepped" diagonal (Fig. 3.3.1) and is therefore the total axial compression along the boundary according to the present definition of the yield criteria.

The acting moment is given by:

$$M_{ACTING} = M_{p} + \int P \omega(s) \, ds$$

Where $M_{p}$ is the moment due to transverse loads and reactions, and the integration represents the moment due to axial compression. If

$$\omega(s) = \bar{w}_{o} f(s)$$

then, in general form:

$$M_{ACTING} = M_{p} + \sum_{n} A_{n} k^{n} \bar{w}_{o} g(s) \quad n \geq 0$$

The total compression is constant throughout the slab, if transverse load only is acting. Thus, it is only a function of the deflected shape and the elastic modulus. If $E$ (for axial compression) is assumed to be constant

$$IP = \sum_{k} A_{k} \bar{w}_{o}^{k} \quad k = 1, 2$$
An upper bound for the stability, taking additional arbitrary variation of the displacement $\overline{\delta \mathbf{w}}$ is:

$$\sum M_{\text{acting}} > \sum M_{\text{resisting}}$$

Assuming that the variation of the applied load due to additional arbitrary variation of the central deflection is zero (this was not the case in the experimental setup, where $\delta \mathbf{w}_c$ leads to $\delta \mathbf{h} < 0$, and therefore second equilibrium is possible), $\delta \mathbf{m}_n = 0$. One then gets:

$$\sum M_{\text{acting}} = \sum_n A_n \left[ n P^{n-1} \frac{d}{d \overline{\mathbf{w}}_o} g(s) \delta \overline{\mathbf{w}}_o + P^n g(s) \delta \overline{\mathbf{w}}_o \right]$$

$$\sum M_{\text{resisting}} = \sum_m m P^{m-1} \frac{d}{d \overline{\mathbf{w}}_o} \delta \overline{\mathbf{w}}_o$$

Thus, an obvious upper bound for stability, however, not necessarily the limiting condition, is given by:

$$\frac{\partial P}{\partial \overline{\mathbf{w}}_o} = 0$$

while the limiting condition which is found in section (4.2) appears to give the same results approximately.

Mathematically, it will be found for the thin slab ($\frac{L}{a_0} = 2.0$), that the condition $\delta \overline{\mathbf{w}} = 0$ is reached when $\frac{\overline{\mathbf{w}}_o}{h} \approx 0.4$, where $\overline{\mathbf{w}}_o$ is the deflection of the center slab. It is well known, that for such relations the so-called small deflection theory can be used, since the
stresses initiated by the stretching of the neutral surface can be neglected in comparison to the bending stresses. However, because the axial compression is a function of the horizontal displacement at the reinforcement level at the boundary, and because that part of the displacement caused by the geometrical movement of the neutral axis resulting from the deflections, is of the same order of magnitude as that part of the displacement caused by the elastic strains of the slabs, therefore the large-deflection theory must be used.

The equilibrium equations are:

\[ \nabla^2 M = f_c + P_x \sigma_{xx} + 2P_y \sigma_{xy} + P_y \sigma_{yy} \]
\[ M = M_x + M_y \]
\[ P_{xx} + P_{xy,y} = 0 \]
\[ P_{yy} + P_{xy,x} = 0 \]

When the stress resultants are:

\[ P = \int f_c \, d\gamma \]
\[ M = \int f_c \gamma \, d\gamma \]

where \( P \) is positive if \( f_c \) is tensile stress. (Note that the opposite is true for the rigid plastic solution).

For:

\[ (\varepsilon_c)_x + \gamma (\varepsilon_c)_y \leq \varepsilon_{cy} \]

get

\[ (f_c)_{ch} = \frac{f_c}{E_{cy}} \left[ (\varepsilon_c)_x + \gamma (\varepsilon_c)_y \right] \]
For:

\[ \varepsilon_u \geq (\varepsilon_c)_x + \nabla (\varepsilon_c)_y \geq \varepsilon_{cy} \]

get

\[ (f_c)_{pl.} = f_c^{''} \]

The strain-displacement relations are:

\[ U = U_0 + \varepsilon_{0x} z \]
\[ \varepsilon_x = \varepsilon_{0x} + \omega_{xx} z \]
\[ \varepsilon_{0x} = \varepsilon_{0x} + \frac{1}{2} \omega_{xx}^2 \]

By definition, the behavior is ideal elasto-plastic. Non-linearity of the material and changes in the effective cross section due to tensile cracks prior to yielding will be neglected. These conditions occur initially only at a limited region near the center of the slab. It is well known that the error introduced will be small. (see Appendix I).

For the linear elastic case, using a stress function where:

\[ P_x = \overline{t} F_{yy} \]
\[ P_y = \overline{t} F_{xx} \]
\[ P_{xy} = -\overline{t} F_{xy} \]

\( \overline{t} \) is the effective cross section to resist axial compression.

The equilibrium equations reduce to:
If the reference axis is the neutral surface of the slab, the boundary conditions are:

\[
\nabla^4 \omega = E \left( \omega_{xxyy}^2 - \omega_{xxx} \omega_{yy} \right)
\]

\[D \nabla^4 \omega = \bar{h} \left( \frac{P}{h} + F_{yy} \omega_{xxx} + F_{xx} \omega_{yy} - 2 F_{xy} \omega_{yxy} \right)
\]

If the reference axis is the neutral surface of the slab, the boundary conditions are:

\[x = \frac{a}{2}
\]

\[F_{xy} = 0
\]

\[M_x = - P_x \delta_o
\]

\[\omega = 0
\]

\[U_o + \omega_{x} x \delta_o = 0
\]

\[y = \frac{a}{2}
\]

\[F_{xy} = 0
\]

\[M_y = - P_y \delta_o
\]

\[\omega = 0
\]

\[U_o + \omega_{y} y \delta_o = 0
\]

For R/C composite materials, poisson ratio is assumed to be zero.

The method of solution is based on an estimation of the deflected shape. For mathematical simplicity, the general form of the deflected shape is taken as:
\[ W = W_1 + W_2 \]  \hspace{1cm} (4.1.1)

where (Fig. 4.1.1)

\[ \omega_i = \sum_{m}^{n} \omega_{mn} \cos \alpha x \cos \alpha y \]
\[ \alpha = \frac{\pi}{a} \quad m, n = 1, 3, 5 \ldots \]  \hspace{1cm} (4.1.2)

\( W_1 \) satisfies all the boundary conditions, except the moment boundary condition, which are satisfied by \( W_2 \).

\[ \nabla^4 \omega_2 = 0 \]  \hspace{1cm} (4.1.3)

\[ \omega_2 = \omega_{2H} + \omega_{2e} \]

Where \( \omega_{2H} \) satisfies the moment boundary conditions, and \( \omega_{2e} \) is a correction in order to satisfy \( \omega_2 = 0 \) for

\[ x = \frac{3a}{2}, \quad y = \frac{a}{2}. \]

Take:

\[ \omega_{2H} = \sum_{n} H_n (m, x \sinh \alpha y - \frac{m\pi}{2} \tanh \frac{m\pi}{2} \cosh \alpha y) \cos \alpha x + \]
\[ + \sum_{m} H_m^1 (m, x \sinh \alpha x - \frac{m\pi}{2} \tanh \frac{m\pi}{2} \cosh \alpha x) \cos \alpha y \]  \hspace{1cm} (4.1.4)

and then correct the boundary conditions with \( \omega_{2e} \).

4.2 \hspace{0.5cm} SOLUTION FOR AXIAL COMPRESSION

Initially, the first governing differential equation is satisfied

\[ \nabla^4 F = E \left( \omega_{xy} - \omega_{xx} \omega_{yy} \right) \]  \hspace{1cm} (4.2.1)

In order to get a first order approximation for the non-linear equation, take

\[ \omega = \bar{\omega} = \bar{\omega} \cos \alpha x \cos \alpha y \]  \hspace{1cm} (4.2.2)
FIG. 4.1.1: THE SQUARE SLAB.

FIG. 4.2.1: THE VARIATION OF THE FUNCTION $\xi(x)$. 
This engineering approximation introduces a small error only;

a) The observed deflected shape of the specimen tested stands with close agreement to this shape.

b) The effect of \( W_2 \) on the deflected shape is small (see also eq. 4.3.31)

Substitute in the differential equation

\[
\nabla^4 F = -\frac{E\alpha^4}{2} (\cos 2\alpha_x X + \cos 2\alpha_y Y) =
\]

\[
= E\alpha^2 \sum_P \sum_D \cos 2\alpha_x X \cos 2\alpha_y Y
\]

\( p, q = 0, 1, 2, \ldots \)

where

\[
D_{10} = D_{01} = -\frac{\omega_0^2}{2}
\]

The stress function \( F \) is given by

\[
F = F_o + F_1
\]

where \( F_o \) is the homogeneous part and \( F_1 \) is the particular part of the solution.

\[
\nabla^4 F = E\alpha^4 \sum_P \sum_D \cos 2\alpha_x X \cos 2\alpha_y Y
\]

\[
F_1 = E \sum_P \sum_{pq} \phi_{pq} \cos 2\alpha_x X \cos 2\alpha_y Y
\]

\( (4.2.3) \)

where

\[
\phi_{pq} = \frac{D_{pq}}{16 (p^2 + q^2)}
\]

\[
\phi_{00} = \frac{-\omega_0^2}{32}
\]

\[
\nabla^4 F_o = 0
\]

\[
F_o = f + g
\]
\[ f = f_n(y) \cosh \alpha x \quad n = 0, 1, 2, \ldots \]
\[ \frac{d^4 f_n}{dy^4} - 2 \frac{d^2 f_n}{dy^2} \alpha_n^2 + \alpha_n^4 f_n = 0 \quad n \neq 0 \]
\[ f_n(y) = A \cosh n \alpha y + B \sinh n \alpha y + C \sinh n \alpha y + D \cosh n \alpha y \quad (4.2.4) \]

F is an even function. Since \( F \) is even, therefore \( F_0 \) must be even.

\[ C = D = 0 \]

At \( x = \frac{a}{2}, y = \frac{a}{2} \), \( F_{xy} = 0 \)
\[ F_{0,xy} = 0; \quad \alpha_n \sinh \frac{n \pi}{2} + B \left( \sinh \frac{n \pi}{2} + \frac{n \pi}{2} \cosh \frac{n \pi}{2} \right) = 0 \]
\[ f_n(y) = C_n \left[ \left( \sinh \frac{n \pi}{2} + \frac{n \pi}{2} \cosh \frac{n \pi}{2} \right) \cosh n \alpha y - \alpha_n y \sinh \frac{n \pi}{2} \sinh n \alpha y \right] \quad (4.2.5) \]

For \( n = 0 \)
\[ \frac{d^4 f_0}{dy^4} = 0 \]
\[ f_0 = A_0 + B_0 y + C_0 y^2 + D_0 y^3 \]

To satisfy the boundary conditions and symmetry, one gets
\[ f_0 = C_0 y^2 \]

By analogy, if
\[ g = g_n(x) \cos \alpha y \]
Get the most general solution
\[ F_0 = c_0 y^2 + C_0 x^2 + \sum_n c_n \left( \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) \cosh n \alpha, y - n \alpha, \sinh \frac{m \pi}{2} x \sinh n \alpha, y \cos \alpha, x + \sum_m c_m \left[ \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right] \cosh m \alpha, x - m \alpha, \sinh \frac{m \pi}{2} x \sinh m \alpha, x \cos m \alpha, y \] (4.2.6)

The constants are found from the boundary conditions

\[ U_0 + \frac{\partial}{\partial x} (\sigma_x) \bigg|_{x = \frac{a}{2}} = 0 \quad \text{and for simplicity), take } \varphi = 0. \]

For reinforced concrete, take \( \varphi = 0 \).

\[ U_0 = \int_0^{a/2} \left[ \frac{1}{E} F_{yy} - \frac{1}{2} (\sigma_y^2) \right] d\alpha \]

\[ F_{yy} = 2c_0 + \sum_n c_n \left[ \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right] n^2 \alpha_x^2 \cosh n \alpha, y - n \alpha, \sinh \frac{m \pi}{2} x \sinh n \alpha, y \cos \alpha, x + n^2 \alpha_y^2 \sinh n \alpha, y \cos \alpha, x - \sum_m m^2 \alpha_x^2 c_m \left[ \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right] \cosh m \alpha, x - m \alpha, \sinh \frac{m \pi}{2} x \sinh m \alpha, x \cos m \alpha, y - \alpha_x^2 \sum_p \phi_p \phi_p \cos 2p \alpha, x \cos 2q \alpha, y \] (4.2.7)

\[ \rho, q = 0,1, \ldots. \]
\[ \int_0^{a/2} \omega_x^2 \, dx = \frac{\omega_0^2}{16} \alpha_i^2 \alpha (1 + \cos 2\alpha y) \]

\[ (\omega_x)_{x=\pm a/2} = -\alpha_i \omega_0 \cos \alpha y \]

\[ \int_0^{a/2} F_{yy} \, dx = \frac{c_0 q}{\varepsilon} + \frac{1}{\varepsilon} \sum_n c_n \left[ \left( \sinh \frac{n \pi}{2} + \frac{n \pi}{2} \cosh \frac{n \pi}{2} \right) \frac{\alpha_i \cosh n \alpha y}{\cosh^2 \frac{n \pi}{2}} - \right. \]

\[ - n \alpha_i \sinh \frac{n \pi}{2} \left( 2 \cosh n \alpha y + \right. \]

\[ + n \alpha_i y \sinh n \alpha y \left\{ \sin \frac{n \pi}{2} - \right. \]

\[ - \frac{1}{\varepsilon} \sum m \alpha_i c_m \left[ \left( \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) \sinh \frac{m \pi}{2} - \right. \]

\[ - \sinh \frac{m \pi}{2} \left( \frac{m \pi}{2} \cosh \frac{m \pi}{2} - \right. \]

\[ - \sinh \frac{m \pi}{2} \right] \cos m \alpha y + \]

\[ + \frac{n^2}{2 \alpha_i} \sum_{n=0}^\infty \frac{1}{a} \frac{1}{n^2 a^2 + n^2 \cos 2n \alpha y} \] (4.28)

Use the series:

\[ \cosh n \alpha y = \sum_m E_{m,n} \cos 2m \alpha y \]

\[ y \sinh n \alpha y = \sum_m E_{m,n} \cos 2m \alpha y \]

\[ m=0,1,\ldots \]

For the interval \( a \), get the Fourier integrals

\[ E_{m,n} = \frac{4}{a} \int_0^{a/2} \cosh n \alpha y \cos 2m \alpha y \, dy = \]

\[ = \frac{4}{a} \frac{n}{\alpha_i} \frac{1}{4m^2 + n^2 \cos n \pi \sinh \frac{n \pi}{2}} \]
Therefore, for $m \neq 0$

$$\cosh h x, y = \sum_{m} \frac{a}{4} \frac{n}{\alpha_1} \frac{\cos \frac{m \pi}{2}}{m^2 + n^2} \sinh \frac{nm}{2} \cos \alpha y$$  \hspace{1cm} (4.2.9)$$

$$F_{m} = \int_{0}^{a/2} y \sinh h x, y \cos 2m \alpha, y dy =$$

$$= \frac{4}{a} \frac{\alpha_1}{(4m^2 + n^2)} \left( \frac{nm}{2} \frac{\cos \frac{m \pi}{2}}{m^2 + n^2} \cosh \frac{nm}{2} - \frac{n^2 - 4m^2}{4m^2 + n^2} \cos \frac{m \pi}{2} \sinh \frac{nm}{2} \right)$$

For $m \neq 0$

$$y \sinh h x, y = \sum_{m} \frac{a}{\alpha_1} \frac{4}{n} \left( \frac{nm}{2} \frac{\cos \frac{m \pi}{2}}{m^2 + n^2} \cosh \frac{nm}{2} - \frac{n^2 - 4m^2}{m^2 + n^2} \cos \frac{m \pi}{2} \sinh \frac{nm}{2} \right)$$  \hspace{1cm} (4.2.10)$$

for $m = 0$

$$F_{0} = \frac{a}{2} \frac{\alpha_1}{4} \int_{0}^{a/2} \cosh h x, y dy = \frac{2}{a} \frac{\alpha_1}{4} \sinh \frac{nm}{2}$$  \hspace{1cm} (4.2.11)$$

$$F_{0} = \frac{a}{2} \frac{\alpha_1}{4} \int_{0}^{a/2} y \sinh h x, y dy = \frac{2}{n^2 \alpha_1} \left( \frac{\alpha_1}{2} \cos \frac{nm}{2} - \sinh \frac{nm}{2} \right)$$  \hspace{1cm} (4.2.12)$$

Get for $x$ direction.

$$C_{0} = \frac{a}{4} + \frac{1}{E} \sum_{n} C_{n} \left\{ \left( \sinh \frac{nm}{2} + \frac{\pi}{2} \cosh \frac{nm}{2} \right) \left( \frac{4}{a} \sinh \frac{nm}{2} + \frac{1}{\alpha_1} \left( \frac{nm}{2} \frac{\cosh \frac{nm}{2}}{m^2 + n^2} - 2 \sinh \frac{nm}{2} \right) \right) \right\} \sin \frac{nm}{2} +$$

$$+ \frac{1}{E} \sum_{n} C_{n} \left\{ \left( \sinh \frac{nm}{2} + \frac{\pi}{2} \cosh \frac{nm}{2} \right) \sum_{m} \frac{4n^2}{a} \frac{\cos \frac{m \pi}{2}}{m^2 + n^2} \sinh \frac{nm}{2} \cos \alpha y -

- \frac{\alpha_1}{n^2} \frac{\sinh \frac{nm}{2}}{2} \left[ \sum_{m} \frac{4n^2}{a} \frac{\cos \frac{m \pi}{2}}{m^2 + n^2} \sinh \frac{nm}{2} \cos \alpha y +

+ \sum_{m} \frac{4n^2}{a} \frac{\cos \frac{m \pi}{2}}{m^2 + n^2} \sinh \frac{nm}{2} \cos \alpha y \right] \right\} \sin \frac{nm}{2} +$$

$$- \frac{n^2 - 4m^2}{n^2 + m^2} \cos \frac{m \pi}{2} \cosh \frac{nm}{2} \cos \alpha y -

- \frac{n^2 - m^2}{n^2 + m^2} \cos \frac{m \pi}{2} \sinh \frac{nm}{2} \cos \alpha y \right\} \sin \frac{nm}{2} +$$
\[-\frac{1}{E} \sum_{n} m \alpha_{n} C_{m} \left\{ (\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} ) \sinh \frac{m \pi}{2} - \sinh \frac{m \pi}{2} \left( \frac{m \pi}{2} \cosh \frac{m \pi}{2} - \sinh \frac{m \pi}{2} \right) \right\} \cos \alpha_{i} y + \]
\[+ \frac{\pi^{2}}{a} \bar{\omega}_{0}^{2} \pi^{2} \alpha_{i} \cos \alpha_{i} y - \frac{\bar{\omega}_{0}^{2} \pi^{2}}{16} (1 + \cos \alpha_{i} y) + \alpha_{i} \bar{\omega}_{0}^{2} \bar{\omega}_{0} \cos \alpha_{i} y = 0 \]

\[C_{0} = \frac{E \bar{\omega}_{0}^{2} \pi^{2}}{16} \alpha_{i}^{2} \tag{4.2.13}\]

for a given \( m \)

\[C_{m} = \frac{-4m \cos \frac{m \pi}{2}}{\pi \sinh \frac{m \pi}{2}} \sum_{n} C_{n} m \sin \frac{m \pi}{2} \sinh \frac{m \pi}{2} - \frac{\phi_{n} E \alpha}{2m \pi \sinh \frac{m \pi}{2}} \tag{4.2.14}\]

where \[\phi_{1} = -\frac{\pi}{a} \bar{\omega}_{0} \bar{\omega}_{0} \quad \phi_{m>1} = 0 \]

By analogy, for \( y \) direction:

\[C_{n} = \frac{-4n \cos \frac{n \pi}{2}}{\pi \sinh \frac{n \pi}{2}} \sum_{m} C_{m} m \sin \frac{n \pi}{2} \sinh \frac{n \pi}{2} - \frac{\phi_{n} E \alpha}{2n \pi \sinh \frac{n \pi}{2}} \tag{4.2.15}\]

Since \[\phi_{n} = \phi_{m=1} \]

therefore \[C_{1} = C_{1}', \quad C_{2} = C_{2}' \quad \text{etc.} \]

In the limits of the approximation, the first four terms are of interest:

\[C_{1} = -\frac{\phi_{1} E \alpha}{2\pi \sinh \frac{\pi}{2}} = \frac{\bar{\omega}_{0} \bar{\omega}_{0} E}{2 \sinh \frac{\pi}{2}} \tag{4.2.16}\]
\[ C_2 = -\frac{4 \Phi, E a}{25 \pi^2 \sinh^2 \pi} = -\frac{4}{25} \frac{\omega_0 \beta, E}{\pi \sinh^2 \pi} \]  
\[ C_3 = 0 \]  
\[ C_4 = \frac{8}{289} \frac{\Phi, E a}{\pi^2 \sinh^2 \pi} = -\frac{8}{289} \frac{\omega_0 \beta, E}{\pi \sinh^2 \pi} \]  

If \( P \) is the total compression along a boundary

\[ \frac{P}{2h} = \int_0^{a/2} F_{yy} dy \]  

Therefore

\[ \frac{P}{2h} = C_0 q + \sum_n C_n \left[ \left( \sinh \frac{\pi}{2} + \frac{n \pi}{2} \cosh \frac{n \pi}{2} \right) \sinh \frac{\pi}{2} \right] \sin \frac{n \pi}{2} \]

\[ = -n \alpha, \sinh \frac{n \pi}{2} \left( \sinh \frac{n \pi}{2} + \frac{n \pi}{2} \cosh \frac{n \pi}{2} \right) \cos n \alpha, x - \]

\[ - \sum_{n} m \alpha, C_m \left[ \left( \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) \cosh m \alpha, x - \right] \sin \frac{m \pi}{2} \sinh m \alpha, x \]

\[ = -m \alpha, \sinh \frac{m \pi}{2} \times \sinh m \alpha, x \]

\[ \sin \frac{m \pi}{2} = C_0 q - \alpha, C_j(x) \]  

Where

\[ \xi(x) = \frac{\pi}{2} \cosh \frac{\pi}{2} \cosh \alpha, x + \sinh \frac{\pi}{2} \left( \cosh \alpha, x - \alpha, x \sinh \alpha, x \right) \]

Actually, \( P \) should not be a function of \( x \). Observing the nature of the function \( \xi(x) \) (fig. 4.2.1), it is seen that within the limits of the approximation

\[ \xi(x) \approx \text{CONST.} \]

Here take the value for \( x = \pm \frac{a}{2} \)

\[ \frac{P}{2h} = C_0 q - \alpha, C_j \left( \frac{\pi}{2} + \frac{1}{2} \sinh \pi \right) \]  

\( \frac{\Phi, E a}{25 \pi^2 \sinh^2 \pi} = -\frac{4}{25} \frac{\omega_0 \beta, E}{\pi \sinh^2 \pi} \]  

\[ \frac{8}{289} \frac{\Phi, E a}{\pi^2 \sinh^2 \pi} = -\frac{8}{289} \frac{\omega_0 \beta, E}{\pi \sinh^2 \pi} \]
If \( \bar{\omega} \) is assumed to be the deflection at the center of the slab, from the calculation of the stress function \( F \), get

\[
\frac{P}{a} = \frac{h a^2 E}{2} \left( 0.25 \bar{\omega}^2 - 0.885 \bar{\omega} \right)
\]

(4.2.23)

Thus, the maximum value for \( P \) is obtained when

\[
\frac{dP}{\alpha/\bar{\omega}^2} = 0 \quad ; \quad \bar{\omega} = 1.77 \bar{\omega}_0
\]

(4.2.24)

The limiting equilibrium condition is given by

\[
\delta M_{\text{ACTING}} = \delta M_{\text{RESISTING}}
\]

(4.2.25)

While, using the equations of state suggested in section (4.1),

\[
M_{\text{RESISTING}} = M_{\text{PLASTIC}} = 2 \int_0^{\pi/2} M_{\text{PLASTIC}} dX =
\]

\[
= 2 M_0 \int_0^{\pi/2} \left[ 1 + \alpha \frac{P}{h} - (1-\alpha) \left( \frac{P}{h} \right)^2 \right] dX =
\]

\[
= 2 M_0 \left[ \frac{a}{2} + \alpha \frac{P}{h} - (1-\alpha) \int_0^{\pi/2} \left( \frac{P}{h} \right)^2 dX \right] \approx
\]

\[
= 2 M_0 \left[ \frac{a}{2} + \frac{\alpha h a^2 E^2 a}{4 h} \left( 0.25 \bar{\omega}_0^2 - 0.885 \bar{\omega} \right) \right] - \frac{1-\alpha}{h} \left[ \frac{h^2 a^4 E^2 a}{16} \left( 0.25 \bar{\omega}_0^2 - 0.885 \bar{\omega} \right)^2 \right]
\]

\[
= A_1 + A_2 \bar{\omega}_0^2 + A_3 \bar{\omega}_0^4 + A_4 \bar{\omega}_0^6 + A_5 \bar{\omega}_0^8
\]

(4.2.26)

\[
M_{\text{ACTING}} = M_0 + 2 \int_0^{\pi/2} (P \omega)|_{\omega=\bar{\omega}} dX
\]

(4.2.27)

Since

\[
(\omega)|_{\omega=\bar{\omega}} = \bar{\omega} \cos^2 \alpha X
\]

\[
(P)|_{\omega=\bar{\omega}} = 2 \bar{h} c_0 - \bar{h} \sum \frac{2 c_n h \alpha^2 \sinh \frac{N}{2} \cosh \alpha X \cos \alpha X}{N}
\]

\[
- 4 \frac{h^2}{\bar{h}} \alpha^2 E \cos 2 \alpha X
\]

(4.2.28)
Therefore

\[ M_{\text{act,net}} = M_1 + \bar{w}_0 C_0 \bar{h} - \frac{3}{2} h \bar{w}_0 \alpha_1 \sinh \frac{\pi}{2} \left( \frac{1}{1 + n^2} - \frac{1}{\pi + n^2} \right) - \bar{h} Y_0 E \alpha_2 \bar{w}_0 \alpha = - \frac{h}{E} \bar{h} - 2.8 \bar{w}_0^2 + \frac{\bar{h}}{\rho} - 0.93 \bar{w}_0^2 = \]

\[ = A_6 \bar{w}_0^2 + A_7 \bar{w}_0^3 \]  

(4.2.27)

For additional variation of the central displacement, the critical deflection for limiting equilibrium is given by:

\[ (A_2 + 2A_3 \bar{w}_o + 3A_4 \bar{w}_0^2 + 4A_5 \bar{w}_0^3) \delta \bar{w}_0 = (2A_6 \bar{w}_0 + 3A_7 \bar{w}_0^2) \delta \bar{w}_0 \]  

(4.2.30)

Since \[ \delta \bar{w}_0 \neq 0 \], this equation can be solved numerically for the ultimate deflection. If the values calculated from \[ \delta \bar{P} = 0 \] are checked, they appear to be in close agreement, but they always occur at some larger deflection. (see also experimental load-restraining force curves).

4.3 SOLUTION FOR LOAD-DEFLECTION RELATIONS

The expressions for \( \bar{W} \) and \( F \) satisfy the first governing differential equation, and the displacement boundary conditions. However, it is obvious that the second governing differential equation can not be exactly satisfied.

Therefore, using the stress function obtained, the deflection is taken as \( \bar{w} = \bar{w}_1 + \bar{w}_2 \) to satisfy also the force boundary conditions. Galerkin method is used to minimize the error in the second differential equation.

In order to solve for \( \bar{w}_2 \), use the boundary conditions
\[-D (\omega_{xxx})_{x=\frac{a}{2}} = \bar{H} \gamma_0 (F_{yy})_{x=\frac{a}{2}} \quad (4.3.1)\]

where \( \bar{H} \) is the effective depth.

\[-D (\omega_{yy})_{y=\frac{a}{2}} = \bar{H} \gamma_0 (F_{xx})_{y=\frac{a}{2}} \quad (4.3.2)\]

For the \( x \) direction use eq. (4.1.4) when \( H_n = 0 \)

\[\left( \omega_{xx} \right)_{x=\frac{a}{2}} = \sum_m H_m m^2 \alpha_i^2 2 \cosh \frac{m \pi}{2} \cos \alpha x, y \]

from eq. (4.2.7)

\[\left( F_{yy} \right)_{x=\frac{a}{2}} = 2c_0 + \sum_n C_n \left[ \left( -\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) n^2 \alpha_i^2 \cosh \alpha x, y - \right. \]

\[-\frac{n^3 \alpha_i^2 y \sinh \frac{m \pi}{2} \sinh \alpha x, y \cosh \frac{m \pi}{2}}{\cos \frac{m \pi}{2}} - \]

\[-\sum_m C_m m^2 \alpha_i^2 \left( \sinh \frac{m \pi}{2} \cosh \frac{m \pi}{2} + \frac{m \pi}{2} \right) \cos \alpha x, y - \]

\[-4 \alpha_i^2 E \varphi, \cos 2 \alpha x, y \quad (4.3.3)\]

Use

\[c_0 = \frac{4c_0}{\pi} \sum_m \frac{1}{m} \sin \frac{m \pi}{2} \cos \alpha x, y \]

with eq. (4.2.9 + 12) and (4.3.1) get:

\[-\frac{\bar{H}}{h} \sum_m H_m m^2 \alpha_i^2 2 \cosh \frac{m \pi}{2} \cos \alpha x, y = \]

\[= \frac{\bar{H}}{h} \frac{8c_0}{\pi} \sin \frac{m \pi}{2} \sum_m \frac{1}{m} \cos \alpha x, y + \]

\[+ \frac{3}{h} \sum_n C_n \left( \left( -\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) n^2 \alpha_i^2 \sum_m E_{mn} \cos \alpha x, y - \right. \]
\[-n^2 \alpha^2 \sinh \frac{m \pi}{2} \left( \sum_m 2E_{m} \cos \alpha y + n \chi \sum_m F_{m} \cos \alpha y \right) \] 
\[\cos \frac{m \pi}{2} - \]

\[-2e n \chi \sum_m m^2 \alpha^2 \left( \sinh \frac{m \pi}{2} \cosh \frac{m \pi}{2} + \frac{m \pi}{2} \right) \cos \alpha y - \]

\[-2e 4 \alpha^2 E \Psi, \cos 2 \alpha y \] \hspace{1cm} (4.3.4)

For a given m

\[\frac{H_4}{\hbar} = -2e 4 \alpha^2 E \cos \frac{m \pi}{2} + \frac{\Phi}{2 D} \left( \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) \]

\[+ \frac{m \pi}{2 \cosh \frac{m \pi}{2}} \sum_m \frac{2 \alpha^2}{D \pi} \cosh \frac{m \pi}{2} \cos \alpha y \cos \frac{m \pi}{2} \] \hspace{1cm} (4.3.5)

By analogy, for y direction

\[\frac{H_4}{\hbar} = -2e 4 \alpha^2 E \sin \frac{m \pi}{2} + \frac{\Phi}{2 D} \left( \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) \]

\[+ \frac{m \pi}{2 \cosh \frac{m \pi}{2}} \sum_m \frac{2 \alpha^2}{D \pi} \cosh \frac{m \pi}{2} \cos \alpha y \sin \frac{m \pi}{2} \] \hspace{1cm} (4.3.6)

Where \[\overline{\Phi} = 0 \quad \overline{\Phi} = -2e E \overline{\omega} \overline{G}^2 \]

For the first four terms:

\[H_4 = -2e \alpha^2 E \cosh \frac{m \pi}{2} + \frac{\Phi}{2 D} \left( \sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right) = \]

\[= -2e \overline{E} \overline{h} \left( 0.0316 \overline{\omega}^2 - 0.1390 \overline{\omega} \overline{G} \right) \] \hspace{1cm} (4.3.7)
\[ H_2 = \frac{\hbar \phi_2}{8D \cosh \pi} + \frac{\hbar \phi_2 C_2 \sinh^2 \pi}{2D \pi \cosh \pi} + \frac{\hbar \phi_2 C_2 (\sinh \pi + \frac{\pi}{3 \cosh \pi})}{2D} = \]
\[ = \frac{3\phi_2 \hbar E}{D} \left(-0.0015 \bar{w}_0^2 + 0.0019 \bar{w}_0 \gamma_0 \right) \quad (4.3.8) \]

\[ H_3 = \frac{4\phi_2 C_0 \hbar}{27D \pi \cosh \frac{3\pi}{4} \pi} = \frac{3\phi_2 E \hbar \bar{w}_0^2}{D} - 0.000021 \quad (4.3.9) \]

\[ H_4 = -\frac{\hbar C_2 \sinh^2 \pi}{2D \pi \cosh 2\pi} + \frac{\hbar C_2 \sinh^2 \pi}{2D \pi \cosh 2\pi} = \]
\[ = \frac{3\phi_2 E \hbar \bar{w}_0}{D} - 0.000003 \quad (4.3.10) \]

The correction \( \omega_{2\varepsilon} \) is given by:

\[ \nabla^2 \omega_{2\varepsilon} = 0 \quad (4.3.11) \]

\[ y = \pm \frac{a}{2}; \quad \omega_{2\varepsilon} = -\omega_{2\varepsilon} = \sum_{s} H_s (6 \alpha, x \sinh \alpha x - \frac{s \pi}{2} \tanh \frac{s \pi}{2} \cosh \alpha x) \cos \frac{s \pi}{2} \]
\[ M_y = -D \omega_{2\varepsilon, y} = 0 \quad (4.3.12) \]
\[ (4.3.13) \]

By analogy, for \( x = \pm \frac{a}{2} \) \( x \) and \( y \) are interchanged.

The solution of eq. (4.3.11) can be represented by:

\[ \omega_{2\varepsilon} = \sum_n \left( A_n \cos n \alpha, y + B_n \sin n \alpha, y \sinh \alpha x \right) \cos n \alpha, x + \]
\[ + \sum_m \left( A_m^\prime \cos m \alpha, x + B_m^\prime \sin m \alpha, x \sinh m \alpha, x \right) \cos m \alpha, y \quad n, m = 1, 3, 5 \ldots (4.3.14) \]

From symmetry \( A_n = A_m^\prime \quad B_n = B_m^\prime \)

Use the Fourier expansion in order to solve for the
coefficients

\[ \cosh s\alpha, x = \sum_n G_n \cos n\alpha, x \]
\[ G_n = \frac{4n}{\alpha n} \frac{\sin \frac{n\pi}{2}}{n^2 + s^2} \cosh \frac{s\pi}{2} \quad n = 1, 3, 5, \ldots \]

\[ x \sinh s\alpha, x = \sum_n K_n \cos n\alpha, y \]
\[ K_n = \frac{4\sin \frac{n\pi}{2}}{\alpha n^2 (n^2 + s^2)} \left( \frac{n\pi}{2} \sinh \frac{s\pi}{2} - \frac{2ns}{n^2 + s^2} \cosh \frac{s\pi}{2} \right) \quad n = 1, 3, 5, \ldots \]

get

\[ A_n \cosh \frac{n\pi}{2} + B_n \frac{n\pi}{2} \tanh \frac{n\pi}{2} = \sum_h \sum_s \frac{4s^2 n}{\pi (n^2 + s^2)} \sin \frac{n\pi}{2} \cosh \frac{s\pi}{2} \cos \frac{s\pi}{2} \quad (4.3.15) \]

\[ A_n \cosh \frac{n\pi}{2} + B_n \left( \frac{n\pi}{2} \sinh \frac{n\pi}{2} + 2 \cosh \frac{n\pi}{2} \right) = 0 \quad (4.3.16) \]

\[ B_n = - \sum_s \sum_h \frac{4s^2 n}{\pi (n^2 + s^2)} \cosh \frac{s\pi}{2} \left( \sin \frac{n\pi}{2} \cos \frac{s\pi}{2} \right) \quad (4.3.17) \]

\[ A_n = -B_n \left( 2 - \frac{n\pi}{2} \tanh \frac{n\pi}{2} \right) \quad (4.3.18) \]

In the limits of the accuracy one gets

\[ B_1 = 0.94H_2 + 7.50H_4 \quad (4.3.19) \]
\[ B_3 = -0.05H_2 - 1.28H_4 \quad (4.3.20) \]

The additional moment for the points \( x = y \) is given by:

\[ |x| = |y| \quad (M_x)_{2\varepsilon} = -D \sum_n 2n^2 \alpha^2 B_n \cosh n\alpha, x \cos n\alpha, x \quad (4.3.21) \]
Use Galerkin method for the solution of the second governing differential equation.

\[
\int_0^{\alpha_0} \int_0^{\beta_0} L(\omega, \bar{F}) \delta \omega_{mn} \, dx \, dy = 0
\]

(4.3.22)

\[
L(\omega, \bar{F}) = D \nabla^4 \omega + \bar{h}
\begin{pmatrix}
F_{yy} \bar{\omega}_{xx} + F_{xx} \bar{\omega}_{yy} - 2F_{xy} \bar{\omega}_{xy}
\end{pmatrix} - \rho
\]

(4.3.23)

Where \( \bar{F} \) is given by eq. (4.1.1 + 4), (4.3.14) and \( \bar{\omega} \)
is given by (4.2.2).

Therefore, taking for \( \omega \), the first term only

\[
\omega = \omega_0 \cos \alpha \times \cos \alpha y
\]

(4.3.24)

\[
\bar{\omega} = \omega_0 + (\delta \omega_0)_{x = \eta = 0}
\]

(4.3.25)

\[
\delta \omega = \delta \omega_0 \cos \alpha \times \cos \alpha y
\]

since \( \nabla^4 \omega = 0 \), get from eq. (4.3.22)

\[
\int_0^{\alpha_0} \int_0^{\beta_0} \left[ D \nabla^4 \omega + \bar{h}
\begin{pmatrix}
F_{yy} \bar{\omega}_{xx} + F_{xx} \bar{\omega}_{yy} - 2F_{xy} \bar{\omega}_{xy}
\end{pmatrix} - \rho \right] \cos \alpha \times \cos \alpha y \delta \omega_0 \, dx \, dy = 0
\]

(4.3.26)

\[
\int_0^{\alpha_0} \int_0^{\beta_0} \left\{ -4D \omega_0 \alpha_0^4 \cos \alpha \times \cos \alpha y + \bar{h}
\begin{pmatrix}
2c_0 + \sum_n \left[ (\sinh \frac{nr \pi}{2} + \frac{nr \pi}{2} \cosh \frac{nr \pi}{2}) \right] \alpha_0^2 \cosh \alpha y
\end{pmatrix} - \right.
\]

\[
\left. - \sum_m \left[ \cosh \alpha_0 x \alpha_0^2 \cosh \alpha_0 x \right] \sinh \frac{nr \pi}{2} \cosh \alpha_0 x \right)\alpha_0^2 \cosh \alpha y
\]

\[
- \sum_m \left[ \cosh \alpha_0 x \alpha_0^2 \cosh \alpha_0 x \right] \sinh \frac{nr \pi}{2} \cosh \alpha_0 x \right)\alpha_0^2 \cosh \alpha y
\]

\[
- \sum_m \left[ \cosh \alpha_0 x \alpha_0^2 \cosh \alpha_0 x \right] \sinh \frac{nr \pi}{2} \cosh \alpha_0 x \right)\alpha_0^2 \cosh \alpha y
\]
\[
- m \alpha, \sinh \frac{m \pi}{2} x \sinh \alpha, x \int \cos \alpha, y \wedge \bar{\omega}, \alpha, \cos \alpha, x \cos \alpha, y +
+ \hbar \left\{ 2 c i - \sum_n c_n n^2 a_i \left[ (\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2}) \cosh \alpha, y \right] -
- m \alpha, \sinh \frac{m \pi}{2} y \sinh \alpha, y \right\} \cos \alpha, x +
+ \sum_n c_n^{\dagger} \left[ (\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2}) m^2 \alpha, \cosh \alpha, x \right] -
- m \sinh \frac{m \pi}{2} \left( 2 \alpha, x \cosh \alpha, x + m^2 \alpha, x \sinh \alpha, x \right) \int \cos \alpha, y \wedge \bar{\omega}, \alpha, \cos \alpha, x \cos \alpha, y +
+ 2 \hbar \left\{ - \sum_n \alpha, c_n \left[ (\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2}) \alpha, \sinh \alpha, y \right] \sin \alpha, x -
- \sum_n \alpha, c_n^{\dagger} \left[ (\sinh \frac{m \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2}) \alpha, \sinh \alpha, x \right] - \alpha, \sinh \frac{m \pi}{2} \sinh \alpha, x -
- m^2 \alpha, x \sinh \frac{m \pi}{2} \cosh \alpha, x \int \sin \alpha, y \wedge \bar{\omega}, \alpha, \sin \alpha, x \sin \alpha, y -
- \alpha, \sinh \frac{m \pi}{2} y \sinh \alpha, y \right\} \cos \alpha, x \cos \alpha, y -
- \sum_q \gamma, q, \cos \gamma, x \cos \gamma, y + \sum_p \gamma, p, \cos \gamma, x \cos \gamma, y -
- \int \cos \alpha, x \cos \alpha, y d x d y = 0
\]

After integration, one gets finally:
\[ \frac{Dn^2}{4h} - \frac{h \alpha^4}{\omega_0 \pi^4} + \frac{\bar{w}}{\omega_0} \left( C_0 \frac{\alpha^4}{4} - C_1 \frac{88}{75} \sinh^2 \frac{\pi}{2} - C_2 \frac{\pi^4}{4} \sinh^2 \pi - \frac{E \pi^2 \psi_0}{4} \right) = 0 \]

\[ \frac{\pi \alpha^4}{E \pi \pi^4} = \frac{I}{4h} + 0.0234 \bar{w}_o^3 - 0.0638 \bar{w}_o^2 \bar{z}_o \] (4.3.27)

From eq. (4.3.25), for the center of the slab

\[ \bar{w}_o = w_0 - \sum \gamma_n n \sin \theta \tanh \frac{h \pi}{2} + \sum A_n \left( 2 - \frac{n \pi}{2} \tanh \frac{n \pi}{2} \right) \] (4.3.28)

\[ \bar{w}_o = \bar{w}_o + \frac{3 \alpha^2 h}{4} \left( 0.4128 \bar{w}_o \bar{z}_o^2 - 0.0825 \bar{w}_o^2 \right) \] (4.3.29)

Finally, the load deflection relations are given by the non-dimensional equations:

\[ \frac{\pi \alpha^4}{h^4 E} = \pi^6 \left[ \frac{I}{4 \pi h^3} \frac{\bar{w}_o}{h} + 0.0234 \left( \frac{\bar{w}_o}{h} \right)^3 - \frac{3 \alpha^2}{h} \cdot 0.0638 \left( \frac{\bar{w}_o}{h} \right)^2 \right] \] (4.3.30)

\[ \frac{\bar{w}_o}{h} = \frac{\bar{w}_o}{h} + \frac{3 \alpha^2 h}{4} \left[ \frac{\bar{w}_o}{h} \frac{2 \alpha}{h} 0.4128 - 0.0825 \left( \frac{\bar{w}_o}{h} \right)^2 \right] \] (4.3.31)
4.4 NUMERICAL SOLUTION

(1) The properties of the cross-section, Fig. (4.4.1), are given for $a_i = h = \frac{3}{4}$

![Diagram of cross-section with dimensions labeled]

**Fig. 4.4.1: Dimensions of cross section.**

\[ d = 0.75 d_i \]
\[ r = \frac{A_s}{d} \]
\[ \bar{h} = h + \left( \frac{E_s}{E_c} - 1 \right) A_s = h \left[ 1 + 0.75 (t-r) \right] \] \hspace{1cm} (4.4.1)
\[ \beta_i = \frac{\frac{1}{2} d_i^2 + \left( \frac{E_s}{E_c} - 1 \right) 0.25 d_i A_s}{d_i + \left( \frac{E_s}{E_c} - 1 \right) A_s} = \frac{0.5 + 0.188 (t-r)}{1 + 0.75 (t-r)} \] \hspace{1cm} (4.4.2)
\[ \beta_o = \beta_i - 0.25 h \]
\[ I = \frac{1}{3} \bar{h}^3 - \bar{h} \beta^2_i + A_s \left( \frac{E_s}{E_c} - 1 \right) (\beta_i - 0.25 h)^2 = \]
\[ = \bar{h}^3 \left\{ \frac{1}{3} - \left[ \frac{0.5 + 0.188 (t-r)}{1 + 0.75 (t-r)} \right]^2 + 0.75 (t-r) \left[ \frac{0.5 + 0.188 (t-r)}{1 + 0.75 (t-r)} - 0.25 \right] \right\} \] \hspace{1cm} (4.4.3)

Following the stress path, when the state of stress is such that the zone $\beta_i - \beta_o$ is cracked, the effective depth used for the stiffness when moments are calculated is $\bar{d}$. For axial compression, the total cross section is available, and effective depth $\bar{h}$ is used.

The numerical solution is compared with test results for
two reinforcement ratios: \( r=1\% \) and \( r=3\% \). The actual stress path is calculated for points along the diagonal \( x=y \) of the square slab, since the values are used later for total equilibrium conditions, in order to calculate the force deflection relations in the elasto-plastic mode.

(2) The moment-axial force interaction curve.

The stress path that defines the moment-axial load history of a given point gives an insight into the elasto-plastic behavior, as was shown earlier.

For \( x=0, y=0 \), the moment as a function of the displacement (eq. 4.1.1) is given by:

\[
M_x = D \omega_x \frac{q_{35}}{\alpha_z} \left( -D \sum_m (H_n + B_m) a_n \alpha_i \frac{m \gamma}{2} \tanh \frac{m \pi}{2} + 
\right.

\[
- D \sum_m (H_n + B_m) a_n \alpha_i \left( 2 - \frac{m \gamma}{2} \tanh \frac{m \pi}{2} \right) =
\]

\[
= D \omega_x \frac{q_{35}}{\alpha_z} - 2D \sum_m (H_n + B_m) a_n \alpha_i \] 

\[
= \alpha_z \left[ 0.18 l \omega_x - \beta_h \left( 0.3055 \ t_x - 0.0375 \ t_x \right) \right] \quad (4.4.4)
\]

Where \( \omega_x \) is given by (4.3.3).

The axial compression

\[
P_x = \bar{h} f_{yy} =
\]

\[
= \bar{h} \left\{ 2c + \sum_n \sum_m \left[ \sin h \frac{m \pi}{2} \cosh n \frac{m \pi}{2} \right] a_n \alpha_i \frac{m \gamma}{2} - 2h^2 a_n^2 \sinh \frac{m \pi}{2} \right\} \]
\[- \sum_{m} C_{m} (\sinh \frac{m \pi r}{2} + \frac{m \pi}{2} \cosh \frac{m \pi r}{2}) \alpha_{i}^{2} - 4 \alpha_{i}^{2} E \nu_{i} \{ \}
\]

\[= \bar{h} \alpha_{i}^{2} \frac{\bar{\omega}_{0}^{2}}{4} - \bar{h} \sum_{m} C_{m} 2m^{2} \alpha_{i}^{2} \sinh \frac{m \pi r}{2} = \bar{h} \alpha_{i}^{2} E \left( 0.2840 \bar{\omega}_{0}^{2} - 0.4684 \bar{\omega}_{0} \nu_{0} \right) \] (4.4.5)

For \( x = y = \frac{a}{4} \)

\[(M)_{x=y=} \frac{\alpha_{i}^{2}}{4} = D (\frac{\bar{\omega}_{0}}{\alpha_{i}^{2}}) - D \sum_{m} H_{m} 2m^{2} \alpha_{i}^{2} \cosh \frac{m \pi r}{2} \cos \frac{m \pi r}{4} - D \sum_{m} 2m^{2} \alpha_{i}^{2} \cosh \frac{m \pi r}{4} \cos \frac{m \pi r}{4} =
\]

\[= E \alpha_{i}^{2} \left[ \frac{\sinh \frac{\pi \nu_{0} \alpha_{i}^{2}}{2}}{\alpha_{i}^{2}} I \nu_{0} - \bar{\nu}_{0} \nu_{0} \right] \left( 0.2617 \bar{\omega}_{0} \nu_{0} - \bar{\omega}_{0}^{2} 0.0581 \right) \] (4.4.6)

\[(P)_{x=y=} \frac{\alpha_{i}^{2}}{4} = \bar{h} \left( 2c_{\nu_{0}} \sum_{m} C_{m} 2m^{2} \alpha_{i}^{2} \sinh \frac{m \pi r}{2} \cosh \frac{m \pi r}{4} \cos \frac{m \pi r}{4} \right) =
\]

\[= \bar{h} \alpha_{i}^{2} E \left( \bar{\omega}_{0}^{2} 0.1250 - 0.4187 \bar{\omega}_{0} \nu_{0} \right) \] (4.4.7)

The moment with respect to the reinforcement axis

\[(M_{x=1})_{x} = M_{x} + P \nu_{0} \]

If the properties of the cross-section are given, the dimensionless stress path is calculated. The important
parameters are summarized in table (4.4.1).

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>(\frac{3}{4}'')</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>0.562''</td>
</tr>
<tr>
<td>$f_c''(\text{ksi})$</td>
<td>4350</td>
</tr>
<tr>
<td>$f_y''(\text{ksi})$</td>
<td>60000</td>
</tr>
<tr>
<td>$r$</td>
<td>0.01</td>
</tr>
<tr>
<td>$t$</td>
<td>0.138</td>
</tr>
<tr>
<td>$\frac{h}{h}$</td>
<td>1.100</td>
</tr>
<tr>
<td>$\frac{b_0}{h}$</td>
<td>0.228</td>
</tr>
<tr>
<td>$d^3/h^3 (h'/t)$</td>
<td>7.35 (9.65)</td>
</tr>
<tr>
<td>$\left(\frac{M_{1-x}}{M_o}\right)_{x=y=0}$</td>
<td>$17.8 \bar{w}_o - 22.6 \bar{w}_o^2$ (4.4.8)</td>
</tr>
<tr>
<td>$\left(\frac{P}{T_o}\right)_{x=y=0}$</td>
<td>$70.0 \bar{w}_o^2 - 19.7 \bar{w}_o$ (4.4.9)</td>
</tr>
<tr>
<td>$\left(\frac{M_{1-x}}{M_o}\right)_{x=y=\frac{a}{4}}$</td>
<td>$11.1 \bar{w}_o - 8.1 \bar{w}_o^2$ (4.4.10)</td>
</tr>
<tr>
<td>$\left(\frac{P}{T_o}\right)_{x=y=\frac{a}{4}}$</td>
<td>$30.9 \bar{w}_o^2 - 17.7 \bar{w}_o$ (4.4.11)</td>
</tr>
<tr>
<td>$\bar{w}_o (\text{in.})$</td>
<td>$1.228 \bar{w}_o - 0.268 \bar{w}_o^2$ (4.4.16)</td>
</tr>
</tbody>
</table>

**TABLE 4.4.1**: PARAMETERS AND EQUATIONS FOR ELASTIC SOLUTION, $d_1 = \frac{3}{4}''$
The stress path of the points \((0,0)\) and \(\left( \pm \frac{a}{4}, \pm \frac{a}{4} \right)\) is given in Fig. (4.4.2) for \(r=0.01\) and Fig. (4.4.3) for \(r=0.03\). Plastic conditions start in both cases when \(\bar{w} \approx 0.15\). With increasing deflection the stress path \((0,0)\) moves along the yield function. The location of the path can be determined by the axial compression (eq. 4.4.5). When ultimate load \(P_{ul}\) is reached, the plastic zone extends to \(x = y = \pm \frac{a}{4}\). This analysis of the stress path appears to be useful for the elasto-plastic solution (section 5). The change in moments and axial forces once plastic zone has developed, is neglected.

(3) Axial compression vs. load.

The load-deflection curve is given by eq. (4.4.18) from eq. (4.3.30) and (4.3.31). This relation is valid for elastic deflections only \(\varepsilon < 1\), i.e. for \(0 < \bar{w} \leq \bar{w}_{pl}\), when \(\bar{w}_{pl}\) is the deflection at the center of the slab at the start of plastic conditions there.

For the range \(\bar{w}_{pl} < \bar{w} \leq \bar{w}_{max}\), elasto-plastic solution to account for the plastic zone is used as a correction \(\Delta p\) for the elastic equation (4.4.18)

\[
\bar{p} = \frac{E}{2} \left[ \frac{w}{a^4} + \frac{3r}{I} \left( 0.4128 \bar{w} \bar{z}_o - 0.0825 \bar{w}^2 \right) \bar{z}_o \right] + \frac{1}{\bar{z}_o} w \bar{w}^3 \bar{z}_o 0.0234 - \bar{d} \bar{w} \bar{z}_o 0.0638 \right]
\]

The elasto-plastic correction \(\Delta p\) is given in section 5.

The average axial compression along the boundary \(\mathbb{F}\)
is given as a function of the central deflection \( \bar{w}_z \) by eq. (4.2.23), for the interval \( 0 < \bar{w}_z < (\bar{w}_z)_{F_{\text{max}}} \).

The elasto-plastic change of shape toward the conical configuration is neglected for the computation of axial compression.

For the interval \( \bar{w}_z > (\bar{w}_z)_{F_{\text{max}}} \), the results of the modified-plastic solutions are used, where the load is given by eq. (3.3.9) and the restraining force as a function of the central deflection is given by eq. (3.3.8).

For the cases studied ( \( r=1\% \) and \( r=3\% \)), the important parameters for the load-restraining force relations are given in table (4.4.2). Interaction load-restraining force curves are given for the cases studied in Fig. (5.2.2).
<table>
<thead>
<tr>
<th>$r$</th>
<th>$0.01$</th>
<th>$0.03$</th>
</tr>
</thead>
</table>
| $0 \leq \bar{\omega} \leq (\bar{\omega}_o)_{pl}$ | \[
\frac{\pi}{a} = 310 \bar{\omega}_o - 460 \bar{\omega}_o^2 + 840 \bar{\omega}_o^3
\] (4.4.21) | \[
\frac{\pi}{a} = 415 \bar{\omega}_o - 484 \bar{\omega}_o^2 + 973 \bar{\omega}_o^3
\] (4.4.27) |
| $0 \leq \bar{\omega}_o \leq (\bar{\omega}_o)_{pl}$ | $\Delta \bar{\omega}$ (ch. 5) | $\Delta \bar{\omega}$ (ch. 5) |
| $0 \leq \bar{\omega}_o \leq (\bar{\omega}_o)_{pl}$ | \[
\frac{\pi}{a} = 104.00 \bar{\omega}_o^2 - 6250 \bar{\omega}_o
\] (4.4.22) | \[
\frac{\pi}{a} = 12200 \bar{\omega}_o^2 - 6250 \bar{\omega}_o
\] (4.4.28) |
| $\bar{\omega}_o > (\bar{\omega}_o)_{pl}$ modified | \[
\frac{\pi}{a} = 2080 \bar{\omega}_o - 1070 \bar{\omega}_o^2 - \frac{\bar{\omega}_o}{\bar{\omega}_o + 0.125}
\] (4.4.23) | \[
\frac{\pi}{a} = 1435 \bar{\omega}_o - \frac{\bar{\omega}_o}{\bar{\omega}_o + 0.125} - 1360 \bar{\omega}_o^2 - \frac{\bar{\omega}_o}{\bar{\omega}_o + 0.125}
\] (4.4.29) |
| $\bar{\omega}_o > (\bar{\omega}_o)_{pl}$ rigid-plastic | $\bar{\omega}_o = 2950 - 1190 \bar{\omega}_o$ (4.4.25) | $\bar{\omega}_o = 1810 - 1160 \bar{\omega}_o$ (4.4.31) |

**NOTE:** FORCES IN LBS, LENGTH IN INS.

**TABLE 4.4.2:** LOAD-DEFORMATION-RESTRAINING FORCE RELATIONS FOR $d_1 = \frac{3}{4}''$. 
FIG. 4.4.2: INTERACTION CURVE FOR $\theta_1=34^\circ$; $r=1\%$. 

STRESS PATH OF: 
1. $x=y=0$ 
2. $x=y=\frac{a}{4}$ 

LINEAR THEORY 

MECHANISM
FIG. 4.43: INTERACTION CURVE FOR $d_1 = \frac{3}{4}\text{"}, r = 3\%$. 

STRESS PATH OF: 

1. $x = y = 0$ ——— Δ
2. $x = y = \frac{a}{4}$ ——— ○

(A01)
CHAPTER 5

"SPREADING MECHANISM" MODE (ELASTO-PLASTIC SOLUTION)

5.1 SOLUTION AS A CONTINUATION OF THE ELASTIC STAGE

If the stress path of the points along the diagonal x=y is considered, the initial plastic conditions are achieved when the stress path of the center of the slab meets the yield line (Fig. 4.4.2). With additional load, the plastic zone spreads, and x₀ is increased (Fig. 5.1.1). The plastic zone is a function of the force-moment distribution in the slab. Any discontinuity at the boundary of the elastic and plastic regions is a result of the discontinuity assumed for the stress strain relations (Fig. 2.2.3). However, the existence of boundary layers will not influence the macro-equilibrium conditions for the total regions.

The exact solution of the problem is too involved to be tried here. It should consist of an elastic solution for square slabs with a central hole as a function of the stress resultants applied to the edge of the hole. The hole is occupied by a plastic region that satisfies the additive elastic-perfectly plastic strain law, the equilibrium equations and the plastic stress resultants on the boundary. The compatibility of stress along the boundary of the region leads to a solution. However, such a solution is possible for cases with rotational symmetry only. In order to estimate the reduction in loading capacity due to the plastic condition, an
FIG. 5.11: ELASTO-PLASTIC CONDITIONS FOR A SQUARE SLAB.

FIG. 5.12: "STEPED" YIELD LINES IN THE PLASTIC ZONE.
approximate solution is used. For a given elastic deflection \( \bar{\omega} \), the axial forces and moments can be calculated. The axial force is a function of the deflected shape and the modulus of elasticity. If plastic conditions are developed, and additional changes of shape are small with respect to the elastic shape, the axial force is assumed to be unchanged. The error introduced is also proportional to changes in \( E \), that are small since the entire cracked cross section resists the axial compression. The moment that corresponds to the deflection \( \bar{\omega} \) is calculated for the elastic conditions. However, only a part of this moment can be supported by the axial force, while the other part is "dissipated" through plastic rotation, that leads to additional deflection (Fig. 5.1.1) and therefore to additional change of stress resultants \( (\Delta \rho)_p \) and \( (\Delta M)_p \). Thus, if the acting distributed load was initially \( \rho_0 \), a reduction in the load \( \Delta \rho \) results.

For the boundary of the plastic zone, one gets

\[
q_i = q_e - \Delta q_p. \tag{5.1.1}
\]

\[
Q_i = Q_e - \Delta Q_p. \tag{5.1.2}
\]

where \( q \) and \( Q \) represent all the components of displacement and forces, respectively.

Compatibility conditions between the elastic and the plastic regions, require that along the boundary, stress and strain components are continuous.
Note:

\[ \mu = \mu_0 - \Delta \rho \]

If \( \omega \) is the deflection of the elastic region, then the governing equilibrium equation for the elastic region is given by:

\[ L(\omega, F, \mu) = 0 \]  \hspace{1cm} (5.1.3)

where \( L(\ ) \) is given by eq. (4.3.23) and the boundary conditions are given by (5.1.1 + 2).

Solving the equations simultaneously for the elastic and the plastic regions, the final load and shape are obtained.

The approximation here is based on the following procedure:

1. A kinematic field for the additional plastic strains is chosen so that the additional displacement field will be compatible with the shape of the elastic region.

2. The plastic deformation \( (\Delta \omega)_p \) is calculated by considering the elastic energy "dissipated" by the yield lines pattern, that satisfy the kinematic field.

3. The displacement boundary conditions between the regions (eq. 5.1.1) are assumed to be satisfied by a proper choice of the kinematic field, while eq. (5.1.2) is determined from total equilibrium of the plastic region, as a function of \( \Delta \rho \).

4. Using eq. (5.1.3), (5.1.2), \( \Delta \rho \) is determined. The final deflection is given by:
\[ \omega(0,0) = \omega_i(x_o, y_o) + \left[ \bar{\omega}_0 - \bar{\omega}_e(x_o, y_o) \right] + (\Delta \bar{\omega}_e)_{pl}. \quad (5.1.4) \]

By this procedure one obtains a final deflection field which is compatible on the average to the final stress field and is close to the true solution.

Initially, \( x_o \) is determined. For \( x = y \), \( 0 \leq x \leq \frac{a}{2} \) for \( f = 1 \) one gets

\[ P = \bar{F}_{yy} = 2c_o h - 2h \sum \frac{n^2 \alpha^2 \sinh \frac{nh}{2} \cosh n\alpha x}{n} \cos \alpha x, x = 4h \alpha^2 E \varepsilon_0 \cos 2\alpha x \quad (5.1.5) \]

For \( x = y \), \( x_o < x < \frac{a}{2} \) and \( f < 1 \)

\[ (M_{z-1})_e = D \alpha^2 \omega_o \cos^2 \alpha x - D \sum \frac{h^2 \alpha^2 \cosh n\alpha x}{h} \alpha^2 \cos \alpha x + P^2 \frac{P}{P_o}. \quad (5.1.6) \]

For \( 0 \leq x \leq x_o \) and \( f = 1 \)

\[ \frac{(M_{z-1})}{(M_o)_{pl.}} = 1 + \alpha \frac{P}{P_o} - (1 - \alpha) \left( \frac{P}{P_o} \right)^2 \quad (5.1.7) \]

where \( P \) is substituted from eq. (5.1.5).

The set of equations (5.1.5 \& 7) yield a non-linear equation for \( x_o \) and \( \bar{\omega}_e \).

Next, the plastic deformation \( (\Delta \bar{\omega}_e)_{pl.} \) is calculated. The difference in moments is dissipated by transfer of the corresponding elastic energy into a plastic rotation along yield lines that are assumed to develop.
The "excess" of elastic energy is given by

\[ U_e = \int \left[ \frac{1}{E A} \left( (M_{r-I})_{c}^2 - (M_{r-I})_{pl}^2 \right) \right] dA \tag{5.1.8} \]

where \((A)\) is the area of the plastic region, and \((*)\) denotes maximum principal moments. For simplicity, diagonal yield lines only are assumed (Fig. 5.1.2). Thus, continuity and restraints along AB are maintained, while "release" of excess principal moments is possible. For further simplification, rotation along the yield lines is assumed to be rigid plastic for the region AOB. The plastic work done by rotation along "stepped" diagonal yield lines limited by \((\bar{x}_p, \bar{y}_p), \bar{x} = \bar{y}\) (Fig. 5.1.2) is given by

\[ U_{pl} = \frac{\Delta \bar{y}_p}{\bar{x}_p} 8 \int_{0}^{\bar{x}_p} \left[ (M_{r-I})_{c} + (M_{r-I})_{pl} \right] dX \tag{5.1.9} \]

From \( U_e = U_{pl}, \ (\Delta \bar{y}_p)_{pl} \) can be calculated.

Next, neglecting the change in axial force \((\Delta P)_{pl} \), and using moment equilibrium along ABO, one finds

\[ \int_{0}^{\bar{x}_p} (M_{r-I})_y (\bar{y}_p, X, Y) dX = \int_{0}^{\bar{x}_p} (M_{r-I})_y (\bar{y}_p, X, Y) dX + \frac{\Delta P X_0^3}{E} - \int_{0}^{\bar{x}_p} \left[ M_{r-I} (\bar{y}_p, \bar{x} = \bar{y}) - (M_{r-I})_{pl} \right] dX \tag{5.1.10} \]

Further approximation is done by using the equations developed for the square slab in order to solve eq. (5.1.3).
The load $\Delta_f$ is found by iteration, so that eq. (5.1.10) will be satisfied.

For numerical solution, eq. (5.1.8) is calculated if the yield region is bounded by $-x_e < x, y < x_e, y = y_e$. This yields for the dissipation of energy

$$U_e = 4 \int_{x_e}^{x_e} \int_{y_e}^{y_e} \frac{1}{E I} \left[ (M_t - M_{t-1})_e^2 - (M_{t-1})_{pl}^2 \right] dx dy = 2 St.*$$

where $St.*$ is the first moment of the maximum principal moment area divided by the stiffness with respect to the base of the moment diagram.

Numerical results for the cases studied when $x_e = \frac{a}{4}$ are given in table (5.1.1). Comparison between experimental data and analysis for load-deflection relations is shown in Fig. (5.2.1).

5.2 MODIFIED-PLASTIC SOLUTION AS A CONTINUATION OF ELASTO-PLASTIC SOLUTION.

The effect of the axial strain was taken into account by modification of the rigid-plastic solution, where the simplification $P_y = 0$ was suggested.

Instead of a modification of the rigid-plastic solution to account for the curvature strain of the "rigid" regions, the effect of the curvature and axial strains for $\overline{\delta}_e > (\overline{\delta}_e)_{P_{max}}$ are considered by introducing initial strains as initial conditions.
<table>
<thead>
<tr>
<th>( r )</th>
<th>0.01</th>
<th>0.03</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\omega}_0 )</td>
<td>0.48 d</td>
<td>0.47 d</td>
</tr>
<tr>
<td>( P(0,0) )</td>
<td>+0.2T₀</td>
<td>+0.15T₀</td>
</tr>
<tr>
<td>((m_{x-x})_{pl}(0,0))</td>
<td>0.81M₀</td>
<td>0.88M₀</td>
</tr>
<tr>
<td>((m_{x-x})_{el}(\bar{\omega}_0,0,0))</td>
<td>3.13M₀</td>
<td>1.76M₀</td>
</tr>
<tr>
<td>( x_0 )</td>
<td>0.25a</td>
<td>0.25a</td>
</tr>
<tr>
<td>((m_{x-x})_{el}(\bar{\omega}_0,x_0,y_0))</td>
<td>2.41M₀</td>
<td>1.19M₀</td>
</tr>
<tr>
<td>( \bar{\omega}_c (x_0,y_0) )</td>
<td>0.19 d</td>
<td>0.18 d</td>
</tr>
<tr>
<td>( \int_0^{x_0} \left[ (m_{x-x})<em>{el}(\bar{\omega}<em>0,x,y) - (m</em>{x-x})</em>{pl} \right] dx )</td>
<td>4.06M₀</td>
<td>2.98M₀</td>
</tr>
<tr>
<td>( \int_0^{x_0} (m_{x-x})_{el}(\bar{\omega}_0,x,y_0) dx )</td>
<td>9.75M₀</td>
<td>8.70M₀</td>
</tr>
<tr>
<td>( \int_0^{x_0} (m_{x-x})_{el}(\bar{\omega}_0,x,y_0) dx )</td>
<td>6.85M₀</td>
<td>6.27M₀</td>
</tr>
<tr>
<td>( \omega_1(x_0,y_0) )</td>
<td>0.11 d</td>
<td>0.13 d</td>
</tr>
<tr>
<td>( \Delta \rho_{el} (\rho_{si}) )</td>
<td>22</td>
<td>28</td>
</tr>
<tr>
<td>( \rho_{el} (\rho_{si}) )</td>
<td>66</td>
<td>96</td>
</tr>
<tr>
<td>( \rho_{pl} - \Delta \rho_{pl} (\rho_{si}) )</td>
<td>44</td>
<td>68</td>
</tr>
<tr>
<td>( U_{el} )</td>
<td>8( \Delta \bar{\omega}_0 M_0 )</td>
<td>5.28( \Delta \bar{\omega}_0 M_0 )</td>
</tr>
<tr>
<td>( U_{pl} )</td>
<td>( \frac{m^2 q^2}{EI} 0.835 )</td>
<td>( \frac{m^2 q^2}{EI} 0.186 )</td>
</tr>
<tr>
<td>( (\Delta \bar{\omega}<em>0)</em>{pl} )</td>
<td>0.13 d</td>
<td>0.06 d</td>
</tr>
<tr>
<td>( \phi(0,0) )</td>
<td>0.55 d</td>
<td>0.48 d</td>
</tr>
</tbody>
</table>

**TABLE 5.1.1**: ELASTO-PLASTIC NUMERICAL RESULTS FOR \( x = \frac{a}{4}, d_i = \frac{3}{4} \)
For deflections larger than the deflection at ultimate load, it can be assumed that \( \chi = r \frac{a}{2} \). This means that a full yield line is developed, and the additional change of curvature is concentrated at this line. On the other hand, the axial strain occurs at each point of the slab.

Thus, if the initial condition for a plastic solution is taken from the elasto-plastic solution at \( \bar{\omega}_0 = \left( \bar{\omega}_0 \right)_{P_{max}} \) and a continuation with the modified-plastic solution is used, the kinematic and stress fields are described continuously.

From eq. (3.3.8)

\[
\frac{n_P}{n_{o_{L}}} = \frac{P_{max}}{n_{o_{L}}} + \frac{\alpha}{2(1-\alpha)} \left[ \frac{\bar{\omega}_0}{\omega_0 + \mu} - \frac{\left( \bar{\omega}_0 \right)_{P_{max}}}{\left( \bar{\omega}_0 \right)_{P_{max}} + \mu} \right] - \frac{(2-\alpha)}{8d(1-\alpha)} \left[ \frac{\bar{\omega}_0^2}{\omega_0 + \mu} - \frac{\left( \bar{\omega}_0 \right)_{P_{max}}^2}{\left( \bar{\omega}_0 \right)_{P_{max}} + \mu} \right] (6.21)
\]

From eq. (3.3.9)

\[
\frac{n^2}{n_{o_{L}}^2} = \frac{n^2_{ust}}{n_{o_{L}}^2} + \frac{\alpha^2}{4(1-\alpha)} \left[ \frac{\bar{\omega}_0^2 + 2M\bar{\omega}_0}{\left( \bar{\omega}_0 + \mu \right)^2} - \frac{\left( \bar{\omega}_0 \right)_{P_{max}}^2 + 2M\left( \bar{\omega}_0 \right)_{P_{max}}}{\left( \bar{\omega}_0 \right)_{P_{max}} + \mu} \right] + \frac{\alpha(2-\alpha)}{8d(1-\alpha)} \left[ \frac{\bar{\omega}_0^2 + 2M\bar{\omega}_0}{d\left( \bar{\omega}_0 + \mu \right)^2} - \frac{\left( \bar{\omega}_0 \right)_{P_{max}}^2 + 2M\left( \bar{\omega}_0 \right)_{P_{max}}}{d\left( \bar{\omega}_0 \right)_{P_{max}} + \mu} \right] + \frac{(2-\alpha)^2}{48d(1-\alpha)} \left[ \frac{\bar{\omega}_0^2}{d\left( \bar{\omega}_0 + \mu \right)^2} - \frac{\bar{\omega}_0^2}{d\left( \bar{\omega}_0 + \mu \right)^2} \right]
\]
where \( P_{\text{max}} \) AND \( /2_{\text{ult}} \) are given by the elasto-plastic solution.

If the continuation of the elasto-plastic field is satisfied by the rigid-plastic solution (i.e. no change of axial strain of the rigid regions for \( \vec{\omega}_{e} > (\vec{\omega}_{e})_{P_{\text{max}}} \))

Note: \( \Delta \vec{\omega}_{e} = \vec{\omega}_{e} - (\vec{\omega}_{e})_{P_{\text{max}}} \)

\[
\frac{P}{\bar{P}_{0}} = \frac{P_{\text{max}}}{\bar{P}_{0}} - \frac{(2-\alpha)}{4(1-\alpha)} \frac{\Delta \vec{\omega}_{e}}{d} \left( 1 - \frac{x}{L} \right) \tag{5.2.3}
\]

\[
\frac{f_{L}}{f_{2L}} = \frac{f_{\text{ult}}}{f_{2L}} - \frac{\alpha(2-\alpha)}{4(1-\alpha)} \frac{\Delta \vec{\omega}_{e}}{d} + \frac{1}{16} \frac{(2-\alpha)^{2}}{(1-\alpha)} \frac{\Delta \vec{\omega}_{e}^{2} + \Delta \vec{\omega}_{e}(\vec{\omega}_{e})_{P_{\text{max}}}}{d^{2}} \tag{5.2.4}
\]

The load-deflection relations and the load-restraining force relations are given in Fig (5.2.1 + 2) for \( 0 \leq \vec{\omega}_{e} \leq d \).
FIG. 5.2.1: LOAD DEFLECTION RELATIONS. \( d_1 = \frac{3}{4}'' \)
Fig. 5.2.2: Load-Restraining Force Relations. \(d_1 = 3/4"\), \(\gamma = 1\%\).
6.1 YIELD CRITERIA FOR THE THICK SLAB

The general yield surface for a three dimensional state of stress contains all possible stress resultants (moments, membrane and shear forces), that effect the energy dissipation during the plastic deformation.

However, the analysis is simplified due to the fact that only principal directions are considered. For a R/C slab, the directions of the orthogonal reinforcement do not coincide with those of the principal directions of the stresses. Thus, it is assumed that \( m_{xy} \) AND \( P_{xy} \) are carried by the concrete without any effect on the uniaxial moment-curvature relations. More basic research should be done in order to find the effect of the mixed stress resultants on uniaxial energy dissipation.

In addition, it is assumed that conditions are set, and only such modes are considered in which tensile failure due to diagonal tension is not possible. Since along the yield line of a square slab, the shear force is zero, the shear force is assumed to be inactive as far as the yield criteria is considered. Kirchhoff's assumption is still valid along the yield line.

Thus, the three principal normal stresses are considered for the yield criteria. From section 2.2. one gets (Fig. 6.1.1).
FIG. 6.1.1: THREE DIMENSIONAL YIELD CRITERIA.
\[
\left[ \frac{f_{c_{\text{max}}}}{f_c''} \leq 1 + \eta \right] \max(f_{c_{\text{max}}}, f_{c_{\text{min}}})
\]  

(6.1.1)

where

\[
\eta = \frac{f_{c_{\text{min}}}}{f_c''}
\]

The moment and the membrane forces are used as coordinates of the yield surface with stress distribution as in Fig. (2.2.4), but \( f_{c_{3}} \) is some average vertical stress given by

(1) For axial load

\[
(f_{c_3})_{m_1} = \frac{r^2}{k} \int_{d/(1-k)} \left[ d - \frac{(d-\bar{x})^2}{2kd^2(1-\frac{1}{2}k)} \right] \, 3 \, 3 = \\
= \frac{1}{2} \left[ 1 - \frac{k}{h(1-\frac{1}{2}k)} \right] = m_1, \quad \frac{2}{3} \leq m_1 \leq 1
\]

(6.1.2a)

(2) For moments

\[
(f_{c_3})_{m_2} = \frac{r^2}{k} \int_{d/(1-k)} \left[ d - \frac{(d-\bar{x})^2}{2kd^2(1-\frac{1}{2}k)} \right] \, 3 \, \bar{y} = m_2, \quad \frac{5}{2} \leq m_2 \leq 1
\]

(6.1.2b)

In the case of a square slab, in order to get yielding along the diagonal, both perpendicular directions of reinforcement (\( x, y \)) are assumed to yield. Thus, for this case

\( f_{c_{3}} = f_{c_{\text{min}}} \).
\[ \eta_{m_i} = \frac{(f_{c3})_{m_i}}{f_c} \quad \eta_{m_2} = \frac{(f_{c3})_{m_2}}{f_c} \]

get

\[ M_{i-1} = \frac{1 + \eta_{m_2}}{1 + \eta_{m_1}} \left[ T_0 d \left( 1 - \frac{1}{2} t_i \right) + P d (1 - t_i) - \frac{p^2 t_i d}{2 T_0} \right] \]

Simplify \( \frac{1 + \eta_{m_2}}{1 + \eta_{m_1}} \approx 1 \) and note

\[ t_L = \frac{t}{1 + \eta_L} \]

\[ \eta_L = \frac{(f_{c3})_{L}}{f_c} \]

also note

\[ \eta_{m_1} = \eta = \frac{m_i / \beta}{f_c} \]

GET

\[ M_{i-1} = T_0 d \left( 1 - \frac{1}{2} t_i \right) + P d (1 - t_i) - \frac{p^2 t_i d}{2 T_0} \] (6.1.3)

\[ M_o = T_0 d \left( 1 - \frac{1}{2} t_L \right) \] (6.1.4)

\[ \frac{M_{i-1}}{M_o} = \gamma + \left( \frac{P}{T_0} \right) \beta + \left( \frac{P}{T_0} \right)^2 (\beta - \gamma^*) \] (6.1.5)

where

\[ \beta = \frac{1 + \eta_L - \frac{t}{2}}{1 + \eta_L - \frac{t}{2} t} \quad \frac{1 + \eta L}{1 + \eta} \]

\[ \gamma^* = \frac{1 + \eta_L - \frac{t}{2}}{1 + \eta_L - \frac{t}{2} t} \quad \frac{1 + \eta L}{1 + \eta} \]

when

\[ \frac{M_{i-1}}{M_o} \rightarrow 1 \quad \text{get} \quad \gamma^* \rightarrow 1 \].

Since the yield criteria is a function of the applied load, upper and lower bounds for \( \eta \) are established.
The lower bound is given by \( \eta = \eta_L \). Using eq. (3.2.6) and (6.1.4), get for a square slab

\[
f_c^2 t d^2 \left[1 - \frac{t}{2(1 + \eta_L)}\right] = \frac{t \eta_L t^2}{24}
\]

since

\[\left(\frac{f_{c3}}{f_{c4}}\right) \approx \eta_L\]

GET

\[
\eta_L = -\frac{1}{2} \left[1 - \frac{24 t d^2}{L^2}\right] ^\frac{1}{2} \left[\frac{1}{4} (1 - \frac{24 t d^2}{L^2})^2 + \frac{24 t d^2 (l - \frac{1}{2} t)}{L^2}\right]
\]

(6.1.6)

The upper bound \( \eta = \eta_m \) corresponds to the condition

\[
\eta = \eta_m , \quad \text{see section 2.3.}
\]

\[
\frac{\beta f}{\beta (1 + \beta)} = 0 ;
\]

\[
\frac{f}{L_0} = \frac{\beta}{2(1 - \beta)} ; \quad \frac{m_{r-1}}{m_0} = \gamma + \frac{\beta^2}{4(1 - \beta)}
\]

(6.1.7)

The corresponding values of \( \eta_m \) are given from the upper bound rigid-plastic solution (section 6.3) for the case \( \tilde{\omega}_0 = 0 \)

\[
\frac{f_{r-m}}{f_{r-L}} = \gamma + \frac{\beta^2}{4(1 - \beta)} = \frac{1 + \eta_L}{1 + \eta_L - \frac{1}{2} t} \frac{1 + \eta_m}{2 t}
\]

(6.1.8)

Due to the uncertainty associated with the shear distribution for the thick slab, the value \( \eta = \eta_m = 1 \) is suggested for simplicity, thus \( \eta_m = \frac{f_{r-m}}{f_{c3}} \) and \( \frac{f_{r-m}}{f_{r-L}} = \frac{\eta_m}{\eta_L} \).
The values for the cases studied are given in table (6.1.1).

<table>
<thead>
<tr>
<th>$d_L$</th>
<th>1½&quot;</th>
<th>3&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.01</td>
<td>0.03</td>
</tr>
<tr>
<td>$t$</td>
<td>0.154</td>
<td>0.462</td>
</tr>
<tr>
<td>$P_L$</td>
<td>0.021</td>
<td>0.057</td>
</tr>
<tr>
<td>$\eta_L$</td>
<td>0.920</td>
<td>0.720</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_m$</td>
<td>0.084</td>
<td>0.088</td>
</tr>
<tr>
<td>$P_m$</td>
<td>0.930</td>
<td>0.739</td>
</tr>
<tr>
<td>$\gamma_m$</td>
<td>1.005</td>
<td>1.010</td>
</tr>
</tbody>
</table>

**TABLE 6.1.1**: PARAMETERS FOR RIGID PLASTIC SOLUTION, $d_L=1\frac{1}{2}$", 3"

The compression failure for the thick slab is given by (see eq. (2.3.2) for comparison)

For $k < 1$

\[
\frac{m_{t-1}}{m_r} = \frac{(1 + \eta)0.85k(2 - 0.85k)}{(1 + \eta)L (2 - t_L)} = \\
= \left\{ \frac{1}{2} \frac{P}{t} - 1 + \left[ \frac{1}{4} \left( \frac{P}{t} \right)^2 - \frac{P}{t} + 1 \right] + \frac{1.7}{t} (1 + \eta) \right\} \left\{ \frac{1 + \eta}{1 + \eta - \frac{1}{2} t} \right\} - \\
- (n - 1) \left\{ \frac{1}{2} \frac{P}{t} - 1 + \left[ \frac{1}{4} \left( \frac{P}{t} \right)^2 - \frac{P}{t} + 1 \right] + \frac{1.7}{t} (1 + \eta) \right\}^{1/2} \right\} \right\} \right\} 
\] (6.1.10)
FOR \( K = 1 \)
\[
\frac{P}{I_c} = \frac{1 + \nu}{t} \quad \quad \quad \frac{M_{z,z}}{M_o} = \gamma + \frac{\beta^2}{\eta (\eta - \beta)} \quad (6.1.1)
\]

FOR \( K = 2 \)
\[
\frac{P}{I_c} = \left(1 + \frac{d}{a} \right) \frac{1 + \nu}{t} \quad \quad \quad \frac{M_{z,z}}{M_o} = 0.96 \left( \frac{1 + \nu}{1 + \nu} - \frac{1}{t} \right) - \frac{1 + \nu}{2t} \quad (6.1.2)
\]

The graphical representation of the yield criteria for the various cases studied is given, together with the stress path in the next section.

6.2 STRESS PATH FOR \( f < 1 \) (ELASTIC SOLUTION).

The elastic differential equation for bending including shear effect (Reissner) is given for the case when poisson ratio \( \nu = 0 \)
\[
\nabla^4 \psi = \frac{h^2}{E} \nabla^2 \psi \quad (6.2.1)
\]

where
\[
\frac{h^2}{E} = h + P_x \omega_{xx} + P_y \omega_{yy} + 2P_{xy} \omega_{xy}
\]
\[
\nabla^2 \psi - \frac{10}{h^2} \psi = 0 \quad (6.2.2)
\]

\( \psi \) = stress function.

Here a linear stress distribution is assumed for the inplane normal stresses, and a parabolic distribution for the vertical shear stresses. The error introduced by this assumption for the thick slab is negligible. The centroid coincides with the mid-surface of the slab. With the reinforcement ratios considered \(( \rho_{\text{max}} = 3\% \)) the centroid is sufficiently close to the mid-depth. \(( \frac{h}{h/2} > 0.8 \)).

Within the degree of accuracy used in the analysis, the additional effect of shear is calculated if the centroid is at mid-depth. The stiffness, however, is related to the centroid.
For a thick slab, the order of magnitude of the horizontal displacement at the restraint level due to rotation is much higher than that due to shortening because of non-linear geometry. Thus, the inplane forces are not coupled with the deflections, and "small deflection" equations are used.

\[ P_{x,x} + P_{y,y} = 0 \]  \hspace{1cm} (6.2.3)

\[ P_{x,y} + P_{y,x} = 0 \]  \hspace{1cm} (6.2.4)

If Poisson ratio \( \nu = 0 \), the stress resultants are given by

\[ M_x = -D \omega_{xx} + \frac{h^2}{5} V_{x,y} \]  \hspace{1cm} (6.2.5)

\[ M_y = -D \omega_{yy} + \frac{h^2}{5} V_{y,x} \]  \hspace{1cm} (6.2.6)

\[ M_{xy} = D \omega_{xy} - \frac{h^2}{10} \left( V_{x,y} + V_{y,x} \right) \]  \hspace{1cm} (6.2.7)

\[ V_x - \frac{h^2}{10} \nabla^2 V_x = -D (\nabla^2 \omega)_{xx} - \frac{h^2}{10} \bar{F}_{xx} \]  \hspace{1cm} (6.2.8)

\[ V_y - \frac{h^2}{10} \nabla^2 V_y = -D (\nabla^2 \omega)_{yy} - \frac{h^2}{10} \bar{F}_{yy} \]  \hspace{1cm} (6.2.9)

The boundary conditions are

\[ x = \pm \frac{a}{2} \rightarrow \omega = 0 \leftarrow \quad M_x = P_x \beta \leftarrow \quad M_{xy} = 0 \]  \hspace{1cm} (6.2.10)
For membrane forces, neglecting the shear effect,

\[ \frac{1}{E_h} \int_0^{a/2} P_x \, dx + \omega_2 \beta_n \gamma_n = 0 \]  

(6.2.11)

For \( y = \frac{a}{2} \) get symmetrical equations.

In order to solve for the stress resultants, the deflections are taken as

\[ \omega = \omega_1 + \omega_2 \]  

(6.2.12)

where

\[ \omega_1 = \sum \sum \omega_{mn} \cos \alpha x \cos \alpha y \]  

(6.2.13)

\[ \nabla^4 \omega_2 = 0 \; \] 

\[ \omega_2 = \sum_n H_n (n \alpha x \sinh \alpha y - \frac{n \pi}{2} \tanh \frac{n \pi}{2} \cosh \alpha x) \cos \alpha y + \] 

\[ + \sum_m H_m (m \alpha y \sinh m \alpha x - \frac{m \pi}{2} \tanh \frac{m \pi}{2} \cosh m \alpha y) \cos m \alpha x \]  

(6.2.14)

\( m, n = 1, 3, 5, \ldots \)

\( P \) is calculated from the first order approximation discussed in Chapter 4.

\[ \bar{\omega} \propto \bar{\omega}_0 \cos \alpha x \cos \alpha y \]

where

\[ \bar{\omega}_0 = (\omega_1)_{x=0} + (\omega_2)_{x=0} \]

using stress function \( F \)

\[ \nabla^4 F = 0 \; \]

\[ P_x = \hat{h} F_{yy} \quad P_y = \hat{h} F_{xx} \quad P_{xy} = -\hat{h} F_{xy} \]

since the geometry is linear, get the solution

\[ F = \sum_n C_n \left[ \sinh \frac{n \pi}{2} + \frac{m \pi}{2} \cosh \frac{m \pi}{2} \right] \cos n \alpha y \]


\[-na_y \sinh \frac{n\pi}{2} \sinh \alpha y \cosh \alpha x +
+ \sum_m c_m \left( \sinh \frac{m\pi}{2} + \frac{m\pi}{2} \cosh \frac{m\pi}{2} \right) \cosh \alpha x -$ \n
\[-\alpha x \sinh \frac{m\pi}{2} \sinh \alpha x \right] \cos \alpha y \quad m, n = 1, 2, 3, \ldots \quad (6.2.15)

using eq. (6.2.11), get

\[-\frac{1}{\mu} \sum_n c_n \left( \sinh \frac{n\pi}{2} + \frac{n\pi}{2} \cosh \frac{n\pi}{2} \right) \n \cosh \alpha x -

\[-na \sinh \frac{n\pi}{2} \left( 2 \cosh \alpha x + na \sinh \alpha x \right) \left( -1 \right)^{\frac{n-1}{2}} -

\[- \frac{1}{\mu} \sum_m m \alpha c_m \sinh \frac{m\pi}{2} \cos \alpha y \phi_m. \quad m, n = 1, 3, 5, \ldots \]

where

\[m = 1 \quad \phi_1 = - \frac{n \alpha c_0}{q} \]

\[m > 1 \quad \phi_m = 0 \]

Use Fourier series

\[\cosh \alpha y = \sum_m E_{nm} \cos m \alpha y \quad n, m = 1, 3, 5, \ldots \quad (6.2.16)\]

\[\sinh \alpha y = \sum_m E_{nm} \cos m \alpha y \quad n, m = 1, 3, 5, \ldots \quad (6.2.17)\]

For the interval 2\alpha, get the Fourier integrals

\[E_{nm} = \frac{4m}{\pi \alpha^2 (n^2 + m^2)} \cosh \frac{n\pi}{2} (\pi \frac{n-1}{2}) \quad (6.2.18)\]
For a given \( m \)

\[
F_{nm} = \frac{4 (-i)^{\frac{m-1}{2}}}{\alpha_i^2 (m^2 + n^2)} \left( \frac{\pi}{2} \sinh \frac{n \pi}{2} - \frac{2n m}{m^2 + n^2} \cosh \frac{n \pi}{2} \right) \tag{6.2.19}
\]

Consider \( m = n = 1 \) only

\[
C_i = \frac{E \bar{\psi}_s}{2 \sinh \frac{n \pi}{2}} \tag{6.2.21}
\]

In order to solve for the moments, and since for the range of deflections that are considered here \( \frac{\bar{\psi}_s}{d} \ll \frac{1}{10} \)

the terms with the second derivatives are small with respect to the load.
Therefore, one term approximation by a single trigonometric wave is used

\[ \tilde{\alpha} = \alpha_0 \cos \alpha_x \times \cos \alpha_y, \gamma \]  

\[ \text{Thus, eq. (6.2.13) reduces to} \]

\[ \omega_{\alpha} = \omega_0 \cos \alpha_x \times \cos \alpha_y \]  

\[ \text{The problem reduces to} \]

\[ D \nabla^4 \omega_{\alpha} = \overline{\nu} - \frac{h^2}{2} \nabla^2 \tilde{\alpha} \]

\[ \nabla^4 \omega_{\alpha} = 0 \]  

\[ \nabla \omega_{\alpha} = \nabla \omega_{\alpha} - D (\nabla^2 \omega_{\alpha})_x + \Psi_{\omega_{\alpha}} \]  

\[ \nabla \omega_{\alpha} = \nabla \omega_{\alpha} - D (\nabla^2 \omega_{\alpha})_y - \Psi_{\omega_{\alpha}} \]  

\[ \overline{\nabla}_x - \frac{h^2}{10} \nabla^2 \overline{\nabla}_x = -D (\nabla^2 \omega_{\alpha})_x - \frac{h^2}{10} \overline{\nu}_{\omega_{\alpha}} \]  

\[ \overline{\nabla}_y - \frac{h^2}{10} \nabla^2 \overline{\nabla}_y = -D (\nabla^2 \omega_{\alpha})_y - \frac{h^2}{10} \overline{\nu}_{\omega_{\alpha}} \]  

get

\[ \tilde{\nu}_0 = \frac{D \alpha_0^4 4 \omega_0}{1 + \frac{3 \nu_2^2}{5} (\alpha_0)^2} \]  

\[ \tilde{\nabla}_x = V_1 \sin \alpha_x \times \cos \alpha_y \]  

\[ \tilde{\nabla}_y = V_2 \cos \alpha_x \times \sin \alpha_y \]
where
\[ V_i = -2 \omega_0 \alpha_i^3 D + \frac{j_0}{h_2} \frac{h_i^2}{1 + \frac{\pi^2}{5} (\frac{h_i}{a})^2} = V_2 = V \]

\[-j_0 = 2V \alpha_i\]  \hspace{1cm} (6.2.30)

The stress function is given by
\[ \psi = \sum_n \left( A_n \sin \alpha_i x \sinh \alpha y + A_n^i \sin \alpha_i x \sinh \alpha_i y \right) \]  \hspace{1cm} (6.2.31)

\[ n = 1, 3, 5, \ldots \]

From eq. (6.2.2) get
\[ \lambda = \lambda' = \left( \alpha_i^2 + \frac{\pi^2}{h_i^2} \right)^{1/2} \]  \hspace{1cm} (6.2.32)

In order to solve for \( H_n \) \( AND \) \( A_n \), the boundary conditions (6.2.10) are used. Since due to symmetry
\[ H_n = H_n^* \quad A_n = A_n^* \]

get
\[ \lambda = \pm \frac{\alpha}{2} \]

\[ M_x = \bar{m}_{x} \bar{F}_{yy} \]

\[ -D \sum_n H_n n^2 \alpha_i^2 2 \cosh \frac{\pi \alpha}{2} \cos \alpha y + \frac{j_0^2}{h_2^2} \sum_n H_n n^2 \alpha_i^2 2 \cosh \frac{\pi \alpha}{2} \cos \alpha y + \]

\[ + \sum_n A_n n^2 \alpha_i^2 \cos \alpha y \cosh \left( \lambda \frac{\alpha}{2} \right) = \]

\[ = -\frac{1}{h_2^2} c_{\alpha_i^2} (\sinh \frac{\pi \alpha}{2} \cosh \frac{\pi \alpha}{2} + \frac{\pi}{2}) \cos \alpha y \]

For a given \( n \), get
\[ A_n = \frac{H_n D n^2 \alpha_i^2 2 \cosh \frac{\pi \alpha}{2} \left[ 1 + \frac{\pi^2}{5} \left( \frac{h_i}{a} \right)^2 \right]}{\frac{\pi}{h_2^2} n \lambda \cosh \left( \lambda \frac{\alpha}{2} \right)} \]  \hspace{1cm} (6.2.33)

\[ X = \pm \frac{\alpha}{2} \]

\[ M_{xy} = 0 \]
\[
\begin{align*}
D\omega_0 \alpha^2 \sin\alpha y - D\sum_n H_n n^2 \alpha^2 (\sinh \frac{n\pi}{2} + \frac{n\pi}{2} \cosh \frac{n\pi}{2}) &= \\
- \frac{n\pi}{2} \tanh h \frac{n\pi}{2} \sinh \frac{n\pi}{2} \sinh \alpha y - D\sum_n H_n n^2 \alpha^2 (\sinh \alpha y + \\
&+ n\alpha y \cosh \alpha y, y - \frac{n\pi}{2} \tanh h \frac{n\pi}{2} \sinh \alpha y) (-1) \frac{n\pi}{2} + \\
+ \sum_n A_n \frac{A}{\lambda} \frac{1}{\lambda^2} (\lambda^2 + \frac{1}{\lambda^2}) \sin \lambda (\lambda \frac{\alpha}{\lambda^2}) \sinh \alpha y - \\
- \frac{\frac{2n\pi^2}{n\pi} \frac{\Delta y^2}{\pi}}{1 + \frac{\frac{2n\pi^2}{n\pi} \frac{\Delta y^2}{\pi}}{5}} - \frac{2n\pi^2}{n\pi} \sum_n H_n n^4 \alpha^2 \sinh \frac{n\pi}{2} \sinh \alpha y - \\
- D\sum_n H_n \frac{2n\pi^2}{n\pi} n^4 \alpha^2 \sinh \alpha y (-1) \frac{n\pi}{2} - \\
- \sum_n \frac{1}{\lambda^2} (\lambda^2 + \frac{1}{\lambda^2}) \sinh \alpha y (-1) \frac{n\pi}{2} = 0
\end{align*}
\]

From eq. (6.2.16) and (6.2.17) get

\[
\begin{align*}
\sinh \alpha y &= - \sum_n \frac{4n^2}{n^2} \frac{\sin \frac{n\pi}{2}}{n^2 + (\frac{\Delta y}{\lambda})^2} \cosh (\lambda \frac{\alpha}{\lambda^2}) \sin \alpha y \\
\sinh \alpha y &= - \sum_n \frac{2n}{\pi \Delta y} \sin \frac{n\pi}{2} \cosh \frac{n\pi}{2} K \sin \alpha y \\
\alpha \alpha y \cosh \alpha y &= \sum_n \frac{2n}{\pi \Delta y} \sin \frac{n\pi}{2} (2 \cosh \frac{n\pi}{2} - \frac{n\pi}{2} \sinh \frac{n\pi}{2})
\end{align*}
\]

got for a given n

\[
\begin{align*}
-D\omega_0 \alpha^2 \frac{2n^2}{\pi} \frac{(\frac{h}{q})^2}{1 + \frac{2n^2}{\pi} \frac{(\frac{h}{q})^2}{5}} + \sum_n H_n n^2 \alpha^2 \left[ \frac{2n^2}{\pi} \frac{(\frac{h}{q})^2}{5} \cosh \frac{n\pi}{2} + \frac{n\pi}{2} \cosh \frac{n\pi}{2} \right] + \\
+ \frac{2n}{\pi \Delta y} \cosh \frac{n\pi}{2} \left( \frac{n\pi}{2} \tanh \frac{n\pi}{2} - \frac{2n\pi^2}{5} \frac{(\frac{h}{q})^2}{\alpha} \right)
\end{align*}
\]
\[-A_n \frac{n^2}{10} \left( \alpha^2 + \alpha \beta \right) \cosh(\frac{\beta}{2}) \left[ \tanh(\frac{\alpha}{2}) \right] + \]

\[+ \frac{4n^2}{a \lambda (n^2 + \frac{\alpha}{\beta})^2} = 0 \]

with eq. (6.2.33) get

\[H_n = \frac{n^2 \sigma_c (\sinh \frac{n}{2} \cosh \frac{\pi}{2} \left[ \tanh(\frac{\alpha}{2}) + \frac{4n^2}{a \lambda (n^2 + \frac{\alpha}{\beta})^2} \right] - \frac{\omega_n}{1 - \frac{2\pi}{5}} n^2 \left[ \tanh \frac{n}{2} \left[ 1 + \frac{2 \pi n^2 (h)^2}{\frac{5}{\pi}} \right] - \frac{2 \pi n^2 (h)^2}{\cosh \frac{n}{2}} \left[ \frac{n}{2} \tanh \frac{n}{2} - \frac{2 \pi^2 (h)^2}{\frac{5}{\pi}} \right] \right] (6.2.34) \]

Where

\[\omega_n = \omega_0 + \sum_n H_n \pi \tanh \frac{n\pi}{2} \]

Thus it is seen that the shear effect reduces the effect of the boundary moment. The moment along the diagonal \(x=y\) is given by eq. (6.2.5).

\[M_x = D \omega \sigma_c^2 \cos^2 \alpha \left[ 1 - \frac{\pi^2 (h)^2}{\frac{5}{\pi}} \right] - DH \sigma_c^2 \cos \alpha \cos \alpha \lambda \] 

(6.2.35)

And the membrane force from eq. (6.2.15), when \(P_x = \frac{\pi h}{2} E_{xx}\)

\[P_x = -\alpha \sigma_c \frac{h^2 \cosh \frac{\pi}{2} \cosh \alpha x \cos \alpha x}{\frac{5}{\pi}} \]

(6.2.36)

The elastic stress path for the thick slab can be calculated from the above equations.

A thickness to span ratio of \(\frac{a}{\lambda} = \frac{1}{10}\) appears to be a border case for the effect of non-linear geometry on bending.
<table>
<thead>
<tr>
<th>$d_i$</th>
<th>$1\frac{1}{2}''$</th>
<th>$3''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>1.2''</td>
<td>2.4''</td>
</tr>
<tr>
<td>$f_{c_i}^n$ (psi)</td>
<td>3570</td>
<td>3000</td>
</tr>
<tr>
<td>$f_y$ (psi)</td>
<td>55000</td>
<td>53500</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.01</th>
<th>0.03</th>
<th>0.01</th>
<th>0.03</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{h}{h}$</td>
<td>1.115</td>
<td>1.346</td>
<td>1.145</td>
<td>1.437</td>
</tr>
<tr>
<td>$\frac{h}{h}$</td>
<td>0.269</td>
<td>0.222</td>
<td>0.318</td>
<td>0.254</td>
</tr>
<tr>
<td>$h_i/\ell (d_i^{1/3})$</td>
<td>8.25 (7.05)</td>
<td>5.81 (4.73)</td>
<td>7.05 (6.7)</td>
<td>4.76 (4.47)</td>
</tr>
<tr>
<td>$H_i$</td>
<td>$0.0622 \bar{w}_o$</td>
<td>$0.0218 \bar{w}_o$</td>
<td>$0.0695 \bar{w}_o$</td>
<td>$0.0292 \bar{w}_o$</td>
</tr>
<tr>
<td>I $\frac{m_{z+0}}{m_o} \mid_{xy=0}$</td>
<td>41.3 $\bar{w}_o$</td>
<td>19.4 $\bar{w}_o$</td>
<td>83.2 $\bar{w}_o$</td>
<td>36.8 $\bar{w}_o$</td>
</tr>
<tr>
<td>I $\frac{p_{xy}}{e_o} \mid_{xy=0}$</td>
<td>$-40.0 \bar{w}_o$</td>
<td>$-13.3 \bar{w}_o$</td>
<td>$-77.8 \bar{w}_o$</td>
<td>$-26.0 \bar{w}_o$</td>
</tr>
<tr>
<td>II $\frac{m_{z+0}}{m_o} \mid_{xy=0}$</td>
<td>29.3 $\bar{w}_o^2$ - 22.4 $\bar{w}_o^2$</td>
<td>18.7 $\bar{w}_o^2$ - 7.9 $\bar{w}_o^2$</td>
<td>$-26.0 \bar{w}_o$</td>
<td>$-26.0 \bar{w}_o$</td>
</tr>
<tr>
<td>II $\frac{p_{xy}}{e_o} \mid_{xy=0}$</td>
<td>59.0 $\bar{w}_o^2$ - 39.6 $\bar{w}_o^2$</td>
<td>23.7 $\bar{w}_o^2$ - 13.1 $\bar{w}_o^2$</td>
<td>$-26.0 \bar{w}_o$</td>
<td>$-26.0 \bar{w}_o$</td>
</tr>
<tr>
<td>I $\frac{m_{z+0}}{m_o} \mid_{xy=\frac{3}{4}}$</td>
<td>26.1 $\bar{w}_o$</td>
<td>11.3 $\bar{w}_o$</td>
<td>52.2 $\bar{w}_o$</td>
<td>21.9 $\bar{w}_o$</td>
</tr>
<tr>
<td>I $\frac{p_{xy}}{e_o} \mid_{xy=\frac{3}{4}}$</td>
<td>$-37.5 \bar{w}_o$</td>
<td>$-12.5 \bar{w}_o$</td>
<td>$-72.5 \bar{w}_o$</td>
<td>$-24.4 \bar{w}_o$</td>
</tr>
<tr>
<td>II $\frac{m_{z+0}}{m_o} \mid_{xy=\frac{3}{4}}$</td>
<td>21.5 $\bar{w}_o^2$ - 12.0 $\bar{w}_o^2$</td>
<td>10.6 $\bar{w}_o^2$ - 4.4 $\bar{w}_o^2$</td>
<td>$-26.0 \bar{w}_o$</td>
<td>$-26.0 \bar{w}_o$</td>
</tr>
<tr>
<td>II $\frac{p_{xy}}{e_o} \mid_{xy=\frac{3}{4}}$</td>
<td>26.2 $\bar{w}_o^2$ - 25.4 $\bar{w}_o^2$</td>
<td>10.5 $\bar{w}_o^2$ - 11.7 $\bar{w}_o^2$</td>
<td>$-26.0 \bar{w}_o$</td>
<td>$-26.0 \bar{w}_o$</td>
</tr>
</tbody>
</table>

**TABLE 6.2.1:** GOVERNING PARAMETERS AND EQUATIONS FOR ELASTIC SOLUTION, $d_i=1\frac{1}{2}, 3''$

**Note:** I. = **LINEAR GEOMETRY, INCLUDING THE EFFECT OF SHEAR.**

II. = **NON LINEAR GEOMETRY, EXCLUDING SHEAR EFFECT.**
Fig. 6.2.1: Interaction curve for $d_1 = 1/4''$, $r = 1\%$. 

Stress path of:
1. $x = y = 0$—△
2. $x = y = \frac{a}{4}$—○
3. $x = y = \frac{3a}{8}$—☆

$P$ is the applied load.
FIG. 6.2.2: INTERACTION CURVE FOR $d_i=1/2"$, $r=3\%$. 

STRESS PATH OF: 
1. $x=y=0$ 
2. $x=y=\frac{3}{4}$ 
3. $x=y=\frac{3}{8}$. 

$\text{S}_{\text{STRESS PATH}}$
FIG. 6.23: INTERACTION CURVE FOR $d = 3''$; $r = 1\%$. 

Stress Paths of:
1. $x = y = 0$  
2. $x = y = \frac{a}{2}$  
3. $x = y = \frac{3}{5}a$
Fig. 6.2.4: Interaction Curve for $d_1 = 3''$. $r = 3\%$.
from one side, and the effect of shear on bending from the other side. Thus, the set of eq. (6.2.34 \pm 36) and (4.4.4 \pm 8) gives approximately the same values for small deflections, and therefore for the case \( f < 1 \). However, with larger deflections the effect of non linear geometry should be considered.

The stress path for the cases studied are given in Fig. (6.2.1 \pm 4). The important parameters are summarized in Table (6.2.1).

6.3 UPPER BOUND SOLUTION (RIGID PLASTIC SOLUTION)

From the stress path it is seen that along the diagonal, yielding may result due to so called "compression failure" or "tensile failure".

For unrestrained slab, the modified criteria for balanced cross section if 3 dimensional state of stress is considered is given by

\[
(t_L)_{BAL.} = \frac{t}{t_1 + \gamma_L} = 0.536
\]  

(6.3.1)

Thus, for over-reinforced unrestrained slabs, if

\( t_L > (t_L)_{BAL.} \)

the yield moment without axial compression corresponds to compression failure. With additional restraint that causes axial compression in the elastic regions, more of the concrete area is active, and the capacity of the slab increases, since the yield criteria is a function of the applied load. (Fig. 6.2.1 \pm 4).

For under-reinforced restrained slabs, as is the case
for all the slabs tested, eq. (6.3.1), compression failure may or may not occur. [Fig (6.2.1) + (6.2.4)]. However, even if compression failure occurs initially, with increasing deflections the neutral axis moves toward the compression face of the slab, and complete equipotential plastic yielding follows.

(1) Equipotential yielding (tensile failure $\varepsilon_s > \varepsilon_y$).

Following the same procedure as in section 3.1, get

$$\frac{\varepsilon}{E} = -\frac{4P}{\lambda \frac{\beta}{3(\gamma - \beta)}} = -\frac{4P}{\lambda \frac{\beta}{3(\gamma - \beta)}} = -\frac{4P}{\lambda \frac{\beta}{3(\gamma - \beta)}} = \frac{\sigma_0}{E} \left( 1 - 2\frac{1 + \nu}{1 - \nu} \right).$$

Using eq. (6.1.5) get

$$\frac{P}{\gamma_e} = \frac{\beta}{3(\gamma - \beta)} + \frac{1 + \nu}{2} \frac{\sigma_e}{2d} =$$

$$= \frac{\beta}{3(\gamma - \beta)} + \frac{\sigma_e}{2d} \left( \gamma \frac{1 + \eta}{\eta - \eta_L} - \frac{1 + \eta_L}{\eta - \eta_L} \right) \frac{1}{\gamma - \beta}$$

Again, if equilibrium of a rigid part is considered as in section 3.2, when $\eta$ is assumed to be constant along the diagonal of the square slab (actually $\eta$ varies from $\eta = \frac{1 + \nu}{3\nu}$ when $k=0$ to $\eta = \frac{2}{3} \frac{1 + \nu}{\nu}$ when $k=1$), get from the force
equilibrium (eq. 3.2.2),

\[
\frac{P}{L_0} = \frac{\beta}{2(\gamma^\prime - \beta)} - \frac{\sigma_0}{E} \frac{\gamma^\prime - \frac{1}{2}n \frac{n}{n-L}}{\gamma - \beta}
\]  \hspace{1cm} (6.3.4)

and from the moment equilibrium (eq. 3.2.3),

\[
\frac{P}{L_0} = \gamma + \frac{\beta^2}{4(\gamma^\prime - \beta)} - \frac{\beta}{4} \left( \gamma^\prime - \frac{1}{2}n \frac{n}{n-L} - \frac{1}{2} \frac{n}{n-L} \right) \frac{\sigma_0}{d} + \frac{1}{16} \left( \gamma^\prime - \frac{1}{2}n \frac{n}{n-L} - \frac{1}{2} \frac{n}{n-L} \right)^2 \left( \frac{\sigma_0}{d} \right)^2
\]  \hspace{1cm} (6.3.5)

In order to solve for \( P \), \( \eta = \frac{F_{c_3}}{t_c} \) is substituted in eq. (6.3.5), where \( F_{c_3} \) is given by eq. (6.1.2). Since \( k \) is a function of \( \sigma_0 \),

\[
k = 1 - \frac{\sigma_0}{2d}
\]

an average value \( \bar{m} \) is used. Get

\[
\frac{P}{L_0} = \frac{1}{(1+\frac{n}{n-L})^2} + \frac{(\bar{\sigma}_0)^2}{4} \frac{1}{1+\frac{n}{n-L}} \eta_0 \bar{m} + \frac{\sigma_0}{d} \eta_0 \bar{m} - (\bar{\sigma}_0)^2 \frac{n}{n-L} \frac{\bar{m}}{4}
\]  \hspace{1cm} (6.3.6)

(2) Modified rigid-plastic solution.

Following the same procedure as in section (3.3),

\[
\frac{\varepsilon}{\eta} = \frac{\bar{\sigma}_0}{2} \left( 1 - 2 \frac{L}{L} \right) + \frac{P L^2}{4 \bar{E} \bar{n} \bar{\sigma}_0} = \frac{P}{L_0} + \frac{2(\gamma^\prime - \beta)}{L_0} \frac{1}{\bar{m}_0}
\]
Using equilibrium conditions as in section 3.2

\[
\frac{p}{T_0 l} = \frac{\beta}{2(y - \beta)} \frac{\omega_0^2 + 3 \omega_0 \gamma}{(\omega_0 + \gamma)^2} - \frac{\beta(y - \beta)}{8(y - \beta)} \frac{1 + \eta}{n_0 - n_L} \frac{\omega_0^2}{\omega_0 + \gamma} + \gamma - \beta \frac{1 + \eta}{n_0 - n_L} \frac{\omega_0^2}{\omega_0 + \gamma} + \frac{3 \omega_0^2 + 4 \omega_0 \gamma}{(\omega_0 + \gamma)^2} \left( \frac{\omega_0}{d} \right) \frac{2 \omega_0^2 + 3 \omega_0 \gamma}{(\omega_0 + \gamma)^2} + \frac{\omega_0^2}{4d(y - \beta)} \left( \frac{\omega_0}{d} \right)\frac{\omega_0^2}{\omega_0 + \gamma} \frac{3 \omega_0^2 + 4 \omega_0 \gamma}{(\omega_0 + \gamma)^2}
\]
Numerical results for the cases studied are given in Fig. (6.3.2), (6.3.3) for $m = 1$.

(3) Compression failure ($\varepsilon_s < \varepsilon_y$)

For slabs that are associated with initial compression failure (Fig. 6.2.1 - 6.2.4), movement along the failure line AB (Fig. 6.3.1) is not equipotential, and elastic work is done along this line (stretching of the reinforcement).

Line AB represents compression failure without advanced crushing of concrete. If due to the given field crushing occurs, the behavior is limited by line BB', where $k=1$. Thus, approaching point B' along BB' corresponds to $\varepsilon_s \to \infty$. Actual crushing corresponds to $\varepsilon_s > \varepsilon_c$, thus the actual crushing path is inside ABB'.

The entry from compression failure to equipotential yielding is given for some elastic deflection $(\bar{\omega}_o)_{ult}$. Thus with rigid-plastic conditions, for $0 \leq \bar{\omega}_o < (\bar{\omega}_o)_{ult}$

$$\frac{M_x-x}{M_o} = \left(\frac{M_x-x}{M_o}\right)_m = \text{CONST.}$$

$$\frac{f_m}{f_c} = \frac{1 + \frac{t_c}{t}}{1 + \frac{t_c}{2t}} \frac{1 + \frac{t}{2t}}{2t}$$

(6.3.12)

For $\bar{\omega}_o > (\bar{\omega}_o)_{ult}$ the procedure of section (5.2) is used, when

$$\Delta \bar{\omega}_o = \bar{\omega}_o - (\bar{\omega}_o)_{ult}$$

$$\frac{f_m}{f_c} = \frac{f_m}{f_c} - \frac{\beta}{4(\gamma - 1)} \frac{1 + \frac{t}{t_c}}{1 - \frac{t}{2t}} \frac{\Delta \bar{\omega}_o}{d} +$$

$$+ \frac{1 + \frac{t}{t_c}}{1 - \frac{t}{2t}} \frac{\Delta \bar{\omega}_o^2 + \Delta \bar{\omega}_o (\bar{\omega}_o)_{ult}}{d^2}$$

(6.3.13)
(a) OVERREINFORCED (WHITNEY) SLAB.

(b) UNDERREINFORCED (WHITNEY) SLAB.

FIG. 6.3.1: SCHEMATIC COMPRESSION-TENSION FAILURE INTERACTION.
The rigid plastic solution for $\bar{m} = m_i = 1$ are compared with the experimental results in Fig. (6.3.2) and (6.3.3). The values of $\frac{\phi}{f_{\text{cc}}} \cdot n_i$ are given in Table (6.3.1).

<table>
<thead>
<tr>
<th>$d_i$</th>
<th>1/2&quot;</th>
<th>3&quot;</th>
</tr>
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<tbody>
<tr>
<td>$\tau$</td>
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<td>D.03</td>
</tr>
<tr>
<td>$\left( \frac{h}{f_{\text{cc}}} \right)_{\bar{\omega}_i = 0}$</td>
<td>3.88</td>
<td>1.51</td>
</tr>
<tr>
<td>$\left( \frac{h}{f_{\text{cc}}} \right)_{\bar{\omega}_i = 0.034}$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\left( \frac{h}{f_{\text{cc}}} \right)_{\bar{\omega}_i = 0.07}$</td>
<td>2.35</td>
<td>1.15</td>
</tr>
</tbody>
</table>

**Table 6.3.1: Numerical Results for Rigid-Plastic Relations.**

For $d_i = 3\"$, $\tau = 3\%$ the rigid-plastic or modified rigid-plastic solutions are not shown in Fig (6.3.3), since from the stress path the initial compression failure is obvious. The non-linear geometry effect is not present initially, thus $\left( \bar{\omega}_s \right)_{\text{ult}}$ is expected to be relatively large. This is not so for $d_i = 1/2\"$, where the reduction in loading capacity is not associated with significant value of $\left( \bar{\omega}_s \right)_{\text{ult}}$ (section 6.4).

The value of $\bar{m} = 1$ was used in the numerical analysis, due to the uncertain shear distribution for heavily reinforced thick slabs. Thus an upper bound solution results. However, for $d_i = 3\"$, $\tau = 1\%$, the rigid plastic result for $\bar{m} = \frac{2}{3}$ is shown, where the "elastic-cracked" variation of $f_{\text{cc}}$ along the cross section is taken into account, as shown in Fig. (2.2.4).
Fig. 6.32: Load deflection relations - upper bound. $d_i = \frac{1}{2}$"
Fig. 6.3.3: Load Deflection Relations - Upper Bound. $d_1=3$
6.4 "SPREADING MECHANISM" MODE (ELASTO PLASTIC SOLUTION)

The relation between the particular part of the central deflection $\omega'_z$ and the acting load, if the deflected shape is approximated to one trigonometric wave is given by eq. (6.2.29).

From similar analysis with the same approximations for the deflected shape $\omega'$, given by

$$\omega' = \omega_c \cos \alpha_x \cos \alpha_y$$

but without including, the shear effect, one gets

$$\frac{\omega'_c}{\omega_c} = \frac{1}{1 + \frac{2 \pi^2}{5} \left( \frac{y}{a} \right)^2} \tag{6.4.1}$$

With a uniformly distributed load $\rho$, using (4.3.22 + 26) when $n=1$ only, get

$$\frac{\alpha^2 D \omega_c \alpha^2}{4} - \rho - \frac{1}{2} \alpha^4 \omega_c \sigma \frac{8 \pi}{75} \sinh \frac{\pi}{2} = 0$$

With the shear effect including the homogeneous part $\omega_c = \omega_0 + H \tanh \frac{h}{2}$, get

$$\rho = \frac{D \pi^6}{4 \alpha^4} \left( \omega_0 + \frac{H \tanh \frac{h}{2}}{1 + \frac{2 \pi^2}{5} \left( \frac{h}{a} \right)^2} \right) - \frac{1}{2} \alpha^4 \omega_c \sigma \frac{8 \pi}{75} \sinh \frac{\pi}{2} \tag{6.4.2}$$

For the various cases studied, the elastic load-deflection relations are given in Table (6.4.1)

As was discussed in section 6.3, two kinds of failure may exist along the diagonal:

1. equipotential or tensile failure
2. compression failure
From the stress path (Fig. 6.2.1-4) it is seen that for a thick slab, initially compression failure occurs along the diagonal (or at least at its major part). However, for low reinforcement ratios since the capacity increased rapidly with increasing load due to the restraint and to the 3 dimensional yield criteria, then with an increased deflection the compression failure may develop into complete tensile failure with equipotential characteristics. Thus, the ultimate condition may be associated with compression or tensile failure at ultimate load. (Fig. 6.3.1). An obvious result may be seen when the stress path is followed: For compression failure \( M_{z-1} \) remains approximately unchanged, while \( P \) reduces rapidly. For tensile failure \( M_{z-2} \) is reduced more rapidly than \( P \).

The location of the stress path on the failure condition curve indicates whether ultimate load is reached under compression or tensile conditions. The stress path is determined by the elastic equations if \( f < 1 \). However, once the path satisfies initial yield condition for some \( \eta \), the additional axial compression should be compatible with the deflection field: the only possible movement of the neutral axis with additional deflection is toward the compression face.

The criteria for compression failure of a point under axial compression \( \left( \frac{P}{\gamma_0} \right)_{\eta} \) is given from eq. (6.1.10) (6.1.5), if \( t_\eta = 0.05 k_\gamma \)

\[
2t_\eta - t_\eta^2 > \frac{(1+\eta_\gamma) t_\eta (2-t_\gamma)}{1+\eta} \left[ \left( \frac{P}{\gamma_0} \right)_{\beta} + \left( \frac{P}{\gamma_0} \right)^2 (\beta - \gamma) \right]_{\eta} \quad (6.4.3)
\]
(1) Equipotential yielding.

For a thin slab, the ultimate load is reached due to non linear geometry. Thus, when yield is reached in a given point, there is no dissipation of axial compression due to crushing of concrete. The ultimate conditions are given by

\[
\sum M_{\text{acting}} = \sum M_{\text{resisting}}
\]

\[
\frac{dP}{d\omega} = 0
\]

For the thick slab, these conditions are not satisfied by the non linear geometry effect ("geometrical instability") but rather by the capacity of the R/C cross section under the deflection field.

(2) Compression failure.

If the slab is "under-reinforced" at failure, i.e. if for a given \( \eta_m \),

\[
t_m = \frac{t}{1 + \eta_m} < 0.536
\]

then the compression failure is imposed only because of the restraint (Fig. 6.3.1). When ultimate load is reached while the major part of the yield line is under compression failure, then with increasing plastic deflection the restraining force is reduced due to geometrical changes, and tensile failure follows.

From this discussion it appears that the difference between initial compression or equipotential failures becomes more important when the depth/span ratio is increased, since
for compression failure the ultimate load is associated with some plastic-non linear geometric changes. Thus, if the thick slabs studied with \( r=1\% \) show more of the equipotential mode of failure, and those with \( r=3\% \) show compression type of failure, this distinction is much more important for \( d_i=\frac{3}{4}'' \) than for \( d_i=\frac{1}{2}'' \). Actually, test results show that the ultimate deflection is approximately the same for \( d_i=\frac{1}{2}'' \), regardless of the reinforcement ratio, while they give significant difference for \( d_i=3'' \).

The reduction in loading capacity \( \Delta P \) is calculated here with the same procedure used in section 5. Thus, the solution is an approximate one. Here, however, in addition to moment dissipation, axial force is also dissipated due to crushing.

For points in the interval \(-x_0 \leq x = y \leq x_0\), the limiting axial compression \( P_{pl} \) can be calculated from eq. (6.2.35 & 6) and (6.1.5). If the criteria set by eq. (6.4.3) holds, eq. (6.1.10) should be used instead (6.1.5). If the elastic axial force \( P_{e} \) is calculated from eq. (6.2.36), then the axial compression dissipated at a point along the diagonal of the plastic zone is given by

\[
\Delta P = P_e - P_{pl} + \Delta P_{pl} \tag{6.4.4}
\]

Where \( \Delta P_{pl} \) is additional plastic deflection caused by the dissipation of excess strain energy (see Ch. 5).

Following the procedure of Ch. 5, the "dissipation" of axial compression should be considered once equilibrium of the plastic region is considered. Eq. (5.1.10) is modified here to
be
\[ \int_{x_0}^{x_c} (M_{x-1})(\omega, x, y) dx = \int_{x_0}^{x_c} (M_{x-1})(\omega, x, y) dx + \frac{\Delta P x_c^2}{\beta} - \int \left[ M_{x-1}(\omega, x = y) - (M_{x-1})_{pl} \right] dx + \int \Delta P \left[ (\omega(x = y)) - \omega(x, y) \right] dx \]
\[ (6.4.5) \]

For \( f < 1 \), \( \omega < (\omega)_{pl} \), the total axial compression is given by
\[ P_c = -4\alpha^2 c, \sinh \frac{\pi}{2} \int_0^{a/2} \cosh q x \cos \alpha x dx = -2\alpha c, \sinh \frac{\pi}{2} \cosh \frac{a}{2} \]
\[ (6.4.6) \]

For \( f < 1 \), \( (\omega)_{pl} < \omega(0, 0) < (\omega)_{u.tr} \)

\[ P = \int_{x_0}^{x_c} (P_{pl} - \Delta P_{pl}) dx + \int_{x_0}^{a/2} P_c (\omega) dx \]
\[ (6.4.7) \]

For \( f = 1 \), \( \omega(0, 0) > (\omega)_{u.tr} \), if rigid plastic conditions are assumed

\[ \frac{P}{T_o Q} = \frac{P_{u.tr}}{T_o Q} \frac{\beta - \eta - \eta_1}{\eta - \eta_1} \frac{1 + \eta_1}{(\eta - \beta)} \Delta \tilde{\omega} \]
\[ (6.4.8) \]

The load deflection relations for \( f = 1 \) are given by eq. (6.3.13).

If modified rigid plastic solution is considered

\[ \frac{P}{T_o Q} = \frac{(P)_{u.tr}}{T_o Q} + \frac{\beta}{2(\eta - \beta)} \left[ \frac{\tilde{\omega}_0}{\tilde{\omega}_0 + \beta} + \frac{(\tilde{\omega})_{u.tr}}{(\tilde{\omega})_{u.tr} + \beta} \right] \]
Numerical equations and results for the case studied are given in Table (6.4.1). Comparison with tests results is given in Fig. (6.4.1 + 2) for load-deflection relations, and Fig. (6.4.3. + 4) for load-restraining force relations.

For a given load, the experimental restraining force is less than predicted by the theory. For \( r=3\% \) the difference is larger than for \( r=1\% \). The reason for this is due to the use of the same loading cells to restrain each one of the slabs. Thus, while the analysis is based on infinitely rigid cells, the relative horizontal slab/cell rigidity increased with the thickness and the reinforcement ratio. (see section 9.1)
<table>
<thead>
<tr>
<th>$d_i$</th>
<th>$1/2^\circ$</th>
<th>$3^\circ$</th>
<th>$0.01$</th>
<th>$0.03$</th>
<th>$0.01$</th>
<th>$0.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{f}_w &lt; \left( \bar{f}<em>w \right)</em>{pl.}$</td>
<td>$h=2600 \bar{f}_w - 3160 \bar{f}_w^2$</td>
<td>$h=3450 \bar{f}_w - 3180 \bar{f}_w^2$</td>
<td>$h=1900 \bar{f}_w - 12800 \bar{f}_w^2$</td>
<td>$h=2870 \bar{f}_w - 12800 \bar{f}_w^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_0/\alpha$</td>
<td>$0.3$</td>
<td>$0.33$</td>
<td>$0.34$</td>
<td>$0.425$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{f}_w / d$</td>
<td>$0.166$</td>
<td>$0.187$</td>
<td>$0.071$</td>
<td>$0.115$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m(\bar{f}_w, 0, 0)$</td>
<td>$8.26 M_o$</td>
<td>$4.35 M_o$</td>
<td>$15.0 M_o$</td>
<td>$11.1 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m(\bar{f}_w, x_0, y_0)$</td>
<td>$3.90 M_o$</td>
<td>$1.55 M_o$</td>
<td>$4.50 M_o$</td>
<td>$1.80 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_{pl.}(0, 0)$</td>
<td>$3.25 M_o$</td>
<td>$1.45 M_o$</td>
<td>$3.50 M_o$</td>
<td>$1.50 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sum \frac{m_y(x, y)^2}{d^2}$</td>
<td>$13.50 M_o$</td>
<td>$3.90 M_o$</td>
<td>$21.40 M_o$</td>
<td>$3.50 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sum \frac{m_y(x, y)^2}{d^2}$</td>
<td>$21.00 M_o$</td>
<td>$7.75 M_o$</td>
<td>$21.40 M_o$</td>
<td>$10.40 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sum \frac{M(x, y)^2}{d^2}$</td>
<td>$11.03 M_o$</td>
<td>$7.65 M_o$</td>
<td>$29.00 M_o$</td>
<td>$29.50 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sum \frac{\Delta_{pl}[M(x, y) - M_{pl}]}{d^2}$</td>
<td>$0.76 T_o$</td>
<td>$0.52 T_o$</td>
<td>$2.2 T_o$</td>
<td>$6.2 T_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega(0, 0)$</td>
<td>$0.043$</td>
<td>$0.038$</td>
<td>$0.008$</td>
<td>$0.016$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho (\rho_s i)$</td>
<td>$392$</td>
<td>$610$</td>
<td>$3320$</td>
<td>$7440$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta \rho (\rho_s i)$</td>
<td>$150$</td>
<td>$300$</td>
<td>$2170$</td>
<td>$5800$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho - \Delta \rho (\rho_s i)$</td>
<td>$242$</td>
<td>$310$</td>
<td>$1150$</td>
<td>$1640$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_e$</td>
<td>$\Delta \bar{f}_w 23.0 M_o$</td>
<td>$\Delta \bar{f}_w 11.6 M_o$</td>
<td>$\Delta \bar{f}_w 36.0 M_o$</td>
<td>$\Delta \bar{f}_w 23.8 M_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U_{pl.}$</td>
<td>$5.96 \frac{m^2 a^2}{EI}$</td>
<td>$2.06 \frac{m^2 a^2}{EI}$</td>
<td>$22.0 \frac{m^2 a^2}{EI}$</td>
<td>$22.0 \frac{m^2 a^2}{EI}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\Delta \bar{f}<em>w)</em>{pl.}$</td>
<td>$0.100$</td>
<td>$0.092$</td>
<td>$0.046$</td>
<td>$0.118$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega(0, 0)$</td>
<td>$0.242$</td>
<td>$0.241$</td>
<td>$0.097$</td>
<td>$0.230$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{h_{w, ur}}{h}$</td>
<td>$3.65 T_o$</td>
<td>$1.15 T_o$</td>
<td>$4.0 T_o$</td>
<td>$1.31 T_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega(0, 0) &lt; \left( \bar{f}<em>w \right)</em>{pl.}$</td>
<td>$\frac{P}{a} = 21000 \bar{f}_w$</td>
<td>$\frac{P}{a} = 20900 \bar{f}_w$</td>
<td>$\frac{P}{a} = 86700 \bar{f}_w$</td>
<td>$\frac{P}{a} = 86.600 \bar{f}_w$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| $\omega(0, 0) > \left( \bar{f}_w \right)_{pl.}$ | $\frac{P}{a} = 3.65 - 1474 \bar{f}_w$ | $\frac{P}{a} = 1.15 - 0.97 \bar{f}_w$ | $\frac{P}{a} = 4.0 - 0.84 \bar{f}_w$ | $\frac{P}{a} = 1.31 - 0.28 \bar{f}_w$

**TABLE 6.4.1: Elasto-Plastic Numerical Results for $d_i = 1/2^\circ$, 3°.**
FIG. 6.4.1: LOAD-DEFLECTION RELATIONS. $d_1 = 1\frac{1}{2}''$. 
FIG. 6.4.2: LOAD DEFLECTION RELATIONS. $d_1 = 3"$. 

- **A** EXPERIMENT 
- **B** ELASTIC 
- **C** ELASTO-PLASTIC 
- **D** RIGID-PLASTIC 
- **E** ELASTO-PLASTIC
FIG. 6.43: LOAD RESTRAINING FORCE RELATIONS. $d_1 = \frac{1}{2}''$. 
FIG. 6.4.1: LOAD - RESTRAINING FORCE RELATIONS. $d_1 = 3''$. 
In the experimental results, once the ultimate load is obtained, the shape of the load restraining force relations confirms the theoretical discussion on the effect of reinforcement ratio. For \( r=1\% \), the average restraining force decreases more "slowly" than the load, while for \( r=3\% \) the average restraining force decreases more "rapidly". It should be mentioned, however, that the results shown in Fig. (6.4.3 : 4) are averages of forces along the boundary that are given for each separate cell in Appendix II. It is seen that for \( d_i=1/2'' \) the restraining force in cell "A" (located near the corner of the slab) increases after ultimate load. The author interpreted it as a local effect due to the rigid edge of the steel frame used to protect the thick slab against local shear failure (see Appendix II). This detail caused higher resistance against plastic rotation at the corner of the slab.

The significant difference between "compression" and tensile failures for \( \frac{\alpha}{d_i}=5 \) can be seen in Table (6.4.1). The ultimate load of the slab \( d_i=3'' \), \( r=3\% \) ("compression failure") is achieved under different conditions than for "tensile failures":

1. Much more "dissipation of axial force due to crushing."

2. Larger plastic region \( (x_c = 0.425 \alpha) \)

3. Much more plastic deflection \( (\Delta \bar{\omega})_{pl} \) is needed in order to "dissipate the excessive moments". Again, the reason for this is that under compression failure development of a full plastic hinge is associated with sufficient change of
geometry. The "stress hardening" effect is larger for "compression failure" since much more of the concrete is active, thus the resisting moment increases rapidly with the load.
CHAPTER 7

THE UNREINFORCED SLABS

The yield criteria for unreinforced slabs is a special case of the application of the non-tension theory of concrete to analyze restrained slabs. For this case, the compression failure coincides with the tensile failure.

Since the elastic change of curvature is small for the unreinforced slab, the modified rigid plastic solution that corrects the rigid plastic solution in order to account for elastic inplane strains is expected to give results that are close to the actual behavior.

Using rectangular stress block and 3 dimensional state of stress, the yield criteria is given by

\[ f = \frac{Pd}{M_{t-I}} \left[ 1 - \frac{P}{2d f_c' (1+\eta)} \right] = 1 \]  \hspace{1cm} (7.1.1)

The graphical representation of the yield criteria is given in Fig. (7.1.1) for the case \( \eta = \eta_s \ll 1 \).

Using the same procedure as in Ch. 2 and 3 one gets

\[ \frac{\sigma}{f_y} = \frac{\lambda}{\lambda} \frac{\partial f}{\partial P} = \frac{P}{f_c' (1+\eta)} = \frac{P}{2} \left( 1 - 2 \frac{|x|}{L} \right) + \frac{P^2}{4Ed_s \omega_s} \]  \hspace{1cm} (7.1.2)

Note:

\[ N = \frac{f_c' (1+\eta)^2}{4Ed_s} \]  \hspace{1cm} (7.1.3)
FIG. 7.1: YIELD CRITERIA FOR UNREINFORCED SLAB. ($\eta \ll 1$).
Then

\[
\frac{P}{t_e}\frac{d}{d} = \frac{\bar{\omega}_o (1 + \eta)}{\bar{\omega}_o + M_o} \left[ 1 - \frac{\bar{\omega}_o}{2 d} \left( 1 - 2 \frac{L}{L} \right) \right]
\]

\[
\frac{m}{t_e} = \frac{d}{d} = (1 + \eta) \left[ \frac{\bar{\omega}_o^2 + 2 \bar{\omega}_o M_o}{(\bar{\omega}_o + M_o)^2} - \frac{1}{2 d} \frac{\bar{\omega}_o M_o}{(\bar{\omega}_o + M_o)^2} \left( 1 - 2 \frac{L}{L} \right)^2 \right]
\]

Using moment equilibrium conditions along the yield lines, one gets

\[
\int_0^{L/2} M_1 \, dx = \frac{P L^3}{48} + \int_0^{L/2} P \omega \, dx
\]

Substitute

\[
\eta = -\frac{m}{t_e}
\]

\[
\frac{d}{d} = \frac{d^2}{d} \left[ \frac{\bar{\omega}_o^2 + 2 \bar{\omega}_o M_o}{(\bar{\omega}_o + M_o)^2} - \frac{1}{d} \frac{2 \bar{\omega}_o^3 + 3 \bar{\omega}_o^2 M_o}{(\bar{\omega}_o + M_o)^2} \right]
\]

The load deflection relations are given in Fig. (7.1.2), for \( d_1 = 3/4" \), \( 1/2" \), \( 3" \). The rigid plastic solution \( (M_o = 0) \) as well as the modified rigid plastic solutions are used. For \( d_1 = 3/4" \), \( 1/2" \), two dimensional yield criteria (i.e. \( \eta_o \ll 1 \))
is used. For \( d_i = 3 \), the three dimensional yield criteria is used. Since the considered deflections are small with respect to the thickness, the value \( \bar{n} = \frac{2}{3} \) is used.
Fig. 7.1.2: Load deflection relations for unreinforced slabs.
CHAPTER 8

CORRELATION BETWEEN THE IDEALIZED CASE STUDIED AND RESTR AI NT CONDITIONS IN ACTUAL PRACTICE (THIN SLAB)

8.1 THE CLAMPED SLAB

For thin slab with \( f<1 \) everywhere, the effect of non-linear geometry is larger than the effect of upward movement of the neutral axis due to cracking of the concrete. Thus, if the rotation of the slab is zero along the boundary, there is no compression membrane action, since at the boundary \( \psi_x = \psi_y = 0 \).

If a clamped R/C slab is considered with bottom reinforcement at the center, and top and bottom along the boundary, once the conditions \( M=M_0', P \approx 0 \) are achieved along the boundary, a plastic rotation is allowed. Hence, a compression membrane starts, and the moment can be increased farther along the plastic path.

While a "spreading hinge" is developed along the boundary, the plastic region is spreading at the center of the slab. As was assumed for the simply supported slab, the effect of the spreading central mechanism on the deflected shape, and therefore on the compressive membrane action, is negligible for the thin slab.

The stress path for the condition \( M_x (\pm \frac{a}{2}, 0) = -M_0' \) is given in Fig. 8.1.1.

If most of the slab remains elastic when
then at the same time only a small rotation occurs along the boundary, since the plastic region along the boundary cannot spread into the slab.

When the equality (8.1.1) is satisfied, the deflection is taken as the sum

$$\omega = \omega_1 + \omega_2$$  (8.1.2)

$\omega_1$ is given by the equilibrium equation and boundary conditions

$$\nabla^4 \omega_1 = p_x,$$

$$\omega_1(x, y) = \omega_1(x, y) = 0$$

$$\omega_{i x} (x, y) = \omega_{i y} (x, y) = 0$$  (8.1.3)

$P = 0$ everywhere. If $E_m$ is a known term, one gets

(see reference 5)

$$m_1 = m_1(x, y) = -\frac{4 \pi a^2}{\kappa^3} \sum (-1)^{m-i} E_m \cos \kappa x$$  (8.1.4)

For $x = y = 0$

$$\bar{\omega}_{01} = \frac{\pi a^4}{D} 0.00126$$  (8.1.5)
Using
\[ 2 \int m_i dx = -M_e q \]
eq (8.1.5) becomes

\[ \tilde{\omega}_{01} = \frac{M_e a^2}{D \sum \frac{1}{m_i} E_m} - \frac{0.00126 \pi^4}{8} \quad (8.1.6) \]

\( \omega_2^2 \) is a correction due to the plastic moment distribution along the boundary

\[ \nabla^4 \omega_2 = 0 \]

\[ \omega_2(\pm \frac{a}{2}, y) = \omega_2(x, \pm \frac{a}{2}) = 0 \]

\[ \omega_{2,xx}(\pm \frac{a}{2}, y) = \omega_{2,yy}(0, \pm \frac{a}{2}) = M_2 \]

where

\[ M_2 = m_0' - m, \quad 2 \int_0^{\pi/2} M_2 dx = 0 \]

here assume \( p \approx 0 \).

If

\[ m_2 = \sum m_i \cos m_1 \]

Where \( F_m \) is a known term, one gets

\[ \tilde{\omega}_{02} = \frac{a^2}{\pi^2 D} \sum \frac{F_m}{m \cosh \frac{m \pi}{2}} \left( \frac{m \pi}{2} \tan \frac{m \pi}{2} \right) \quad (8.1.7) \]

The actual stress path for the equality (8.1.1) is given in Fig. (8.1.2).

For additional load \( \Delta \rho \), plastic rotation occurs along the boundary and a solution for simply supported slabs is
**Fig. 8.1.1:** CLAMPED SLAB; STRESS PATH FOR $f<1$.

**Fig. 8.1.2:** CLAMPED SLAB; STRESS PATH FOR $f=1$ ALONG THE BOUNDARY.

**Fig. 8.2.1:** INTERNAL PANEL RESTRAINED BY ELASTIC BEAMS.
superimposed on the above solution. If the increment of deflection with plastic rotation at the boundary is $\Delta \omega$, then

$$\Delta \omega = \Delta \bar{\omega}, \cos \alpha, x \cos \alpha, y$$

$$\sum \omega = \omega + \Delta \omega \quad (8.1.8)$$

The complete procedure for the solution of compressive membrane action of simply supported is added. Thus, an approximate solution for the effect of compression membrane forces of clamped slabs is obtained.

8.2 ELASTIC RELEASE OF THE RESTRAINT

In actual practice, partial restraint against horizontal movement is present due to the existence of elastic members at the boundary of the single panel. Since the restraint slab, especially the thin slab, appears to be sensitive to horizontal movement on the boundary, the degree of restraint is a major factor in the resistance of the slab considered.

Consider the case of a panel surrounded by horizontal elastic beams. For simplicity, a square panel with symmetrical boundary conditions is assumed. However, the same procedure can be applied to a rectangular panel with different conditions for each boundary. Consider the case where only bottom reinforcement exists everywhere, or where a plastic hinge has already developed along the boundary (section 8.1).

The internal panel of a two way slab is one example. However, this problem represents a complicated stress distribution
problem for the surrounding panels. Since exact solution for this problem is not the scope of this research, only cases where simple elastic beam theory is applicable are considered here (Fig. 8.2.1).

Assume that the restraint is provided by an elastic beam with the rigidity ($B$). Two cases are considered:

1. Fix support at $(\frac{a}{2}, \frac{a}{2})$. 
   $$u\left(\frac{a}{2}, \frac{a}{2}\right) = 0 \quad (8.2.1)$$
   This condition occurs in practice when the internal panel is simply supported by a shear wall along its boundaries.

2. Complete release at $(\frac{a}{2}, \frac{a}{2})$.

Due to inplane forces in the surrounding panels that are assumed to be uniformly distributed in the uncracked cross section $\frac{h}{na}/q$, the horizontal movement at $(\frac{a}{2}, \frac{a}{2})$ if the effect of shear forces is neglected is given by

$$u\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{p}{4n_h B E} \quad (8.2.2)$$

A second boundary condition involves load-deflection relations for the horizontal beam

$$P(\frac{a}{2}, y) = -Bu\left(\frac{a}{2}, y\right)_{yy}\quad (8.2.3)$$

For case (1), using the results of section 4 and eq. (8.2.1)

$$\frac{C_o q}{E} + \frac{i}{E} \sum_n C_n \left(\sinh \frac{n\pi}{2} + \frac{n\pi}{2} \cosh \frac{n\pi}{2} \sum_m \frac{4\pi^2}{a} \frac{\cos \frac{m\pi}{2}}{\pi^2 + m^2} \frac{\sinh \frac{n\pi}{2}}{a} \right)$$
Using the boundary conditions given by eq. (8.2.3)

\[
\frac{1}{E} \sum_m c_n \left\{ \sinh \frac{n\pi}{2} + \frac{n\pi}{2} \cosh \frac{n\pi}{2} \right\} \sum_m \frac{4n^2}{\pi^2} \frac{n\pi}{m^2 + n^2} \sinh \frac{n\pi}{2} +
\]

\[
- \frac{n\alpha}{\sinh \frac{n\pi}{2}} \left[ 2 \sum_m \frac{4n}{\pi} \frac{n\pi}{m^2 + n^2} \sinh \frac{n\pi}{2} \right] +
\]

\[
+ \sum_m \frac{4n}{\pi (m^2 + n^2)} \left( \frac{n\pi}{2} \cos \frac{n\pi}{2} \cosh \frac{n\pi}{2} - \frac{n^2 - m^2}{m^2 + n^2} \cos \frac{n\pi}{2} \sinh \frac{n\pi}{2} \right) \right\} \sin \frac{n\pi}{2} \cos m \alpha y -
\]

\[
- \frac{1}{E} \sum_m 2m \alpha c_m \sinh \frac{m\pi}{2} \cos m \alpha y + \frac{\alpha_k}{m^2} \bar{\omega}_0 \frac{\pi^2}{\pi^2} \cos \alpha y =
\]

\[
= - \frac{\bar{\tau}}{B \alpha_k} \sum_m \frac{1}{m^4} \left\{ \frac{\delta \cos \sin \frac{m\pi}{\alpha_k}}{m^4 \alpha_k^2} + \frac{E}{8} \bar{\omega}_0^2 \right\}
\]
\[ - \frac{4m^2}{\pi} \sum_n C_n \frac{e^{-h^2 \cos \frac{m\pi}{2}}}{(n^2 + m^2)^2} \sinh^2 \frac{m\pi}{2} \cos \frac{m\pi}{2} \]
\[ - c_n m^2 \left( \sinh^2 \frac{m\pi}{2} \cosh \frac{m\pi}{2} + \frac{m\pi}{2} \right) \cos \alpha \phi \]

(8.2.6)

For a given \( m \)

\[ C_m \left[ \frac{2m\pi \sinh^2 \frac{m\pi}{2}}{E^2} + \frac{m^2}{m^2} \left( \sinh \frac{m\pi}{2} \cosh \frac{m\pi}{2} + \frac{m\pi}{2} \right) \right] = \]
\[ = - \frac{8m^2 \cos \frac{m\pi}{2}}{E} \sum_n C_n \frac{n \sin \frac{m\pi}{2}}{(n^2 + m^2)^2} \sinh^2 \frac{n\pi}{2} - \]
\[ \phi_m + \frac{\Gamma}{\beta \alpha c^2 m^4} \left[ \frac{8c \cos \frac{m\pi}{2}}{m\pi \alpha^2} - \phi_m \right] \]
\[ - \frac{4m^2}{\pi} \sum_n C_n \frac{2h^2 \cos \frac{m\pi}{2}}{(n^2 + m^2)^2} \sinh^2 \frac{n\pi}{2} \cos \frac{m\pi}{2} \]

(8.2.7)

where
\[ \phi_1 = - \frac{\pi}{\alpha} \bar{\omega} \phi_0 \quad \phi_{m>1} = 0 \]
\[ \bar{\phi}_2 = - \frac{E \bar{\omega}^2}{8} \quad \bar{\phi}_{m \neq 2} = 0 \]

of interest are

\[ C_i = \frac{\pi}{\alpha} \bar{\omega} \frac{\phi_0}{\beta \alpha T \phi_0} + \frac{\Gamma}{\beta \alpha \pi c \phi_0} \]
\[ = \frac{E \bar{\omega}^2}{10.6 + 0.236 \xi} + \frac{E \bar{\omega}^2}{2060 + 46.1 \xi} \]

(8.2.8)
The total horizontal restraining force along the boundary is obtained from eq. (4.2.20).

\[
\frac{P}{2h} = C_0 q - c_1 \alpha_1 \xi_1(x) + 3c_3 \alpha_1 \xi_3(x)
\]  

(8.2.9)

where

\[
\xi = \frac{h q^2 E}{b}
\]

(8.2.10)

The maximum axial compression is given for

\[
\left( \tilde{\omega} \right)_{\text{Pmax}} = 2 \left( \frac{0.458 + 0.0102 \xi}{0.458 + 0.0102 \xi} \right) \left( \frac{1.23 - \frac{2 \xi}{89.3 + 2 \xi}}{89.3 + 2 \xi} \right) \left( \frac{2 \xi}{3020 + 180 \xi} \right)
\]

(8.2.13)
For $B \rightarrow \infty$ and eq. (8.2.13) reduce to eq. (4.2.24). The ultimate load $p_{vlt}$ is a function of the maximum axial compression. Thus, for $B \rightarrow \infty$ $p_{vlt}$ is given in Fig. (5.2.1). The lower bound is given for $B \rightarrow 0$, $P \rightarrow 0$, $p_{vlt} \rightarrow p_L$

Variation of the maximum axial compression as a function of $(\xi)$ is given in Fig. (8.2.2). For the case illustrated in Fig. (8.2.1)

$$\xi = \frac{iZ}{k n^2}$$

$k, n$ are limited to the applicability of the simple beam theory.

The deflection at ultimate load as function of $\xi$ is given in Table (8.2.1)

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>0</th>
<th>1.5</th>
<th>12</th>
<th>40.5</th>
<th>324</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\frac{iZ}{k}) \frac{P_{max}}{\lambda_0}$</td>
<td>1.77</td>
<td>1.76</td>
<td>1.69</td>
<td>1.51</td>
<td>0.76</td>
</tr>
</tbody>
</table>

**Table 8.2.1: Case (i) deflection at ultimate load as a function of $(\xi)$.
For case (2), inplane strains are taken into account, if all the assumptions that lead to eq. (8.2.2) are valid. Thus, using eq. (8.2.4) one gets

$$\frac{C_o}{E} \frac{\bar{W}_e^2 \pi^2}{16 \alpha} = \frac{P}{4 Kh h E} \quad (8.2.14)$$

From eq. (8.2.11)

$$C_o = \frac{2 k n}{2 Kh n (1)} \frac{E \bar{W}_e^2 \pi^2}{16 \alpha^2} + \frac{1}{2 Kh n + 1} \left( C_1 \frac{\pi^2}{q^2} 7.34 - C_3 \frac{\pi}{q^2} 30.4 \right) \quad (8.2.15)$$

Using the second boundary condition (eq. 8.2.6), where $C_3$ is neglected without a change of the results' accuracy

$$C_1 \left[ \frac{2 \pi \sinh \frac{\pi}{2}}{E \varphi} + \frac{\pi \alpha^2}{1.5 \pi^2} \left( \sinh \frac{\pi}{2} \cosh \frac{\pi}{2} + \frac{\pi}{2} \right) \right] =$$

$$= \frac{E \bar{W}_e^2 \varphi}{1.5} + \frac{1}{2n \varphi q^2 + 1} \left[ \frac{2 \pi k}{2 n \varphi q^2 + 1} \frac{E \bar{W}_e^2 \pi^2}{q^2} + \frac{1}{2 Kh n + 1} C_1 \frac{\pi^2}{q^2} 7.34 \right] \quad (8.2.16)$$

$$C_1 = \frac{E \bar{W}_e^2 \varphi}{10.6 + 0.236 \xi - \frac{\xi}{2 Kh n + 1} 0.193} +$$

$$+ \frac{2 k n}{2 Kh n + 1} \frac{\xi E \bar{W}_e^2}{2060 + 46.15 \xi - \frac{\xi}{2 Kh n + 1} 37.6} \quad (8.2.17)$$

From eq. (8.2.15)
\[ C_0 = \frac{2\,k_n}{2\,k_{n+1}} \frac{E\,\delta_0^2}{16\,q^2} + \frac{1}{2\,k_{n+1}} \frac{E\,\delta_0^3}{q^2} \frac{\frac{7.34}{10.6 + 0.236\,\xi - \frac{\xi}{2\,k_{n+1}}}}{0.193} + \]

\[ + \frac{2\,k_n}{(2\,k_{n+1})^2} \frac{\int E\,\delta_0^2}{q^2} \frac{\frac{7.34}{2060 + 46.1\,\xi - \frac{\xi}{2\,k_{n+1}}}}{0.193} \]  

\[(8.2.18)\]

Substituting in eq. (8.2.11)

\[ \frac{\dot{P}}{a} = \frac{E\,h}{a^2} \frac{2\,k_n}{2\,k_{n+1}} \left[ \bar{\omega}_0^2 \left( 1.23 - \frac{2\,k_n}{2\,k_{n+1}} \frac{2\,\xi}{89.3 + 2\,\xi - \frac{\xi}{2\,k_{n+1}} 1.63} \right) \right. \]

\[ \left. - \bar{\omega}_0^2 \frac{2}{0.458 + 0.0102\,\xi - \frac{\xi}{2\,k_{n+1}} 0.0084} \right] \]  

\[(8.2.19)\]

\[ \frac{d\dot{P}}{d\bar{\omega}_0} = 0 ; \]

\[ (\bar{\omega}_0)_{\text{max}} = \frac{3\,\pi}{0.458 + 0.0102\,\xi - \frac{\xi}{2\,k_{n+1}} 0.0084} \frac{1.23 - \frac{2\,k_n}{2\,k_{n+1}} \frac{2\,\xi}{89.3 + 2\,\xi - \frac{\xi}{2\,k_{n+1}} 1.63}}{1} \]

\[(8.2.20)\]

For \( k_n \rightarrow \infty \), eq. (8.2.20) reduces to eq. (8.2.13).

For \( k=1 \), the case is one of an internal panel of a two way slab. For \( n \rightarrow 1 \), the assumed behavior of the restraint is changed to that of a deep beam, and for \( n \rightarrow \infty \) to semi-infinite plane. In spite of this fact, the case \( k=1 \) is studied in order to illustrate the sensitivity of the restraint.
to elastic deformation in two-way slabs, and some general conclusions can be made from this study (see section 9).

The ultimate deflection as a function of \( \frac{(\bar{\omega})}{\bar{\omega}_{p,\infty}} \) is given in Table (8.2.2).

<table>
<thead>
<tr>
<th>( \frac{(\bar{\omega})}{\bar{\omega}_{p,\infty}} )</th>
<th>0</th>
<th>1.5</th>
<th>12</th>
<th>40.5</th>
<th>324</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{(\bar{\omega})}{\bar{\omega}_{p,\infty}} )</td>
<td>1.77</td>
<td>1.77</td>
<td>1.71</td>
<td>1.53</td>
<td>0.77</td>
</tr>
</tbody>
</table>

**TABLE 8.2.2: CASE (2); ULTIMATE DEFLECTION AS A FUNCTION OF \( \frac{(\bar{\omega})}{\bar{\omega}_{p,\infty}} \).**

It is seen that the ultimate deflection is not greatly affected under the conditions assumed. However, the ultimate inplane compression is more sensitive to values of \( n < 1 \) as is seen in Fig. (8.2.2).
FIG. 8.22: VARIATION OF THE RESTRAINING FORCE WITH ELASTIC SUPPORTS.
9.1 COMPARISON WITH TEST RESULTS

The analysis is based on experimental investigation of uniformly loaded square R/C slabs restrained with horizontal steel cells. The restraint was provided at the reinforcement level. A description of the test procedure results and the choice of the material parameters of the small scale models, is given in Appendix II.

The results of the tests on simply supported panels, restrained at the bottom reinforcement level, are compared with the various solutions where they appear in the text. Only three cases were compared: 1) \( r=0\% \), 2) \( r=1\% \), 3) \( r=3\% \). The tests however, covered a wide range of reinforcement ratios \( (r=0\%, 0.5\%, 1\% , 2\% , 3\% ) \). The cases compared appear to be in good agreement with the results. The principle parameters, as for example \( f_{\text{cr}} \) for the rigid-plastic solution, were checked for all reinforcement ratios tested (Appendix II). The agreement is shown to be satisfactory. Other parameters, for example, the ultimate load \( f_{\text{u,r}} \) were not checked for each slab separately. However, they vary in accordance with the prediction of the solutions.

The elasticity of the steel elements used as restraining cells is checked in the following. If the displacement at the boundary is \( U \left( \pm \frac{a}{2}, y \right) \), then the boundary condition (eq. 8.2.3)
is,

\[ P \left( \frac{t_a}{2}, y \right) = -KH \cdot U \left( \frac{t_a}{2}, y \right) \]  \hspace{1cm} (9.1.1)

Where \( K \) is the stiffness of the steel cell \( l = 7^{\frac{1}{2}} \)" along with \( \tilde{h} = 0.51" \) average thickness.

\[ KH = \frac{\tilde{h}}{l} E_s = \frac{1}{14.7} E_s \]  \hspace{1cm} (9.1.2)

Following the same procedure as in section (8.2), with the boundary conditions given by eq. (9.1.1), and if the rigidity \( B \) is given by

\[ B = KH \frac{Q^4}{\pi^4} \]  \hspace{1cm} (9.1.3)

one gets for the average axial compression along the boundary

\[ \frac{P_{\text{av}}}{\alpha} = \frac{E_s}{\alpha^2} \left[ \tilde{w}_o \left( 1.23 - \frac{2 \gamma_h}{89.3 + 2 \gamma_h} - \frac{2 \gamma_h}{37.3 + 180 \gamma_h} \right) \right] \]

\[ - \tilde{w}_o \tilde{b} = \frac{2}{0.455 + 0.0102 \gamma_h} \]  \hspace{1cm} (9.1.4)

From eq. (8.2.10), (9.1.2), (9.1.3), and if \( E_s \approx 12 \) \( E_c \)
\( d_i = 1^{\frac{1}{2}} \)" , \( r = 1\% \)

\[ \tilde{\gamma}_h = 7.0 \hspace{1cm} \Rightarrow \frac{(P_h)_{\text{max}} Q}{E_c b_3^2} = 3.18 \]  \hspace{1cm} (9.1.5)

Thus, for a thin slab tested with the apparatus used in the experiment

\[ \frac{P_h}{P_\infty} \rightarrow 0.82 \]  \hspace{1cm} (9.1.6)
This value stands in agreement to the test results (Fig. 5.2.2).

For thick slab \((d_s = 3''\), \(r = 3\%\)) \(\frac{f_y}{f_c} = 36.5\) \(\frac{P_s}{P_{cr}} = 0.49\).

Continuous slabs, with bottom reinforcement only continuing over the support, were also tested. (slabs \#42 - \#49, see Appendix II). The monolithic R/C edges of the slab replaced the loading cells, while the length available for elastic shortening was reduced considerably by the forcing of the moveable head against the edges of the slab. The results obtained for ultimate load and deflection are in close agreement with the corresponding tests with steel cells, since the final stiffness was approximately the same. After the formation of a crack along the boundary, the natural restraining point appears to be at the reinforcement level.

The effect of cracking of the concrete was tested by changing the restraint location from the bottom to the center of the slab. Although an ideal case for this effect was chosen \((d_s = 3'', r = 0\%\). See Appendix II, slabs \#25, \#26), the most significant effect of the restraining phenomena appears to be the location of the restraint at the bottom of the slab, while at mid depth the resistance is of a slab with half the depth. Thus, the upper movement of the neutral axis prior to rigid-plastic kinematic motion is a secondary phenomena (slab \#26, with bottom restraint failed due to shear at \(P \approx 800 \ p.s.i\) with 0.15" central deflection, while \#25 with mid-depth restrained failed at \(P_{ult} \approx 300 \ p.s.i\) with 0.25" deflection). Brief discussion of the cracking effect is given in Appendix I.
9.2 CONCLUSIONS

The effect of the restraint is to increase the capacity of the slab by "prestressing" action. This fact was confirmed by several previous investigations. Aside from a development of a procedure for analysis of ultimate loads and deflections in the present investigation, the following conclusions can be drawn:

(1) Restraint differs from reinforcement in that it is a sensitive function of the deflection

a) For thin under-reinforced slab, the ultimate strength is associated with "geometrical instability", which results in a loss of the restraining force and therefore to loss of the carrying capacity.

b) The location of the restraint has a major effect on the behavior of restrained slabs. Addition of restraint at the bottom of the slab does not merely over-reinforce the slab or increase the "prestressing" effect, but also preserves the effect of restraint with increasing deflection. In a thin R/C slab, location of the restraint at the mid-depth may result in no-restraining effect, since the variation of the horizontal displacement on the boundary due to non-linear geometry is higher than the cracking effect even at small deflections, prior to rigid-plastic motion.

(2) For thick slabs, the 3-dimensional yield condition
leads to "stress hardening". Thus, the criteria for over-reinforcement is changed to include this effect. An additional result is that thick heavily reinforced slabs show "ductile" behavior.

(3) While rotational restraint may reduce the horizontal restraining effect considerably \( f<1 \), the effect appears with development of a plastic hinge \( f=1 \) along the boundary. Elastic release of the horizontal displacement results in reduction of the restraining force.

I. Effect of Span/depth ratio

The behavior of the thin slabs was shown to be a typical non-linear geometry problem where the capacity of the slab is limited by non-linear geometry effect. For thick slabs, the capacity is a result of concrete strength. Generally, slabs with \( \frac{L}{d_i} \approx 10 \) represent border cases between the two categories.

For a thick slab, the effect of shear while \( f<1 \) reduces the restraining effect. The axial compression introduced by the restraint causes lowering of the neutral axis at failure, so that the effect of the "stress hardening" due to the 3 dimensional state of stress is significant. The result is much higher loading capacity than expected if compression failure governed the behavior, while the failure is more ductile than expected \( (d_i=3^"; r=2\%, 3\%) \).

II. Effect of reinforcement ratio

If elastic conditions are maintained in the slab under an infinitely rigid restraint against horizontal movement, the restraining force is not a sensitive function of the reinforce-
ment ratio. With elastic restraint, the relative slab-restraint stiffness has a considerable effect on the restraining force.

For thin slabs the magnification of the ultimate load due to the restraining effect is much higher for an under-reinforced slab with a low reinforcement ratio. With balanced slabs, the restraint causes higher strains in the concrete due to the enforced deflection field, thus a small increase in the capacity results from the restraint.

The definition of balanced sections for the restrained thick slabs is revised to include the "stress hardening" effect. For under-reinforced slabs within this revised definition, low reinforcement ratios lead to equipotential yielding, while high ratios lead to initial compression failure followed by equipotential yielding.

Ultimate load under equipotential yielding is associated with instability, since the axial compression which is the reason for the increasing loading capacity, reaches an extremal value, due to change of geometry (thin slab) or crushing of concrete (thick slab). During "mechanism" stage, the loading capacity is reduced more "rapidly" than the restraining force. Ultimate load under compression failure is associated with a sufficient change in geometry while the restraining force is dissipated. Finally, a full plastic yielding \( (M_{ult}) \) is developed. During "mechanism" stage, a "rapid" reduction of the restraining force is accompanied with a "slow" reduction of the loading capacity. Due to the "stress hardening" effect resulting from the hydrostatic component of stress considering a 3 dimensional state of stress, over reinforced (by Whitney's criteria) thick slabs exhibit greater capacity than is predicted for an over-
reinforced slab and a ductile type of failure rather than brittle.

III. Application to slabs used in actual practice.

For continuous slabs with top and bottom reinforcement at the supports, the effect of the restraint against horizontal movement is insignificant, unless plastic rotation is possible along the boundary. Mathematically, it can be proven that the hinge at the boundary cannot spread into the slab. Thus, when a plastic hinge is formed, the slab is jammed at the bottom of the rotation point. The total effect can be approximated by a simply supported restrained slab with plastic moment applied at the boundary.

In actual practice the horizontal restraint is provided by an elastic member. The compression axial force is sensitive to a release of the complete horizontal restraint at the boundary. A relationship between the rigidity of the restraint and the ultimate compression and deflection for the thin slab was developed. From the results, one can conclude that for two way slab, an internal panel loses its restraining effect if surrounded by a strip whose width is less than half the span of the panel. If the width of the surrounding panels is more than two times the span, 75% of the absolute value of the maximum restraining force can be assumed to act. This, however, represents a limiting value for two way slabs, since the elasticity of a semi-infinite plane is a limiting value.

9.3 SUGGESTIONS FOR FUTURE RESEARCH

The following points are suggested for future research, as they are important for more advanced analysis of horizontally restrained slabs.
(1) Analysis and tests in order to find a generalized yield criteria, including torsional moments, transverse shear forces and membrane shear forces. The results should be available in convenient form for coordinate transformation.

(2) Exact elasto-plastic solutions for square slabs, if linear elastic-perfectly plastic material is assumed. At the present time exact solutions are available for axi-symmetric cases only, linear geometry and no membrane forces.

(3) Stress analysis for rigidity of the horizontal support provided by two way slabs to an internal panel. Solution for semi-infinite as well as finite surrounding panels is desired.

(4) Tests on continuous slabs with top and bottom reinforcement and with horizontal elastic restraint, in order to check the validity of the present analysis and to extend it.

(5) Analysis and tests on restrained slabs, when a shear mode of failure is allowed to occur. This is important in particular for over-reinforced thin slabs and for thick slabs.
BIBLIOGRAPHY


APPENDIX I

Effect of Cracking on Horizontal Restraint at the Boundary

The differential equation for cracked section was given by Brotchie

\[ \bar{D} \nabla^4 \omega + \nabla_1(F, \omega) + \nabla_2(F, \omega) = \mu \]  

(A1.1)

where: \( \bar{D} \) is the stiffness of the cracked section
\( \nabla_1(F, \omega) \) = effect of non-linear geometry
\( \nabla_2(F, \omega) \) = effect of limited tensile strength (cracking)
\( \mu \) = stress function for axial force.

\( F \) is a function of the boundary conditions. Thus, to find the order of magnitude of the various terms in the differential equation and their effect on \( F \), the boundary condition should be considered.

However, it is not possible to solve exactly the governing differential equation, mainly because of the difficulty associated with the solution for \( \bar{D} \) in a slab. Therefore, the simple uniaxial case of simply supported beam is considered. The order of magnitude of the displacement at the boundary is calculated as a function of the displacement. If \( u \) is the horizontal displacement at the reinforcement level,

\[
u = \omega_{jx} \left( \frac{a}{2} \right) \cdot \frac{P}{E} \cdot \frac{h}{2} \int_{0}^{\frac{a}{2}} x \cdot dx - \frac{1}{2} \int_{0}^{\frac{a}{2}} (\omega_{jx} - t_{jx})^2 + \frac{1}{2} \int_{0}^{\frac{a}{2}} t_{jx}^2 dx = 0
\]

(A1.2)

\[ u = u_1 + u_2 - u_3 + u_4 \]

(A1.2)

\[ u_1 = \omega_{jx} \left( \frac{a}{2} \right) \cdot \frac{P}{E} \cdot \frac{h}{2} \int_{0}^{\frac{a}{2}} x \cdot dx \]

\[ u_2 = \frac{a}{2} \int_{0}^{\frac{a}{2}} \frac{P}{E} \cdot \frac{h}{2} \cdot dx \]

\( \bar{h} \cdot E = \text{effective depth and modulus.} \)
\[ U_3 = \frac{1}{2} \int_0^a (\alpha_x^2 \, dx) \]

\[ U_4 = \int_0^a \omega_{yx} \, t_x \, dx \]

\[ t(x) = \text{the neutral axis curve.} \]

Consider the two limiting cases:

1. \( P = 0 \);
   \( u = u_1 - u_3 + u_4 \)  \hspace{1cm} (Upper bound)  \hspace{1cm} (A1.3)

2. \( P(f_c^+) \)
   where \( P(f_c^+) \) is the compression needed to prevent
tensile cracks. Then if \( u_i \) and \( u_4 \) are assumed to
remain approximately the same for a given deflection, get

\[ u = u_1 + u_2 - u_3 \]  \hspace{1cm} (Lower bound)  \hspace{1cm} (A1.4)

1. Upper bound \( P = 0 \)
   
   a. For complete linear range  \( E_c \approx E_{co} \)  \hspace{1cm} (Fig. A1.2)

\[ f_c^r \leq \frac{1}{3} f_c^w \quad f_c \leq \frac{1}{3} f_c^w \]

If cracked (Fig. A1.1(a))

\[ M_{max}' \approx \frac{f_c^w d^2 t^a}{c} (1 - \frac{1}{3}t) \]  \hspace{1cm} (A1.5)

\[ t' = -r h + \left( r^2 h^2 + 2 r h \right)^{1/2} \quad h = \frac{E_5}{E_{co}} \]  \hspace{1cm} (A1.6)

If uncracked \( d = 0.75d_1 \) (Fig. A1.1(b))

\[ M_{max}'' = \frac{f_c^w}{E(1-t')d_1} \left[ \frac{1}{3} d_1^3 - d_1^3 (1-t')^2 + A_3(n-1) \left( (1-t')d_1 - 0.25d_1 \right)^2 \right] \]  \hspace{1cm} (A1.7)

For \( r = 1\% \quad n = 10 \) get

\[ t' = 0.36 \quad t'' = 0.52 \]

\[ M_{max}' = f_c^w d^2 0.0528 \quad M_{max}'' = f_c^w d^2 0.048 \]

IF \( \alpha = \frac{\pi}{a} \)

\[ \theta \cdot \alpha \]

\[ \end{align*} \]
**Fig. A1.1:** Stress and strain distribution for the linear elastic stage.

**Fig. A1.2:** Linear elastic simply supported beam.

**Fig. A1.3:** Simply supported beam under ultimate load.
b. For ultimate moment (Fig. A1.3)

For

\[ t_0 = \pi \frac{f_y}{f_c} \]

\[ m_m = M_0 = f_y d^2 t_0 (1 - t) \]

For \( r = 1\% \); \( t_0 = 0.138 \)

\[ M = M_0 \cos \alpha \]

\[ t \approx (0.69 - t_0) \cos \alpha \]

\[ \alpha \approx \frac{4 \pi}{3} \]

IF

\[ \varepsilon_c = E_\text{c} \cos \alpha \]

\[ E_\text{c} = 0.38\% \]

GET

\[ \omega_{xx} = \frac{1}{R} = \frac{M}{EI} = \frac{E_c}{(69 - t_0) d} - \frac{E_\text{c} \cos \alpha X}{0.69 d - d(0.69 - t_0) \cos \alpha X} \]

\[ \approx \frac{0.055}{a d} \left( \frac{a}{2} - X \right) \]

\[ \omega = \frac{0.055}{b} \frac{a}{d} \left( \frac{a}{2} - X \right) - \frac{0.055}{b} \frac{1}{4 d} \left( \frac{a}{2} - X \right)^3 \]

FOR \( X = 0 \)

\[ \omega_0 = \frac{0.055}{24} \frac{a^2}{d} \]

SINCE \( \frac{3}{a} = 0.3 d \)

\[ U_1 = \omega_{xx} \left( \frac{a}{2} \right) \frac{2}{3} = \frac{0.055}{b} \frac{a}{d} \frac{1}{3} \times \frac{1}{98.5} \]

\[ U_3 = \frac{1}{2} \int_0^{a/2} \omega_{xx}^2 \, dx = \frac{1}{2} \left[ \frac{0.055}{8} \frac{a}{d} + \frac{0.055}{2 a d} \left( \frac{a}{2} - X \right)^2 \right] \, dx = \frac{1}{39.5} \frac{a^2}{d} \frac{a}{400} \]

\[ U_4 = \frac{3}{2} \left[ \frac{0.055}{8} \frac{a}{d} + \frac{0.055}{2 a d} \left( \frac{a}{2} - X \right)^2 \right] \alpha_2 d \left( 0.69 - t_0 \right) \sin \alpha \, dx = \frac{1}{380} \frac{a}{q} \]
2. Lower bound \( P = P(f_c^+) \)

\[
\begin{align*}
U_1 &= \frac{1}{485} a \\
U_2 &= \frac{P(f_c^+)}{\frac{h}{E}} \\
U_3 &= \frac{1}{395(d^2)} \frac{a}{400} \\
U_4 &= 0
\end{align*}
\]

Comparison of the different components are given in Fig. (A1.4)
The displacement due to elastic shortening is not shown.

**Fig. A1.4:** The various components of the horizontal displacement \( u \) as a function of the central deflection for a simply supported beam.
It is seen that the calculations of $u_4$ are involved with solution for the actual stiffness,

$$\bar{D} = E(x,y,p) I(x,y,p)$$  \hspace{1cm} (A1.14)

where $E$, $I$ are variables. The calculation of $u_4$ with uncracked stiffness introduces inconsistent error.

Thus, for the horizontally restrained R/C slab, instead of such a solution (which is not available even for small deflection theory), ideal elasto-plastic moment-curvature relations are used. Fig. (A1.5).

In the ideal elastic range,

$$\bar{D} = D = \text{CONST}.$$  

$$u_4 = 0$$

For the ideal plastic range, the effect of cracking is considered in the rigid-plastic solution.

$$\Delta P = f(\Delta \theta)$$

where $P = \text{axial compression}$. Actually, this assumption introduced a small error only, since:

1) For the thick slab the condition $P > P(f_c^+)$ is almost satisfied and therefore $u_4 = 0$ up to the yield criteria.

2) For the thin slab the condition $P > P(f_c^+)$ is satisfied through the major part of the stress path, and only a limited region near the center of the slab is cracked. However, even here the axial compression is high enough, so that the cracking effect is small in comparison to the other factors, while the effect of non-linear geometry is always higher.
**Fig A1.5:** Idealized Moment Curvature Relations
3) Even at a later stage, where the neutral axis at the center of the slab is advanced toward the compression face of the slab, and arching action is developed at the center, the surrounding region remains uncracked and acts as restraining frame. On the boundary, the horizontal movement due to cracking is small for loads that are below the ultimate load.

The cracks that occur prior to the yielding of the steel are not associated with significant rotation of the cross section, or a rise of the neutral axis. On the other hand, significant plastic rotation occurs due to the yielding of the steel which is accompanied by a rise of the neutral axis. The plastic rotation is associated therefore with significant horizontal expansion at the reinforcement level, which lead to compression membrane action.

When plastic conditions \((f=1)\) are developed in the vicinity of the center of the slab, the effect of cracking is taken into account by using an additional rigid-plastic solution for the dissipation of moments that are higher than the yield moment.

Thus, instead of a solution for the complete non-linear equations, including variable stiffness and non-linear geometry, for different boundary conditions, the problem is reduced to the solution of a fundamental system, consisting of ideal linear elastic-perfectly plastic material, and a simply supported slab. Any desired boundary conditions are solved by using the fundamental force-displacement relations.
APPENDIX II

Experimental Data

The experimental aspect of the research was carried under the supervision of Professor M. J. Holley and Professor J. F. Brotchie. The research group included Mr. S. Okubo, Mr. S. Kokkins, Mr. R. Fowler and the author, as research assistants, and Mr. E. McCaffrey and Mr. J. Jones as laboratory assistants. The author wishes to thank the supervisors and the members of the group for their help and suggestions, and for the opportunity to use the test results for his thesis.

A2.1. Test apparatus and equipment. (Fig. A2.1)

The equipment used for the experiments is as follows:
A2.1.1. Test apparatus, which provides means for loading and support of the model (Fig. A2.2)
A2.1.2. Support and restraining equipment (Fig. A2.3)
A2.1.3. Hydraulic loading system (Fig. A2.4)
A2.1.4. Recording system. (Fig. A2.5)

A2.1.1 Test apparatus, consists of: 1. Service stand for the loading device, built from mild steel sections. 2. Test loading device, built from welded plates machined where necessary.

The Test loading device includes:

a. A rigid bed fabricated from plates welded into a box shape, with an upper machined platform for vertical and horizontal support of the slabs.

b. A movable head, consisting of a rigid welded plate, with provisions for connection of the loading cell.

The movable head was connected to the base bed with 4 high-strength 2" bolts, that served as a guide for the movable head, and
transmitted a tensile force between the head and the base during testing so that a self contained loading system was created. The connection of the loading device to the stand was not rigid, and allowed rotation.

A2.1.2 Support and restraining equipment.

The slab was supported vertically on a 3/4" solid steel shaft, that allowed free rotation and translation along the boundary. The horizontal restraint was provided by 24 restraining cells reacting against the sides of the base bed, 6 along each boundary. Six of the cells along one boundary in each direction were loading cells, with active strain gages. The cells were fitted into a groove at the steel edge of the slab by taking up on an adjusting nut on the cell.

A2.1.3 Hydraulic loading system.

The hydraulic fluid was stored in a reservoir, from which it was pumped by a hand pump through high-pressure hoses into the loading cell. The loading cell contained a top steel plate bolted to the movable head, and a bottom plate with an opening for a soft rubber membrane. The bottom plate was bolted to the top plate with high strength bolts, while the rubber membrane, glued to the base plate with epoxy resin, was locked between the two plates. An access pipe led to low and high range pressure guages, and a bleed release was provided to release the hydraulic fluid back into the reservoir.

A2.1.4 Recording system.

Two kinds of recording systems were used:

a. Recording of deflections by dial gages. One central gage to measure the vertical deflection, and two others located near the supports, in order to indicate support settlements. Where unrestrained slabs were tested, two horizontal dial gages were located at the bottom reinforcement on the center of the slab.
b. Recording of axial compression at each individual cell along the boundary, by means of strain gages mounted on the cell. Two Wheatstone bridges were used, so that the active gages were located on the thinner part of the cell, while the dummy gages - for temperature compensation - were located on the thick part. The gages were located so that stresses caused by bending were cancelled. In order to increase the sensitivity, two separate readings were taken for each cell. The cells were connected to a power supply set of 30 volts, and the recording was done by recording oscillograph on C.E.C. paper, from which the results were reduced later.

Each one of the loading-cells was calibrated in order to establish load-displacement relations for the cell.

A2.2. Materials

A2.2.1 Similitude requirements

A technique for modeling reinforced concrete structures with small scale reinforced mortar (micro-concrete) models, was developed in the M.I.T. Civil Engineering model laboratory. Since similitude conditions indicate that the strain components must be the same in model and prototype once non-linear elasto-plastic material is considered, it was found that from all the materials available for modeling of R/C structures, reinforced mortar will function the best.

A2.2.2 The micro-concrete

The mix was designed by a trial and error procedure, according to the general ideas suggested by the M.I.T. research group. High early strength cement - "Velo type III" was used. The aggregate used was New Jersey fine-mortar sand, graded between sieve No. 8 and No. 200,
so that actual concrete aggregate is modeled to 1:10 scale. The proportions by weight were: water 0.70-0.72, cement 1.00, sand 3.6-3.7. ASTM C305 mixing method was used. Curing was done in a moisture room at 75°F, 95% humidity. Molds were stripped after 24 hours. The slabs were tested after 21 days, and were removed from the moisture room 24 hours before testing.

Since the problem of small scale models involves considerations of size effect, and the strength of the concrete is estimated by means of control specimens (cylinders with 1:2 diameter to length ratio), a few cylinder sizes were tried: 1/2", 3/4", 1", 1 1/2", 2", 3". Six specimens were tested for each diameter. From the results of these tests it was seen that only 4 sizes for each slab are sufficient to describe the variation of the strength as a function of the cylinder size: 1", 1 1/2", 2", 3". Using statistical analysis, the ultimate strength of each cylinder for a given slab was determined.

The average ultimate strength for each slab as a function of the cylinder size is given in Fig. (A2.6 + A2.7.).

Stress strain relations for the concrete cylinders were measured with SR-4 gages located on the cylinders. The results for slab No. 4 are given in Fig. (A2.8.).

2.3 The reinforcement

Uncoated mild steel wires were used as a reinforcement. Since the wires were cold drawn prior to installation as reinforcement, the yield range was eliminated from the stress strain relations. Thus, in order to model the actual properties of the reinforcement, the wires were annealed at high temperature of 1200° 1300°F for 20 minutes. The strain-stress relation for the wires that were used are given in Fig. (A2.9). The results are an average of 4 samples tested. The results were checked randomly in the process of preparation of tests, and the variations were found to be negligible. Since spot-welding
technique was used for the fabrication of the reinforcement, the effect of welding was tested, and also found to be negligible.

The wires used were S.W.G #13 (d = 0.090 in.) for 3/4" slabs, S.W.G #10 (d = 0.135") for 1 1/2" slabs, and S.W.G #5 (d = 0.210") for 3" slabs. The spacing and the location of the wires is given in the List of Slabs, table (A2.2).

2.4 Fabrication method (Fig. A2.10)

The edge-piece, designed to serve as a vertical and horizontal hinge, was mounted in a steel molding box. The reinforcement wire was spot welded at a few intersections and welded to the edge piece to prevent bond failure. The edge piece was designed so as to prevent local shear failure due to stress distribution near the support.

The desired concrete mix was poured upright in the oiled steel forms and vibrated by hand vibrator on the steel form until consistency of the mix was uniform. At the same time the cylinder specimens were prepared and rodded with steel rod. The forms were stripped after 24 hours. The slab and cylinders were in curing room up to 24 hours before testing. The total time from casting to testing was 21 days.

A.2.3. Test procedure

A2.3.1 Loading method and observations

The prepared slab was put into the apparatus, supported on the vertical supporting system. The restraining pieces were put freely in place without being tightened. Then recording of restraining cells without any load was done, and marked as first "0". Next the restraining cells were made finger tight. The slab was covered on the top with teflon and neoprene sheets in order to protect the rubber membrane as shown in Fig. (A2.11).
The movable head was lowered to contact the slab, while free rotations of the slab are possible. In this stage deflections were recorded on the dial gages for "0" load. At the same time second "0" reading was recorded for the restraining cells. Pressure was applied slowly with the hand pump, to ensure low rate of loading. The pressure increments for \( d_1 = 3/4" \) was 4 psi, for \( d_1 = 1 1/2" \) 10 psi, and for \( d_1 = 3" \) 50 psi. Each increment was done over an interval of 2 - 3 min. approximately. At ultimate load a special reading was recorded. Since there was some difference between the load at the beginning of a reading operation and the load at the end of such reading (the time interval for reading was 10 sec. approximately) the load was recorded at the beginning and at the end of the reading interval. A simple interpolation procedure was used for the interpretation of this effect.

When the deflection at the center of the slab was 2.3" approximately, the load was released, and "0" load was recorded. The restraining cells were released, and a second final zero was recorded for these cells.

During the tests, observation of the crack patterns was done and recorded by free drawing at certain critical stages.

Once the movable head was lifted up, photos of the top and bottom of the slab were taken.

The results were reduced by using the I.B.M. 1620. Plotting was done on a Gerber VP600 Plotter.

A2.4. Interpretation of the results.

Only the results associated with tests without shear failure are considered here. In the first series of tests (slabs #1=#26) shear failure occurred due to diagonal tension at the internal regions, or due to local failures near the supports caused by stress concentrations. Special precautions (reinforcement and details) were taken from slab #27 on.
A2.4.1 Sources of error

1. The load was applied perpendicular to the slab surface, thus with large deflections the load is not representative of vertical distributed load.

2. The rate effect associated with the loading of the small scale model was dictated by the desire to obtain results for static load. Due to the limited means, no additional investigations were done to correlate this effect.

3. Creep deformations were small for the elastic stage. However, it appears to be more significant in the elasto-plastic and plastic stages. Recording was done in too short an interval to achieve complete equilibrium with zero velocity.

4. The kinematic motion of the supporting and restraining system was associated with some small changes of the restraining cell geometry.

5. There was small rotational restraint due to some contact between the loading-cell and the edges of the slabs.

All these errors resulted from simplification of the test procedures, and have a systematic nature. The magnitude of these errors, however, is estimated to be very small for the range of behavior that was studied here and in comparison with the order of magnitude of the measured data.

The accuracy of the recording system was designed so, that the "noise level" of the recording equipment will be reduced to a minimum. Thus, the random errors that occurred due to the recording method is negligible. Random errors that are associated with the statistical nature of the physical model or the measured properties are not known.
A2.4.2 The statistical problem.

The mean value used for the material properties was obtained by statistical analysis. While the standard deviation for the stress-strain relations of the reinforcement wires was \( \sigma_m \leq \pm 3\% \), it varied for the concrete according to the cylinder sizes, being approximately \( \sigma_m \leq \pm 5\% \) for 3" \( \phi \) cylinders, and \( \sigma_m \leq \pm 10\% \) for 1" \( \phi \) cylinders.

Unfortunately, due to time and cost considerations, the model study was limited to one test for a particular type of slab and edge details. Thus, statistical analysis is impossible. Due to this fact, much effort was spent in order to eliminate the blunders and major systematic errors. Those errors listed in section 4.1 are hopefully the least of the worst that one can expect.

A2.4.3 The choice of parameters

In addition to the above mentioned considerations, the parameters are chosen as follows.

1. Results from the recording system are accepted as they appear. Precautions taken during the tests hopefully reduced the amount of error to a minimum.

2. Stress strain relations for steel wires are obtained by statistical analysis, with a satisfactory degree of uniformity. The mean values are used in the analysis.

3. The mean value of concrete for each slab is given in Fig. A2.6 - A2.7. Interpretation of these results lead to an unverified conclusion, that the increment of unconfined strength of the concrete cylinders from 3" \( \phi \) to 1 1/4" \( \phi \) is associated with some statistical size effect. The reduction of the strength from 1" \( \phi \) to 1/2" \( \phi \) is interpreted as some systematic error introduced due to the microscopic nature of the cylinder size and the nature of the test. One remembers, however, that these results represent merely unconfined compression test of concrete cylinder.
This interpretation leads to the decision to use the mean value of 3" \( \phi \) cylinder for \( d_1 = 3" \), 1 1/2" \( \phi \) for \( d_1 = 1 1/2" \), and 1" \( \phi \) for \( d_1 = 3/4" \). The values used for the analysis are averages for slabs with \( r = 1\% \) and \( r = 3\% \), for series of tests starting at slab #27. (Table A2.1)

<table>
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<th></th>
<th>( d_1 = 3/4&quot; )</th>
<th>( d_1 = 1 1/2&quot; )</th>
<th>( d_1 = 3&quot; )</th>
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<tr>
<td>( f'_{c} ) (ksi)</td>
<td>5100</td>
<td>4200</td>
<td>3500</td>
</tr>
<tr>
<td>( f'' \approx 0.85 f'_{c} )</td>
<td>4350</td>
<td>3570</td>
<td>3000</td>
</tr>
<tr>
<td>( f_y ) (ksi)</td>
<td>60000</td>
<td>55000</td>
<td>53500</td>
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Table A2.1: Average parameters of the mechanical properties of the materials used for the analysis.
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<tr>
<th>SLAB No.</th>
<th>r%</th>
<th>d₁ (in.)</th>
<th>REINF. SIZE (S.W.G.)</th>
<th>REINF. SPACING (in.)</th>
<th>P₂₅ ULT (kips</th>
<th>(β₂₅) ULT (in.)</th>
<th>φ₀ max/α₀ (kip/in.)</th>
<th>FAILURE MODE AT ULT. LOAD</th>
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<td>--</td>
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<td>220.0</td>
<td>0.17</td>
<td>**</td>
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</tr>
</tbody>
</table>

(*) = "CONTINUOUS" SLABS WITH BOTTOM REINFORCEMENT ONLY.

(*** ) = TWO WIRES LOCATED TOGETHER

**TABLE A2.2:** LIST OF SLABS.
A2.5. Data

A2.5.1 LOAD Deflection Relations

The curves are grouped according to the thickness and the support condition (simply supported unrestrained, simply supported restrained and "continuous" with bottom reinforcement only). The results are given in Fig. (A2.12) - (A2.17).

A2.5.2 Load-restraining force relations

The figures are given separately for each simply supported restrained slab. The three curves on each figure represent average of 4 similar restraining cells.

The origin of the curves is associated with three different procedures:

(1) The curves start with non-zero restraining force. This corresponds to "prestressing" of the cells without the applied load.

(2) The curves start from (0) restraining force, since initial "0" reading was not taken.

(3) The curves start with dotted line. This corresponds to arbitrary value introduced while plotting, in order to eliminate a final negative value, obtained due to errors of the recording system. The results are given in Fig. (A2.20) - (A2.36).
FIG. A2.1: TEST APPARATUS AND EQUIPMENT.
FIG. A2.2: TEST APPARATUS.
FIG. A2.3: SUPPORT AND RESTRAINING EQUIPMENT.
FIG A2.4: HYDRAULIC LOADING SYSTEM.

FIG A2.5: RECORDING SYSTEM.
**Fig. A26:** Compression strength - cylinder size relations.
FIG A2.7: COMPRESSION STRENGTH - CYLINDER SIZE RELATIONS
FOR "CONTINUOUS" SLABS.
Fig. A2.8: Typical stress-strain relations for various cylinder sizes.
FIG. A2.9: MEAN ENGINEERING STRESS-STRAIN CURVE FOR THE REINFORCEMENT.
FIG. A2.10: ARRANGEMENT OF REINFORCEMENT.
FIG A2.11: SLAB-LOADING CELL DETAILS.
FIG. A2.12: LOAD-DEFLECTION RELATION
(3/4 INCH THICKNESS SIMPLY SUPPORTED SLABS)
FIG. A2.13: LOAD-DEFLECTION RELATION
(3/4 INCH THICKNESS RESTRAINED SLABS II)
FIG. A2.14: LOAD-DEFLECTION RELATION
(3/4 INCH THICKNESS CONTINUOUS SLABS)
FIG. A2.15: LOAD-DEFLECTION RELATION
(1.5 INCH THICKNESS SIMPLY SUPPORTED SLABS)
FIG. A2.16: LOAD-DEFLECTION RELATION
(1.5 INCH THICKNESS RESTRAINED SLABS II)
FIG. A2.17: LOAD-DEFLECTION RELATION
(1.5 INCH THICKNESS CONTINUOUS SLABS)
FIG. A2.18: LOAD-DEFLECTION RELATION
(3.0 INCH THICKNESS SIMPLY SUPPORTED SLABS)
FIG. A2.19: LOAD-DEFLECTION RELATION (3.0 INCH THICKNESS RESTRAINED SLABS II)
SLAB NO. 25

A. AVERAGE OF CELLS 1.6.7.12

B. AVERAGE OF CELLS 2.5.8.11

C. AVERAGE OF CELLS 3.4.9.10

APPLIED LOAD (PSI)

LOAD-RESTRAINING FORCE RELATION

FIG A2.20: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 26
(3.0 INCH 0.0 R)
A. AVERAGE OF CELLS 1, 6, 7, 12
B. AVERAGE OF CELLS 2, 5, 8, 11
C. AVERAGE OF CELLS 3, +, 9, 10

FIG. A2.21: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 27
(1.5 INCH 0.0 R)

A. AVERAGE OF CELLS 1, 6, 7, 12
B. AVERAGE OF CELLS 2, 5, 8, 11
C. AVERAGE OF CELLS 3, 4, 9, 10

RESTRRAINING FORCE (LBS)

APPLIED LOAD (PSI)

FIG. A2.22: LOAD-RESTRRAINING FORCE RELATION
SLAB NO. 28

(AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

APPLIED LOAD (PSI)

FIG. A2.23: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 29
(1.5 INCH 1.0 R)
A. AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

FIG. A2.24: LOAD-RESTRAINING FORCE RELATION
FIG. A2.25: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 31

(1.5 INCH 2.0 R)

A. AVERAGE OF CELLS 1, 6, 7, 12
B. AVERAGE OF CELLS 2, 5, 8, 11
C. AVERAGE OF CELLS 3, 4, 9, 10

APPLIED LOAD (PSI)

RESTRAINING FORCE (LBS)

FIG A2.26: LOAD-RESTRAINING FORCE RELATION
FIG. A2.27: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 33

(.75 INCH 0.5 R)

A. AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

APPLIED LOAD (PSI)

FIG. A2.28: LOAD–RESTRAINING FORCE RELATION
SLAB NO. 34
(.75 INCH 1.0 R)

A. AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

FIG. A2.29: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 35
(.75 INCH 2.0 R)

A. AVERAGE OF CELLS 1, 6, 7, 12
B. AVERAGE OF CELLS 2, 5, 8, 11
C. AVERAGE OF CELLS 3, 9, 10

FIG. A2.30: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 36

A. AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

APPLIED LOAD (PSI)

RESTRAINING FORCE (LBS)

FIG. A2.3f: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 37
(3.0 INCH 0.0 R)

A. AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

FIG. A2.32: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 38

(3.0 INCH 0.5 R)

A. AVERAGE OF CELLS 1.6.7.12
B. AVERAGE OF CELLS 2.5.8.11
C. AVERAGE OF CELLS 3.4.9.10

Fig. A2.33: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 39

AVERAGE OF CELLS 1.6.7.12

AVERAGE OF CELLS 2.5.8.11

AVERAGE OF CELLS 3.4.9.10

FIG. A2.34: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 40
(3.0 INCH 2.0 R)

A. AVERAGE OF CELLS 1, 6, 7, 12
B. AVERAGE OF CELLS 2, 5, 8, 11
C. AVERAGE OF CELLS 3, 4, 9, 10

APPLIED LOAD (PSI)

RESTRAINING FORCE (LBS.)

FIG.A2.35: LOAD-RESTRAINING FORCE RELATION
SLAB NO. 41
(3.0 INCH 3.0 R)
A. AVERAGE OF CELLS 1, 6, 7, 12
B. AVERAGE OF CELLS 2, 5, 8, 11
C. AVERAGE OF CELLS 3, 4, 3, 10

FIG. A2.36: LOAD-RESTRAINING FORCE RELATION
The author was born in Tel-Aviv, Israel, on January 15, 1931. In 1948 he was graduated from New High School in this city. In the same year he joined the Israeli Army serving on active duty in the Israeli Independence War and continues to serve as a Captain of Artillery in the reserves. In 1951 he entered "Technion, Israel Institute of Technology", where he received a B.Sc. degree in Civil Engineering in 1954, and a C.E. degree in 1955. He was then employed as a Design Engineer by "Fertilizers and Chemicals," Haifa, Israel. In 1956 he was employed as a senior design engineer by "Hamat", Israel where he became chief engineer in 1958. During this period, he practiced as a consulting engineer with many of the leading organizations in Israel.

In 1960 he began studies at "Technion", under the auspices of a "Hamat" grant. In 1961 he was invited by the faculty of Civil Engineering to serve as a part time instructor for structural design. In 1961 he received the degree of M.Sc. In 1962 he came to M.I.T. and since 1963 has been a research assistant in the Department of Civil Engineering. The author is a member of Tau-Beta-Pi and Sigma Xi.